Mathematical Models for Red Blood Cell

2018 Summer AM-SURE

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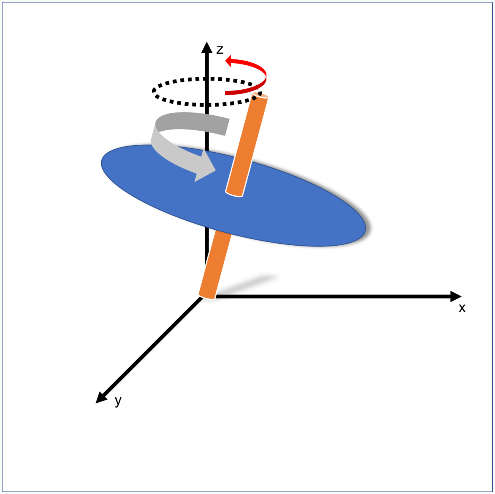
Red Blood Cell



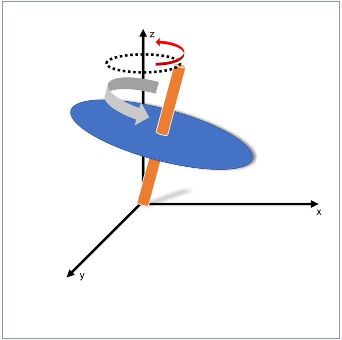
Goal

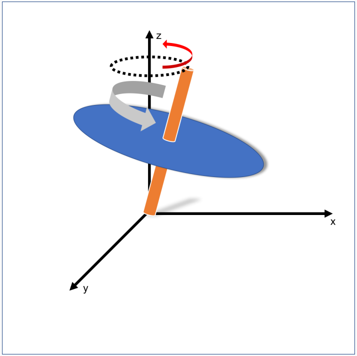
Find the most simplest equation and use discrete methods to generate a biconcave shape from a perfect sphere.

Mathematical Modeling

Triangulated Surface

We start from a sphere (codes from Professor Peskin).





The more triangles we use, the more smooth the surface is.

Model Setup

Discovered by Canham in 1970, the biconcave shape requires the least amount of bending energy E.

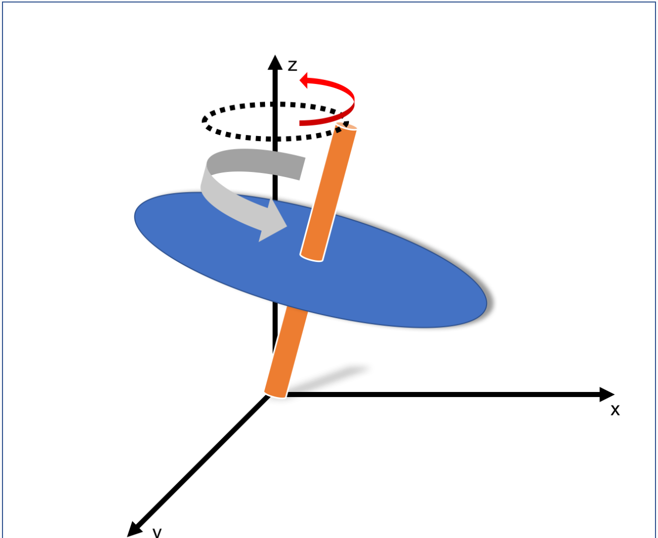
- Minimize E.
- Minimize $\frac{dE}{dt}$, where t refers to time.
- Minimize $\frac{dE}{dt} + \frac{1}{2}\sum_k \frac{dX_k}{dt}\frac{dX_k}{dt}$, where X_k is the coordinate vector of the k^{th} point.

Constraints

- Area: Surface area should remain constant: $\frac{dA}{dt} = 0$.
- Volume: Force the volume to decrease: $\frac{dV}{dt} = \frac{dV_0}{dt}$

Constraints

$V_0(t) = V\min + (V\max - V\min)e^{-at}$, where V_{max} refers to the initial volume, V_{min} refers to the final volume, and a is a positive constant.



- In summary:

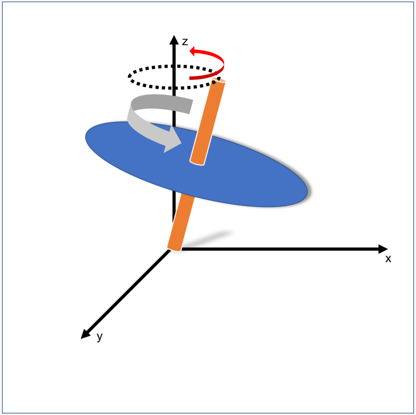
$$\text{Minimize} \quad \frac{dE}{dt}+\frac{1}{2}\sum_k \frac{dX_k}{dt}\frac{dX_k}{dt}, \quad \text{subject to} \quad \frac{dV}{dt}=\frac{dV_0}{dt} \quad \text{and} \quad \frac{dA}{dt}=0.$$

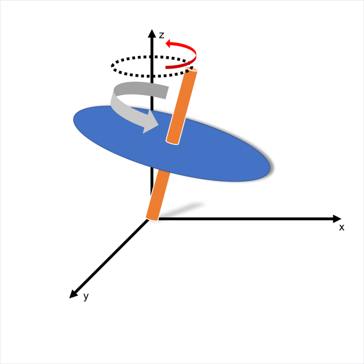
- Translate the above equation into the following ODE equation:

$$\frac{dX_k}{dt}=-\frac{\partial E}{\partial X_k}-\lambda\frac{\partial V}{\partial X_k}-\mu\frac{\partial A}{\partial X_k}, \quad \text{subject to} \quad \frac{dV}{dt}=\frac{dV_0}{dt} \quad \text{and} \quad \frac{dA}{dt}=0,$$

where λ and μ are lagrange multipliers for the constraints.

Calculation

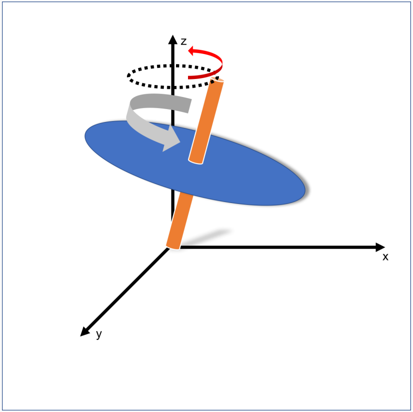




Calculation

Volume
$V = \sum_f V_f$, where
$V_f = \frac{1}{6} X_{f_1}(X_{f_2} \times X_{f_3})$.

Surface area
$A = \sum_f a_f$, where $a_f =$
$[\frac{1}{2}(X_{f_2} - X_{f_1}) \times (X_{f_3} - X_{f_1})] \cdot n$,
where n is the normal vector to the surface f.



Calculation

On a smooth surface:
(1) Bending energy $E = \int H^2 da$, where H and a represent mean curvature and area, respectively.

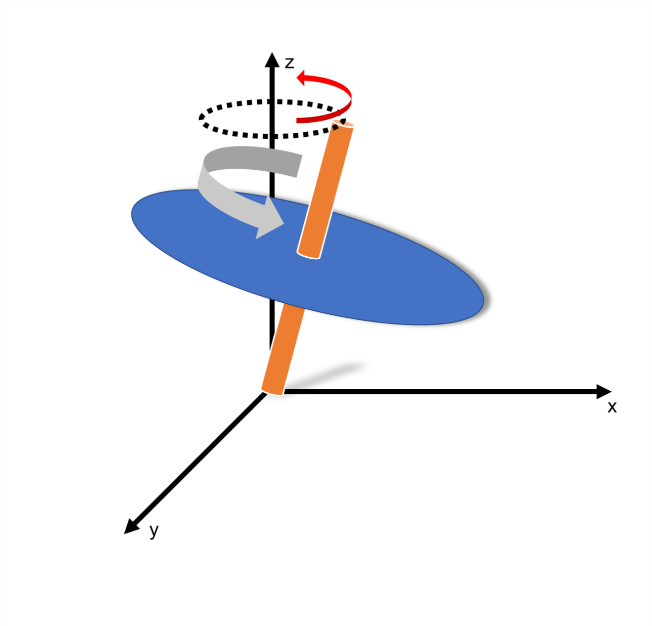
(2) Mean curvature H of a surface is the rate of change of its area, per unit area, when this surface is moving in the normal direction to itself.

On discrete surface (face f) :
(1) Mean curvature h_f is the rate of change of face f's area, to its area, when face f is moving in the normal direction to itself.

(2) $h_f = (da_f/dt) \cdot a_f^{-1}$, where a_f is the area of the face f.

(3) Then $E = \int H^2 da \approx \sum_f h_f^2 a_f$.

Non-Smoothness



Refinement

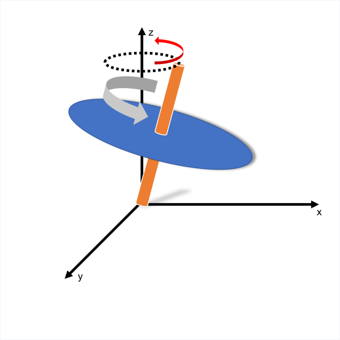
Minimize $E + \sum_{e=1}^{\#\text{ of edges}} l_e^3$, where l_e represents the length of the e^{th} edge.

Limit the possibility that one edge will get extremely long compared to other edges, and therefore limit the possibility that triangles will get deformed.

This gives us

$$\frac{dX_k}{dt} = -\frac{\partial E}{\partial X_k} - \frac{\partial \sum_e l_e^3}{\partial X_k} - \lambda \frac{\partial V}{\partial X_k} - \mu \frac{\partial A}{\partial X_k}.$$

Results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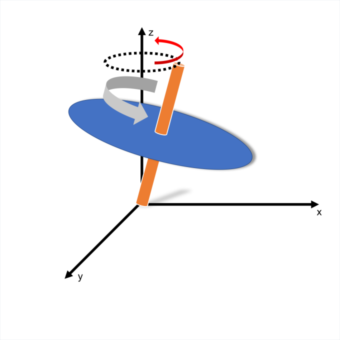
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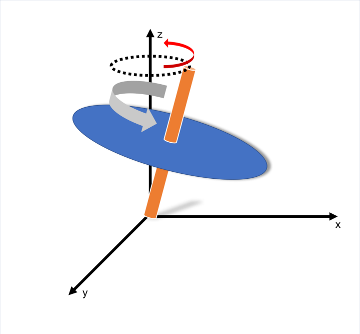
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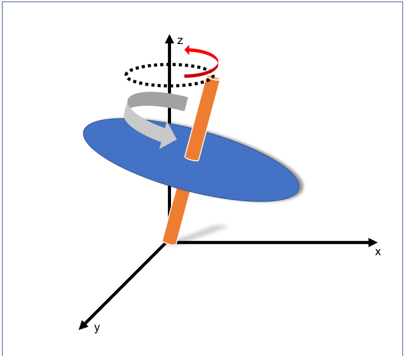
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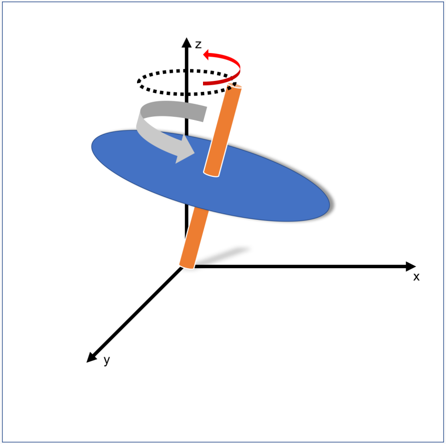
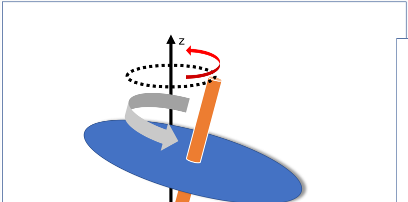
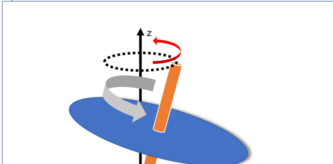
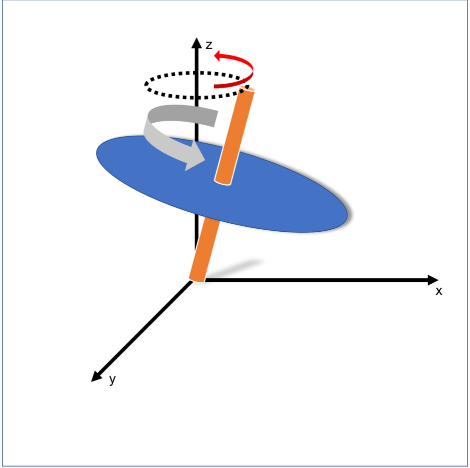
At time t = 1.2,





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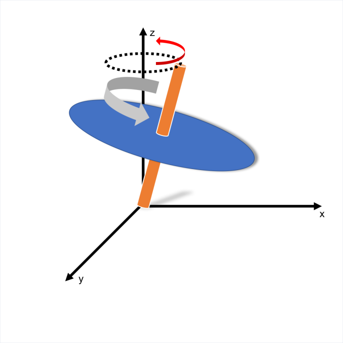
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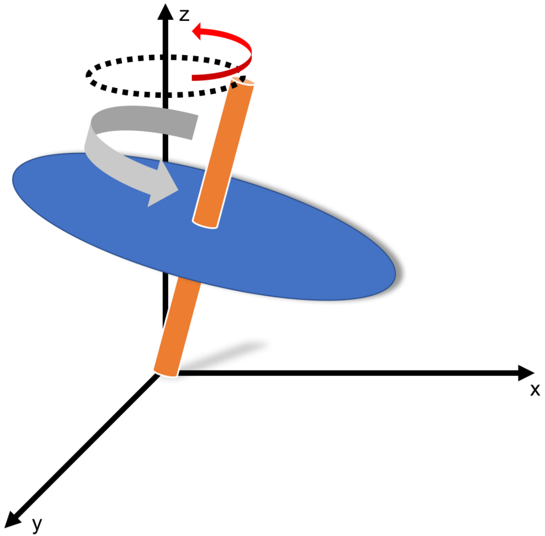
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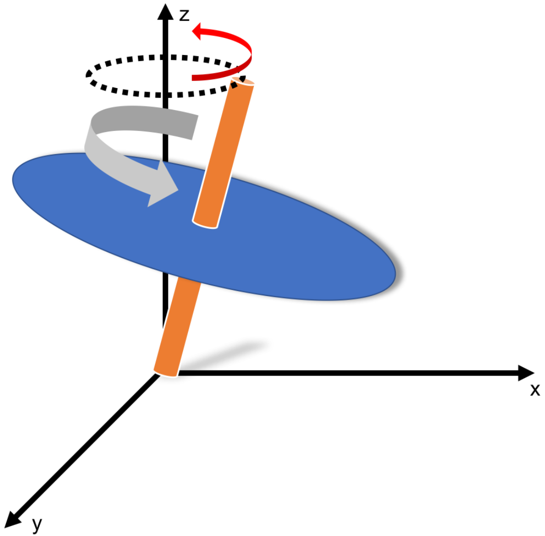
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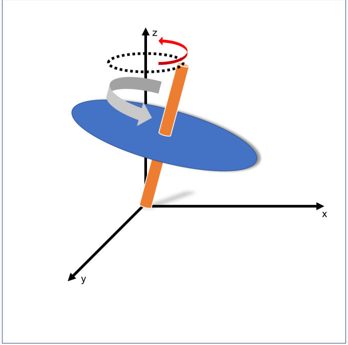
Conclusion

(1) Generate a shape that is close to biconcave, but not exactly the same.

(2) Possible explanations for the instabilities.

(3) Look into the future.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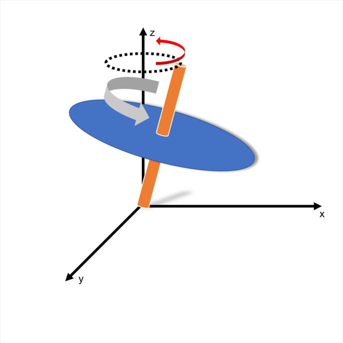
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