Differential Equations Review Sheet

Exam II

4.1 Linear Equations

General form of a nth order linear differential equation:

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = g(x)$$

Thm: 4.1.1 Existence of a Unique Solution:
For an interval $I : a < x < b$, if the functions $a_n(x), a_{n-1}(x), \ldots, a_0(x)$ and $g(x)$ are continuous on I and $a_n(x) \neq 0$ for all x in I, then there exists a unique solution $y = y(x)$ of the differential equation.

If $x = x_o$ is in I, then a solution that satisfies the initial conditions exists on the interval and is unique.
Note: If $a_n(x) = 0$ for some x in the interval I then there **may not** exist a unique solution.

Boundary Value Problems
If the constraints of a linear differential equation are at *different points* instead of using derivatives at the *same* point then it is known as a **boundary value problem (BVP)** and the constraints are known as **boundary conditions**.

Homogeneous vs Nonhomogeneous nth order ODE
Homogeneous: $a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = 0$

Nonhomogeneous: $a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = g(x)$, where $g(x) \neq 0$

Thm 4.1.2 Superposition Principle - Homogeneous Equations
If $y_1, y_2, \ldots, y_k$ are solutions to the *homogeneous* nth-order linear differential equation on an interval I, then the linear combination where $c_1, c_2, \ldots, c_n$ are arbitrary constants.

$$y = c_1y_1(x) + c_2y_2(x) + \cdots + c_ky_k(x)$$

is *also* a solution on the interval.

Fundamental Set of Solutions
There exists a fundamental set of solutions $\{y_1, y_2, \dots, y_n\}$ to the homogeneous nth-order linear differential equation. The fundamental set of solutions has to be *linearly independent*.

We test for linear independence using the **Wronskian**:

The solutions $\{y_1, y_2, \dots, y_n\}$ are linearly independent on the interval I if and only if

$$W(y_1, y_2, ..., y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ Dy_1 & Dy_2 & \cdots & Dy_n \\ \vdots & \vdots & & \vdots \\ D^{n-1}y_1 & D^{n-1}y_2 & \cdots & D^{n-1}y_n \end{vmatrix} \neq 0$$

for every x in the interval.

Note: D is the differentiation operator, i.e. $Dy = \frac{dy}{dx}$, $D^2y = \frac{d^2y}{dx^2}$, and so on.

Thm 4.1.5 General Solution — Homogeneous Equations
If $y_1, y_2, \dots, y_n$ is a fundamental set of solutions to the homogeneous nth-order linear differential equation on an interval I, then the general solution is given by

$$y = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x)$$

where $c_1, c_2, \dots, c_n$ are arbitrary constants.

Thm 4.1.6 General Solution — Nonhomogeneous Equations
If y_p is a particular solution to the nonhomogeneous nth-order linear differential equation on an interval I, and if y_h is the general solution to the corresponding homogeneous equation, then the general solution to the nonhomogeneous equation is given by

$$y = y_h + y_p$$

4.2 Reduction of Order
Homogeneous, standard form rewrite for formula

We know that the general solution of a homogeneous linear second-order differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

is a linear combination

$$y = c_1y_1 + c_2y_2$$

where y_1 and y_2 are solutions that constitute a linearly independent set on some interval I.

Given that the differential equation only has constant coefficients, if we know one solution y_1, we can find a second solution y_2 using the method of **reduction of order**.

Method:
Since y_1 and y_2 are linearly independent, then their quotient y_2/y_1 hat to be nonconstant on I. In other words,

$$\frac{y_2(x)}{y_1(x)} = u(x)$$
$$\implies y_2(x) = u(x)y_1(x)$$

We can find the function $u(x)$ by substituting $y_2(x) = u(x)y_1(x)$ into the original differential equation and solving for $u(x)$.

<u>Note:</u> This method reduces the differential equation from second-order to first-order in terms of $w = u'(x)$. But, it only works for second order ODE.

Solving for $u(x)$ for a General Case:
For a second-order linear differential equation with constant coefficients:

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

We put into general form by dividing through by $a_2(x)$:

$$y'' + P(x)y' + Q(x)y = 0$$

Assuming we know one solution y_1, we let $y_2 = u(x)y_1(x)$. Then we compute the first and second derivatives of y_2:

$$y_2' = u'y_1 + uy_1'$$
$$y_2'' = u''y_1 + 2u'y_1' + uy_1''$$

Substituting y_2, y_2', and y_2'' into the original differential equation gives:

$$u\left[y_1'' + P(x)y_1' + Q(x)y_1\right] + u''y_1 + u'(2y_1' + P(x)y_1) = 0$$

Since y_1 is a solution to the original equation, $y_1'' + P(x)y_1' + Q(x)y_1 = 0$.

$$\implies u''y_1 + u'(2y_1' + P(x)y_1) = 0$$

Letting $w = u'$, we have a linear first order differential equation in w:

$$w'y_1 + w(2y_1' + P(x)y_1) = 0$$

We can use an integrating factor to solve for w:

<u>Recall:</u> Given a first-order linear ODE of the form

$$\frac{dy}{dx} + p(x)y = g(x)$$

the integrating factor is given by

$$\mu(x) = e^{\int p(x)\,dx}$$

and the solution is

$$y(x) = \frac{1}{\mu}\int \mu g(x)\,dx$$

Thus,

$$w' + w\frac{2y_1' + P(x)y_1}{y_1} = 0$$

$$\mu(x) = e^{\int \frac{2y_1' + P(x)y_1}{y_1}\,dx}$$

$$\int \frac{2y_1' + P(x)y_1}{y_1}\,dx = \int 2\frac{y_1'}{y_1}\,dx + \int P(x)\,dx, \quad u = y_1, du = y_1'dx$$

$$2\int \frac{1}{u}\,dx + \int P(x)\,dx$$

$$2ln(y_1) + \int P(x)\,dx$$

$$\implies \mu(x) = e^{2\ln(y_1) + \int P(x)\,dx} = y_1^2 e^{\int P(x)\,dx}$$

Finally, we can solve for w:

$$w(x) = \frac{1}{\mu(x)}\int \mu(x)\cdot 0\,dx = \frac{c_1}{\mu(x)} = c_1\frac{e^{-\int P(x)\,dx}}{y_1^2}$$

Then, we can find $u(x)$ by integrating w:

$$u(x) = \int w(x)\,dx = c_1\int \frac{e^{-\int P(x)\,dx}}{y_1^2}\,dx + c_2$$

Choosing $c_1 = 1$ and $c_2 = 0$ for the fundamental set:

$$\boxed{u(x) = \int \frac{e^{-\int P(x)\,dx}}{y_1^2}\,dx}$$

Where $y_2(x) = u(x)y_1(x)$.

4.3 Homogeneous Linear Equations with Constant Coefficients

Homogeneous

For higher order homogeneous linear differential equations with constant coefficients of the form, we can use the characteristic equation to find the general solution.

Method:
Given a homogeneous linear differential equation with constant coefficients:

$$ay'' + by' + cy = 0$$

We assume the solution of the form:

$$\begin{aligned} y(t) &= e^{rt} \\ y'(t) &= re^{rt} \\ y''(t) &= r^2e^{rt} \end{aligned}$$

Then,

$$\begin{aligned} ay'' + by' + cy &= 0 \\ ar^2e^{rt} + bre^{rt} + ce^{rt} &= 0 \\ e^{rt}(ar^2 + br + c) &= 0 \\ ar^2 + br + c &= 0 \end{aligned}$$

We can solve for r using the quadratic formula:

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Note: For higher order differential equations, we would get a polynomial of degree n, and we would solve for the roots by factoring.

Case 1: Real and Unique Roots
For roots r_1 and r_2, the general solution is:

$$\boxed{y(t) = c_1e^{r_1t} + c_2e^{r_2t}}$$

Case 2: Complex Roots
For roots $\alpha \pm \beta i$, the general solution is:

$$\boxed{y(t) = c_1e^{\alpha t}\cos\beta t + c_2e^{\alpha t}\sin\beta t}$$

Case 3: Repeated Roots
For repeated root r, the general solution is:

$$\boxed{y(t) = c_1e^{rt} + c_2te^{rt}}$$

4.4 Undetermined Coefficients (Superposition Approach)
Particular

To find a particular solution Y_p to the higher order nonhomogeneous differential equation with constant coefficients, we will guess the form of Y_p based on the form of $g(x)$.

Method:

- Guess a form of $Y_P(t)$ leaving the coefficient(s) undetermined (and hence the name of the method).
- Plug the guess into the differential equation and see if we can determine values of the coefficients.
- If we can determine values for the coefficients then we guessed correctly, if we can't find values for the coefficients then we guessed incorrectly.

In general, we will guess

$g(t)$	$Y_P(t)$
ae^{bt}	Ae^{bt}
$a\cos(\beta t)$	$A\cos(\beta t) + B\sin(\beta t)$
$a\sin(\beta t)$	$A\cos(\beta t) + B\sin(\beta t)$
$a\cos(\beta t) + b\sin(\beta t)$	$A\cos(\beta t) + B\sin(\beta t)$
n-th degree polynomial	$A_n t^n + \cdots + A_1 t + A_0$

Note: If any term of Y_P is duplicated in y_h then the duplicated term must be multiplied by the minimum + integer power of t required to make all terms in general solution linearly independent. (So find the general homogeneous solution first before guessing the particular solution!)

Then we will substitute our guess for $Y_P(t)$ into the original differential equation and solve for the undetermined coefficients.

4.5 Undetermined Coefficients (Annihilator Approach)
Particular

Another way to solve for a particular solution Y_p to the higher order nonhomogeneous differential equation with constant coefficients is to use the **differential operator** D to create an **annihilator operator** to reduce the ODE.

Annihilator
If L is a differential operator such that

$$L(g(x)) = 0$$

then L is called an **annihilator** of $g(x)$.

Ex:

$$y'' + y' - 6y = 0$$

$$D^2y + Dy - 6y = 0 \implies (D^2 + D - 6)y = 0$$

Here, the differential operator is $D^2 + D - 6$.

Method:

Given a nonhomogeneous differential equation:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = g(x)$$

We can rewrite it using the differential operator L on y such that:

$$L(y) = g(x)$$

Next, we find an annihilator L_1 of $g(x)$ such that:

$$L_1(g(x)) = 0$$

Then, we apply L_1 to both sides of the equation:

$$L_1(L(y)) = L_1(g(x))$$

$$\implies L_1L(y) = 0$$

So by solving the *homogeneous higher-order equation* $L_1L(y) = 0$ we can discover the form of a particular solution y_p for the original *nonhomogeneous equation* $L(y) = g(x)$.

List of Annihilators:

- D^n annihilates polynomials of degree $n - 1$ or less. Thus, it annihilates each of the following:

$$1, x, x^2, \ldots, x^{n-1}$$

- $(D - \alpha)^n$ annihilates each of the following:

$$e^{\alpha x}, xe^{\alpha x}, x^2e^{\alpha x}, \ldots, x^{n-1}e^{\alpha x}$$

- $[D^2 - 2\alpha D + (\alpha^2 + \beta^2)]^n$ annihilates each of the following:

$$e^{\alpha x}\cos\beta x, xe^{\alpha x}\cos\beta x, \ldots, x^{n-1}e^{\alpha x}\cos\beta x$$

$$e^{\alpha x}\sin\beta x, xe^{\alpha x}\sin\beta x, \ldots, x^{n-1}e^{\alpha x}\sin\beta x$$

When going backwards from the annihilator to the function $g(t)$, we look at the roots of the characteristic equation of the annihilator.

For example, the characteristic equation of $D^2 - 2\alpha D + (\alpha^2 + \beta^2)$ is:

$$r^2 - 2\alpha r + (\alpha^2 + \beta^2) = 0$$

$$r = \frac{2\alpha \pm \sqrt{4\alpha^2 - 4\alpha^2 - 4\beta^2}}{2} = \frac{2\alpha \pm 2\beta i}{2}$$

$$\implies g(t) = e^{(\alpha \pm \beta i)t} = e^{\alpha t}(\cos \beta t + i \sin \beta t)$$

<u>Ex:</u> Finding the particular solution for $y'' + y' - 6y = e^{4t}$
Using $y = e^{rt}$, we find the homogeneous solution:

$$y_h = c_1 e^{-3t} + c_2 e^{2t}$$

The annihilator for e^{4t} is $(D - 4)$. Thus, we apply the annihilator to both sides of the equation:

$$(D - 4)(D^2 + D - 6)y = 0$$

Solving the characteristic equation:

$$(D - 4)(D - 2)(D + 3)y = 0$$

The general solution to the higher order homogeneous equation is (matching each annihilator factor to its corresponding $g(t)$ term):

$$y = c_1 e^{4t} + c_2 e^{2t} + c_3 e^{-3t}$$

<u>Note:</u> List EVERY function that the differential operator annihilates. (Ex: $D^2 : Ax + B$)

Thus, we can choose the particular solution to be:

$$y_p = Ae^{4t}$$

Substituting y_p into the original differential equation we can solve for A.

4.6 Variation of Parameters

Particular, standard form rewrite to use formulas

For nonhomogeneous linear differential equations where the method of undetermined coefficients does NOT work (ie. $g(x)$ is not of the right form, and NONconstant coefficients), we can use the method of **variation of parameters** to find a particular solution.

<u>Note:</u> Variation of parameters works for higher order ODEs, but the formulas get more complicated so most examples will be second order ODEs.
IF we are given y_h, the homogeneous solution, we can find y_p, the particular solution, using variation of parameters. However, if we are not given y_h, we must find it first using the characteristic equation method. (Thus, unless given y_h the problems we do will still only be constant coefficients)

Method:

- Obtain general homogeneous solution:

$$y_h = c_1 y_1 + c_2 y_2$$

  For the second order ODE:

$$y'' + p(x)y' + q(x)y = g(x)$$

  Note: the coefficient of y'' **must** be 1, so divide through if necessary.

- The particular solution is of the form:

$$y_p = u_1(x)y_1 + u_2(x)y_2$$

  where we need to solve for $u_1(x)$ and $u_2(x)$.

Where $W(y_1, y_2)$ is the Wronskian of y_1 and y_2:

$$\boxed{u_1 = -\int \frac{y_2 g(t)}{W(y_1, y_2)}\, dt, \quad u_2 = \int \frac{y_1 g(t)}{W(y_1, y_2)}\, dt}$$

Finally, the particular solution is:

$$\boxed{Y_P = -y_1 \int \frac{y_2 g(t)}{W(y_1, y_2)}\, dt + y_2 \int \frac{y_1 g(t)}{W(y_1, y_2)}\, dt}$$

4.7 Cauchy-Euler Equation

Nonhomogeneous differential equations with variable coefficients of the form:

$$a_n x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + \cdots + a_1 x y' + a_0 y = g(x)$$

where $a_n, a_{n-1}, \ldots, a_1, a_0$ are constants, are known as **Cauchy-Euler equations**.

The method for solving Cauchy-Euler equations can be applied to higher order ODEs, but we will focus on second order ODEs.

$$ax^2 y'' + bxy' + cy = g(x)$$

Note: The coefficient ax^2 is zero at $x = 0$. To guarantee uniqueness we will assume we are looking for general solutions defined on the interval $(0, \infty)$.

Method to solve for homogeneous solution:
Given:

$$ax^2y'' + bxy' + cy = 0$$

We assume the solution of the form:

$$\begin{aligned} y(t) &= x^m \\ y'(t) &= mx^{m-1} \\ y''(t) &= m(m-1)x^{m-2} \end{aligned}$$

Plugging into the differential equation:

$$\begin{aligned} ax^2y'' + bxy' + cy &= 0 \\ ax^2m(m-1)x^{m-2} + bxmx^{m-1} + cx^m &= 0 \\ x^m(am(m-1) + bm + c) &= 0 \\ am(m-1) + bm + c &= 0 \end{aligned}$$

Thus, the characteristic equation for m is:

$$am^2 + (b-a)m + c = 0$$

Case 1: Real and Unique Roots
For roots m_1 and m_2, the general solution is:

$$\boxed{y_h = c_1 x^{m_1} + c_2 x^{m_2}}$$

Case 2: Repeated Roots
If we have a repeated root then we know the discriminant of the quadratic formula must be zero. Thus, the root is of the form $m_1 = -(b-a)/(2a)$ To get the second solution we can do a **reduction of order**.

Rewriting the Cauchy-Euler equation in standard form:

$$y'' + \frac{b}{ax} y' + \frac{c}{ax^2} y = 0$$

Letting $y_1 = x^{m_1} \implies y_2 = u(x) y_1$.

Recall that (reduction of order):

$$u(x) = \int \frac{e^{-\int P(x)\,dx}}{y_1^2}\,dx$$

Where $P(x) = \frac{b}{ax}$ and

$$\int P(x)\,dx = \int \frac{b}{ax}\,dx = \frac{b}{a}\ln(x)$$

Thus,

$$\begin{aligned}
y_2 &= x^{m_1} \int \frac{e^{-b/a \ln(x)}}{(x^{m_1})^2}\,dx \\
&= x^{m_1} \int x^{-b/a} \cdot x^{-2m_1}\,dx \\
&= x^{m_1} \int x^{-b/a} \cdot x^{(b-a)/a}\,dx \qquad \leftarrow m_1 = -\frac{b-a}{2a} \\
&= x^{m_1} \int x^{-1}\,dx \\
&= x^{m_1} \ln(x)
\end{aligned}$$

The general solution is then:

$$\boxed{y_h = c_1 x^{m_1} + c_2 x^{m_1} \ln(x)}$$

Case 3: Complex Roots
If the roots are complex then they come in conjugate pairs

$$m = \alpha \pm i\beta$$

so then a homogeneous solution is

$$y_h = c_1 x^{\alpha + i\beta} + c_2 x^{\alpha - i\beta}$$

But then this solution is complex. To fix first note the identity

$$x^{i\beta} = (e^{\ln x})^{i\beta} = (e^{i\beta \ln x})$$

and by Euler's formula

$$x^{i\beta} = \cos(\beta \ln x) + i \sin(\beta \ln x)$$
$$x^{-i\beta} = \cos(\beta \ln x) - i \sin(\beta \ln x)$$

So adding and subtracting the two gives

$$y_1 = x^{i\beta} + x^{-i\beta} = 2\cos(\beta \ln x)$$

$$y_2 = x^{i\beta} - x^{-i\beta} = 2i \sin(\beta \ln x)$$

But since we are dealing with just real solutions we can drop the i in y_2 by assuming it is integrated into the constant c_2. Thus the homogeneous solution is

$$\boxed{y_h = c_1 x^{\alpha} \cos(\beta \ln x) + c_2 x^{\alpha} \sin(\beta \ln x)}$$

Note: In general, to find the particular solution y_p for Cauchy-Euler equations, method of undetermined coefficients **does not** apply since the coefficients are not constant. Thus, we must use variation of parameters to find y_p using the homogeneous solution y_h.

Eigenvalues and Eigenvectors

For a matrix $\mathbf{A} \in \mathcal{R}^{n\times n}$. An element $\mathbf{x} \in \mathcal{R}^n$, where $\mathbf{x} \neq 0$ is called an **eigenvector** of $\mathbf{A}$ if there exists a scalar λ, known as an **eigenvalue** such that

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$$

In order to calculate the eigenvalues/vectors note that if

$$\begin{aligned} \mathbf{A}\mathbf{x} &= \lambda\mathbf{x} \\ \mathbf{A}\mathbf{x} - \lambda\mathbf{x} &= \mathbf{O} \\ (\mathbf{A} - \lambda\mathbf{I})\mathbf{x} &= \mathbf{O} \end{aligned}$$

Since $\mathbf{x} \neq \mathbf{O}$ then $\mathbf{A} - \lambda\mathbf{I}$ must **not** be invertible. Thus,

$$\boxed{\det(\mathbf{A} - \lambda\mathbf{I}) = 0}$$

Method:

- Calculate the **characteristic polynomial** by looking for the eigenvalues λ where

  $$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

- Calculate the **eigenspace** associated with λ by looking at

  $$\text{null}(\mathbf{A} - \lambda\mathbf{I}) \implies (\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0 \quad (\text{where } \mathbf{x} \text{ is the nullspace})$$

  Any non-zero $\mathbf{x} \in \text{null}(\mathbf{A} - \lambda\mathbf{I})$ is an **eigenvector** of $\mathbf{A}$ associated with λ.

  Note: The null space is all vectors $\mathbf{x}$ such that $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$.

Defective Matrix
A matrix $\mathbf{A}$ is said to be **defective** if it cannot be diagonalized. This occurs when there are not enough linearly independent eigenvectors to form a basis for $\mathcal{R}^n$.

We can check if a matrix is defective by comparing the **algebraic multiplicity** and **geometric multiplicity** of each eigenvalue.

- **Algebraic Multiplicity:** The algebraic multiplicity of an eigenvalue λ is the number of times λ appears as a root of the characteristic polynomial.

- **Geometric Multiplicity:** The geometric multiplicity of an eigenvalue λ is the dimension of the eigenspace associated with λ, which is given the by number of linearly independent eigenvectors associated with λ.

Note: For characteristic polynomials with complex roots the eigenvalues will **always** come in pairs: $a \pm ib$. The eigenvectors will also come in pairs $\mathbf{x}, \overline{\mathbf{x}}$. So we can solve for just the plus case and get the minus case by taking the complex conjugate.

Diagonalization

If a matrix $\mathbf{A}$ is diagonalizable (non-defective) then we can write it as:

$$\mathbf{A} = \mathbf{C}\mathbf{D}\mathbf{C}^{-1}$$

Where $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are the linearly independent eigenvectors of $\mathbf{A}$ and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the corresponding eigenvalues, then

$$\mathbf{C} = \begin{pmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \\ | & | & & | \end{pmatrix} \quad \text{and} \quad \mathbf{D} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

Matrix Inverses
For a 2x2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \quad \text{if } ad - bc \neq 0.$$

For higher order matrices, we can use row reduction on the augmented matrix

$$\left[\,\mathbf{A} \,\middle|\, \mathbf{I}\,\right] = \left[\begin{array}{ccc|ccc} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{row reduce}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & b_{11} & b_{12} & b_{13} \\ 0 & 1 & 0 & b_{21} & b_{22} & b_{23} \\ 0 & 0 & 1 & b_{31} & b_{32} & b_{33} \end{array}\right] = \left[\,\mathbf{I} \,\middle|\, \mathbf{A}^{-1}\,\right]$$

Diagonalization can be useful in cases where we need to find $\mathbf{A}^n$ for some large n. Instead of multiplying $\mathbf{A}$ by itself n times we can use diagonalization:

$$\mathbf{A}^n = (\mathbf{C}\mathbf{D}\mathbf{C}^{-1})^n = \mathbf{C}\mathbf{D}^n\mathbf{C}^{-1}$$

Where

$$\mathbf{D}^n = \begin{bmatrix} d_{11}^n & 0 & 0 & \cdots & 0 \\ 0 & d_{22}^n & 0 & \cdots & 0 \\ 0 & 0 & d_{33}^n & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_{nn}^n \end{bmatrix}$$

4.9 Solving Systems by Elimination

Simultaneous ordinary differential equations involve two or more equations that contain derivatives of two or more dependent variables (the unknown functions) **with respect to a single independent variable** can be solved by using differential operators to eliminate one of the dependent variables.

Ex: Suppose we have the system of equations:

$$\begin{aligned} \frac{dx}{dt} &= 4x + 7y \\ \frac{dy}{dt} &= x - 2y \end{aligned}$$

Rewriting with differential operators:

$$\begin{aligned} Dx - 4x - 7y &= 0 \\ Dy - x + 2y &= 0 \end{aligned}$$

Now let's rearrange a little

$$\begin{aligned} (D-4)x - 7y &= 0 \\ -x + (D+2)y &= 0 \end{aligned}$$

Now let's operate on the second equation by (D-4)

$$\begin{aligned} (D-4)x - 7y &= 0 \\ -(D-4)x + (D-4)(D+2)y &= 0 \end{aligned}$$

and add

$$(D-4)(D+2)y - 7y = D^2y - 2Dy - 15y = (D-5)(D+3)y = 0$$

Thus

$$\boxed{y(t) = c_1e^{5t} + c_2e^{-3t}}$$

Now if we operated on the first equation by $(D+2)$ and multiplied the second equation by 7

$$\begin{aligned} (D+2)(D-4)x - 7(D+2)y &= 0 \\ -7x + 7(D+2)y &= 0 \end{aligned}$$

and added we would get

$$(D+2)(D-4)x - 7x = D^2x - 2Dx - 15x = (D-5)(D+3)x = 0$$

so

$$\boxed{x(t) = c_3e^{5t} + c_4e^{-3t}}$$

Finally we substitute both solutions back into one of the original equations to the constants.

Note: For non-homogeneous systems, we must also operate on the non-homogeneous part when we apply the differential operator to both sides of the equation. This way we can get a differential equation in terms of only 1 of the variables so we can apply undetermined coefficients or other methods to find a particular solution.

8.1 Preliminary Theory - Linear Systems

A **first order system** is made up of differential equations of the form:

$$\begin{aligned}
\frac{dx_1}{dt} &= g_1(t, x_1, x_2, \ldots, x_n) \\
\frac{dx_2}{dt} &= g_2(t, x_1, x_2, \ldots, x_n) \\
&\vdots \\
\frac{dx_n}{dt} &= g_n(t, x_1, x_2, \ldots, x_n)
\end{aligned}$$

If the functions $g_1, g_2 \ldots g_n$ are linear in terms of the dependent variables $x_1, x_2, \ldots x_n$, then the system is called a **linear system**.

$$\begin{aligned}
\frac{dx_1}{dt} &= a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n + f_1(t) \\
\frac{dx_2}{dt} &= a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n + f_2(t) \\
&\vdots \\
\frac{dx_n}{dt} &= a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n + f_n(t)
\end{aligned}$$

This can be written in matrix form as:

$$\mathbf{X}' = \mathbf{A}\mathbf{X} + \mathbf{F}(t)$$

$$\begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{pmatrix} = \begin{pmatrix} a_{1,1}(t) & a_{1,2}(t) & \cdots & a_{1,n}(t) \\ a_{2,1}(t) & a_{2,2}(t) & \cdots & a_{2,n}(t) \\ \vdots & \vdots & \vdots & \\ a_{n,1}(t) & a_{n,2}(t) & \cdots & a_{n,n}(t) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{pmatrix}$$

Where the **solution vector** is

$$\mathbf{X} = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

whose entries are differentiable functions satisfying the system on the interval.

Thm 8.1.1 Existence of a Unique Solution:

If the entries of $\mathbf{A}(t)$ and $\mathbf{F}(t)$ are continuous on an open interval I containing t_0, then there exists a unique solution of the initial- value problem on the interval.

$$\begin{aligned}
\mathbf{X}' &= \mathbf{A}\mathbf{X} + \mathbf{F}(t) \\
\mathbf{X}(0) &= \mathbf{X}_0
\end{aligned}$$

Thm 8.1.2 Superposition Principle:
Let $\mathbf{X}_1, \mathbf{X}_2, ..., \mathbf{X}_k$ be a set of solution vectors of the homogeneous system. Then the linear combination

$$\mathbf{X} = c_1\mathbf{X}_1 + c_2\mathbf{X}_2 + \cdots + c_k\mathbf{X}_k$$

where the c_i, $i = 1, 2, \ldots, k$ are arbitrary constants, is also a solution on the interval.

Thm 8.1.3 Criterion for Linearly Independent Solutions:
For systems of first order differential equations, the Wronskian determinant is defined differently than for single differential equations. Given n solution vectors:

$$\mathbf{X}_1 = \begin{pmatrix} x_{11}(t) \\ x_{21}(t) \\ \vdots \\ x_{n1}(t) \end{pmatrix} \mathbf{X}_2 = \begin{pmatrix} x_{12}(t) \\ x_{22}(t) \\ \vdots \\ x_{n2}(t) \end{pmatrix}, \ldots, \mathbf{X}_n = \begin{pmatrix} x_{1n}(t) \\ x_{2n}(t) \\ \vdots \\ x_{nn}(t) \end{pmatrix}$$

The Wronskian determinant is defined as:

$$W(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n) = \begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{vmatrix}$$

Note: This is just the determinant of the matrix formed by placing the solution vectors as columns.

Similarly, the solution vectors are linearly independent on the interval I if and only if:

$$W(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n) \neq 0 \quad \text{(for all t in } I)$$

Note: Any set $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ of n linearly independent solution vectors to the homogeneous system on the interval I is said to be a **fundamental set of solutions** on the interval.

General Solution of a Nonhomogeneous System
Similar to before, the general solution is:

$$\mathbf{X} = \mathbf{X}_c + \mathbf{X}_p$$

Where $\mathbf{X}_c$ is the general solution to the homogeneous system (**the complementary function**) and $\mathbf{X}_p$ is a particular solution to the nonhomogeneous system.

8.2 Homogeneous Linear Systems

Given a homogeneous linear system of the form:

$$\mathbf{X}' = \mathbf{A}\mathbf{X}$$

Where $\mathbf{A}$ is a constant matrix, we can solve for the general solution using eigenvalues and eigenvectors.

Method:
Find the eigenvalues and eigenvectors of $\mathbf{A}$ by solving:

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be the corresponding eigenvectors. Then the general solution depends on the type of eigenvalues we have.

Case 1: Real and Distinct Eigenvalues

$$\boxed{\mathbf{X}(t) = k_1 e^{\lambda_1 t}\mathbf{v}_1 + k_2 e^{\lambda_2 t}\mathbf{v}_2}$$

Case 2: Complex Eigenvalues
We know that complex eigenvalues and eigenvectors come in conjugate pairs, so we can just use the $+$ case and get the $-$ case by taking the complex conjugate.

Looking at the $+$ case, let $\lambda_1 = \alpha + i\beta$ be the eigenvalue and $\mathbf{v}_1 = \mathbf{p} + i\mathbf{q}$ be the eigenvector. Then the general solution is:

$$\boxed{\mathbf{X}(\mathbf{t}) = k_1[\mathbf{B_1}\cos(\beta t) - \mathbf{B_2}\sin(\beta t)]e^{\alpha t} + k_2[\mathbf{B_2}\cos(\beta t) + \mathbf{B_1}\sin(\beta t)]e^{\alpha t}}$$

Where $\mathbf{B}_1$ and $\mathbf{B}_2$ are the real and imaginary parts of the eigenvector $\mathbf{v}_1$.

$$\mathbf{B_1} = Re(\mathbf{v_1}) = \mathbf{p}$$

$$\mathbf{B_2} = Im(\mathbf{v_1}) = \mathbf{q}$$

Note: This formula comes from Euler's formula and separating the real and imaginary parts of the solution $e^{\lambda_1 t}\mathbf{v}_1 = e^{\alpha t}(\cos\beta t + i\sin\beta t)\mathbf{v}_1$.

Case 3: Real and Repeated Eigenvalues
We will have two cases for eigenvalue λ:

1. If
$$\mathbf{A} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$$
Then we can choose **any** two linearly independent eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$. We can keep it simple and choose:
$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
Thus, the general solution is:
$$\boxed{\mathbf{X}(t) = k_1 e^{\lambda t}\mathbf{v}_1 + k_2 e^{\lambda t}\mathbf{v}_2}$$

2. If not, we can use the initial conditions and set
$$\mathbf{v}_0 = \begin{pmatrix} x(0) \\ y(0) \end{pmatrix}$$
Then set
$$\mathbf{v}_1 = (\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_0$$
And the solution becomes
$$\boxed{\mathbf{X} = e^{\lambda t}\mathbf{v}_0 + t e^{\lambda t}\mathbf{v}_1}$$
This method automatically solves for the initial conditions!

   If $\mathbf{v_1} \neq \mathbf{0}$ then $\mathbf{v_1}$ is an eigenvector with eigenvalue λ. If $\mathbf{v_1} = \mathbf{0}$, then $\mathbf{v_0}$ is the eigenvector.

The textbook solution for case 2 *without* initial conditions is:

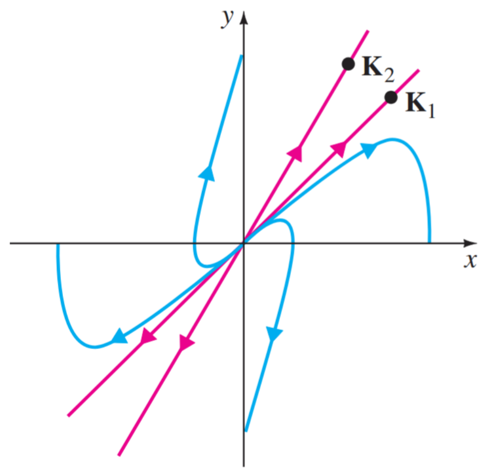
Suppose we found the one eigenvector $\mathbf{v}_1$ associated with the repeated eigenvalue λ. Then we can find a second, linearly independent solution vector $\mathbf{v}_2$ by solving:
$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_2 = \mathbf{v}_1$$
Then the general solution is:
$$\boxed{\mathbf{X}(t) = k_1 e^{\lambda t}\mathbf{v}_1 + k_2[\mathbf{v_1} t e^{\lambda t} + \mathbf{v_2} e^{\lambda t}]}$$

Phase Portraits

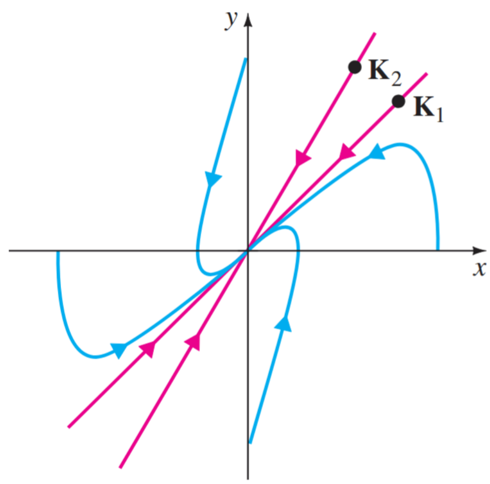
By analyzing the eigenvalues and eigenvectors of the system, we can see the dynamics of the solutions across time.

Real Eigenvalues

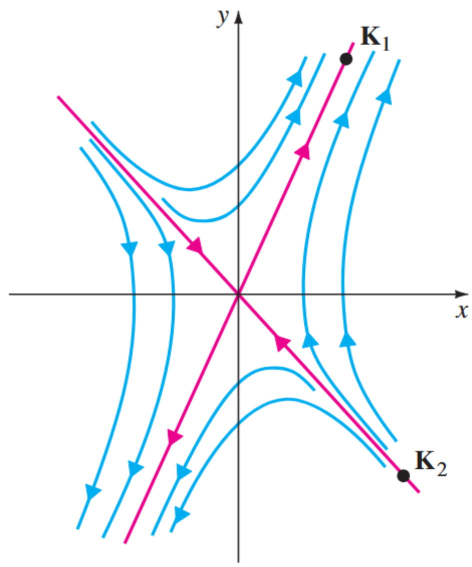
- Both eigenvalues positive: **Unstable Node** (all solutions move away from origin)



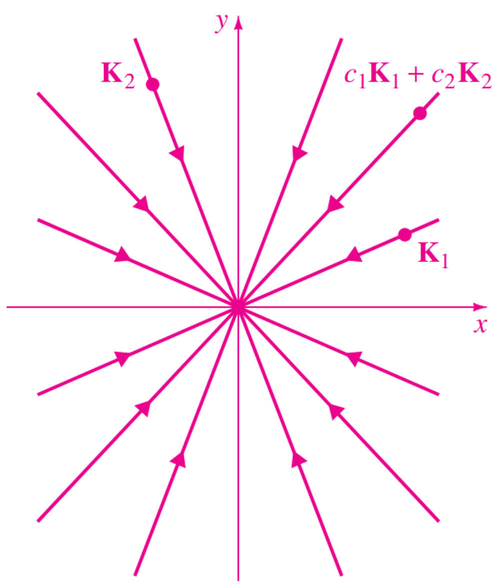
- Both eigenvalues negative: **Stable Node** (all solutions move toward origin)



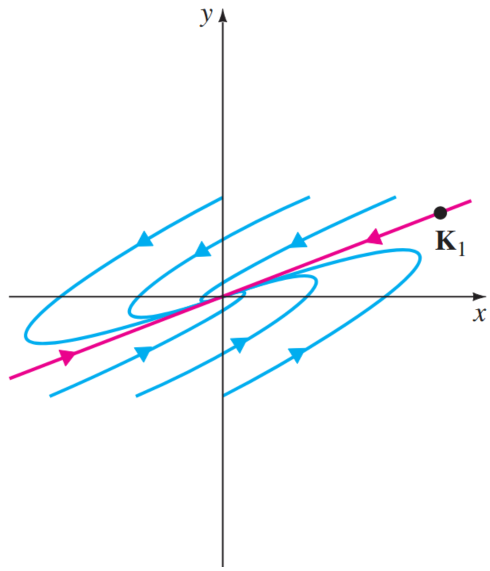
- Eigenvalues of opposite sign: **Saddle Point (Semi-Unstable)** (solutions move away from origin along one eigenvector and toward origin along the other eigenvector)



- Repeated eigenvalues: All solutions move toward or away from origin along eigenvector direction (depending on sign of eigenvalue). There are two cases for the two general solutions:
  - For a diagonal matrix, **all** vectors in $\mathbb{R}^2$ are eigenvectors, so solutions move directly toward or away from origin along straight lines in all directions.



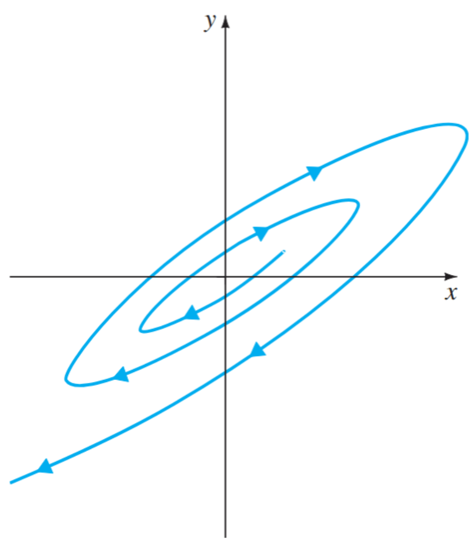
– For a non-diagonal matrix, there is only **one** eigenvector, so solutions move toward or away from origin along the eigenvector direction, and other trajectories curve toward this line.



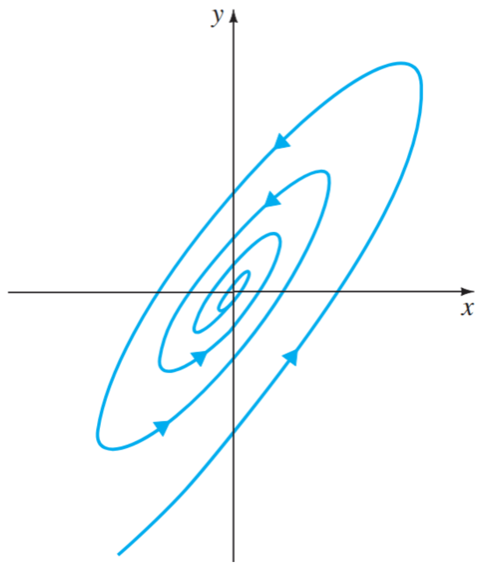
The eigenvectors are straight line solutions that indicate the direction of the flow in the phase plane where all other trajectories are asymptotic to these lines (to satisfy uniqueness of solutions).

Complex Eigenvalues

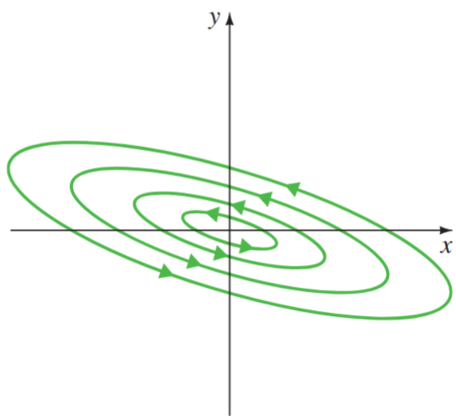
- Real part positive: **Unstable Spiral** (solutions spiral away from origin)



- Real part negative: **Stable Spiral** (solutions spiral toward origin)



- Real part zero: **Center** (solutions circle around origin)



By plugging in the initial conditions into the differential equation, we can see the direction of motion in terms of dx/dt and dy/dt at that point. Therefore, we can determine if the motion is *clockwise* or *counterclockwise* around the origin.

8.3 Non-Homogeneous Linear Systems

To solve non-homogeneous linear systems of the form:

$$\mathbf{X}' = \mathbf{A}\mathbf{X} + \mathbf{F}(t)$$

We find the particular solution $\mathbf{X}_p$ using either **undetermined coefficients** or **variation of parameters**, and then add it to the complementary function $\mathbf{X}_c$ found from solving the homogeneous system.

Undetermined Coefficients
Similar to before, we guess the form of $\mathbf{X}_p$ based on the form of $\mathbf{F}(t)$. However, now we must guess a vector function.

<u>Ex:</u> Given the system:

$$\begin{aligned} \frac{dx}{dt} &= 2x + 3y - 7 \\ \frac{dy}{dt} &= -x - 2y + 5 \end{aligned}$$

This can be rearranged as:

$$\frac{d\mathbf{X}}{dt} = \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix} \mathbf{X} + \begin{pmatrix} -7 \\ 5 \end{pmatrix}$$

To solve the particular solution. Look at $\mathbf{F}(t) = \begin{pmatrix} -7 \\ 5 \end{pmatrix}$ which is just a constant solution so guess

$$\mathbf{X}_p = \begin{pmatrix} A \\ B \end{pmatrix}$$

Now plug in and we get

$$\begin{aligned} 0 = 2A + 3B - 7 \quad &\Rightarrow\ 7 = 2A + 3B \qquad &&\Rightarrow\ 7 = 2(5 - 2B) + 3B = 10 - B, \\ 0 = -A - 2B + 5 \quad &\Rightarrow\ -5 = -A - 2B \qquad &&\Rightarrow\ A = 5 - 2B. \end{aligned}$$

Thus, $B = 3$ and $A = -1$

If we wanted to see the phase plane for this system, we can look at the
x-nullclines:

$$0 = x' = 2x + 3y - 7 \Rightarrow y = \frac{7 - 2x}{3}$$

the y-nullclines:

$$0 = y' = -x - 2y + 5 \Rightarrow y = \frac{5 - x}{2}$$

The intersection of these two lines is the equilibrium point of the system. By analyzing the eigenvalues of the homogeneous system we can determine the stability of this equilibrium point.

$$\begin{aligned}
\frac{7-2x}{3} &= \frac{5-x}{2} \\
14-4x &= 15-3x \\
x &= -1 \Rightarrow y = 3
\end{aligned}$$

Variation of Parameters

If $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ is a fundamental set of solutions on an interval I to the homogeneous system $\mathbf{X}' = \mathbf{A}\mathbf{X}$, then its general solution is the linear combination:

$$\mathbf{X} = c_1\mathbf{X}_1 + c_2\mathbf{X}_2 + \ldots + c_n\mathbf{X}_n$$

Rewriting it as a matrix:

$$\mathbf{X} = c_1 \begin{pmatrix} x_{11} \\ x_{12} \\ \vdots \\ x_{1n} \end{pmatrix} + c_2 \begin{pmatrix} x_{21} \\ x_{22} \\ \vdots \\ x_{2n} \end{pmatrix} + \cdots + c_n \begin{pmatrix} x_{n1} \\ x_{n2} \\ \vdots \\ x_{nn} \end{pmatrix} = \begin{pmatrix} c_1x_{11} + c_2x_{12} + \cdots + c_nx_{1n} \\ c_1x_{21} + c_2x_{22} + \cdots + c_nx_{2n} \\ \vdots \\ c_1x_{n1} + c_2x_{n2} + \cdots + c_nx_{nn} \end{pmatrix}$$

In other words,

$$\mathbf{X} = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \mathbf{\Phi}(t)\mathbf{C}$$

Where $\mathbf{\Phi}(t)$ is called the **fundamental matrix** of the system (columns are the solution vectors).

There are two properties of the fundamental matrix:

- The matrix $\mathbf{\Phi}(t)$ is nonsingular
- If $\mathbf{\Phi}(t)$ is a fundamental matrix of the system $\mathbf{X}' = \mathbf{A}\mathbf{X}$ then

$$\mathbf{\Phi}(t)' = \mathbf{A}\mathbf{\Phi}(t)$$

By applying these properties we can find a particular solution to the non-homogeneous system using variation of parameters:

$$\mathbf{X}' = \mathbf{A}\mathbf{X} + \mathbf{F}(t)$$

Method: Similar to variation of parameters before, we are looking to replace $\mathbf{C}$ with a vector of functions

$$\mathbf{U} = \begin{pmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_n(t) \end{pmatrix}$$

so that $\mathbf{X}_p = \mathbf{\Phi}(t)\mathbf{U}(t)$ is a particular solution to the nonhomogeneous system.

$$\boxed{\mathbf{X}_p = \mathbf{\Phi}(t) \int \mathbf{\Phi}^{-1}(t)\mathbf{F}(t)\ dt}$$

<u>Ex:</u> Solve the system

$$\begin{aligned} \frac{dx}{dt} &= 2x + 3y - 7 \\ \frac{dy}{dt} &= -x - 2y + 5 \end{aligned}$$

We find the homogeneous solution to be:

$$\mathbf{X}_c(t) = k_1 e^t \begin{pmatrix} -3 \\ 1 \end{pmatrix} + k_2 e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Using *variation of parameters*:

$$\mathbf{\Phi}(t) = \begin{pmatrix} -3e^t & -e^{-t} \\ e^t & e^{-t} \end{pmatrix} \Rightarrow \mathbf{\Phi}^{-1}(t) = \frac{1}{-3+1} \begin{pmatrix} e^{-t} & e^{-t} \\ -e^t & -3e^t \end{pmatrix}$$

$$\begin{aligned} \mathbf{X}_p &= \mathbf{\Phi}(t) \int \mathbf{\Phi}^{-1}(t)\mathbf{F}(t)\ dt \\ &= \begin{pmatrix} -3e^t & -e^{-t} \\ e^t & e^{-t} \end{pmatrix} \int \frac{1}{-3+1} \begin{pmatrix} e^{-t} & e^{-t} \\ -e^t & -3e^t \end{pmatrix} \begin{pmatrix} -7 \\ 5 \end{pmatrix}\ dt \\ &= \begin{pmatrix} -3e^t & -e^{-t} \\ e^t & e^{-t} \end{pmatrix} \int \frac{1}{-3+1} \begin{pmatrix} -2e^{-t} \\ -8e^t \end{pmatrix}\ dt \\ &= \begin{pmatrix} -3e^t & -e^{-t} \\ e^t & e^{-t} \end{pmatrix} \int \begin{pmatrix} e^{-t} \\ 4e^t \end{pmatrix}\ dt \\ &= \begin{pmatrix} -3e^t & -e^{-t} \\ e^t & e^{-t} \end{pmatrix} \begin{pmatrix} -e^{-t} \\ 4e^t \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ 3 \end{pmatrix} \end{aligned}$$

Therefore

$$\mathbf{X} = \mathbf{X}_c + \mathbf{X}_p = k_1 e^t \begin{pmatrix} -3 \\ 1 \end{pmatrix} + k_2 e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

8.4 Matrix Exponentials

Just like how for linear first order equations where the solution for $x' = \alpha x$, where α is constant, as:

$$x = Ce^{\alpha t}$$

We can do the same for matrices where $\mathbf{X}' = \mathbf{AX}$, where $\mathbf{A}$ is a constant matrix. The solution is:

$$\mathbf{X} = e^{\mathbf{A}t}$$

Thm:

The matrix exponential $\mathbf{X} = e^{\mathbf{A}t}\mathbf{c}$ solves

$$\mathbf{X}' = \mathbf{AX}$$

for every vector of constants $\mathbf{c}$ since

$$\mathbf{X}' = \frac{d}{dt}e^{\mathbf{A}t}\mathbf{c} = \mathbf{A}e^{\mathbf{A}t}\mathbf{c} = \mathbf{AX}$$

Thm:

The general solution of

$$\mathbf{X}' = \mathbf{AX} + \mathbf{F}(t)$$

is

$$\mathbf{X} = \mathbf{X}_h + \mathbf{X}_p = e^{\mathbf{A}t}\mathbf{c} + e^{\mathbf{A}t}\int_{t_0}^{t} e^{-\mathbf{A}s}\mathbf{F}(s)\ ds$$

Def: For any $n \times n$ matrix $\mathbf{A}$ the **matrix exponential** is defined as

$$e^{\mathbf{A}t} = \sum_{k=0}^{\infty} \mathbf{A}^k \frac{t^k}{k!} = \mathbf{I} + \mathbf{A}t + \frac{(\mathbf{A}t)^2}{2!} + \frac{(\mathbf{A}t)^3}{3!} + \cdots$$

Note: The series converges for all $t \in \mathbb{R}$ and any $n \times n$ matrix $\mathbf{A}$. Also,

$$\mathbf{A}^2 = \mathbf{AA}, \mathbf{A}^3 = \mathbf{A}(\mathbf{A}^2), \cdots$$

Rcll:

$$e^{\alpha t} = \sum_{k=0}^{\infty} \alpha^k \frac{t^k}{k!} = 1 + \alpha t + \frac{(\alpha t)^2}{2!} + + \frac{(\alpha t)^3}{3!} + \cdots$$

Def: The derivative of $e^{\mathbf{A}t}$ is

$$\frac{d}{dt}e^{\mathbf{A}t} = \frac{d}{dt}\sum_{k=0}^{\infty} \mathbf{A}^k \frac{t^k}{k!} = \sum_{k=1}^{\infty} \mathbf{A}^k \frac{t^{k-1}}{(k-1)!} = \mathbf{A}\sum_{k=1}^{\infty} \mathbf{A}^{k-1} \frac{t^{k-1}}{(k-1)!} = \mathbf{A}\sum_{k=0}^{\infty} \mathbf{A}^k \frac{t^k}{k!}$$

$$\frac{d}{dt}e^{\mathbf{A}t} = \mathbf{A}e^{\mathbf{A}t}$$

Ex:

$$\begin{aligned}
e^{\begin{pmatrix}2&0\\0&3\end{pmatrix}t} &= \mathbf{I}+\begin{pmatrix}2&0\\0&3\end{pmatrix}t+\frac{1}{2!}\begin{pmatrix}2&0\\0&3\end{pmatrix}^2 t^2++\frac{1}{3!}\begin{pmatrix}2&0\\0&3\end{pmatrix}^3 t^3+\cdots\\
&= \sum_{k=0}^{\infty}\frac{t^k}{k!}\begin{pmatrix}2&0\\0&3\end{pmatrix}^k\\
&= \begin{pmatrix}\sum_{k=0}^{\infty}\frac{t^k}{k!}2^k & 0\\ 0 & \sum_{k=0}^{\infty}\frac{t^k}{k!}3^k\end{pmatrix}\\
&= \begin{pmatrix}e^{2t}&0\\0&e^{3t}\end{pmatrix}
\end{aligned}$$

Ex: Solve

$$\mathbf{X}'=\begin{pmatrix}2&0\\0&3\end{pmatrix}\mathbf{X}+\begin{pmatrix}2\\3\end{pmatrix}$$

From above we see

$$\mathbf{X}_h=\begin{pmatrix}e^{2t}&0\\0&e^{3t}\end{pmatrix}\mathbf{c}=\begin{pmatrix}c_1e^{2t}\\c_2e^{3t}\end{pmatrix}=c_1e^{2t}\begin{pmatrix}1\\0\end{pmatrix}+c_2e^{3t}\begin{pmatrix}0\\1\end{pmatrix}$$

Now to calculuate a particular solution we first see that

$$e^{\mathbf{A}t}=\begin{pmatrix}e^{2t}&0\\0&e^{3t}\end{pmatrix}\Rightarrow e^{-\mathbf{A}t}=\frac{1}{e^{5t}}\begin{pmatrix}e^{3t}&0\\0&e^{2t}\end{pmatrix}=\begin{pmatrix}e^{-2t}&0\\0&e^{-3t}\end{pmatrix}$$

Therefore the particular solution

$$\begin{aligned}
\mathbf{X}_p &= e^{\mathbf{A}t}\int_{t_0}^{t}e^{-\mathbf{A}s}\mathbf{F}(s)\,ds\\
&= \begin{pmatrix}e^{2t}&0\\0&e^{3t}\end{pmatrix}\int_{t_0}^{t}\begin{pmatrix}e^{-2s}&0\\0&e^{-3s}\end{pmatrix}\begin{pmatrix}2\\3\end{pmatrix}\,ds\\
&= \begin{pmatrix}e^{2t}&0\\0&e^{3t}\end{pmatrix}\int_{t_0}^{t}\begin{pmatrix}2e^{-2s}\\3e^{-3s}\end{pmatrix}\,ds\\
&= \begin{pmatrix}e^{2t}&0\\0&e^{3t}\end{pmatrix}\left(\begin{pmatrix}-e^{-2s}\\-e^{-3s}\end{pmatrix}\Bigg|_{t_0}^{t}\right)\\
&= \begin{pmatrix}e^{2t}&0\\0&e^{3t}\end{pmatrix}\left(\begin{pmatrix}-e^{-2t}\\-e^{-3t}\end{pmatrix}+\begin{pmatrix}e^{-2t_0}\\e^{-3t_0}\end{pmatrix}\right)\\
&= \begin{pmatrix}-1\\-1\end{pmatrix}+\begin{pmatrix}e^{2t}&0\\0&e^{3t}\end{pmatrix}\begin{pmatrix}e^{-2t_0}\\e^{-3t_0}\end{pmatrix}\\
&= \begin{pmatrix}-1\\-1\end{pmatrix}+\left(e^{2t}\begin{pmatrix}e^{-2t_0}\\0\end{pmatrix}+e^{3t}\begin{pmatrix}0\\e^{-3t_0}\end{pmatrix}\right)
\end{aligned}$$

Notice the second part of the sum can be incorporated into the homogeneous solution $\mathbf{X}_h$. Thus the general solution is

$$\mathbf{X}=c_1e^{2t}\begin{pmatrix}1\\0\end{pmatrix}+c_2e^{3t}\begin{pmatrix}0\\1\end{pmatrix}-\begin{pmatrix}1\\1\end{pmatrix}$$

Applications: 5.1 Linear Models

We study spring–mass systems modeled by the second-order linear ODE

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = g(t)$$

With initial conditions $y(t_0) = y_0, \quad y'(t_0) = y_1$

Free Undamped
From Hooke's law, where the positive direction is downward:

$$\sum F_x = -kx = ma_x$$

As a differential equation:

$$\boxed{\frac{d^2x}{dt^2} + \frac{k}{m}x = 0}$$

If we let $\omega^2 = \frac{k}{m}$, then the general solution is:

$$x(t) = c_1 \cos \omega t + c_2 \sin \omega t$$

The motion is periodic with period

$$T = \frac{2\pi}{\omega}$$

With frequency

$$f = \frac{1}{T} = \frac{\omega}{2\pi}$$

Free Damped
Now we add a damping force proportional to velocity:

$$\sum F_x = -kx - \beta\frac{dx}{dt} = ma_x$$

As a differential equation:

$$\boxed{\frac{d^2x}{dt^2} + \frac{\beta}{m}\frac{dx}{dt} + \frac{k}{m}x = 0}$$

Like before, let $\omega^2 = \frac{k}{m}$ and let $2\lambda = \frac{\beta}{m}$.

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = 0$$

Then the characteristic equation is:

$$m^2 + 2\lambda m + \omega^2 = 0$$

and the corresponding roots are then

$$m = -\lambda \pm \sqrt{\lambda^2 - \omega^2}$$

There are three cases:

- **Case 1:** $\lambda^2 - \omega^2 > 0$
  This system is said to be **overdamped** because the damping coefficient β is large compared to the spring constant ω so the solution is of the form

$$x(t) = e^{-\lambda t}\left(c_1 e^{\sqrt{\lambda^2-\omega^2}t} + c_2 e^{-\sqrt{\lambda^2-\omega^2}t}\right)$$

- **Case 2:** $\lambda^2 - \omega^2 = 0$
  This system is said to be **critically damped** because any slight decrease in the dampling force would result in oscillatory motion. The solution is of the form

$$x(t) = e^{-\lambda t}(c_1 + c_2 t)$$

- **Case 3:** $\lambda^2 - \omega^2 < 0$
  This system is said to be **underdamped** because the damping coefficient β is small compared to the spring constant ω so the solution is of the form

$$x(t) = e^{-\lambda t}\left(c_1 \cos\sqrt{\omega^2-\lambda^2}t + c_2 \sin\sqrt{\omega^2-\lambda^2}t\right)$$

Driven Motion
Now we add an external force $f(t)$ to the system:

$$\sum F_x = -kx - \beta\frac{dx}{dt} + f(t) = ma_x$$

As a differential equation, $f(t)$ makes it a non-homogeneous equation:

$$\boxed{m\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = f(t)}$$

Basically, once we have the differential eqation set up, we can solve for the homogenous and particular solutions normally.