Probability Axioms

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1 Basics

Some definitions and terms:

- Set : Collection of elements(without ordering)
- Experiment : Empirical procedure
- Random experiment : Experiment with random/uncertain outcome
- Sample space(Ω) : Set of all possible outcomes
- Event : subset of sample space
- Operations on event : eg A-B $\Rightarrow$ (A but not B)
- Event Space(β) : $\beta \subseteq$ power set of Ω(elaborated later)

2 Event Space

Rules that event space must have:

1. $\Phi \in \beta$
2. Closed under countable unions
3. Closed under complementation

Set β that satisfies the above axioms is called σ-algebra.
Borel σ- algebra:
Briefly, borel sigma algebra is sigma algebra of an infinite set.

3 Probability function

Probability function is a function $P : \beta \Rightarrow [0, 1]$ Axioms:

1. $P(\Phi) = 0$
2. $P(\Omega) = 1$
3. Probability of union of countable disjoint events is sum of probabilities of individual events.

A probability space is (Ω, β, P).

3.1 Joint probability

$P(A, B) = P(A \cap B)$

3.2 Conditional probability

$P(A|B) = \frac{P(A \cap B)}{P(B)}$
Applications:

1. Image Generation:P(pixel takes specific color| object label)
2. Image acquisition (Perturbation model):P(pixel takes specific color| true color)
3. Image Generation:P(pixel takes specific color| color of adjacent pixel)

4 Partition

Set of mutually exclusive and exhaustive subsets of sample space.
Total probability theorem Let $B_1, B_2, \ldots, B_n$ be partition of a certain sample space and let A be any event,

$$P(A) = \sum_{i}^{n} P(A|B_i)P(B_i)$$

5 Independence

Two events A and B are independent iff $P(A \cap B) = P(A)P(B)$

6 Conditional Independence

Two events A and B are conditionally independent given event C iff

$$P(A, B|C) = P(A|C)P(B|C)$$

which is equivalent to,

$$P(A|B, C) = P(A|C)$$

Interpretation:Probability of A given C does not depend on occurrence of B.
Example: In an image of constant color(given, say red) the probability of i^{th} pixel being red does not depend on color of some other pixel.

7 Random Number Generator

Generally, random numbers that computer software produce are pseudo random, meaning the random numbers are deterministic if the internal working and the seed to the function is known.
Mersenne twister is state-of-art pseudo-random number generator.
Random number generators are important for **Monte Carlo methods**. (Getting good approximations to analytically-intractable problems using simulated random experiments). Eg: Finding value of π using probability of a random point falling within a circle given that the point is within a square circumscribing the circle.