

Problem Set #4

Smooth and convex optimization

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Exercise 6.6

We first find the critical points. Observe that $f(x, y) = 3x^2y + 4xy^2 + xy$

let $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$

FOC: $Df(\mathbf{x}) = [f_x, f_y] = [6xy + 4y^2 + y, 3x^2 + 8xy + x] = [0, 0]$

$$\Rightarrow \begin{cases} 6xy + 4y^2 + y = 0 \\ 3x^2 + 8xy + x = 0 \end{cases}$$

The critical points are:

$$\mathbf{x}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 0 \\ -\frac{1}{4} \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} -\frac{1}{3} \\ 0 \end{bmatrix}, \mathbf{x}_4 = \begin{bmatrix} -\frac{1}{9} \\ -\frac{1}{12} \end{bmatrix}$$
$$D^2f = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 6y & 6x + 8y + 1 \\ 6x + 8y + 1 & 8x \end{bmatrix}$$

Then,

$$D^2f(\mathbf{x}_1) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow \text{saddle point}$$

$$D^2f(\mathbf{x}_2) = \begin{bmatrix} -\frac{3}{2} & -1 \\ -1 & 0 \end{bmatrix} \Rightarrow \text{saddle point}$$

$$D^2f(\mathbf{x}_3) = \begin{bmatrix} 0 & -1 \\ -1 & -\frac{8}{3} \end{bmatrix} \Rightarrow \text{saddle point}$$

$$D^2f(\mathbf{x}_4) = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{8}{9} \end{bmatrix} < 0 \Rightarrow \text{local maximum}$$

Exercise 6.7

(1) Since $Q = A^T + A$, and A is a square matrix, $Q^T = (A^T)^T + A^T = A + A^T = Q$. So Q is symmetric. Observe that $\mathbf{x}^T A \mathbf{x} = \langle \mathbf{x}, A \mathbf{x} \rangle$, and $\mathbf{x}^T A^T \mathbf{x} = (A \mathbf{x})^T \mathbf{x} = \langle A \mathbf{x}, \mathbf{x} \rangle$. Since here we restrict the field to be \mathbb{R} , we have $\langle \mathbf{x}, A \mathbf{x} \rangle = \langle A \mathbf{x}, \mathbf{x} \rangle$. So it follows that

$$\mathbf{x}^T Q \mathbf{x} = \mathbf{x}^T (A^T + A) \mathbf{x} = \mathbf{x}^T A^T \mathbf{x} + \mathbf{x}^T A \mathbf{x} = 2\mathbf{x}^T A \mathbf{x}.$$

Thus we have

$$f(\mathbf{x}) = \frac{1}{2} \mathbf{x}^T Q \mathbf{x} - \mathbf{b}^T \mathbf{x} + c.$$

(2) The first order necessary condition implies that if \mathbf{x}^* is a minimizer, then it must be $Df(\mathbf{x}^*) = Q^T \mathbf{x}^* - \mathbf{b} = 0$. Hence $Q^T \mathbf{x}^* = \mathbf{b}$.

(3) Observe that $D^2f(\mathbf{x}^*) = Q$, and since Q is positive definite, it follows from the second order sufficient condition that \mathbf{x}^* is a minimizer, which is also the solution to the linear system $Q^T \mathbf{x}^* = \mathbf{b}$.

Exercise 6.11

Proof. Observe that $f''(x) = 2a > 0$, so the x^* that satisfies $f'(x^*) = 2ax^* + b = 0$ is

the minimizer. $\forall x_0$, by Newton's method,

$$x_1 = x_0 - \frac{f'(x_0)}{f''(x_0)} = x_0 - \frac{2ax_0 + b}{2a} = -\frac{b}{2a}.$$

Since the quadratic function can also be expressed as $f(x) = a(x + \frac{b}{2a})^2 + \frac{4ac-b^2}{4a}$, it follows that $x_1 = -\frac{b}{2a}$ is the unique minimizer. \square

Exercise 7.1

Proof. Since $\text{conv}(S)$ is the set of all convex combinations of vectors in S , it follows immediately that this is a convex set. \square

Exercise 7.2

Proof. A hyperplane is a set of the form $P = \{\mathbf{x} \in V \mid \langle \mathbf{a}, \mathbf{x} \rangle = b\}$, for some $b \in \mathbb{R}$ and $\mathbf{a} \neq \mathbf{0}$. Take $\mathbf{x}, \mathbf{y} \in P$. Let $\lambda \in [0, 1]$. Observe that

$$\langle \mathbf{a}, \lambda \mathbf{x} + (1 - \lambda) \mathbf{y} \rangle = \lambda \langle \mathbf{a}, \mathbf{x} \rangle + (1 - \lambda) \langle \mathbf{a}, \mathbf{y} \rangle = \lambda b + (1 - \lambda) b = b.$$

So $\lambda \mathbf{x} + (1 - \lambda) \mathbf{y} \in P, \forall \mathbf{x}, \mathbf{y}$. Thus a hyperplane is a convex set. \square

Proof. A half space is a set of the form $H = \{\mathbf{x} \in \mathbb{R}^n \mid \langle \mathbf{a}, \mathbf{x} \rangle \leq b\}$, for some $b \in \mathbb{R}$ and $\mathbf{a} \neq \mathbf{0}$. Take $\mathbf{x}, \mathbf{y} \in H$. Let $\lambda \in [0, 1]$. Observe that

$$\langle \mathbf{a}, \lambda \mathbf{x} + (1 - \lambda) \mathbf{y} \rangle = \lambda \langle \mathbf{a}, \mathbf{x} \rangle + (1 - \lambda) \langle \mathbf{a}, \mathbf{y} \rangle \leq \lambda b + (1 - \lambda) b = b.$$

So $\lambda \mathbf{x} + (1 - \lambda) \mathbf{y} \in H, \forall \mathbf{x}, \mathbf{y}$. Thus a half space is a convex set. \square

Exercise 7.2

(1) Since we restrict the field to be \mathbb{R} , we have $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle, \forall \mathbf{x}, \mathbf{y}$.

$$\begin{aligned} RHS &= \langle \mathbf{x} - \mathbf{p}, \mathbf{x} - \mathbf{p} \rangle + \langle \mathbf{p} - \mathbf{y}, \mathbf{p} - \mathbf{y} \rangle + 2\langle \mathbf{x} - \mathbf{p}, \mathbf{p} - \mathbf{y} \rangle \\ &= (\langle \mathbf{x} - \mathbf{p}, \mathbf{x} - \mathbf{p} \rangle + \langle \mathbf{x} - \mathbf{p}, \mathbf{p} - \mathbf{y} \rangle) + (\langle \mathbf{p} - \mathbf{y}, \mathbf{p} - \mathbf{y} \rangle + \langle \mathbf{x} - \mathbf{p}, \mathbf{p} - \mathbf{y} \rangle) \\ &= \langle \mathbf{x} - \mathbf{p}, \mathbf{x} - \mathbf{p} + \mathbf{p} - \mathbf{y} \rangle + \langle \mathbf{p} - \mathbf{y}, \mathbf{p} - \mathbf{y} + \mathbf{x} - \mathbf{p}, \mathbf{p} - \mathbf{y} \rangle \\ &= \langle \mathbf{x} - \mathbf{p}, \mathbf{x} - \mathbf{y} \rangle + \langle \mathbf{x} - \mathbf{y}, \mathbf{p} - \mathbf{y} \rangle \\ &= \langle \mathbf{x} - \mathbf{y}, \mathbf{x} - \mathbf{p} \rangle + \langle \mathbf{x} - \mathbf{y}, \mathbf{p} - \mathbf{y} \rangle \\ &= \langle \mathbf{x} - \mathbf{y}, \mathbf{x} - \mathbf{p} + \mathbf{p} - \mathbf{y} \rangle \\ &= \langle \mathbf{x} - \mathbf{y}, \mathbf{x} - \mathbf{y} \rangle = \|\mathbf{x} - \mathbf{y}\|^2 = LHS \end{aligned}$$

(2) Since $2\langle \mathbf{x} - \mathbf{p}, \mathbf{p} - \mathbf{y} \rangle > 0$ and $\|\mathbf{p} - \mathbf{y}\|^2 \geq 0$, we have

$$\begin{aligned} \|\mathbf{x} - \mathbf{y}\|^2 &= \|\mathbf{x} - \mathbf{p}\|^2 + \|\mathbf{p} - \mathbf{y}\|^2 + 2\langle \mathbf{x} - \mathbf{p}, \mathbf{p} - \mathbf{y} \rangle \\ &> \|\mathbf{x} - \mathbf{p}\|^2 + 0 + 0 \\ &= \|\mathbf{x} - \mathbf{p}\|^2, \forall \mathbf{y} \neq \mathbf{p} \end{aligned}$$

(3) Let $\mathbf{z} = \lambda\mathbf{y} + (1 - \lambda)\mathbf{p}$. Then

$$\begin{aligned} LHS &= \langle \mathbf{x} - \lambda\mathbf{y} - (1 - \lambda)\mathbf{p}, \mathbf{x} - \lambda\mathbf{y} - (1 - \lambda)\mathbf{p} \rangle \\ &= \langle \mathbf{x}, \mathbf{x} \rangle - 2\lambda\langle \mathbf{x}, \mathbf{y} \rangle - 2(1 - \lambda)\langle \mathbf{x}, \mathbf{p} \rangle + 2\lambda(1 - \lambda)\langle \mathbf{p}, \mathbf{y} \rangle + \lambda^2\langle \mathbf{y}, \mathbf{y} \rangle + (1 - \lambda)^2\langle \mathbf{p}, \mathbf{p} \rangle \end{aligned}$$

$$\begin{aligned} RHS &= \langle \mathbf{x} - \mathbf{p}, \mathbf{x} - \mathbf{p} \rangle + 2\lambda\langle \mathbf{x} - \mathbf{p}, \mathbf{p} - \mathbf{y} \rangle + \lambda^2\langle \mathbf{y} - \mathbf{p}, \mathbf{y} - \mathbf{p} \rangle \\ &= \langle \mathbf{x}, \mathbf{x} \rangle - 2\langle \mathbf{x}, \mathbf{p} \rangle + \langle \mathbf{p}, \mathbf{p} \rangle + 2\lambda\langle \mathbf{x}, \mathbf{p} \rangle - 2\lambda\langle \mathbf{p}, \mathbf{p} \rangle + 2\lambda\langle \mathbf{p}, \mathbf{y} \rangle \\ &\quad - 2\lambda\langle \mathbf{x}, \mathbf{y} \rangle + \lambda^2\langle \mathbf{y}, \mathbf{y} \rangle + \lambda^2\langle \mathbf{p}, \mathbf{p} \rangle - 2\lambda^2\langle \mathbf{p}, \mathbf{y} \rangle \\ &= \langle \mathbf{x}, \mathbf{x} \rangle - 2\lambda\langle \mathbf{x}, \mathbf{y} \rangle - 2(1 - \lambda)\langle \mathbf{x}, \mathbf{p} \rangle + 2\lambda(1 - \lambda)\langle \mathbf{p}, \mathbf{y} \rangle + \lambda^2\langle \mathbf{y}, \mathbf{y} \rangle + (1 - 2\lambda + \lambda^2)\langle \mathbf{p}, \mathbf{p} \rangle \\ &= \langle \mathbf{x}, \mathbf{x} \rangle - 2\lambda\langle \mathbf{x}, \mathbf{y} \rangle - 2(1 - \lambda)\langle \mathbf{x}, \mathbf{p} \rangle + 2\lambda(1 - \lambda)\langle \mathbf{p}, \mathbf{y} \rangle + \lambda^2\langle \mathbf{y}, \mathbf{y} \rangle + (1 - \lambda)^2\langle \mathbf{p}, \mathbf{p} \rangle \\ &= LHS \end{aligned}$$

(4) Suppose \mathbf{p} is the projection of \mathbf{x} onto convex set C . Pick $\forall \mathbf{y} \in C$. Since C is convex, it follows that $\mathbf{z} = \lambda\mathbf{y} + (1 - \lambda)\mathbf{p} \in C$, $\lambda \in [0, 1]$. Since \mathbf{p} is the projection of \mathbf{x} , by definition $\|\mathbf{x} - \mathbf{z}\| \geq \|\mathbf{x} - \mathbf{p}\|$, and hence $\|\mathbf{x} - \mathbf{z}\|^2 \geq \|\mathbf{x} - \mathbf{p}\|^2$.

By (3), when $\lambda \neq 0$,

$$2\langle \mathbf{x} - \mathbf{p}, \mathbf{p} - \mathbf{y} \rangle = \lambda \|\mathbf{y} - \mathbf{p}\|^2 = \frac{\|\mathbf{x} - \mathbf{z}\|^2 - \|\mathbf{x} - \mathbf{p}\|^2}{\lambda} \leq 0, \forall \mathbf{y}.$$

Since $\|\mathbf{y} - \mathbf{p}\|^2 \geq 0$, and λ can be arbitrarily small, we have $2\langle \mathbf{x} - \mathbf{p}, \mathbf{p} - \mathbf{y} \rangle \geq 0$, and hence $\langle \mathbf{x} - \mathbf{p}, \mathbf{p} - \mathbf{y} \rangle \geq 0$.

When $\lambda = 0$, we have $\mathbf{z} = \mathbf{y}$ and $\|\mathbf{x} - \mathbf{y}\|^2 = \|\mathbf{x} - \mathbf{p}\|^2$. Since the projection is unique, we have $\mathbf{y} = \mathbf{p}$. In this case $\langle \mathbf{x} - \mathbf{p}, \mathbf{p} - \mathbf{y} \rangle = 0$.

Hence, $\mathbf{y} \in C$, $\langle \mathbf{x} - \mathbf{p}, \mathbf{p} - \mathbf{y} \rangle \geq 0$.

To show the converse direction, we can see that by (2), if $\langle \mathbf{x} - \mathbf{p}, \mathbf{p} - \mathbf{y} \rangle \geq 0$, then $\|\mathbf{x} - \mathbf{y}\| > \|\mathbf{x} - \mathbf{p}\|$, $\mathbf{y} \in C$, $\mathbf{y} \neq \mathbf{p}$. It then follows that \mathbf{p} is the projection of \mathbf{x} onto C .

Exercise 7.8

Proof. Let $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^n$ be two arbitrary vectors. Let $\lambda \in [0, 1]$. Observe that

$$\begin{aligned} g(\lambda\mathbf{x}_1 + (1 - \lambda)\mathbf{x}_2) &= f(A[\lambda\mathbf{x}_1 + (1 - \lambda)\mathbf{x}_2] + b) \\ &= f(\mathbf{x}_1 + (1 - \lambda)A\mathbf{x}_2 + b) \\ &= f(\lambda[A\mathbf{x}_1 + b] + (1 - \lambda)[A\mathbf{x}_2 + b]) \\ &\leq (A\mathbf{x}_1 + b) + (1 - \lambda)f(A\mathbf{x}_2 + b) \\ &= \lambda g(\mathbf{x}_1) + (1 - \lambda)g(\mathbf{x}_2) \end{aligned}$$

Hence g is a convex function. □

Exercise 7.8

(1)

Proof. Suppose A, B are positive definite matrices. Let $\lambda \in [0, 1]$. Observe that $\forall \mathbf{x}$, $\mathbf{x}^T(\lambda A) \mathbf{x} + \mathbf{x}^T(1 - \lambda)B \mathbf{x} = \lambda \mathbf{x}^T A \mathbf{x} + (1 - \lambda)\mathbf{x}^T B \mathbf{x} > 0$. So the set of positive definite matrices is a convex set. □

(2)

Proof. (a) This follows immediately from Lemma 7.2.7.

(b) Observe that

$$\begin{aligned}
g(t) &= -\log\{\det[tA + (1-t)B]\} = -\log\{\det[tS^H S + (1-t)S^H(S^H)^{-1}BS^{-1}S]\} \\
&= -\log\{\det[S^H(tI + (1-t)(S^H)^{-1}BS^{-1})S]\} \\
&= -\log\{\det[S^H]\det[tI + (1-t)(S^H)^{-1}BS^{-1}]\det[S]\} \\
&= -\log\{\det(S^H S)\det[tI + (1-t)(S^H)^{-1}BS^{-1}]\} \\
&= -\log\{\det(A)\det[tI + (1-t)(S^H)^{-1}BS^{-1}]\} \\
&= -\log(\det(A)) - \log(\det[tI + (1-t)(S^H)^{-1}BS^{-1}])
\end{aligned}$$

(c) We need the following facts:

1. If λ_i is an eigenvalue of M , then $t + (1-t)\lambda_i$ is an eigenvalue of $tI + (1-t)M$.
2. $\det(M) = \prod_i \lambda_i$, where each λ_i is an eigenvalue of M .

Hence, $\det[tI + (1-t)(S^H)^{-1}BS^{-1}] = \prod_{i=1}^n t + (1-t)\lambda_i$. Therefore we have

$$g(t) = -\log(\det(A)) - \log\left(\prod_{i=1}^n t + (1-t)\lambda_i\right) = -\log(\det(A)) - \sum_{i=1}^n \log(t + (1-t)\lambda_i).$$

(d)

$$\begin{aligned}
g'(t) &= \sum_{i=1}^n -\frac{1 - \lambda_i}{t + (1-t)\lambda_i}. \\
g''(t) &= \sum_{i=1}^n -\frac{(1 - \lambda_i)(\lambda_i - 1)}{(t + (1-t)\lambda_i)^2} = \frac{(\lambda_i - 1)^2}{(t + (1-t)\lambda_i)^2} \geq 0.
\end{aligned}$$

Since $g''(t) \geq 0, \forall t \in [0, 1]$, $g(t)$ is convex. So $f(X)$ is convex. \square

Exercise 7.13

Proof. By contradiction, assume f is not constant. Then there exist $a \neq b$ such that $f(a) \neq f(b)$. Without loss of generality we assume $a < b$ and $f(a) < f(b)$. Now pick any point c such that $c > b$. Let $\lambda = \frac{c-b}{c-a}, 1 - \lambda = \frac{b-a}{c-a}$. Observe that $\lambda a + (1 - \lambda)c = b$. Since f is a convex function, it follows that $\lambda f(a) + (1 - \lambda)f(c) \geq f(\lambda a + (1 - \lambda)c) = f(b)$. So we have

$$\begin{aligned}
f(c) &\geq \frac{f(b) - \lambda f(a)}{1 - \lambda} = \frac{f(b) - \frac{c-b}{c-a}f(a)}{\frac{b-a}{c-a}} = \frac{(c-a)f(b) - (c-b)f(a)}{b-a} \\
&= \frac{(c-a)(f(b) - f(a)) + (b-a)f(a)}{b-a} \\
&= f(a) + (c-a)\frac{f(b) - f(a)}{b-a}
\end{aligned}$$

Let $c \rightarrow \infty$, we see that $f(c) \rightarrow \infty$. This is contradicted to the fact that f is bounded above. Hence f must be a constant function. \square

Exercise 7.20

Proof. Since f is convex and $-f$ is convex, we have $\forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n, \forall \lambda \in [0, 1]$,

$$f(\lambda \mathbf{x} + (1 - \lambda)\mathbf{y}) \leq \lambda f(\mathbf{x}) + (1 - \lambda)f(\mathbf{y}) \quad (1)$$

$$-f(\lambda \mathbf{x} + (1 - \lambda)\mathbf{y}) \leq -\lambda f(\mathbf{x}) - (1 - \lambda)f(\mathbf{y}) \quad (2)$$

Multiply the second equation by (-1) ,

$$f(\lambda \mathbf{x} + (1 - \lambda)\mathbf{y}) \geq \lambda f(\mathbf{x}) + (1 - \lambda)f(\mathbf{y}).$$

This implies

$$f(\lambda \mathbf{x} + (1 - \lambda)\mathbf{y}) = \lambda f(\mathbf{x}) + (1 - \lambda)f(\mathbf{y}).$$

Hence f is affine. □

Exercise 7.21

Proof. We first show that the second claim implies the first claim.

Let Ω denote the feasible set. Suppose $\mathbf{x}^* \in \Omega$ is a local minimizer of $f(\mathbf{x})$, then in this neighborhood, $\forall \mathbf{x}, f(\mathbf{x}) \geq f(\mathbf{x}^*)$. Since ϕ is a strictly increasing function, it follows that $\phi(f(\mathbf{x})) \geq \phi(f(\mathbf{x}^*))$. So \mathbf{x}^* is a local minimizer of $\phi(f(\mathbf{x}))$.

Then we show that the first claim implies the second claim.

Suppose \mathbf{x}^* is a local minimizer of $\phi(f(\mathbf{x}))$. Then $\forall \mathbf{x}, \phi(f(\mathbf{x})) \geq \phi(f(\mathbf{x}^*))$ in its neighborhood. By definition this means that \mathbf{x}^* is a local minimizer of $f(\mathbf{x})$. □