## CHAPTER 9

# Nonlinear Equations

One of the most frequently occurring problems in practical applications is to find the roots of equations of the form

$$f(x) = 0, \tag{9.1}$$

where $f : [a, b] \to \mathbb{R}$ is a given nonlinear function. Recall that a root of a nonlinear equation of the form (9.1) is a real number $r \in [a, b]$ such that $f(r) = 0$. It is well-known that not all nonlinear equations can be solved explicitly to obtain the exact value of the roots and hence, we need to look for methods to compute approximate values of the roots. By approximate root to (9.1), we mean a point $x^* \in [\mathbb{R}[a, b]$ such that $|r - x^*|$ is 'very near' to zero and $f(x^*) \approx 0$.

In this chapter, we introduce various iterative methods to obtain an approximation to a real root of equations of the form (9.1) with $f$ being a continuous nonlinear function. The key idea in approximating the real roots of (9.1) consists of two steps:

1. **Starting/Initialization Step:** Take one or more points (arbitrarily or following a procedure) $x_i \in [a, b]$ $(i = 0, 1, \cdots, m,\ m \in \mathbb{N})$. Consider $x_m$ as an approximation to the root of (9.1).
2. **Improvisation Step:** If $x_m$ is not 'close' to the required root, then devise a procedure to obtain another point $x_{m+1}$ that is 'more close' to the root than $x_m$.

   Repeat the improving step until we obtain a point $x_n$ $(n \geq m)$ which is 'sufficiently close' to the required root.

This process of improving the approximation to the root is called the ***iterative process*** (or ***iterative procedure***), and such methods are called ***iterative methods***. In an iterative method, we obtain a sequence of numbers $\{x_n\}$ which is expected to converge to the root of (9.1) as $n \to \infty$. We investigate conditions on the function $f$, its domain, and co-domain under which the sequence of iterates converge to a root of the equation (9.1).

We classify the iterative methods discussed in this chapter into two types, namely,

1. ***Closed Domain Methods:*** As the starting step, these methods need the knowledge of an interval in which at least one root of the given nonlinear equation exists. Further iterations include the restriction of this interval to smaller intervals in which root lies. These methods are also called ***bracketing methods***.

2. ***Open Domain Methods:*** The $x_i$'s mentioned in the starting step above are chosen arbitrarily and the consecutive iterations are based on a formula.

In the case of closed domain methods, the difficult part is to locate an interval containing a root. But, once this is done, the iterative sequence will surely converge as we will see in Section 9.1. In this section, we discuss two closed domain methods, namely, the *bisection method* and the *regula-falsi method*. In the case of open domain methods, it is easy at the starting step as we can choose the $x_i$'s arbitrarily. But, it is not necessary that the sequence converges. We discuss the *secant method*, the *Newton-Raphson method* and the *fixed point method* in Section 9.3, which are some of the open domain methods.

## 9.1 Closed Domain Methods

The idea behind the closed domain methods is to start with an interval (denoted by $[a_0, b_0]$) in which there exists at least one root of the given nonlinear equation and then reduce the length of this interval iteratively with the condition that there is at least one root of the equation at each iteration.

Note that the initial interval $[a_0, b_0]$ can be obtained using the intermediate value theorem (as we always assume that the nonlinear function $f$ is continuous) by checking the condition that

$$f(a_0)f(b_0) < 0.$$

That is, $f(a_0)$ and $f(b_0)$ are of opposite sign. The closed domain methods differ from each other only by the way they go on reducing the length of this interval at each iteration.

In the following subsections we discuss two closed domain methods, namely, (i) the *bisection method* and (ii) the *regula-falsi* method.

### 9.1.1 Bisection Method

The most simple way of reducing the length of the interval is to sub-divide the interval into two equal parts and then take the sub-interval that contains a root of the equation and discard the other part of the interval. This method is called the ***bisection method***. Let us explain the procedure of generating the first iteration of this method.

**Step 1:** Define $x_1$ to be the mid-point of the interval $[a_0, b_0]$. That is,

$$x_1 = \frac{a_0 + b_0}{2}.$$

**Step 2:** Now, exactly one of the following two statements hold.

1. $x_1$ solves the nonlinear equation. That is, $f(x_1) = 0$.
2. Either $f(a_0)f(x_1) < 0$ or $f(b_0)f(x_1) < 0$.

If case (1) above holds, then $x_1$ is a required root of the given equation $f(x) = 0$ and therefore we stop the iterative procedure. If $f(x_1) \neq 0$, then case (2) holds as $f(a_0)$ and $f(b_0)$ are already of opposite signs. In this case, we define a subinterval $[a_1, b_1]$ of $[a_0, b_0]$ as follows.

$$[a_1, b_1] = \begin{cases} [a_0, x_1], & \text{if } f(a_0)f(x_1) < 0, \\ [x_1, b_0], & \text{if } f(b_0)f(x_1) < 0. \end{cases}$$

The outcome of the first iteration of the bisection method is the interval $[a_1, b_1]$ and the first member of the corresponding iterative sequence is the real number $x_1$. Observe that

- the length of the interval $[a_1, b_1]$ is exactly half of the length of $[a_0, b_0]$ and
- $[a_1, b_1]$ has at least one root of the nonlinear equation $f(x) = 0$.

Similarly, we can obtain $x_2$ and $[a_2, b_2]$ as the result of the second iteration and so on.

We now present the algorithm for the bisection method.

**Algorithm 9.1.1 [Bisection Method].**

**Hypothesis:** There exists an interval $[a_0, b_0]$ such that the function $f : [a_0, b_0] \to \mathbb{R}$ is continuous, and the numbers $f(a_0)$ and $f(b_0)$ have opposite signs.

**Algorithm:**
**Step 1:** For $n = 0, 1, 2, \cdots$, define the iterative sequence of the bisection method as

$$x_{n+1} = \frac{a_n + b_n}{2},$$

which is the midpoint of the interval $[a_n, b_n]$.

**Step 2:** One of the following two cases hold.

1. $x_{n+1}$ solves the nonlinear equation. That is, $f(x_{n+1}) = 0$.
2. Either $f(a_n)f(x_{n+1}) < 0$ or $f(b_n)f(x_{n+1}) < 0$.

Define the subinterval $[a_{n+1}, b_{n+1}]$ of $[a_n, b_n]$ as follows.

$$[a_{n+1}, b_{n+1}] = \begin{cases} [a_n, x_{n+1}], & \text{if } f(a_n)f(x_{n+1}) < 0, \\ [x_{n+1}, b_n], & \text{if } f(b_n)f(x_{n+1}) < 0. \end{cases}$$

**Step 3:** Stop the iteration if one of the following happens:

- the case (1) in step 2 holds. Then declare the value of $x_{n+1}$ as the required root.
- $(b_{n+1} - a_{n+1})$ is sufficiently small (less than a pre-assigned positive quantity). Then declare the value of $x_{n+2}$ as the required root up to the desired accuracy.

If any of the above stopping criteria does not hold, then repeat step 1 with $n$ replaced by $n + 1$. Continue this process till one of the above two stopping criteria is fulfilled.

**Remark 9.1.2.**

In practice, one may also use any of the stopping criteria listed in Section 9.2, either single or multiple criteria.

Assuming that, for each $n = 1, 2, \cdots$, the number $x_n$ is not a root of the nonlinear equation $f(x) = 0$, we get a sequence of real numbers $\{x_n\}$. The question is whether this sequence converges to a root of the nonlinear equation $f(x) = 0$. We now discuss the *error estimate* and convergence of the iterative sequence generated by the bisection method.

**Theorem 9.1.3 [Convergence and Error Estimate of Bisection Method].**

**Hypothesis:** Let $f : [a_0, b_0] \to \mathbb{R}$ be a continuous function such that the numbers $f(a_0)$ and $f(b_0)$ have opposite signs.

**Conclusion:**

- There exists an $r \in (a_0, b_0)$ such that $f(r) = 0$ and the iterative sequence $\{x_n\}$ of the bisection method converges to $r$.

- For each $n = 0, 1, 2, \cdots$ , we have the following *error estimate*

$$|x_{n+1} - r| \le \left(\frac{1}{2}\right)^{n+1} (b_0 - a_0). \tag{9.2}$$

**Proof.**

It directly follows from the construction of the intervals $[a_n, b_n]$ that

$$b_n - a_n = \frac{1}{2}(b_{n-1} - a_{n-1}) = \cdots = \left(\frac{1}{2}\right)^n (b_0 - a_0).$$

As a consequence, we get

$$\lim_{n\to\infty} (b_n - a_n) = 0.$$

By algebra of limits of sequences, we get

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n.$$

Since for each $n = 0, 1, 2, \cdots$, the number $x_{n+1}$ is the mid-point of the interval $[a_n, b_n]$, we also have

$$a_n < x_{n+1} < b_n.$$

Now by sandwich theorem for sequences, we conclude that the sequence $\{x_n\}$ of mid-points also converges to the same limit as the sequences of end-points. Thus we have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \lim_{n\to\infty} x_n = r \text{ (say)}. \tag{9.3}$$

Since for each $n = 0, 1, 2, \cdots$, we have $f(a_n)f(b_n) < 0$, applying limits on both sides of the inequality and using the continuity of $f$, we get

$$f(r)f(r) \le 0, \tag{9.4}$$

from which we conclude that $f(r) = 0$. That is, the sequence of mid-points $\{x_n\}$ defined by the bisection method converges to a root of the nonlinear equation $f(x) = 0$.

Since the sequences $\{a_n\}$ and $\{b_n\}$ are non-decreasing and non-increasing, respectively, for each $n = 0, 1, 2, \cdots$ , we have $r \in [a_n, b_n]$. Also, $x_{n+1}$ is the mid-point of the interval $[a_n, b_n]$. Therefore, we have

$$|x_{n+1} - r| \leq \frac{1}{2}(b_n - a_n) = \cdots = \left(\frac{1}{2}\right)^{n+1}(b_0 - a_0),$$

which is the required estimate.

**Corollary 9.1.4.**

Let $\epsilon > 0$ be given. Let $f$ satisfy the hypothesis of bisection method with the interval $[a_0, b_0]$. Let $x_n$ be as in the bisection method, and $r$ be the root of the nonlinear equation $f(x) = 0$ to which bisection method converges. Then Then

$$|x_n - r| \leq \epsilon \tag{9.5}$$

whenever $n$ satisfies

$$n \geq \frac{\log(b_0 - a_0) - \log \epsilon}{\log 2} \tag{9.6}$$

**Proof.**

By the error estimate of bisection method given by (9.2), we are sure that

$$|x_n - r| \leq \epsilon,$$

whenever $n$ is such that

$$\left(\frac{1}{2}\right)^{n}(b_0 - a_0) \leq \epsilon.$$

By taking logarithm on both sides of the last inequality, we get the desired estimate on $n$.

**Remark 9.1.5.**

The Corollary 9.1.4 tells us that if we want an approximation $x_n$ to the root $r$ of the given equation such that the absolute error is less than a pre-assigned positive quantity, then it is enough to perform $n$ iterations, where $n$ is the least integer that satisfies the inequality (9.6). It is interesting to observe that to obtain this $n$, we don't need to know the root $r$.

However, it is not always true that the smallest $n$ satisfying the inequality (9.6) is the smallest $n$ for which (9.5) is satisfied (see Example 9.1.6).

**Example 9.1.6.**

Let us find an approximate root to the nonlinear equation

$$\sin x + x^2 - 1 = 0$$

using bisection method so that the resultant absolute error is at most $\epsilon = 0.125$.

To apply bisection method, we must choose an interval $[a_0, b_0]$ such that the function

$$f(x) = \sin x + x^2 - 1$$

satisfies the hypothesis of bisection method. Note that $f$ satisfies hypothesis of bisection on the interval $[0, 1]$. In order to achieve the required accuracy, we should first decide how many iterations are needed. The inequality (9.6), says that required accuracy is achieved provided $n$ satisfies

$$n \geq \frac{\log(1) - \log(0.125)}{\log 2} = 3$$

Thus we have to compute $x_3$. We will do it now.

**Iteration 1:** We have $a_0 = 0$, $b_0 = 1$. Thus $x_1 = 0.5$. Since,

$$f(x_1) = -0.27 < 0, \ f(0) < 0, \text{ and } f(1) > 0,$$

we take $[a_1, b_1] = [x_1, b_0] = [0.5, 1]$.

**Iteration 2:** The mid-point of $[0.5, 1]$ is $x_2 = 0.75$. Since

$$f(x_2) = 0.24 > 0, \ f(0.5) < 0, \text{ and } f(1) > 0,$$

we take $[a_2, b_2] = [a_1, x_2] = [0.5, 0.75]$.

**Iteration 3:** The mid-point of $[0.5, 0.75]$ is $x_3 = 0.625$.

The true value of the root $r \approx 0.636733$ and $|x_3 - r| \approx 0.011733 < 0.125$.

However, the required accuracy is achieved by $x_2$, *i.e.*, $|x_2 - r| \approx 0.1133$, which is also less than 0.125.

**Remark 9.1.7 [Comments on Bisection method].**

1. Note that the mid-point of an interval $[a, b]$ is precisely the $x - coordinate$ of the point of intersection of the line joining the points $(a, \operatorname{sgn}(f(a)))$ and $(b, \operatorname{sgn}(f(b)))$ with the $x$-axis.

2. Let $f$, and the interval $[a_0, b_0]$ satisfy the hypothesis of bisection method. Even if the nonlinear equation $f(x) = 0$ has more than one root in the interval $[a, b]$, the bisection method chooses the root that it tries to approximate. In other words, once we fix the initial interval $[a_0, b_0]$, the bisection method takes over and we cannot control it to find some specific root than what it chooses to find.

3. Given a function $f$, choosing an interval $[a, b]$ such that $f(a)f(b) < 0$ is crucial to bisection method. There is no general procedure to find such an interval. This is one of the main drawbacks of bisection method.

4. Bisection method cannot be used to find zero of functions for which graph touches $x$-axis but does not cross $x$-axis.

5. There is a misconception about bisection method's order of convergence (see Chapter 1 for the definition), which is claimed to be 1. However there is no proof of $|x_{n+1} - r| \le c|x_n - r|$. If the sequence $\{x_n\}$ converges linearly to $r$, then we get
$$|x_{n+1} - r| \le c^{n+1}|x_0 - r|.$$
In the case of bisection method, we have this inequality with $c = 1/2$, which is only a necessary condition for the convergence to be linear but not a sufficient condition.

### 9.1.2 Regula-falsi Method

The regula-falsi method is similar to the bisection method. Although the bisection method (discussed in the previous subsection) always converges to the root, the convergence is very slow, especially when the length of the initial interval $[a_0, b_0]$ is very large and the equation has the root very close to the one of the end points. This is because, at every iteration, we are subdividing the interval $[a_n, b_n]$ into two equal parts and taking the mid-point as $x_{n+1}$ (the $(n+1)^{\text{th}}$ member of the iterative sequence). Therefore, it takes several iterations to reduce the length of the interval to a very small number, and as a consequence the distance between the root and $x_{n+1}$.

The *regula-falsi method* differs from the bisection method only in the choice of $x_{n+1}$ in the interval $[a_n, b_n]$ for each $n = 0, 1, 2, \cdots$. Instead of taking the midpoint of the interval, we now take the $x$-coordinate of the point of intersection of the line joining the points $(a_n, f(a_n))$ and $(b_n, f(b_n))$ with the $x$-axis. Let us now explain the procedure of generating the first iteration of this method.

**Step 1:** Assume the hypothesis of the bisection method and let $[a_0, b_0]$ be the initial interval. The line joining the points $(a_0, f(a_0))$ and $(b_0, f(b_0))$ is given by

$$y = f(a_0) + \frac{f(b_0) - f(a_0)}{b_0 - a_0}(x - a_0),$$

The first member $x_1$ of the regula-falsi iterative sequence is the $x$-coordinate of the point of intersection of the above line with the $x$-axis. Therefore, $x_1$ satisfies the equation

$$f(a_0) + \frac{f(b_0) - f(a_0)}{b_0 - a_0}(x_1 - a_0) = 0$$

and is given by

$$x_1 = a_0 - f(a_0)\frac{b_0 - a_0}{f(b_0) - f(a_0)},$$

which can also be written as

$$x_1 = \frac{a_0 f(b_0) - b_0 f(a_0)}{f(b_0) - f(a_0)}.$$

**Step 2:** Now, exactly one of the following two statements hold.

1. $x_1$ solves the nonlinear equation. That is, $f(x_1) = 0$.
2. Either $f(a_0)f(x_1) < 0$ or $f(b_0)f(x_1) < 0$.

If case (1) above holds, then $x_1$ is a required root of the given equation $f(x) = 0$ and therefore we stop the iterative procedure. If $f(x_1) \neq 0$, then case (2) holds as $f(a_0)$ and $f(b_0)$ are already of opposite signs. We now define a subinterval $[a_1, b_1]$ of $[a_0, b_0]$ as follows.

$$[a_1, b_1] = \begin{cases} [a_0, x_1], & \text{if } f(a_0)f(x_1) < 0, \\ [x_1, b_0], & \text{if } f(b_0)f(x_1) < 0. \end{cases}$$

The outcome of the first iteration of the regula-falsi method is the interval $[a_1, b_1]$ and the first member of the corresponding iterative sequence is the real number $x_1$. Observe that

- the length of the interval $[a_1, b_1]$ may be (although not always) much less than half of the length of $[a_0, b_0]$ and
- $[a_1, b_1]$ has at least one root of the nonlinear equation $f(x) = 0$.

We now summarize the regula-falsi method.

**Algorithm 9.1.8 [Regula-falsi method].**

**Hypothesis:** Same as bisection method.

**Algorithm:**

**Step 1:** For $n = 0, 1, 2, \cdots$, define the iterative sequence of the regula-falsi method as

$$x_{n+1} = a_n - f(a_n)\frac{b_n - a_n}{f(b_n) - f(a_n)} \tag{9.7}$$

or

$$x_{n+1} = \frac{a_n f(b_n) - b_n f(a_n)}{f(b_n) - f(a_n)}, \tag{9.8}$$

which is the $x$-coordinate of the point of intersection of the line joining the points $(a_n, f(a_n))$ and $(b_n, f(b_n))$ (obtained at the $n^{\text{th}}$ iteration) with the $x$-axis.

**Step 2:** One of the following two cases hold.

1. $x_{n+1}$ solves the nonlinear equation. That is, $f(x_{n+1}) = 0$.
2. Either $f(a_n)f(x_{n+1}) < 0$ or $f(b_n)f(x_{n+1}) < 0$.

Define the subinterval $[a_{n+1}, b_{n+1}]$ of $[a_n, b_n]$ as follows.

$$[a_{n+1}, b_{n+1}] = \begin{cases} [a_n, x_{n+1}], & \text{if } f(a_n)f(x_{n+1}) < 0, \\ [x_{n+1}, b_n], & \text{if } f(b_n)f(x_{n+1}) < 0. \end{cases}$$

**Step 3:** Stop the iteration if the case (1) in step 2 holds and declare the value of $x_{n+1}$ as the required root. Otherwise repeat step 1 with $n$ replaced by $n + 1$.

Continue this process till a desired accuracy is achieved.

**Remark 9.1.9 [Stopping criteria].**

Unlike in the case of bisection method, there is no clear way of stopping the iteration of regula-falsi method as the length of the interval $[a_n, b_n]$ obtained at the $(n+1)^{\text{th}}$ iteration may not converge to zero as $n \to \infty$. This situation occurs especially when the function $f$ is concave or convex in the interval $[a_0, b_0]$ as illustrated in Example 9.1.11.

**Remark 9.1.10 [Important observations].**

Assuming that, for each $n = 1, 2, \cdots$, the number $x_n$ is not a root of the nonlinear equation $f(x) = 0$, we get a sequence of real numbers $\{x_n\}$. The question is whether this sequence converges to a root of the nonlinear equation $f(x) = 0$. Before addressing this question, let us consolidate the information that we have so far.

1. The sequence of left end points of the intervals $[a_n, b_n]$ is a non-decreasing sequence that is bounded above by $b$. That is,
$$a_0 \leq a_1 \leq \cdots \leq a_n \cdots \leq b_0.$$
Hence the sequence $\{a_n\}$ has a limit, *i.e.,* $\lim_{n\to\infty} a_n$ exists.

2. The sequence of right end points of the intervals $[a_n, b_n]$ is a non-increasing sequence that is bounded below by $a$. That is,
$$b_0 \geq b_1 \geq \cdots \geq b_n \cdots \geq a_0.$$
Hence the sequence $\{b_n\}$ has a limit, *i.e.,* $\lim_{n\to\infty} b_n$ exists.

3. Since $a_n < b_n$ for all $n = 0, 1, 2, \cdots$, we conclude that
$$\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n.$$
If the lengths of the intervals obtained in regula-falsi method tend to zero, then we have
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n,$$
in which case, we also have by sandwich theorem
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} b_n,$$
as $a_n < x_{n+1} < b_n$ for all $n = 0, 1, 2, \cdots$. In this case, the common limit will be a root of the nonlinear equation, as is the case with bisection method.

It is important to observe that, it may happen that the lengths of the subintervals chosen by regula-falsi method do not go to zero. In other words, if $a_n \to \alpha$ and $b_n \to \beta$, then it may happen that $\alpha < \beta$. Let us illustrate this case by the following example.

**Example 9.1.11.**

Consider the nonlinear equation

$$e^x - 2 = 0.$$

Note that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = e^x - 2$ satisfies the hypothesis of the regula-falsi method on the interval $[0, 1]$. Let us start the iteration with the initial interval $[a_0, b_0] = [0, 1]$.

**Iteration 1:** Using the formula (9.7), we get

$$x_1 = \frac{1}{e - 1}.$$

Since $f(x_1) \approx -0.21043$, the subinterval chosen by regula-falsi method is the interval $[a_1, b_1] = [x_1, 1]$.

**Iteration 2:** We have $x_2 \approx 0.67669$. Since $f(x_2) \approx -0.032645$, the subinterval chosen by regula falsi method is the interval $[a_2, b_2] = [x_2, 1]$.

By continuing these computations, we see that the subinterval at every iteration chosen by regula-falsi method is the right half of the interval defined in the previous iteration. In other words, for $n = 1, 2, \cdots$, we have $[a_n, b_n] = [x_n, 1]$. This is clearly seen geometrically as the function $f$ is convex. A similar situation occurs when the given function $f$ is concave (at least in the initial interval $[a_0, b_0]$). We checked this to be true for $n = 1, 2$ and a proof for a general $n$ is left as an exercise.

Note in this example that $\beta = 1$ and $\alpha \leq r \neq 1$, where $r$ is the root of the given equation $f(x) = 0$ to which the regula-falsi method is expected to converge.

The last example ruled out any hopes of proving that $\alpha = \beta$. However, it is true that the sequence $\{x_n\}$ converges to a root of the nonlinear equation $f(x) = 0$. We omit the theorem and its proof for this course.

**Example 9.1.12.**

Let us find an approximate root to the nonlinear equation

$$\sin x + x^2 - 1 = 0$$

using regula-falsi method. Exact root is approximately 0.637.

We choose the initial interval as $[a_0, b_0] = [0, 1]$ as done in Example 9.1.6.

**Iteration 1:** We have $a_0 = 0$, $b_0 = 1$. Thus $x_1 = 0.54304$. Since $f(x_1) = -0.18837 < 0$, $f(0) < 0$, and $f(1) > 0$, we take $[a_1, b_1] = [x_1, b_0] = [0.54304, 1]$.

**Iteration 2:** Using the regula-falsi formula (9.7), we get $x_2 = 0.62662$. Since $f(x_2) = -0.020937 < 0$, $f(0.54304) < 0$, and $f(1) > 0$, we take $[a_2, b_2] = [x_2, 1] = [0.62662, 1]$.

**Iteration 3:** Using the regula-falsi formula (9.7), we get $x_3 = 0.63568$. Since $f(x_3) = -0.0021861 < 0$, $f(0.62662) < 0$, and $f(1) > 0$, we take $[a_3, b_3] = [x_3, 1] = [0.63658, 1]$.

**Iteration 4:** Using the regula-falsi formula (9.7), we get $x_4 = 0.63662$.
and so on.

In this example also, we observe that the lengths of the intervals $[a_n, b_n]$ do not seem to be tending to zero as $n \to \infty$. However, we see that the sequence $\{x_n\}$ is approaching the root of the given equation.

## 9.2 Stopping Criteria

The outcome of any iterative method for a given nonlinear equation is a sequence of real numbers that is expected to converges to a root of the equation. When we implement such a methods on a computer, we cannot go on computing the iterations indefinitely and needs to stop the computation at some point. It is desirable to stop computing the iterations when the $x_n$'s are reasonably close to an exact root $r$ for a sufficiently large $n$. In other words, we want to stop the computation at the $n^{\text{th}}$ iteration when the computed value is such that

$$|x_n - r| < \epsilon$$

for some pre-assigned positive number $\epsilon$.

In general, we don't know the root $r$ of the given nonlinear equation to which the iterative sequence is converging. Therefore, we have no idea of when to stop the iteration as we have seen in the case of regula-falsi method. In fact, this situation will be there for any open domain methods discussed in the next section. An alternate way is to look for some criteria that does not use the knowledge of the root $r$, but gives a rough idea of how close we are to this root. Such a criteria is called a ***stopping criteria***. We now list some of the commonly used stopping criteria for iterative methods to nonlinear equations.

**Stopping Criterion 1:** Fix a $K \in \mathbb{N}$, and ask the iteration to stop after finding $x_K$.

This criterion is borne out of fatigue, as it clearly has no mathematical reason why the $K$ fixed at the beginning of the iteration is more important than any other natural number! If we stop the computation using this criterion, we will declare $x_K$ to be the approximate root to the nonlinear equation $f(x) = 0$.

**Stopping Criterion 2:** Fix a real number $\epsilon > 0$ and a natural number $N$. Ask the iteration to stop after finding $x_k$ such that

$$|x_k - x_{k-N}| < \epsilon.$$

One may interpret this stopping criterion by saying that there is 'not much' improvement in the value of $x_k$ compared to a previous value $x_{k-N}$. If we stop the computation using this criterion, we will declare $x_k$ to be the approximate root of the nonlinear equation $f(x) = 0$.

It is more convenient to take $N = 1$ in which case, we get the stopping criteria

$$|x_k - x_{k-1}| < \epsilon.$$

**Stopping Criterion 3:** Fix a real number $\epsilon > 0$ and a natural number $N$. Ask the iteration to stop after finding $x_k$ such that

$$\left|\frac{x_k - x_{k-N}}{x_k}\right| < \epsilon.$$

If we stop the computation using this criterion, we will declare $x_k$ to be the approximate root to the nonlinear equation $f(x) = 0$.

As in the above case, it is convenient to take $N = 1$.

**Stopping Criterion 4:** Fix a real number $\epsilon > 0$ and ask the iteration to stop after finding $x_k$ such that

$$|f(x_k)| < \epsilon.$$

If we stop the computation using this criterion, we will declare $x_k$ to be the approximate root to the nonlinear equation $f(x) = 0$. Sometimes the number $|f(x_k)|$ is called the ***residual error*** corresponding to the approximate root $x_k$ of the nonlinear equation $f(x) = 0$.

In practice, one may use any of the stopping criteria listed above, either single or multiple criteria.

**Remark 9.2.1.**

We can also use any of the above stopping criteria in bisection method.

## 9.3 Open Domain Methods

In the closed domain methods described in the previous section, we have seen that the iterative sequences always converge to a root of the nonlinear equation. However, initially to start the iteration, we need to give an interval where at least one root of the equation is known to exist. In many practical situations it may be very difficult to obtain this interval manually. Also, it will be very expensive to find this interval using a computer program as we have to adopt a trial and error algorithm for this. Therefore, it is highly desirable to devise an iterative method that does not need this information and gives freedom to choose any arbitrary starting point as done in the iterative method for linear systems. This is the objective of an open domain method.

The open domain methods do not pre-suppose that a root is enclosed in an interval of type $[a, b]$. These methods always start with a set of initial guesses $x_0, x_1, \cdots, x_m$ distributed anywhere on the real line and tell us how to compute the values $x_{m+1}$, $x_{m+2}$, $\cdots$ . However, we may not be able to construct the sequence $\{x_n\}$ and even if the sequence is constructed, it may not converge.

In this section we study three open domain iterative methods for approximating a root of a given nonlinear equation.

### 9.3.1 Secant Method

The straightforward modification of the regula-falsi method is the well-known ***secant method***. The only difference in the secant method (when compared to the regula-falsi method) is that here we do not demand the initial guesses $a_0$ and $b_0$ to be on the either side of a root. Let us now present the algorithm of the secant method.

**Algorithm 9.3.1 [Secant method].**

**Hypothesis:** Given any initial values $x_0$ and $x_1$ (not necessarily on the either side of a root) such that $f(x_0) \neq f(x_1)$.

**Algorithm:**

**Step 1:** For $n = 1, 2, \cdots$ , the iterative sequence for secant method is given by

$$x_{n+1} = \frac{f(x_n)x_{n-1} - f(x_{n-1})x_n}{f(x_n) - f(x_{n-1})}. \tag{9.9}$$

This expression can also be written as

$$x_{n+1} = x_n - f(x_n)\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} \tag{9.10}$$

**Step 2:** Choose any one of the stopping criteria (or a combination of them) discussed in Section 9.2. If this criterion is satisfied, stop the iteration. Otherwise, repeat the step 1 by replacing $n$ with $n + 1$ until the criterion is satisfied.

Recall that $x_{n+1}$ for each $n = 1, 2, \cdots$ given by (9.9) (or (9.10)) is the $x$-coordinate of the point of intersection of the secant line joining the points $(x_{n-1}, f(x_{n-1}))$ and $(x_n, f(x_n))$ with the $x$-axis and hence the name ***secant method***.

**Remark 9.3.2.**

It is evident that the secant method fails to determine $x_{n+1}$ if we have $f(x_{n-1}) = f(x_n)$. Observe that such a situation never occurs in regula-falsi method.

**Example 9.3.3.**

Consider the equation

$$\sin x + x^2 - 1 = 0.$$

Let $x_0 = 0$, $x_1 = 1$. Then the iterations from the secant method are given by

| $n$ | $x_n$ | $\epsilon$ |
|---|---|---|
| 2 | 0.543044 | 0.093689 |
| 3 | 0.626623 | 0.010110 |
| 4 | 0.637072 | 0.000339 |
| 5 | 0.636732 | 0.000001 |

Figure 9.1 shows the iterative points $x_2$ and $x_3$ in black bullet. Recall that the exact value of the root (to which the iteration seems to converge) unto 6 significant digits is $r \approx 0.636733$. Obviously, the secant method is much faster than bisection method in this example.

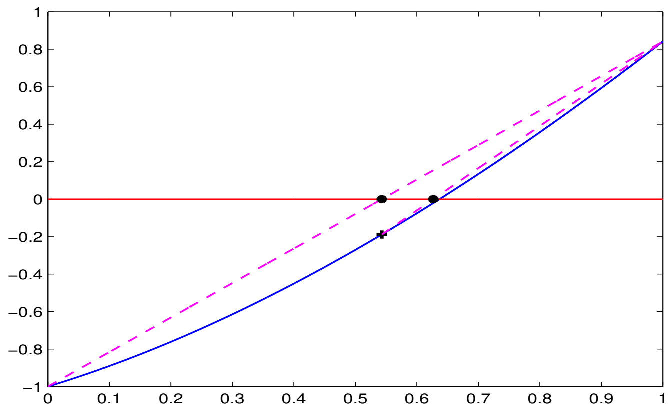


Figure 9.1: Iterative points of secant method.

We now state the convergence theorem on secant method.

**Theorem 9.3.4 [Convergence of Secant Method].**

**Hypothesis:**

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a $C^2(\mathbb{R})$ function.
2. Let $r$ be a simple root of the nonlinear equation $f(x) = 0$, that is, $f'(r) \neq 0$.

**Conclusion:** Then there exists a $\delta > 0$ such that for every $x_0, x_1 \in [r - \delta, r + \delta]$,

1. the secant method iterative sequence $\{x_n\}$ is well-defined;
2. the sequence $\{x_n\}$ belongs to the interval $[r - \delta, r + \delta]$;
3. $\lim_{n\to\infty} x_n = r$; and
4. we have

$$\lim_{n\to\infty} \frac{|x_{n+1} - r|}{|x_n - r|^{\alpha}} = \left| \frac{f''(r)}{2f'(r)} \right|^{\alpha/(\alpha+1)}, \tag{9.11}$$

where $\alpha = (\sqrt{5} + 1)/2 \approx 1.62$.

The proof of the above theorem is omitted for this course.

**Remark 9.3.5.**

The expression (9.11) implies that the order of convergence (see Chapter 1 for the definition) of the iterative sequence of secant method is 1.62.

### 9.3.2 Newton-Raphson Method

In Theorem 9.3.4, we have seen that the secant method has more than linear order of convergence. This method can further be modified to achieve quadratic convergence. To do this, we first observe that when the iterative sequence of the secant method converges, then as $n$ increases, we see that $x_{n-1}$ approaches $x_n$. Thus, for a sufficiently large value of $n$, we have

$$f'(x_n) \approx \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}},$$

provided $f$ is a $C^1$ function. Thus, if $f(x)$ is differentiable, then on replacing in (9.10), the slope of the secant by the slope of the tangent at $x_n$, we get the iteration formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{9.12}$$

and is called the ***Newton-Raphson method***.

**Remark 9.3.6 [Geometric Motivation].**

If a function $f$ is differentiable at a point $x_0$, then the tangent $(y = g(x))$ to the graph of $f$ at the point $(x_0, f(x_0))$ is given by

$$g(x) = f'(x_0)(x - x_0) + f(x_0).$$

We may assume that for $x \approx x_0$, $f(x) \approx g(x)$. This can also be interpreted as

"If a function $f$ is differentiable at a point, then the graph of $f$ looks like a straight line, for $x \approx x_0$ on zooming".

Now, if we choose the initial guess $x_0$ very close to the root $r$ of $f(x) = 0$. That is., if $r \approx x_0$, we have $g(r) \approx f(r) = 0$. This gives (approximately)

$$0 \approx f'(x_0)(r - x_0) + f(x_0).$$

Up on replacing $r$ by $x_1$ and using '=' symbol instead of '≈' symbol, we get the first iteration of the Newton-Raphson's iterative formula (9.12).

Recall in secant method, we need two initial guesses $x_0$ and $x_1$ to start the iteration. In Newton-Raphson method, we need one initial guess $x_0$ to start the iteration. The consecutive iteration $x_1$ is the $x$-coordinate of the point of intersection of the $x$-axis and the tangent line at $x_0$, and similarly for the other iterations. This geometrical interpretation of the Newton-Raphson method is clearly observed in Figure 9.2.

We now derive the Newton-Raphson method under the assumption that $f$ is a $C^2$ function.

Let $x_0$ be given. The Taylor's polynomial of degree $n = 1$ with remainder is given by

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{(x - x_0)^2}{2!} f''(\xi),$$

where $\xi$ lies between $x_0$ and $x$. When $x_0$ is very close to $x$, the last term in the above equation is smaller when compared to the other two terms on the right hand side. By neglecting this term we have

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0). \tag{9.13}$$

Notice that the graph of the function $g(x) = f(x_0) + f'(x_0)(x - x_0)$ is precisely the tangent line to the graph of $f$ at the point $(x_0, f(x_0))$. We now define $x_1$ to be the $x$-coordinate of the point of intersection of this tangent line with the $x$-coordinate. That is, the point $x_1$ is such that $g(x_1) = 0$, which gives

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}. \tag{9.14}$$

This gives the first member of the iterative sequence of the Newton-Raphson's method.

We now summarize the Newton-Raphson's method.

**Algorithm 9.3.7.**

**Hypothesis:**

1. Let the function $f$ be $C^1$ and $r$ be the root of the equation $f(x) = 0$ with $f'(r) \neq 0$.
2. The initial guess $x_0$ is chosen sufficiently close to the root $r$.

**Algorithm:**
**Step 1:** For $n = 0, 1, 2, \cdots$, the iterative sequence of the Newton-Raphson's method is given by (9.12)

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

**Step 2:** Choose any one of the stopping criteria (or a combination of them) discussed in Section 9.2. If this criterion is satisfied, stop the iteration. Otherwise, repeat the step 1 by replacing $n$ with $n + 1$ until the criterion is satisfied.

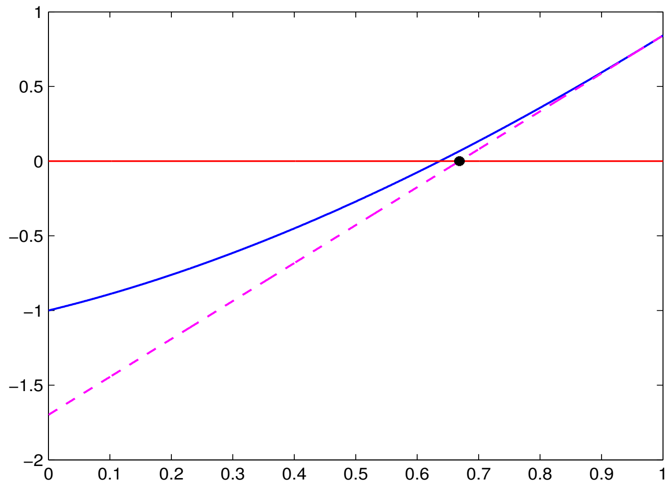


Figure 9.2: Iteration Procedure of Newton-Raphson's method for $f(x) = \sin(x) + x^2 - 1$.

**Remark 9.3.8.**

It is evident that if the initial guess $x_0$ is such that $f'(x_n) = 0$, for some $n \in \mathbb{N}$, then the Newton-Raphson method fails. Geometrically, this means that the tangent line to the graph of $f$ at the point $(x_n, f(x_n))$ is parallel to the $x$-axis. Therefore, this line never intersects $x$-axis and hence $x_{n+1}$ never exists. See Remark 9.3.2 for the failure of secant method and compare it with the present case.

**Example 9.3.9.**

Consider the equation

$$\sin x + x^2 - 1 = 0.$$

Let $x_0 = 1$. Then the iterations from the Newton-Raphson method gives

| $n$ | $x_n$ | $\epsilon$ |
|---|---|---|
| 1 | 0.668752 | 0.032019 |
| 2 | 0.637068 | 0.000335 |
| 3 | 0.636733 | 0.000000 |

Figure 9.2 shows the iterative points $x_2$ and $x_3$ in black bullet. Recall that the exact root is $x^* \approx 0.636733$. Obviously, the Newton-Raphson method is much faster than both bisection and secant methods in this example.

Let us now discuss the convergence of the Newton-Raphson method.

**Theorem 9.3.10 [Convergence of Newton-Raphson method].**

**Hypothesis:**

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a $C^2(\mathbb{R})$ function.
2. Let $r$ be a simple root of the nonlinear equation $f(x) = 0$, that is $f'(r) \neq 0$.

**Conclusion:** Then there exists a $\delta > 0$ such that for every $x_0 \in [r - \delta, r + \delta]$,

1. each term of the Newton-Raphson iterative sequence $\{x_n\}$ is well-defined;
2. the sequence $\{x_n\}$ belongs to the interval $[r - \delta, r + \delta]$;
3. $\lim_{n\to\infty} x_n = r$; and
4. we have

$$\lim_{n\to\infty} \frac{|x_{n+1} - r|}{|x_n - r|^2} = \left| \frac{f''(r)}{2f'(r)} \right|. \tag{9.15}$$

Proof of the above theorem is omitted for this course.

Theorem on Newton-Raphson method says that if we start near-by a root of the nonlinear equation, then Newton-Raphson iterative sequence is well-defined and converges. For increasing convex functions, we need not be very careful in choosing the initial guess. For such functions, the Newton-Raphson iterative sequence always converges, whatever may be the initial guess. This is the content of the next theorem.

**Theorem 9.3.11 [Convergence Result for Convex Functions].**

**Hypothesis:** Let $f : \mathbb{R} \to \mathbb{R}$ be a twice continuously differentiable function such that

1. $f$ is convex, *i.e.,* $f''(x) > 0$ for all $x \in \mathbb{R}$.
2. $f$ is strictly increasing, *i.e.,* $f'(x) > 0$ for all $x \in \mathbb{R}$.
3. there exists an $r \in \mathbb{R}$ such that $f(r) = 0$.

**Conclusion:** Then

1. $r$ is the unique root of $f(x) = 0$.
2. For every choice of $x_0$, the Newton-Raphson iterative sequence converges to $r$.

**Proof.**

**Proof of** (1)**:** Since $f$ is strictly increasing, the function cannot take the same value more than once. Thus $f(x) = 0$ has exactly one root.

**Proof of** (2)**:** The Newton-Raphson sequence is defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

The error $e_{n+1} := r - x_{n+1}$ can be written as

$$e_{n+1} \quad = \quad \frac{(r - x_n)f'(x_n) + f(x_n)}{f'(x_n)}$$

Using Taylor's theorem, we have

$$0 = f(r) = f(x_n) + (r - x_n)f'(x_n) + \frac{f''(\xi_n)}{2}(r - x_n)^2, \tag{9.16}$$

where $\xi_n$ lies between $r$ and $x_n$. Using the information from the last equation, we get the following expression for $e_{n+1}$:

$$e_{n+1} = -\frac{f''(\xi_n)}{2f'(x_n)}(r - x_n)^2. \tag{9.17}$$

Thus, we obtained

$$e_{n+1} = -\frac{f''(\xi_n)}{2f'(x_n)}e_n^2. \tag{9.18}$$

It follows that $e_{n+1} \le 0$. This implies that $r \le x_{n+1}$. Since $f$ is a strictly increasing function, we get $f(r) \le f(x_{n+1})$. Thus $f(x_{n+1}) \ge 0$ for every $n \in \mathbb{N}$. From (9.12), we get $x_{n+1} \le x_n$. That is, the sequence $\{x_n\}$ is a non-increasing sequence, and is bounded below by $r$, and hence converges. Let us denote the limit by $x^*$.

On the other hand, the sequence $\{e_n\}$ is a non-decreasing sequence, bounded above by 0. Let $e^*$ denote the limit of the sequence $\{e_n\}$.

Passing to the limit as $n \to \infty$ in the equation

$$e_{n+1} \quad = \quad e_n + \frac{f(x_n)}{f'(x_n)},$$

we get the desired result.

To following example illustrates the quadratic convergence of the Newton-Raphson method.

**Example 9.3.12.**

Start with $x_0 = -2.4$ and use Newton-Raphson iteration to find the root $r = -2.0$ of the polynomial

$$f(x) = x^3 - 3x + 2.$$

The iteration formula is

$$x_{n+1} = \frac{2x_n^3 - 2}{3x_n^2 - 3}.$$

It is easy to verify that $|r - x_{n+1}|/|r - x_n|^2 \approx 2/3$, which shows the quadratic convergence of Newton-Raphson's method.

### 9.3.3 Fixed-Point Iteration Method

In fixed point iteration method, the problem of finding a root of a nonlinear equation $f(x) = 0$ is equivalently viewed as the problem of finding a fixed point of a suitably defined function $g$.

Let us first define the notion of *fixed point* of a function.

**Definition 9.3.13 [Fixed point].**

Let $g$ be a continuous function. A point $\alpha \in \mathbb{R}$ is said to be a ***fixed point*** of $g$ if

$$\alpha = g(\alpha).$$

The idea behind the choice of $g$ is to rewrite the given nonlinear equation $f(x) = 0$ in the form $x = g(x)$ for some function $g$. In general, there may be more than one choice of $g$ with this property as illustrated by the following example.

**Example 9.3.14.**

Note that $\alpha \in \mathbb{R}$ is a root of the equation $x^2 - x - 2 = 0$ if and only if $\alpha$ is a root of each of the following equations.

1. $x = x^2 - 2$
2. $x = \sqrt{x + 2}$
3. $x = 1 + \dfrac{2}{x}$.

In other words, obtaining a root of the equation $f(x) = 0$, where $f(x) = x^2 - x - 2$ is equivalent to finding the fixed point of any one of the following functions:

1. $g_1(x) = x^2 - 2$
2. $g_2(x) = \sqrt{x+2}$
3. $g_3(x) = 1 + \dfrac{2}{x}$.

The ***fixed-point iteration method*** for finding a root of $g(x) = x$ consists of a sequence of iterates $\{x_n\}$, starting from an initial guess $x_0$, defined by

$$x_n = g(x_{n-1}) \tag{9.19}$$

The function $g$ is called the ***iteration function***.

As we saw in Example 9.3.14, for a given nonlinear equation, the iteration function is not unique. The crucial point in this method is to choose a good iteration function $g(x)$. A good iteration function should satisfy the following properties:

1. For the given starting point $x_0$, the successive approximations $x_n$ given by (9.19) can be calculated.
2. The sequence $x_1$, $x_2$, $\cdots$ converges to some point $\xi$.
3. The limit $\xi$ is a fixed point of $g(x)$, ie., $\xi = g(\xi)$.

Not every iteration function has all these properties. The following example shows that for certain iteration functions, even the sequence of iterates need not be defined.

**Example 9.3.15.**

Consider the equation

$$x^2 - x = 0.$$

This equation can be re-written as $x = \pm\sqrt{x}$. Let us take the iterative function

$$g(x) = -\sqrt{x}.$$

Since $g(x)$ is defined only for $x > 0$, we have to choose $x_0 > 0$. For this value of $x_0$, we have $g(x_0) < 0$ and therefore, $x_1$ cannot be calculated.

Therefore, the choice of $g(x)$ has to be made carefully so that the sequence of iterates can be calculated.

How to choose such an iteration function $g(x)$?

Note that $x_1 = g(x_0)$, and $x_2 = g(x_1)$. Thus $x_1$ is defined whenever $x_0$ belongs to the domain of $g$. Thus we must take the initial guess $x_0$ from the domain of $g$. For defining

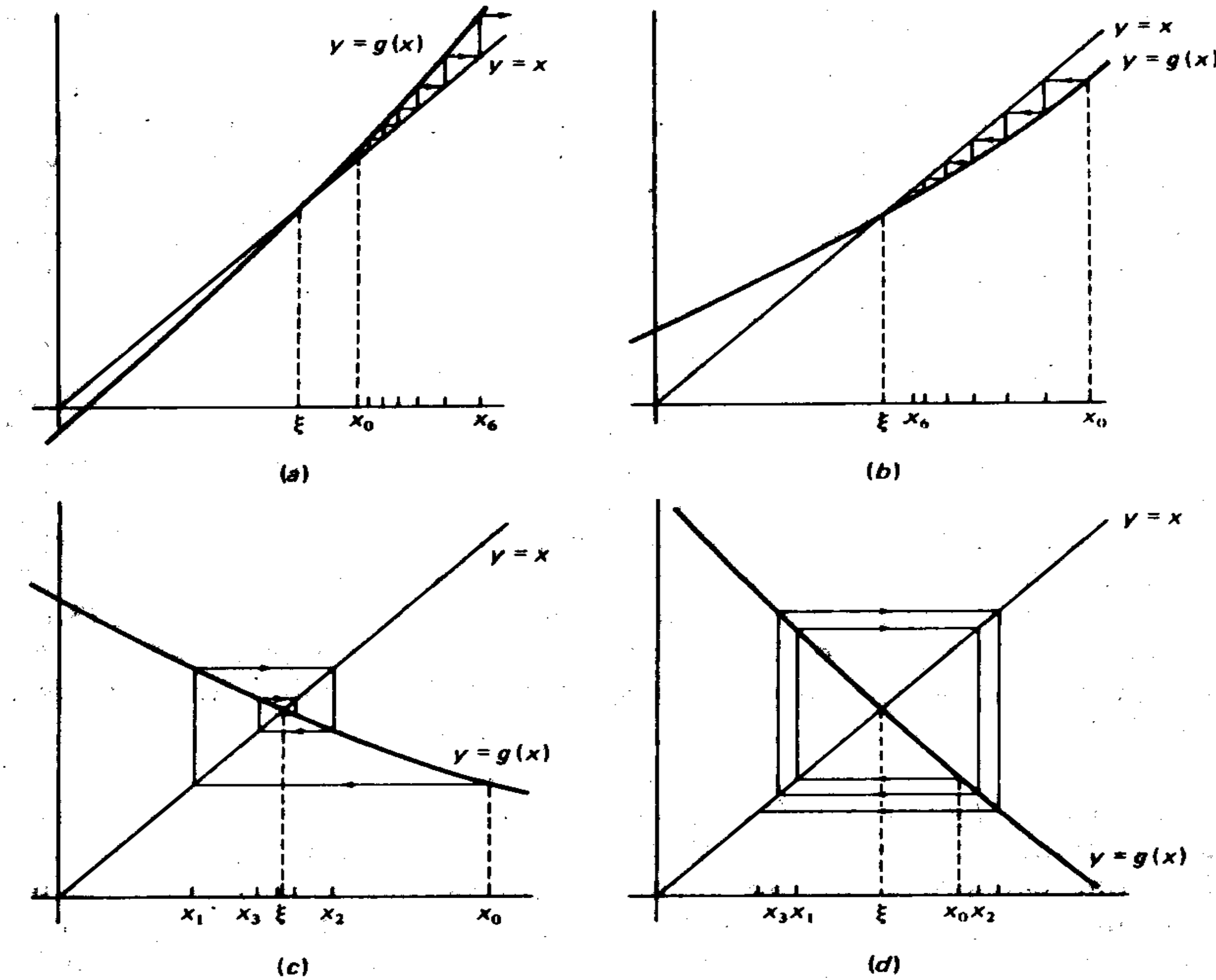


Figure 9.3: Fixed-point Iteration Procedure.

$x_2$, we need that $x_1$ is in the domain of $g$ once again. In fact the sequence is given by

$$x_0,\ g(x_0),\ g \circ g(x_0),\ g \circ g \circ g(x_0),\ \cdots$$

Thus to have a well-defined iterative sequence, we require that

Range of the function $g$ is contained in the domain of $g$.

A function with this property is called a ***self map***. We make our first assumption on the iterative function as

**Assumption 1:** $a \leq g(x) \leq b$ for all $a \leq x \leq b$.

It follows that if $a \leq x_0 \leq b$, then for all $n$, $x_n \in [a, b]$ and therefore $x_{n+1} = g(x_n)$ is defined and belongs to $[a, b]$.

Let us now discuss about the point 3. This is a natural expectation since the expression $\xi = g(\xi)$. To achieve this, we need $g(x)$ to be a continuous function. For if $x_n \to \xi$ then

$$\xi = \lim_{n\to\infty} x_n = \lim_{n\to\infty} g(x_{n-1}) = g(\lim_{n\to\infty} x_{n-1}) = g(\xi)$$

Therefore, we need

**Assumption 2:** The iterative function $g$ is continuous.

It is easy to prove that a continuous self map on a bounded interval always has a fixed point. However, the question is whether the sequence (9.19) generated by the iterative function $g$ converges, which is the requirement stated in point 2. This point is well understood geometrically. Figure 9.3(a) and Figure 9.3(c) illustrate the convergence of the fixed-point iterations whereas Figure 9.3(b) and Figure 9.3(d) illustrate the diverging iterations. In this geometrical observation, we see that when $|g'(x)| < 1$, we have convergence and otherwise, we have divergence. Therefore, we make the assumption

**Assumption 3:** The iteration function $g(x)$ is differentiable on $I = [a, b]$. Further, there exists a constant $0 < K < 1$ such that

$$|g'(x)| \le K, \quad x \in I. \tag{9.20}$$

Such a function is called the ***contraction map***.

Let us now present the algorithm of the fixed-point iteration method.

**Algorithm 9.3.16.**

**Hypothesis:** Let $g : [a, b] \to [a, b]$ be an iteration function such that Assumptions 1, 2, and 3 stated above hold.

**Algorithm:**

**Step 1:** Choose an initial guess $x_0 \in [a, b]$.

**Step 2:** Define the iteration methods as

$$x_{n+1} = g(x_n), \quad n = 0, 1, \cdots$$

**Step 3:** For a pre-assigned positive quantity $\epsilon$, check for one of the (fixed) stopping criteria discussed in Section 9.2. If the criterion is satisfied, stop the iteration. Otherwise, repeat the step 1 by replacing $n$ with $n + 1$ until the criterion is satisfied.

**Theorem 9.3.17 [Convergence Result for Fixed-Point Iteration Method].**

**Hypothesis:** Let the iterative function $g$ be chosen so that

1. $g$ is defined on the interval $[a,b]$ and $a \leq g(x) \leq b$. That is, $g$ is a self map on $[a,b]$
2. $g$ is continuously differentiable on $[a,b]$
3. $g$ is a contraction map. That is,

$$\lambda = \max_{a\leq x\leq b}|g'(x)| < 1. \tag{9.21}$$

**Conclusion:** Then

1. $x = g(x)$ has a unique root $r$ in $[a,b]$.
2. For any choice of $x_0 \in [a,b]$, with $x_{n+1} = g(x_n)$, $n = 0, 1, \cdots$,

$$\lim_{n\to\infty} x_n = r.$$

3. We further have

$$|x_n - r| \leq \lambda^n |x_0 - r| \leq \frac{\lambda^n}{1-\lambda}|x_1 - x_0| \tag{9.22}$$

and

$$\lim_{n\to\infty} \frac{r - x_{n+1}}{r - x_n} = g'(r). \tag{9.23}$$

**Proof.**

Proof for (1) is easy.

From mean-value theorem and (9.20), we have

$$|r - x_{n+1}| = |g(r) - g(x_n)| \leq \lambda |r - x_n|. \tag{9.24}$$

By induction, we have

$$|r - x_{n+1}| \leq \lambda^n |r - x_0|, \quad n = 0, 1, \cdots .$$

Since, as $n \to \infty$, $\lambda^n \to 0$, we have $x_n \to r$. Further, we have

$$\begin{aligned} |r - x_0| &= |r - x_1 + x_1 - x_0| \\ &\leq |r - x_1| + |x_1 - x_0| \\ &\leq \lambda |r - x_0| + |x_1 - x_0|. \end{aligned}$$

Then solving for $|r - x_0|$, we get (9.22).

Now we will prove the rate of convergence (9.23). From Mean-value theorem

$$\begin{aligned} r - x_{n+1} &= g(r) - g(x_n) \\ &= g'(\xi_n)(r - x_n), \quad n = 0, 1, \cdots. \end{aligned} \tag{9.25}$$

with $\xi_n$ an unknown point between $r$ and $x_n$. Since $x_n \to r$, we must have $\xi_n \to r$ and therefore,

$$\lim_{n\to\infty} \frac{r - x_{n+1}}{r - x_n} = \lim_{n\to\infty} g'(\xi_n) = g'(r).$$

This completes the proof.

**Remark 9.3.18.**

From the inequality (9.24), we see that the fixed point iteration method has linear convergence. In other words, the order of convergence of this method is at least 1.

**Example 9.3.19.**

The nonlinear equation

$$x^3 + 4x^2 - 10 = 0$$

has a unique root in $[1, 2]$. Note that the solution of each of the following fixed-point problems is a root to the given nonlinear equation.

1. $x = g_1(x)$, where $g_1(x) = x - x^3 - 4x^2 + 10$
2. $x = g_2(x)$ where $g_2 = \sqrt{\dfrac{10}{x} - 4x}$
3. $x = g_3(x)$ where $g_3 = \dfrac{1}{2}\sqrt{10 - x^3}$
4. $x = g_4(x)$ where $g_4 = \sqrt{\dfrac{10}{4 + x}}$

5. $x = g_5(x)$ where $g_5 = x - \dfrac{x^3 + 4x^2 - 10}{3x^2 + 8x}$.

We now show that among the five equivalent fixed-point formulations of the given nonlinear equation, only some of them turn out to be good iterative functions. Let us implement fixed-point iteration method with each of the five iterating functions, and compare the results which are tabulated below.

| $n$ | $x = g_1(x)$ | $x = g_2(x)$ | $x = g_3(x)$ | $x = g_4(x)$ | $x = g_5(x)$ |
|---|---|---|---|---|---|
| 0 | 1.5 | 1.5 | 1.5 | 1.5 | 1.5 |
| 1 | -0.875 | 0.8165 | 1.286953768 | 1.348399725 | 1.373333333 |
| 2 | 6.732 | 2.9969 | 1.402540804 | 1.367376372 | 1.365262015 |
| 3 | -469.7 | $\sqrt{-8.65}$ | 1.345458374 | 1.364957015 | 1.365230014 |
| 4 | $1.03 \times 10^8$ | $\times$ | 1.375170253 | 1.365264748 | 1.365230013 |
| 5 | $\times$ | $\times$ | 1.360094193 | 1.3652 | $\times$ |
| 6 | $\times$ | $\times$ | 1.367846968 | 1.365230576 | $\times$ |
| 7 | $\times$ | $\times$ | 1.363887004 | 1.365229942 | $\times$ |
| 8 | $\times$ | $\times$ | 1.365916734 | 1.365230022 | $\times$ |
| 9 | $\times$ | $\times$ | 1.364878217 | 1.365230012 | $\times$ |
| 10 | $\times$ | $\times$ | 1.365410062 | 1.365230014 | |
| 15 | $\times$ | $\times$ | 1.365223680 | 1.365230013 | |
| 20 | $\times$ | $\times$ | 1.365230236 | $\times$ | $\times$ |
| 25 | $\times$ | $\times$ | 1.365230006 | $\times$ | $\times$ |
| 30 | $\times$ | $\times$ | 1.365230013 | $\times$ | $\times$ |

From the above table, we conclude that the iterative functions $g_1$ and $g_2$ are very bad, while that given by $g_5$ is the best. However iterative functions $g_3$ and $g_4$ are also good but requires more number of iterations compared to $g_5$.

1. Let us start analyzing the iterative sequence generated by the iteration function $g_1$. In this case, the iterative sequence seems to be diverging. The main reason behind this is the fact that $g_1$ is not a contraction map for $x \in [1, 2]$, which can be seen easily. Thus the successive iterates using $g_1$ have increasing moduli, which also shows that $g_1$ is not a self map.

2. Let us consider the iterative function $g_2$. It is easy to check that $g_2$ is not a self map of $[1, 2]$ to itself. In our computation above, we see that the entire iterative sequence is not defined as one of the iterates becomes negative, when the initial guess is taken as 1.5. The exact root is approximately equal to $r = 1.365$. There is no interval containing $r$ on which $|g_2'(x)| < 1$. In fact, $g_2'(r) \approx 3.4$ and as a consequence $|g_2'(x)| > 3$ on an interval containing $r$. Thus we don't expect a convergent iterative sequence even if the sequence is well-defined!

3. Regarding the iterative function $g_3$, note that this iteration function is a decreasing function on $[1, 2]$ as

$$g_3'(x) = -\frac{3x^2}{4\sqrt{10 - x^3}} < 0$$

on $[1, 2]$. Thus maximum of $g_3$ is attained at $x = 1$, which is 1.5; and the minimum is attained at $x = 2$ which is approximately equal to 0.707. Thus $g_3$ is a self map of $[1, 2]$. But $|g_3'(2)| \approx 2.12$. Thus the condition

$$|g_3'(x)| \leq \lambda < 1$$

is violated on the interval $[1, 2]$. However by restricting to a smaller interval $[1, 1.5]$, we get that $g_3$ is a self map of $[1, 1.5]$ as $g_3$ is still decreasing function on $[1, 1.5]$ and $g_3(1) = 1.5$, $g_3(1.5) \approx 1.28$, and also

$$|g_3'(x)| \leq |g_3'(1.5)| \approx 0.66.$$

Thus $g_3$ satisfies the hypothesis of theorem on fixed-point iteration, and as expected the sequence of iterates converge.

4. Note that the iterative function $g_4$ is given by

$$g_4(x) = \sqrt{\frac{10}{4 + x}}.$$

We have

$$|g_4'(x)| = \left|\frac{-5}{\sqrt{10}(4 + x)^{3/2}}\right| \leq \frac{5}{\sqrt{10}(5)^{3/2}} < 0.15 \text{ for all } x \in [1, 2].$$

The bound obtained on the derivative of $g_4$ is considerably smaller when compared to that of $g_3$, which explains why the convergence is faster for the iterates obtained using $g_4$, when compared with those obtained by using $g_3$.

5. Finally, let us consider the iterative sequence generated by $g_5$. This sequence converges much more faster compared to $g_3$ and $g_4$. Note that the fixed-point iterative sequence generated by $g_5$ is nothing but the iterative sequence of Newton-Raphson method for the root of the nonlinear equation $f(x) = 0$.

**Example 9.3.20.**

Consider the equation

$$\sin x + x^2 - 1 = 0.$$

Take the initial interval as $[0, 1]$. There are at least three possible choices for the iteration function, namely,

1. $g_1(x) = \sin^{-1}(1 - x^2)$,
2. $g_2(x) = -\sqrt{1 - \sin x}$,
3. $g_3(x) = \sqrt{1 - \sin x}$.

Here we have

$$g_1'(x) = \frac{-2}{\sqrt{2 - x^2}}.$$

We can see that $|g_1'(x)| > 1$. Taking $x_0 = 0.8$ and denoting the absolute error as $\epsilon$, we have

| $n$ | $g_1(x)$ | $\epsilon$ |
|---|---|---|
| 0 | 0.368268 | 0.268465 |
| 1 | 1.043914 | 0.407181 |
| 2 | -0.089877 | 0.726610 |
| 3 | 1.443606 | 0.806873 |

The sequence of iterations is diverging as expected.

If we take $g_2(x)$, clearly the assumption 1 is violated and therefore is not suitable for the iteration process.

Let us take $g_3(x)$. Here, we have

$$g_3'(x) = \frac{-\cos x}{2\sqrt{1 - \sin x}}.$$

Therefore,

$$\begin{aligned} |g_3'(x)| &= \frac{\sqrt{1 - \sin^2 x}}{2\sqrt{1 - \sin x}} \\ &= \frac{\sqrt{1 + \sin x}}{2} \\ &\leq \frac{1}{\sqrt{2}} < 1. \end{aligned}$$

Taking $x_0 = 0.8$ and denoting the absolute error as $\epsilon$, we have

| $n$ | $g_3(x)$ | $\epsilon$ |
|---|---|---|
| 0 | 0.531643 | 0.105090 |
| 1 | 0.702175 | 0.065442 |
| 2 | 0.595080 | 0.041653 |
| 3 | 0.662891 | 0.026158 |

The sequence is converging.

## 9.4 Comparison and Pitfalls of Iterative Methods

**Closed domain methods: Bisection and Regula falsi methods**

1. In both these methods, where we are trying to find a root of the nonlinear equation $f(x) = 0$, we are required to find an interval $[a, b]$ such that $f(a)$ and $f(b)$ have opposite signs. This calls for a complete study of the function $f$. In case the function has no roots on the real line, this search for an interval will be futile. There is no way to realize this immediately, thus necessitating a full fledged understanding of the funtion $f$.

2. Once it is known that we can start these methods, then surely the iterative sequences converge to a root of the nonlinear equation.

3. In bisection method, we can keep track of the error by means of an upper bound. But such a thing is not available for regula falsi method. In general convergence of bisection method iterates is slower compared to that of regula falsi method iterates.

4. If the initial interval $[a, b]$ is such that the equation $f(x) = 0$ has a unique root in it, then both the methods converge to the root. If there are more than one roots in $[a, b]$, then usually both methods find different roots. The only way of finding the desired root is to find an interval in which there is exactly one root to the nonlinear equation.

**Open domain methods: Secant, Newton-Raphson, and Fixed point methods**

1. The main advantage of the open domain methods when compared to closed domain methods is that we don't need to locate a root in an interval. Rather, we can start the iteration with an arbitrarily chosen initial guess(es).
2. The disadvantage of the open domain methods is that the iterative sequence may not be well-defined for all initial guesses. Even if the sequence is well-defined, it may not converge. Even if it converges, it may not converge to a specific root of

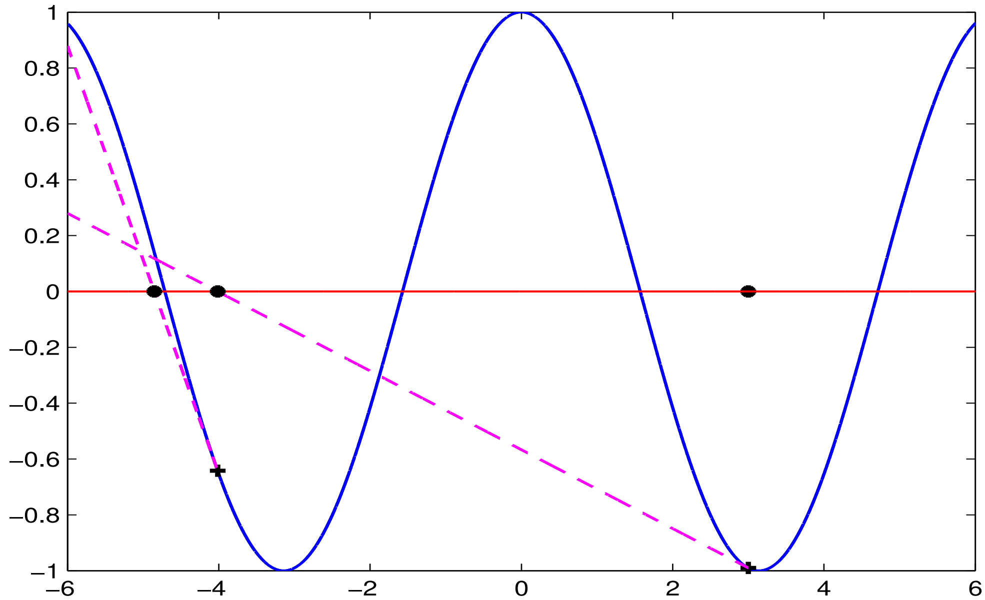


Figure 9.4: Iteration Procedure of Newton-Raphson's method for $f(x) = \cos(x)$.

interest.

3. In situations where both open and closed domain methods converge, open domain methods are generally faster compared to closed domain methods. Especially, Newton-Raphson's method is faster than other methods as the order of convergence of this method is 2. In fact, this is the fastest method known today.

4. In these methods, it may happen that we are trying to find a particular root of the nonlinear equation, but the iterative sequence may converge to a different root. Thus we have to be careful in choosing the initial guess. If the initial guess is far away from the expected root, then there is a danger that the iteration converges to another root of the equation.
In the case of Newton-Raphson's method, this usually happens when the slope $f'(x_0)$ is small and the tangent line to the curve $y = f(x)$ is nearly parallel to the $x$-axis. Similarly, in the case of secant method, this usually happens when the slope of the secant joining $(x_0, f(x_0))$ and $(x_1, f(x_1))$ is nearly parallel to the $x$-axis.
For example, if

$$f(x) = \cos x$$

and we seek the root $x^* = \pi/2$ and start with $x_0 = 3$, calculation reveals that

$$x_1 = -4.01525, \quad x_2 = -4.85266, \cdots ,$$

and the iteration converges to $x = -4.71238898 \approx -3\pi/2$. The iterative sequence for $n = 1, 2$ is depicted in Figure 9.4.

5. Suppose that $f(x)$ is positive and monotone decreasing on an unbounded interval $[a, \infty)$ and $x_0 > a$. Then the sequence might diverge. For example, if $f(x) = xe^{-x}$ and $x_0 = 2$, then

$$x_1 = 4.0, \quad x_2 = 5.333333..., \quad \cdots , p_{15} = 19.72354..., \cdots .$$

and the sequence diverges to $+\infty$. This particular function has another suprising problem. The value of $f(x)$ goes to zero rapidly as $x$ gets large, for example $f(x_{15}) = 0.0000000536$, and it is possible that $p_{15}$ could be mistaken for a root as per the residual error. Thus, using residual error for iterative methods nonlinear equations is often not preferred.

6. The method can stuck in a cycle. For instance, let us compute the iterative sequence generated by the Newton-Raphson's method for the function $f(x) = x^3 - x - 3$ with the initial guess $x_0 = 0$. The iterative sequence is

$$x_1 = -3.00, \quad x_2 = -1.961538, \quad x_3 = -1.147176, \quad x_4 = -0.006579,$$

$$x_5 = -3.000389, \quad x_6 = -1.961818, \quad x_7 = -1.147430, \cdots$$

and we are stuck in a cycle where $x_{n+4} \approx x_k$ for $k = 0, 1, \cdots$. But if we start with a value $x_0$ sufficiently close with the root $r \approx 1.6717$, then the convergence is obtained. The proof of this is left as an exercise.

7. If $f(x)$ has no real root, then there is no indication by these methods and the iterative sequence may simply oscillate. For example compute the Newton-Raphson iteration for

$$f(x) = x^2 - 4x + 5.$$

## 9.5 Exercises

### Bisection Method and Regula-falsi Method

In the following problems on bisection method, the notation $x_n$ is used to denote the mid-point of the interval $[a_{n-1}, b_{n-1}]$, and is termed as the bisection method's $n^{\text{th}}$ iterate (or simply, the $n^{\text{th}}$ iterate, as the context of bisection method is clear)

1. Let bisection method be used to solve the nonlinear equation

$$2x^6 - 5x^4 + 2 = 0$$

   starting with the initial interval $[0, 1]$. In order to approximate a solution of the nonlinear equation with an absolute error less than or equal to $10^{-3}$, what is the number of iterations required as per the error estimate of the bisection method? Also find the corresponding approximate solution.

2. Let bisection method be used to solve the nonlinear equation ($x$ is in radians)

$$x \sin x - 1 = 0$$

   starting with the initial interval $[0, 2]$. In order to approximate a solution of the nonlinear equation with an absolute error less than or equal to $10^{-3}$, what is the number of iterations required as per the error estimate of the bisection method? Also find the corresponding approximate solution.

3. Let bisection method be used to solve a nonlinear equation $f(x) = 0$ starting with the initial interval $[a_0, b_0]$ where $a_0 > 0$. Let $x_n$ be as in the bisection method, and $r$ be the solution of the nonlinear equation $f(x) = 0$ to which bisection method converges. Let $\epsilon > 0$. Show that the absolute value of the relative error of $x_n$ w.r.t. $r$ is at most $\epsilon$ whenever $n$ satisfies

$$n \geq \frac{\log(b_0 - a_0) - \log \epsilon - \log a_0}{\log 2}.$$

   What happens if $a_0 < 0 < b_0$?

4. Consider the nonlinear equation

$$10^x + x - 4 = 0$$

   i) Find an interval $[a_0, b_0]$ such that the function $f(x) = 10^x + x - 4$ satisfies the hypothesis of bisection method.
   ii) Let $r$ be the solution of the nonlinear equation to which bisection method iterative sequence converges. Find an $n$ such that $x_n$ (notation as in the bisection method) approximates $r$ to two significant digits. Also find $x_n$.

5. If bisection method is used with the initial interval $[a_0, b_0] = [2, 3]$, how many iterations are required to assure that an approximate solution of the nonlinear equation $f(x) = 0$ is obtained with an absolute error that is at most $10^{-5}$?
6. Assume that in solving a nonlinear equation $f(x) = 0$ with the initial interval $[a_0, b_0]$, the iterative sequence $\{x_n\}$ given by bisection method is never an exact solution of $f(x) = 0$. Let us define a sequence of numbers $\{d_n\}$ by

$$d_n = \begin{cases} 0 & \text{if } [a_n, b_n] \text{ is the left half of the interval } [a_{n-1}, b_{n-1}], \\ 1 & \text{if } [a_n, b_n] \text{ is the right half of the interval } [a_{n-1}, b_{n-1}]. \end{cases}$$

   Using the sequence $\{d_n\}$ defined above, express the solution of $f(x) = 0$ to which the bisection method converges. (**Hint:** Try the case $[a_0, b_0] = [0, 1]$ first and think of binary representation of a number. Then try for the case $[a_0, b_0] = [0, 2]$, then for the case $[a_0, b_0] = [1, 3]$, and then the general case!)

7. In the notation of bisection method, determine (with justification) if the following are possible.
   i) $a_0 < a_1 < a_2 < \cdots < a_n < \cdots$
   ii) $b_0 > b_1 > b_2 > \cdots > b_n > \cdots$
   iii) $a_0 = a_1 < a_2 = a_3 < \cdots < a_{2m} = a_{2m+1} < \cdots$ (**Hint:** First guess what should be the solution found by bisection method in such a case, and then find the simplest function having it as a root! Do not forget the connection between bisection method and binary representation of a number, described in the last problem)

8. Draw the graph of a function that satisfies the hypothesis of bisection method on the interval $[0, 1]$ and having exactly one root in $[0, 1]$ such that the errors $e_1, e_2, e_3$ satisfy $|e_1| > |e_2|$, and $|e_2| < |e_3|$. Give formula for one such function.

9. Draw the graph of a function for which bisection method iterates satisfy $x_1 = 2$, $x_2 = 0$, and $x_3 = 1$ (in the usual notation of bisection method). Indicate in the graph why $x_1 = 2$, $x_2 = 0$, and $x_3 = 1$ hold. Also mention precisely the corresponding intervals $[a_0, b_0]$,$[a_1, b_1]$, $[a_2, b_2]$.

10. Draw the graph of a function (there is no need to give a formula for the function) for which $a_0, a_1, a_2, a_3$ (in the usual notation of bisection method) satisfy $a_0 < a_1 = a_2 < a_3$. (Mark these points clearly on the graph.)

## Secant Method and Newton-Raphson Method

11. Discuss some instances where the secant method fails. Note that failure of secant

method results from one of the following two situations: (i) the iterative sequence is not well-defined, and (ii) the iterative sequence does not converge at all.

12. Let $\alpha$ be a positive real number. Find formula for an iterative sequence based on Newton-Raphson method for finding $\sqrt{\alpha}$ and $\alpha^{1/3}$. Apply the methods to $\alpha = 18$ to obtain the results whcih are correct to two significant digits when compared to their exact values.

13. Consider the nonlinear equation

$$\frac{1}{3}x^3 - x^2 + x + 1 = 0.$$

Show that there exists an initial guess $x_0 \in (0, 4)$ for which $x_2$ of the Newton-Raphson method iterative sequence is not defined.

14. Let $a$ be a real number such that $0 < a \leq 1$. Let $\{x_n\}_{n=1}^{\infty}$ be the iterative sequence of the Newton-Raphson method to solve the nonlinear equation $e^{-ax} = x$. If $x^*$ denotes the exact root of this equation and $x_0 > 0$, then show that

$$|x^* - x_{n+1}| \leq \frac{1}{2}(x^* - x_n)^2.$$

15. Newton-Raphson method is to be applied for approximating a root of the nonlinear equation $x^4 - x - 10 = 0$.
    i) How many solutions of the nonlinear equation are there in $[1, \infty)$? Are they simple?
    ii) Find an interval $[1, b]$ that contains the smallest positive solution of the nonlinear equation.
    iii) Compute five iterates of Newton-Raphson method, for each of the initial guesses $x_0 = 1$, $x_0 = 2$, $x_0 = 100$. What are your observations?
    iv) A solution of the nonlinear equation is approximately equal to 1.85558 Find a $\delta$ as in the proof of theorem on Newton-Raphson method, so that iterative sequence of Newton-Raphson method always converges for every initial guess $x_0 \in [1.85558 - \delta, 1.85558 + \delta]$.
    v) Can we appeal to the theorem for convex functions in the context of Newton-Raphson method? Justify.

16. Newton-Raphson method is to be applied for approximating a root of the equation $\sin x = 0$.
    i) Find formula for the Newton-Raphson iterative sequence.
    ii) Let $\alpha \in (-\pi/2, \pi/2)$ and $\alpha \neq 0$ be such that if $x_0 = \alpha$, then the iteration becomes a cycle *i.e.*,

    $$\alpha = x_0 = x_2 = \cdots = x_{2k} = x_{2k+2} = \cdots, \quad x_1 = x_3 = \cdots = x_{2k+1} = x_{2k+3} = \cdots$$

    Find a non-linear equation $g(y) = 0$ whose solution is $\alpha$.

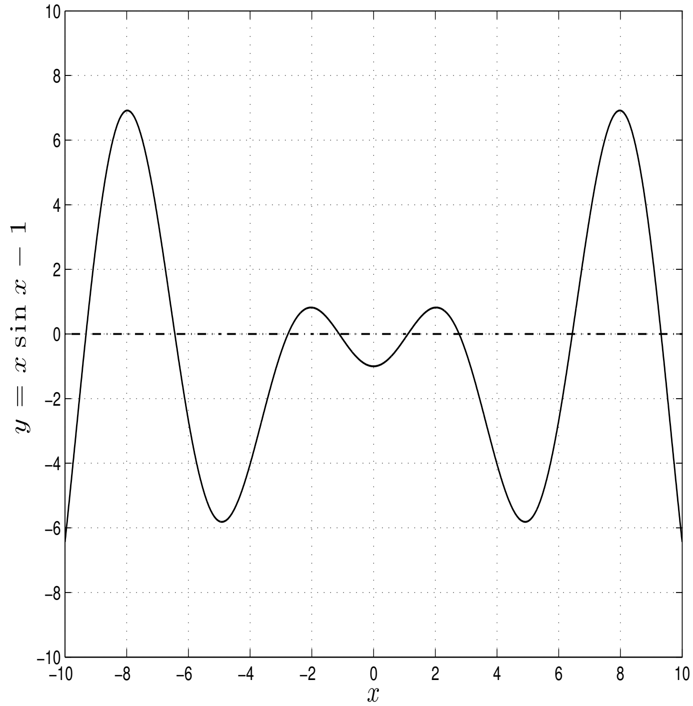


Figure 9.5: Graph of $x \sin x - 1$

iii) Starting with the initial guess $x_0 = \alpha$, compute $x_1, x_2, x_3, x_4, x_5$ using Newton-Raphson method for the equation $\sin x = 0$.

iv) Starting with the initial guess $y_0 = 1$, compute $y_1, y_2, y_3, y_4, y_5$ using Newton-Raphson method for the equation $g(y) = 0$ to find an approximate value of $\alpha$.

v) Starting with initial guess $x_0 = y_5$, where $y_5$ is obtained in (iv) above, compute $x_1, x_2, x_3, x_4, x_5$ using Newton-Raphson method for the equation $\sin x = 0$.

17. Consider the nonlinear equation $x \sin x - 1 = 0$.

i) Find an initial guess $x_0$ such that $x_0 > 1$, with the help of the graph depicted in Figure 9.5, such that the Newton-Raphson method is likely to converge to the solution $x^*$ of the given nonlinear equation lying in the interval $(-10, -9)$. Compute $x_1, x_2, x_3, x_4$ of the corresponding Newton-Raphson iterative sequence. Explain why the Newton-Raphson iterative sequence so obtained would converge to the desired $x^*$. This example shows that even though the initial guess is close to a solution, the corresponding Newton-Raphson iterative sequence might converge to a solution that is far off!

ii) Find another initial guess $x_0$ such that $x_0 > 1$, with the help of the graph, such that the Newton-Raphson method is likely to converge to the smallest positive solution of the given nonlinear equation. Compute $x_1, x_2, x_3, x_4$ of the corresponding Newton-Raphson iterative sequence. Explain why the Newton-Raphson iterative sequence so obtained would converge to the desired solution.

18. Draw the graph of a function for which the Newton-Raphson iterates satisfy $x_0 = x_2 = x_4 = \cdots = 0$, and $x_1 = x_3 = x_5 = \cdots = 2$. Indicate in the graph why this happens.

19. Draw the graph of a function for which secant method iterates satisfy $x_0 = 0$, $x_1 = 3$, and $x_2 = 1$, $x_3 = 2$ (in the usual notation of secant method). Indicate in the graph why $x_2 = 1$, $x_3 = 2$ hold.

## Fixed-Point Iteration Method

20. The nonlinear equation $f(x) = x^2 - 2x - 8 = 0$ has two solutions $x = -2$ and $x = 4$. Consider the three fixed-point formulations
    i) $x = \dfrac{8}{x-2}$,
    ii) $x = \sqrt{2x+8}$,
    iii) $x = \dfrac{x^2-8}{2}$.

    Carry out fixed-point iteration method with two initial guesses $x_0 = -1$, $x_0 = 3$, and for all the three iteration functions. Discuss the convergence or divergence of all the iterative sequences. Can you justify theoretically?

21. To solve the nonlinear equation $x - \tan x = 0$ by fixed-point iteration method, the following fixed-point formulations may be considered.
    i) $x = \tan x$
    ii) $x = \tan^{-1} x$

    Discuss about convergence of the fixed-point iterative sequences generated by the two formulations.

22. To solve the nonlinear equation $e^{-x} - \cos x = 0$ by fixed-point iteration method, the following fixed-point formulations may be considered.
    i) $x = -\ln(\cos x)$
    ii) $x = \cos^{-1}(e^{-x})$

    Discuss about convergence of the fixed-point iterative sequences generated by the two formulations.

23. Show that $g(x) = \pi + \frac{1}{2}\sin(x/2)$ has a unique fixed point in $[0, 2\pi]$. Use fixed-point iteration method with $g$ as the iteration function and $x_0 = 0$ to find an approximate solution for the equaton $\frac{1}{2}\sin(x/2) - x + \pi = 0$ with the stopping criterion that the residual error is less than $10^{-4}$.

24. Let $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$ be the roots of $x^2 + ax + b = 0$, and such that $|\alpha| > |\beta|$. Let $g$ and $h$ be two iterating functions satisfying the hypothesis of the theorem on fixed-point method on some intervals. Consider the iterative sequences $\{x_n\}$ and $\{y_n\}$ corresponding to the iterative functions $g$ and $h$ given by
$$x_{n+1} = -\frac{ax_n + b}{x_n}, \text{ and } y_{n+1} = -\frac{b}{y_n + a}$$
respectively. Show that the iterative sequences $\{x_n\}$ and $\{y_n\}$ converge to $\alpha$ and $\beta$ respectively.

25. Let $\{x_n\} \subset [a, b]$ be a sequence generated by a fixed point iteration method with a continuously differentiable iteration function $g(x)$. If this sequence converges to $x^*$, then show that
$$|x_{n+1} - x^*| \leq \frac{\lambda}{1-\lambda}|x_{n+1} - x_n|,$$
where $\lambda := \max\limits_{x\in[a,b]}|g'(x)| < 1$. (This estimate helps us to decide when to stop iterating if we are using a stopping criterion based on the distance between successive iterates.)

26. Explain why the sequence of iterates $x_{n+1} = 1 - 0.9x_n^2$, with initial guess $x_0 = 0$, does not converge to any solution of the quadratic equation $0.9x^2 + x - 1 = 0$? [Hint: Observe what happens after 25 iterations, may be using a computer.]

27. Let $x^*$ be the smallest positive root of the equation $20x^3 - 20x^2 - 25x + 4 = 0$. The following question is concerning the fixed-point formulation of the nonlinear equation given by $x = g(x)$, where $g(x) = x^3 - x^2 - \frac{x}{4} + \frac{1}{5}$.
    i) Show that $x^* \in [0, 1]$.
    ii) Does the function $g$ satisfy the hypothesis of theorem on fixed-point method? If yes, we know that $x^*$ is the only fixed point of $g$ lying in the interval $[0, 1]$. In the notation of fixed-point iteration method, find an $n$ such that $|x^* - x_n| < 10^{-3}$, when the initial guess $x_0$ is equal to 0.

28. Let $f : [a, b] \to \mathbb{R}$ be a function such that $f'$ is continuous, $f(a)f(b) < 0$, and there exists an $\alpha > 0$ such that $f'(x) \geq \alpha > 0$.

i) Show that $f(x) = 0$ has exactly one solution in the interval $[a, b]$.

ii) Show that with a suitable choice of the parameter $\lambda$, the solution of the nonlinear equation $f(x) = 0$ can be obtained by applying the fixed-point iteration method applied to the function $F(x) = x + \lambda f(x)$.

29. Let $p > 1$ be a real number. Show that the following expression has a meaning and find its value.

$$x = \sqrt{p + \sqrt{p + \sqrt{p + \cdots}}}$$

Note that the last equation is interpreted as $x = \lim_{n\to\infty} x_n$, where $x_1 = \sqrt{p}$, $x_2 = \sqrt{p + \sqrt{p}}, \cdots$ . (**Hint:** Note that $x_{n+1} = \sqrt{p + x_n}$, and show that the sequence $\{x_n\}$ converges using the theorem on fixed-point method.)

30. Let $p$ be a positive real number. Show that the following expression has a meaning and find its value.

$$x = \cfrac{1}{p + \cfrac{1}{p + \cfrac{1}{p + \cdots}}}$$

Note that the last equation is interpreted as $x = \lim_{n\to\infty} x_n$, where $x_1 = \frac{1}{p}$, $x_2 = \frac{1}{p+\frac{1}{p}}, \cdots$ . (**Hint:** You may have to consider the three cases $0 < p < 1/4$, $p = 1/4$, and $p > 1/4$ separately.)

31. Draw the graph of a function having the following properties: (i) The function has exactly TWO fixed points. (ii) Give two choices of the initial guess $x_0$ and $y_0$ such that the corresponding sequences $\{x_n\}$ and $\{y_n\}$ have the properties that $\{x_n\}$ converges to one of the fixed points and the sequence $\{y_n\}$ goes away and diverges. Point out the first three terms of both the sequences on the graph.