Dynamics Formula Sheet

Quiz I

Kinematics

$$\vec{v} = \frac{dx}{dt}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d\vec{x}}{dt}$$

Note: Using the chain rule, $\vec{a}$ can be rewritten

$$\vec{a} = \vec{v}\frac{d\vec{x}}{dt}$$

Constant Acceleration Kinematics

$$\vec{v} = \vec{v}_o + \vec{a}_o t$$

$$\vec{v}_f^2 = \vec{v}_o^2 + 2\vec{a}_o \Delta\vec{x}$$

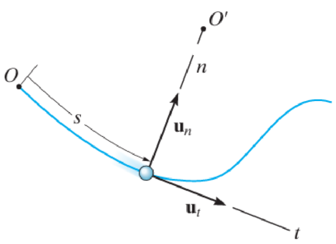
$$\vec{x} = \vec{x}_o + \vec{v}_o t + \frac{1}{2}\vec{a}_o t^2$$

Relative Motion

$x_{B/A} = x_B - x_A$ B relative to A

Note: Subscripts cancel $x_B = x_A + x_{B/A}$

Normal and Tangential



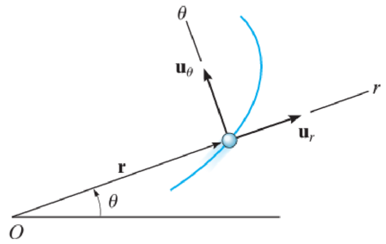
$$\vec{v} = v\hat{u}_t$$

$$\vec{a} = \frac{dv}{dt}\hat{u}_t + \frac{v^2}{\rho}\hat{u}_n$$

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left|\frac{d^2y}{dx^2}\right|} \quad \text{(Radius of curvature)}$$

Note: In 3d, define $\hat{u}_b = \hat{u}_t \times \hat{u}_n$

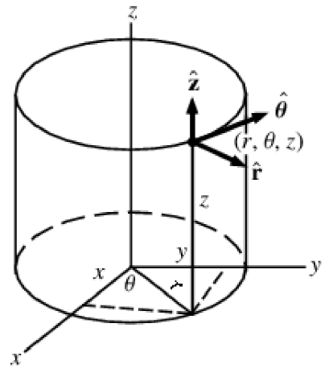
Cylindrical (Radial and Transverse)



$$\vec{v} = \dot{r}\hat{u}_r + r\dot{\theta}\hat{u}_\theta$$

$$\vec{a} = \left[\ddot{r} - r\dot{\theta}^2\right]\hat{u}_r + \left[r\ddot{\theta} + 2\dot{r}\dot{\theta}\right]\hat{u}_\theta$$

Note: In 3d, add a $\hat{u}_z$ where $\vec{r}_z = z\hat{u}_z$,$\vec{v}_z = \dot{z}\hat{u}_z$ and $\vec{a}_z = \ddot{z}\hat{u}_z$



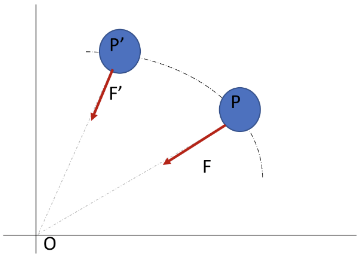
Orbital Mechanics

Newton's Law of Gravitation

$$F_g = \frac{GMm}{r^2}$$

Differential equation (for shape of orbital trajectory):

Derived by applying **central force motion**



$$\begin{aligned} \sum F_r &= ma_r \\ -F &= m(\ddot{r} - r\dot{\theta}^2) \end{aligned} \qquad \begin{aligned} \sum F_\theta &= ma_\theta \\ 0 &= m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) \end{aligned}$$

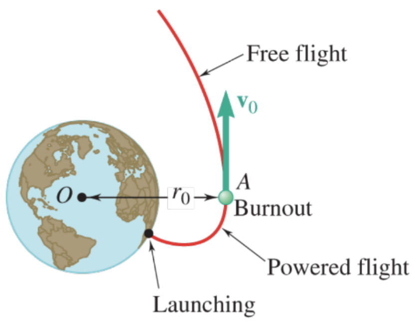
Rewriting $-F = m(\ddot{r} - r\dot{\theta}^2)$ only in terms of θ using $h = r^2\dot{\theta}$:

$$\boxed{\frac{d^2u}{d\theta^2} + u = \frac{GM}{h^2}}$$

Where:

- $u = 1/r$, r is distance from centers of mass
- $h = r^2\dot{\theta}$, angular momentum per unit mass (where angular momentum $= H_o = hm = rmv_\theta$)
- $G = 6.67 \times 10^{-11} m^3/kgs^2$
- M is the mass of the central body

Solutions to this differential equation are conic paths:



$$\boxed{u = \frac{1}{r} = \frac{GM}{h^2} + C\cos\theta = \frac{GM}{h^2}(1 + \varepsilon\cos\theta)}$$

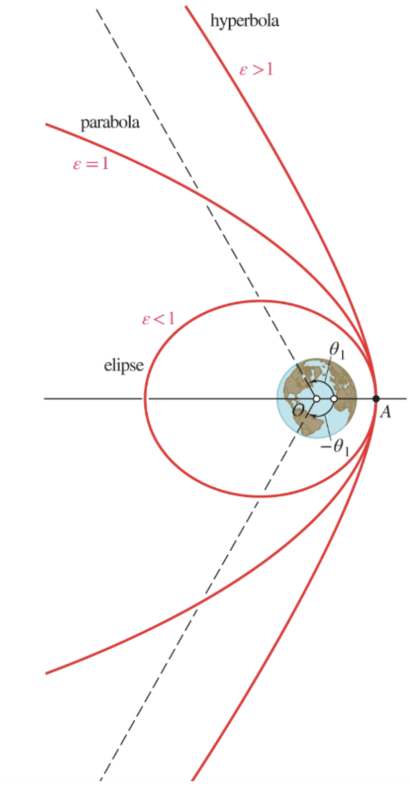
Where C and h are constants defined by the instantaneous r and v at any point along the orbit:

- $h = r^2\dot{\theta} = r_0 v_0$
- $C = \frac{1}{r_0} - \frac{GM}{h^2} = \frac{1}{r_0} - \frac{GM}{r_0^2 v_0^2}$

With **eccentricity**:

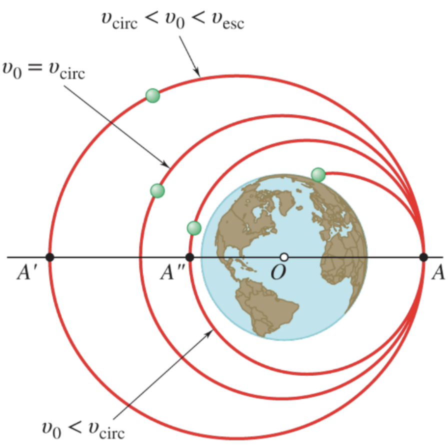
$$\varepsilon = \frac{C}{GM/h^2} = \frac{Ch^2}{GM}$$

<u>Note:</u>



$\varepsilon > 1:$	Hyperbola
$\varepsilon = 1:$	Parabola
$\varepsilon < 1:$	Ellipse
$\varepsilon = 0:$	Circle

Initial conditions for each path



Parabolic path (barely escaping orbit) $\varepsilon = 1$:

$$u = \frac{1}{r} = \frac{GM}{h^2}(1 + \cos\theta)$$

At $\theta = 0$ and $r = r_o$ (at start of free flight):

$$\frac{1}{r_o} = \frac{GM}{r_o^2 v_o^2}(1+1) \quad (h = r_o v_o)$$

$$v_{esc} = v_o = \sqrt{\frac{2GM}{r_o}}$$

<u>Note:</u>

$$v_{esc,Earth} = \sqrt{\frac{2gR^2}{r_o}} \quad \text{where R is the radius of Earth}$$

Circular orbit $\varepsilon = 0$

$$u = \frac{1}{r} = \frac{GM}{h^2}(1)$$

At $\theta = 0$ and $r = r_o$ (at start of free flight):

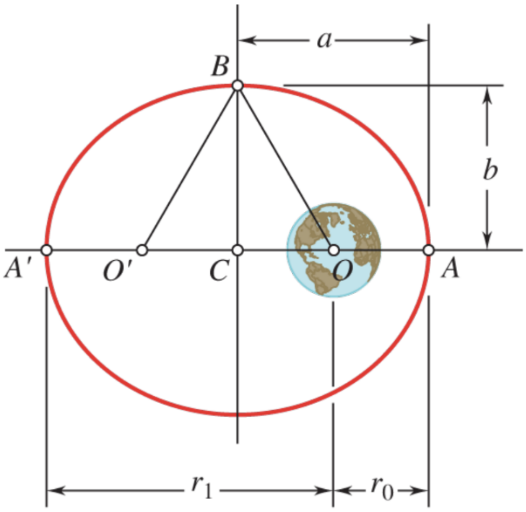
$$\frac{1}{r_o} = \frac{GM}{r_o^2 v_o^2} \quad (h = r_o v_o)$$

$$v_{circ} = v_o = \sqrt{\frac{GM}{r_o}}$$

Note:

$$v_{circ,Earth} = \sqrt{\frac{gR^2}{r_o}} \quad \text{where R is the radius of Earth}$$

Periodic time (τ):



Time to complete 1 orbit:

$$\tau = \frac{\text{Area of elliptical orbit}}{\text{Areal velocity}}$$

From geometry, the major and minor axes are:

$$a = \frac{1}{2}(r_0 + r_1)$$

$$b = \sqrt{r_0 r_1}$$

The area of an ellipse is:

$$A = \pi ab$$

Areal velocity (constant):

$$dA = \frac{1}{2} r^2 d\theta$$

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{h}{2} \quad (h = r^2 \dot{\theta})$$

Thus periodic time is:

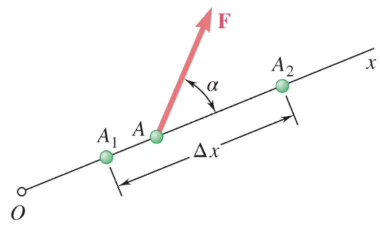
$$\boxed{\tau = \frac{A}{h/2} = \frac{2A}{h} = \frac{2\pi ab}{h}}$$

Energy

Work

$$U_{1\to 2} = \int_{r_1}^{r_2} \vec{F} \cdot d\vec{r}$$

Constant force in rectilinear motion:



$$U_{1\to 2} = Fcos(\alpha)\Delta x$$

Work by gravity/weight

$$U_{g,1\to 2} = -W\Delta y$$

Work by gravity (space)

$$U_{g,1\to 2} = \frac{GMm}{r_2} - \frac{GMm}{r_1}$$

Work by spring

$$U_{s,1\to 2} = -\frac{1}{2}k(x_2^2 - x_1^2) = \frac{1}{2}k(x_1^2 - x_2^2)$$

Work and Energy

$$U_{1\to 2} = T_2 - T_1$$

$$T_2 = T_1 + U_{1\to 2}$$

Work of a force = change in kinetic Energy
Kinetic energy = capacity to do work
<u>Note:</u> $T = \frac{1}{2}mv^2$

Power and Efficiency

Power

$$P = \frac{dU}{dt} = \vec{F} \cdot \vec{v}$$

Efficiency

$$\eta = \frac{P_{out}}{P_{in}}$$

Potential Energy

Gravitational Potential

On Earth

$$V_g = Wy = mgh$$

In space

$$V_g = -\frac{GMm}{r} = -\frac{WR^2}{r}$$

Elastic Potential

$$V_e = \frac{1}{2}kx^2$$

Conservation of Energy

$$T_1 + V_{g1} + V_{e1} + U_{1\to 2}^{NC} = T_2 + V_{g2} + V_{e2}$$