Probability I (SI 427)
Department of Mathematics, IIT Bombay
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Problem set 2

1. We define the *upper limit* or *limit superior* of a sequence $A_1, A_2, \ldots \subset \Omega$ of sets by

$$\limsup_{n\to\infty} A_n = \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} A_k$$

and the *lower limit* or *limit inferior* by

$$\liminf_{n\to\infty} A_n = \cup_{n=1}^{\infty} \cap_{k=n}^{\infty} A_k.$$

We say that the sequence $\{A_n\}$ converges to a limit A if $\limsup_{n\to\infty} A_n = \liminf_{n\to\infty} A_n$.

Let $A_1, A_2, \ldots \in \mathcal{F}$, where $\mathcal{F}$ is a σ-field.

   (a) Show that
   $$\liminf_{n\to\infty} A_n \in \mathcal{F} \quad \text{and} \quad \limsup_{n\to\infty} A_n \in \mathcal{F}.$$

   (b) Show that
   $$\liminf_{n\to\infty} A_n = \{\omega : \omega \in A_n \text{ for all but finitely many } n\},$$
   $$\limsup_{n\to\infty} A_n = \{\omega : \omega \in A_n \text{ for infinitely many } n\}.$$

   (c) Show that $\liminf_{n\to\infty} A_n \subseteq \limsup_{n\to\infty} A_n$.

   (d) Show that $\mathrm{P}(\liminf_{n\to\infty} A_n) = 1 - \mathrm{P}(\limsup_{n\to\infty} A_n^c)$.

   (e) Suppose $A_n \to A$. Show that $\mathrm{P}(A_n) \to \mathrm{P}(A)$.

2. Two fair dice are rolled. Show that the event that their sum is 7 is independent of the score shown by the first dice.

3. There are two roads from A to B and two roads from B to C. Each of the roads has probability p of being blocked by landslide, independently of all others. What is the probability that there is an open road from A to C?

4. Urn 1 contains two white balls and one black ball, while urn 2 contains one white ball and five black balls. One ball is drawn at random from urn 1 and placed in urn 2. A ball is then drawn from urn 2. It happens to be white. What is the probability that the transferred ball was white?

5. A urn contains b black balls and r red balls.One of the ball is drawn at random, but when it is put back in the urn c additional balls of the same colour are put in with it. Now suppose that we draw another ball . Show that the probability that the first ball drawn was black given that the second ball drawn was red is $b/(b + r + c)$.

6. Urn 1 has 5 white and 7 black balls. Urn 2 has 3 white and 12 black balls. We flip a fair coin. If the outcome is head, then a ball from urn 1 is selected, while if outcome is tail, then a ball from urn 2 is selected. Suppose that a white ball is selected. What is the probability that the coin landed tails?

7. Suppose that an experiment can result in one of r possible outcomes, the ith outcome having probability p_i, $i = 1, 2, \ldots, r$, $\sum_{i=1}^{r} p_i = 1$. If n of these experiments are performed, and if the outcome of any one of the n does not affect the outcome of the other $n-1$ experiments, then find the probability of that the first outcome appears x_1 times, the second x_2 times, and the rth x_r times.

8. Suppose n couples have arrived in a party. The boys and girls are then paired off at random to dance. What is the probability that no boys end up with their own girls? What is the probability that only the 1st boy get his own girl?

9. Let X, Y be two independent *Geometric* random variables with parameter p. That is
$$\mathrm{P}(X = k) = (1-p)^{k-1}p, \text{ for } k = 1, 2, 3, \ldots$$
Find the following:
$\mathrm{P}(X = Y)$, $\mathrm{P}(X > Y)$, $\mathrm{P}(X < Y)$, $\mathrm{P}(X = k|X > Y)$, $\mathrm{P}(X = k|X = Y)$, $\mathrm{P}(X = k|X + Y = l)$.

10. Let X and Y be two independent random variables each having uniform mass on $\{0, 1, 2, \ldots, n\}$, that is, $\mathrm{P}(X = i) = \frac{1}{n+1}$, $\mathrm{P}(Y = i) = \frac{1}{n+1}$ for $i = 0, 1, 2, \ldots, n$. Find $\mathrm{P}(X > Y)$, $\mathrm{P}(X = Y)$. Find the probability mass function of $U = \max\{X, Y\}$ and $V = \min\{X, Y\}$.

11. Suppose that independent trials, each of which results in any of m possible outcomes with respective probabilities $p_1, p_2, \ldots, p_m$, $\sum_{i=1}^{n} p_i = 1$, are continually performed. Let X denote the number of trials needed until each outcome has occurred at least once. Find the probability mass function of X.

12. Find a collection of events such that they are pairwise independent but not independent.