CS207 Sets and relations

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Contents

1 Basics

- Set : Unordered collection of elements
- Set membership(belongs to) : $73 \in \mathbb{Z}$
- Set inclusion (subset) : $\mathbb{Z} \subseteq \mathbb{R}$
- Set operations : union, intersection,...

1.1 Sets as predicate

Sets can be considered as predicates which maps all the elements present in it to true and all the remaining elements in the domain to false.
Eg, wingset=$\{x|wing(x)\}$

1.2 Binary/unary operators for set operations

Let S and T be sets and inS(.) and inT(.) be their corresponding predicates. Then

- $in\bar{S}(x) \equiv \sim inS(x)$
- $inS \cup T(x) \equiv inS(x) \vee inT(x)$
- $inS \cap T(x) \equiv inS(x) \wedge inT(x)$
- $inS - T(x) \equiv inS(x) \nrightarrow inT(x)$
- $inS\Delta T(x) \equiv inS(x) \oplus inT(x)$

1.3 Set inclusion

- $S \subseteq T \equiv inS(x) \rightarrow inT(x)$
- If $S \subseteq T$ and $T \subseteq R$, then $S \subseteq R$
- $S \subseteq T \equiv \bar{T}(x) \subseteq \bar{S}(x)$

General template to prove S=T: Prove that $S \subseteq T$ and $T \subseteq S$
Inclusion-Exclusion principle: $|S \cup T| = |S| + |T| - |S \cap T|$
where $|X|$ is the number of elements in set X.

1.4 Cartesian Product

Let S and T be sets.
$S \times T = \{(s,t) | s \in S; t \in T\}$
$|S \times T| = |S| \cdot |T|$ Properties:

- $(A \cup B) \times C = (A \times C) \cup (B \times C)$
- $\overline{S \times T} = (\bar{S} \times T) \cup (S \times \bar{T}) \cup (\bar{S} \times \bar{T})$

2 Relation

A relation can be defined as a predicate over $S \times S$.
(Note: We are considering binary homogeneous relations)

Ways of representation of relations:

1. **Set:** $\{(r,s) | r \sqsubset s\}$
2. **Boolean Matrix:** $M_{r,s} = True/1$ iff $r \sqsubset s$
3. **Directed graphs:** Edge directed from node r to node s iff $r \sqsubset s$

Note: $r \sqsubset s$ is same as r is related to s.

Since relation is a set, all the set operations can be extended to relation. Other operations can also be defined, like:

- Transpose: $R^T = (x,y) | (y,x) \in R$
- Composition: $R \circ R' = (x,y) | \exists w[(x,w) \in R \wedge (w,y) \in R']$

Composition can be considered as matrix multiplication. $(M \circ M')_{x,y} = \vee_w (M_{x,w} \wedge M'_{w,y})$

2.1 Types of relations

2.1.1 Reflexive and Irreflexive relation

A relation R, defined on set S, is said to be reflexive iff $\forall x \in S \ (x,x) \in R$.
In terms of boolean matrix, all diagonal entries should be true.
In terms of directed graphs, each node should have a self-loop.

A relation R, defined on set S, is said to be **ir**reflexive iff $\forall x \in S \ (x,x) \notin R$.
In terms of boolean matrix, all diagonal entries should be false.
In terms of directed graphs, no node should have a self-loop.

2.1.2 Symmetric and anti-symmetric relations

A relation R, defined on set S, is said to be symmetric iff $(x,y) \in R \implies (y,x) \in R$.
In terms of boolean matrix, the matrix is symmetric.
In terms of directed graphs, all edges are bidirectional.

A relation R, defined on set S, is said to be anti-symmetric iff $(x,y) \in R \wedge x \neq y \implies (y,x) \notin R$.
In terms of directed graphs, none of the edges are bidirectional.

2.1.3 Transitive relation

A relation R, defined on set S, is said to be transitive relation, iff $(x, y) \in R \wedge (y, z) \in R \implies (x, z) \in R$.
Equivalently, R is transitive iff $R \circ R \subseteq R \iff R^k \subseteq R \forall k > 1$. R^k is k times composition of R.
In terms of directed graphs, if there is a path from a to z, there is an edge (a,z).

A relation R, defined on set S, is said to be intransitive relation, iff $(x, y) \in R \wedge (y, z) \in R \implies (x, z) \notin R$.
A complete relation is reflexive, symmetric and transitive.
Various closures on R:

1. Reflexive closure on R: R' is minimal relation such that $R \subseteq R' \wedge R'$ is reflexive
2. Symmetric closure on R: R' is minimal relation such that $R \subseteq R' \wedge R'$ is symmetric
3. Transitive closure on R: R' is minimal relation such that $R \subseteq R' \wedge R'$ is transitive

Note: Minimal relation means, removing any edge violates subset condition or relation condition.
Each of these closures are unique for a given relation R.

2.1.4 Equivalence relation

A relation is called equivalence iff it is reflexive, symmetric and transitive.
Equivalence class(Eq(.)): $Eq(x) = \{y | x \sim y\}$
Every element is in its own equivalence class.(which means every element is present in atleast one equivalence class)
Claim: If $Eq(x) \cap Eq(y) \neq \phi$, then $Eq(x) = Eq(y)$
Proof:Let $z \in Eq(x) \cap Eq(y)$. Let $w \in Eq(x)$. Now, $x \sim z$ hence $z \sim x$. Also, $x \sim w$ and $w \sim x$. Hence, $z \sim w$. And $y \sim w$.
Therefore, $w \in Eq(y) \implies Eq(x) \subseteq Eq(y)$.
Similarly, $Eq(y) \subseteq Eq(x)$.

Equivalence classes partition the domain.(The relation set).
This means that their union is the relation set, and intersection of any distinct pair of equivalence classes is unique.

3 Partially ordered sets

A relation that is reflexive, transitive and anti-symmetric is a partially ordered relations. Eg: $\leq$
A relation that is **ir**reflexive, transitive and anti-symmetric is a strictly partially ordered relations.
A relation that i transitive and anti-symmetric has no closed loops(acyclic).

A poset is a partial order relation defined over a non-empty set. It can be denoted as $(S, \leq)$.

3.1 Extremal and Extremum

3.1.1 Minimal and Maximal elements

$x \in S$ is minimal if $\nexists y$ such that $y \neq x$ and $y \leq x$.(all arrows except self-loops directed out)
Similarly, $x \in S$ is maximal if $\nexists y$ such that $y \neq x$ and $x \leq y$.(all arrows except self-loops directed in)
Minimal and maximal elements may not exists(eg, $S = \mathbb{Z}$) and if they exist, they may not be unique(eg, for $(\mathbb{Z}^+ - 1, |(divisibility))$ all prime numbers are minimal).

Claim: Every finite poset has atleast one minimal and one maximal element.
Proof: Induction on $|S|$
True for $|S| = 1$. Le it be true for $|S| = k$.
Then for $|S| = k + 1$, let S be parted into a k-element set and 1-element set(having element z).
Let x be a minimal in the prior set.
If $x \leq z$ or x and z are not related, then minimal of set S is x.
If $z \leq x$, the z is minimal of S using transitivity.
Similarly, for maximal.
(QED)

3.1.2 Greatest and least element

$x \in S$ is the greatest element if $\forall y \in S\ y \leq x$.
$x \in S$ is the least element if $\forall y \in S\ x \leq y$.
There may not exist least or greatest elements even for finite poset. However if they exist they are unique.

3.2 Reductions

- Reflexive reduction: $<$ is called reflexive reduction of $\leq$ if $\leq$ is reflexive closure of $<$ and $<$ is irreflexive. In other words, remove all self-loops.
- Transitive reduction:$\sqsubseteq$ is called transitive reduction of $\leq$, if $\leq$ is transitive closure of $\sqsubseteq$, and $\forall a, b\ (a \sqsubseteq b \implies \nexists m$ s.t.$m \notin \{a, b\}\ a \leq m \leq b)$

For finite posets, transitive reductions are well-defined, but may or may not exist for infinite posets.eg:(R,$\leq$)
Divisibility poset(on Z^+) has a well-defined transitive reduction even though it is infinite.

3.3 Hasse diagram

For a poset (S,$\leq$), the transitive reduction of its reflexive reduction(if they exists) has all information of the original poset(that is the poset can be reconstructed from the new relation).

3.4 Lower and upper bound

Let $T \subseteq S$. Define minimal, maximal, least and, greatest element on T.
Lower bound of T: $x \in S$ such that $x \leq y\ \forall y \in T$
Upper bound of T: $x \in S$ such that $y \leq x\ \forall y \in T$

Least upper bound for T: Least in $\{x|x$ is u.b. of T$\}$
Greatest lower bound for T: Greatest in $\{x|x$ is l.b. of T$\}$

3.5 Total/Linear order

If in a poset, every pair of elements are comparable(that is there is an edge between all pairs), then the poset is totally ordered.
Such a poset can be represented in linear fashion with all possible right pointing arrows.
Finite total order posets have unique maximal and unique minimal element.

Order Extension:
A poset (s,$\sqsubseteq$) is an extension of poset (S,$\leq$), iff
$a \leq b \implies a \sqsubseteq b$
Any finite poset can be extended to total ordering.
For infinite posets, "order extension principle" is taken as an axiom.

4 Chains and anti-chains

For a poset defined by (S,$\leq$)

Chain: $C \in S$ is said to be a chain if $\forall a, b \in C$, either $a \leq b$ or $b \leq a$. That is, $(C, \leq)$ is a total order.

Anti-chain: $A \in S$ is said to be a chain if $\forall a, b \in A$, neither $a \leq b$ nor $b \leq a$ unless $a = b$.
Subset of chain is a chain. Similar result for anti-chain.

Any chain and anti-chain can have atmost one element common($|A \cap C| = 1$). Also, a singleton set is both chain and anti-chain.

4.1 Height in a poset

Definition: For any element $a \in S$, height(a)=size of maximum chain with a as maximum. Note that height(.)$geq1$.
Eg: For poset $(Z^+, |)$, if $m = p_1^{d1} \ldots p_i^{di}$, then height(m)=1 + $\sum_{j=1}^{j=i} d_j$.
We can also define height of poset (S,$\leq$) as max{height(a)|$a \in S$}=max{$|C|; C$ is chain}.

4.2 Anti-chains from height

Let $A_h = \{a|height(a) = h\}$
Claim: A_h form anti-chains.
Proof: If any two elements are related then the height of one of them would be greater than h.
Claim: $max\{h|A_h \neq \phi\}$ = height of poset
Proof: Can be easily proved by definition of height of poset.
Note: In a finite poset since every element has a finite height, every element occurs in atleast A_h.

4.3 Mirsky's Theorem

Least number of chains required to partition a set S is the size of largest chain.
Proof: The anti-chains A_h make one such partition.
Now, consider the largest chain C. Any anti-chain cannot have more than one elements in common with C. Hence, atleast $|C|$ anti-chains are required to have elements of the C.