CS207 Counting

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October 2021

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1 Permutations and Combinations

1.1 String

A string is an ordered collection of elements from an alphabet(a finite set).
Mathematically, it is a mapping of each position in the string to an element in alphabet,

$$String(\sigma) : \{1, 2, \dots k\} \rightarrow B$$

Number of strings of length k from an alphabet of size n: n^k

If n=2, then the string is called a **binary string**.
Binary strings can be used to represent subsets of $[k] = \{1, 2 \dots k\}$.
Let the alphabet be {0,1}. Then the subset of $[k]$ corresponding to a binary sting is:

$$S_\sigma = \{x | \sigma_x = 1\}$$

1.2 Permutations

Permutations refer to the arrangements of elements of alphabet without repetitions of the elements.
The number of permutations of length k from a alphabet of size n is denoted as P(n,k).

$$P(n,k) = \begin{cases} 0 & if \quad k > n \\ \frac{n!}{(n-k)!} & if \quad k \leq n \end{cases}$$

The above expression can be proved by induction on n and using the relation

$$P(n,k) = n \cdot P(n-1, k-1)$$

in the induction step.

1.3 Combinations

Combinations can considered as subsets of a given set. The number of subsets of size k from a set of size n is given by:

$$C(n,k) = \frac{P(n,k)}{k!}$$

Important property: **C(n,k)=C(n-1,k-1)+C(n-1,k)**
The above property can be used find the coefficients of x^k in expansion of $(1+x)^k$, inductively.
The property also gives a **recursive definition** of C(n,k) with base cases $C(n,0) = C(n,n) = 1$

2 Balls and Bins

Let the number of balls be k and number of bins be n.
We need to allot each ball to exactly one bin.
Consider the following different cases(N is the number of ways):

2.1 Labelled balls and labelled bins

2.1.1 No restriction

N = number of functions from set of size k to set of size n = n^k

2.1.2 Atmost one ball in every bin

N=number of injective functions = $P(n, k)$

2.1.3 No bin empty

N=number of onto functions=N(k,n)

$$N(k,n) = \begin{cases} \sum_{i=0}^{i=n} (-1)^i \cdot C(n,i) \cdot (n-i)^k & if \quad n \leq k \\ 0 & if \quad n > k \end{cases}$$

The above equation can be proved by inclusion-exclusion principle.

2.2 Unlabelled balls labelled bins

2.2.1 No restriction

This case can be represented by a multi-set, which is a set in which multiple entries of an element can occur, but is unordered. Each multi-set of length k having elements from the bin set represents one way of distribution.
In this case, N can be found by partitioning the the set of identical balls. The problem can be reformulated as ways of arranging k identical balls and n-1 identical sticks in a row, which is

$$N = \frac{(k+n-1)!}{(n-1)!(k)!} = C(k+n-1, n-1)$$

2.2.2 Atmost one ball in every bin

N=ways of selecting k bins out of n bins (remaining will be empty)=C(n,k)

2.2.3 No bin empty

This case is similar to no restriction case after giving one ball to each bin. Hence,

$$N = C(k-1, n-1)$$

2.3 Labelled balls and unlabelled bins

2.3.1 No restriction

This case can be reformulated as number of ways of partitioning a set of length k. The number of ways of partitioning a set of length k into n non-empty subsets is given by **Stirling's number of second kind** and is denoted by S(k,n).
Hence N is given by,

$$N = B_k(\text{Bell number}) = \sum_{i=1}^{i=k} S(k,i) \quad [= \sum_{i=1}^{i=n} S(k,i))]$$

where,

$$S(k,n) = \frac{N(k,n)}{n!}$$

2.3.2 Atmost one ball in every bin

$$N = \begin{cases} 1 & if \quad n \geq k \\ 0 & if \quad n < k \end{cases}$$

2.3.3 No bin empty

$$N = S(k,n)$$

2.4 Unlabelled balls and unlabelled bins

2.4.1 No restriction

The problem cn be reformulated as number of integer solutions $(x_1, x_2, \ldots x_n)$ such that,

$$x_1 + x_2 \ldots x_n = k$$

and

$$0 \leq x_1 \leq x_2 \cdots \leq x_n$$

If the bins are non-empty ,i.e., 1 instead of 0 in the above relation(the no bin empty case) then the number of such ways has a name, partition number(denoted by $P_n(k)$).
Therefore in this case, just add n 1's on both sides of the equation. This given,

$$N = P_n(n+k)$$

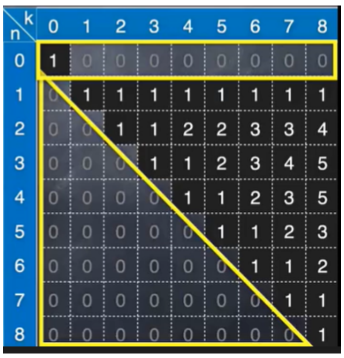
2.4.2 Atmost one ball in every bin

$$N = \begin{cases} 1 & if \quad n \geq k \\ 0 & if \quad n < k \end{cases}$$

2.4.3 No bin empty

$$N = P_n(k)$$

How to calculate $P_n(k)$?
$P_n(k)$ can be calculated recursively as follows:
Base case: $P_n(k) = 0$ if $n > k$; and $P_0(0) = 1$; and $P_0(k) = 0$ if $k > 0$
Recursive relation: $P_n(k) = P_n(k-n) + P_{n-1}(k-1)$
Above relation can be proved by considering exhaustive cases $x_1 > 1$ or $x_1 = 1$.



n \ k	0	1	2	3	4	5	6	7	8
0	1	0	0	0	0	0	0	0	0
1	0	1	1	1	1	1	1	1	1
2	0	0	1	1	2	2	3	3	4
3	0	0	0	1	1	2	3	4	5
4	0	0	0	0	1	1	2	3	5
5	0	0	0	0	0	1	1	2	3
6	0	0	0	0	0	0	1	1	2
7	0	0	0	0	0	0	0	1	1
8	0	0	0	0	0	0	0	0	1

Figure 1: Partition number; Source : CS207 Lectures 2021