Mathematical Methods

1. Use the Taylor series expansion to find approximations. The ones for sin, cos, tan, and $(1+x)^n$ are especially useful.

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\tan x = \sum_{n=1}^{\infty} \frac{B_{2n}(-4)^n(1-4^n)}{(2n)!} x^{2n-1} = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \cdots$$

$$(1+x)^m = \sum_{n=0}^{\infty} \binom{m}{n} x^n = 1 + mx + \frac{m(m-1)}{2}x^2 + \frac{m(m-1)(m-2)}{6}x^3 + \cdots$$

   Note: For small x, higher order terms reduce to zero

2. Use complex exponentials to manipulate complicated trig functions.

$$e^{ix} = \cos x + i \sin x$$

3. Solve differential equations by substituting in trial solutions. Especially you should recognize the differential equation for a simple harmonic oscillator and be able to come up with solutions to that ODE that satisfy any initial conditions you are given.

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

$$x(t) = A\cos(\omega t) + B\sin(\omega t)$$

   Wave equation:

$$v^2 \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2}$$

4. Useful integration formulas:

$$\int \frac{1}{(x^2+a^2)^{\frac{3}{2}}} dx = \frac{x}{a^2\sqrt{x^2+a^2}} + C$$

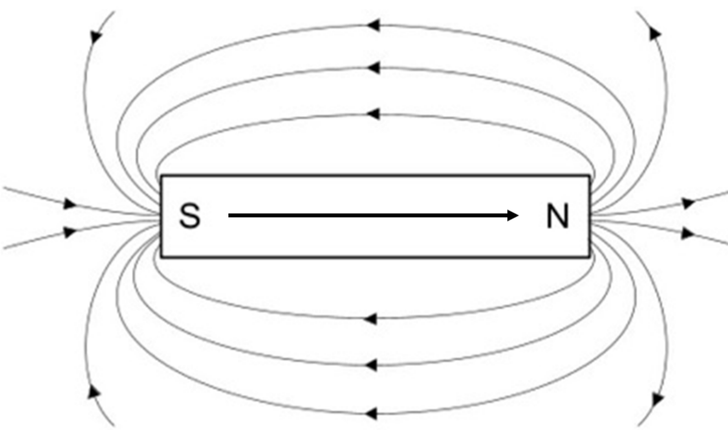
$$\int \frac{x}{(x^2+a^2)^{\frac{3}{2}}} dx = -\frac{1}{\sqrt{x^2+a^2}} + C$$

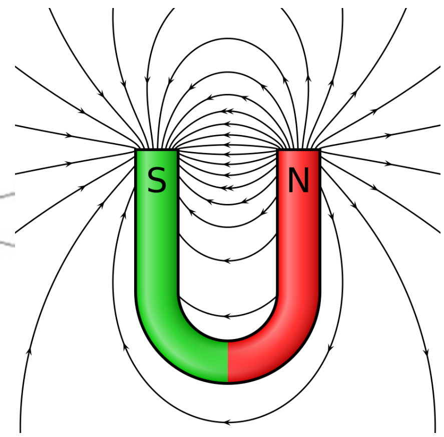
Chapter 28 - Magnetic Fields

Types of magnets

- **Current loop:** a current carrying loop of wire creates an electromagnet.
- **Permanent Magnet:** the magnetic fields of the electrons within the material do not cancel out, resulting in a net magnetic field.

All magnets are **magnetic dipoles** with a **north** and **south** pole (the magnetic monopole doesn't exist, sadly). Opposite magnetic poles attract each other, and like magnetic poles repel each other. Magnetic field lines are *closed loops* that exit through the north pole and enter through the south pole.





Note: Inside the bar magnet, the magnetic field lines point from south to north, completing the closed loop.

Magnetic field lines and the magnetic field are related by:

- The direction of the magnetic field is tangent to the field lines.
- The spacing of the field lines represents the strength (magnitude) of the magnetic field. Closer lines = stronger field.

Also, analogous to Gauss's law for electric fields, we have **Gauss's law for magnetism:**

$$\int \vec{B} \cdot \hat{n}\, d\vec{A} = 0$$

Since there are no magnetic monopoles, the net magnetic flux through any closed surface is zero (there are no sources or sinks of magnetic field lines).

1. Solve Newton's second law to determine the motion of charged particles acting under the influence of a magnetic field and any other forces (e.g., gravity, electric fields...).

   Stationary charges do not interact with the magnetic field. Moving charges with a component of velocity perpendicular to the magnetic field experiences a force:

   $$\boxed{\vec{F}_B = q\vec{v} \times \vec{B}}$$

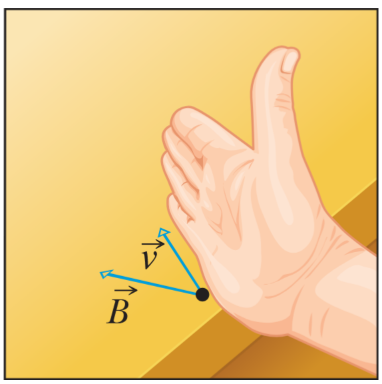
   Note: This force is *always* perpendicular to the velocity of the particle, so it does **no work** on the particle and cannot change its speed, only its direction.

   Note: The magnetic force is zero when the velocity is along the magnetic field lines (i.e., parallel or antiparallel) or when stationary.

   The unit for the magnetic field $\vec{B}$ is the Tesla (T):

   $$1\,\mathrm{T} = 1\,\frac{\mathrm{N}}{\mathrm{C}\cdot\mathrm{m/s}} = 1\,\frac{\mathrm{N}}{\mathrm{A}\cdot\mathrm{m}}$$

   Recall: **Right hand rule**

   - Point fingers in the direction of the velocity $\vec{v}$.
   - Curl fingers toward the direction of the magnetic field $\vec{B}$, sweeping through the smaller angle.
   - Thumb points in the direction of the force $\vec{F}_B$ for **a positive charge**. For a negative charge, the force is in the opposite direction.

   Note: When $\vec{B}$ and $\vec{v}$ are orthogonal, we can just multiply the magnitudes to find the force and use the right hand rule to find the direction.

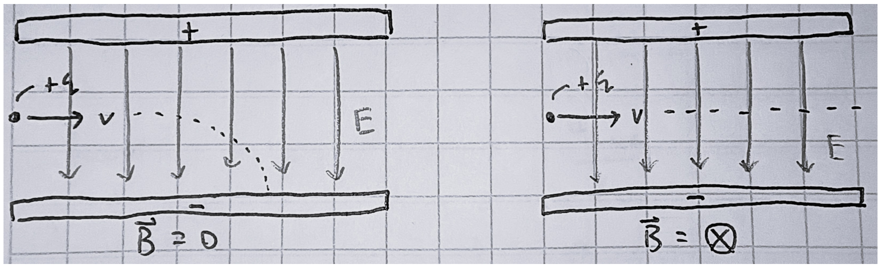
   Note: A magnetic force exists even if there is *relative velocity* between charges and a magnetic field. For example, a moving magnet will exert a magnetic force on stationary charges.

   Note to self: Bring dynamics formula sheet for kinematics equations.

2. Explain the Hall effect and describe its applications.

Here are several interesting applications where both the magnetic field and electric field acts on a moving charge.

Wien Filter (Velocity Selector)



Suppose we have a source charged particles ($+q$) with random velocities. If it passes through a region with *only* an $\vec{E}$ field, it will be pushed onto the negative plate, following a parabolic trajectory. However, if there is a $\vec{B}$ field in addition to the $\vec{E}$ field, then the forces will cancel for particles with a specific velocity:

$$\sum \vec{F}_y = \vec{F}_B - \vec{F}_E = 0 \implies \vec{F}_B = \vec{F}_E$$

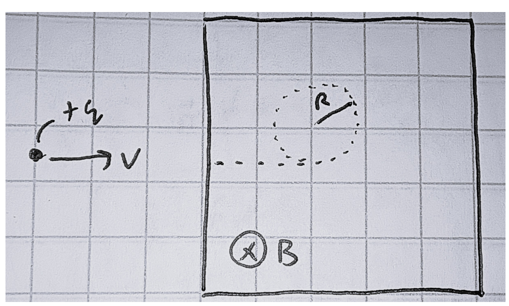
$$q\vec{v} \times \vec{B} = q\vec{E}$$

$$\boxed{v = \frac{E}{B}}$$

Thus, only particles with velocity $v = E/B$ will pass straight through the filter.

By combining the Wien filter with another region of magnetic fields, we create a mass spectrometer that can separate particles based on their charge-to-mass ratio.

Mass Spectrometer



Since we know both the charge and velocity entering the magnetic field region, we can find the particle's mass by measuring the radius of its circular path:

$$\sum \vec{F}_n = \vec{F}_B = \frac{mv^2}{R} \implies qvB = \frac{mv^2}{R}$$

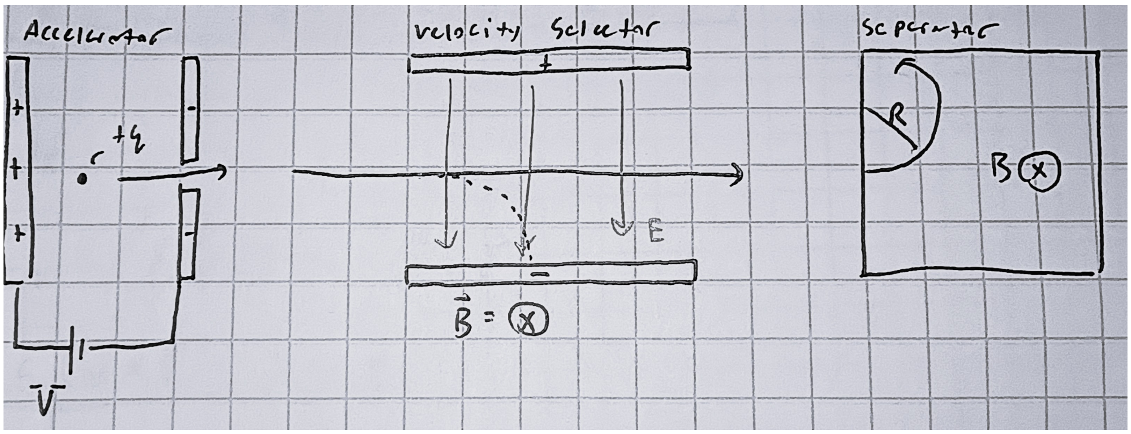
$$\boxed{m = \frac{BRq}{v}}$$

Note: Typically, the charges are accelerated through a potential difference V before entering the velocity selector, so we can find their velocity using energy conservation:

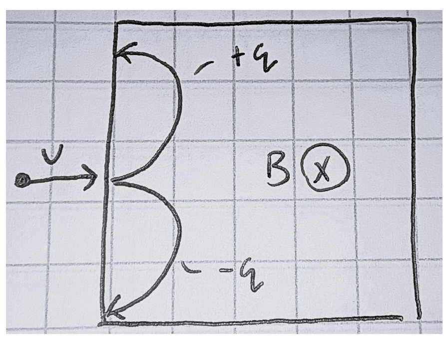
$$W_{nc} = \Delta E = 0 \implies \Delta U = \Delta K$$

$$qV = \frac{1}{2}mv^2 \implies v = \sqrt{\frac{2qV}{m}}$$

The full set up looks like this:

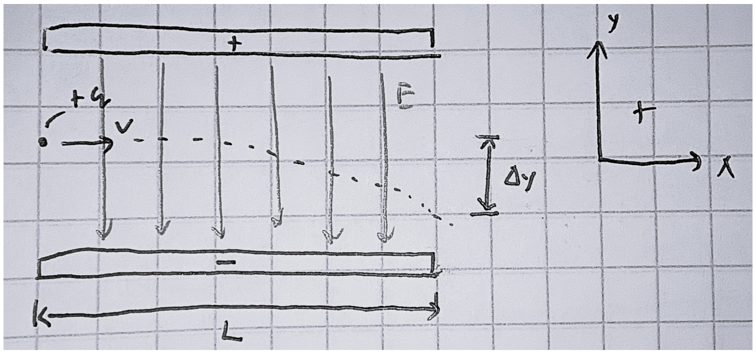


Also note that since the magnetic force is opposite for negative charges, they will curve in the opposite direction in the magnetic field region:



Let's consider a setup similar to a Wien filter, but where the parallel plates are designed to deflect the particle beam rather than selectively filter it.

Cathode Ray Tube



First, without a $\vec{B}$ field, the particles will be deflected by the $\vec{E}$ field:

$$\sum \vec{F} = -qE\hat{j} \implies |a_y| = \frac{|q|E}{m}$$

The time spent in the field is:

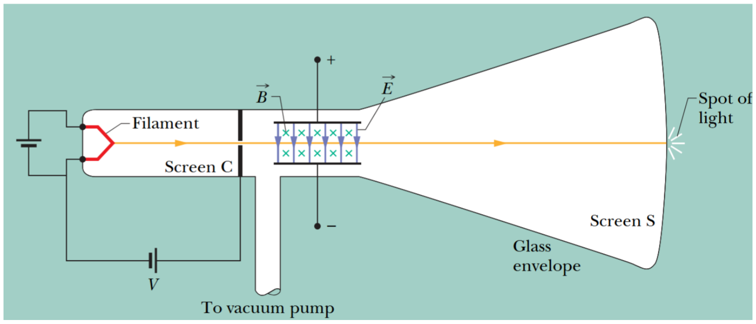
$$\Delta x = vt = L \implies t = \frac{L}{v}$$

The vertical displacement upon exiting the plates is:

$$\Delta y = v_{oy}t + \frac{1}{2}a_y t^2 = 0 + \frac{1}{2}\left(\frac{|q|E}{m}\right)\left(\frac{L}{v}\right)^2$$

$$\boxed{\Delta y = \frac{|q|EL^2}{2mv^2}}$$

Now consider adding a magnetic field like this:



We know from the Wien filter that the forces will cancel when:

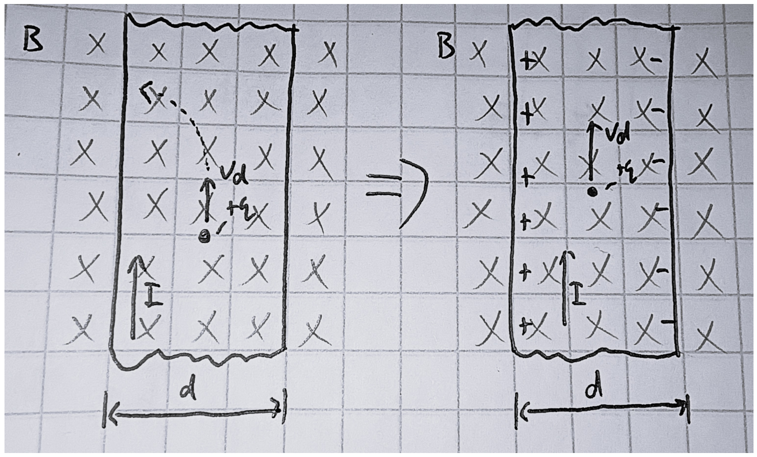
$$v = \frac{E}{B}$$

Thus, plugging this into our previous equation for vertical displacement:

$$\frac{m}{|q|} = \frac{BL^2}{2\Delta y E}$$

Finally, let's talk about the Hall effect!

Hall Effect



Consider a conductor with a current I flowing through it in a region with a $\vec{B}$ field. The moving charges will be pushed to one side of the conductor by the magnetic force, creating a **Hall potential difference** (ΔV) and an electric field ($\vec{E}$) inside the conductor.

$$\Delta V = Ed$$

Eventually, when the electric force balances the magnetic force, the charges stop accumulating.

$$\sum \vec{F} = \vec{F}_E - \vec{F}_B = 0 \implies qE = qv_d B$$

Thus, by measuring the Hall potential difference, we can find the magnetic field strength:

$$\boxed{B = \frac{\Delta V}{v_d d}}$$

We can also find the number of charge carriers per unit volume (n) in the conductor, letting $q = e$ for electrons and plugging in for v_d from before:

$$I = nev_d A, \quad \text{(A is cross-sectional area of conductor)}$$

$$n = \frac{IBd}{eA\Delta V}$$

Note: It is also possible to determine the drift velocity using the Hall effect, by mechanically moving the conductor such that there is no relative velocity between the charges and the magnetic field. Therefore, there will be **zero** Hall potential difference (since there is no magnetic force).

3. Explain the principle of operation of a cyclotron

4. Determine the forces and/or torques on various arrangements of current carrying wires (straight, circular loops, square loops, etc...) located in a given magnetic field.

Chapter 29 - Magnetic Fields due to Currents

1. Use the Biot-Savart law to calculate the magnetic field due to a current-carrying wires of arbitrary (but tractable) geometry. e.g., a loop.

2. Describe Ampere's law and the conditions under which it can be used to solve for a magnetic field from a given current distribution. Use Ampere's law to find the field in those situations with sufficient symmetry to apply it. e.g., a long wire.

3. Find the magnetic force on a current carrying wire due to another current carrying wire.

4. Describe the forces and torques on magnetic dipoles in terms of their magnetic moment.

Chapter 30 & 31.11 - Induction and Inductance

1. Understand the meaning Faraday's law of induction and be able to use it to determine the electromotive force. Explain and apply the equivalence of Faraday's law and the Lorentz force law for motional emfs.

2. Apply Lenz's law to determine the sign of induced currents or electromotive forces.

3. Define inductance and be able to calculate the self or mutual inductance for various (usually simple) current distributions. Know and use the reciprocity relation for mutual inductances.

4. Be able to calculate the energy stored in magnetic fields for a given inductor using either the formula involving inductance or the energy density of the magnetic field.

5. Describe the principle of operation of a generator and a motor.

Chapter 32 - Maxwell's Equations

1. Describe Gauss's law for Magnetism and its applications.

2. Describe the Ampere-Maxwell law, the displacement current, and calculate the magnetic field in a charging or discharging capacitor.

Chapter 33 - Electromagnetic Waves

1. Test whether a given arrangement of propagating electromagnetic fields satisfies Maxwell's Equations.

2. Understand the principles of electromagnetic waves and their basic properties, such as phase speed, relation between $\vec{k}$, $\vec{E}$, and $\vec{B}$.

3. Define the Poynting vector or Poynting flux and be able to use it to calculate the energy carried by electromagnetic fields.

4. Qualitatively describe how electromagnetic waves are generated. For an accelerating charge, identify the directions in which one will see the largest/smallest electric fields and describe the polarization of the electric field vector.

Chapters 27, 30.12, & 31 - Circuits

1. Know and be able to apply the Voltage Loop Rule and the Current Node Rule (conservation of energy and conservation of charge, respectively) to analyze a circuit. A good example is to prove the rules for adding resistors in series or in parallel.

2. Know the relation between current and voltage for the basic circuit elements of resistor, capacitor, and inductor.

3. Write differential equations for RLC circuits (or RL, RC, LC). Understand qualitatively the behavior of resistors, inductors, and capacitors at high and low frequencies. Be able to describe how electromagnetic energy is stored, supplied, and dissipated in such circuits.

4. Solve the differential equations for such circuits, identify characteristic timescales, frequencies, and resonances. If initial conditions are given, be able to find solutions that satisfy those initial conditions.

5. Know or be able to derive the complex impedances for resistors, capacitors, and inductors. Know or derive the rules for combining parallel and series impedances. Use those impedances to describe the relations among currents and voltages in circuits, both in magnitude and phase.

End Final Exam