Mathematical Methods

1. Use the Taylor series expansion to find approximations. The ones for sin, cos, tan, and $(1+x)^n$ are especially useful.

$$\sin x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\tan x = \sum_{n=1}^{\infty}\frac{B_{2n}(-4)^n(1-4^n)}{(2n)!}x^{2n-1} = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \cdots$$

$$(1+x)^m = \sum_{n=0}^{\infty}\binom{m}{n}x^n = 1 + mx + \frac{m(m-1)}{2}x^2 + \frac{m(m-1)(m-2)}{6}x^3 + \cdots$$

2. Use complex exponentials to manipulate complicated trig functions.

$$e^{ix} = \cos x + i\sin x$$

3. Solve differential equations by substituting in trial solutions. Especially you should recognize the differential equation for a simple harmonic oscillator and be able to come up with solutions to that ODE that satisfy any initial conditions you are given.

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$
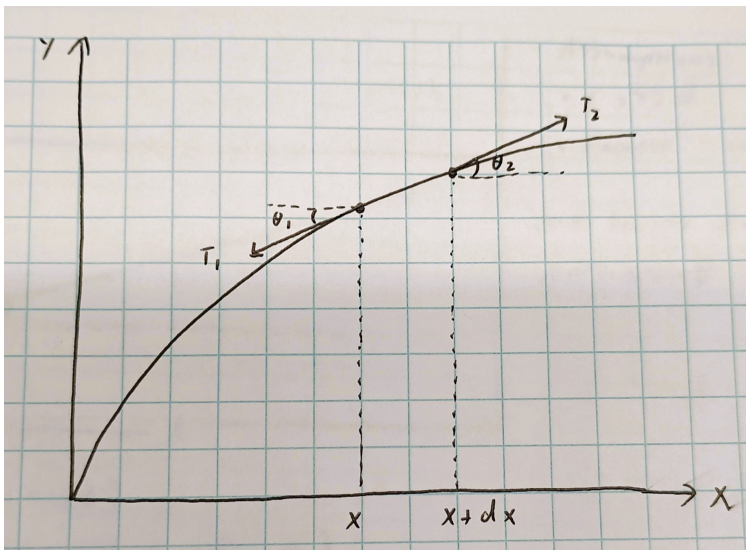
$$x(t) = A\cos(\omega t) + B\sin(\omega t)$$

Wave equation:

$$v^2\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2}$$

Chapter 16, 17, & parts of 35 - Waves

1. Set up the equation of motion and thereby find the wave equation for mechanical waves like waves on a string, sound waves, or similar systems. Explain the approximations made in the derivation of the wave equation and when they are valid.

   - Waves on a string

$$\text{where } \theta_1 = \theta(x), \quad \theta_2 = \theta(x + dx), \quad dm = \mu \, dx$$

<u>In y:</u>

$$\sum F_y = dm \, a_y$$

$$T_2 \sin \theta_2 - T_1 \sin \theta_1 = (\mu \, dx)\frac{d^2 y}{dt^2}$$

By small angle approx $\sin \theta \approx \theta \approx \tan \theta$ and $\tan \theta = dy/dx$

$$T_2 \left.\frac{dy}{dx}\right|_{x+dx} - T_1 \left.\frac{dy}{dx}\right|_x = \mu \, dx \frac{d^2 y}{dt^2}$$

<u>In x</u>

$$\sum F_x = dm \, a_x, \quad a_x = 0$$

$$T_2 \cos \theta_2 - T_1 \cos \theta_1 = 0 \quad \text{with small-angle approx } \cos \theta \approx 1$$

$$T_1 = T_2$$

Thus,

$$T \left(\frac{\left.\frac{dy}{dx}\right|_{x+dx} - \left.\frac{dy}{dx}\right|_x}{dx} \right) = \mu \frac{d^2 y}{dt^2}$$

$$\boxed{T\frac{d^2 y}{dx^2} = \mu \frac{d^2 y}{dt^2}}$$

- Sound Waves

  The physics of the phenomenon of sound waves involves three features:

  (a) The gas moves and changes its density.

  (b) The change in density corresponds to a change in pressure.

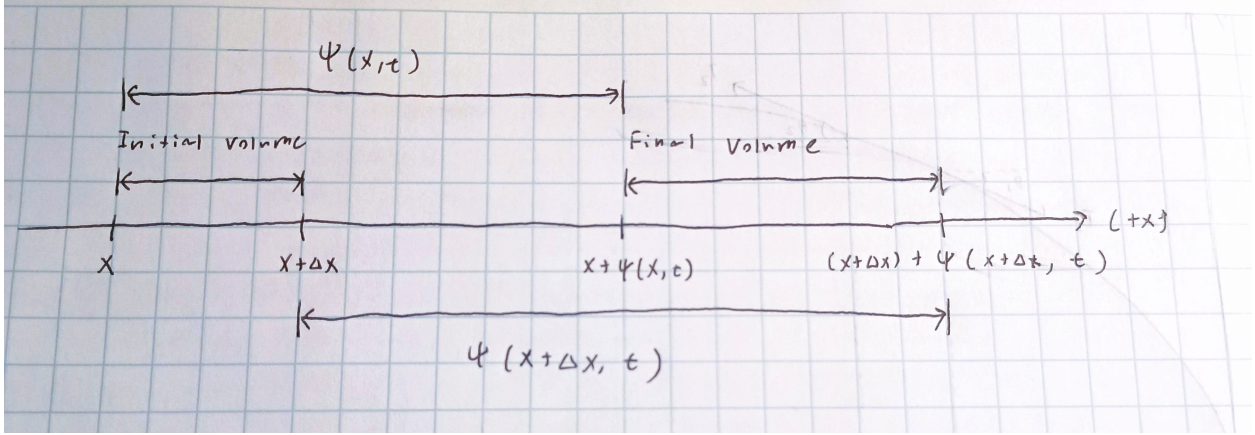 (c) Pressure differences generate gas motion.

Figure 1: The displacement of the air particles initially at x is $\psi(x, t)$, and at $x + \Delta x$ it is $\psi(x + \Delta x, t)$.

$$\text{Initial Volume} = A\big[(x + \Delta x) - x\big] = A\,\Delta x$$

$$\text{Final Volume} = A\Big[(x + \Delta x + \psi(x + \Delta x, t)) - (x + \psi(x, t))\Big]$$

$$= A(\Delta x + \Delta \psi)$$

By conservation of mass (mass before and after the wave passes are equal)

Where:

- $A =$ Cross-sectional area of air
- $V_0 =$ Initial volume
- $\rho_0 =$ Initial density

$$\rho_0 A\, dx = (\rho_0 + d\rho)A\,(dx + d\psi)$$

$$\rho_0\, dx = (\rho_0 + d\rho)\left(1 + \frac{d\psi}{dx}\right) dx$$

$$\rho_0 = (\rho_0 + d\rho)\left(1 + \frac{d\psi}{dx}\right)$$

$$\rho_0 = \rho_0 + \rho_0\frac{d\psi}{dx} + d\rho + d\rho\frac{d\psi}{dx} \quad d\rho \text{ is small so it reduces to } 0$$

$$\boxed{d\rho = -\rho_0\frac{d\psi}{dx}} \tag{1}$$

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This equation is what we would expect physically. If the displacements vary with x, then there will be density changes. The sign is also correct: if the displacement ψ increases with x, so that the air is stretched out, the density must decrease.

Also by conservation of mass

$$\rho_0 V_0 = (\rho_0 + \Delta\rho)(V_0 + \Delta V)$$

$$\Delta\rho\, V_0 = -\rho_0\,\Delta V$$

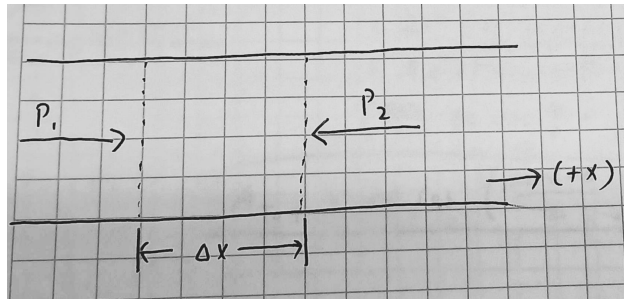$$\frac{\Delta\rho}{\rho_0} = -\frac{\Delta V}{V_0}$$

Substitute into

$$B = -\frac{\Delta P}{\Delta V/V}$$

$$B = \frac{\Delta P}{\Delta\rho/\rho_0}$$

$$\boxed{\Delta P = \frac{B}{\rho_0}\Delta\rho} \tag{2}$$

We find in this way that the change pressure ΔP is proportional to the change in density $\Delta\rho$, and we may call the proportionality factor B/ρ_o.

Force analysis on a region of air

Where P_o = Pressure in region

$$\sum F = ma$$

$$P_1 A - P_2 A = \rho_0 A \, dx \, \frac{d^2 \psi}{dt^2}$$

$$(P_0 + \Delta P_1)A - (P_0 + \Delta P_2)A = \rho_0 A \, dx \, \frac{d^2 \psi}{dt^2}$$

$$\Delta P_1 - \Delta P_2 = \rho_0 \, dx \, \frac{d^2 \psi}{dt^2}, \quad \left(\Delta P_2 - \Delta P_1 = \frac{d(\Delta P)}{dx} \, dx \right)$$

$$-\frac{d(\Delta P)}{dx} \, dx = \rho_0 \, dx \, \frac{d^2 \psi}{dt^2}$$

$$\rho_0 \frac{d^2 \psi}{dt^2} = -\frac{d}{dx} \left(\frac{B}{\rho_0} d\rho \right) \quad \text{Plug in eqn (1)}$$

$$\rho_0 \frac{d^2 \psi}{dt^2} = -\frac{d}{dx} \left(\frac{B}{\rho_0} \left(-\rho_0 \frac{d\psi}{dx} \right) \right) \quad \text{Plug in eqn (2)}$$

$$\boxed{\rho_0 \frac{d^2 \psi}{dt^2} = B \frac{d^2 \psi}{dx^2}}$$

2. Understand the properties of the wave equation and be able to prove the general solution to the wave equation works. Determine whether a given function is a solution to the wave equation, prove the principal of superposition for waves, and describe qualitatively and quantitatively such superpositions.

3. Use phasor diagrams (or algebraically use complex exponentials) to calculate the superposition of sinusoidal waveforms.

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4. Express solutions to the wave equation in terms of trigonometric functions and be able to define the transverse velocity, phase velocity, angular frequency, wavenumber, phase shift of a wave, wave speed, wavelength, period, etc.

- Waves on a string has the form

$$y(x,t) = y_m \sin(kx \pm \omega t + \phi)$$

Where

- y_m is the amplitude
- k is the angular wave number (spatial frequency: measures how many oscillations per wavelength in $[rad/m]$
- ω is the angular frequency in $[rad/s]$
- $(kx \pm \omega t + \phi)$ is the phase with a phase shift of ϕ

$$k = \frac{2\pi}{\lambda}$$

$$\frac{\omega}{2\pi} = f = \frac{1}{T} \text{ [Hz]}$$

The wave travels with a phase velocity

$$v = \frac{\omega}{k} = \frac{\lambda}{T} = \lambda f$$

$$v_{string} = \sqrt{\frac{\tau}{\mu}}$$

and has a transverse velocity of

$$\frac{\partial y}{\partial t} = \pm y_m \, \omega \, \cos(kx \pm \omega t + \phi)$$

- Similarly, sound waves has the form

$$s(x,t) = s_m \cos(kx \pm \omega t + \phi)$$

where s_m is the displacement amplitude from the equilibrium.
The sound wave also carries a pressure change of the medium from equilibrium pressure

$$\Delta P = \Delta P_m \sin(kx \pm \omega t + \phi)$$

where the pressure amplitude ΔP_m is

$$\Delta P_m = v \, \rho \, \omega \, s_m$$

$$\implies s_m = \frac{\Delta p_m}{v \, \rho \, \omega} = \frac{\Delta p_m}{Bk}$$

The sound wave travels with a phase velocity

$$v_{sound} = \sqrt{\frac{B}{\rho}}, \quad \text{where} \quad B = -\frac{\Delta P}{\Delta V/V}$$

Note: Speed of sound in air at 20°C is 343 m/s

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5. Define the impedance of a wave, describe its physical significance and relation to reflection and transmission of waves.

- Impedance is a physical property of a wave medium that relates the force (or pressure) to the particle velocity in a wave. When a wave encounters a boundary between two media with different impedances, part of the wave is reflected. This reflection occurs because the difference in impedance ensures that the forces and wave speeds at the boundary are balanced, maintaining continuity of motion and energy across the interface.

  For strings:
  $$Z = \mu\, v = \sqrt{\mu T}$$
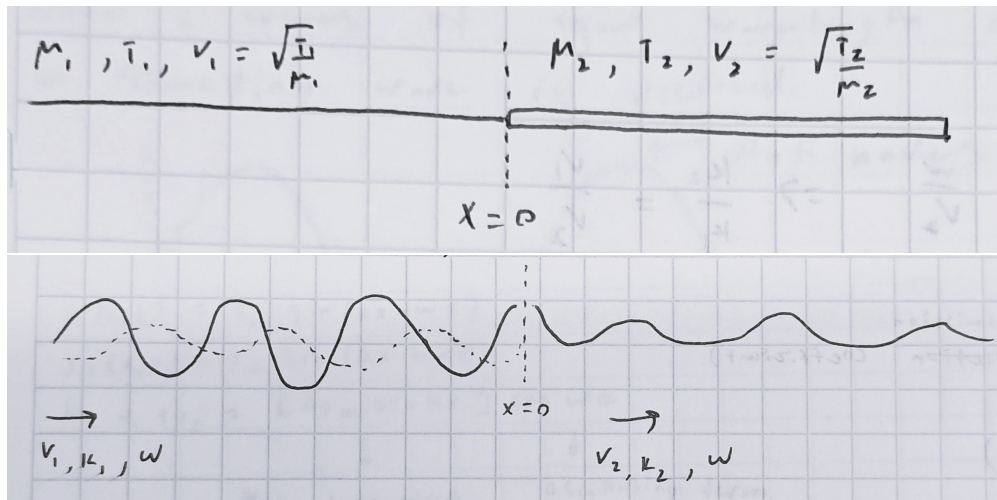 $$F_y = Z\frac{\partial y}{\partial t}$$

  For sound waves moving through air:
  $$Z = \rho\, v = \sqrt{\rho B}$$
  $$\Delta P = Z\frac{\partial s}{\partial t}$$

6. Describe the boundary conditions that must be satisfied an interface between two media. Use those conditions to determine the reflectance and transmittance of a wave through the interface.

- **Set up:**

  - Tension must be equal or else the string will be pulled to one side
  - $\omega_1 = \omega_2$ as is $f_1 = f_2$ because it is determined by the source of the wave
  - $\lambda_1 \neq \lambda_2$ since wave velocity is not equal but frequency is equal $(v = \lambda f)$
  - $k_1 \neq k_2$ since wavelengths are not equal

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Wave in medium 1: $y_{incident} + y_{reflected}$

$$y_1(x, t) = y_I cos(k_1 x - \omega t) + y_R cos(k_1 x + \omega t)$$

(y_R has the same wavenumber since wave speeds are equal within the same medium and f)

Wave in medium 2: $y_{transmitted}$

$$y_2(x, t) = y_T cos(k_2 x - \omega t)$$

- **Boundary conditions at x=0:**
  - String is continuous (total transverse displacement at x=0 for all time is equal for both mediums)
  $$y_1(0, t) = y_2(0, t)$$
  $$y_I(0 - \omega t) + y_R cos(0 + \omega t) = y_T cos(0 - \omega t)$$
  $$y_I + y_R = y_T \quad cos((\omega t) \text{ cancels out})$$
  - The slope is continuous at x=0 (if slopes are not equal, tension will not cancel out)
  $$\frac{dy_1}{dx}\bigg|_{x=0} = \frac{dy_2}{dx}\bigg|_{x=0}$$
  $$\left(-y_I k_1 \sin(k_1 x - \omega t) - y_R k_1 \sin(k_1 x + \omega t)\right)\big|_{x=0} = \left(-y_T k_2 \sin(k_2 x - \omega t)\right)\big|_{x=0}$$
  $$-y_I k_1 \sin(-\omega t) - y_R k_1 \sin(\omega t) = -y_T k_2 \sin(-\omega t)$$
  $$y_I k_1 - y_R k_1 = y_T k_2 \quad (sin(\omega t) \text{ cancels out})$$

- **Solve system of equations for y_T/y_I and y_R/y_I**

$$\boxed{y_T = \frac{2v_2}{v_1 + v_2} y_I} \quad \text{(Transmission coefficient)}$$

$$\boxed{y_R = \frac{v_2 - v_1}{v_1 + v_2} y_I} \quad \text{(Reflection coefficient)}$$

Note: If T=transmission coefficient and R=reflection coefficient then 1+R=T

7. Determine the resonance condition for standing waves on strings, closed ended pipes, open-ended pipes, etc. For the same situations be able to relate wavelengths, frequencies, phase velocities and solve for unknown quantities.

- **Standing waves** are formed when two waves of the same frequency and amplitude travel in opposite directions and interfere.
  Let
  $$y_1(x, t) = y_m sin(kx - \omega t)$$
  $$y_2(x, t) = y_m sin(kx + \omega t)$$
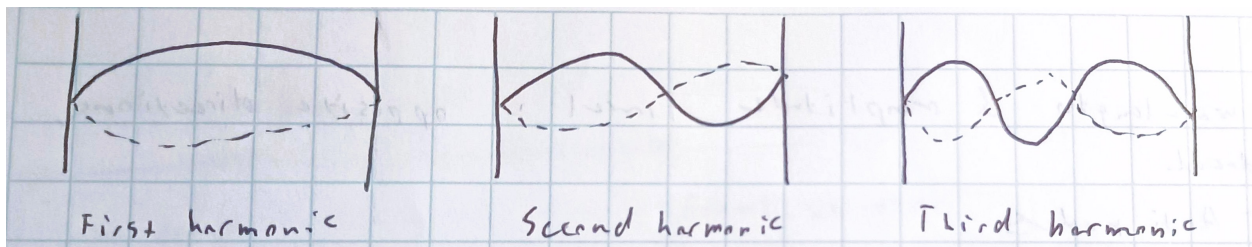 $$y_1 + y_2 = [2y_m sin(kx)]cos(\omega t)$$

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The amplitude is determined by $2y_m sin(kx)$ thus there will be a node when $sin(kx) = 0$ and an antinode when $sin(kx) = 1$

Setting $kx = n\pi$

$$x = n\frac{\lambda}{2} \quad \text{x positions for nodes}$$

Setting $kx = (n + \frac{1}{2})\pi$

$$x = \left(n + \frac{1}{2}\right)\frac{\lambda}{2} \quad \text{x positions for antinodes}$$
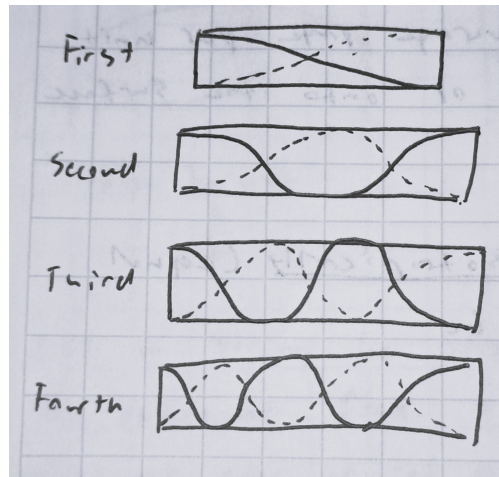
- **Strings fixed at both ends separated by length L**

First harmonic   Second harmonic   Third harmonic

$$\boxed{\lambda = \frac{2L}{n}} \quad \text{Where n is the harmonic number}$$

$$\boxed{f_{resonant} = \frac{v}{\lambda} = \frac{nv}{2L}}$$

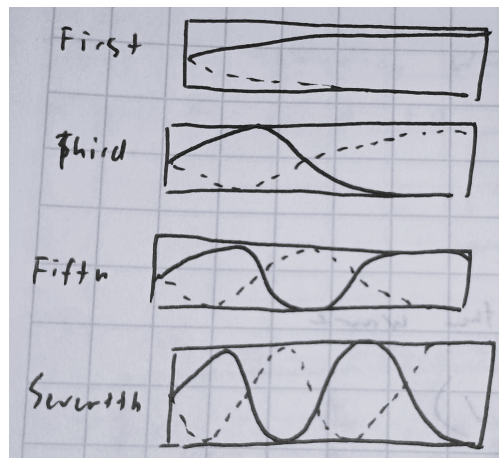The first harmonic is also called the fundamental frequency.

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- **Pipes open at both ends separated by length L (same as strings fixed at both ends)**

$$\lambda = \frac{2L}{n}$$

$$f_{resonant} = \frac{v}{\lambda} = \frac{nv}{2L}$$

- **Pipes closed at one end separated by length L (ONLY odd harmonics)**



$$\lambda = \frac{4L}{n}$$ Where n is odd (1,3,5...)

$$f_{resonant} = \frac{v}{\lambda} = \frac{nv}{4L}$$

8. Explain what a beat is, when it occurs, and derive the beat frequency.

- A beat is formed when two waves of slightly different frequencies interfere. The resulting wave oscillates at a frequency that is the average of the two original frequencies, but its amplitude varies at a rate equal to the difference between the two frequencies.

Let

$$s_1(t) = s_m cos(\omega_1 t)$$

$$s_2(t) = s_m cos(\omega_2 t)$$
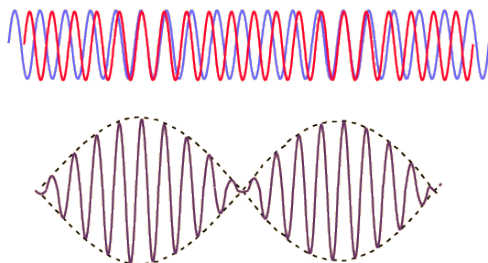
$$s_1 + s_2 = [2s_m cos(\omega' t)] \, cos(\omega t)$$

Where $\omega' = (\omega_1 - \omega_2)/2$ (half the difference in angular freq) and $\omega = (\omega_1 + \omega_2)/2$ (the average angular freq)

$$\boxed{f_{beat} = f_1 - f_2}$$

9. Know the average wave power carried by a wave on a string.

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$$W = F_y dy$$

$$P = \frac{W}{\Delta t} = F_y \frac{dy}{dt}$$

For a wave on a string:

$$P = \left(Z\frac{\partial y}{\partial t}\right)\frac{\partial y}{\partial t}$$

Say $y = y_m cos(kx - \omega t)$

$$P = Zy_m^2 \omega^2 sin^2(kx - \omega t)$$

$$P_{average} = \frac{1}{2} Zy_m^2 \omega^2$$

Note: $Z = \mu v$ and the average of sin^2 and cos^2 over one period is $1/2$

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10. Describe the relation between intensity and amplitude. Write an equation giving the variation of the intensity with distance from the source for waves on a string or sound waves in air.

Note: For a wave on a string, use cross sectional area for power per unit area and length for power per unit length.

$$\text{Intensity} = \frac{\text{Average rate of energy delivered (Avg Power)}}{\text{Area that wave spreads out}}$$

$$I \propto (\text{Amplitude})^2$$

- For sound waves:
$$\text{Intensity} = I = \frac{P_{average}}{4\pi r^2} = \frac{1}{2}\rho v \omega^2 s_m^2$$

- Intensity level (in decibels)

$$\beta = (10\,dB)\log\left(\frac{I}{I_0}\right)$$

Where $I_0 = 10^{-12}\,W/m^2$ is reference intensity (usually the threshold of hearing)

11. By considering the effective wavelength and the relative velocity between the waves and the observer, determine the Doppler shift for any combination of moving source, moving detector, and "wind".

Consider the following variables:

(a) $v_{relative}$: Relative speed between the receiver and the wave fronts. $(v_{sound}\pm v_{receiver})$

(b) $\lambda_{effctive}$: The distance between two successive wave fronts as measured by the receiver.
$$\lambda_{effctive} = (v_{sound} \pm v_{source})T = \frac{v_{sound} \pm v_{source}}{f_{source}}$$

Thus the Doppler frequency can be described by:

$$f'_{Doppler} = \frac{v_{rel}}{\lambda_{eff}} = \frac{v_{sound} \pm v_{receiver}}{v_{sound} \pm v_{source}}f_{source}$$

Note: Wind either speeds up or slows down the sound wave fronts, consider this when solving cases with wind.

General rule of thumb:
If receiver moves towards the incoming sound waves $\implies$ higher received frequency.
If receiver moves away from incoming sound waves $\implies$ lower received frequency.

12. Understand the effects of interference on waves. Calculate a phase difference from path length or delay time differences. Use a phasor diagram to quantitatively and qualitatively describe the observed intensities at any given location. For example, describe the pattern of waves produced by two or more "slits". Explain applications of interference, for example the interferometer, spectrometer, and thin films like soap bubbles. (Note - these are in the textbook but may not all have been discussed in class).

- Interference due to path length difference
  Suppose in 1D, two waves travel in the same direction but y_2 starts a distance d behind y_1.

  $$y_1 = A\cos(kx - \omega t)$$

  $$y_2 = A\cos(k(x + d) - \omega t) = A\cos(kx - \omega t + kd)$$

  Thus, y_2 is out of phase with y_1 by

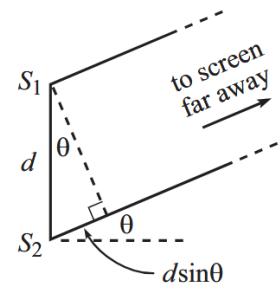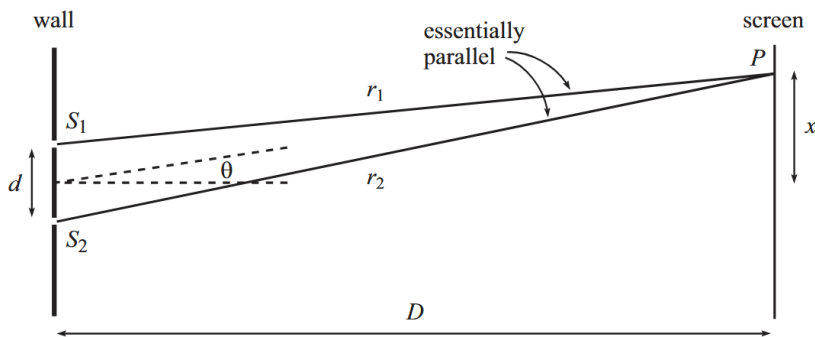 $$\boxed{\phi = kd = \frac{2\pi}{\lambda}d}$$

- Interference due to time delay difference
  Similarly, if y_2 starts a time Δt after y_1 then

  $$y_2 = A\cos(kx - w(t + \Delta t)) = A\cos(kx - \omega t - w\Delta t)$$

  Thus, y_2 is out of phase with y_1 by

  $$\boxed{\phi = \omega \Delta t}$$

- Interference from double slits

Assumptions:

- $L \gg d$ (distance to screen is much greater than slit separation)
- Parallel lines meet at infinity

Path length difference:
$$\Delta L = d\sin\theta$$

Phase difference:
$$\boxed{\phi = k\Delta L = \frac{2\pi}{\lambda}d\sin\theta}$$

Complete constructive interference (bright fringes):
$$\phi = kd\sin\theta = 2\pi n$$
$$d\sin\theta = n\lambda$$

Complete destructive interference (dark fringes):
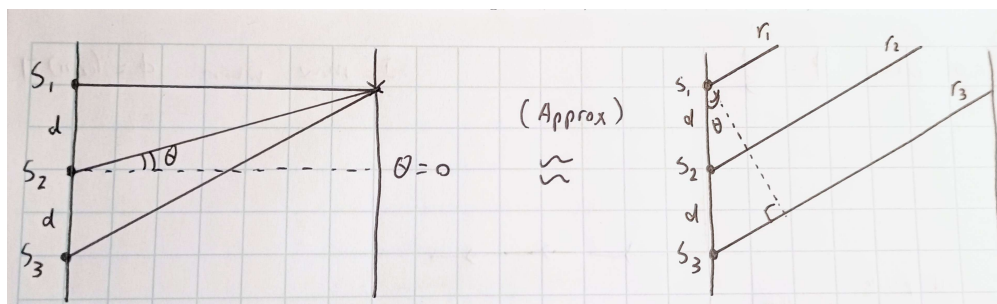$$\phi = kd\sin\theta = (2n+1)\pi$$
$$d\sin\theta = \left(n + \frac{1}{2}\right)\lambda$$

Where n is the nth bright/dark fringe from the center

- When asked for intensity, use phasor diagram to find resultant amplitude and then use
$$I \propto A^2$$

- **Example: Triple slit**
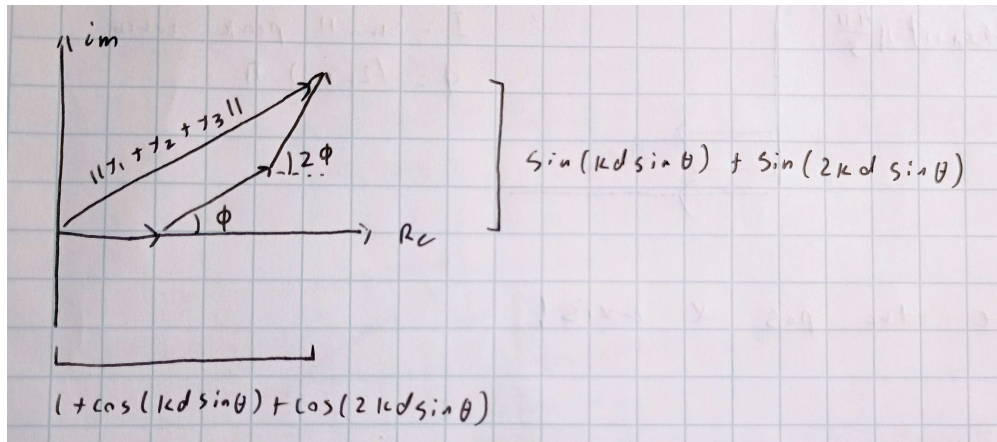

$$r_2 - r_1 = d\sin\theta$$
$$r_3 - r_1 = 2d\sin\theta$$

Thus y_2 and y_3 are phase shifted from y_1

$$y_1 = A\cos(kx - \omega t)$$
$$y_2 = A\cos(kx - \omega t + kd\sin\theta)$$
$$y_3 = A\cos(kx - \omega t + 2kd\sin\theta)$$
$$y_1 + y_2 + y_3 = Ae^{i(kx-\omega t)}\left(1 + e^{ikd\sin\theta} + e^{i2kd\sin\theta}\right)$$

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$$\|y_1+y_2+y_3\| = \sqrt{(1 + cos(kdsin\theta) + cos(2kdsin\theta))^2 + (sin(kdsin\theta) + sin(2kdsin\theta))^2}$$

Let α be the phase shift of the superposition

$$\alpha = arctan\left(\frac{(kdsin\theta) + sin(2kdsin\theta)}{1 + cos(kdsin\theta) + cos(2kdsin\theta)}\right)$$
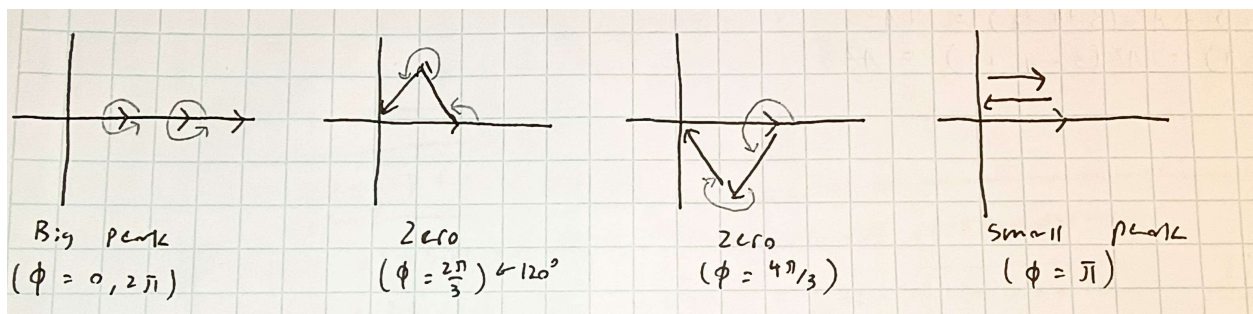
$$y_1 + y_2 + y_3 = A\|y_1 + y_2 + y_3\| \operatorname{Re}(e^{i(kx-\omega t+\alpha)})$$

$$y_1 + y_2 + y_3 = A\|y_1 + y_2 + y_3\|cos(kx - \omega t + \alpha)$$

The intensity as a function of θ is

$$I = (A\|y_1 + y_2 + y_3\|)^2$$

$$I = A^2\left[(1 + cos(kdsin\theta) + cos(2kdsin\theta))^2 + (sin(kdsin\theta) + sin(2kdsin\theta))^2\right]$$

13. Find the Fourier series representation for any given periodic function. Describe applications of Fourier analysis. Know the properties of Fourier series. For example, you can reduce the work you need to do for odd versus even functions.

(Almost) any periodic f(x) with period λ can be written as a fourier series:

$$f(x) = \frac{A_o}{2} + \sum_{m=1}^{\infty} A_m \cos\left(\frac{2\pi}{\lambda}mx\right) + \sum_{m=1}^{\infty} B_n \sin\left(\frac{2\pi}{\lambda}mx\right)$$

Where

$$A_o = \frac{2}{\lambda} \int_0^{\lambda} f(x)\, dx = 2 \cdot (\text{Average value of f(x)})$$

$$A_n = \frac{2}{\lambda} \int_0^{\lambda} f(x)\cos\left(\frac{2\pi}{\lambda}mx\right) dx$$

$$B_n = \frac{2}{\lambda} \int_0^{\lambda} f(x)\sin\left(\frac{2\pi}{\lambda}mx\right) dx$$

Note: Bounds of intergration don't have to start at 0 but has to cover 1 full period.
Recall: Odd fxn: f(-x)=-f(x), even fxn: f(-x)=f(x).
(Even × Even = Even, Odd × Odd = Even, Even × Odd = Odd)

Even functions only have cosine terms, odd functions only have sine terms

Ex: Fourier series of $f(x) = |x|$ with $-\pi < x < \pi$ and repeating

$$f(x) = |x|, \quad x \in (-\pi, \pi) \implies \lambda = 2\pi$$

$$f(x) = \begin{cases} -x, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$$

$$A_0 = \frac{2}{2\pi}\left[\int_{-\pi}^0 -x\, dx + \int_0^{\pi} x\, dx\right] = \frac{1}{\pi}\left[-\tfrac{1}{2}x^2\Big|_{-\pi}^0 + \tfrac{1}{2}x^2\Big|_0^{\pi}\right] = \frac{1}{\pi}\left[(0 - -\tfrac{1}{2}\pi^2) + (\tfrac{1}{2}\pi^2 - 0)\right] = \pi$$

$$A_n = \frac{2}{2\pi}\left[\int_{-\pi}^0 -x\cos(mx)\, dx + \int_0^{\pi} x\cos(mx)\, dx\right]$$

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Using Integration by parts

$$\int x\cos(mx)\, dx = \tfrac{x}{m}\sin(mx) - \tfrac{1}{m}\int \sin(mx)\, dx = \tfrac{x}{m}\sin(mx) + \tfrac{1}{m^2}\cos(mx)$$

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So

$$\int_{-\pi}^{0} -x\cos(mx)\,dx = -\left[\frac{x}{m}\sin(mx) + \frac{1}{m^2}\cos(mx)\right]_{-\pi}^{0} = \left(-\frac{1}{m^2}\right) - \left(\frac{-\pi}{m}\sin(-m\pi) - \frac{1}{m^2}\cos(-m\pi)\right)$$

$$\int_{0}^{\pi} x\cos(mx)\,dx = \left[\frac{x}{m}\sin(mx) + \frac{1}{m^2}\cos(mx)\right]_{0}^{\pi} = \left(\frac{\pi}{m}\sin(m\pi) + \frac{1}{m^2}\cos(m\pi)\right) - \left(\frac{1}{m^2}\right)$$

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$$A_n = \frac{1}{\pi}\left[\left(-\frac{1}{m^2}\right) - \left(\frac{-\pi}{m}\sin(-m\pi) - \frac{1}{m^2}\cos(-m\pi)\right) + \left(\frac{\pi}{m}\sin(m\pi) + \frac{1}{m^2}\cos(m\pi)\right) - \left(\frac{1}{m^2}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{2}{m^2}\cos(m\pi) - \frac{2}{m^2}\right]$$

$$= \frac{2}{\pi}\left(\frac{1}{m^2}(-1)^m - \frac{1}{m^2}\right) = \frac{2}{\pi m^2}\left((-1)^m - 1\right)$$

$$B_n = 0 \quad (f(x) \text{ is even})$$

So the Fourier series is:

$$f(x) = |x| \approx \frac{\pi}{2} + \sum_{m=1}^{\infty} \frac{2}{\pi m^2}\left((-1)^m - 1\right)\cos(mx)$$

For m even: $(-1)^m - 1 = 0$.
For m odd: $(-1)^m - 1 = -2$.

So let $m = 2k - 1$:

$$f(x) \approx \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{\cos((2k-1)x)}{(2k-1)^2}$$

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14. Know, qualitatively, what makes dispersive waves, like water waves, different from linear waves, like waves on a string. This includes the definition of dispersion as applied to waves and the distinction between phase and group velocity.

Unlike linear waves, dispersive waves are a superposition of waves that breaks apart as it propagates. (Ex: Water waves with $v_w = \sqrt{g\lambda}$ so longer wavelength waves will spread out more)

- Phase velocity: speed of individual wave fronts

$$v_{phase} = \frac{\omega}{k}$$

- Group velocity: the speed of a traveling interference pattern (the overall shape), sometimes called a wave packet. (Ex: the "beat" velocity)

$$v_{group} = \frac{d\omega}{dk}$$

Ex: Water waves

$$v_w = \frac{\omega}{k} = \sqrt{\frac{g\lambda}{2\pi}}$$

$$\omega = \sqrt{gk}$$

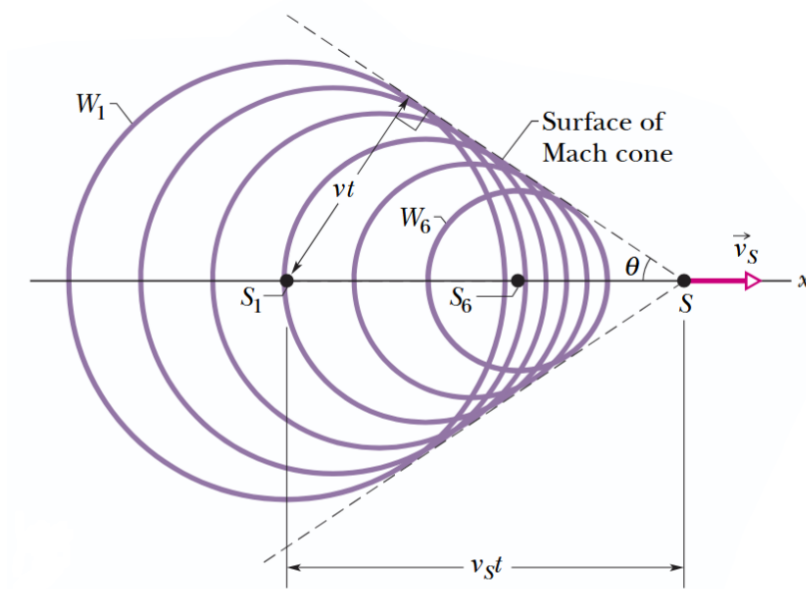$$v_{group} = \frac{d\omega}{dk} = \frac{1}{2}\sqrt{\frac{g}{k}}$$

15. Misc.

- Supersonic shock waves
  When an object moves faster than the speed of sound, it forms a mach cone with a mach cone angle:

$$sin\theta = \frac{v_{sound}t}{v_{source}t} = \frac{v_{sound}}{v_{source}}$$

Where t is the time elapsed between the emission of wave fronts

$$\text{Mach Number} = \frac{v_{source}}{v_{sound}}$$

End Exam 1

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