Prop. n even. The largest possible L_1 norm between two $[1, \cdots n]$ permutations is $\frac{n^2}{2}$.

prf. Wlog we fix one permutation to be $(1, 2, 3, 4, \cdots n)$, denoted $\vec{I}$.

Claim: $\|\vec{I} - (n, n-1, n-2, \cdots, 2, 1)\| = \frac{n^2}{2}$, i.e. the reverse order yields the maximum distance.

Pf. prf by straightforward calculation. ☒

Now, it suffices to show that any other permutation $\vec{a} \neq (n, n-1, \cdots, 3, 2, 1)$ would yield a distance <u>smaller than or equal to</u> $\|\vec{I} - (n, n-1, \cdots, 2, 1)\|$.

By contradiction, suppose $\exists \vec{a} \neq (n, n-1, \cdots 3, 2, 1)$ yields the largest distance, larger than the reverse order, to $\vec{I}$. Then $\exists\ i < j$, $a_i < a_j$.

We claim that swapping a_i and a_j would at least make you not worse off.

Observe that

$$\Delta = \|(a_1, a_2, \cdots, a_n) - \vec{I}\| - \|(a_1, a_2, \cdots a_{i-1}, \underline{a_j}, a_{i+1}, \cdots, a_{j-1}, \underline{\underline{a_i}}, a_{j+1}, \cdots a_n) - \vec{I}\|$$

$$= |a_i - i| + |a_j - j| - |a_j - i| - |a_i - j|.$$

The four numbers i, j, a_i, a_j has in total $\frac{4!}{2!\,2!} = 6$ ways of ordering, since we have $i < j$ and $a_i < a_j$.

① $i \quad j \quad a_i \quad a_j$

$$\Delta = (a_i - i) + (a_j - j) - (a_j - i) - (a_i - j)$$
$$= 0.$$

② $i \quad a_i \quad j \quad a_j$

$$\Delta = (a_i - i) + (a_j - j) - (a_j - i) - (j - a_i)$$
$$= a_i \cancel{- i} \cancel{+ a_j} - j \cancel{- a_j} \cancel{+ i} - j + a_i$$
$$= 2(a_i - j) \leq 0.$$

③ $i \quad a_i \quad a_j \quad j$

$$\Delta = (a_i - i) + (j - a_j) - (a_j - i) - (j - a_i)$$
$$= a_i \cancel{- i} \cancel{+ j} - a_j - a_j \cancel{+ i} \cancel{- j} + a_i$$
$$= 2(a_i - a_j) < 0.$$

④ $a_i \quad i \quad j \quad a_j$

$$\Delta = (i - a_i) + (a_j - j) - (a_j - i) - (j - a_i)$$
$$= i \cancel{- a_i} \cancel{+ a_j} - j \cancel{- a_j} + i - j \cancel{+ a_i}$$
$$= 2(i - j) < 0.$$

⑤ $a_i \quad i \quad a_j \quad j$

⑥ $a_i \quad a_j \quad i \quad j.$

$$\Delta = (i-a_i)+(j-a_j)-(j-a_i)-(a_j-i)$$
$$= i-\cancel{a_i}+\cancel{j}-a_j-\cancel{j}+\cancel{a_i}-a_j+i$$
$$= 2(i-a_j) \leq 0.$$

$$\Delta = (i-a_i)+(j-a_j)-(i-a_j)-(j-a_i)$$
$$= \cancel{i}-\cancel{a_i}+\cancel{j}-\cancel{a_j}-\cancel{i}+\cancel{a_j}-\cancel{j}+\cancel{a_i}$$
$$= 0.$$

Hence if we do a swap, $\Delta = \|\vec{a}-\vec{T}\| - \|\vec{a}'-\vec{T}\| \leq 0.$

$\Rightarrow \|\vec{a}'-\vec{T}\| \geq \|\vec{a}-\vec{T}\|.$

In words, swapping two originally increasing numbers to get the reverse decreasing order would at least make you (weakly) better off. Hence to get the maximum possible distance you should keep swapping until you get the complete reverse ordering $(n, n-1, \ldots 3, 2, 1)$. ∎