Covariance & Correlation

Let X, Y be random variables on $(\Omega, \mathcal{F}, P)$.

$$\text{Cov}(X,Y) = E\left[(X-E(X))(Y-E(Y))\right]$$

$$= E(XY) - E(X)E(Y) \qquad \mu_X = E(X) \quad \mu_Y = E(Y)$$

$$\rho(X,Y) = \text{Corr}(X,Y) = \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)}\sqrt{\text{Var}\,Y}} = \frac{\text{Cov}(X,Y)}{\sigma_X \sigma_Y}$$

Observation

① Suppose X and Y are independent, then $\text{Cov}(X,Y)=0$ and hence $\text{Corr}(X,Y)=0$

Equivalently, if $\text{Cov}(X,Y) \neq 0$, then X & Y are not independent.

Cov and Corr ~~are~~ measures some kind of dependence of X and Y.

② $\text{Cov}(X,Y) = 0 \not\Rightarrow$ X and Y are independent.

③ Sign of $\text{Cov}(X,Y)$ tell us a relation between X and Y.

If X has tendancy of taking higher values (lower values) with higher values of Y (lower value of Y), then $X-\mu_X$ is positive with $Y-\mu_Y$ ~~with~~ positive. ( )

In this $\text{cov}(X,Y) > 0$.

If X takes higher (lower) values with lower values (higher) of Y, then $\text{cov}(X,Y) < 0$.

Sign of covariance gives you the tendancy of changing Y with X.

④ Absolute value of covariance does not give any information about the amount of dependancy between X and Y.

⑤ $|\text{Corr}(X,Y)| \leq 1$

(it follows from cauchy schwartz inequality)

Suppose $Y = aX + b$

$$Corr(X,Y) = \frac{E(XY) - E(X)E(Y)}{\sigma_X \sigma_Y} = \frac{E(aX^2 + bX) - E(X)(aE(X)\ldots}{\sigma_X \sigma_Y}$$

$$= \frac{a\,E(X^2) - a\,[E(X)]^2 + 0}{\sigma_X \sigma_Y}$$

$$= \frac{a \cdot Var\,X}{\sqrt{Var\,X}\,\sqrt{Var\,Y}} = \frac{a\sqrt{Var\,X}}{\sqrt{Var\,Y}} = \begin{cases} +1 & \text{if } a>0 \\ -1 & \text{if } a<0 \end{cases}$$

$$Var\,Y = a^2\,Var\,X$$

$aE(X)+b$

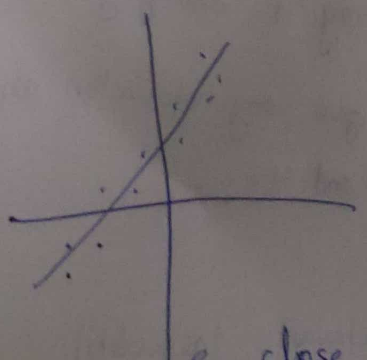
$$Var(Y) = E\left((aX+b)^2\right) - \left(E(aX+b)\right)^2$$

$$= a^2E(X^2) + 2abE(X) + b^2 - a^2(E(X))^2 - 2abE(X) - b^2$$

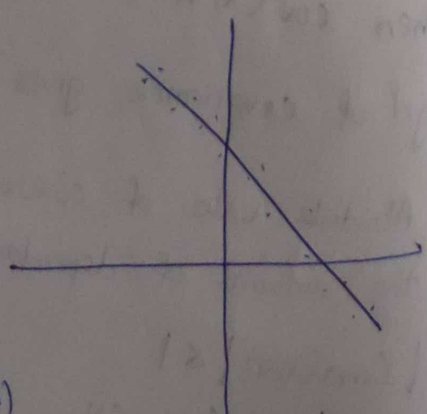
$$= a^2\,Var\,X$$

From Cauchy Schwartz inequality, X and Y linearly related if and only if $|e_{X,Y}| = 1$

Heureshcally, if X and Y are close to linearly dependent, then Corr (X,Y) is close to +1 or −1.



e_{xy} close to 1 ($\because X\uparrow \Rightarrow Y\uparrow$)



e_{xy} close to −1

In other situation, $e_{X,Y} \underset{\text{close to}}{\approx} 0$ (No Proof, only examples)

Examples

1. X is uniform on $\{1, 2, \dots n\}$

$P(X=i) = \frac{1}{n}$ $\quad 1 \le i \le n$

Y is uniform on $\{1, 2, \dots k\}$

Suppose X and Y are independent.

Define $Z = X + Y$

$$\text{Corr}(X, Z) = \frac{\text{Cov}(X, Z)}{\sqrt{\text{Var} X}\,\sqrt{\text{Var} Z}}$$

($\because X, Y$ are independent)

$$\text{Cov}(X, Z) = E(X^2 + XY) - (E(X))^2 - E(X)E(Y) = E(X^2) - (E(X))^2$$

$$= \text{Var}(X) = \frac{n^2-1}{12}$$

$$E(X) = \frac{n+1}{2}$$

$$E(Y) = \frac{k+1}{2}$$

$$\text{Var}(X) = E(X^2) - (E(X))^2$$

$$= \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4} = \frac{n+1}{2}\left[\frac{2n+1}{3} - \frac{n+1}{2}\right]$$

$$= \left(\frac{n+1}{2}\right)\left(\frac{n-1}{6}\right) = \frac{n^2-1}{12}$$

$$\text{Var}(Y) = \frac{k^2-1}{12}$$

$$\text{Cov}(X, Z) = \frac{n^2-1}{12}$$

$$\text{Corr}(X, Z) = \frac{\frac{n^2-1}{12}}{\sqrt{\frac{n^2-1}{12}}\sqrt{\frac{k^2-1}{12} + \frac{n^2-1}{12}}} = \frac{\sqrt{n^2-1}}{\sqrt{k^2-1+n^2-1}} = \begin{cases} \frac{1}{\sqrt{2}} & \text{if } k = n \\ \approx 1 & \text{if } k \ll n \\ \approx 0 & \text{if } k \gg n \end{cases}$$

Given $X = j$,

How Z is distributed?

Z is uniformly distributed on $\{ j+1, j+2, \dots, j+k \}$

$(Z = j + Y)$

↓

this match with $\begin{cases} \frac{1}{\sqrt{2}} \\ 1 \\ 0 \end{cases}$

on last page.

(prediction possibility)

Example 2

X uniform on $\{-n, -n+1, \dots 0, 1, 2, \dots n\}$

$P(X = j) = \frac{1}{2n+1}$ $\quad -n \le j \le n$

Y is uniform on $\{1, 2, \dots k\}$

X, Y are independent.

$Z = X^2 + Y \rightarrow X, Z$ are dependent.

$\text{Corr}(X, Z) = ?$

$\text{Cov}(X, Z) = E(XZ) - E(X)E(Z)$

$E(X) = 0$ $\quad (\because \text{symmetry})$

$\text{Cov}(X, Z) = E(X^3 + XY)$

$= E(X^3) + E(XY)$

$= E(X^3)$

$= 0$

$(\because X, Y$ are independent

$(\because E(XY) - E(X)E(Y) = 0$

$\Rightarrow E(XY) = 0$ as $E(X) = 0$.

& $\text{Corr}(X, Z) = 0 = \text{Cov}(X, Z)$

Plot (X,Z) will lie around parabola.

$Corr(X,Z) = 0$

But (X,Z) are heavily dependent.

Continuous Random Variable

$(\Omega, \mathcal{F}, P) \to$ probability space

$X : \Omega \to \mathbb{R}$

$\{\omega : X(\omega) \leq x\} \in \mathcal{F} \quad \forall x \in \mathbb{R}$ $\Big\}$ random variable.

Discrete Random Variable: A random variable X is called discrete if X takes at most countably many values.

Range of X is $\{x_1, x_2, \ldots\}$.

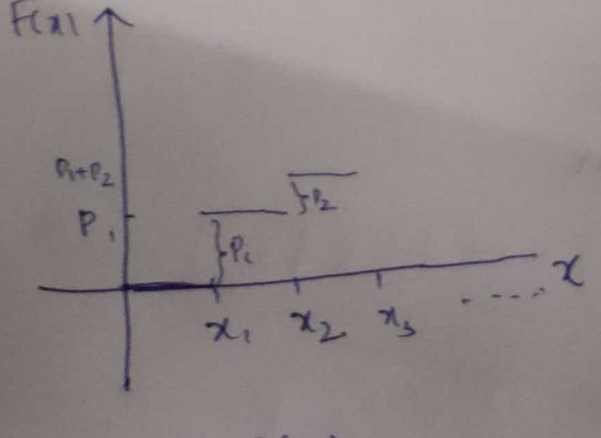
Probability mass f^n

$$p(x_i) = P(X = x_i) = P(\{\omega : X(\omega) = x_i\})$$

$$\begin{cases} p : \mathbb{R} \to [0,1] \\ p(x) > 0 \quad \text{for atmost countably many } x. \\ \sum P(x_i) = 1 \end{cases}$$

$$F_X(x) = \sum_{x_i \leq x} p(x_i)$$

Distribution of discrete random variable is not continuous. It has jump discontinuities at x_i if $P(X = x_i) > 0$



$p_n \to P(x_n)$

For any distribution function F, we have:

(i) F is right continuous

$$\lim_{y \to x^+} F(y) = F(x)$$

(ii) left limit of F(x) exist.

$$\lim_{y \to x^-} F(y) \text{ exists.}$$

$$P(X=x) = F(x) - \lim_{y \to x^-} F(y)$$

$$= F(x) - \lim_{h \to 0^+} F(x-h)$$

Continuous RV

$(\Omega, \mathcal{F}, P)$ $X: \Omega \to \mathbb{R}$ $\{w : X(w) \le x\} \in \mathcal{F}$ $\forall x$

A random variable X is called continuous RV if its distribution function F is continuous.

Equivalently, X is called continuous RV if $P(X=x) = 0$ $\forall x \in \mathbb{R}$.

Observation:

① If X is continuous RV, then range of X is uncountable.

eg. Range of $X = \mathbb{R}$ or $[0,1]$ or $[a,b]$ or $[0,\infty)$

② Ω is uncountable.

eg: $\Omega = [0,1]$ or $[a,b]$ or $\mathbb{R}$ or (a,b)

Discrete situation:

$\Omega = \mathbb{N}$ $\quad P: \mathcal{F} \to [0,1]$

$\mathcal{F} = P(\mathbb{N})$

$P(\{n\})$ $\forall n$ known, then

$$P(A) = \sum_{i \in A} P(i) \quad A \in \mathcal{F}$$

$\Omega = [0,1]$ $\mathcal{F}$ = Borel sigma algebra (sigma algebra generated by open intervals of $[0,1]$).
↓
smaller than power set of $[0,1]$

$P([a,b]) = b-a.$

where $[a,b] \subseteq [0,1]$

Ω

S is a set of subsets of Ω.

$\mathcal{F}_S = \cap \mathcal{G}$ where $\mathcal{G}$ is a σ algebra containing S.

Absolutely Continuous RV

X is a continuous RV. It is called absolutely continuous if $\exists f : \mathbb{R} \to \mathbb{R}_{\geq 0}$

s.t. $F(x) = \int_{-\infty}^{x} f(t)dt \quad \forall x \in \mathbb{R}.$

f is called the density of random variable X or distribution F.

$f(x) \neq P(X=x)$ → for continuous, right side is always zero.

$p(x_i) = P(X = x_i)$
↑ pmf. ↑ discrete

Tut

2) Given finite additivity of P.

$$P(\phi) = 1 - P(\Omega) = 0$$

Continuity $\Rightarrow$ Countable additivity

let B_i be increasing : $B_i \subset B_{i+1}$

$$P\left(\bigcup_{i=1}^{\infty} B_i\right) = \lim_{n\to\infty} P(B_n)$$

Let $\{A_i\}$ is ~~piece~~ pairwise disjoint sets,

$$B_j = \bigcup_{i=1}^{j} A_i$$ Taking P on this

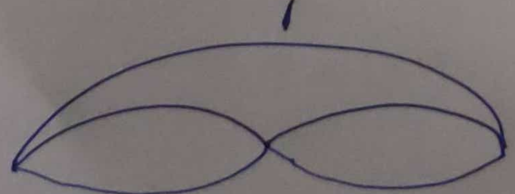
B_j is increasing.

$$P\left(\bigcup_{j=1}^{\infty} B_j\right) = \lim_{n\to\infty} P(B_n) = \lim_{n\to\infty} P\left(\bigcup_{i=1}^{n} A_i\right)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} P(A_i)$$ by countable additivity

$$= \lim_{n\to\infty} P(B_n)$$

3)



$F \to 1$ open

$$P(A \text{ to } C) = P(A \text{ to } C \mid F) + P(A \text{ to } C \mid F^c)$$

$$= 1 - p + (1-p^2)^2 \cdot p$$

4) $\{X_n\}_{i=1}^{\infty}$

X_i takes values in $\mathbb{N}$

$\Omega = \mathbb{N}^{\mathbb{N}}$ (countable cartesian product of natural numbers)

$\mathcal{F} = P(\mathbb{N}^{\mathbb{N}})$

$N = \min\{n > 0 ; X_n = X_0\}$

Is this a RV?

$G : \Omega \to N$ if $G^{-1}(j) \in \mathcal{F} \quad \forall j \in \mathbb{N}$

$\{N = 1\} = \{X_1 = X_0\}$

$= \bigcup_{i=1}^{\infty} \{X_1 = X_0 = i\} \in$

$N^{-1}(1) = \{ --, N N N -- \} \in \mathcal{F}$

$N^{-1}(i)^{i>1} = \{ X_n \neq X_0, n < i, X_n = X_0 \}$

$= \bigcap_{j=1}^{\infty} \{X_n \neq X_0\} \cap \{X_i = X_0\}$

$= \left(\bigcap_{j=1}^{\infty} \{X_n = X_0\}^c \right) \cap \{X_i = X_0\} \in \mathcal{F}$

$\boxed{?}$

$\{X_2 = X_0\} = \{ (1, 1, N, N, \ldots) \} \cup \{ (2, 2, N, N, \ldots) \}$

Continuous RV

Let $(\Omega, \mathcal{F}, P)$ be a probability space and X be a random variable of Ω.

X is called continuous rv if its distribution function F is continuous

Equivalently, X is cont. rv if $P(X=x) = 0 \quad \forall x \in \mathbb{R}$.

Range of X is uncountable.

Ω is uncountable.

If Ω is countable/ finite, then we have a clear understanding of $\mathcal{F}$. But Ω is uncountable, then it is not so straight forward.

Nothing here

Let $\mathcal{C}$ be a collection of subsets of Ω.

Def^n: (σ - field generated by $\mathcal{C}$)

A σ field, denoted by $\sigma(\mathcal{C})$, is called the σ- field generated by $\mathcal{C}$ if it satisfies the following:

(i) $\mathcal{C} \subset \sigma(\mathcal{C})$

(ii) If $\mathcal{F}$ is any other σ-field containing $\mathcal{C}$, then

$$\sigma(\mathcal{C}) \subseteq \mathcal{F}$$

$\sigma(\mathcal{C})$ is also known as minimal sigma field containing $\mathcal{C}$.

Result: Given a collection $\mathcal{C}$ of subsets of Ω, there exist a unique minimal σ- sigma field containing $\mathcal{C}$.

In other words, σ- field generated by $\mathcal{C}$ is unique.

Proof: Let $T = \{\mathcal{F} : \mathcal{F} \text{ is a } \sigma\text{-field and } \mathcal{C} \subseteq \mathcal{F}\}$

observe that T is non empty.

$\because P(\Omega) \in T$

$\Omega = \{1,2,3,4\}$

$\mathcal{C} = \{\{1\},\{2\}\}$

$T = \{P(\Omega), \{\phi, \{1\}, \{2\}, \{2,3,4\}, \{1,3,4\}, \{1,2\}, \Omega\}, \dots$

let $\mathcal{G} = \bigcap_{\mathcal{F} \in T} \mathcal{F}$

$\mathcal{G} \neq \phi$ ($\because \Omega \in \mathcal{G}$

$\because \Omega \in \mathcal{F} \quad \forall \mathcal{F}$

($\because$ sigma field should contain full set))

We have seen that $\mathcal{G}$ is a σ field ($\because$ arbitrary $\cap$ of σ-fields is σ-field).

Claim: $\sigma(\mathcal{C}) = \mathcal{G}$

By definition $\mathcal{C} \subseteq \mathcal{G}$ $(\because \mathcal{C} \in \mathcal{F} \;\; \& \;\; \mathcal{F} \in T)$

$\Rightarrow \mathcal{C} \in \bigcap_{\mathcal{F} \in T} \mathcal{F} = \mathcal{G}$

Condition (i) ✓

Let $\mathcal{F}'$ be a sigma field containing $\mathcal{C}$.

TST: $\mathcal{G} \subseteq \mathcal{F}'$

$\mathcal{F}' \in T$ ($\because$ it is a σ-field containg $\mathcal{C}$)

$\bigcap_{\mathcal{F} \in T} \mathcal{F} \subseteq \mathcal{F}'$ $\therefore \mathcal{G} \subseteq \mathcal{F}'$ ($\because \mathcal{G}$ is intersection of all $\mathcal{F}$ in T)

Condition (ii) ✓

$\therefore \sigma(\mathcal{C}) = \mathcal{G}$ ($\because$ if there is K also $K \subseteq \mathcal{G}$, $\mathcal{G} \subseteq K$ $\therefore K = \mathcal{G}$)

$\therefore$ There exist a unique minimal σ-field containing $\mathcal{C}$.

For example, suppose $\Omega = (0,1)$

$$P((a,b)) = b-a$$

$$\mathcal{C} = \{(a,b) : 0 < a \leq b < 1\}$$

Note $\mathcal{C}$ is not a σ-field.

$(0, \frac{1}{4}) \in \mathcal{C}$ $\quad (\frac{1}{2}, 1) \in \mathcal{C}$

Union, $(0, \frac{1}{4}) \cup (\frac{1}{2}, 1) \notin \mathcal{C}$

Consider $\sigma(\mathcal{C}) \to \sigma$-field

Then $\exists$ a probability measure on $\sigma(\mathcal{C})$ such that $P((a,b)) = b-a$

(no proof here)

Borel σ-field of $\mathbb{R}$ ($\mathcal{B}(\mathbb{R})$)

$$\mathcal{B}(\mathbb{R}) := \sigma(\mathcal{C})$$

where $\mathcal{C} = \{(a,b) : a \leq b ; a,b \in \mathbb{R}\}$.

An element of $\mathcal{B}(\mathbb{R})$ is called borel set.

There are alternate descriptions of borel σ field of $\mathbb{R}$, which are as follows:

$$\mathcal{B}(\mathbb{R}) = \sigma(\{(a,b) : a \leq b , a,b \in \mathbb{R}\})$$

$$= \sigma(\{(a,b] : a \leq b , a,b \in \mathbb{R}\})$$

$$= \sigma(\{[a,b] : \dots\})$$

$$= \sigma(\{(-\infty, x] : x \in \mathbb{R}\})$$

$$= \sigma(\{(x, \infty) : x \in \mathbb{R}\})$$

$$= \sigma(\text{open sets of } \mathbb{R})$$

$\vdots$

Proof: TST: $\sigma(\mathcal{C}) = \sigma(\mathcal{C}_1)$

where $\mathcal{C} = \{(a,b) : a \leq b\}$

$\mathcal{C}_1 = \{(a,b] : a \leq b\}$

Suppose we show $(a,b) \in \sigma(\mathcal{C}_1)$ — (i)

$\mathcal{C} \subseteq \sigma(\mathcal{C}_1)$

$\sigma(\mathcal{C}) \subseteq \sigma(\mathcal{C}_1)$ ($\because \sigma(\mathcal{C}$ is minimal$)$)

Suppose we show $(a,b] \in \sigma(\mathcal{C})$ — (ii)

$\mathcal{C}_1 \subseteq \sigma(\mathcal{C}) \Rightarrow \sigma(\mathcal{C}_1) \subseteq \sigma(\mathcal{C})$

Finished

So, need to show (i) & (ii)

TST: $(a,b) \in \sigma(\mathcal{E}_1)$

$$(a,b) = \bigcup_{n=1}^{\infty} \left(a, b-\frac{1}{n}\right]$$

$$\left(a, b-\frac{1}{n}\right] \in \mathcal{E}_1 \quad \therefore \left(a, b-\frac{1}{n}\right) \in \sigma(\mathcal{E}_1)$$

$$\Rightarrow \bigcup_{n=1}^{\infty} \left(a, b-\frac{1}{n}\right] \in \sigma(\mathcal{E}_1)$$

$$(a,b] = \bigcap_{n=1}^{\infty} \left(a, b+\frac{1}{n}\right) \in \sigma(\mathcal{E}) \quad \text{similarly.}$$

Absolutely Cont. RV

$$P(X \leq x) = F(x) = \int_{-\infty}^{x} f(t)\,dt \quad \forall x \in \mathbb{R}$$

X is absolutely cont. RV if $\exists$ $f \geq 0$ s.t

f - called density function of X.

Remark: ① f is not unique.

② Every continuous random variable does not have a density f^n.

eg: Cantor distribution.

③ $$P(a \leq x \leq b) = \int_a^b f(t)\,dt$$

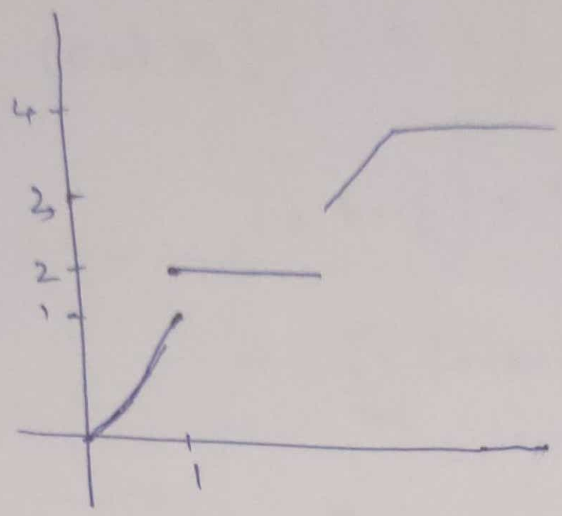
$$= P(a < x \leq b)$$

$$= P(a \leq x < b)$$

$$= P(a < x < b)$$

Neither Discrete nor Continuous:

Mix of both



Q. Suppose X conti. with distribution f^n F. How to find the density?

Working Rule:

Take the derivative of F. (Do not exist always. If exist, this is one way)

Define $f(x) = F'(x)$

X discrete. rv with density f.

$g : \mathbb{R} \longrightarrow \mathbb{R}$

$Y = g(X)$ is a rv.

in discrete case $\rightarrow$ $\{\omega : Y(\omega) = y\} \in \mathcal{F}$

$$= \bigcup_{\substack{i \\ g(x_i) = y}} \{\omega : X(\omega) = x_i\}$$

$(\Omega, \mathcal{F}, P)$

X is a continuous random variable with density f.

$g : \mathbb{R} \to \mathbb{R}$ borel measurable.

$Y = g(X)$ will be a RV.

We want to find, if possible, distribution f^n & density f^n of Y.

Example

not this g.

$g(x) = ax + b \qquad a \neq 0$

$Y = g(X) = aX + b$

$F_Y(y) = P(Y \leq y) = P(aX + b \leq y)$

$= P\left(X \leq \frac{y-b}{a}\right)$ assume $a > 0$

$y = ka + b$

$= F_X\left(\frac{y-b}{a}\right)$

$= \int_{-\infty}^{\frac{y-b}{a}} f(t)\,dt \overset{?}{=} \int_{-\infty}^{y} g(u)\,du$; Do change of variable & find g.

assume $a < 0$.

$F_Y(y) = P\left(X \geq \frac{y-b}{a}\right)$

$= 1 - P\left(X < \frac{y-b}{a}\right) = 1 - P\left(X \leq \frac{y-b}{a}\right)$ [∵ continuous. So equality where doesn't matter]

$= 1 - F_X\left(\frac{y-b}{a}\right)$

$= 1 - \int_{-\infty}^{\frac{y-b}{a}} f(t)\,dt \rightarrow$ try to find density

(ii) $g(x) = x^2$

Find distribution function and density function of $Y = g(X) = X^2$

$F_Y(y) = P(Y \leq y)$

$= P(X^2 \leq y) = 0$ if $y < 0$

If $y > 0$, $F_Y(y) = P(X^2 \leq y) = P(-\sqrt{y} \leq X \leq \sqrt{y})$ (property of continuous used)

$= F_X(\sqrt{y}) - F_X(-\sqrt{y})$

$$F_Y(y) = \begin{cases} 0 & \text{if } y<0 \\ F(\sqrt{y}) - F(-\sqrt{y}) & \text{if } y \geq 0 \end{cases}$$

$$\downarrow$$

$$\int_{-\infty}^{\sqrt{y}} f(t)\,dt - \int_{-\infty}^{-\sqrt{y}} f(t)\,dt \;\approx\; \int_{-\infty}^{y} g(u)\,du$$

after some manipulation

(or back calculation by differentiation).

$$\frac{d}{dy} F_Y(y) = f(\sqrt{y}) \cdot \frac{1}{2\sqrt{y}} + \frac{f(-\sqrt{y})}{2\sqrt{y}} = \frac{f(\sqrt{y}) + f(-\sqrt{y})}{2\sqrt{y}}$$

Density of Y

$$g(t) = \begin{cases} 0 & \text{if } t<0 \\ \frac{1}{2\sqrt{t}} \left(f(\sqrt{t}) + f(-\sqrt{t})\right) & t \geq 0 \end{cases}$$

Show: $\int^{y} g(t)\,dt = F_Y(y)$

Given y_1, y_2 are two lin independent solution of $L(y(x))=0$ on (a,b).

So $W(y_1,y_2) \neq 0$.

let $\exists\, \alpha, \beta \in (a,b)$ for which $y_2(\alpha) = 0 = y_2(\beta)$

we need to find $c \in (\alpha,\beta)$ s.t $y_1(c)=0$.

If not, $y_1(c) \neq 0 \quad \forall\, c \in (\alpha,\beta)$.

Then $\frac{y_2}{y_1}$ is defined in (α,β).

y_1, y_2 must be diff $\longrightarrow$ so is $\frac{y_2}{y_1}$ also diff on (α,β) & conti on $[\alpha,\beta]$.

$$\frac{y_2(\alpha)}{y_1} = 0 = \frac{y_2}{y_1}(\beta)$$

By Rolle's theorem, $\left(\frac{y_2}{y_1}\right)'(c) = 0$ for at least one $c \in (\alpha,\beta)$. $\Rightarrow \frac{W(y_1,y_2)(c)}{y_1^2} = 0$

$\Rightarrow W(y_1,y_2)(c) = 0 \longrightarrow$ not possible.

$y(t) = e^{at}$

So $\exists\, c \in (\alpha,\beta)$ such that $y_1(c)=0$.

$a^2 + ab + c = 0$

If $\exists\, c,d$ in (α,β) such that $y_1(c) = 0 = y_1(d)$. then in similar manner, $\exists\, \xi \in (c,d)$ s.t $y_2(\xi)=0$ which shows $\exists$ only one.

6. Variation of Parameters

Let X be a cont. rv with density f. We need to find density of $Y = X^2$.

$F_Y(y) = P(X^2 \leq y) = 0$ if $y \leq 0$

$F_Y(y) = P(X^2 \leq y) = P(-\sqrt{y} < X < \sqrt{y})$ if $y > 0$

$= F(\sqrt{y}) - F(-\sqrt{y})$

$$F_Y'(y) = \frac{f(\sqrt{y})}{2\sqrt{y}} + \frac{1}{2\sqrt{y}} f(-\sqrt{y})$$

Density f^n of Y

$$g(y) = \begin{cases} 0 & \text{if } y \leq 0 \\ \frac{1}{2\sqrt{y}}\left(f(\sqrt{y}) + f(-\sqrt{y})\right) & \text{if } y > 0 \end{cases}$$

Easy to see.

$$\int_{-\infty}^{y} g(t)\,dt = \int_0^y \frac{1}{2\sqrt{t}} f(\sqrt{t}) + f(-\sqrt{t})\,dt \qquad (\sqrt{t} = s \text{ substi})$$

$$= \int_0^{\sqrt{y}} f(s) + f(-s)\,ds.$$

$$= \int_0^{\sqrt{y}} f(s)\,ds + \int_{-\sqrt{y}}^{0} f(s)\,ds.$$

$$= \int_{-\sqrt{y}}^{\sqrt{y}} f(s)\,ds$$

$$= F(\sqrt{y}) - F(-\sqrt{y})$$

Result: Suppose X is a conti rv ...

$\phi : \mathbb{R} \to \mathbb{R}$

differentiable, strictly monotonically increasing or decreasing on a set $I \subseteq \mathbb{R}$. Suppose $f(x) = 0$ if $x \notin I$

Then the density f^n of random variable $Y = \phi(X)$ is

$$g(y) = \begin{cases} f(\phi^{-1}(y)) \left| \frac{d}{dy} \phi^{-1}(y) \right| & \text{for } y \in \phi(I) \\ 0 & \text{otherwise.} \end{cases}$$

To remeber:

$$g(y) = \begin{cases} f(x) \left| \frac{dx}{dy} \right| & y \in \phi(I) \\ 0 & \text{otherwise} \end{cases}$$

Proof: Assume ϕ is strictly increasing.

$x_1 < x_2 \Rightarrow \phi(x_1) < \phi(x_2)$

ϕ^{-1} is defined on $\phi(I)$ and ϕ^{-1} is also strictly increasing.

$$F_Y(y) = P(\phi(X) \leq y) \quad \text{where } y \in \phi(I)$$

$$= P(\phi(X) \leq y)$$

$$= P(X \leq \phi^{-1}(y))$$

$$= F(\phi^{-1}(y))$$

Take derivative to find density f^n.

If ϕ is strictly decreasing, ϕ^{-1} will be strictly decreasing.

$$F_Y(y) = P(\phi(X) \leq y)$$

$$= P(X \geq \phi^{-1}(y)) = 1 - F(\phi^{-1}(y))$$

Complete it ...

Result: Suppose X is a conti rv with density

$\phi: \mathbb{R} \to \mathbb{R}$

differentiable, strictly monotonically increasing or decreasing on a set $I \subseteq \mathbb{R}$. Suppose $f(x) = 0$ if $x \notin I$

Then the density f^n of random variable $Y = \phi(X)$ is

$$g(y) = \begin{cases} f(\phi^{-1}(y)) \left| \frac{d}{dy} \phi^{-1}(y) \right| & \text{for } y \in \phi(I) \\ 0 & \text{otherwise.} \end{cases}$$

To remeber:
$$g(y) = \begin{cases} f(x) \left| \frac{dx}{dy} \right| & y \in \phi(I) \\ 0 & \text{otherwise} \end{cases}$$

Proof: Assume ϕ is strictly increasing.

$x_1 < x_2 \Rightarrow \phi(x_1) < \phi(x_2)$

ϕ^{-1} is defined on $\phi(I)$ and ϕ^{-1} is also strictly increasing.

$F_Y(y) = P(\phi(X) \leq y)$ where $y \in \phi(I)$

$= P(\phi(X) \leq y)$

$= P(X \leq \phi^{-1}(y))$

$= F(\phi^{-1}(y))$

Take derivative to find density f^n.

If ϕ is strictly decreasing, ϕ^{-1} will be strictly decreasing.

$F_Y(y) = P(\phi(X) \leq y)$

$= P(X \geq \phi^{-1}(y)) = 1 - F(\phi^{-1}(y))$

Complete it take

Symmetric RV

A rv X is called symmetric (around 0) if X and $-X$ have same distribution.

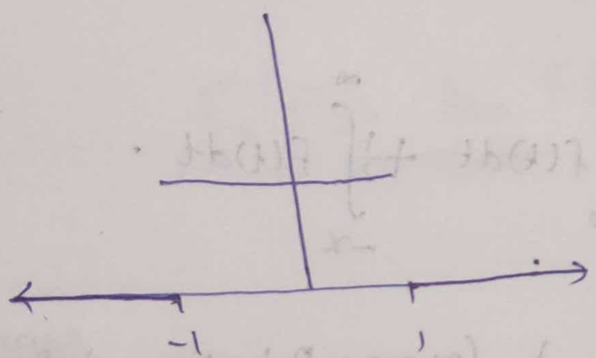
$$F_X(x) = P(X \le x)$$

$$\parallel$$

$$F_{-X}(x) = P(-X \le x) = P(X \geq -x)$$

Examples: $U(-1,1)$ (uniform), Normal $N(0,\sigma^2)$

$U(-1,1)$



$U(-1,1)$

density $f(x) = \begin{cases} \frac{1}{2} & x \in (-1,1) \\ 0 & \text{otherwise} \end{cases}$

A f^n f is called symmetric around 0 if $f(x) = f(-x)$ $\forall x \in \mathbb{R}$

Observe: 1) Density functions are symmetric RV.

Result: A continuous rv X is symmetric iff it has a symmetric density function.

Proof: Suppose X has a symmetric density f.

$$f(x) = f(-x) \quad \forall x \in \mathbb{R}.$$

Check: $P(X \le x) = P(X \geq -x)$ $\quad \forall x \in \mathbb{R}$

Conversely,

X is a symmetric rv, with density f.

We want to show that X has a symmetric density.

$$g(x) = \frac{1}{2}\left(f(x) + f(-x)\right)$$ → normal way to make symmetric f^n.

$g(x) = g(-x)$, g is symmetric.

TST: $F(x) = \int_{-\infty}^{x} g(t)\,dt \quad \forall x \in \mathbb{R}$

$$F(x) = \int_{-\infty}^{x} g(t)\,dt = \int_{-\infty}^{x} \frac{1}{2}\left(f(t) + f(-t)\right) dt$$

$$= \frac{1}{2}\int_{-\infty}^{x} f(t)\,dt + \frac{1}{2}\int_{-x}^{\infty} f(t)\,dt \cdots = F(x)$$

Examples to Read: N , $U(a,b)$, Gamma, Beta, Exponential distributions.

<u>Bivariate Distribution</u>

Suppose X, Y are rvs defined on $(\Omega, \mathcal{F}, P)$

Then joint distribution f^n of (X,Y) is defined as

$$F(x,y) = P(X \le x, Y \le y)$$

(X,Y) are called continuous random vector if $F(x,y)$ is continuous.

If $\exists$ a non negative f^n $f: \mathbb{R}^2 \to \mathbb{R}$ s.t

$$F(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(s,t)\,ds\,dt$$

Then f is called density function of F or the pair (X,Y).

Given bivariate distribution F, we can calculate marginal distribution functions.

$$F_X(x) = P(X \leq x) = P(X \leq x, Y \in \mathbb{R})$$

$\downarrow$ marginal distribution of X

$$= \lim_{y \to \infty} F(x,y)$$

using continuity of probability.

$P\left(\bigcup_{n=1}^{\infty} A_n\right)$ where $A_n \uparrow$

$\shortparallel$

$\lim_{n \to \infty} P(A_n)$

$$F_Y(y) = \lim_{x \to \infty} F(x,y)$$

__Exercise:__

Find marginal density function of X and Y from joint density function $F(x,y)$

__Independence__

X, Y discrete

$P(X = x_i, Y = y_j) = P(X = x_i)\, P(Y = y_j)$ (if X & Y are independent)

If X, Y continuous

null event

$P(X = x) = 0 \quad \forall x$ $\qquad P(Y = y) = 0 \quad \forall y$

$$P\left((X,Y) = (x,y)\right) \leq P(X = x) = 0$$

Two random variable X and Y are independent if

$$P(X \le x)\, P(Y \le y) = P(X \le x, Y \le y)$$

equivalently $F(x,y) = F_X(x)\, F_Y(y)$

equivalently $P(a < X \le b, c < Y \le d) = P(a < X \le b)\, P(c \le Y \le d)$

whenever $a \le b$, $c \le d$.

Exam

Q.4. $N = \min\{n > 0 : X_n = 0\}$

Possible values of N is $1, 2, 3, \dots$

To show:

$$\{N \le x\} \in \mathcal{F} \quad \forall x \in \mathbb{R}$$

$$\{N \le x\} = \phi \in \mathcal{F} \quad \text{if } x < 1.$$

$x \ge 1$

$$\{N \le x\} = \bigcup_{\substack{k \in \mathbb{N} \\ k \le x}} \{N = k\}$$

$$\{N = k\} = \{\omega : X_i \ne X_0 \text{ for } 1 \le i \le k-1, \; X_k = X_0\}$$

$$= \bigcup_{j=1}^{m} \{\omega : X_i \ne j \quad 1 \le i \le k-1, \; X_k = X_0 = j\}$$

$$= \bigcup_{j=1}^{m} \left\{ \bigcap_{i=1}^{k-1} \{X_i \ne j\} \cap \{X_0 = j\} \cap \{X_k = j\} \right\} \in \mathcal{F}$$

$$\boxed{E(N) = \sum_{n=1}^{\infty} P(N \ge n)}$$

$$P(N \ge n) = P(X_i \ne X_0, \; 1 \le i \le n-1)$$

$$= \sum_{j=1}^{m} P(X_i \ne X_0, \; 1 \le i \le n-1, \; X_0 = j)$$

$$= \sum_{j=1}^{m} P(X_i \ne j \quad 1 \le i \le n-1, \; X_0 = j)$$

$$= \sum_{j=1}^{m} \prod_{i=1}^{n-1} P(X_i \ne j) \, P(X_0 = j)$$

$$= \sum_{j=1}^{m} p(j) \, (1 - p(j))^{n-1} = m$$