Introduction to WFSTs

Lecture 2b

CS 753, 2021

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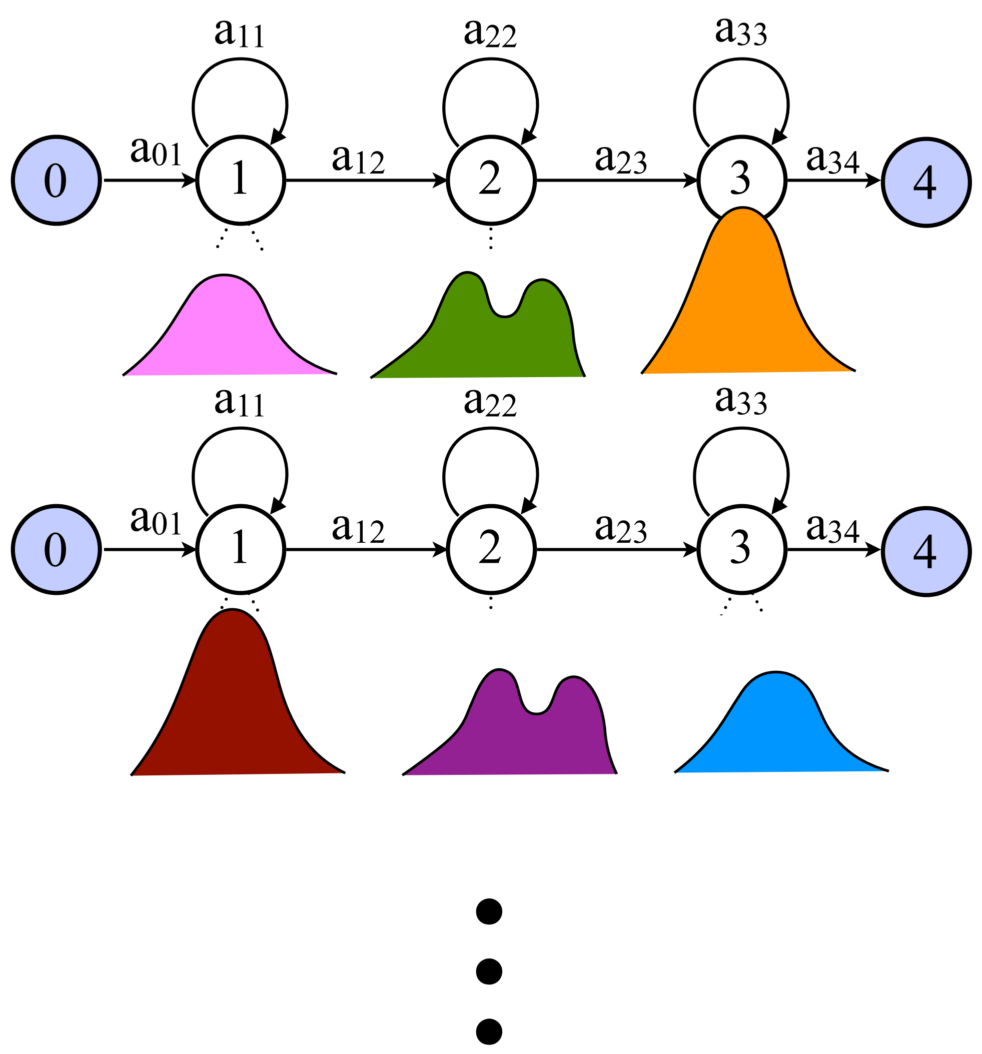
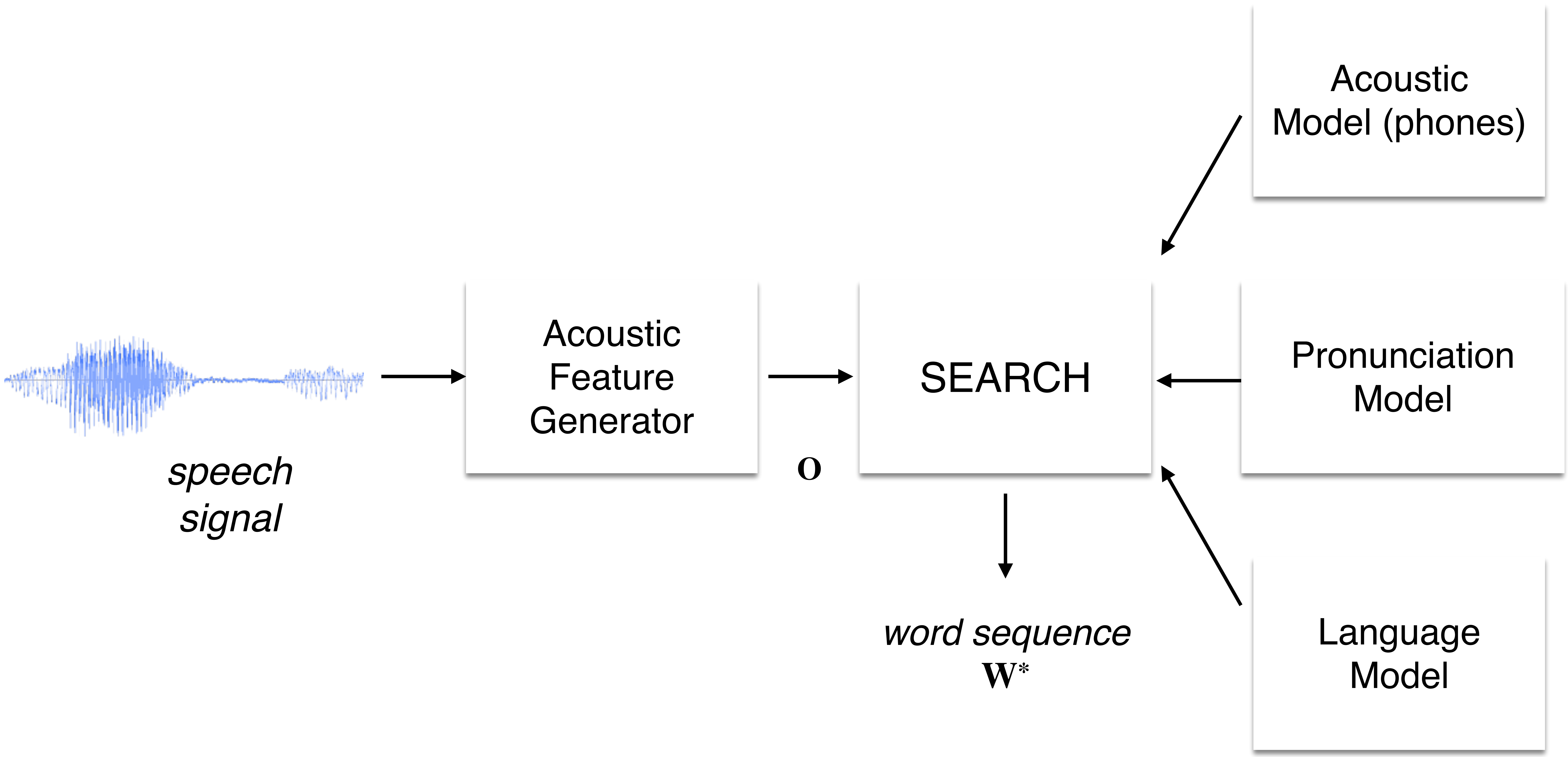
Architecture of an ASR system





How do we combine the acoustic model HMMs with the remaining components? Weighted finite-state transducers (WFSTs)!

(Weighted) Automaton

- Accepts a subset of strings (over an alphabet), and rejects the rest
  - Mathematically, specified by $L \subseteq \Sigma^*$ or equivalently $f : \Sigma^* \rightarrow \{0,1\}$
- Weighted: outputs a "weight" as well (e.g., probability)
  - $f : \Sigma^* \rightarrow W$
- Transducer: outputs another string (over possibly another alphabet)
  - $f : \Sigma^* \times \Delta^* \rightarrow W$

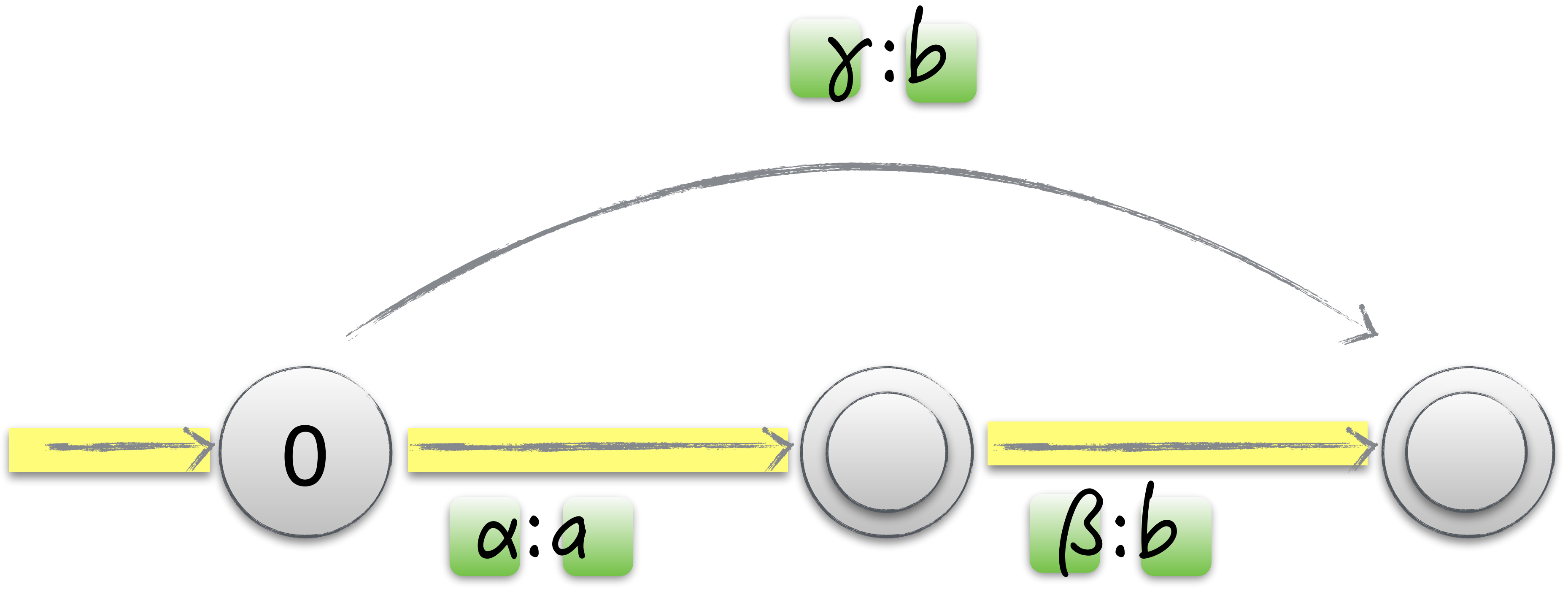
(Weighted) Finite State Automaton

- Functions that can be implemented using a machine which:
  - reads the string one symbol at a time
  - has a fixed amount of memory: so, at any moment, the machine can be in only one of finitely many *states*, irrespective of the length of the input string
- Allows efficient algorithms to reason about the machine
  - e.g., output string with maximum weight for input $\alpha\beta\gamma$

Why WFSTs?

- Powerful enough to (reasonably) model processes in language, speech, computational biology and other machine learning applications
    - Simpler WFSTs can be combined to create complex WFSTs, e.g., speech recognition systems
- If using WFST models, efficient algorithms available to train the models and to make inferences
    - Toolkits that don’t have domain specific dependencies

Structure: Finite State Transducer (FST)

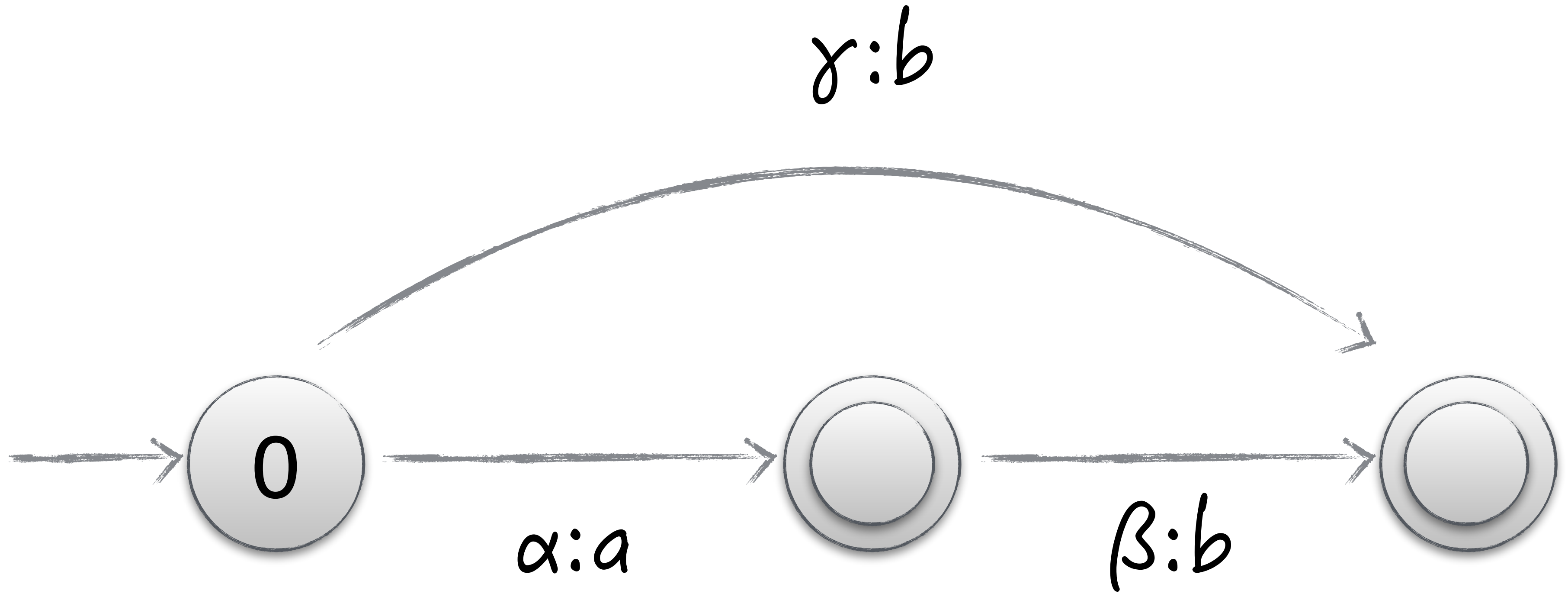


Elements of an FST

- States
- Start state (0)
- Final states (1 & 2)
- Arcs (transitions)
- Input symbols (from alphabet Σ)
- Output symbols (from alphabet Δ)

FST maps input strings to output strings

Path



- A successful “path” → Sequence of transitions from the start state to any final state

- Input label of a path → Concatenation of input labels on arcs. Similarly for output label of a path.

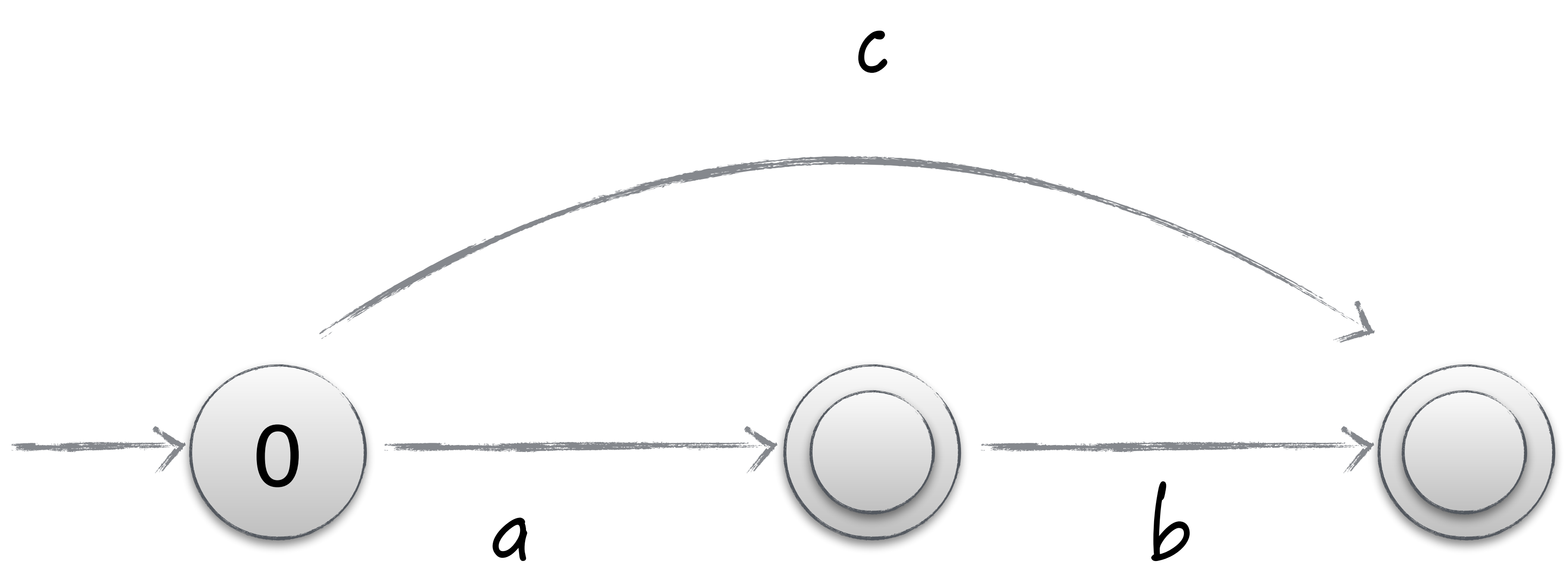
FSAs and FSTs

- Finite state acceptors (FSAs)
  - Each transition has a source & destination state, and **a label**
  - FSA accepts a set of strings, $L \subseteq \Sigma^*$

FSA can be thought of as a special kind of FST

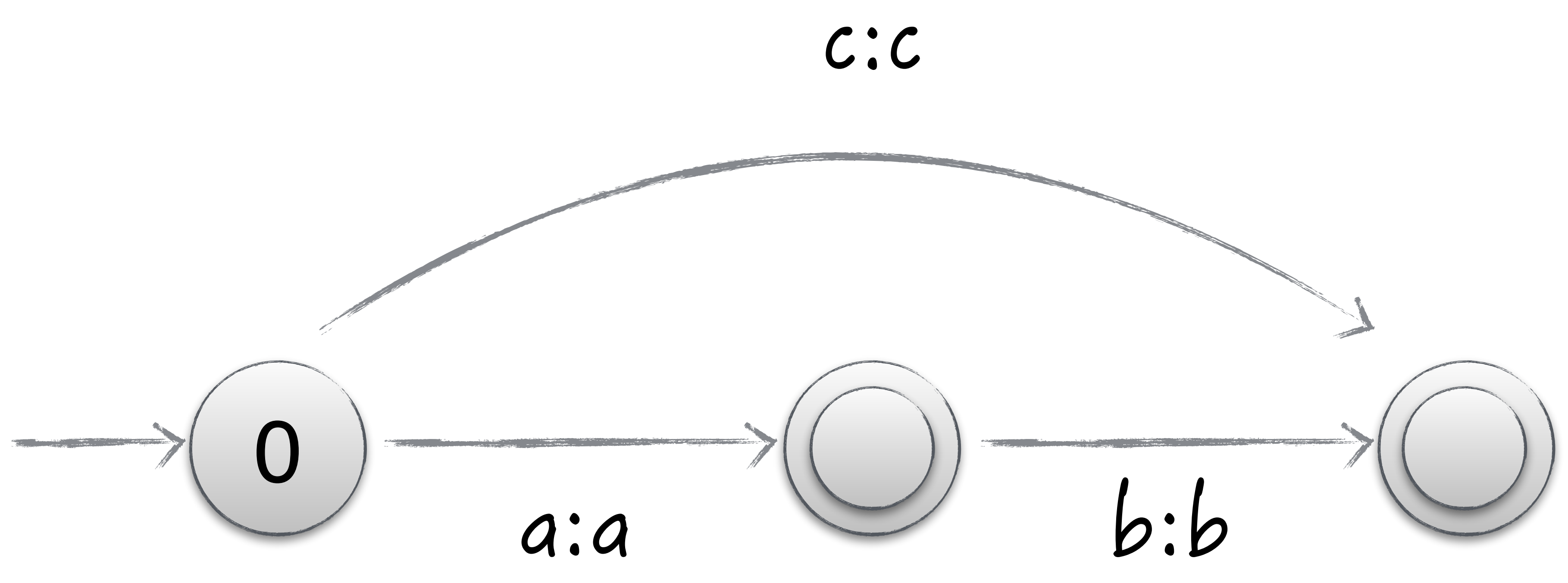
- Finite state transducers (FSTs)
  - Each transition has a source & destination state, **an input label and an output label**
  - FST represents a relation, $R \subseteq \Sigma^* \times \Delta^*$

Example of an FSA



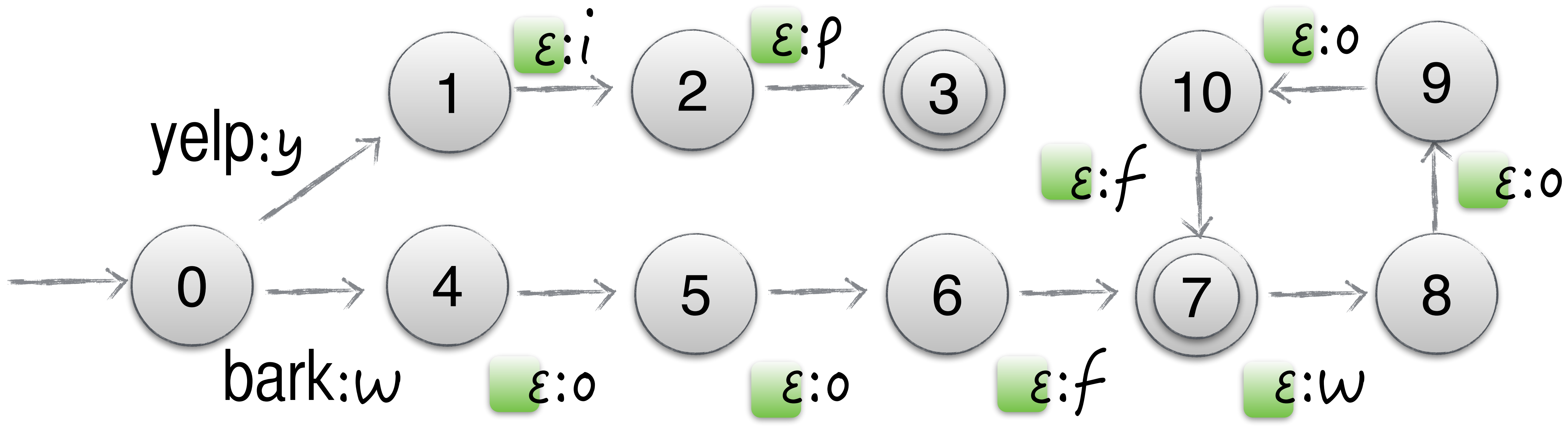
Accepts strings
{c, a, ab}

Equivalent FST



Accepts strings
{c, a, ab}
and outputs identical strings
{c, a, ab}

Barking dog FST



Σ = { yelp, bark }, Δ = { a, …, z }

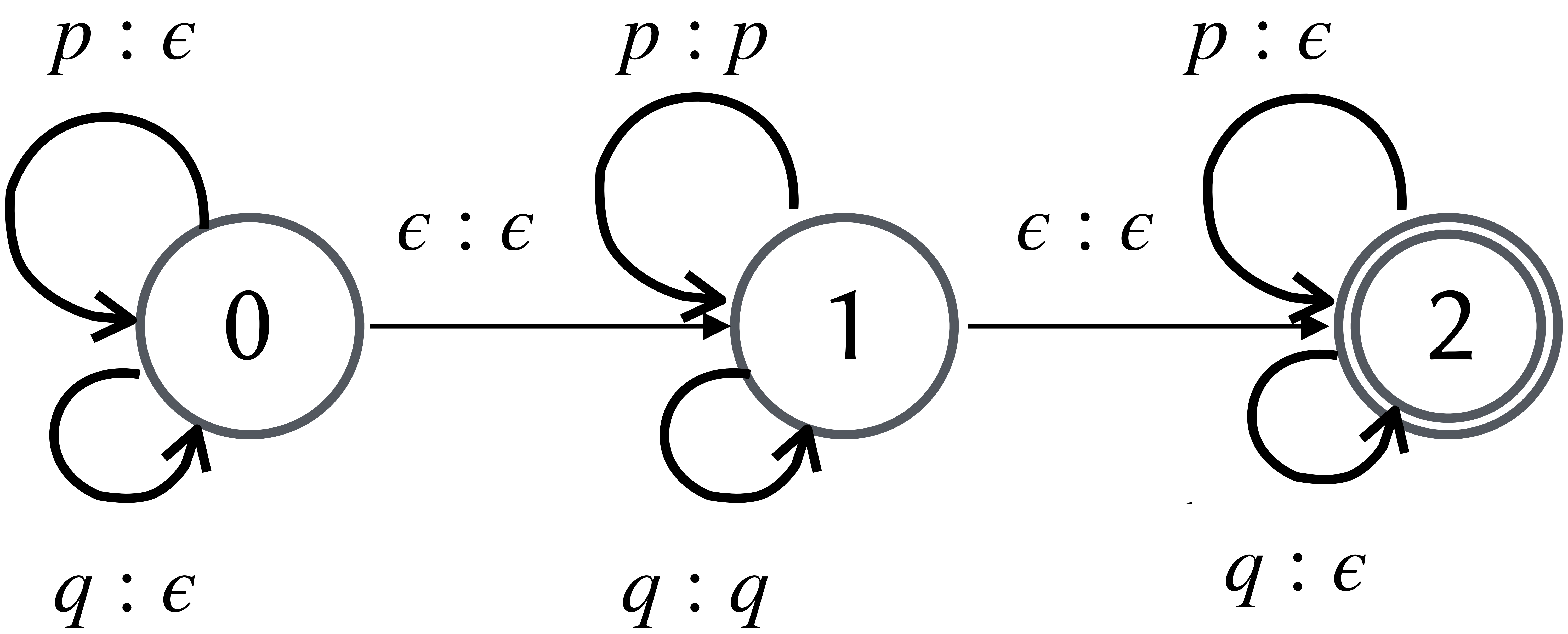
yelp → y i p. bark → w o o f | w o o f w o o f |…

Special symbol, ε (epsilon) : allows to make a move without consuming an input symbol

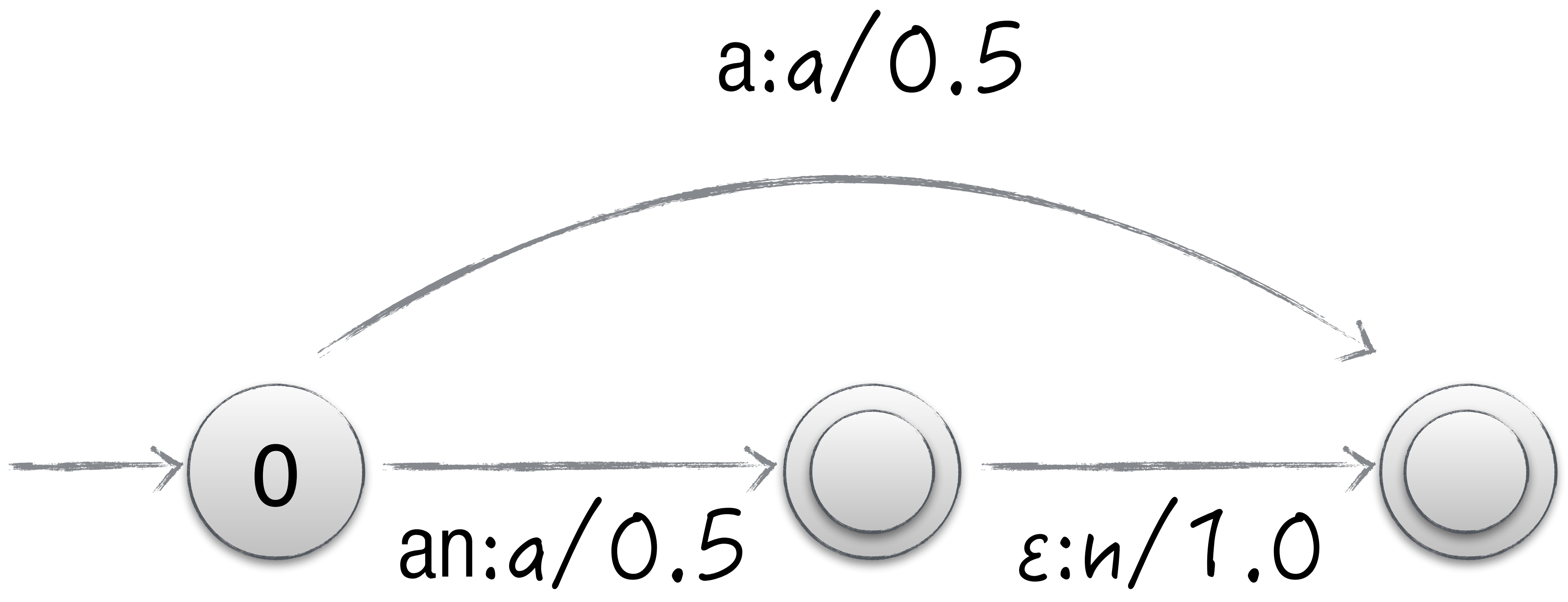
or without producing an output symbol

Problem

Design a finite-state transducer that can map a string in Σ^* to any of its substrings. $s' \in \Sigma^*$ is a substring of $S \in \Sigma^*$ if $S = xs'y$ for some $x, y \in \Sigma^*$. Let $\Sigma = \{p, q\}$.

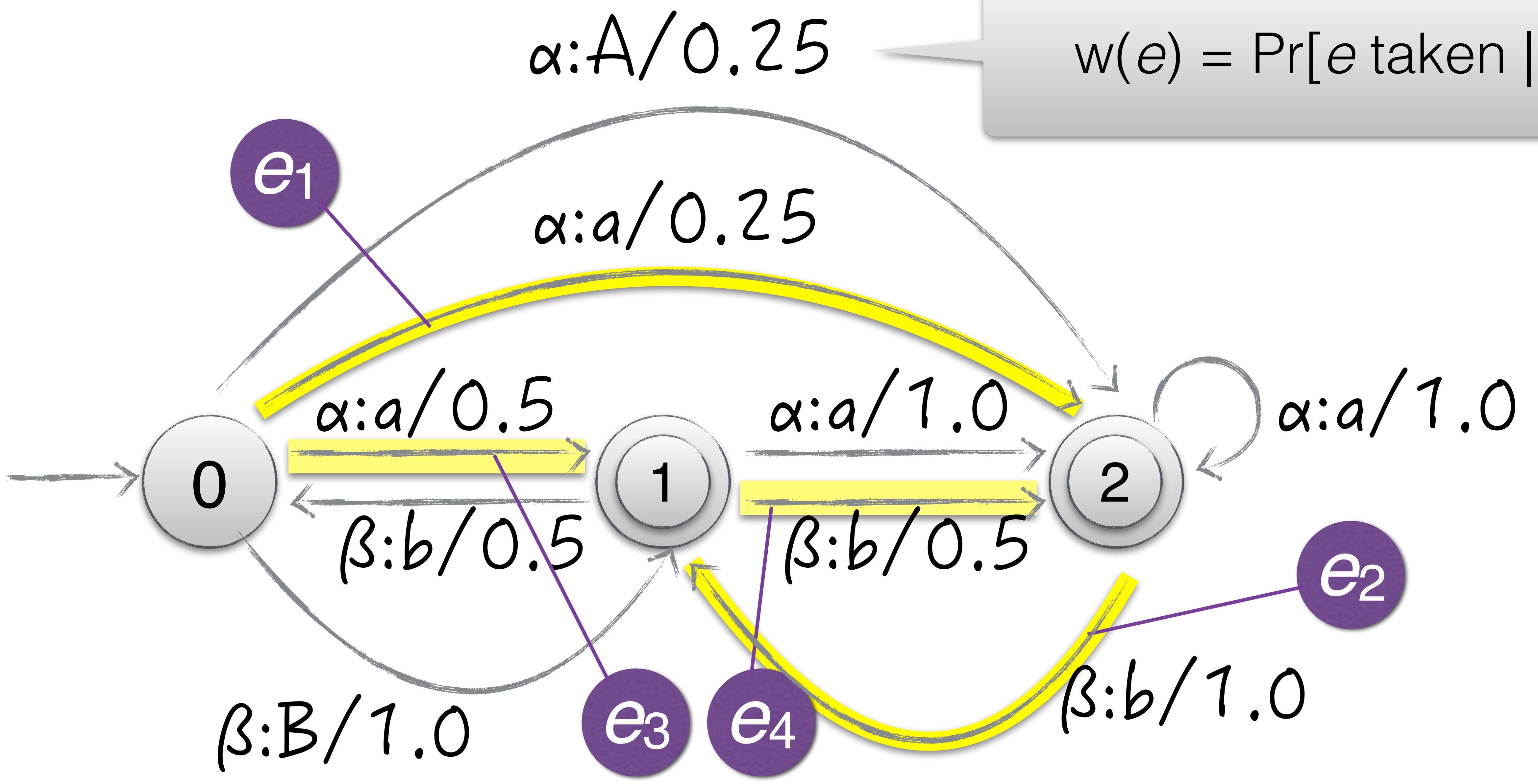


Weighted Path



- “Weights” can be probabilities, negative log-likelihoods, or any cost function representing the cost incurred in mapping an input sequence to an output sequence
- How are the weights accumulated along a path?

Weighted Path: Probabilistic FST



$w(e)$ = Pr[e taken | state=0,in-symbol=α]

$\pi_1 = e_1 e_2$
$\pi_2 = e_3 e_4$

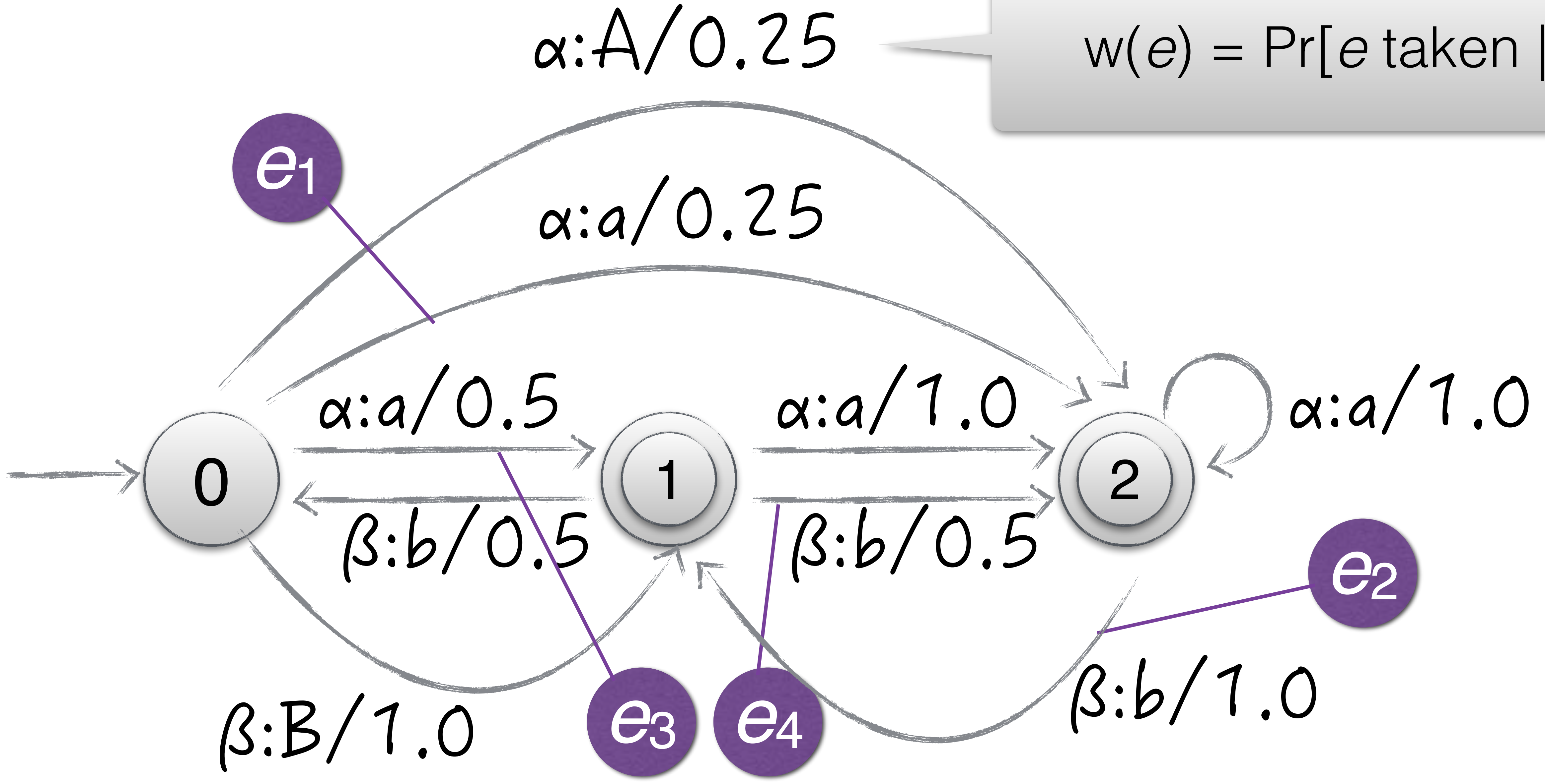
- $T(\alpha\beta, ab)$ = Pr[output=ab, accepts | input=$\alpha\beta$, start]
  = Pr[π_1 | input=$\alpha\beta$, start] + Pr[π_2 | input=$\alpha\beta$, start]

= 0.25 + 0.25 = 0.5

= Pr[e_1 | input=$\alpha\beta$, start] × Pr[e_2 | input=$\alpha\beta$, start, e_1]
= Pr[e_1 | state=0, in-symb=α] × Pr[e_2 | state=2, in-symb=β]
= $w(e_1) \times w(e_2)$ = 0.25×1.0 = 0.25

= w(e3) × w(e4) = 0.5×0.5 = 0.25

Weighted Path: Probabilistic FST



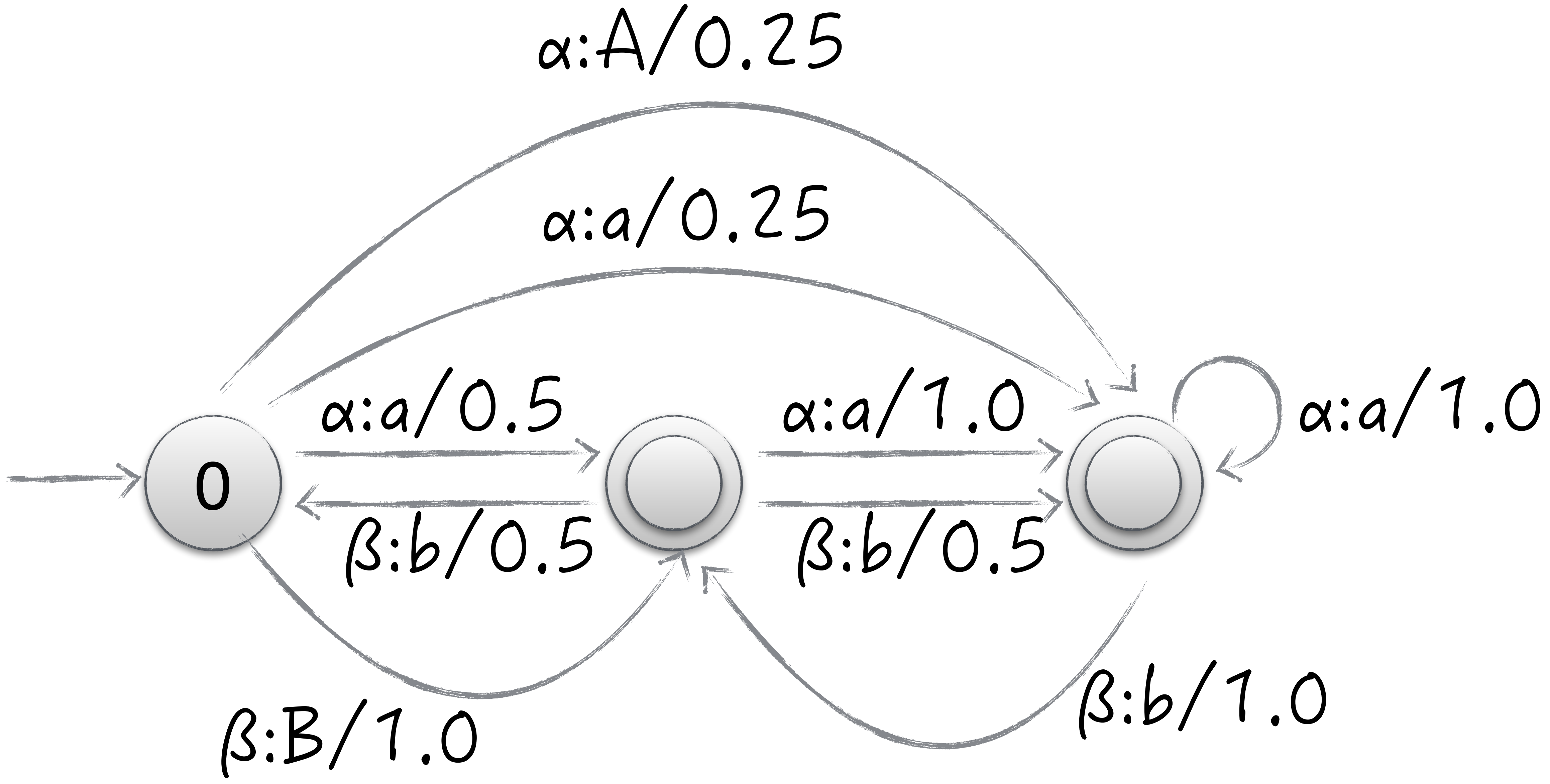
w(e) = Pr[e taken | state=0,in-symbol=α]

$\pi_1 = e_1 e_2$
$\pi_2 = e_3 e_4$

- T(αβ,ab) = Pr[output=ab, accepts | input=αβ, start]
  = Pr[π_1 | input=αβ, start] + Pr[π_2 | input=αβ, start]

- More generally, $T(x,y) = \Sigma_{\pi \in P(x,y)} \; \Pi_{e \in \pi} \; w(e)$
  where $P(x,y)$ is the set of all accepting paths with input x and output y

Weighted Path



- But not all WFSTs are probabilistic FSTs
- Weight is often a “score” and maybe accumulated differently
- But helpful to retain some basic algebraic properties of weights: abstracted as **semirings**

Semirings

A semiring is a set of values associated with two operations $\oplus$ and $\otimes$, along with their identity values $\bar{0}$ and $\bar{1}$

Weight assigned to an input/output pair

$$T(x,y) = \bigoplus_{\pi \in P(x,y)} \bigotimes_{e \in \pi} w(e)$$

where P(x,y) is the set of all accepting paths with input x, output y

(generalizing the weight function for a probabilistic FST)

Semirings

Some popular semirings [M02]

SEMIRING	SET	$\oplus$	$\otimes$	$\bar{0}$	$\bar{1}$
Boolean	{F,T}	$\vee$	$\wedge$	F	T
Real	$\mathbb{R}_+$	+	$\times$	0	1
Log	$\mathbb{R} \cup \{-\infty, +\infty\}$	$\oplus_{\log}$	+	$+\infty$	0
Tropical	$\mathbb{R} \cup \{-\infty, +\infty\}$	min	+	$+\infty$	0

Is there

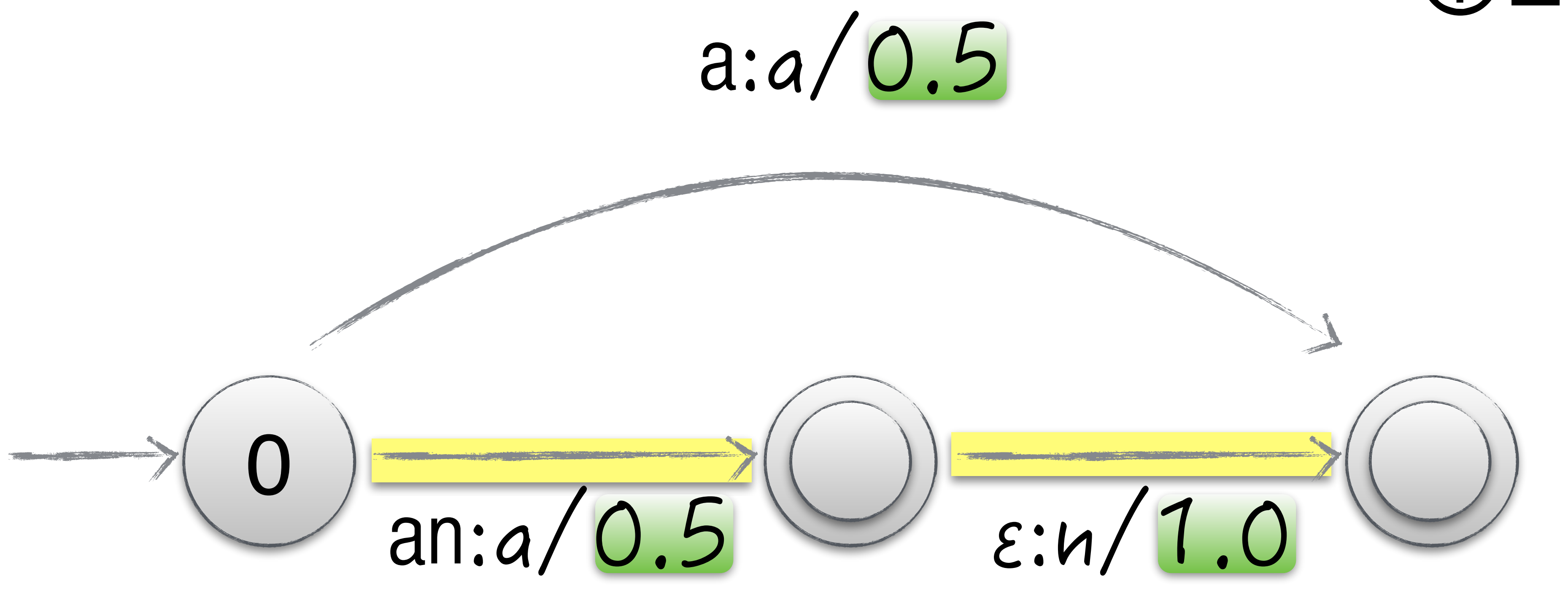
“Viterbi Approx.” of -log Pr[y|x]

Operator $\oplus_{\log}$ defined as: $x \oplus_{\log} y = -\log(e^{-x} + e^{-y})$

[M02] Mohri, M. Semiring frameworks and algorithms for shortest-distance problems, *Journal of Automata, Languages and Combinatorics*, 7(3):321—350, 2002.

Weighted Path: Tropical Semiring

$\oplus$=min | $\otimes$=+



- Weight of a path π is the $\otimes$-product of all the transitions in π

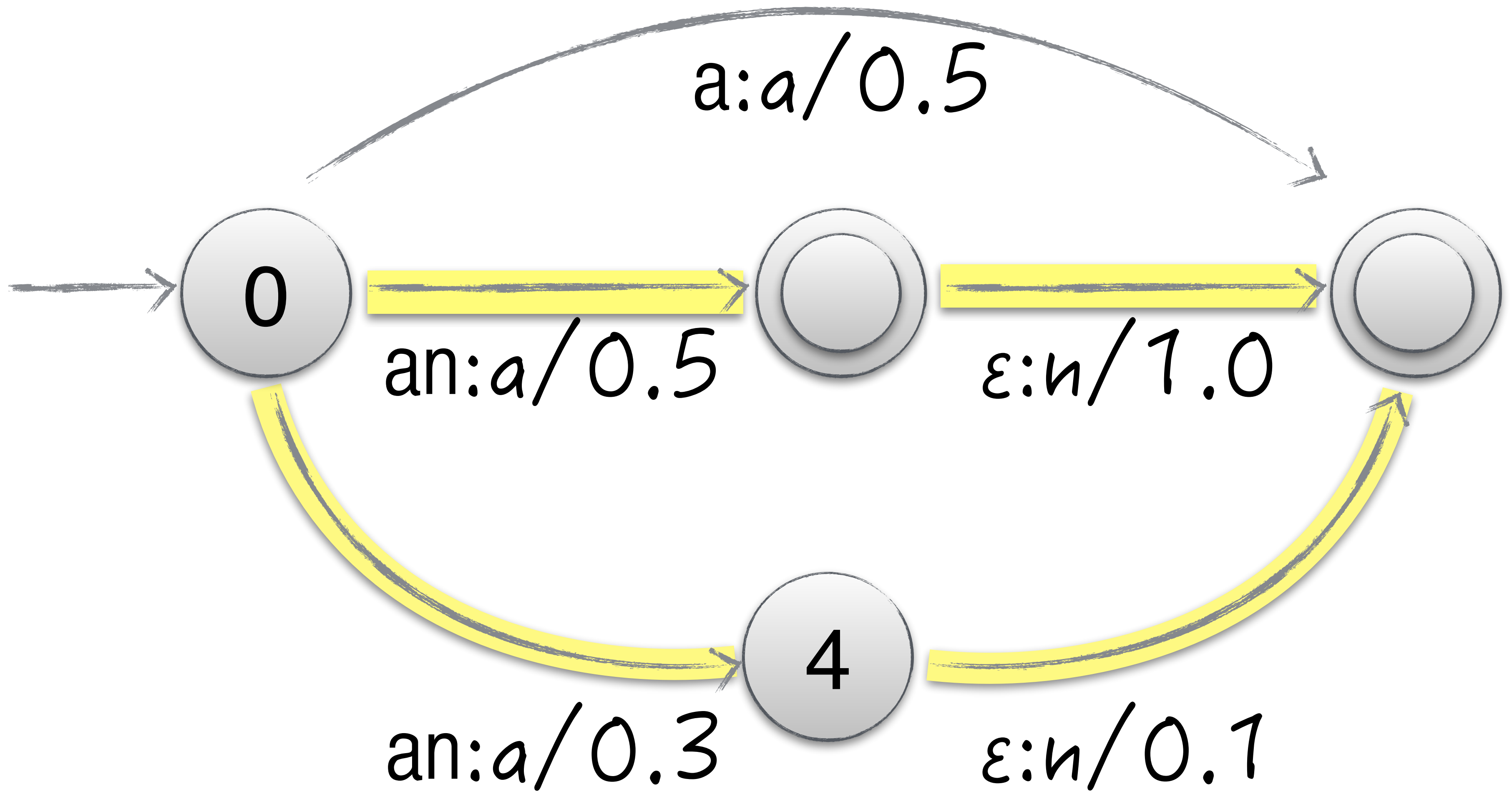
  $w(\pi)$: $(0.5 \otimes 1.0) = 0.5 + 1.0 = 1.5$

- Weight of a sequence "$\underline{x},\underline{y}$" is the $\oplus$-sum of all paths labeled with "$\underline{x},\underline{y}$"

  $w((\text{an}), (a\ n)) = (1.5 \oplus \bar{0}) = \min(1.5, \infty) = 1.5$

Weighted Path: Tropical Semiring

$\oplus$=min | $\otimes$=+



- Weight of a sequence "*x*,*y*" is the $\oplus$-sum of all paths labeled with "*x*,*y*"

  w((an), (a n)) = ?

  Path 1: (0.5 $\otimes$ 1.0) = 1.5

  Path 2: (0.3 $\otimes$ 0.1) = 0.4

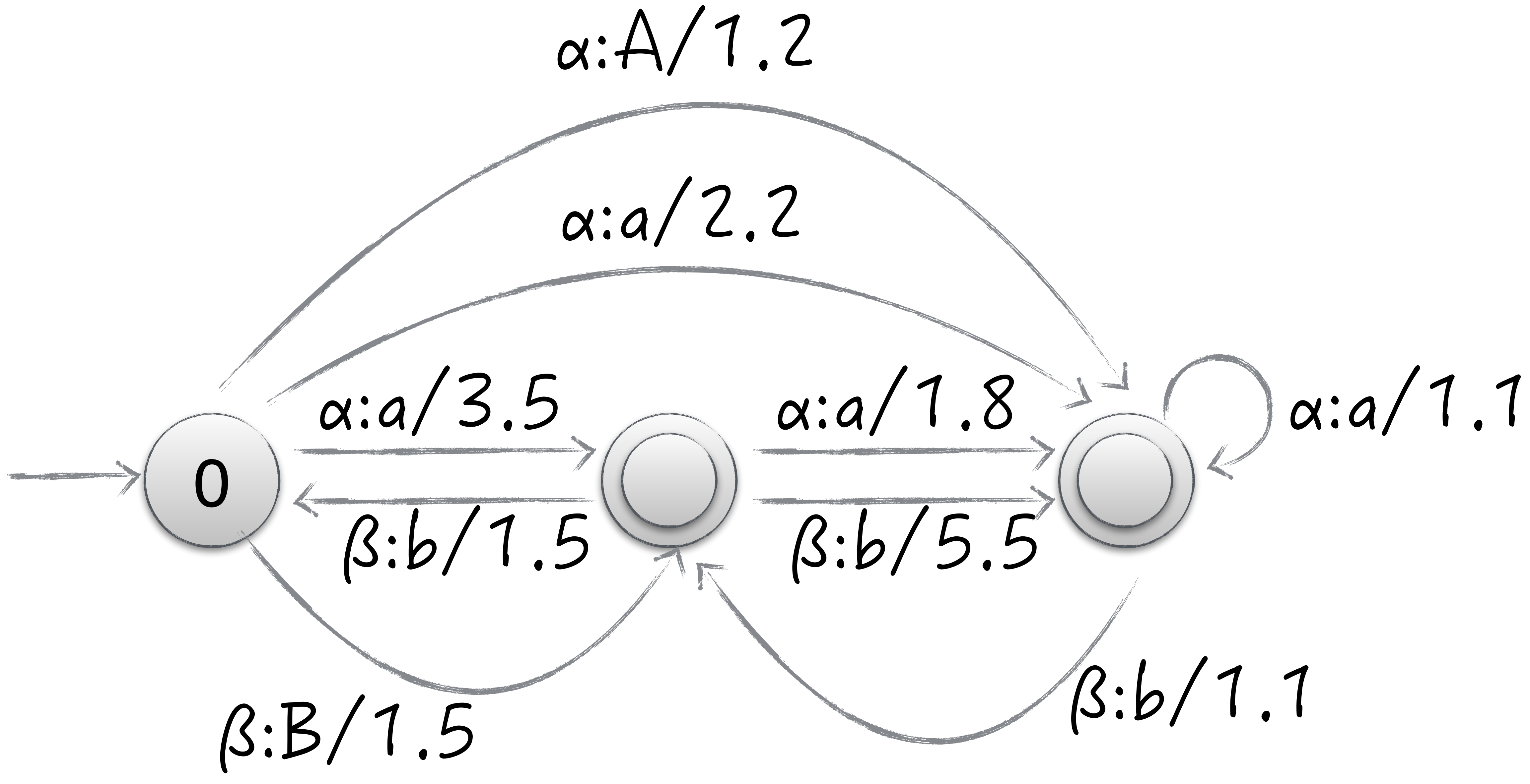
  Weight of "((an), (a n))" = (1.5 $\oplus$ 0.4) = 0.4

Shortest Path

- Recall $T(x,y) = \bigoplus_{\pi \in P(x,y)} w(\pi)$
  where $P(x,y)$ = set of paths with input/output (x,y); $w(\pi) = \bigotimes_{e \in \pi} w(e)$

- In the tropical semiring $\oplus$ is min. $T(x,y)$ associated with a single path in $P(x,y)$: *Shortest Path*
  - Can be found using Dijkstra's algorithm : $\Theta(|E| + |Q| \cdot \log|Q|)$ time

Shortest Path

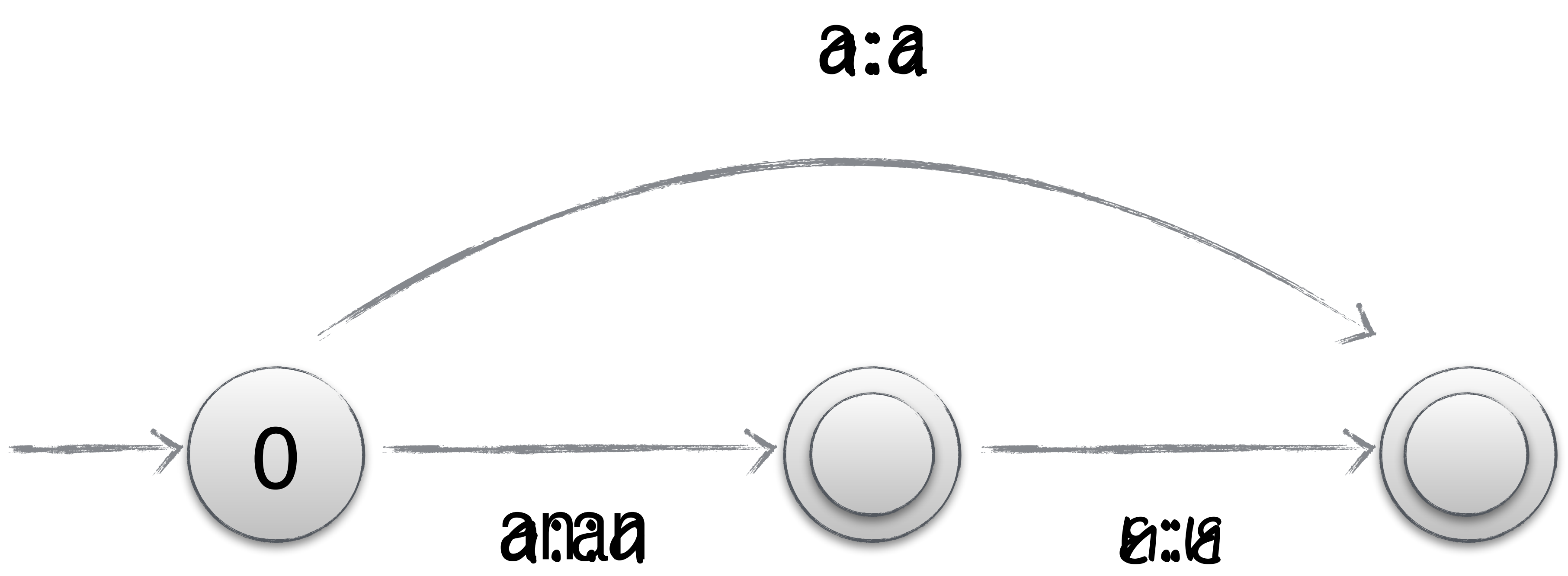


T(“α”, “a”) = ?

T(“αα”, “aa”) = ?

Inversion

Swap the input and output labels in each transition

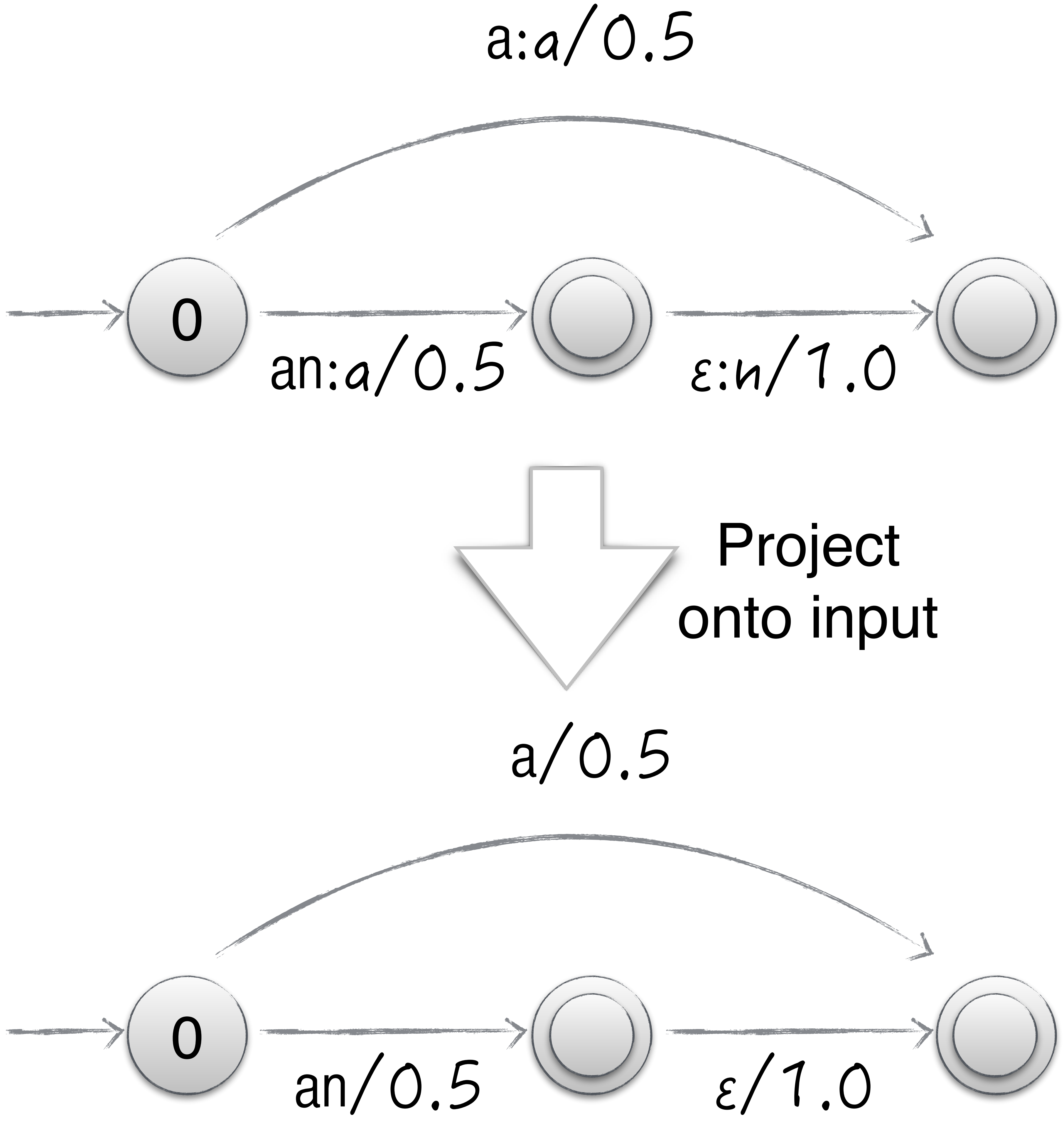


Weights (if they exist) are retained on the arcs

This operation comes in handy, especially during composition!

Projection

Project onto the input or output alphabet



Basic FST Operations (Rational Operations)

The set of weighted transducers are closed under the following operations [Mohri '02]:

1. Sum or **Union**: $(T_1 \oplus T_2)(\underline{x}, \underline{y}) = T_1(\underline{x}, \underline{y}) \oplus T_2(\underline{x}, \underline{y})$

2. Product or **Concatenation**: $(T_1 \otimes T_2)(x, y) = \bigoplus_{\substack{x=x_1x_2 \\ y=y_1y_2}} T_1(x_1, y_1) \otimes T_2(x_2, y_2)$

3. Kleene-**closure**: $T^*(x, y) = \bigoplus_{n=0}^{\infty} T^n(x, y)$

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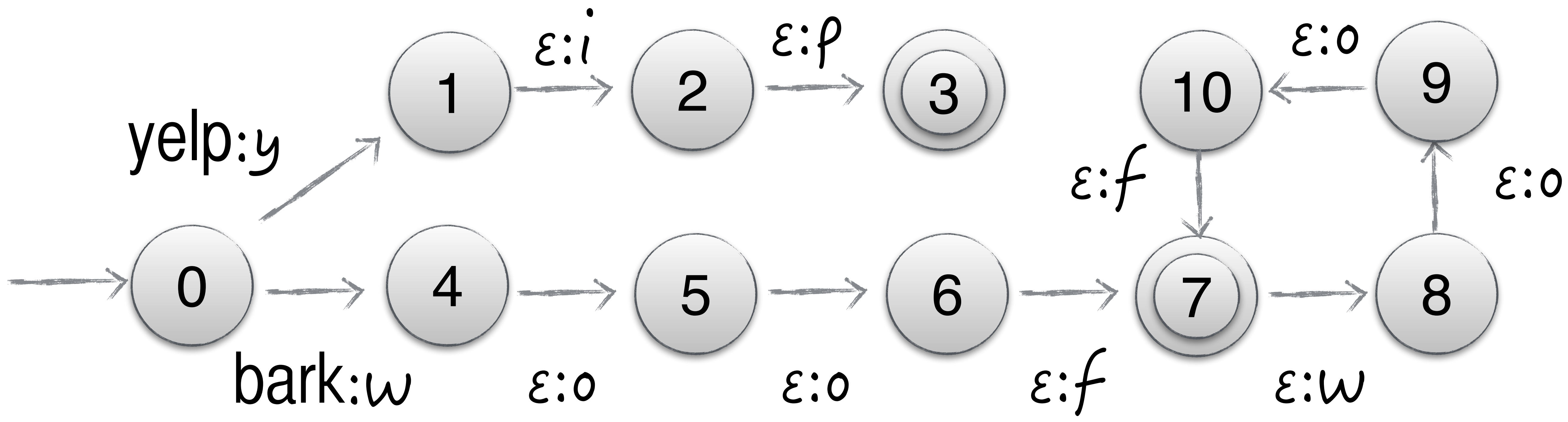
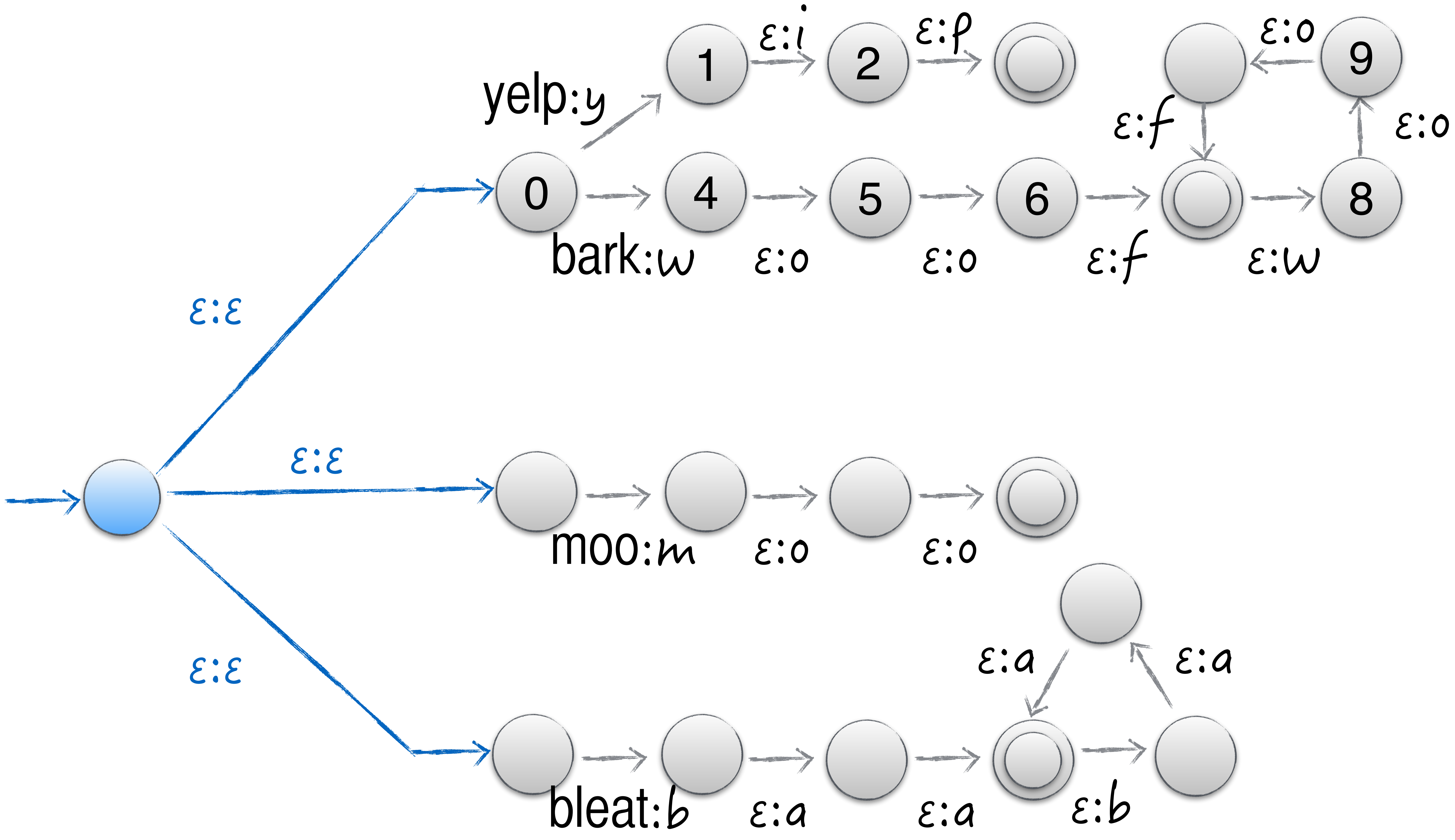
Example: Recall Barking Dog



Example: Union



Animal farm!

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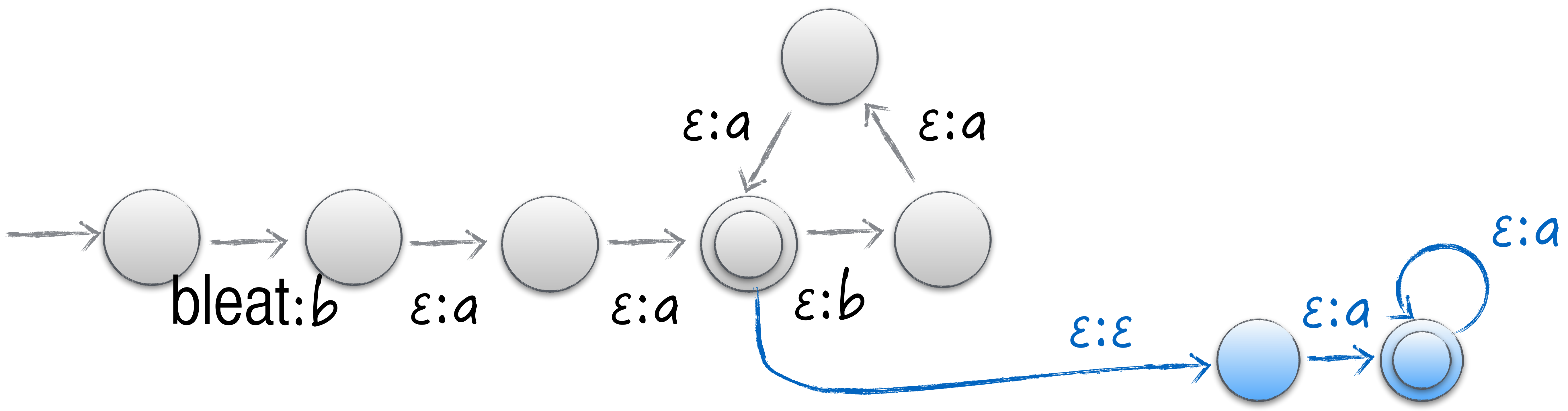
Example: Concatenation



Suppose the last "*baa*" in a bleat should be followed by one or more a's

(e.g., "*baabaa*" is not OK, but "*baaa*" and "*baabaaaaa*" are)

Basic FST Operations (Rational Operations)

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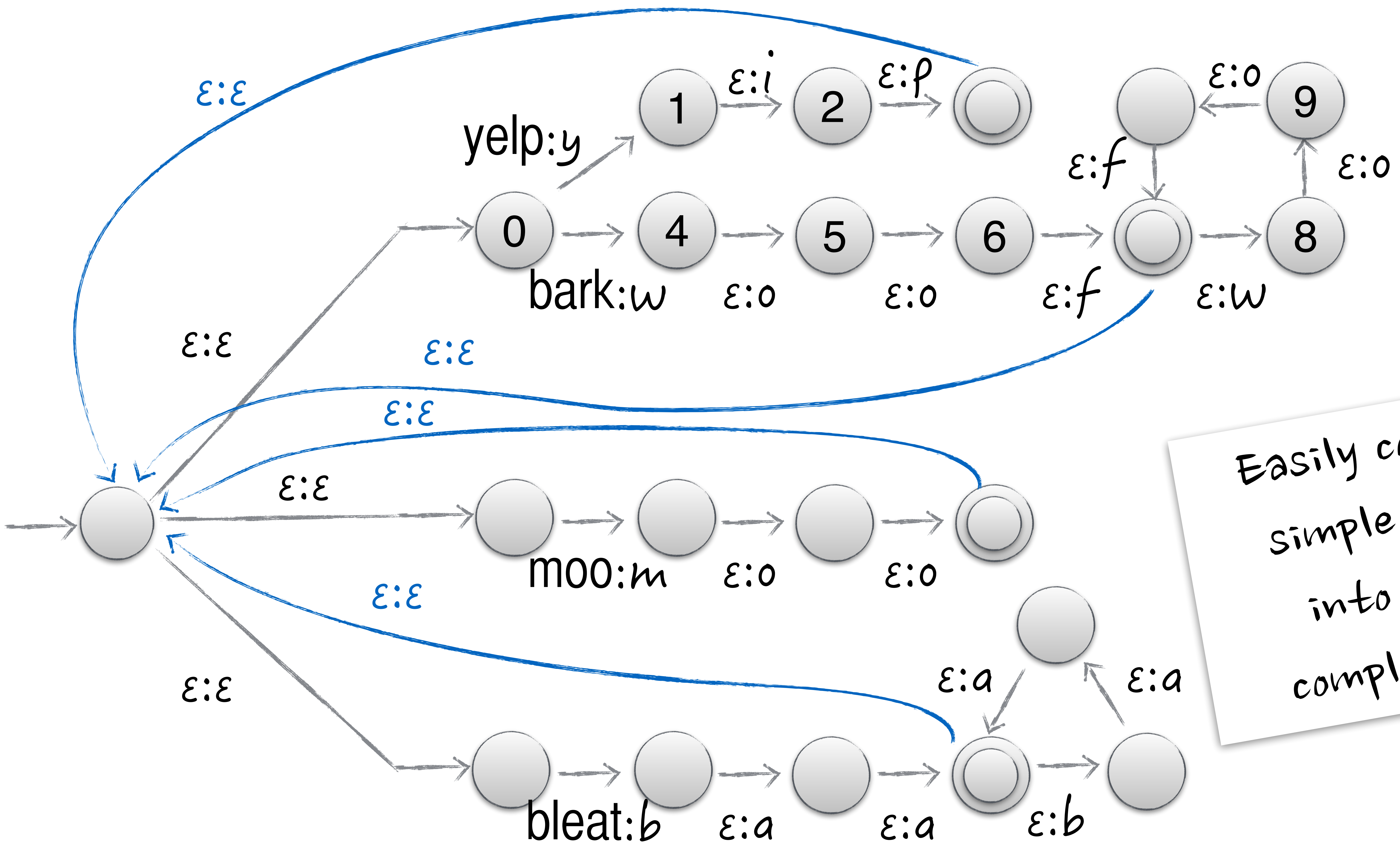
Example: Closure



Easily combine simple FSTs into more complex ones

Animal farm: allow arbitrarily long sequence of sounds!

bark moo yelp bleat → w o o f w o o f m o o y i p b a a b a a

Acoustic Model WFST