Mathematical Methods

1. Use the Taylor series expansion to find approximations. The ones for sin, cos, tan, and $(1+x)^n$ are especially useful.

$$\sin x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\tan x = \sum_{n=1}^{\infty} \frac{B_{2n}(-4)^n(1-4^n)}{(2n)!} x^{2n-1} = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \cdots$$

$$(1+x)^m = \sum_{n=0}^{\infty} \binom{m}{n} x^n = 1 + mx + \frac{m(m-1)}{2}x^2 + \frac{m(m-1)(m-2)}{6}x^3 + \cdots$$

<u>Note:</u> For small x, higher order terms reduce to zero

2. Use complex exponentials to manipulate complicated trig functions.

$$e^{ix} = \cos x + i \sin x$$

3. Solve differential equations by substituting in trial solutions. Especially you should recognize the differential equation for a simple harmonic oscillator and be able to come up with solutions to that ODE that satisfy any initial conditions you are given.

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

$$x(t) = A\cos(\omega t) + B\sin(\omega t)$$

Wave equation:

$$v^2 \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2}$$

4. Useful integration formulas:

$$\int \frac{1}{(x^2+a^2)^{\frac{3}{2}}} dx = \frac{x}{a^2\sqrt{x^2+a^2}} + C$$

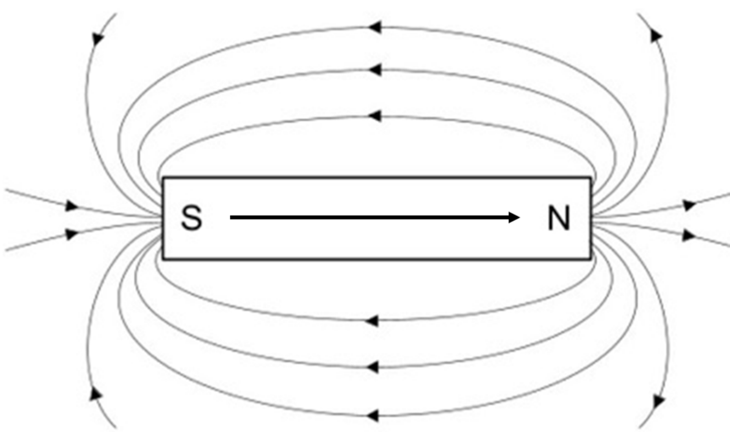
$$\int \frac{x}{(x^2+a^2)^{\frac{3}{2}}} dx = -\frac{1}{\sqrt{x^2+a^2}} + C$$

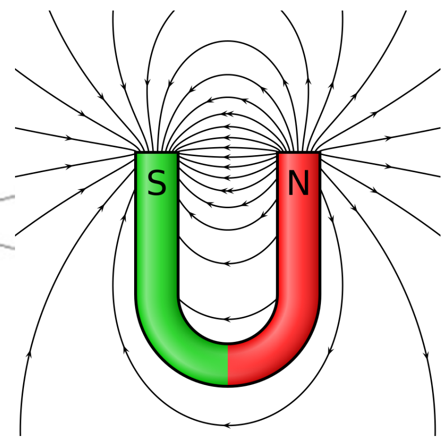
Chapter 28 - Magnetic Fields

Types of magnets

- **Current loop:** a current carrying loop of wire creates an electromagnet.
- **Permanent Magnet:** the magnetic fields of the electrons within the material do not cancel out, resulting in a net magnetic field.

All magnets are **magnetic dipoles** with a **north** and **south** pole (the magnetic monopole doesn't exist, sadly). Opposite magnetic poles attract each other, and like magnetic poles repel each other. Magnetic field lines are *closed loops* that exit through the North pole and enter through the South pole.





Note: Inside the bar magnet, the magnetic field lines point from south to north, completing the closed loop.

Magnetic field lines and the magnetic field are related by:

- The direction of the magnetic field is tangent to the field lines.
- The spacing of the field lines represents the strength (magnitude) of the magnetic field. Closer lines = stronger field.

Also, analogous to Gauss's law for electric fields, we have **Gauss's law for magnetism:**

$$\int \vec{B} \cdot \hat{n}\, d\vec{A} = 0$$

Since there are no magnetic monopoles, the net magnetic flux through any closed surface is zero (there are no sources or sinks of magnetic field lines).

1. Solve Newton's second law to determine the motion of charged particles acting under the influence of a magnetic field and any other forces (e.g., gravity, electric fields...).

   Stationary charges do not interact with the magnetic field. Moving charges with a component of velocity perpendicular to the magnetic field experiences a force:

$$\boxed{\vec{F}_B = q\vec{v} \times \vec{B}}$$

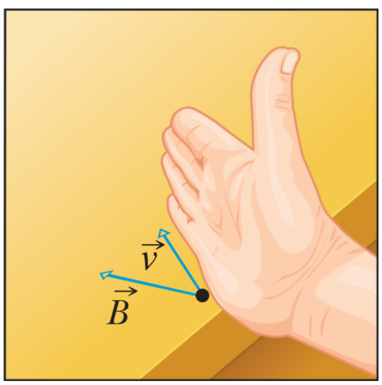
   Note: This force is *always* perpendicular to the velocity of the particle, so it does **no work** on the particle and cannot change its speed, only its direction.

   Note: The magnetic force is zero when the velocity is along the magnetic field lines (i.e., parallel or antiparallel) or when stationary.

   The unit for the magnetic field $\vec{B}$ is the Tesla (T):

$$1\,\mathrm{T} = 1\,\frac{\mathrm{N}}{\mathrm{C}\cdot\mathrm{m/s}} = 1\,\frac{\mathrm{N}}{\mathrm{A}\cdot\mathrm{m}}$$

   Recall: **Right hand rule**



   - Point fingers in the direction of the velocity $\vec{v}$.
   - Curl fingers toward the direction of the magnetic field $\vec{B}$, sweeping through the smaller angle.
   - Thumb points in the direction of the force $\vec{F}_B$ for **a positive charge**. For a negative charge, the force is in the opposite direction.

   Note: When $\vec{B}$ and $\vec{v}$ are orthogonal, we can just multiply the magnitudes to find the force and use the right hand rule to find the direction.

   Note: A magnetic force exists even if there is *relative velocity* between charges and a magnetic field. For example, a moving magnet will exert a magnetic force on stationary charges.

   Note to self: Bring dynamics formula sheet for kinematics equations.

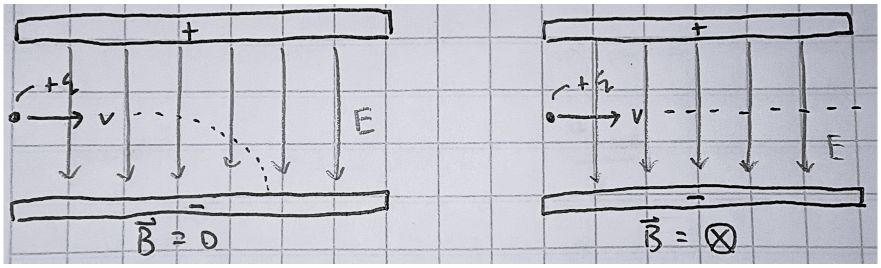
   The total electromagnetic force on a charged particle in both electric and magnetic fields is given by the **Lorentz force:**

$$\boxed{\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}}$$

2. Explain the Hall effect and describe its applications.

Here are several interesting applications where both the magnetic field and electric field acts on a moving charge.

Wien Filter (Velocity Selector)



Suppose we have a source charged particles ($+q$) with random velocities. If it passes through a region with *only* an $\vec{E}$ field, it will be pushed onto the negative plate, following a parabolic trajectory. However, if there is a $\vec{B}$ field in addition to the $\vec{E}$ field, then the forces will cancel for particles with a specific velocity:

$$\sum \vec{F}_y = \vec{F}_B - \vec{F}_E = 0 \implies \vec{F}_B = \vec{F}_E$$

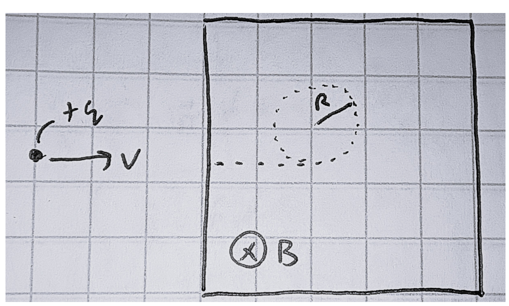
$$q\vec{v} \times \vec{B} = q\vec{E}$$

$$\boxed{v = \frac{E}{B}}$$

Thus, only particles with velocity $v = E/B$ will pass straight through the filter.

By combining the Wien filter with another region of magnetic fields, we create a mass spectrometer that can separate particles based on their charge-to-mass ratio.

Mass Spectrometer



Since we know both the charge and velocity entering the magnetic field region, we can find the particle's mass by measuring the radius of its circular path:

$$\sum \vec{F}_n = \vec{F}_B = \frac{mv^2}{R} \implies qvB = \frac{mv^2}{R}$$

$$\boxed{m = \frac{BRq}{v}}$$

Note: Typically, the charges are accelerated through a potential difference V before entering the velocity selector, so we can find their velocity using energy conservation:

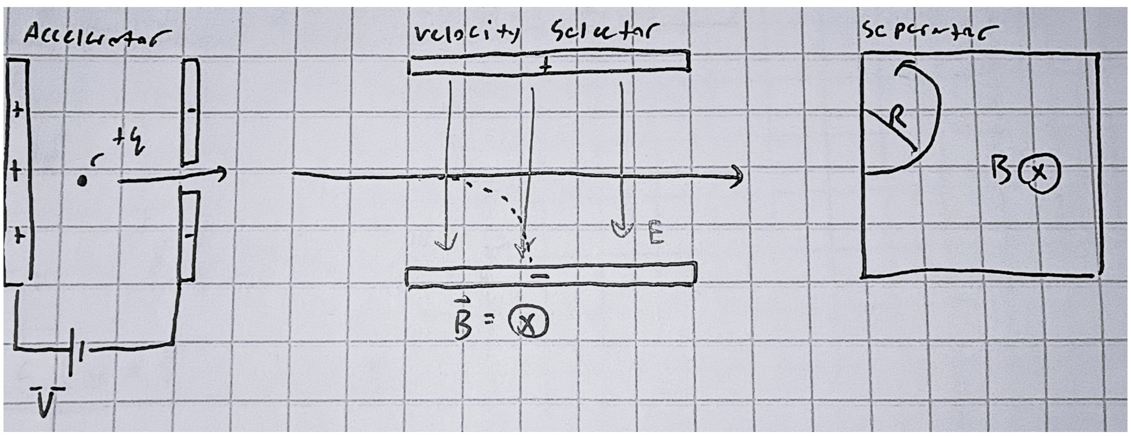
$$W_{nc} = \Delta E = 0 \implies -\Delta U = \Delta K$$

$$\Delta K = -q\Delta V$$

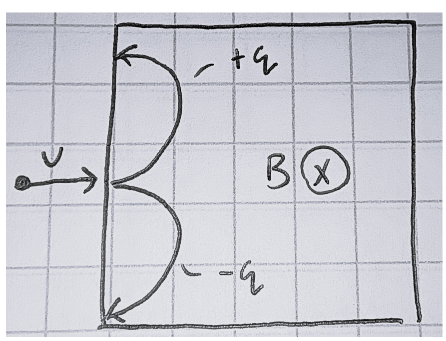
For a positive charge to gain kinetic energy, it must be accelerated through a drop in electric potential (from high to low potential), thus $\Delta V = V_f - V_i = -V$:

$$qV = \frac{1}{2}mv^2 \implies v = \sqrt{\frac{2qV}{m}}$$

The full set up looks like this:

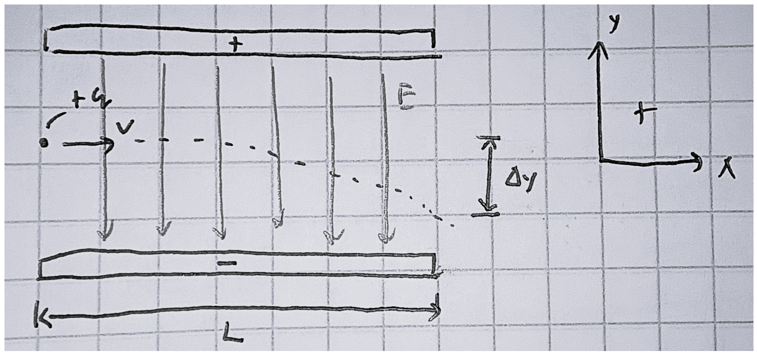


Also note that since the magnetic force is opposite for negative charges, they will curve in the opposite direction in the magnetic field region:



Let's consider a setup similar to a Wien filter, but where the parallel plates are designed to deflect the particle beam rather than selectively filter it.

Cathode Ray Tube



First, without a $\vec{B}$ field, the particles will be deflected by the $\vec{E}$ field:

$$\sum \vec{F} = -qE\hat{j} \implies |a_y| = \frac{|q|E}{m}$$

The time spent in the field is:

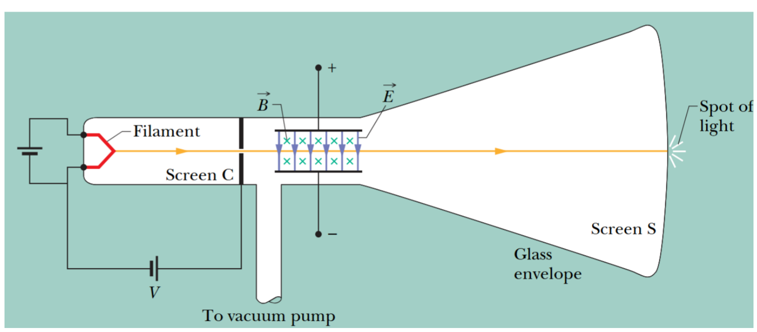
$$\Delta x = vt = L \implies t = \frac{L}{v}$$

The vertical displacement upon exiting the plates is:

$$\Delta y = v_{oy}t + \frac{1}{2}a_y t^2 = 0 + \frac{1}{2}\left(\frac{|q|E}{m}\right)\left(\frac{L}{v}\right)^2$$

$$\boxed{\Delta y = \frac{|q|EL^2}{2mv^2}}$$

Now consider adding a magnetic field like this:



We know from the Wien filter that the forces will cancel when:

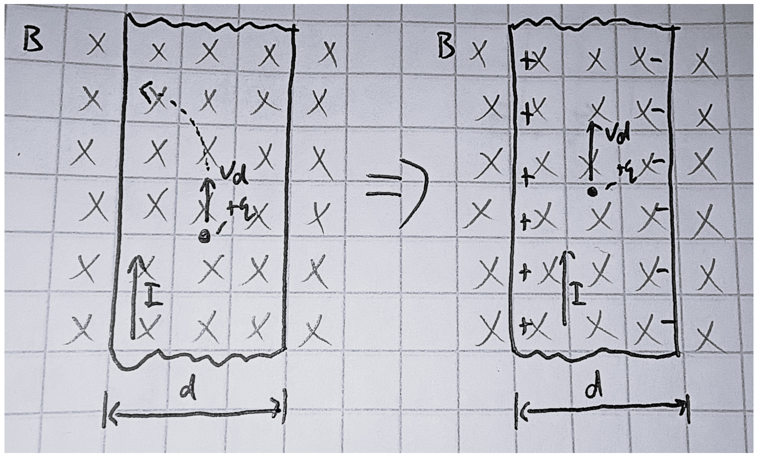
$$v = \frac{E}{B}$$

Thus, plugging this into our previous equation for vertical displacement:

$$\frac{m}{|q|} = \frac{BL^2}{2\Delta y E}$$

Finally, let's talk about the Hall effect!

Hall Effect



Consider a conductor with a current I flowing through it in a region with a $\vec{B}$ field. The moving charges will be pushed to one side of the conductor by the magnetic force, creating a **Hall potential difference** (ΔV) and an electric field ($\vec{E}$) inside the conductor.

$$\Delta V = Ed$$

Eventually, when the electric force balances the magnetic force, the charges stop accumulating.

$$\sum \vec{F} = \vec{F}_E - \vec{F}_B = 0 \implies qE = qv_d B$$

Thus, by measuring the Hall potential difference, we can find the magnetic field strength:

$$\boxed{B = \frac{\Delta V}{v_d d}}$$

We can also find the number of charge carriers per unit volume (n) in the conductor, letting $q = e$ for electrons and plugging in for v_d from before:

$$I = nev_d A, \quad \text{(A is cross-sectional area of conductor)}$$

$$n = \frac{IBd}{eA\Delta V}$$

<u>Note:</u> It is also possible to determine the drift velocity using the Hall effect, by mechanically moving the conductor such that there is no relative velocity between the charges and the magnetic field. Therefore, there will be **zero** Hall potential difference (since there is no magnetic force).

3. Explain the principle of operation of a cyclotron

Circulating Charged Particles

We know that for a particle of charge q moving with $\vec{v}$ in a uniform magnetic field, it will tend towards a circular path due to the magnetic force (no tangential force, only normal force).

$$\sum F_n = |q|vB = \frac{mv^2}{r} \implies r = \frac{mv}{|q|B} \quad \text{(radius)}$$

From which we can define the following quantities:

$$T = \frac{2\pi r}{v} = \frac{2\pi}{v}\frac{mv}{|q|B} = \frac{2\pi m}{|q|B} \quad \text{(period)}$$

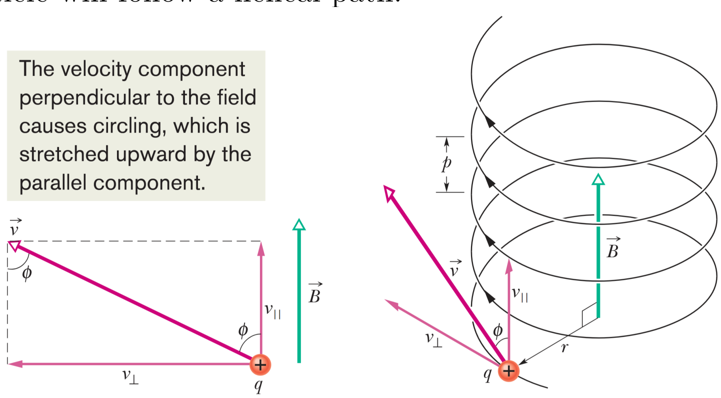
$$f = \frac{1}{T} = \frac{|q|B}{2\pi m} \quad \text{(frequency)}$$

$$\omega = \frac{2\pi}{T} = 2\pi f = \frac{|q|B}{m} \quad \text{(angular frequency)}$$

<u>Note:</u> The quantities $T, f,$ and ω do not depend on the speed of the particle (as long as it isn't moving at relativistic speeds). Fast particles move in large circles and slow ones in small circles, but all particles with the same charge-to-mass ratio $|q|/m$ take the same time T to complete one loop.

Helical Paths

If the velocity of the charged particle has a component parallel to the magnetic field, then the particle will follow a helical path:



Where the angle ϕ is the angle between $\vec{v}$ and $\vec{B}$.

$$v_{\parallel} = v\cos\phi, \quad v_{\perp} = v\sin\phi$$

The radius of the helical path is determined by $v_{\perp}$:

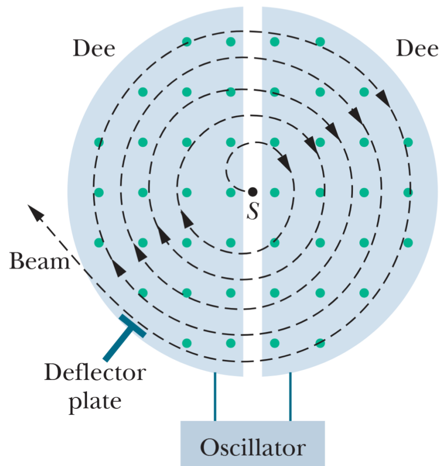
$$r = \frac{mv_{\perp}}{|q|B} = \frac{mv\sin\phi}{|q|B}$$

The pitch of the helix (distance between successive turns) is determined by $v_{\parallel}$ and the period T:

$$p = v_{\parallel}T = v\cos\phi\left(\frac{2\pi m}{|q|B}\right)$$

Cyclotron

A cyclotron is a device that uses a combination of a constant magnetic field and an oscillating electric potential difference to accelerate charged particles. The magnetic field forces the particles to move in circular paths while the potential difference between the dees accelerates them each time they cross the gap.



Suppose a proton is injected at source S. It will be accelerated toward the negatively charged dee and enter it. Once inside, there will be no electric field (shielded by the conducting walls of the dee), and it will move in a semicircular path due to the magnetic field. When it exits the dee, the potential difference is reversed to accelerate it again across the gap. Thus, the frequency f at which the proton circulates (independent of speed) *must* match the frequency of the oscillating potential difference f_{osc}:

$$f = f_{osc} \quad \text{(resonance condition)}$$

$$\frac{|q|B}{2\pi m} = f_{osc}$$

Note: To find the speed of particles exiting the cyclotron, we can use the radius and centripetal force due to the magnetic field, or by using the frequency of oscillation:

$$f = \frac{1}{T} = \frac{v}{2\pi r}$$

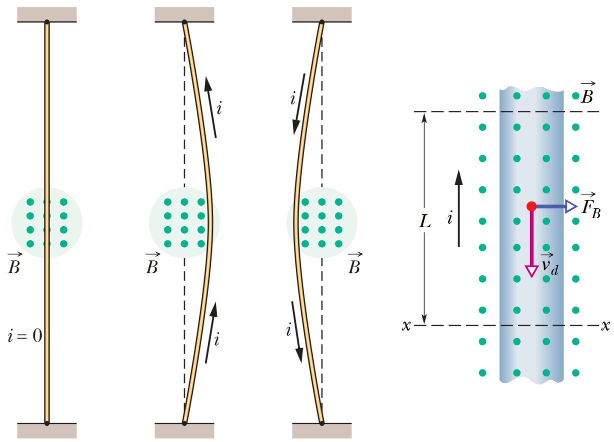
Synchrotron

At relativistic speeds (above 10% of c), the frequency of revolution is now no longer independent of the charged particle's speed. As the speed approaches the speed of light, the frequency of revolution decreases, and is no longer in sync to the fixed f_{osc}. Thus a **synchrotron** is used to vary both the magnetic field and f_{osc} to keep the particle in resonance as it accelerates to higher speeds. The proton also follows a circular path instead of a spiral in a synchrotron.

4. Determine the forces and/or torques on various arrangements of current carrying wires (straight, circular loops, square loops, etc...) located in a given magnetic field.

Magnetic Force on a Current Carrying Wire

We know that moving charges experience a magnetic force in a magnetic field. Thus, a current-carrying wire (which has moving charges) will also experience a magnetic force when placed in a magnetic field.



Note: The motion of electrons is opposite to the direction of conventional current. However, since both the charge and velocity are negative, the magnetic force ends up being in the same direction as if we considered positive charges moving with the current.

We know that the magnetic force on a single charge is:

$$\vec{F}_B = q\vec{v} \times \vec{B}$$

Thus, for N charges in a wire segment of length L, the total magnetic force is:

$$\vec{F}_B = Nq\vec{v}_d \times \vec{B}$$

If we rewrite N in terms of the number of charge carriers per unit volume n and the volume of the wire segment AL (where A is the cross-sectional area), we get:

$$\vec{F}_B = (nAL)q\vec{v}_d \times \vec{B}$$

Recall that current is defined as $I = qnv_dA$, so we can rewrite the magnetic force as:

$$\boxed{\vec{F}_B = I\vec{L} \times \vec{B} \quad \text{(force on a straight wire)}}$$

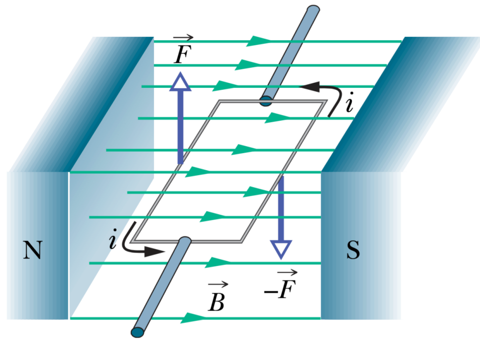
Where $\vec{L}$ is a vector in the direction of the conventional current with magnitude L.

If the wire is not straight or the field is not uniform, we can find the differential force on a small current element Idl and integrate over the length of the wire:

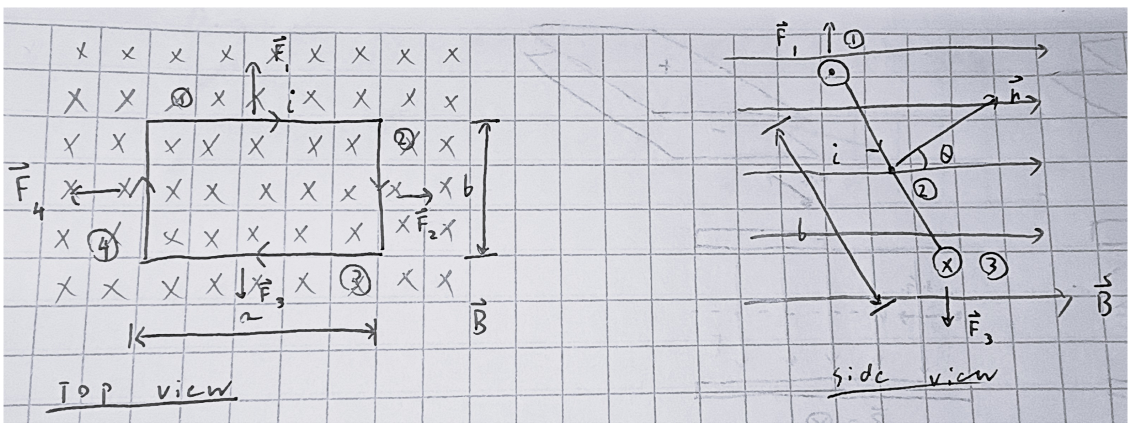
$$\boxed{d\vec{F}_B = Id\vec{L} \times \vec{B}}$$

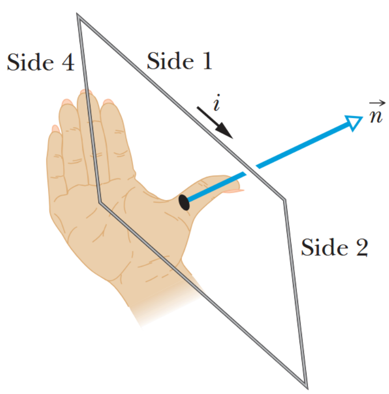
Note: There is no such thing as an isolated current-carrying wire, there must always be a way to introduce current into the wire and take it out at the other end.

Magnetic Torque on a Current Loop
A motor converts current into rotation by using magnetic forces on a current-carrying loop to generate a torque. In this case, the direction of current is reversed every half turn to keep the torque in the same direction using a commutator (not shown).



Let's consider the following rectangular current loop in a uniform magnetic field:





The orientation of the loop is defined using a normal vector $\vec{n}$ that is perpendicular to the plane of the loop and follows the right-hand rule with respect to the current direction.

Curl fingers in the direction of the current and the thumb points in the direction of $\vec{n}$. The angle θ is defined as the angle between $\vec{n}$ and $\vec{B}$.

Finding the magnetic force on each side of the loop using $\vec{F}_B = I\vec{L} \times \vec{B} = ILBsin\phi$ where ϕ is the angle between $\vec{L}$ and $\vec{B}$:

$$||\vec{F}_1|| = ||\vec{F}_3|| = iaB$$

$$||\vec{F}_2|| = ||\vec{F}_4|| = ibBsin(90^\circ - \theta) = ibBcos\theta$$

By symmetry, the forces act in opposite directions on each side, so the net force on the loop is zero. However, there is a torque about the center of the loop due to $\vec{F}_1$ and $\vec{F}_3$ (since their lines of action do not pass through the center):

$$\tau = \vec{r} \times \vec{F}$$

$$\tau = \left(iaB\frac{b}{2}sin\theta\right) + \left(iaB\frac{b}{2}sin\theta\right) = iabBsin\theta$$

Note that $A = ab$ is the area of the loop, so we can rewrite the torque as:

$$\boxed{\tau = iABsin\theta}$$

This relation holds for any shape of current loop, as long as A is the area of the loop and θ is the angle between $\vec{n}$ and $\vec{B}$.

If we have a *coil* with N loops of wire, we can approximate them as N identical current loops stacked together in the same plane. Thus, the total torque on the coil is:

$$\boxed{\sum \tau = NiABsin\theta}$$

<u>Note:</u> The current-carrying coil will tend to rotate such that $\vec{n}$ is aligned with $\vec{B}$, minimizing the potential energy of the system.

Magnetic Dipole Moment
Similar to a bar magnet, a current-carrying loop tends to align itself with an external magnetic field. Thus, the current loop is said to be a **magnetic dipole** with a **magnetic dipole moment** $\vec{\mu}$ defined as:

$$\boxed{\vec{\mu} = NIA\hat{n}}$$

Where N is the number of loops, I is the current, A is the area of the loop, and $\hat{n}$ is the unit normal vector to the plane of the loop.It has units of Amphere-square meter $(\mathrm{A} \cdot \mathrm{m}^2)$.

Using $\vec{\mu}$, we can rewrite the torque on the current loop as:

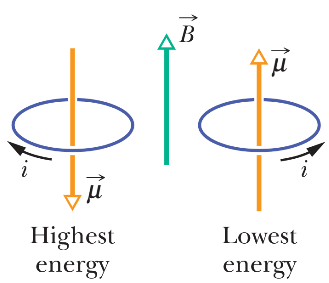
$$\boxed{\vec{\tau} = \vec{\mu} \times \vec{B}}$$

<u>Note:</u> This is only for torque about an axis through the center of the loop.

Similar to the electric dipole in an electric field, the potential energy of a magnetic dipole in a magnetic field is given by integrating the work done by the magnetic torque as it rotates ($\Delta U = -W_c$):

$$\boxed{U = -\vec{\mu} \cdot \vec{B}}$$

The minimum potential energy $(-\mu B)$ occurs when $\vec{\mu}$ is aligned with $\vec{B}$, and the maximum potential energy (μB) occurs when they are anti-aligned.



<u>Note:</u> The work done by an external torque to rotate the dipole is $W_{ext} = \Delta U = U_f - U_i$. If the dipole is stationary before and after the rotation.

We will later see that there is an energy stored in the external magnetic field. The potential energy of the magnetic dipole is related to the change in the energy stored in the magnetic field when the dipole is rotated.

(WORK IN PROGRESS)

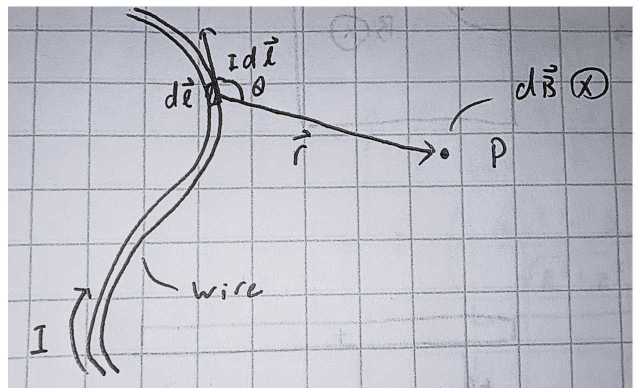
A bar magnet and a rotating sphere of charge are magnetic dipoles as well, and we can approximate the Earth as a big magnetic dipole. Most subatomic particles (like electrons, protons, and neutrons) also have intrinsic magnetic dipole moments due to their spin and charge. Thus, we can model their interactions with magnetic fields using the same equations as above.

When a magnet exerts a magnetic force on a current-carrying wire, Newton's third law requires that the wire exert an equal and opposite force back on the magnet. The only way the wire can interact with the magnet is through magnetic fields, so the wire must itself produce a magnetic field...

Chapter 29 - Magnetic Fields due to Currents

1. Use the Biot-Savart law to calculate the magnetic field due to a current-carrying wires of arbitrary (but tractable) geometry. e.g., a loop.

Like electric fields, magnetic fields obey superposition. Thus, we can find the magnetic field from a wire by summing up the $d\vec{B}$ at point P produced by small current elements $Id\vec{l}$ along the wire.



We can find $d\vec{B}$ by using the **Biot-Savart Law**:

$$\boxed{d\vec{B} = \frac{\mu_0}{4\pi}\frac{Id\vec{l} \times \hat{r}}{r^2} \quad \text{(current carrying wire)}}$$

Where

- $Id\vec{l}$ is the current element that produces the differential magnetic field $d\vec{B}$.
- r is the distance from the current element to point P.
- $\hat{r}$ is the unit vector that points from the current element to point P.
- μ_0 is the permeability of free space and has a value of:

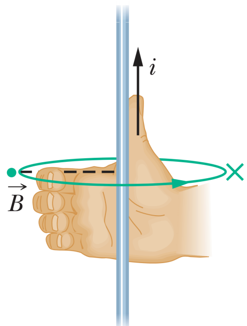
$$\boxed{\mu_0 = 4\pi \times 10^{-7}\,\mathrm{T \cdot m/A}}$$

Note: $Id\vec{l} \times \hat{r} = Idl \sin\theta$ where θ is the angle between $d\vec{l}$ and $\vec{r}$.

For a moving point charge:

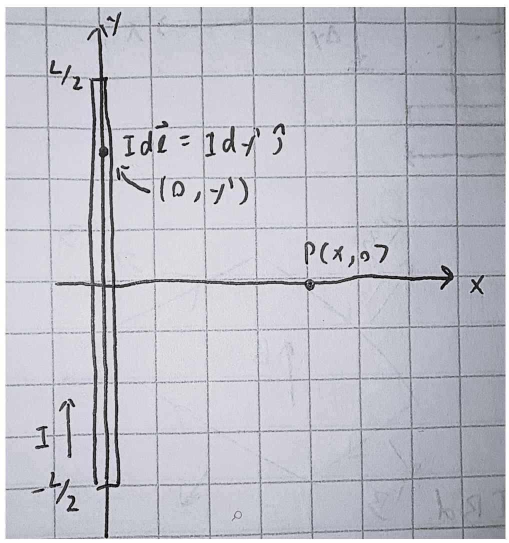
$$\boxed{d\vec{B} = \frac{\mu_0}{4\pi}\frac{q\vec{v} \times \hat{r}}{r^2} \quad \text{(point charge)}}$$

We can also find the direction of the magnetic field using the right-hand rule:



- Grasp the current element and point thumb in the direction of the current.
- Your fingers will then naturally curl around in the direction of the magnetic field lines due to that element.

Magnetic Field Due to a Current in a Straight Wire



Biot-Savart law:

$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{Id\vec{l} \times \hat{r}}{r^2}$$

Vectors:

$$d\vec{l} = dy'\hat{\jmath}$$

$$\vec{r} = \langle x, 0 \rangle - \langle 0, y' \rangle = \langle x, -y' \rangle \,, \quad ||\vec{r}|| = \sqrt{x^2 + y'^2}$$

$$\hat{r} = \frac{\langle x, -y' \rangle}{\sqrt{x^2 + y'^2}}$$

Cross product:

$$\begin{aligned} d\vec{B} &= \frac{\mu_0}{4\pi} \frac{Idy'\hat{\jmath} \times \langle x, -y' \rangle}{(x^2 + y'^2)^{3/2}} \\ &= \frac{\mu_0}{4\pi} \frac{-Ixdy'\hat{k}}{(x^2 + y'^2)^{3/2}} \end{aligned}$$

Integrate:

$$\vec{B} = -\frac{\mu_0 I}{2\pi}\int_{-L/2}^{L/2}\frac{xdy'}{(x^2+y'^2)^{3/2}}\hat{k}$$
$$= -\frac{\mu_0 I}{2\pi}\frac{1}{x\sqrt{x^2+(L/2)^2}}\hat{k}$$

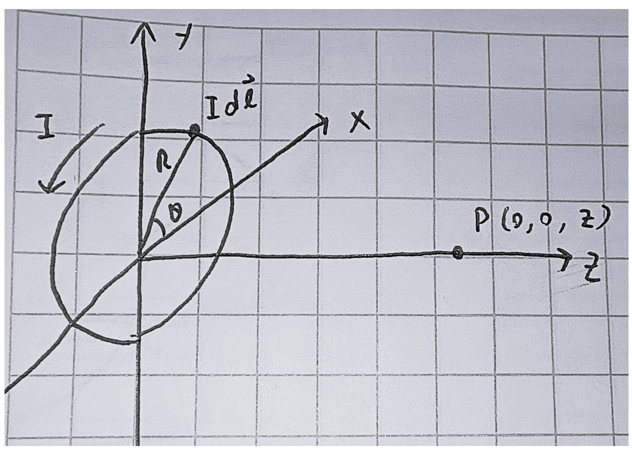
Taking the limit as $L \to \infty$, letting $r = x$:

$$\boxed{||\vec{B}|| = \frac{\mu_0 I}{2\pi r} \quad \text{(infinite straight wire)}}$$

Note: For a semi-infinite wire (from $0 \to \infty$), the magnetic field is half that of an infinite wire at the same distance r from the wire.

$$||\vec{B}|| = \frac{\mu_0 I}{4\pi r} \quad \text{(semi-infinite straight wire)}$$

Magnetic Field Due to a Current in a Circular Loop of Wire



Biot-Savart law:

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{Id\vec{l}\times\hat{r}}{r^2}$$

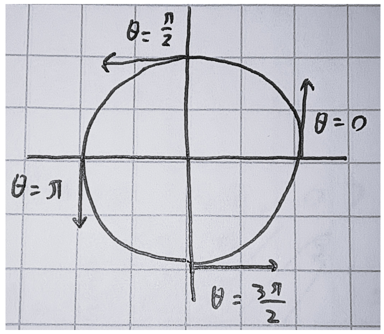
Vectors:

$$d\vec{l} = Rd\theta\,\hat{\theta}$$

$$\vec{r} = \langle 0,0,z\rangle - \langle Rcos\theta, Rsin\theta, 0\rangle = \langle -Rcos\theta, -Rsin\theta, z\rangle\,, \quad ||\vec{r}|| = \sqrt{R^2+z^2}$$

$$\hat{r} = \frac{\langle -Rcos\theta, -Rsin\theta, z\rangle}{\sqrt{R^2+z^2}}$$

Rewriting $\hat{\theta}$ in Cartesian, using a unit circle:



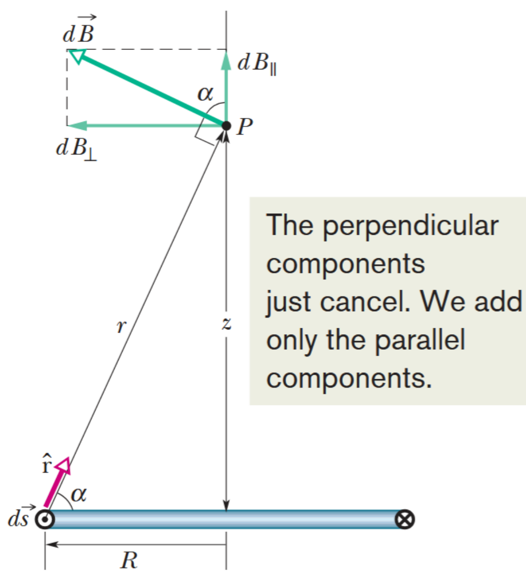
$$\hat{\theta} = \langle -sin\theta, cos\theta, 0\rangle$$

Cross product:

$$d\vec{l} \times \vec{r} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -Rd\theta sin\theta & Rd\theta cos\theta & 0 \\ -Rcos\theta & -Rsin\theta & z \end{vmatrix}$$

$$= (Rzcos\theta d\theta)\hat{\imath} + (Rzsin\theta d\theta)\hat{\jmath} + (R^2sin^2\theta d\theta + R^2cos^2\theta d\theta)\hat{k}$$

$$= (Rzcos\theta d\theta)\hat{\imath} + (Rzsin\theta d\theta)\hat{\jmath} + (R^2 d\theta)\hat{k}$$



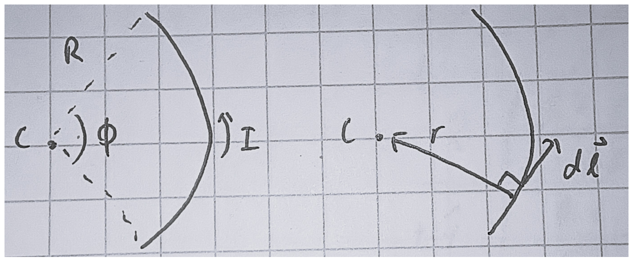
By symmetry, the $\hat{\imath}$ and $\hat{\jmath}$ components will cancel out when integrating over the full loop, so we only need to consider the $\hat{k}$ component:

$$B_z = \int_0^{2\pi} \frac{\mu_0}{4\pi} \frac{IR^2\, d\theta}{(R^2+z^2)^{3/2}} = \frac{\mu_0}{4\pi} \frac{2\pi IR^2}{(R^2+z^2)^{3/2}}$$

Notice that $\mu = NIA = I\pi R^2$, Thus

$$\boxed{\vec{B} = \frac{\mu_0}{4\pi}\frac{2\mu}{(R^2+z^2)^{3/2}}\hat{k}}$$

Magnetic Field due to a Current in a Circular Arc of Wire



Biot-Savart law:

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{Id\vec{l}\times\hat{r}}{r^2}$$

The angle between $d\vec{l}$ and $\vec{r}$ is always 90° for a circular arc, so:

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{Idl}{R^2}\hat{k}, \quad \text{(Direction found by RHR)}$$

$$\vec{B} = \int_0^{R\phi}\frac{\mu_0}{4\pi}\frac{Idl}{R^2}\hat{k}$$

$$\boxed{\vec{B} = \frac{\mu_0 I\phi}{4\pi R}\hat{k} \quad \text{(circular arc of wire)}}$$

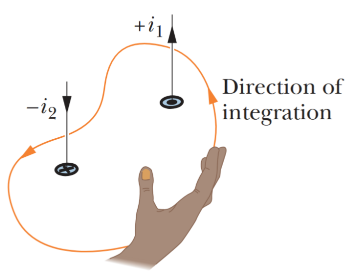
2. Describe Ampere's law and the conditions under which it can be used to solve for a magnetic field from a given current distribution. Use Ampere's law to find the field in those situations with sufficient symmetry to apply it. e.g., a long wire.

Ampere's Law
Ampere's Law states that the circulation of the magnetic field $\vec{B}$ around a closed **Amperian loop** is proportional to the net current I_{enc} piercing the surface bounded by that loop.

$$\boxed{\int \vec{B} \cdot d\vec{l} = \mu_0 I_{\text{enc}}}$$

We can arbitrarily choose a direction for integration around the Amperian loop and assume that the magnetic field $\vec{B}$ is along this direction. If the resulting value of $\vec{B}$ is negative, then the true direction of the magnetic field is opposite to the assumed direction.

The sign for the current enclosed can be determined using the right-hand rule:



- Curl fingers around the Amperian loop, with fingers pointing in the direction of integration ($d\vec{l}$).
- Your thumb will point in the direction of positive current piercing the surface.

Note: Generally, for a CCW integration direction, current coming out of the page is positive and current going into the page is negative.

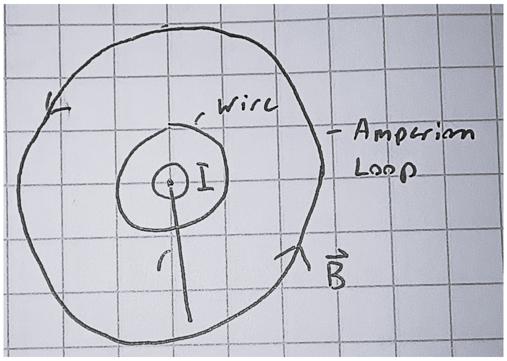
The magnetic field determined using Ampere's law is the superposition of the fields produced by all currents, including those outside the Amperian loop. However, similar to Gauss's law, only the current enclosed by the Amperian loop contributes to the net magnetic circulation around the closed path, while the contribution from external currents results in zero net circulation.

Ampere's law is most useful for finding magnetic fields in situations with high symmetry, such as:

- Infinite straight wire (cylindrical symmetry)
- Infinite solenoid (translational and cylindrical symmetry)
- Toroidal solenoid (cylindrical symmetry)
- Infinite sheet of current (planar symmetry)

In these cases, the magnetic field is uniform along the Amperian loop, allowing us to take $\vec{B}$ outside the integral.

Magnetic Field of a Long Straight Wire with Current
Outside wire:



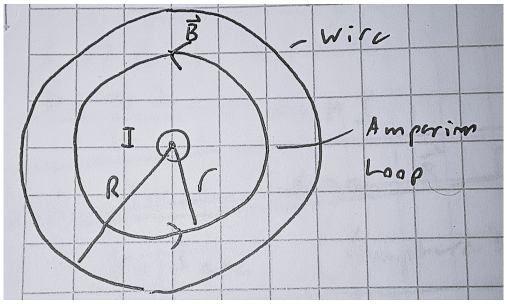
Ampere's law:

$$\int \vec{B} \cdot d\vec{l} = \mu_0 I_{\text{enc}}$$

$$B \int dl = \mu_0 I$$

$$\boxed{B = \frac{\mu_0 I}{2\pi r} \quad \text{(outside wire)}}$$

Inside wire (with uniform current density):



Current density:

$$J = \frac{I}{\pi R^2}$$

$$\implies I_{\text{enc}} = J\pi r^2 = I\frac{r^2}{R^2}$$

Ampere's law:

$$\int \vec{B} \cdot d\vec{l} = \mu_0 I_{\text{enc}}$$

$$B \int dl = \mu_0 I \frac{r^2}{R^2}$$

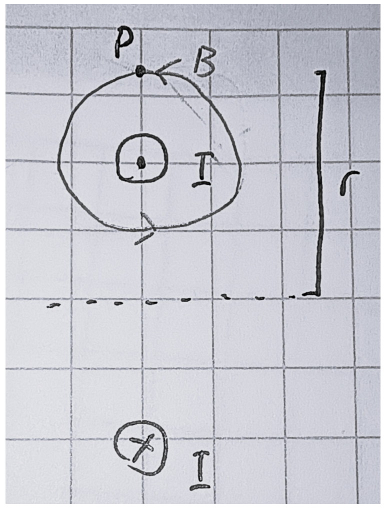
$$\boxed{B = \frac{\mu_0 I r}{2\pi R^2} \quad \text{(inside wire)}}$$

Magnetic Field of an Infinite Solenoid

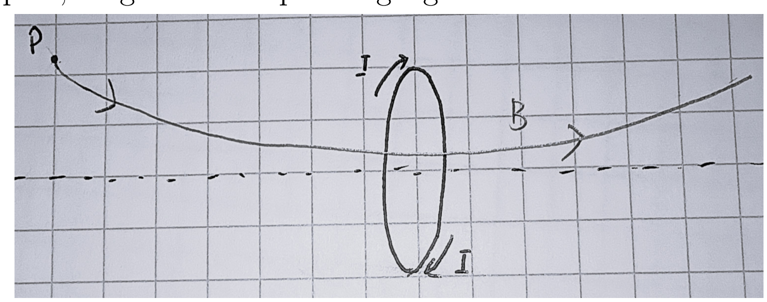
A solenoid is to magnetostatics as capacitors is to electrostatics. An ideal infinite solenoid produces a uniform magnetic field inside and **zero** magnetic field outside.

Why is it zero outside the infinite solenoid?

Consider the magnetic field at a point off axis from a single current loop. When it is close, the current loop looks more like an infinite wire, magnetic field pointing left.

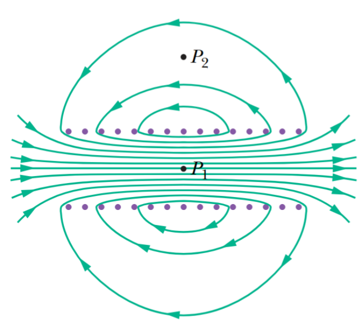


However, as point P moves further away, the current loop starts to look more like a magnetic dipole, magnetic field pointing right.

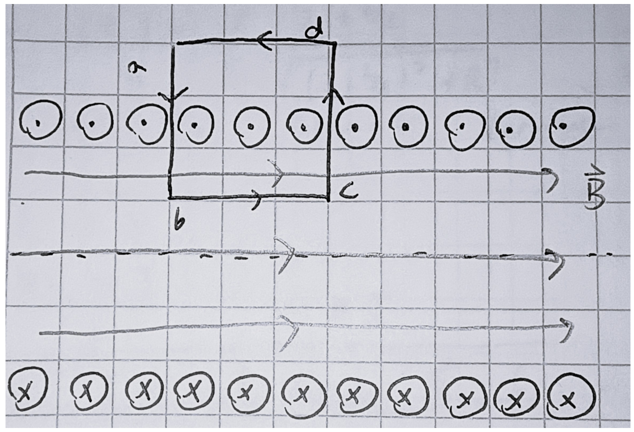


The key idea is that, for an infinite solenoid, only a *finite* set of nearby current loops contributes a strong leftward magnetic field at a given external point, while an *infinite* set of more distant loops contributes a much weaker rightward field. The cumulative effect of these infinitely many weak contributions exactly cancels the finite strong contribution, yielding zero net magnetic field outside the solenoid. The full mathematical proof is given by Pathak (*An elementary argument for the magnetic field outside a solenoid*).

Note that in a **finite** solenoid, it is not long enough for complete cancellation to occur, so there will be a small magnetic field outside the solenoid (similar to the fringe electric fields in a finite capacitor).



Calculating the magnetic field inside an infinite solenoid using Ampere's law:



Suppose an Amperian loop with side length L that encloses N loops of wire, each loop carries current I

$$\int \vec{B} \cdot d\vec{l} = \mu_0 I_{\text{enc}}$$

$$\int_b B dl = \mu_0 N I \quad \text{(Zero magnetic circulation along sides } a, c, d\text{)}$$

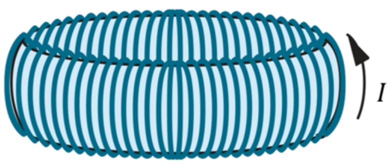
$$B_{\text{inside}} = \frac{\mu_0 N I}{L}$$

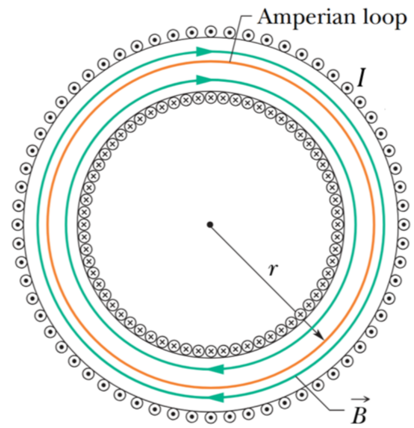
$$\boxed{B_{\text{inside}} = \mu_0 n I \quad \text{(infinite solenoid)}}$$

Where $n = \frac{N}{L}$ is the number of loops per unit length.

Note: If the Amperian loop encloses both ends of the loop, the net current enclosed is zero, so the magnetic field outside the solenoid is zero as well.

Magnetic Field of a Toroidal Solenoid





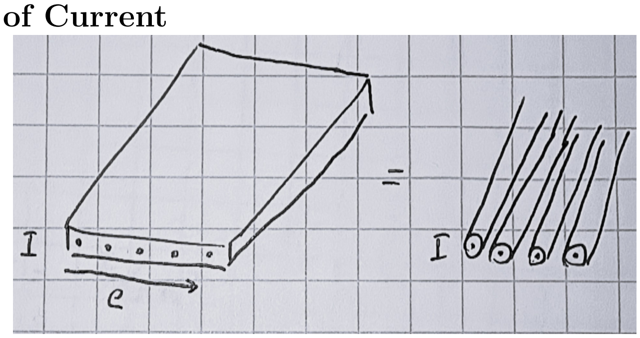
Suppose the toroid has N loops of wire, each carrying current I, we will integrate clockwise around an Amperian loop of radius r:

$$\int \vec{B} \cdot d\vec{l} = \mu_0 I_{\text{enc}}$$

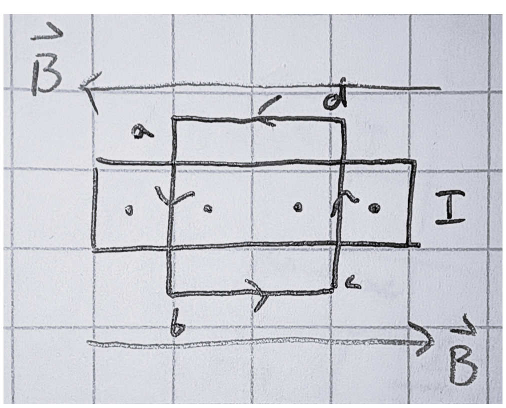
$$B(2\pi r) = \mu_0 IN$$

$$\boxed{B = \frac{\mu_0 IN}{2\pi r} \quad \text{(toroidal solenoid)}}$$

Infinite Sheet of Current



Suppose the sheet carries a current density ρ (current per unit length). We will integrate around a rectangular Amperian loop of width L and height H:



$$\int \vec{B} \cdot d\vec{l} = \mu_0 I_{\text{enc}}, \quad I_{\text{enc}} = \rho L$$

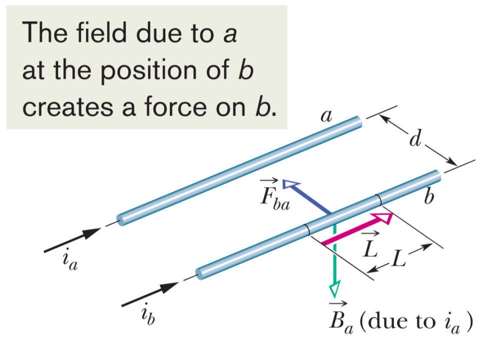
$$B(2L) = \mu_0 \rho L$$

$$\boxed{B = \frac{\mu_0 \rho}{2} \quad \text{(infinite sheet of current)}}$$

<u>Note:</u> Due to superposition, the magnetic field above and below are uniform and point in opposite directions.

3. Find the magnetic force on a current carrying wire due to another current carrying wire.

Force Between Two Parallel Current-Carrying Wires



To find the force on wire b due to wire a, we first find the magnetic field produced by wire a at the location of wire b:

$$B_a = \frac{\mu_0 i_a}{2\pi d}$$

Since the distance between the wires is constant, this magnetic field is unifrom along wire b. Now we can find the force on wire b using

$$\vec{F}_{ba} = i_b \vec{L} \times \vec{B}_a$$

$$F_{ba} = i_b L B_a sin(90^\circ) = \frac{\mu_0 L i_a i_b}{2\pi d}, \quad (\vec{L} \text{ and } \vec{B}_a \text{ are perpendicular})$$

Using the right-hand rule, we find that the force is toward wire a. By Newton's third law, the force on wire a due to wire b is equal in magnitude and opposite in direction.

Note: Parallel currents attract, and anti-parallel currents repel.

4. Describe the forces and torques on magnetic dipoles in terms of their magnetic moment.

Recall that the torque on a magnetic dipole in a magnetic field is given by:

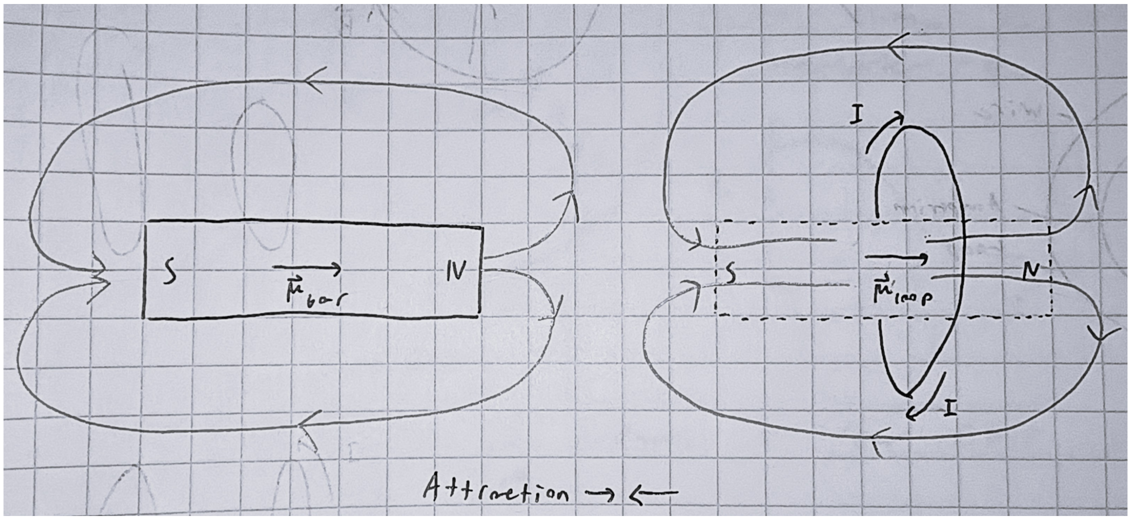
$$\vec{\tau} = \vec{\mu} \times \vec{B} \implies U = -\vec{\mu} \cdot \vec{B}, \qquad \text{where } \vec{\mu} = NIA\hat{n}$$

We know that potential energy is related to force by:

$$\boxed{\vec{F} = -\nabla U}$$

Ex: Force between a bar magnet and a circular loop of current

Since both the bar magnet and the current loop are magnetic dipoles, we can imagine that they will behave like two bar magnets whose like poles repel and opposite poles attract.



Recall that the magnetic field along the axis of a current loop of radius R is:

$$B_z = \frac{\mu_0}{4\pi} \frac{2\mu}{(R^2 + z^2)^{3/2}}$$

Taking the limit $R \ll z$ (far field approximation):

$$B_z = \frac{\mu_0}{4\pi} \frac{2\mu}{z^3}$$

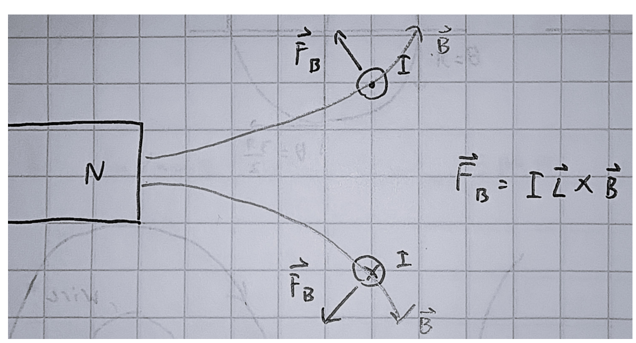
Thus, the potential energy of the current loop in the magnetic field of the bar magnet is:

$$U = -\vec{\mu}_{loop} \cdot \vec{B}_{bar,z} = -\mu_{loop} \frac{\mu_0}{4\pi} \frac{2\mu_{bar}}{z^3}$$

Now let's find the magnitude of the force on the current loop due to the bar magnet:

$$\boxed{||\vec{F}_z|| = \left| -\frac{dU}{dz} \right| = \frac{\mu_0}{4\pi} \frac{6\mu_{bar}\,\mu_{loop}}{z^4}}$$

We can find the direction of the force using the right-hand rule (also by intuition by looking at the poles). Note that this force is equal and opposite for the force on the bar magnet due to the current loop.



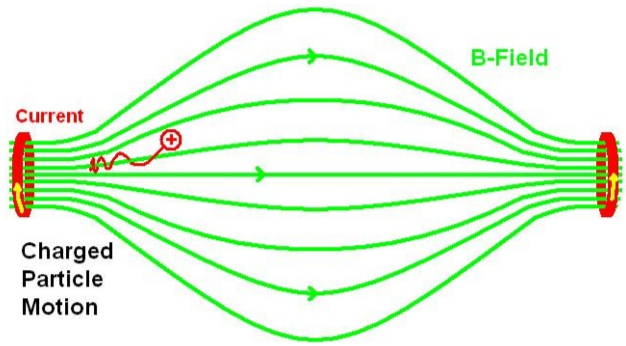
Force from a Magnetic Mirror

A charged particle can be confined to a region of space by a magnetic field that is stronger at the ends than in the middle. As the particle approaches the stronger field (shown by the more closely spaced field lines), a component of the magnetic force pushes it back toward the center of the region, reflecting it back and forth between the two ends.

In a magnetic mirror, a charged particle moves in a helical path with

$$\vec{v} = v_{\parallel}\hat{B} + v_{\perp}\hat{\perp}$$

With a component along the magnetic field and a component perpendicular to it.



Magnetic moment of a positive charged particle (with an axis of rotation tangent to the local magnetic field line):

$$T = \frac{2\pi r_L}{v_\perp} = \frac{2\pi}{\omega}, \qquad \omega = \frac{qB}{m}$$

For a single loop $N = 1$, the magnetic dipole moment magnitude is

$$\|\mu\| = IA, \qquad I = \frac{q}{T} = \frac{q\omega}{2\pi}, \qquad A = \pi r_L^2, \qquad r_L = \frac{mv_\perp}{qB}$$

Substituting,

$$\|\mu\| = \frac{q\omega}{2\pi}\,\pi r_L^2 = \frac{q\omega}{2}\,r_L^2 = \frac{q}{2}\left(\frac{qB}{m}\right)\left(\frac{mv_\perp}{qB}\right)^2$$

$$\boxed{\|\mu\| = \frac{mv_\perp^2}{2B}}$$

Since the magnetic force does no work on a charged particle and there is no electric field, the total kinetic energy is conserved:

$$K = \frac{1}{2}mv_\perp^2 + \frac{1}{2}mv_\parallel^2 = \text{constant}$$

Rewriting the perpendicular kinetic energy in terms of the magnetic moment,

$$\frac{1}{2}mv_\perp^2 = \mu B$$

Thus the total kinetic energy becomes

$$K = \mu B + \frac{1}{2} m v_\parallel^2 = \text{constant}$$

Taking the derivative along the magnetic field lines s,

$$\frac{dK}{ds} = \mu \frac{dB}{ds} + m v_\parallel \frac{dv_\parallel}{ds} = 0$$

Rearranging,

$$m v_\parallel \frac{dv_\parallel}{ds} = -\mu \frac{dB}{ds}$$

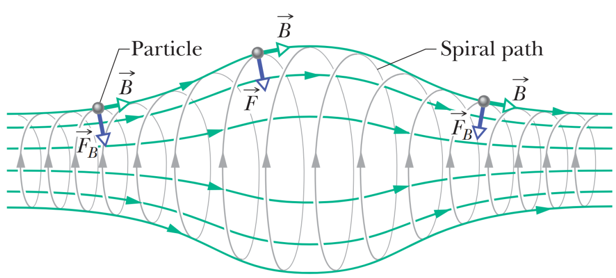
Since $\dfrac{dv_\parallel}{ds} = \dfrac{dv_\parallel}{dt}\dfrac{dt}{ds} = \dfrac{1}{v_\parallel}\dfrac{dv_\parallel}{dt}$, this gives

$$m \frac{dv_\parallel}{dt} = -\mu \frac{dB}{ds}$$

Therefore, the parallel force on the particle is

$$\boxed{F_\parallel = -\mu \frac{dB}{ds}}$$

<u>Note:</u> The magnetic force only reflects the particle as a result of the changing magnetic field strength (shown by the curvature of the field lines), which creates a component of the magnetic force along the direction of motion of the particle.



Chapter 30 & 31.11 - Induction and Inductance

1. Understand the meaning Faraday's law of induction and be able to use it to determine the electromotive force. Explain and apply the equivalence of Faraday's law and the Lorentz force law for motional emfs.

   Faraday's Law of Induction
   Faraday's law states that if an arbitrary closed loop encloses a region with changing magnetic flux, then an electromotive force (emf) will be induced in the loop. If there happens to be a conducting wire along the loop, then an induced current will exist in the wire.

   $$\boxed{\mathcal{E} = -\frac{d\Phi_B}{dt}}$$

   The units for magnetic flux Φ_B are Weber (Wb), where

   $$1\,\mathrm{Wb} = 1\,\mathrm{T}\cdot\mathrm{m}^2$$

   It can also be written in integral form as:

   $$\boxed{\mathcal{E} = \int \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int \vec{B}\cdot\hat{n}dA}$$

   Where

   - $\mathcal{E}$ is the **induced emf** (in volts), representing the work done per unit charge by the induced electric field ($\vec{E}$) around the faradian loop (in a CCW direction).
   - $\hat{n}$ is the unit normal vector to the surface bounded by the faradian loop.
   - dA is the differential area element of that surface.

   Another way to think about Faraday's law is that there will be circulating electric fields induced in regions of changing magnetic flux.

   The induced electric field has the following properties:

   - Do not terminate and originate on charges
   - Form closed loops!
   - Is present whenever the magnetic flux is changing, even in regions where there are no conductors
   - Is a non-conservative field, meaning that the work done by the field around a closed loop is non-zero (and thus cannot be described by a potential energy function)

   In the case where there is a coil of wire with N turns, and the coil is tightly wound so that each turn experiences the same magnetic flux, then the total emf induced in the coil is:

   $$\boxed{\mathcal{E} = -N\frac{d\phi_B}{dt} \quad \text{(coil of N turns)}}$$

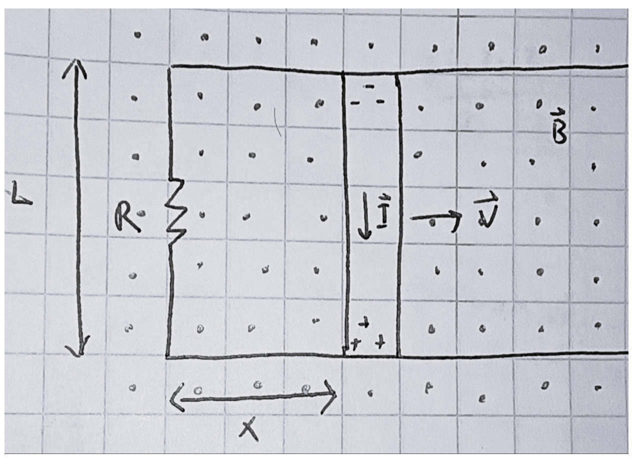
Note: Typically, we don't need to think too hard about the direction of the induced electric field, and just need to consider the magnitude of the induced emf. The direction can be determined using Lenz's law.

By examining the terms in Faraday's law, we can see that there are a few ways to induce an emf in a loop:

- Change the magnitude of the magnetic field B within the coil.
- Change either the total area of the coil or the portion of that area that lies within the magnetic field (ex: by expanding the coil or moving the coil in/out of the magnetic field).
- Change the orientation of the loop (angle between $\vec{B}$ and $\hat{n}$). For example, by rotating the coil within the magnetic field.

Motional emf

A motional emf is induced when there is relative motion between a conductor and a magnetic field. This can be understood using either Faraday's law or the Lorentz force law.



Consider the following circuit formed by a conductor sliding on two stationary rails in the presence of a uniform magnetic field $\vec{B}$ pointing out of the page. By the Lorentz force law, we can find the emf induced in the circuit as follows:

$$\vec{F}_B = q\vec{v} \times \vec{B} \implies F_B = qvB$$

$$F_E = qE$$

The magnetic force pushes positive charges down, creating a potential difference and electric field in the conductor. The electrons stop moving when the magnetic force is balanced by the electric force (Lorentz force equalling zero):

$$F_B = F_E$$

$$qvB = qE \implies E = vB$$

The emf induced in the circuit is then:

$$\boxed{\mathcal{E} = EL = vBL}$$

We can also find the emf using Faraday's law:

$$\mathcal{E} = \int \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \int \vec{B} \cdot \hat{n} dA$$

Only the area is changing as the conductor slides, so:

$$\mathcal{E} = -\frac{d}{dt}(BLx) = -BL\frac{dx}{dt} = -BLv$$

Taking the magnitude, we get the same result as before:

$$\boxed{|\mathcal{E}| = BLv}$$

Power and Energy

Considering the same circuit as before, when there is a current I flowing through the circuit, there is a magnetic force slowing down the conductor:

$$\vec{F}_B = I\vec{L} \times \vec{B} \implies F_B = ILB$$

We can find the current using Ohm's law and the emf:

$$I = \frac{\mathcal{E}}{R} = \frac{BLv}{R}$$

$$\implies F_B = \frac{BLv}{R}LB = \frac{B^2L^2v}{R}$$

If we want the conductor to move at a constant velocity then we need to apply an external force equal in magnitude and opposite in direction to the magnetic force:

$$F_{\text{app}} = F_B = \frac{B^2L^2v}{R}$$

The power supplied by this external force is:

$$P_{\text{app}} = F_{\text{app}}v = \frac{B^2L^2v^2}{R}$$

There is no change in kinetic energy of the conductor so all this power must be dissipated as heat in the resistor. We can verify this by calculating the power dissipated in the resistor:

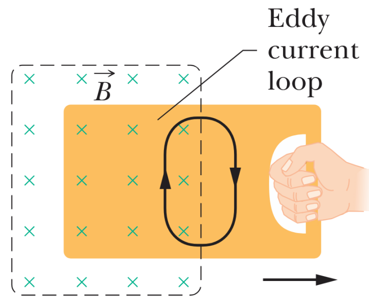
$$P_R = I^2R = \left(\frac{BLv}{R}\right)^2 R = \frac{B^2L^2v^2}{R}$$

<u>Note:</u> Even though there is also current in the upper and lower rails, and thus a magnetic force, these forces cancel out since they point in opposite directions, so they do not contribute to the net force on the moving conductor.

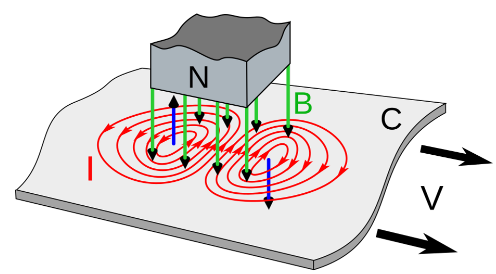
Also note that as a result of Lenz's law, regardless of the direction of motion of the conductor, there will always be a magnetic force opposing the motion and thus requiring the external force to do *positive* work to keep the conductor moving at a constant velocity.

Eddy Currents

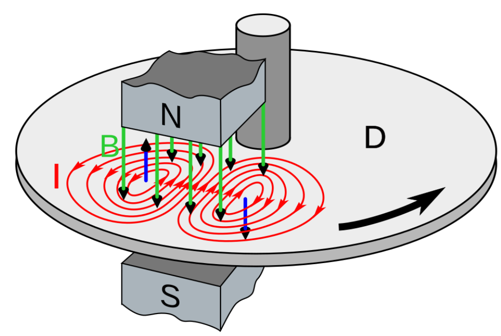
Suppose that we move a solid conducting plate out of a region with a uniform magnetic field $\vec{B}$ pointing into the page. Similar to the motional emf example, the relative motion of the field and the conductor will induce a current and thus a magnetic force that opposes the motion. However, since the conductor solid, the conduction electrons will not follow a single path, but rather will swirl about within the plate, forming eddy currents. However, we can represent the effect of these eddy currents *as if* it were a single induced current loop around the edge of the plate.



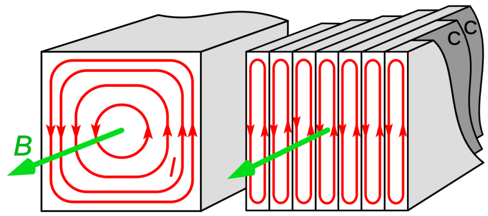
If the plate in the example was longer, it will enclose 2 regions of changing magnetic flux (one where the flux is increasing as the plate enters the field, and one where the flux is decreasing as the plate exits the field). Thus, there will be two sets of eddy currents induced in the plate, each opposing the change in flux in their respective regions.



From in the motional emf example, we know that the interaction between the magnetic field and the eddy currents produces a magnetic force that opposes the motion of the plate. This causes energy to be lost as heat in the conductor due to the resistance of the material, thereby slowing down the motion of the conducting plate. This effect is used in electromagnetic braking systems.



Sometimes we want to minimize the energy loss from eddy current such as in transformers or electric motors. This can be done by using materials with low conductivity (high resistivity) or by using thin sheets of material with insulating layers in between (laminations) to restrict the flow of eddy currents. Similar to the Hall effect, charges build up on the surfaces of the laminations, creating an electric field that opposes the motion of the charges and thus reducing the eddy currents.



Revisiting Electric Potential
In electrostatics, we were able to define an electric potential V as a scalar field from the electric field produced by *static charges*, however, we cannot define a unique electric potential from the circulating electric fields induced by a changing magnetic flux. This is because the induced electric fields are non-conservative as the work done by the field around a closed loop is non-zero ($\oint \vec{E} \cdot d\vec{l} \neq 0$).

However, it is still valid to consider the induced emf as a type of *potential difference*, similar to a battery that adds or removes energy from charges as they move around a circuit. Thus, equations such as Ohm's law, $V = Ed$, $P = IV$, etc. are applicable in circuits with induced emfs.

2. Apply Lenz's law to determine the sign of induced currents or electromotive forces.

Lenz's Law
Lenz's law states that the induced current around a loop will flow in a direction that produces a secondary magnetic field that *opposes* the *change* in the magnetic flux in through the loop. Note that Lenz's law still holds even if there is no conducting wire to actually produce the secondary magnetic field, the direction is the same regardless.

3. Define inductance and be able to calculate the self or mutual inductance for various (usually simple) current distributions. Know and use the reciprocity relation for mutual inductances.

Inductance
Inductance measures the ability of a conductor to induce an emf in itself (self-inductance) or in another conductor (mutual inductance) as a result of a changing current. The inductance depends on only geometry and other constants. For an inductor carrying current I with N turns, the **inductance** is defined as:

$$\boxed{L = N\frac{\Phi_B}{I}}$$

Where ϕ_B is the magnetic flux through a single turn of the inductor. The units for inductance are Henry (H), where

$$1\,\mathrm{H} = 1\,\mathrm{T}\cdot\mathrm{m}^2/\mathrm{A}$$

Self Inductance of a Solenoid
Consider a solenoid with N turns, length l, cross-sectional area A, carrying current I. The magnetic field inside the solenoid is:

$$B = \mu_0 n I, \quad n = \frac{N}{l}$$

The magnetic flux through a single turn of the solenoid is:

$$\Phi_B = BA = \mu_0 n I A$$

Thus, the inductance of the solenoid is:

$$L = \frac{N\Phi_B}{I} = \frac{N(\mu_0 n I A)}{I} = \mu_0 n^2 A l$$

$$\boxed{L = \mu_0 \frac{N^2}{l} A \quad \text{(solenoid)}}$$

Self Induced emf
When the current in an inductor changes, the changing magnetic flux induces an emf in the inductor itself, opposing the change in current (Lenz's law). This is called **self-induction** and the **self-induced emf** can be found using Faraday's law:

We can rewrite emf in terms of inductance:

$$N\Phi_B = LI$$

$$\mathcal{E} = -\frac{d(N\Phi_B)}{dt} = -\frac{d(LI)}{dt}$$

$$\boxed{\mathcal{E} = -L\frac{dI}{dt}}$$

This self-induced emf opposes changes in current, thus an increasing current induces a negative emf (opposing the increase), and this is sometimes called the *back emf.*

We can also define a self-induced potential difference V_L across the inductor (outside the region of changing magnetic flux). For an *ideal* inductor, the magnitude of V_L is equal to the magnitude of the self-induced emf. For *non-ideal* inductors, we can model it as an ideal inductor in series with a resistor (to account for energy loss due to resistance).

Mutual Induction

Suppose we have two inductors (coils of wire) placed close to each other, such that the current in one coil will produce a magnetic flux that passes through the other coil. If the current in coil 1 changes, the changing magnetic flux through coil 2 will induce an emf in coil 2. This is called **mutual induction** and is defined by the **mutual inductance** M_{21} of coil 2 with respect to coil 1 as:

$$\boxed{M_{21} = \frac{N_2 \Phi_{B2}}{I_1}}$$

Note that the mutual inductance is *symmetric*, meaning that

$$M_{21} = M_{12}$$

This is very useful in cases where it is easier to find the flux through one coil than the other.

Similar to self-induction, the emf induced in coil 2 due to a changing current in coil 1:

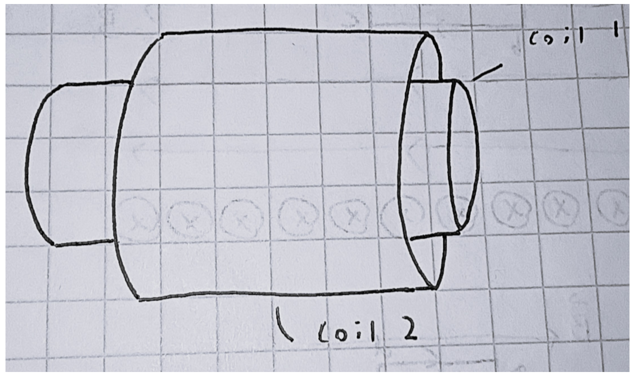
$$\boxed{\mathcal{E}_2 = -M_{21} \frac{dI_1}{dt}}$$

If the current in coil 2 changes, it will induce an emf in coil 1:

$$\boxed{\mathcal{E}_1 = -M_{12} \frac{dI_2}{dt}}$$

Mutual Induction of a Transformer

Suppose there are 2 concentric coils of wire, coil 1 with N_1 turns, length l_1, and area A_1, and coil 2 with N_2 turns, length l_2, and area A_2. Coil 1 carries a current I_1, and coil 2 is an open circuit (not carrying any current).



The magnetic field within coil 1 is the same as that of a solenoid:

$$B_1 = \mu_0 \frac{N_1}{l_1} I_1$$

The magnetic flux through a single turn of coil 2 is:

$$\Phi_{B2} = B_1 A_2 = \mu_0 \frac{N_1}{l_1} I_1 A_2$$

Thus, the mutual inductance of coil 2 with respect to coil 1 is:

$$M_{21} = \frac{N_2 \Phi_{B2}}{I_1} = \frac{N_2(\mu_0 \frac{N_1}{l_1} I_1 A_2)}{I_1} = \mu_0 \frac{N_1 N_2 A_2}{l_1}$$

The voltage across coil 2 when the current in coil 1 is changing is then:

$$V_2 = |\mathcal{E}_2| = M_{21} \frac{dI_1}{dt}$$

$$V_2 = \mu_0 \frac{N_1 N_2 A_2}{l_1} \frac{dI_1}{dt}$$

Since coil 2 doesn't induce a magnetic flux in coil 1 (open circuit), the voltage across coil 1 is just its self-induced emf:

$$V_1 = |\mathcal{E}_1| = L_1 \frac{dI_1}{dt}, \quad L_1 = \mu_0 \frac{N_1^2 A_1}{l_1}$$

$$V_1 = \mu_0 \frac{N_1^2 A_1}{l_1} \frac{dI_1}{dt}$$

The ratio of the voltages across the two coils is then:

$$\frac{V_2}{V_1} = \frac{N_2}{N_1} \quad \text{(transformer equation)}$$

4. Be able to calculate the energy stored in magnetic fields for a given inductor using either the formula involving inductance or the energy density of the magnetic field.

Energy Stored in an Inductor
When current flows through an inductor, energy is stored in the magnetic field created by the current. The energy stored in the magnetic field of an inductor can be found by integrating the power supplied to the inductor as the current increases from 0 to I:

$$|\mathcal{E}| = \frac{d\Phi_B}{dt} = L\frac{dI}{dt}$$

$$P = IV$$

This equation is valid because it is derived using potential difference and work. Thus, the power supplied to the inductor is:

$$P = I|\mathcal{E}| = IL\frac{dI}{dt}$$

The energy stored in the inductor is then:

$$P = \frac{dU}{dt} \implies \int dU = \int_0^I ILdI = \frac{1}{2}LI^2$$

$$\boxed{U = \frac{1}{2}LI^2}$$

Energy Density of a Magnetic Field
Similar to finding the energy density of an electric field, we can find the energy density of a magnetic field by dividing the total energy stored in the magnetic field by the volume over which the field exists. Assuming that the magnetic field is uniform over that volume.

For a solenoid with length l and cross-sectional area A, its volume is $V = Al$, and it stores an energy of:

$$U = \frac{1}{2}LI^2, \quad L_{\text{solenoid}} = \mu_0\frac{N^2}{l}A = \mu_0 n^2 Al$$

The energy density of the magnetic field is then:

$$u_B = \frac{U}{V} = \frac{\frac{1}{2}(\mu_0 n^2 Al)I^2}{Al} = \frac{1}{2}\mu_0 n^2 I^2$$

Since the magnetic field inside the solenoid is $B = \mu_0 nI$, we can rewrite the energy density as:

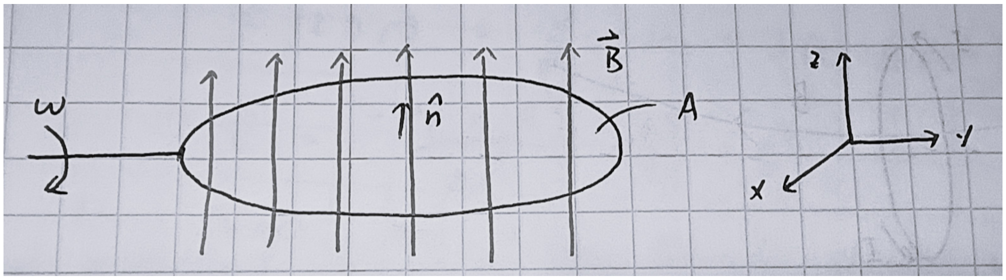
$$\boxed{u_B = \frac{1}{2\mu_0}B^2}$$

Like how we found the general electric field energy density using a parallel plate capacitor. This equation is valid for any magnetic field, not just the uniform field inside a solenoid.

Using the magnetic field energy density, we can find the total energy stored in any magnetic field by integrating the energy density over the volume of the field.

5. Describe the principle of operation of a generator and a motor.

By Faraday's law, we can induce an emf in a coil of wire by changing its orientation in a magnetic field. We can solve for the emf induced in a rotating coil as follows:



As the coil rotates, the magnetic flux through the coil will decrease, thus inducing a ccw current in the coil. First we can define how the orientation of the coil changes by:

$$\hat{n} = \cos(\omega t)\hat{k} + \sin(\omega t)\hat{\imath}$$

The magnetic field is constant and points in the $\hat{k}$ direction $\vec{B} = B\hat{k}$, so the magnetic flux through the coil is:

$$\Phi_B = \vec{B} \cdot \hat{n} A = BA\cos(\omega t)$$

The emf induced in the coil is then:

$$\mathcal{E} = -N\frac{d\Phi_B}{dt}, \quad N = 1$$

$$\mathcal{E} = -\frac{d}{dt}(AB\cos(\omega t)) = AB\omega\sin(\omega t)$$

$$\boxed{\mathcal{E} = AB\omega\sin(\omega t)}$$

This is the principle of operation of an AC generator, where mechanical energy is converted into electrical energy by rotating a coil in a magnetic field. It is AC because the direction of the induced current changes from ccw to cw as the coil continues to rotate.

Similarly, a motor works in reverse by supplying an AC current to a coil in a magnetic field, which produces a torque on the coil that causes it to rotate, converting electrical energy into mechanical energy.

Chapter 32 - Maxwell's Equations

In integral form:

1. **Gauss's Law for Electricity**
   Relates the electric flux through a closed 3D Gaussian surface to the total charge enclosed within that surface.
   $$\int \vec{E} \cdot \hat{n}\, dA = \frac{Q_{\text{enc}}}{\varepsilon_0}$$
2. **Gauss's Law for Magnetism**
   Any 3D Gaussian surface will have zero net magnetic flux (no magnetic monopoles).
   $$\int \vec{B} \cdot \hat{n} dA = 0$$
3. **Ampere-Maxwell Law**
   Relates the magnetic circulation around a closed Amperian loop to the enclosed current and changing electric flux through the surface bounded by the loop.
   $$\oint \vec{B} \cdot d\vec{l} = \mu_0 \left[I_{\text{enc}} + \varepsilon_0 \frac{d\Phi_E}{dt} \right], \quad \Phi_E = \int \vec{E} \cdot \hat{n}\, dA$$
4. **Faraday's Law of Induction**
   Relates the electric circulation around a closed Faradian loop to the changing magnetic flux through the surface bounded by the loop.
   $$\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt}, \quad \Phi_B = \int \vec{B} \cdot \hat{n}\, dA$$
   <u>Note:</u> For coils with N turns, multiply the flux by N.

In differential form:

1. **Gauss's Law for Electricity**
   $$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$
2. **Gauss's Law for Magnetism**
   $$\vec{\nabla} \cdot \vec{B} = 0$$
3. **Ampere-Maxwell Law**
   $$\vec{\nabla} \times \vec{B} = \mu_0 \left(\vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right), \quad I_{\text{enc}} = \int \vec{J} \cdot \hat{n} dA$$
4. **Faraday's Law of Induction**
   $$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

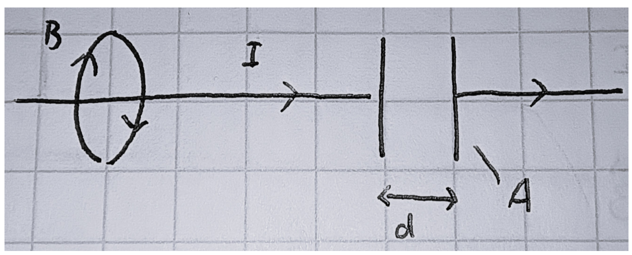
1. Describe Gauss's law for Magnetism and its applications.

   Gauss's law for magnetism states that the net magnetic flux through any closed surface is zero. This is because there only exists magnetic dipoles, so no matter how much or how little magnetic material is enclosed, the net magnetic flux will be zero.

   Even if you split a magnet into pieces, new north and south poles will form on each piece, so there are still no magnetic monopoles. Thus, even if you enclose a section of a magnet, Gauss's law for magnetism still holds.

2. Describe the Ampere-Maxwell law, the displacement current, and calculate the magnetic field in a charging or discharging capacitor.

   Consider the following charging capacitor with rectangular plates of area A and separation distance d. Since it is not fully charged, there is current I in the wires, inducing a magnetic field.

   The magnetic field around the wire can be found using Ampere's law:

   $$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{\text{enc}}$$

   $$B(2\pi r) = \mu_0 I \implies B = \frac{\mu_0 I}{2\pi r}$$

   However in the region between the plates, there is no current, so $I_{\text{enc}} = 0$, and thus Ampere's law would give $B = 0$. However, we can still draw a slanted amperian loop that encloses the wire and passes between the plates. Thus, this indicates that Ampere's law is incomplete.

   To fix this, we have the Ampere-Maxwell law with an additional term called the **displacement current**, which accounts for the changing electric flux between the plates of the capacitor. The displacement current is defined as:

   $$I_d = \varepsilon_0 \frac{d\Phi_E}{dt}, \quad \Phi_E = \int \vec{E} \cdot \hat{n} dA$$

   Thus, the Ampere-Maxwell law is:

   $$\oint \vec{B} \cdot d\vec{l} = \mu_0 \left(I_{\text{enc}} + \varepsilon_0 \frac{d\Phi_E}{dt} \right)$$

To find the magnetic field between the plates, we first know that the electric field between the plates is:

$$E = \frac{V}{d}, \qquad V = \frac{Q}{C}, \qquad C = \varepsilon_0 \frac{A}{d}$$

$$E = \frac{1}{d} Q \frac{d}{\varepsilon_0 A} = \frac{Q}{\varepsilon_0 A}$$

The electric flux through a circular amperian loop that fully encloses the plates is then:

$$\Phi_E = \int \vec{E} \cdot \hat{n} dA = EA = \frac{Q}{\varepsilon_0 A} A = \frac{Q}{\varepsilon_0}$$

The displacement current is then:

$$I_d = \varepsilon_0 \frac{d\Phi_E}{dt} = \varepsilon_0 \frac{d}{dt} \left(\frac{Q}{\varepsilon_0} \right) = \frac{dQ}{dt} = I$$

Note: Q refers to the charge on each plate of the capacitor. The rate at which this charge changes is equal to the current I by conservation of charge.

Thus, the magnetic field around the plates can be found using the Ampere-Maxwell law:

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$$

$$B(2\pi r) = \mu_0 I \implies \boxed{B = \frac{\mu_0 I}{2\pi r}}$$

This is the same magnetic field as outside the plates, showing that the magnetic field is continuous across the capacitor.

Chapter 33 - Electromagnetic Waves

1. Test whether a given arrangement of propagating electromagnetic fields satisfies Maxwell's Equations.

2. Understand the principles of electromagnetic waves and their basic properties, such as phase speed, relation between $\vec{k}$, $\vec{E}$, and $\vec{B}$.

3. Define the Poynting vector or Poynting flux and be able to use it to calculate the energy carried by electromagnetic fields.

4. Qualitatively describe how electromagnetic waves are generated. For an accelerating charge, identify the directions in which one will see the largest/smallest electric fields and describe the polarization of the electric field vector.

Chapters 27, 30.12, & 31 - Circuits

1. Know and be able to apply the Voltage Loop Rule and the Current Node Rule (conservation of energy and conservation of charge, respectively) to analyze a circuit. A good example is to prove the rules for adding resistors in series or in parallel.

2. Know the relation between current and voltage for the basic circuit elements of resistor, capacitor, and inductor.

3. Write differential equations for RLC circuits (or RL, RC, LC). Understand qualitatively the behavior of resistors, inductors, and capacitors at high and low frequencies. Be able to describe how electromagnetic energy is stored, supplied, and dissipated in such circuits.

4. Solve the differential equations for such circuits, identify characteristic timescales, frequencies, and resonances. If initial conditions are given, be able to find solutions that satisfy those initial conditions.

5. Know or be able to derive the complex impedances for resistors, capacitors, and inductors. Know or derive the rules for combining parallel and series impedances. Use those impedances to describe the relations among currents and voltages in circuits, both in magnitude and phase.

End Final Exam