

Mathematical Methods

1. Use the Taylor series expansion to find approximations. The ones for \sin , \cos , \tan , and $(1+x)^n$ are especially useful.

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$\tan x = \sum_{n=1}^{\infty} \frac{B_{2n}(-4)^n(1-4^n)}{(2n)!} x^{2n-1} = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \dots$$

$$(1+x)^m = \sum_{n=0}^{\infty} \binom{m}{n} x^n = 1 + mx + \frac{m(m-1)}{2} x^2 + \frac{m(m-1)(m-2)}{6} x^3 + \dots$$

Note: For small x , higher order terms reduce to zero

2. Use complex exponentials to manipulate complicated trig functions.

$$e^{ix} = \cos x + i \sin x$$

3. Solve differential equations by substituting in trial solutions. Especially you should recognize the differential equation for a simple harmonic oscillator and be able to come up with solutions to that ODE that satisfy any initial conditions you are given.

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

$$x(t) = A \cos(\omega t) + B \sin(\omega t)$$

Wave equation:

$$v^2 \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2}$$

4. Useful integration formulas:

$$\int \frac{1}{(x^2 + a^2)^{\frac{3}{2}}} dx = \frac{x}{a^2 \sqrt{x^2 + a^2}} + C$$

$$\int \frac{x}{(x^2 + a^2)^{\frac{3}{2}}} dx = -\frac{1}{\sqrt{x^2 + a^2}} + C$$

Chapter 21 - Electric Charge

1. Explain what is electric charge, how you can tell if something carries an electric charge, how you could determine what the sign of the charge carriers is, and (qualitatively) how charge is transferred from one object to another.
 - Electric charge is an intrinsic characteristic of the fundamental particles that make up matter. Electrically charged objects will exert electrostatic forces on each other (opposite charges attract and same charges repel).
 - The elementary charge $e = 1.602 \times 10^{-19} C$
 - Electrons has charge $-e$, protons has charge $+e$ and neutrons have no charge
 - Charges are transferred through direct contact and remain fixed once electrically isolated. It is possible to induce a charge on a conductor's surface via an external electric field moving its free electrons.
 - To ground an object, set up a pathway of conductors between the object and the Earth (a big conductor/charge reservoir).

Note: Charge is conserved when charges are transferred.

2. Conductors and insulators:
 - **Conductors:** Contain mobile/free electrons that can move freely within. If an excess charge is placed onto a conductor, it will spread over its surface. Thus, there is zero electric field within a conductor (always true, even with an external electric field, since non zero electric field will create current within the conductor).
 - **Insulators:** Charges cannot move freely and is fixed after getting charged.
3. Use the Coulomb force law to solve mechanics problems involving electric charges.

Charged particles exert forces on each other by:

$$\vec{F}_{1 \rightarrow 2} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r^2} \hat{r}_{1 \rightarrow 2}$$

Where

$$\epsilon_0 = 8.85 \times 10^{-12} \frac{C^2}{Nm^2} \quad (\text{Permittivity constant})$$

$$\vec{r}_{1 \rightarrow 2} = \langle \text{observation location} \rangle - \langle \text{source location} \rangle$$

Chapter 22 - Electric Fields

1. Define a field and identify whether a certain mathematical quantity is a field or not.

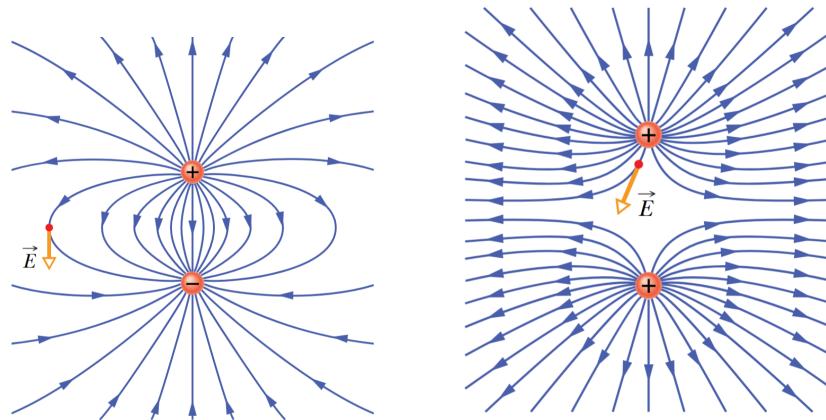
Fields describe how a physical quantity varies throughout a region in space, with a value defined at every point in the region

- **Scalar fields:** assigns a scalar value to each point (Ex: temperature fields)
- **Vector fields:** assigns a vector (magnitude and direction) to each point (Ex: electric field)

2. Electric field lines and the electric field are related by:

- The direction of the electric field at a point is tangent along the electric field line
- The magnitude of the electric field is proportional to the number of field lines per unit area in a plane perpendicular to the lines

Note: The direction of \vec{E} is the direction of the force that acts on a **positive test charge** (negative charges will accelerate antiparallel to the electric field lines)



3. Given an electric field, find the force on a charged object.

Force at point P on an object with charge q_o :

$$\vec{F}_P = \vec{E}_P q_o$$

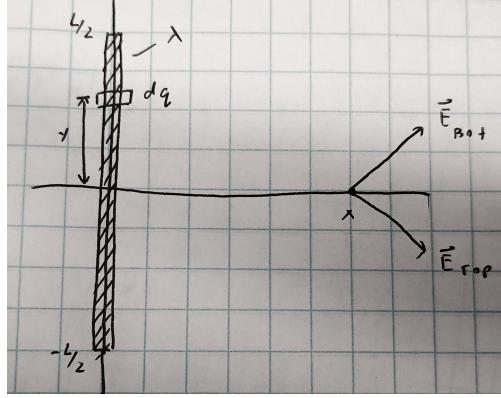
4. Find the electric field at any given location for a set of discrete charges.

Electric field from a point charge q:

$$\vec{E} = \frac{\vec{F}}{q_o} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{r}$$

5. Find the electric field at any given location for a continuous charge distribution. In particular, be able to set up a complete integral for a higher (2 or 3) dimensional object that can be solved to obtain the electric field at a certain location by stacking up simpler (0 or 1) dimensional objects. Be able to solve the integrals for simple cases. Alternatively, use double or triple integrals to perform the same task.

- **Electric field from a line of charge:**



Given:

- Total charge: Q
- Linear charge density λ

Calculating electric field from dq :

$$d\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{dq}{r^2} \hat{r}$$

$$\vec{r} = \langle x, 0 \rangle - \langle 0, y \rangle = \langle x, -y \rangle$$

$$\hat{r} = \frac{\langle x, -y \rangle}{\sqrt{x^2 + y^2}}$$

$$dq = \lambda dy = \frac{Q}{L} dy \quad \left(\lambda = \frac{Q}{L} \right)$$

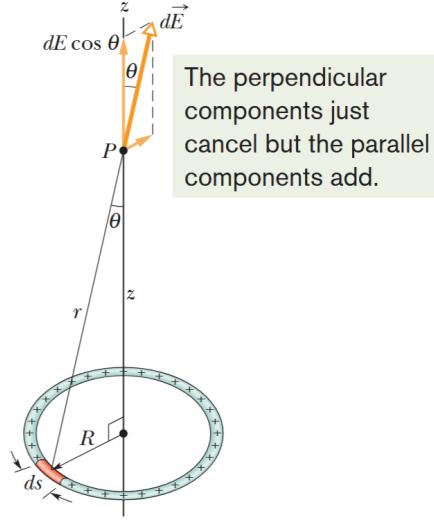
$$\Rightarrow \int d\vec{E} = \int_{-L/2}^{L/2} \frac{1}{4\pi\epsilon_0} \left(\frac{Q}{L} \right) \left(\frac{dy}{(x^2 + y^2)^{3/2}} \right) \langle x, -y \rangle$$

By symmetry, the components of the electric field in the y cancel out so

$$E_x = \frac{1}{4\pi\epsilon_0} \left(\frac{Q}{L} \right) \int_{-L/2}^{L/2} \frac{x}{(x^2 + y^2)^{3/2}} dy$$

$$E_x = \boxed{\frac{1}{4\pi\epsilon_0} \left(\frac{1}{x\sqrt{x^2 + (L/2)^2}} \right)}$$

- Electric field from a ring of charge:



Given:

- Total charge: Q
- Linear charge density λ
- Ring radius R

Calculating electric field from dq :

$$d\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{dq}{r^2} \hat{r}$$

$$\vec{r} = \langle 0, 0, z \rangle - \langle R\cos\theta, R\sin\theta, 0 \rangle = \langle -R\cos\theta, -R\sin\theta, z \rangle$$

$$\hat{r} = \frac{\langle -R\cos\theta, -R\sin\theta, z \rangle}{\sqrt{R^2 + z^2}}$$

$$dq = \lambda ds = \frac{Q}{2\pi R} ds \quad \left(\lambda = \frac{Q}{2\pi R} \right)$$

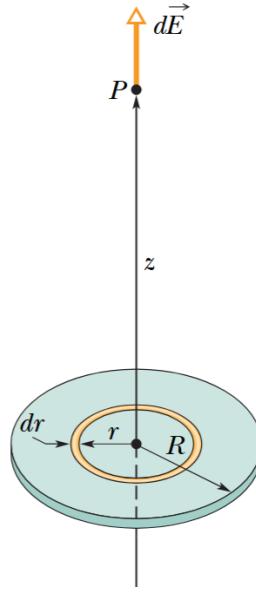
$$\Rightarrow \int d\vec{E} = \int_0^{2\pi R} \frac{1}{4\pi\epsilon_0} \left(\frac{Q}{2\pi R} \right) \left(\frac{ds}{(R^2 + z^2)^{3/2}} \right) \langle -R\cos\theta, -R\sin\theta, z \rangle$$

By symmetry, the components of the electric field in the radial directions cancel out so

$$E_z = \frac{1}{4\pi\epsilon_0} \left(\frac{Q}{2\pi R} \right) \int_0^{2\pi R} \frac{z}{(R^2 + z^2)^{3/2}} ds$$

$$E_z = \boxed{\frac{1}{4\pi\epsilon_0} \left(\frac{Qz}{(R^2 + z^2)^{3/2}} \right)}$$

- Electric field from a disk of charge:



Given:

- Total charge: Q
- Surface charge density σ
- Disk radius R

Using electric field from ring of radius r and charge q :

$$E_{ring,z} = \frac{1}{4\pi\varepsilon_0} \left(\frac{qz}{(r^2 + z^2)^{3/2}} \right)$$

$$dE_{ring,z} = \frac{1}{4\pi\varepsilon_0} \left(\frac{z dq}{(r^2 + z^2)^{3/2}} \right)$$

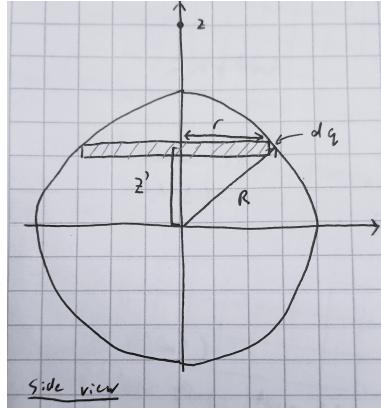
$$dq = \sigma dA = \left(\frac{Q}{\pi R^2} \right) (2\pi r dr)$$

$$\implies E_z = \int_0^R \frac{1}{4\pi\varepsilon_0} \left(\frac{Q}{\pi R^2} \right) \left(\frac{2\pi r z}{(r^2 + z^2)^{3/2}} \right) dr$$

$$E_z = \frac{1}{4\pi\varepsilon_0} \left(\frac{2Q}{R^2} \right) \left(-\frac{z}{\sqrt{R^2 + z^2}} - \frac{z}{\sqrt{z^2}} \right)$$

$$E_z = \frac{1}{4\pi\varepsilon_0} \left(\frac{2Q}{R^2} \right) \left(1 - \frac{z}{\sqrt{R^2 + z^2}} \right)$$

- Electric field from a sphere of charge:



Given:

- Total charge: Q
- Volume charge density ρ
- Sphere radius R

Using electric field from disk of radius r and charge q , from a distance d away:

$$E_{disk,z} = \frac{1}{4\pi\epsilon_0} \left(\frac{2q}{r^2} \right) \left(1 - \frac{d}{\sqrt{r^2 + d^2}} \right)$$

$$r = \sqrt{R^2 - z'^2} \quad (\text{radius of disk as } z' \text{ varies})$$

$$d = z - z' \quad (\text{distance of disk from } z \text{ as } z' \text{ varies})$$

$$\begin{aligned} dq_{disk} &= \rho\pi r^2 dz' = \left(\frac{Q}{\frac{4}{3}\pi R^3} \right) \pi r^2 dz' \\ \implies E_{sphere,z} &= \int_{-R}^R \frac{1}{4\pi\epsilon_0} \left(\frac{2}{r^2} \right) \left(\frac{Q}{\frac{4}{3}\pi R^3} \right) \pi r^2 dz' \left(1 - \frac{d}{\sqrt{r^2 + d^2}} \right) \end{aligned}$$

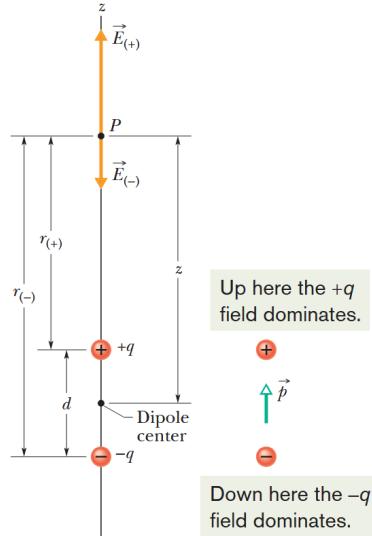
Simplified and plugging in r and d :

$$E_{sphere,z} = \int_{-R}^R \frac{1}{4\pi\epsilon_0} \left(\frac{2Q}{\frac{4}{3}\pi R^3} \right) \pi dz' \left(1 - \frac{z - z'}{\sqrt{R^2 - z'^2 + (z - z')^2}} \right)$$

6. Find the force between objects that have continuous charge distributions, such as two lines of charge.
 - Integrate to find the electric field from one continuous charge distribution at an observation point on the second continuous charge.
 - Integrate again using the electric field on a charge dq on the second continuous charge for total force on it. (Ex: HW 6, Prob 1)

7. Find the torque and work done on an electric dipole by an electric field.

The **Electric dipole moment** ($\vec{P} = qd$) points from the negative to the positive end of the dipole.



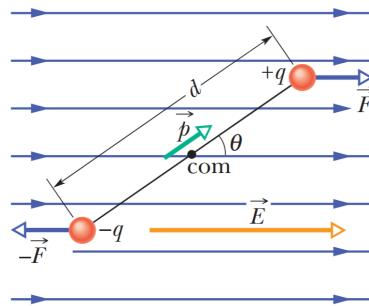
- **Electric field due to an electric dipole**

$$\begin{aligned}\vec{E} &= \vec{E}_+ + \vec{E}_- \\ &= \frac{q}{4\pi\epsilon_0} \left(\frac{1}{(z-d/2)^2} - \frac{1}{(z+d/2)^2} \right)\end{aligned}$$

Assuming $z \gg d$

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{2qd}{z^3}$$

- **Torque on a dipole in an electric field**



In a uniform electric field, the electrostatics on the charged ends of the dipole are equal and opposite, thus there is zero net force and the center of mass does not move.

However, there is non-zero torque:

$$\tau = pE\sin\theta$$

$$\boxed{\vec{\tau} = \vec{p} \times \vec{E}}$$

- **Potential energy of an electric dipole**

- Define the zero potential energy as when $\theta = 90^\circ$
- We can find the potential energy at any θ by calculating the work done by the field when the dipole is rotated from θ to 90° .

$$\Delta U = -W_c$$

$$U = - \int_{90^\circ}^{\theta} \tau d\theta = - \int_{90^\circ}^{\theta} pE\sin\theta d\theta$$

$$U = -pE\cos\theta$$

$$\boxed{U = -\vec{p} \cdot \vec{E}}$$

The dipole has the least potential energy at equilibrium where \vec{p} and \vec{E} are lined up and $\vec{\tau} = \vec{p} \times \vec{E} = 0$. It is like a pendulum where it requires work to rotate the dipole to any other orientation.

Chapter 23 - Gauss's Law

1. Explain Gauss's law and how it relates electric charge to electric flux.

Gauss's law can be applied to find the electric field produced by a symmetric charge distribution (Ex: spherical, cylindrical, or planar symmetry). When other non-symmetric charges are also present, Gauss's law can still be used separately to find the field due to the symmetric charge alone. This is because charges located outside the chosen Gaussian surface contribute no net flux through that surface (the field lines from external charges enter and exit equally) and therefore do not appear in the flux integral. The total electric field at a point is then obtained by superposition of the symmetric field (found via Gauss's law) and the field from the external or asymmetric charges (found using Coulomb's law or direct integration).

Gauss's Law where $d\vec{A}$ is an infinitesimal area element whose direction is normal to the surface.

$$\oint \vec{E} \cdot d\vec{A} = \frac{Q_{\text{enc}}}{\epsilon_0} \quad \text{Integral form}$$

$$\boxed{\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \quad \text{Differential form (local version)}}$$

2. Given an electric field and a surface, calculate the flux of the electric field through the surface.

The electric flux through a surface S is defined as

$$\Phi_E = \iint_S \vec{E} \cdot d\vec{A}$$

For a uniform electric field \vec{E} and a flat surface of area A , the flux simplifies to

$$\Phi_E = EA \cos \theta,$$

where θ is the angle between \vec{E} and the surface normal.

3. Explain the conditions necessary in order to use Gauss's law to solve for an electric field. Identify geometries for which Gauss's law can be used or for which the electric field must be found using Coulomb's law. For applicable cases, construct an appropriate Gaussian surface and determine the electric field using Gauss's law.

We can find the electric field using Gauss's law if we can pick a Gaussian surface such that:

- The electric field is constant over the surface
- The electric field is parallel or normal to the surface everywhere

Symmetries and its respective Gaussian surfaces

- **Spherical**

$$Q_{enc} = \rho V = \frac{4}{3}\rho\pi r^3$$

(Surface Area) = $4\pi r^2$

- **Cylindrical**

Ex: Infinite line of charge (Gaussian cylinder)

$$Q_{enc} = \lambda L$$

$$(Surface Area with non-zero flux) = 2\pi r L$$

Note: The electric field is radial and there is zero flux through the caps.

- **Planer**

Ex: Infinite sheet of charge (Gaussian cylinder)

$$Q_{enc} = \sigma A = \sigma\pi r^2$$

$$(Surface Area with non-zero flux) = 2\pi r^2$$

Note: The electric field is normal and there is zero flux through the sleeve of the cylinder.

4. Use Gauss's law to derive the properties of electric fields in and around conductors. State any required conditions for the standard properties to apply.

Properties of Conductors:

- There is zero electric field inside a conductor (in equilibrium).
- The electric field near the surface of a conductor is always normal with magnitude

$$\vec{E} = \frac{\sigma}{\epsilon_0}$$

- A conductor only has excess charge on its surface.
5. Given an electric field as a function of position, find the charge density.

Ex: A insulating spherical charge with and varying charge density $\rho(r) =$ produces an electric field $\vec{E} = Kr^4$

$$\begin{aligned} \oint \vec{E} \cdot d\vec{A} &= \frac{Q_{\text{enc}}}{\epsilon_0} \\ &= \frac{1}{\epsilon_0} \int \rho(r) dV \\ &= \frac{1}{\epsilon_0} \int_0^{r_o} \rho(r) 4\pi r^2 dr \end{aligned}$$

$$\vec{E}(r_o)(4\pi r_o^2) = \frac{1}{\epsilon_0} \int_0^{r_o} \rho(r) 4\pi r^2 dr$$

$$\frac{d}{dr} \left(\vec{E}(r_o)(4\pi \epsilon_o r_o^2) \right) = \rho(r_o) 4\pi r_o^2$$

$$\frac{\epsilon_o}{r_o^2} \frac{d}{dr} (Kr_o^4(r_o^2)) = \rho(r_o)$$

$$\rho(r_o) = \frac{\epsilon_o}{r_o^2} (6Kr_o^5) = 6K\epsilon_o r_o^3$$

Chapter 24 - Electric Potential

1. Define the electric potential, its relation to electric potential energy, and its relation to the electric field.

- **Electric Potential**

The electric potential at a point in an electric field is:

$$V = \frac{U}{q}$$

The potential difference between any two points in an electric field is:

$$\Delta V = \frac{\Delta U}{q}$$

Since the electrostatic force is a conservative force, it follows the relationship that $\Delta U = -W_{\text{conservative}}$

$$\Delta V = \frac{-W}{q}$$

If we define the potential energy to be zero at ∞ then the potential at any point is:

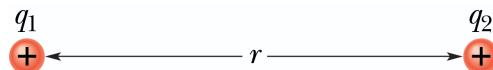
$$V = \frac{-W_\infty}{q}$$

Where W_∞ is the work done by the electric field on a charged particle as it moves from ∞ to a point.

- **Electric potential energy of a system of point charges**

The electric potential of a system of fixed point charges (at rest) is equal to the work that must be done by an external force to assemble the system, bringing in each charge from ∞ .

Ex:

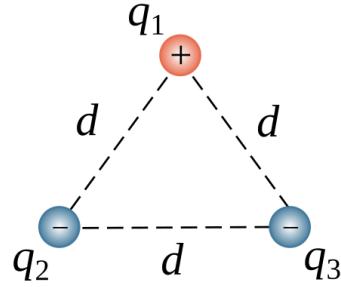


- First bring in q_1 from ∞ : zero work is done because there is no electrostatic force on it, thus force applied is zero.
- Then bring in q_2 from ∞ : there is an electric field from q_1

$$W_{\text{ext}} = -W_{\text{field}} = q_2 \Delta V \quad (V_i = 0)$$

$$U = W_{\text{ext}} = q_2 V_1 = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r}$$

Ex:



- Zero work is done to bring in q_1
- Work done to bring in q_2 is due to potential from q_1

$$U_{12} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{d}$$

- Work done to bring in q_3 is due to the potential from BOTH q_1 and q_2

$$U_{13} + U_{23} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_3}{d} + \frac{1}{4\pi\epsilon_0} \frac{q_2 q_3}{d}$$

$$U = W_{ext} = U_{12} + U_{13} + U_{23}$$

Note: Order doesn't matter when it comes to assembling the system of charges.

- **Relationship Between Potential and the Electric Field**

Since the work done by the electrostatic force and potential is related by:

$$\Delta V = \frac{-W}{q}$$

Work can be calculated by:

$$dW = \vec{F} \cdot d\vec{s} = q\vec{E} \cdot d\vec{s}$$

$$W = q \int_i^f \vec{E} \cdot d\vec{s}$$

Thus,

$$\Delta V = - \int_i^f \vec{E} \cdot d\vec{s}$$

To calculate the electric field given potential:

$$\vec{E} = -\vec{\nabla}V$$

Note:

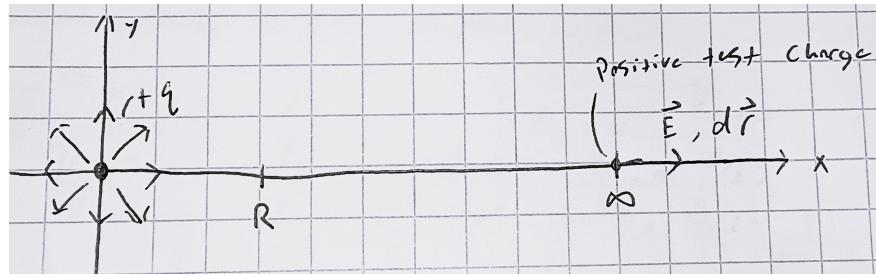
$$\vec{\nabla} = \frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k}$$

2. Given a distribution of charges determine the electric potential in two different ways by either (1) using a line integral and a given electric field or (2) building the potential up by summing discrete charges or integrating over a charge distribution.

Method 1: Integrate from ∞ to observation point

$$\Delta V = - \int_i^f \vec{E} \cdot d\vec{s}$$

Potential of a point charge



We will move a positive test charge from ∞ to R towards a point charge.

$$V_f - V_i = - \int_R^\infty \vec{E} \cdot d\vec{r} \quad (V(\infty) = 0)$$

$$\begin{aligned} V(R) &= - \int_\infty^R \frac{1}{4\pi\epsilon_0 r^2} \hat{r} \cdot d\vec{r} \quad (\hat{r} \cdot d\vec{r} = dr) \\ &= - \frac{q}{4\pi\epsilon_0} \int_\infty^R \frac{1}{r^2} dr \\ &= - \frac{q}{4\pi\epsilon_0} \left(\frac{-1}{r} \right) \Big|_\infty^R \end{aligned}$$

$V = \frac{q}{4\pi\epsilon_0 r} \quad (\text{point charge})$

Note: $d\vec{r}$ points in the same direction as the coordinate axes, the direction of the particle's motion comes from the integration bounds.

Generally, this method is used when we can easily find the electric field.

Method 2: Integrating dV from discrete point charges.

Using the potential from a point charge:

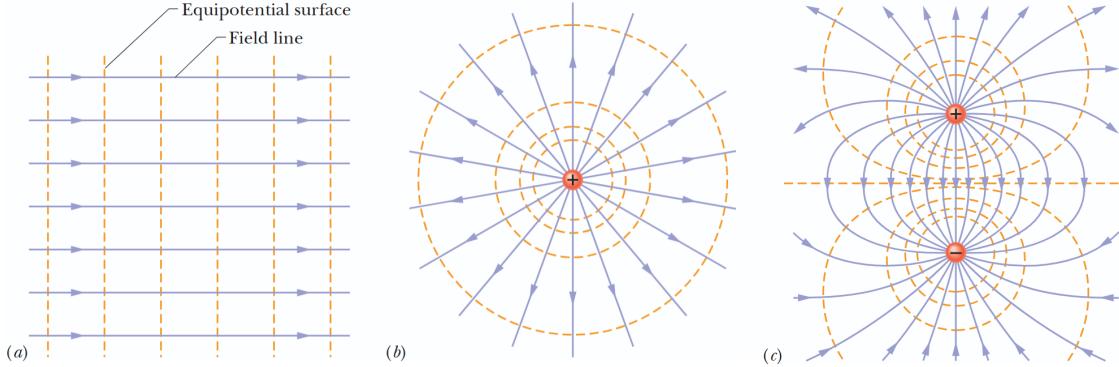
$$V = \frac{q}{4\pi\epsilon_0 r} \implies dV = \frac{dq}{4\pi\epsilon_0 r}$$

$V = \frac{1}{4\pi\epsilon_0} \int \frac{dq}{r}$

3. Know and use the properties of the potential for conductors.

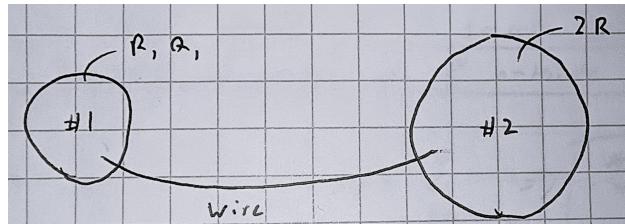
Conductors are equipotential volumes

The potential at every point within an equipotential surface/volume is EQUAL. Thus, no work is done by an electric field on a charge as it moves through an equipotential ($\Delta V = 0 \implies W = 0$).



Note: Equipotential surfaces are always perpendicular to the electric field ($\vec{E} \cdot d\vec{s} = 0$)

Ex: Two conducting spheres are connected with a wire. Initially, sphere 1 is charged with $Q_1 = Q$ and sphere 2 is neutral. How much charge ends up on each sphere after the connection?



$$V_{sphere} = \frac{1}{4\pi\epsilon_0} \frac{q}{r}$$

Since there is no potential difference between the two spheres $V_1 = V_2$

$$\begin{aligned} \frac{1}{4\pi\epsilon_0} \frac{Q_1}{R} &= \frac{1}{4\pi\epsilon_0} \frac{Q_2}{2R} \\ \implies Q_1 &= \frac{Q_2}{2} \end{aligned}$$

Since the total charge is conserved

$$\begin{aligned} Q_1 + Q_2 &= Q \\ \implies Q_1 &= \frac{Q}{3}, \quad Q_2 = \frac{2Q}{3} \end{aligned}$$

Finding the charge distribution on the surface of a conducting sphere

Suppose a charged conducting sphere with radius R and surface charge density σ

$$\int \vec{E} \cdot \hat{n} dA = \frac{Q_{enc}}{\varepsilon_o}$$

$$E(4\pi r^2) = \frac{4\pi R^2 \sigma}{\varepsilon_o}$$

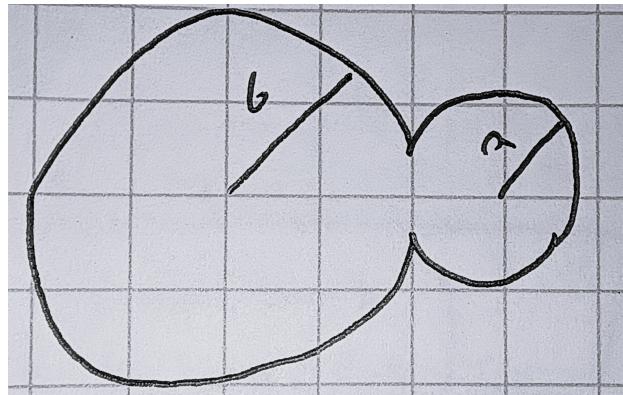
$$E(r) = \frac{\sigma R^2}{\varepsilon_o r^2}$$

$$V = - \int_{\infty}^R \frac{\sigma}{\varepsilon_o} \frac{R^2}{r^2} dr$$

Thus the potential at the surface of a conducting sphere:

$$V = \frac{\sigma R}{\varepsilon_o}$$

Consider the following conductor:



Since we know the potential across the surface is equal, and that each blob is approx a sphere:

$$V = \frac{\sigma_a a}{\varepsilon_o} = \frac{\sigma_b b}{\varepsilon_o}$$

Thus, $\sigma_b > \sigma_a$ since $b > a$. In other words, charges concentrate when the radius of curvature is small (ie. pointy)

4. Use the “method of images” to find the potential for a charge suspended over a grounded infinite conducting plate.

From the differential form of Gauss's Law:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\vec{\nabla} \cdot (-\vec{\nabla}V) = \frac{\rho}{\epsilon_0} \quad (\vec{E} = -\vec{\nabla}V)$$

$$-\nabla^2V = \frac{\rho}{\epsilon_0} \quad (\text{Poisson's Equation})$$

When $\rho = 0$

$$\nabla^2V = 0 \quad (\text{Laplace's Equation})$$

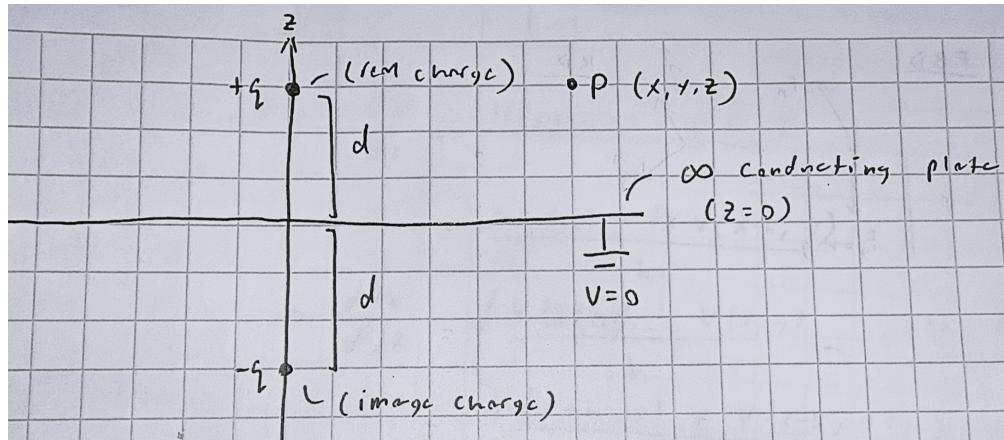
Where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

We use Laplace's equation when solving for the potential in a region without charge and Poisson's equation for charged regions.

Note: Once you fix the boundary conditions, **there can be only one possible potential function (V)** that satisfies both Laplace's equation and those boundaries. This guarantees that any valid method (method of images/numerical method) that finds a solution that matches the boundary conditions **must be the correct solution**.

Set up:



The infinite conducting plate is **grounded**:

- By connecting it with a large charge reservoir (the Earth), it allows excess to charge move onto and out of the plate to maintain a zero potential.
- This gives us the **boundary condition** that $V = 0$ over the plate.

To satisfy Laplace's equation for the region above the plate ($z > 0$), we add a negative **image charge** at the mirror point $(0, 0, -d)$:

- This allows us to define the potential in the upper region such that as $z \rightarrow 0$, $V \rightarrow 0$. Thus satisfying the boundary condition.

Note: To satisfy the Laplace's equation, we need a function for potential that both satisfies the boundary condition (at $z = 0$) and approaches 0 as $r \rightarrow \infty$. If we didn't have the image charge, then V would go to 0 as $r \rightarrow \infty$ but it wouldn't satisfy the boundary condition.

Finding the solution to Laplace's Equation: (V at $P = (x, y, z)$)

$$\vec{r}_+ = \langle x, y, z \rangle - \langle 0, 0, d \rangle, \quad r = \sqrt{x^2 + y^2 + (z-d)^2}$$

$$\vec{r}_- = \langle x, y, z \rangle - \langle 0, 0, -d \rangle \quad r = \sqrt{x^2 + y^2 + (z+d)^2}$$

Thus,

$$V_{(x,y,z)} = \frac{1}{4\pi\epsilon_0} \left(\frac{q}{\sqrt{x^2 + y^2 + (z-d)^2}} - \frac{q}{\sqrt{x^2 + y^2 + (z+d)^2}} \right)$$

Since it satisfies both Laplace's equation and the boundary conditions, this is the **only solution** for potential in the upper region.

Now that we have a function for potential, we can do some fun stuff such as

Finding the z-component of electric field in the upper region

$$E_z = -\frac{d}{dz} \left[\frac{1}{4\pi\epsilon_0} \left(\frac{q}{\sqrt{x^2 + y^2 + (z-d)^2}} - \frac{q}{\sqrt{x^2 + y^2 + (z+d)^2}} \right) \right]$$

$$E_z = \frac{q}{4\pi\epsilon_0} \left[\frac{(z-d)}{(x^2 + y^2 + (z-d)^2)^{3/2}} - \frac{(z+d)}{(x^2 + y^2 + (z+d)^2)^{3/2}} \right]$$

Since we know that the electric field near a conductor is normal to the surface with magnitude $E = \sigma/\epsilon_0$, we can

Find the surface charge density of the conducting plate

$$E_z(z=0) = \frac{q}{4\pi\epsilon_0} \left[\frac{-d}{(x^2 + y^2 + d^2)^{3/2}} - \frac{d}{(x^2 + y^2 + d^2)^{3/2}} \right]$$

Thus,

$$\sigma(x, y, z=0) = \frac{-2qd}{4\pi(x^2 + y^2 + d^2)^{3/2}}$$

Note: We can check our results by integrating

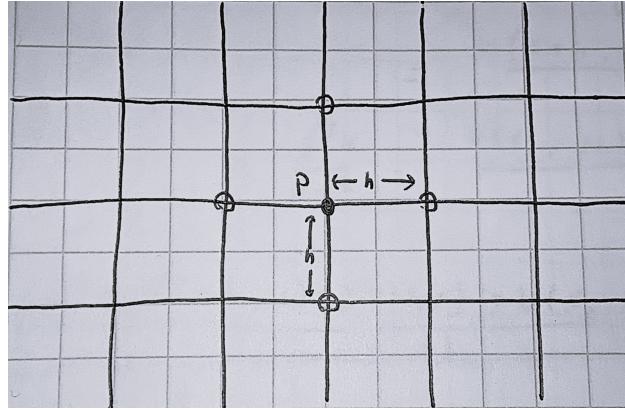
$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sigma dx dy = -q$$

5. Use the “relaxation” method to numerically solve for the electric potential outside of a conductor held at a fixed potential.

Our goal is to solve Laplace’s equation by evaluating derivatives at discrete points using the Finite Difference Method.

$$\nabla^2 V = \frac{d^2 V}{dx^2} + \frac{d^2 V}{dy^2} = 0$$

Set up: $P = (x, y)$



Evaluating Laplace’s equation locally at P:

$$\frac{dV}{dx} = \frac{V(x+h, y) - V(x, y)}{h} = \frac{V(x, y) - V(x-h, y)}{h}$$

Note: These are the forward and backward first-differences for the derivative.

$$\begin{aligned} \frac{d^2 V}{dx^2} &= \frac{\left(\frac{V(x+h, y) - V(x, y)}{h}\right) - \left(\frac{V(x, y) - V(x-h, y)}{h}\right)}{h} \\ &= \frac{V(x+h, y) + V(x-h, y) - 2V(x, y)}{h^2} \end{aligned}$$

Similarly,

$$\frac{d^2 V}{dy^2} = \frac{V(x, y+h) + V(x, y-h) - 2V(x, y)}{h^2}$$

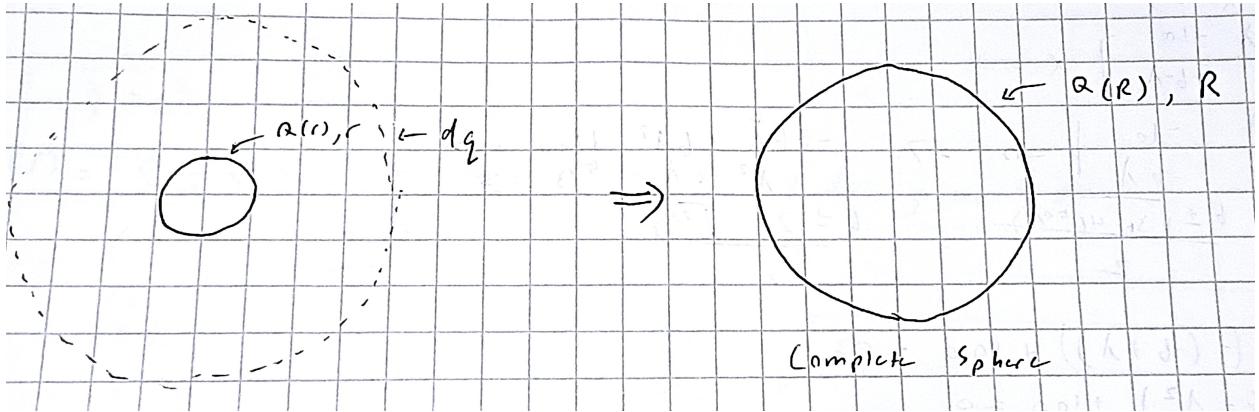
Plugging into Laplace’s Equation we get:

$$V(x, y) = \frac{1}{4} [V(x+h, y) + V(x-h, y) + V(x, y+h) + V(x, y-h)]$$

In other words, the potential at P is the average of the potentials at its 4 adjacent points.

The **Relaxation Method** essentially calculates potentials at every point (not on the boundary) and iterates using previously calculated values until it converges.

Ex: Energy to assemble a system of continuous charges (Solid Sphere)



We build the solid sphere of radius R by bringing in spherical shells of charge from infinity. Suppose a sphere of radius r with charge $q(r)$ already exists.

Charge density of complete sphere:

$$\rho = \frac{Q}{\frac{4}{3}\pi R^3} = \frac{3Q}{4\pi R^3}$$

The voltage at the surface of the sphere of radius r is:

$$V(r) = \frac{1}{4\pi\epsilon_0} \frac{q(r)}{r}$$

The work done to bring in the shell is:

$$dW = V(r) dq$$

$$W = \int \frac{1}{4\pi\epsilon_0} \frac{q(r)}{r} dq$$

Where charge of little sphere with radius r :

$$q(r) = \rho \frac{4}{3}\pi r^3 = Q \frac{r^3}{R^3}$$

Charge of shell (dq):

$$dq = \rho 4\pi r^2 dr = \frac{3Qr^2}{R^3} dr$$

Thus,

$$\begin{aligned} W &= \frac{3Q^2}{4\pi\epsilon_0 R^6} \int_0^R r^4 dr \\ &= \frac{3Q^2}{4\pi\epsilon_0 R^6} \left(\frac{R^5}{5} \right) \end{aligned}$$

$$W = \frac{3}{5} \frac{Q^2}{4\pi\epsilon_0 R}$$

Chapter 25 - Capacitance

1. Define the capacitance and explain its utility.

Generally, separation of charges will result in a potential difference. When a capacitor is charged, charges are effectively moved off of one conductor onto the other. The capacitance is a measure of how much charge can be stored on each conductor for a given potential difference.

$$q = C\Delta V$$

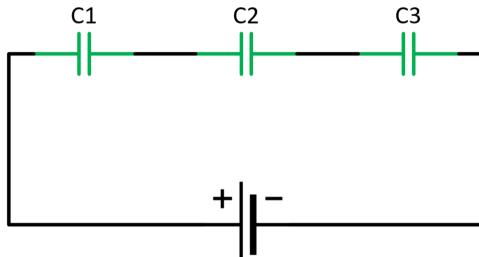
Where:

- C is the capacitance and ONLY depends on the geometry of the conductor (Farads, F or C/V)
- V is the potential difference between the conductors (Volts, V)
- q is the magnitude of charge on each conductor (Coulombs, C)

Note: The greater the capacitance, the more charge can be stored for a given potential difference.

2. Finding the equivalent capacitance for capacitors in series and parallel.

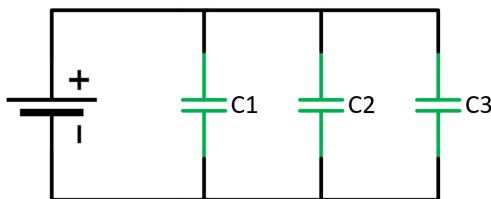
- **Capacitors in Series**



Each capacitor has the same amount of charge q on its plates, and the sum of the potential differences across each capacitor equals the total potential difference.

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \dots$$

- **Capacitors in Parallel**



Each capacitor has the same potential difference across its plates, and the sum of the charges on each capacitor equals the total charge.

$$C_{eq} = C_1 + C_2 + C_3 + \dots$$

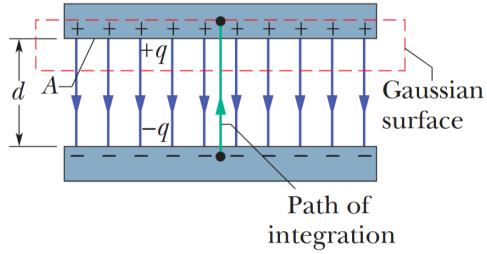
3. Calculate the capacitance for various arrangements of conductors. Especially be familiar with the properties of the ideal parallel plate capacitor.

General steps:

- Assume a charge q on each conductor.
- Calculate the electric field between the conductors using Gauss's law.
- Calculate the potential difference between the conductors using the electric field.
- Calculate the capacitance using

$$C = \frac{q}{\Delta V}$$

- **Parallel Plate Capacitor**



Using Gauss's Law around one plate:

$$\varepsilon_0 \int \vec{E} \cdot \hat{n} dA = Q_{enc}$$

$$\begin{aligned} \varepsilon_0 E A &= q \\ \implies E &= \frac{q}{\varepsilon_0 A} \end{aligned}$$

Integrating from negative to positive plate to find a positive potential difference:

$$\Delta V = - \int_{-}^{+} \vec{E} \cdot d\vec{y}$$

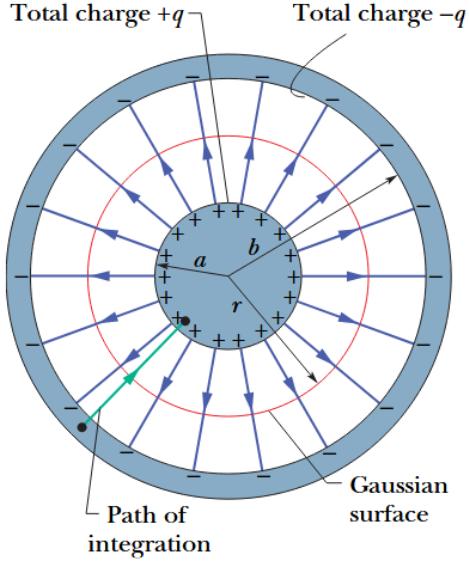
$$\begin{aligned} \Delta V &= \int_0^d E dy \\ \Delta V &= Ed \end{aligned}$$

Thus,

$$C = \frac{\varepsilon_0 A}{d}$$

- Cylindrical Capacitor

Inner cylinder with radius a , outer cylinder with radius b , length L



Using Gauss's Law around the inner cylinder:

$$\varepsilon_0 \int \vec{E} \cdot \hat{n} dA = Q_{enc}$$

$$\varepsilon_0 E (2\pi r L) = q$$

$$\implies E = \frac{q}{2\pi\varepsilon_0 r L}$$

Integrating from inner to outer cylinder to find a positive potential difference:

$$\Delta V = - \int_b^a \vec{E} \cdot d\vec{r}$$

$$\Delta V = - \int_b^a E dr$$

$$\Delta V = - \int_b^a \frac{q}{2\pi\varepsilon_0 r L} dr$$

$$\Delta V = \frac{q}{2\pi\varepsilon_0 L} \ln \left(\frac{b}{a} \right)$$

Thus,

$$C = \frac{2\pi\varepsilon_0 L}{\ln \left(\frac{b}{a} \right)}$$

- **Spherical Capacitor**

Inner sphere (+) with radius a, outer sphere (-) with radius b

Using Gauss's Law around the inner sphere:

$$\begin{aligned}\varepsilon_o \int \vec{E} \cdot \hat{n} dA &= Q_{enc} \\ \varepsilon_o E (4\pi r^2) &= q \\ \Rightarrow E &= \frac{q}{4\pi \varepsilon_o r^2}\end{aligned}$$

Integrating from inner to outer sphere to find a positive potential difference:

$$\begin{aligned}\Delta V &= - \int_b^a \vec{E} \cdot d\vec{r} \\ \Delta V &= - \int_b^a E dr \\ \Delta V &= - \int_b^a \frac{q}{4\pi \varepsilon_o r^2} dr \\ \Delta V &= \frac{q}{4\pi \varepsilon_o} \left(\frac{1}{a} - \frac{1}{b} \right)\end{aligned}$$

Thus,

$$C = 4\pi \varepsilon_o \left(\frac{ab}{b-a} \right)$$

- **Isolated Sphere** To find the capacitance of an isolated sphere, we basically treat it as a spherical capacitor where the outer sphere has infinite radius ($b \rightarrow \infty$).

Thus,

$$C = 4\pi \varepsilon_o R$$

4. Explain the use of dielectrics, their physics principles, and calculate capacitance in the presence of dielectric materials.

Dielectrics are insulating materials that increase the capacitance of a capacitor by changing the permittivity by a factor of κ (dielectric constant).

$$\varepsilon = \kappa \varepsilon_o$$

Thus,

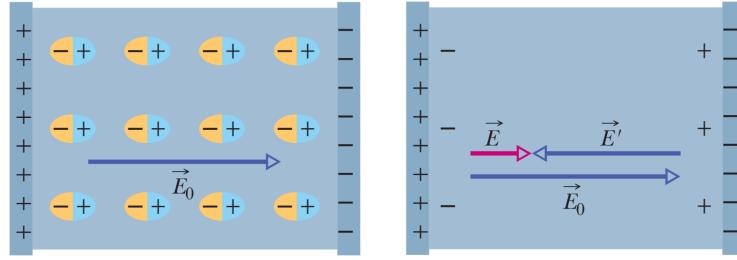
$$C = \kappa C_{\text{air}}$$

where C_{air} is the capacitance with only air between the conductors.

In a region completely filled with a dielectric, electrostatic equations become:

$$\vec{E} = \frac{1}{4\pi \kappa \varepsilon_o} \frac{q}{r^2} \quad (\text{point charge in dielectric})$$

Working mechanism:

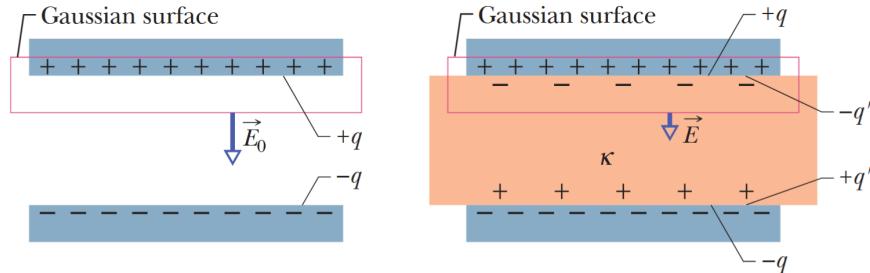


The electric field applied by charging the capacitor plates slightly stretches the atoms, separating the centers of positive and negative charge. These induced dipoles pull surface charges onto the plates that partially cancel the applied field \vec{E}_o .

Note that by decreasing the electric field between the plates, the potential difference also decreases for a given charge. Since $C = q/\Delta V$, the capacitance increases.

Gauss's Law in Dielectrics

Previously, Gauss's Law was used in a vacuum/air, now we modify it for dielectrics.



The original electric field without the dielectric is:

$$E_o = \frac{q}{A\epsilon_o}$$

With the dielectric, the enclosed charge changes:

$$E = \frac{q - q'}{A\epsilon_o} \quad (1)$$

Since the effect of the dielectric is to reduce the electric field by a factor of κ :

$$E = \frac{E_o}{\kappa} = \frac{1}{\kappa} \frac{q}{A\epsilon_o} \quad (2)$$

Equating (1) and (2):

$$\begin{aligned} \frac{q - q'}{A\epsilon_o} &= \frac{1}{\kappa} \frac{q}{A\epsilon_o} \\ \Rightarrow q - q' &= \frac{q}{\kappa} \end{aligned}$$

Substituting $q - q'$ as Q_{enc} in Gauss's Law:

$$\boxed{\varepsilon_o \int \kappa \vec{E} \cdot \hat{n} dA = q}$$

Note: The charge q enclosed by the Gaussian surface is now taken to be the free charge only. The induced surface charge is deliberately ignored because it is accounted by the factor of κ in the permittivity.

- Derive the electric field energy density. Calculate the electrostatic energy stored in a capacitor several different ways i.e., using expressions that directly involve the capacitance or computing the volume integral over the electric field energy density.

First find the energy stored in a capacitor (equal to work done to charge it):

$$dW = Vdq$$

$$C = \frac{q}{V} \implies V = \frac{q}{C}$$

$$W = \int_0^Q \frac{q}{C} dq$$

Thus,

$$\boxed{U_{cap} = \frac{1}{2} \frac{Q^2}{C} = \frac{1}{2} CV^2 = \frac{1}{2} QV}$$

Electric Field Energy Density

In a parallel plate capacitor with plates of area A and separation d , the electric field is uniform between the plates. Thus, the energy stored in the capacitor is:

$$U_{cap} = \frac{1}{2} CV^2$$

and the volume between the plates is:

$$\text{Volume} = Ad$$

The energy density (energy per unit volume) is:

$$u_E = \frac{U_{cap}}{\text{Volume}} = \frac{CV^2}{2Ad}$$

Where

$$C = \frac{\varepsilon_o A}{d}, \quad V = Ed$$

$$\boxed{u_E = \frac{1}{2} \varepsilon_o E^2}$$

Note: Even though this was derived for a parallel plate capacitor, it is valid generally.

Finding Energy Stored in a Capacitor using Energy Density

We can integrate the energy density over a volume to find the total energy stored in a capacitor.

$$U_{cap} = \int u_E dV$$

$$U_{cap} = \frac{\epsilon_0}{2} \int E^2 dV$$

Ex: For a parallel plate capacitor with plate area A and separation d

$$\begin{aligned} U_{cap} &= \frac{\epsilon_0}{2} \frac{V^2}{d^2} \int dV \\ &= \frac{\epsilon_0}{2} \frac{V^2}{d^2} Ad \\ &= \frac{1}{2} CV^2 \end{aligned}$$

Chapter 26 - Current and Resistance

1. Define resistance and know Ohm's law. Use it to find the electric field in and potential difference across a resistor.
2. Know the relations among power, potential, and resistance. Calculate the power dissipated in a resistor.

End Exam 2