# Dynamics Final Exam

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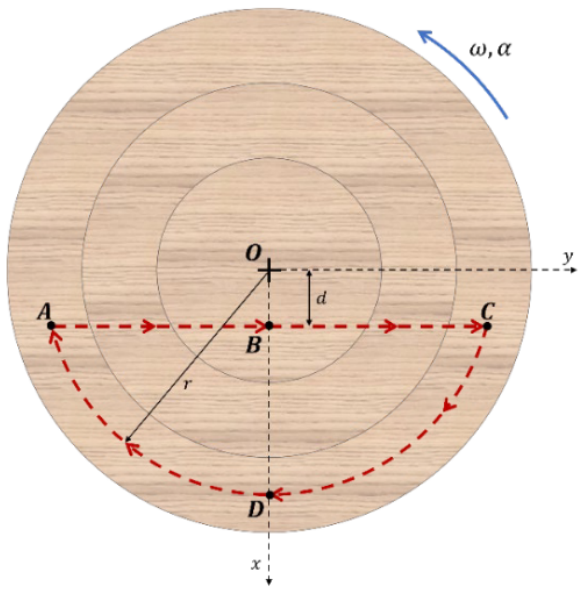
## Problem 1. Way Down Hadestown

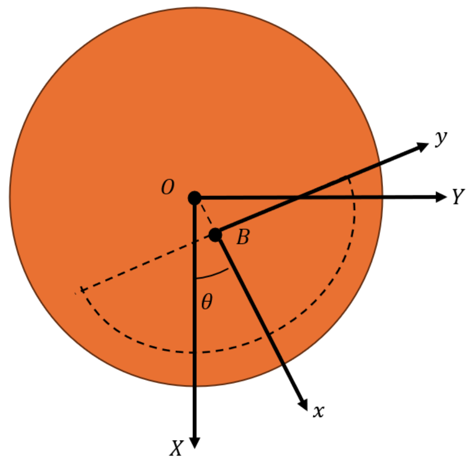



**a)** Given:

- Path $ABC$ is a straight line a distance $d = 0.5\,m$ from $O$
- Path $CDA$ is an arc of radius $r = 2\,m$
- When Orpheus reaches point $A$, the turntable has $\omega = 0.5\,rad/s$ and $\alpha = 0.1\,rad/s^2$ (counterclockwise).
- Orpheus starts from rest at point $A$ and walks with velocity $v = 0.5t\,m/s$ relative to the turntable, where $t$ is the time in seconds since he started walking.

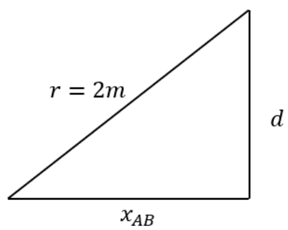
Find:

- His velocity and acceleration at point $B$




**Kinematics**

Finding time $t$ when Orpheus reaches point $B$:




$$x_{AB} = \sqrt{r^2 - d^2} \implies x_{AB} = 1.9364m$$

$$\Delta x = x_{AB} = \int_0^t v dt'$$

$$\int_0^t 0.5t' dt' = \left[0.25t'^2\right]_0^t = 0.25t^2$$

$$1.9364m = 0.25t^2 \implies t = 2.783s$$

Finding $\theta_B$ when Orpheus reaches point $B$:

$$\theta_B = \omega t + \frac{1}{2}\alpha t^2$$

$$\theta_B = 0.5(2.783) + 0.5(0.1)(2.783)^2 = 1.7787\, rad$$

Finding $\omega$ at time $t$:

$$\omega = \omega_0 + \alpha t = 0.5 + 0.1(2.783) = 0.7783\, rad/s$$

**Velocity**

$$\mathbf{v_P} = \mathbf{v_B} + \mathbf{\Omega} \times \mathbf{r_{P/B}} + \left(\mathbf{v_{P/B}}\right)$$

Where:

- $\mathbf{v_P}$: Velocity of Orpheus relative to $XYZ$
- $\mathbf{v_B}$: Velocity of point $B$ relative to $XYZ$

$$\mathbf{v_B} = \langle -d\omega sin\theta_B, d\omega cos\theta_B, 0\rangle = \langle -0.5(0.7783)sin(1.7787), 0.5(0.7783)cos(1.7787), 0\rangle$$

$$\mathbf{v_B} = \langle -0.3807, -0.0803, 0\rangle$$

- $\mathbf{\Omega}$: Angular velocity of turntable

$$\mathbf{\Omega} = \omega\mathbf{k}$$

- $\mathbf{r_{P/B}}$: Position of Orpheus relative to point $B$

$$\mathbf{r_{P/B}} = \langle 0, 0, 0\rangle$$

- $\mathbf{v_{P/B}}$: Velocity of Orpheus relative to turntable (point $B$)

$$\mathbf{v_{P/B}} = \langle 0, 0.5t, 0\rangle$$

Plugging in values:

$$\mathbf{v_P} = \langle -0.3807, -0.0803, 0\rangle + \langle 0, 1.3915, 0\rangle$$

$$\boxed{\mathbf{v_P} = \langle -0.3807, 1.3112, 0\rangle\ m/s}$$

**Acceleration**

$$\mathbf{a_P} = \mathbf{a_B} + \dot{\mathbf{\Omega}} \times \mathbf{r_{P/B}} + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r_{P/B}}) + 2\mathbf{\Omega} \times \left(\mathbf{v_{P/B}}\right)_{xyz} + \left(\mathbf{a_{P/B}}\right)_{xyz}$$

$$\mathbf{a_P} = \mathbf{a_B} + 2\mathbf{\Omega} \times \left(\mathbf{v_{P/B}}\right)_{xyz} + \left(\mathbf{a_{P/B}}\right)_{xyz}$$

Find $\mathbf{a_B}$
where the normal direction is $\hat{n} = \langle -cos\theta, -sin\theta \rangle$ and the tangential direction is $\hat{t} = \langle -sin\theta, cos\theta \rangle$:

$$a_{Bn} = r\omega^2$$

$$a_{Bt} = r\alpha$$

Note that at $t = 2.783s$, $\omega = 0.7783\, rad/s$ and $\alpha = 0.1\, rad/s^2$:

$$\mathbf{a_B} = (0.5)(0.7783)^2 \langle -cos\theta_B, -sin\theta_B, 0 \rangle + (0.5)(0.1) \langle -sin\theta_B, cos\theta_B, 0 \rangle$$

$$\mathbf{a_B} = 0.3028 \langle -cos(1.7787), -sin(1.7787), 0 \rangle + 0.05 \langle -sin(1.7787), cos(1.7787), 0 \rangle$$

$$\mathbf{a_B} = \langle 0.01357, -0.30659, 0 \rangle$$

Cross product:

$$2\mathbf{\Omega} \times \left(\mathbf{v_{P/B}}\right)_{xyz} = 2(\omega\hat{k}) \times (1.3915\hat{\jmath}) = -2(0.7783)(1.3915\hat{\imath}) = -2.166\hat{\imath}$$

Find $\left(\mathbf{a_{P/B}}\right)_{xyz}$:

$$\left(\mathbf{a_{P/B}}\right)_{xyz} = \frac{d}{dt}v = \frac{d}{dt}0.5t = 0.5\hat{\jmath}$$

Plugging in values:

$$\mathbf{a_P} = \langle 0.01357, -0.30659, 0 \rangle + \langle -2.166, 0, 0 \rangle + \langle 0, 0.5, 0 \rangle$$

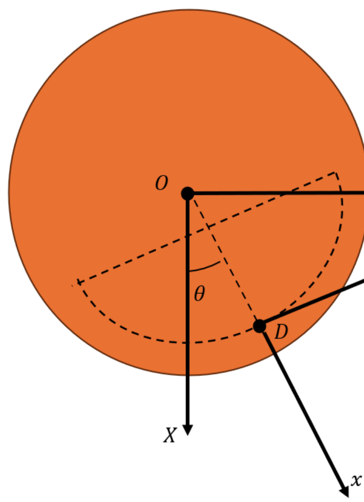
$$\boxed{\mathbf{a_P} = \langle -2.1524, 0.1934, 0 \rangle \ m/s^2}$$

**b)** Given:

- After point $B$ Orpheus continues walking along path $CDA$ at constant speed $v = 1.5\,m/s$ relative to the turntable.
- At point $D$, the turntable has $\omega = 0.5\,rad/s$ and $\alpha = 0.1\,rad/s^2$ (counterclockwise).

Find:

- His velocity and acceleration at point $D$




**Kinematics from point A to point D**
From part A, we know that the distance $AB$ is 1.9364m. Thus, the distance $AC$ is $2(1.9364) = 3.8728m$.

Also from part A, we can find the time $t_{AC}$ it takes for Orpheus to reach point $C$:

$$3.8728\,m = 0.25t^2 \implies t_{AC} = 3.935\,s$$

To find the arc length $s_{CD}$ from point $C$ to point $D$, we need to find the angle in the sector $AOC$. Using the same triangle from part A:

$$\theta_{AOC} = 2 \times \theta_{OAB} = 2 \times cos^{-1}\left(\frac{d}{r}\right) = 2 \times cos^{-1}\left(\frac{0.5}{2}\right) = 2.63623\,rad$$

Thus the arclength $s_{CD}$ is:

$$s_{CD} = r\theta_{AOC} = 2(2.63623) = 5.2724\,m$$

Finding the total time since Orpheus started walking from point $A$ to point $D$:

- Time from $A$ to $C$: $t_{AC} = 3.935\,s$

- Time from $C$ to $D$:

$$s_{CD} = v_{CD}t_{CD} \implies t_{CD} = \frac{s_{CD}}{v_{CD}} = \frac{5.2724}{1.5} = 3.5149\,s$$

- Total time from $A$ to $D$:

$$t_{AD} = t_{AC} + t_{CD} = 3.935 + 3.5149 = 7.4499\,s$$

Finding $\theta_D$ when Orpheus reaches point $D$:

- From $A$ to $D$:

$$\theta_D = \omega_0 t_{AD} + \frac{1}{2}\alpha t_{AD}^2$$
$$\theta_D = 0.5(7.4499) + 0.5(0.1)(7.4499)^2 = 6.5\,rad$$

**Velocity**
Using the same velocity equation from part A:

$$\mathbf{v_P} = \mathbf{v_D} + \mathbf{\Omega} \times \mathbf{r_{P/D}} + \left(\mathbf{v_{P/D}}\right)$$

Where:

- $\mathbf{v_D}$: Velocity of point $D$ relative to $XYZ$

$$\mathbf{v_D} = \langle -r\omega sin\theta_D, r\omega cos\theta_D \rangle$$

$$\mathbf{v_D} = \langle -(2)(0.5)sin(6.5), (2)(0.5)cos(6.5) \rangle$$
$$\mathbf{v_D} = \langle -0.21511, 0.97658, 0 \rangle$$

- $\mathbf{r_{P/D}}$: Position of Orpheus relative to point $D$

$$\mathbf{r_{P/D}} = \langle 0, 0, 0 \rangle$$

- $\mathbf{v_{P/D}}$: Velocity of Orpheus relative to turntable (point $D$)

$$\mathbf{v_{P/D}} = \langle 1.5sin\theta_D, -1.5cos\theta_D, 0 \rangle$$

$$\mathbf{v_{P/D}} = \langle 1.5sin(6.5), -1.5cos(6.5), 0 \rangle = \langle 0.3226, -1.4648, 0 \rangle$$

Plugging in values:

$$\mathbf{v_P} = \langle -0.21511, 0.97658, 0 \rangle + \langle 0.3226, -1.4648, 0 \rangle$$

$$\boxed{\mathbf{v_P} = \langle 0.10748, -0.48822, 0 \rangle \; m/s}$$

**Acceleration**
Using the same acceleration equation from part A:

$$\mathbf{a_P} = \mathbf{a_D} + \dot{\boldsymbol{\Omega}} \times \mathbf{r_{P/D}} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r_{P/D}}) + 2\boldsymbol{\Omega} \times \left(\mathbf{v_{P/D}}\right)_{xyz} + \left(\mathbf{a_{P/D}}\right)_{xyz}$$

$$\mathbf{a_P} = \mathbf{a_D} + 2\boldsymbol{\Omega} \times \left(\mathbf{v_{P/D}}\right)_{xyz} + \left(\mathbf{a_{P/D}}\right)_{xyz}$$

Where:

- $\mathbf{a_D}$: Acceleration of point $D$ relative to $XYZ$
  where the normal direction is $\hat{n} = \langle -cos\theta, -sin\theta \rangle$ and the tangential direction is $\hat{t} = \langle -sin\theta, cos\theta \rangle$:

  $$a_{Dn} = r\omega^2$$

  $$a_{Dt} = r\alpha$$

  Note that at $D$, $\omega = 0.5\, rad/s$ and $\alpha = 0.1\, rad/s^2$:

  $$\mathbf{a_D} = (0.5)(0.5)^2 \langle -cos\theta_D, -sin\theta_D, 0 \rangle + (0.5)(0.1) \langle -sin\theta_D, cos\theta_D, 0 \rangle$$

  $$\mathbf{a_D} = 0.125 \langle -cos(6.5), -sin(6.5), 0 \rangle + 0.05 \langle -sin(6.5), cos(6.5), 0 \rangle$$

  $$\mathbf{a_D} = \langle -0.1328, 0.021939, 0 \rangle$$

- $\mathbf{v_{P/D}}$: Velocity of Orpheus relative to turntable (point $D$)

  $$\mathbf{v_{P/D}} = \langle 0.3226, -1.4648, 0 \rangle$$

- $\mathbf{a_{P/D}}$: Acceleration of Orpheus relative to turntable (point $D$)

  $$\mathbf{a_{P/D}} = \langle 0, 0, 0 \rangle$$

Cross product:

$$2\boldsymbol{\Omega} \times \left(\mathbf{v_{P/D}}\right)_{xyz} = 2(\omega\hat{k}) \times \langle 0.3226, -1.4648, 0 \rangle = (2\omega \cdot 1.4648)\hat{\imath} + (2\omega \cdot 0.3226)\hat{\jmath}$$

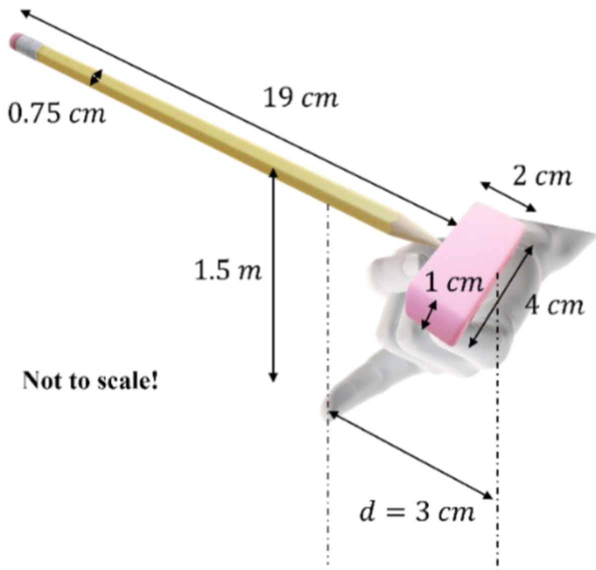
$$2\boldsymbol{\Omega} \times \left(\mathbf{v_{P/D}}\right)_{xyz} = \langle 1.4648, 0.3226, 0 \rangle$$

Plugging in values:

$$\mathbf{a_P} = \langle -0.1328, 0.021939, 0 \rangle + \langle 1.4648, 0.3226, 0 \rangle + \langle 0, 0, 0 \rangle$$

$$\boxed{\mathbf{a_P} = \langle 1.332, 0.3445, 0 \rangle \ m/s^2}$$

# Problem 2. Nervous Fidgeting




Given:

- The pencil is a uniform rod of density $\rho_p = 0.9 g/cm^3$ and the eraser is a rectangular prism of density $\rho_e = 1.4 g/cm^3$.
- The coefficient of restitution between the pencil and finger is $e = 0.1$.
- Impact is impulsive and the finger does not move, only hitting tangentially

**a)** Find the 3D center of mass of the pencil-eraser system.

Finding the masses:

- Mass of pencil:

$$v_p = \pi \left(\frac{.75}{2}\right)^2 (19) = 8.3939\, cm^3$$

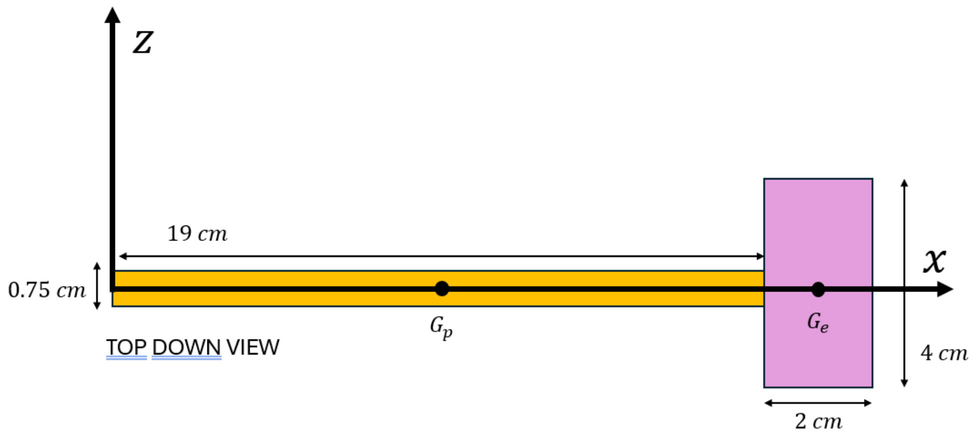
$$m_p = \rho_p v_p = (0.9)(8.3939) = 7.5545\, g$$

- Mass of eraser:

$$v_e = (1)(4)(2) = 8\, cm^3$$

$$m_e = \rho_e v_e = (1.4)(8) = 11.2\, g$$

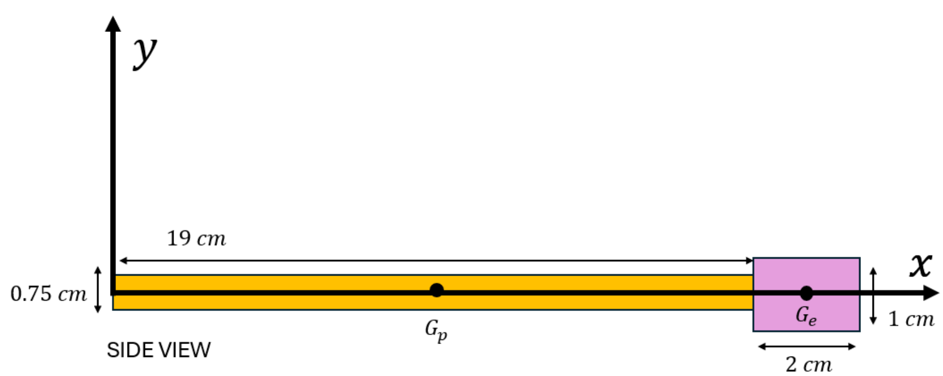
Finding the center of mass:




$$\bar{x} = \frac{m_p(9.5) + m_e(20)}{m_p + m_e}$$

$$\bar{x} = \frac{7.5545(9.5) + 11.2(20)}{7.5545 + 11.2} = 15.7704\,cm$$

$$\bar{z} = \frac{m_p(0) + m_e(0)}{m_p + m_e} = 0\,cm$$




$$\bar{y} = \frac{m_p(0) + m_e(0)}{m_p + m_e} = 0\,cm$$

$$\boxed{G_{\text{sys}} = \langle 15.7704, 0, 0 \rangle\ cm}$$

**b)** Calculate the moment of inertia of the pencil and eraser system about the relevant centroidal axis.

Axis of rotation: z-axis through center of mass

Using the parallel axis theorem:

$$I_{G_{sys}} = I_{G_p} + m_p d_p^2 + I_{G_e} + m_e d_e^2$$

Where:

- $I_{G_p}$: Moment of inertia of pencil about its centroidal axis

$$I_{G_p} = \frac{1}{12} m_p L^2 = \frac{1}{12}(7.5545)(19^2) = 227.279\, g \cdot cm^2$$

- $d_p$: Distance from pencil centroid to system centroid

$$d_p = 15.7704 - 9.5 = 6.2704\, cm$$

- $I_{G_e}$: Moment of inertia of eraser about its centroidal axis

$$I_{G_e} = \frac{1}{12} m_e (b^2 + c^2) = \frac{1}{12}(11.2)(1^2 + 2^2) = 4.666\, g \cdot cm^2$$

- $d_e$: Distance from eraser centroid to system centroid

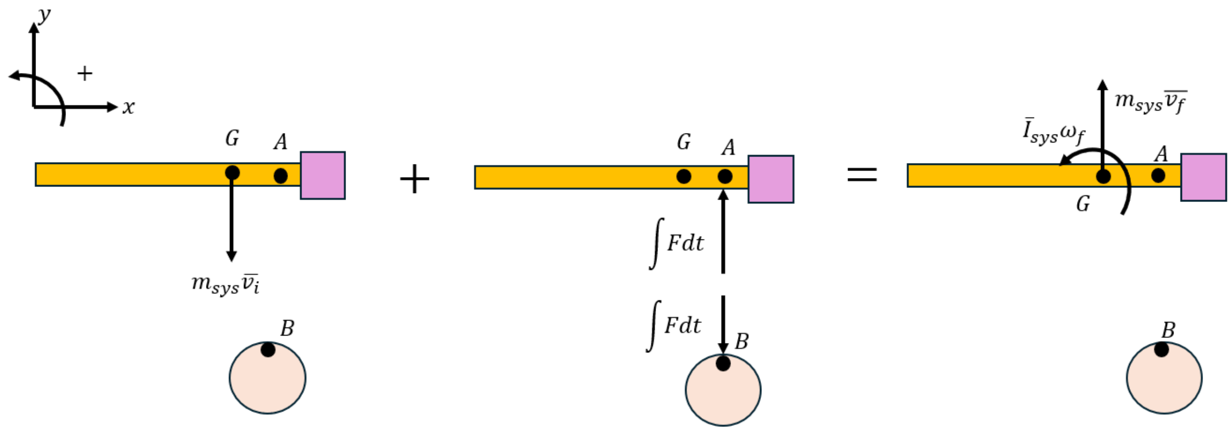
$$d_e = 20 - 15.7704 = 4.2296\, cm$$

Plugging in values:

$$I_{G_{sys}} = 227.279 + 7.5545(6.2704)^2 + 4.666 + 11.2(4.2296)^2$$

$$\boxed{I_{G_{sys}} = 729.334\, g \cdot cm^2}$$

**c)**




**d)** After the pencil has fallen 1.5 m, it hits your finger. Find the angular velocity and the velocity of the center of mass just before and just after the impact.

Finding velocity of center of mass just before impact:

$$mgh = \frac{1}{2}mv^2$$

$$v = \sqrt{2gh} = \sqrt{2(9.81)(1.5)} = 5.4249\,m/s = 542.49\,cm/s$$

Distance from center of mass to point of impact:

$$d = (21-3)cm - (15.7704)6cm = 2.2296\,cm$$

Conservation of angular momentum about point of impact:

$$m_{\text{sys}}\bar{v}_i(2.2296) = -m_{\text{sys}}\bar{v}_f(2.2296) + \bar{I}_{\text{sys}}\omega_f$$

$$41.815\bar{v}_i = -41.815\bar{v}_f + 729.334\omega_f$$

$$22678.79 = -41.815\bar{v}_f + 729.334\omega_f$$

Coefficient of restitution:

$$v'_B - v'_A = e[v_A - v_B]$$

$$-v'_A = (0.1)v_A$$

$$\implies v'_A = -0.1(-542.49), \quad v_A = \bar{v}$$

$$v'_A = 54.249\hat{\jmath}$$

Kinematics:

$$v'_A = \bar{v}_f + \omega\hat{k} \times r_{A/G}$$

$$v'_A = \bar{v}_f + 2.2296\omega\hat{\jmath}$$

$$\implies \bar{v}_f = 54.249\hat{\jmath} - 2.2296\omega\hat{\jmath}$$

Plugging in $\bar{v}_f$ into the angular momentum equation:

$$22678.79 = -41.815(54.249 - 2.2296\omega) + 729.334\omega$$

$$\implies \boxed{\omega_f = 30.3285\, rad/s}$$

Finding $\bar{v}_f$:

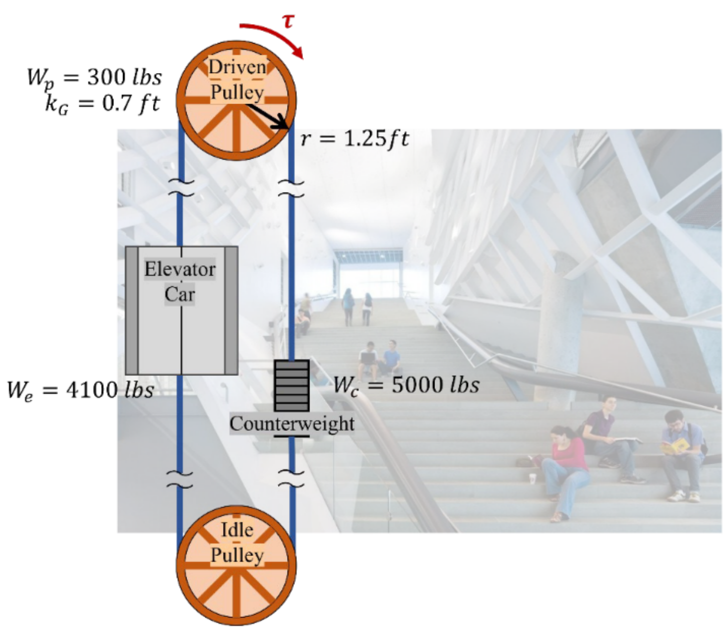
$$\bar{v}_f = 54.249\hat{\jmath} - 2.2296(30.3285)\hat{\jmath}$$

$$\boxed{\bar{v}_f = -13.3714\,(cm/s)\,\hat{\jmath}}$$

Recall that just before impact:

$$\boxed{\bar{v}_i = -542.49\,(cm/s)\,\hat{\jmath} \quad \omega_i = 0\, rad/s}$$

# Problem 3: 41CS Elevator




Given:

- The cable does not slip
- Each pulley has weight $W_p = 300lbs$, radius $r_p = 1.25ft$, and radius of gyration $k_G = 0.7ft$
- Elevator car weighs $W_e = 4100lbs$
- Counterweight weighs $W_c = 5000lbs$
- One of the pulleys is driven with torque $\tau$

**a)** Find the moment of inertia of each pulley about its axle.

$$\bar{I}_P = mk_G^2$$

$$\boxed{\bar{I}_P = \frac{300}{32.2}(0.7)^2 = 4.565\,lb\cdot ft^2}$$

**b)** Find the *constant* torque required for the elevator car to have speed 10 $ft/s$ after moving up 15 $ft$ from the ground floor.

Let the datum be at the ground floor and up be positive. Position 1 is at the ground floor and position 2 is at 15 ft above the ground floor. Assuming at position 1, the elevator car and the pulleys are at rest and the counterweight is at $y_0$ ft above the ground floor.

$$T_1 + V_{g1} + V_{e1} + U_{1\to 2}^{NC} = T_2 + V_{g2} + V_{e2}$$

$$m_c g y_0 + \tau\Delta\theta = \frac{1}{2}m_e v_f^2 + \frac{1}{2}m_c v_f^2 + 2\left(\frac{1}{2}I_p\omega_f^2\right) + m_e g(15\,ft) + m_c g(y_0 - 15\,ft)$$

$$\tau\Delta\theta = \frac{1}{2}m_e(10\,ft/s)^2 + \frac{1}{2}m_c(10\,ft/s)^2 + I_p\omega_f^2 + m_e g(15\,ft) + m_c g(-15\,ft)$$

Kinematics:

$$\Delta y_e = -\Delta y_c \implies v_e = -v_c$$

$$v = 10\,ft/s = \omega_f r \quad \text{(cable does not slip)}$$

$$\implies \omega_f = \frac{10}{1.25} = 8\,rad/s$$

Finding $\Delta\theta$:

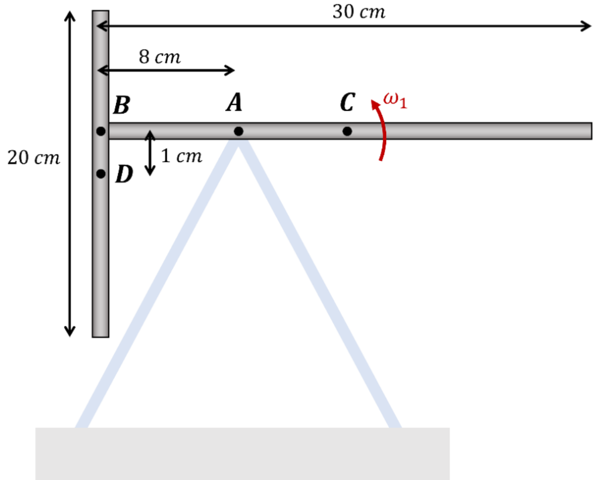
$$\Delta\theta = \frac{\Delta y}{r} = \frac{15\,ft}{1.25\,ft} = 12\,rad$$

Plugging in values:

$$\tau(12) = \frac{1}{2}\left(\frac{4100}{32.2}\right)(10)^2 + \frac{1}{2}\left(\frac{5000}{32.2}\right)(10)^2 + 4.565(8^2) + (4100)(15) + (5000)(-15)$$

$$\implies \boxed{\tau = 76.8828\,lb\cdot ft}$$

# Problem 4: The Swinging Sticks




Given:

- Rod BAC has mass $m_1 = 64g$
- Rod BD has mass $m_2 = 43g$
- BAC has center at C, BD has center at D
- At the instant, rod BAC has angular velocity $\omega_1 = 2\,rad/s$ (CCW) and rod BD has no angular velocity $\omega_2 = 0\,rad/s$