Differential Equations Review Sheet

Laplace transforms and Fourier series

Laplace Transform Formulas

$f(t)$	$F(s)$
1	$\frac{1}{s}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$\sin(at)$	$\frac{a}{s^2+a^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$
$\sinh(at)$	$\frac{a}{s^2-a^2}$
$\cosh(at)$	$\frac{s}{s^2-a^2}$
$\mathscr{U}(t-a)$	$\frac{e^{-as}}{s}$
$\mathscr{U}(t-a)f(t-a)$	$e^{-as}F(s)$
$e^{at}f(t)$	$F(s-a)$
$t^n f(t)$	$(-1)^n \frac{d^n}{ds^n}F(s)$
$\sin(at) - at\cos(at)$	$\frac{2a^3}{(s^2+a^2)^2}$
$t\sin(at)$	$\frac{2as}{(s^2+a^2)^2}$
$f * g$	$F(s)G(s)$
$\delta(t-a)$	e^{-sa}
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$

7.1 Definition of the Laplace Transform

Laplace Transform
For a function $f(t)$ defined for $t \geq 0$, the Laplace transform of $f(t)$ is defined as

$$\boxed{\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\ dt}$$

The Laplace transform is a linear transformation meaning that

$$\mathscr{L}\{\alpha f(t) + \beta g(t)\} = \alpha \mathscr{L}\{f(t)\} + \beta \mathscr{L}\{g(t)\}$$

Exponential order
A function f is said to be of exponential order if there exist constants M, c, and T such that

$$|f(t)| \leq Me^{ct} \text{ for all } t > T$$

Basically, this means f is *bounded* by an exponential function as t approaches infinity.

Thm: Sufficient Condition for Existence
If f is piecewise continuous on $[0, \infty)$ and of exponential order, then the Laplace transform $\mathscr{L}\{f(t)\}$ exists for $s > c$.

7.2 Inverse Transforms and Transforms of Derivatives

Inverse Laplace Transform
Let $F(s)$ be the Laplace transform of $f(t)$ ($\mathscr{L}\{f(t)\} = F(s)$). Then we say that $f(t)$ is the inverse Laplace transform of $F(s)$ and write

$$f(t) = \mathscr{L}^{-1}\{F(s)\}$$

The inverse Laplace transform is also linear:

$$\mathscr{L}^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha \mathscr{L}^{-1}\{F(s)\} + \beta \mathscr{L}^{-1}\{G(s)\}$$

Thm: Transform of a Derivative
If $f, f', \cdots, f^{(n-1)}$ are continuous on $[0, \infty]$ and are of exponential order and $f^{(n)}(t)$ is piecewise continuous on $[0, \infty)$ then we define the Laplace transform of $f^{(n)}(t)$ as

$$\boxed{\mathscr{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)}$$

where $F(s) = \mathscr{L}\{f(t)\}$ and $f(0), f'(0), \cdots, f^{(n-1)}(0)$ are the initial conditions.

Method:
To solve a differential equation with given initial conditions using Laplace transforms:

1. Apply the Laplace transform $\mathscr{L}$ to both sides of the differential equation. This transforms the differential equation to an algebraic equation in the s-domain in terms of $Y(s) = \mathscr{L}\{y(t)\}$.

2. Solve the algebraic equation for $Y(s)$.

3. Apply the inverse Laplace transform $\mathscr{L}^{-1}$ to find $y(t)$, converting from the s-domain back to the time domain.

Note: The Laplace transform already solves for the initial conditions when transforming derivatives.

Typically, partial fraction decomposition is needed to find the inverse Laplace transform.

Partial fraction decomposition
For the method of partial fraction decomposition, the degree of the numerator must be less than the degree of the denominator.

1. Factor the denominator completely into:
   - Linear factors: $(ax + b)$
   - Repeated linear factors: $(ax + b)^n$
   - Irreducible quadratics: $(ax^2 + bx + c)$

2. Write the Decomposition

   Distinct linear factors:

$$\frac{P(x)}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$$

   Repeated linear factors:

$$\frac{P(x)}{(x-a)^n} = \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_n}{(x-a)^n}$$

   Irreducible quadratic factors:

$$\frac{P(x)}{x^2+ax+b} = \frac{Ax+B}{x^2+ax+b}$$

3. Solve for coefficients

7.3 Operational Properties I

First Translation Theorem
If $\mathscr{L}\{f(t)\} = F(s)$ and a is any real number then

$$\boxed{\mathscr{L}\{e^{at}f(t)\} = F(s-a)}$$

Note: This basically shifts the function in the s-domain, so instead of s we have $s - a$. Another way to write this is

$$\mathscr{L}\{e^{at}f(t)\} = F(s-a) = \mathscr{L}\{f(t)\}|_{s \to s-a}$$

Unit Step Function
The unit step function or Heaviside function is defined as

$$\mathscr{U}(t-a) = \begin{cases} 0 & 0 \le t < a \\ 1 & t \ge a \end{cases}$$

Using this function, we can represent piecewise functions as a single function. For example,

$$f(t) = \begin{cases} g(t) & 0 \le t < a \\ h(t) & t \ge a \end{cases}$$

We can define this in the following way

$$f(t) = g(t)(1 - \mathscr{U}(t-a)) + h(t)\mathscr{U}(t-a)$$

Note: We can do this using the fact that $\mathscr{U}(t-a)$ is 0 before $t = a$ and 1 after $t = a$. Thus, $1 - \mathscr{U}(t-a)$ is 1 before $t = a$ and 0 after $t = a$.

Second Translation Theorem
If $F(s) = \mathscr{L}\{f(t)\}$ and $a > 0$, then

$$\boxed{\mathscr{L}\{f(t-a)\mathscr{U}(t-a)\} = e^{-as}F(s)}$$

If $f(t) = \mathscr{L}^{-1}\{F(s)\}$ the inverse form for $a > 0$ is

$$\mathscr{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathscr{U}(t-a)$$

Note that $f(t-a)$ shifts the function to the right by a and $\mathscr{U}(t-a)$ makes the function equal to 0 for $t < a$, thus delaying the function by a units.

Note: The first translation theorem multiplies $f(t)$ by an exponential in the time domain, which produces a shift of the transform in the s-domain. The second translation theorem multiplies $F(s)$ by an exponential in the s-domain, which produces a shift of the function in the time domain.

7.4 Operational Properties II

Derivatives of Transforms
If $F(s) = \mathscr{L}\{f(t)\}$ and n is a positive integer, then

$$\boxed{\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)}$$

Each multiplication by t in the time domain corresponds to a differentiation with respect to s in the s-domain, along with a factor of $(-1)^n$.

Convolution
If functions f and g are piecewise continuous on the interval $[0, \infty)$, then the convolution of f and g is defined as

$$\boxed{f * g = \int_0^t f(\tau) g(t - \tau)\, d\tau}$$

<u>Convolution Theorem</u>
If $f(t)$ and $g(t)$ are piecewise continuous on $[0, \infty)$ and of exponential order, then

$$\boxed{\mathscr{L}\{f * g\} = \mathscr{L}\{f(t)\}\mathscr{L}\{g(t)\}}$$

$$\boxed{\mathscr{L}^{-1}\{F(s)G(s)\} = f * g}$$

Meaning that a convolution in the time domain corresponds to multiplication in the s-domain, and vice versa.

Laplace Transform of Periodic Functions
If $f(t)$ is piecewise continuous on $[0, \infty)$, of exponential order, and periodic with period T, then its Laplace transform is given by

$$\boxed{\mathscr{L}\{f(t)\} = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t) dt}$$

<u>Thm: Laplace Transform of an Integral</u>
Like how the laplace transform of a derivative turns differentiation into multiplication by s, the laplace transform of an integral turns integration into division by s. If $f(t)$ is piecewise continuous on $[0, \infty)$ and of exponential order, then

$$\boxed{\mathscr{L}\left\{\int_0^t f(\tau)\, d\tau\right\} = \frac{1}{s} F(s)}$$

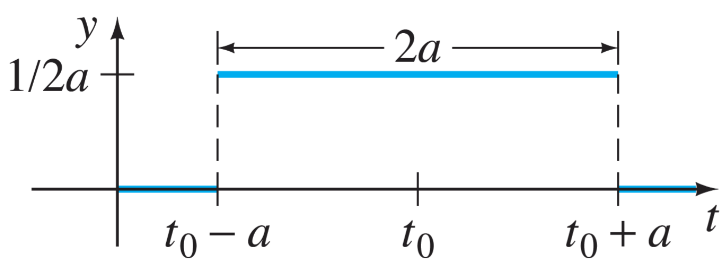
where $F(s) = \mathscr{L}\{f(t)\}$.

7.5 Dirac Delta Function

The Dirac delta function models an instantaneous impulse, where there is a very large force applied over a very short time interval, resulting in a sharp peak at a specific point in time. It is not a function in the traditional sense, but rather a distribution.

Unit Impulse
The unit impulse function $\delta(t - t_0)$ is defined as a piecewise function for $a, t_0 > 0$.

$$\delta_a(t - t_0) = \begin{cases} 0, & 0 \le t < t_0 - a \\ \frac{1}{2a}, & t_0 - a \le t < t_0 + a \\ 0, & t \ge t_0 + a \end{cases}$$



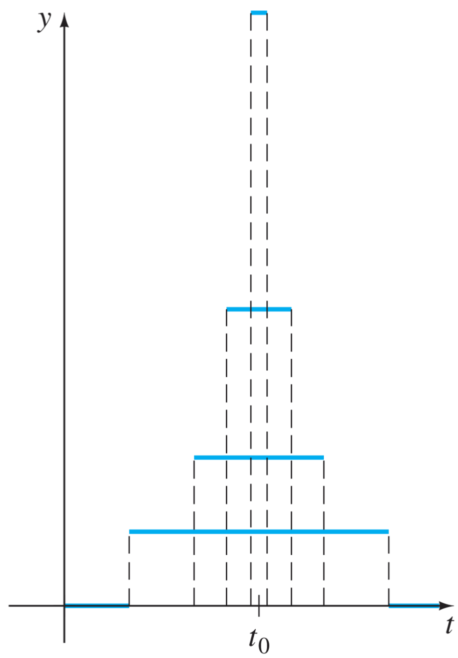
It is called a unit impulse because the area under the curve is equal to 1:

$$\int_0^\infty \delta_a(t - t_0)\ dt = 1$$

Dirac Delta Function
The Dirac delta function $\delta(t-t_0)$ is defined as the limit of the unit impulse as a approaches 0:

$$\delta(t-t_0) = \lim_{a\to 0} \delta_a(t-t_0)$$

Visually, as a approaches 0, the rectangle becomes narrower and taller, approaching an infinitely tall and narrow spike at $t = t_0$. The area under the curve remains equal to 1.



Note that the Dirac delta function is not a function in the traditional sense and can be characterized by the two properties:

$$\delta(t-t_0) = \begin{cases} \infty, & t = t_0 \\ 0, & t \neq t_0 \end{cases}$$

and

$$\int_0^\infty \delta(t-t_0)\, dt = 1$$

Basically, it is zero everywhere except at $t = t_0$, where it is infinitely large, but the area under the curve is equal to 1.

Laplace Transform of the Dirac Delta Function
For $t_0 > 0$,

$$\boxed{\mathscr{L}\{\delta(t-t_0)\} = e^{-st_0}}$$

End of Laplace Transform notes