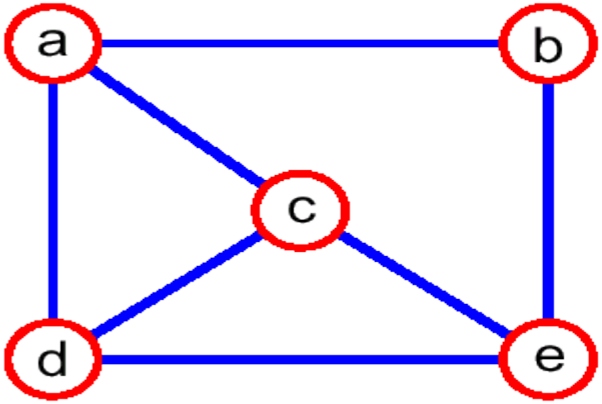
Graphs – Definition

- A **graph G = (**V,E**)** is composed of:
  - V: set of **vertices**
  - $E \subset V \times V$: set of **edges** connecting the **vertices**
- An **edge e** = (u,v) is a pair of vertices
- (u,v) is ordered, if **G** is a **directed** graph

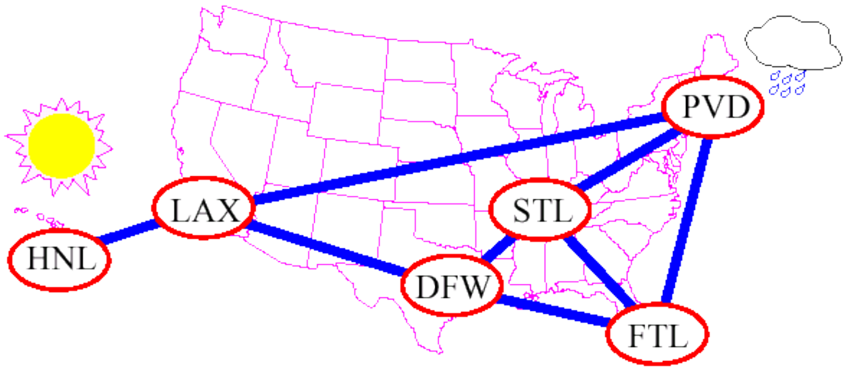


$\mathbf{V} = \{a,b,c,d,e\}$

$\mathbf{E} = \{(a,b),(a,c),(a,d),(b,e),(c,d),(c,e),(d,e)\}$

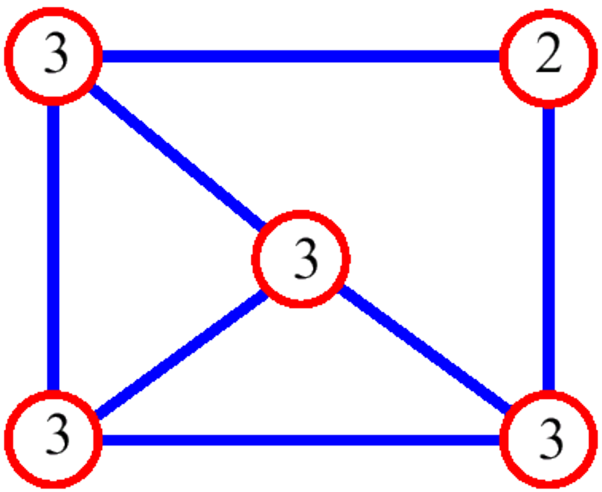
Applications

- Electronic circuits, pipeline networks
- Transportation and communication networks
- Modeling any sort of relationtionships (between components, people, processes, concepts)



Graph Terminology

- adjacent vertices: connected by an edge
- degree (of a **vertex**): # of adjacent vertices



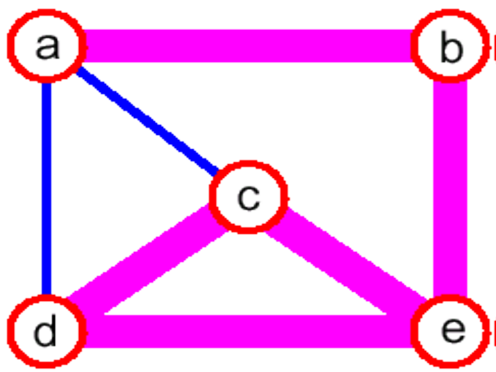
$$\sum_{v \in V} \deg(v) = 2(\#\ \text{of edges})$$

Since adjacent vertices each count the adjoining edge, it will be counted twice

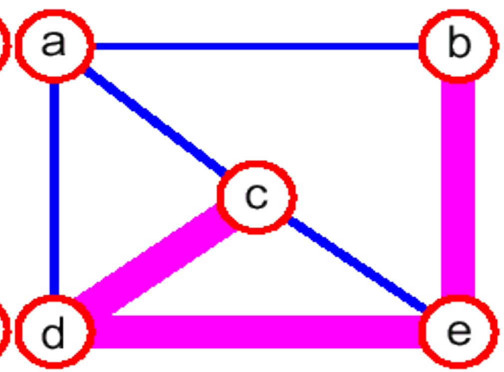
- path: sequence of vertices $v_1, v_2, \ldots v_k$ such that consecutive vertices v_i and v_{i+1} are adjacent

Graph Terminology (2)

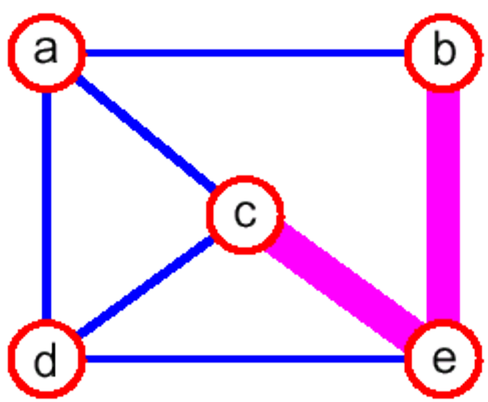
- simple path: no repeated vertices



a b e d c



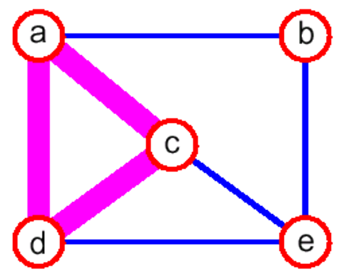
b e d c



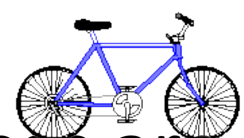
b e c

Graph Terminology (3)

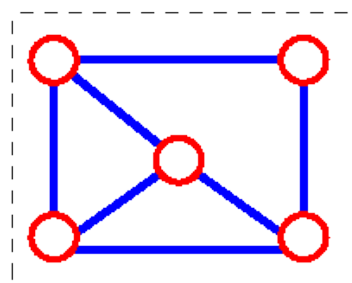
- cycle: simple path, except that the last vertex is the same as the first vertex



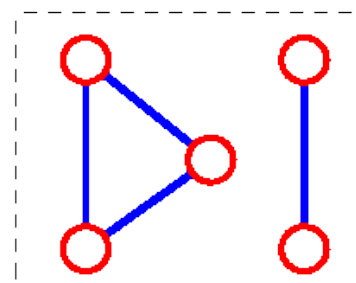
a c d a



- connected graph: any two vertices are connected by some path



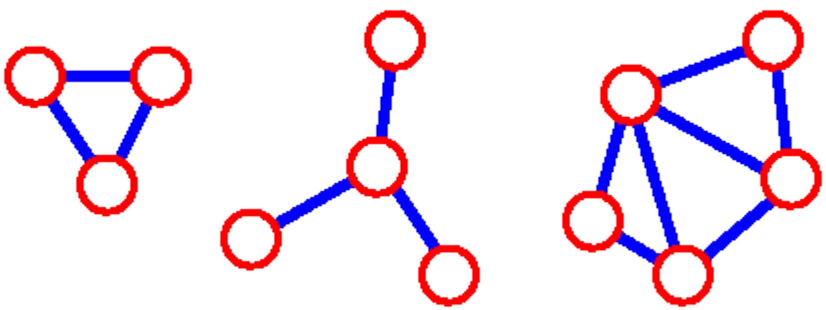
connected



not connected

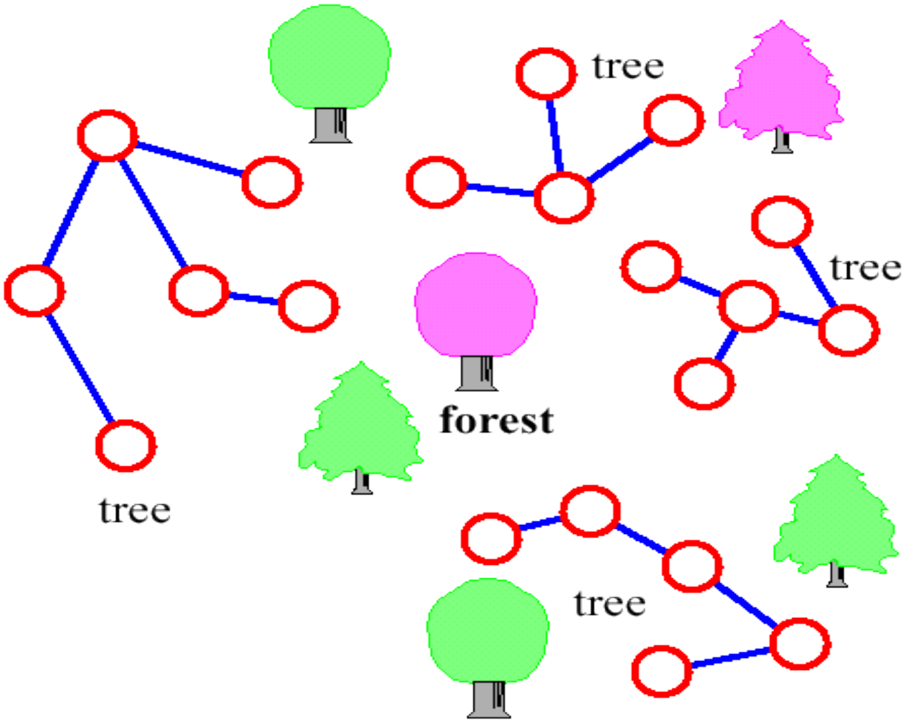
Graph Terminology (4)

- subgraph: subset of vertices and edges forming a graph
- connected component: maximal connected subgraph. E.g., the graph below has 3 connected components



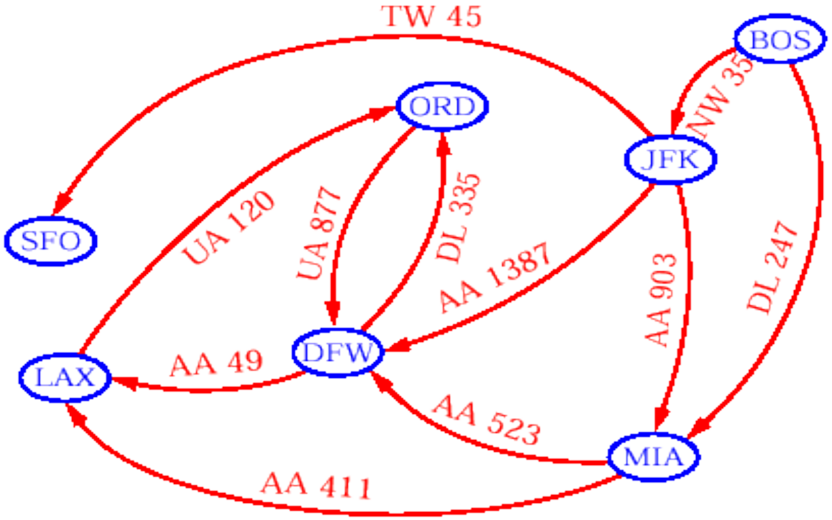
Graph Terminology (5)

- (free) tree - connected graph without cycles
- forest - collection of trees



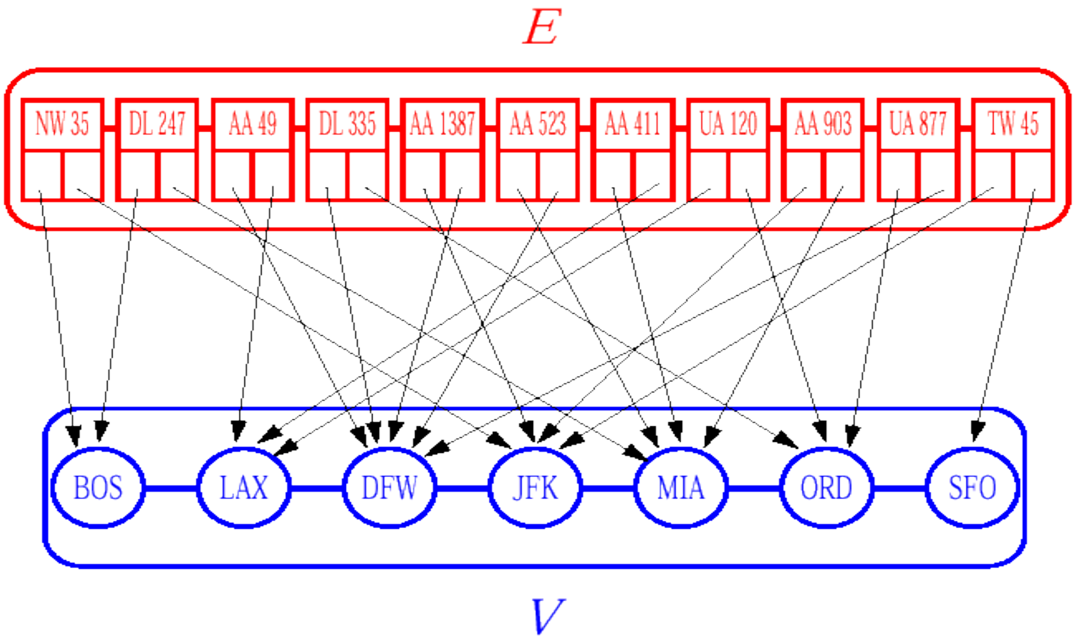
Data Structures for Graphs

- How can we represent a graph?
  - To start with, we can store the vertices and the edges in two containers, and we store with each edge object references to its start and end vertices



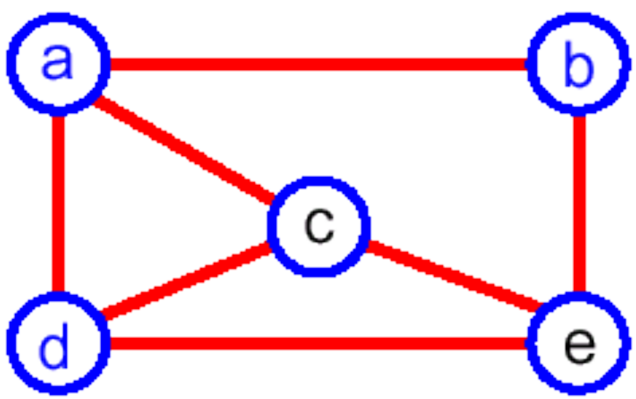
Edge List

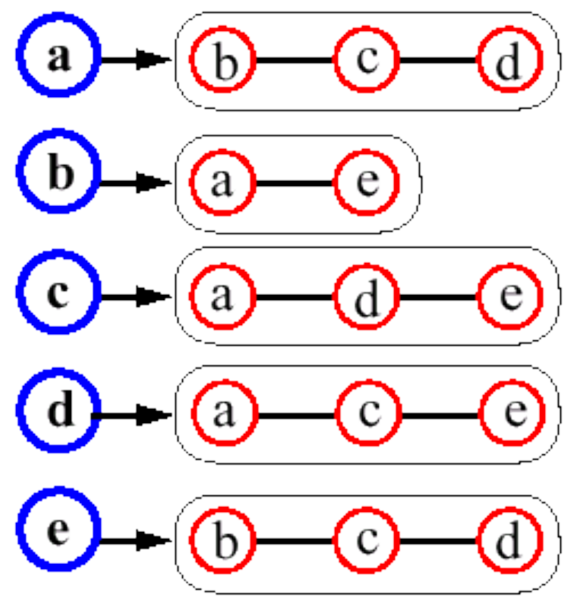
- The **edge list**
  - Easy to implement
  - Finding the edges incident on a given vertex is inefficient since it requires examining the entire edge sequence



Adjacency List

- The **Adjacency list** of a vertex v: a sequence of vertices adjacent to v
- Represent the graph by the adjacency lists of all its vertices

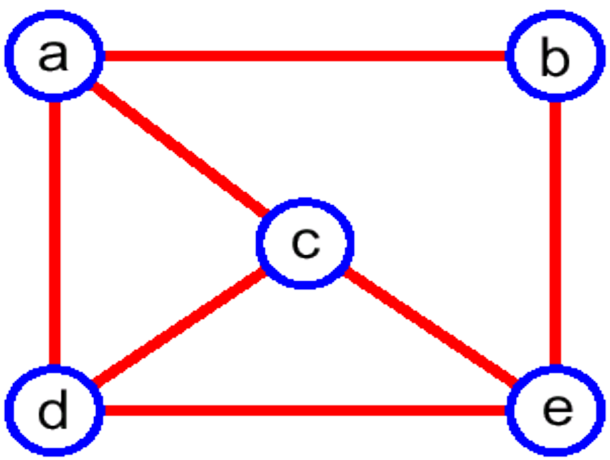




$$\text{Space } = \Theta(n + \sum \deg(v)) = \Theta(n + m)$$

Adjacency Matrix

- Matrix M with entries for all pairs of vertices
- M[i,j] = true – there is an edge (i,j) in the graph
- M[i,j] = false – there is no edge (i,j) in the graph
- Space = $O(\boldsymbol{n}^2)$



	a	b	c	d	e
a	F	T	T	T	F
b	T	F	F	F	T
c	T	F	F	T	T
d	T	F	T	F	T
e	F	T	T	T	F

Graph Searching Algorithms

- Systematic search of every edge and vertex of the graph
- Graph G = (V,E) is either directed or undirected
- Today's algorithms assume an adjacency list representation
- Applications
  - Compilers
  - Graphics
  - Maze-solving
  - Mapping
  - Networks: routing, searching, clustering, etc.

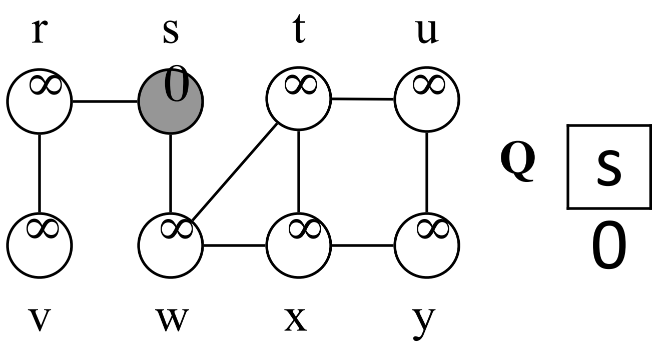
Breadth First Search

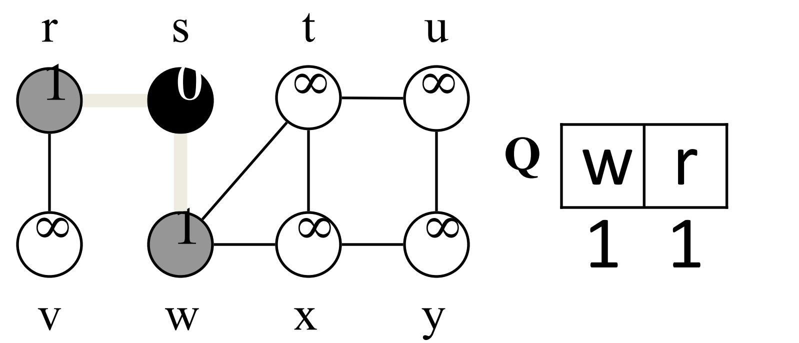
- A **Breadth-First Search (BFS)** traverses a **connected component** of a graph, and in doing so defines a **spanning tree** with several useful properties
- BFS in an **undirected** graph G is like wandering in a labyrinth with a string.
- The starting vertex *s* is assigned a distance 0.
- In the first round, the string is unrolled the length of one edge, and all of the edges that are only one edge away from the anchor are visited (**discovered**), and assigned distances of 1

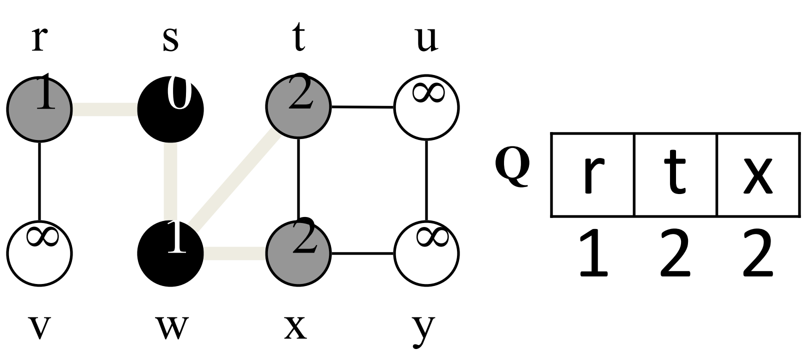
Breadth-First Search (2)

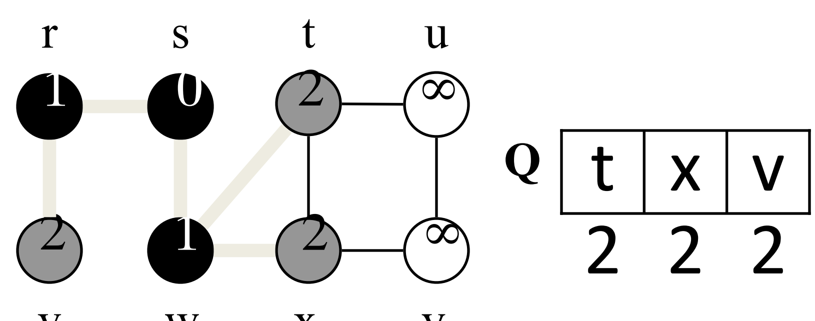
- In the second round, all the new edges that can be reached by unrolling the string 2 edges are visited and assigned a distance of 2
- This continues until every vertex has been assigned a level
- The label of any vertex *v* corresponds to the length of the shortest path (in terms of edges) from *s* to *v*

BFS Example

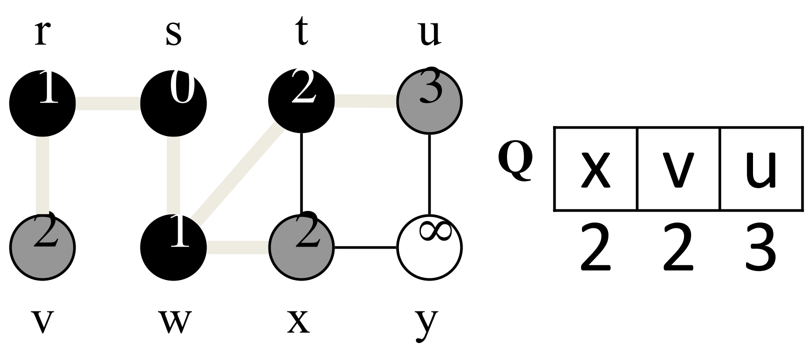

r s t u
∞ 0 ∞ ∞
∞ ∞ ∞ ∞
v w x y
Q
s
0

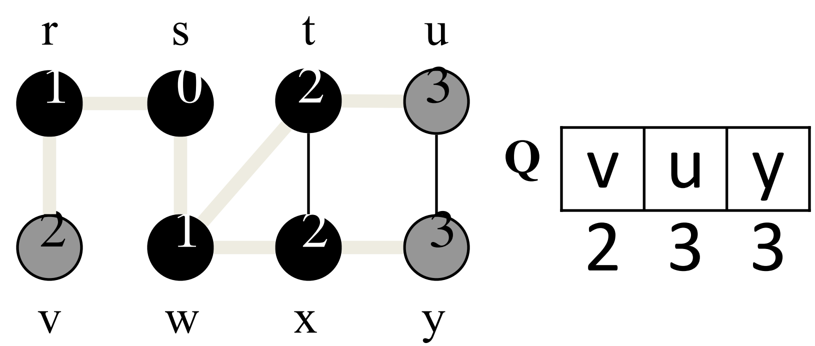

r s t u
1 0 ∞ ∞
∞ 1 ∞ ∞
v w x y
Q
w r
1 1

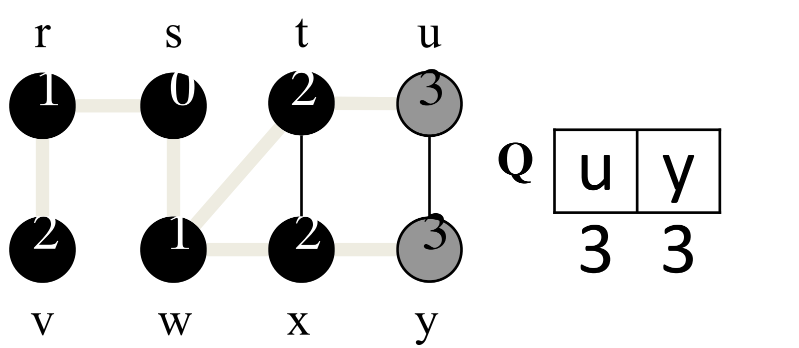

r s t u
1 0 2 ∞
∞ 1 2 ∞
v w x y
Q
r t x
1 2 2

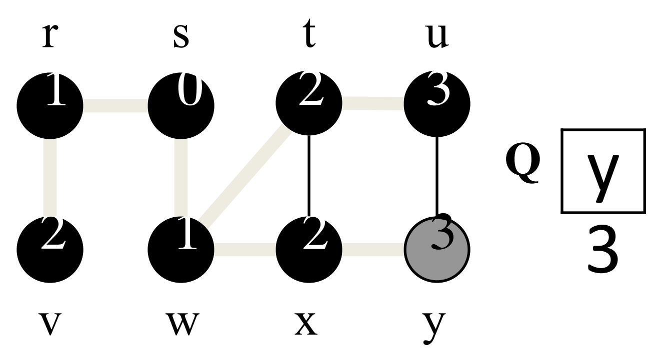

r s t u
1 0 2 ∞
2 1 2 ∞
v w x y
Q
t x v
2 2 2

BFS Example

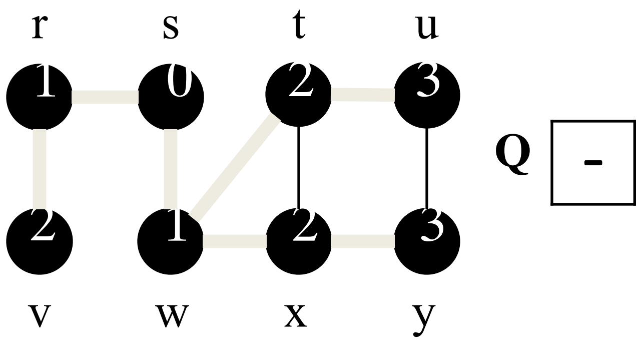

r s t u
1 0 2 3
2 1 2 ∞
v w x y
Q
x v u
2 2 3


r s t u
1 0 2 3
2 1 2 3
v w x y
Q
v u y
2 3 3


r s t u
1 0 2 3
2 1 2 3
v w x y
Q
u y
3 3


r s t u
1 0 2 3
2 1 2 3
v w x y
Q
y
3

BFS Example: Result



BFS Algorithm

```
BFS(G,s)
for each vertex u ∈ V[G]-{s}
   color[u] ← white
   d[u] ← ∞
   π[u] ← NIL
color[s] ← gray
d[s] ← 0
π[u] ← NIL
Q ← {s}
while Q ≠ ∅ do
   u ← head[Q]
   for each v ∈ Adj[u] do
      if color[v] = white then
         color[v] ← gray
         d[v] ← d[u] + 1
         π[v] ← u
         Enqueue(Q,v)
   Dequeue(Q)
   color[u] ← black
```

Init all vertices

Init BFS with *s*

Handle all *u*'s children before handling any children of children

BFS Running Time

- Given a graph $G = (V,E)$
  - Vertices are enqueued if there color is white
  - Assuming that en- and dequeuing takes $O(1)$ time the total cost of this operation is $O(V)$
  - Adjacency list of a vertex is scanned when the vertex is dequeued (and only then…)
  - The sum of the lengths of all lists is $\Theta(E)$. Consequently, $O(E)$ time is spent on scanning them
  - Initializing the algorithm takes $O(V)$
- **Total running time $O(V+E)$** (linear in the size of the adjacency list representation of G)

BFS Properties

- Given a graph G = (V,E), BFS **discovers all vertices reachable from a source vertex *s***
- It computes the **shortest distance** to all reachable vertices
- It computes a **breadth-first tree** that contains all such reachable vertices
- For any vertex *v* reachable from *s*, the path in the breadth first tree from s to v, corresponds to a **shortest path** in G

Breadth First Tree

- Predecessor subgraph of G

$$G_\pi = (V_\pi, E_\pi)$$
$$V_\pi = \{v \in V : \pi[v] \neq NIL\} \cup \{s\}$$
$$E_\pi = \{(\pi[v], v) \in E : v \in V_\pi - \{s\}\}$$

- G_π is a breadth-first tree
  - V_π consists of the vertices reachable from s, and
  - for all $v \in V_\pi$, there is a unique simple path from *s* to *v* in G_π that is also a shortest path from *s* to *v* in G
- The edges in G_π are called tree edges

Depth-First Search

- **A depth-first search (DFS)** in an undirected graph G is like wandering in a labyrinth with a **string** and a **can of paint**
  - We start at vertex *s*, tying the end of our string to the point and painting *s* "visited (discovered)". Next we label *s* as our current vertex called *u*
  - Now, we travel along an arbitrary edge (*u*,*v*).
  - If edge (*u*,*v*) leads us to an already visited vertex *v* we return to *u*
  - If vertex *v* is unvisited, we unroll our string, move to *v*, paint *v* "visited", set *v* as our current vertex, and repeat the previous steps

Depth-First Search (2)

- Eventually, we will get to a point where **all incident edges on *u* lead to visited vertices**
- We then **backtrack** by unrolling our string to a previously visited vertex *v*. Then *v* becomes our current vertex and we repeat the previous steps
- Then, if all incident edges on *v* lead to visited vertices, we backtrack as we did before. We **continue to backtrack along the path we have traveled**, finding and exploring unexplored edges, and repeating the procedure

DFS Algorithm

- Initialize – color all vertices white
- Visit each and every white vertex using DFS-Visit
- Each call to DFS-Visit(u) roots a new tree of the depth-first forest at vertex u
- A vertex is **white** if it is undiscovered
- A vertex is **gray** if it has been discovered but not all of its edges have been discovered
- A vertex is **black** after all of its adjacent vertices have been discovered (the adj. list was examined completely)

DFS Algorithm (2)

DFS(G)

```
for each vertex u ∈ V[G]
    do color[u] ← WHITE
time ← 0
for each vertex u ∈ V[G]
    do if color[u] = WHITE
        then DFS-VISIT(u)
```

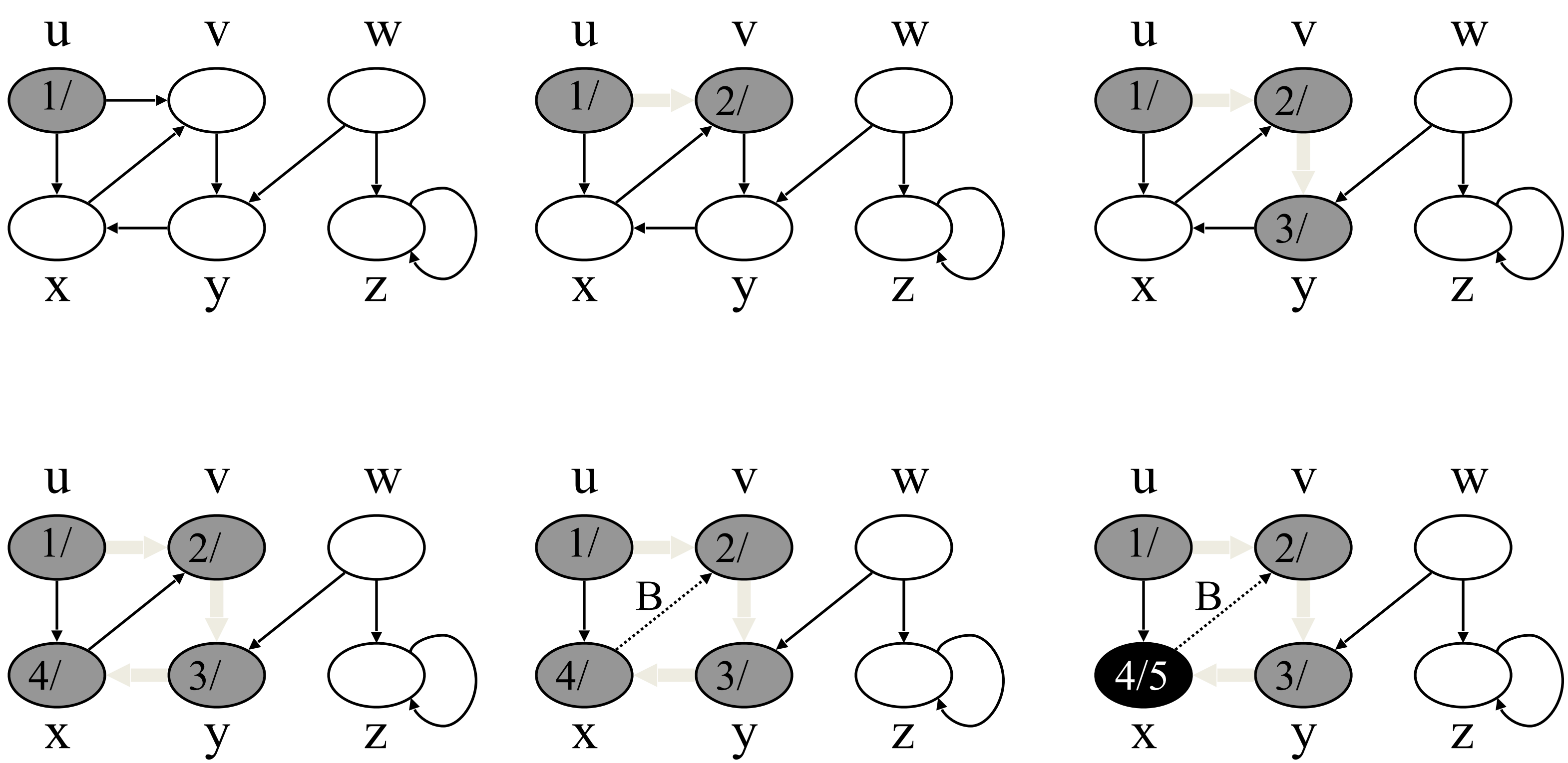
Init all vertices

DFS-VISIT(u)

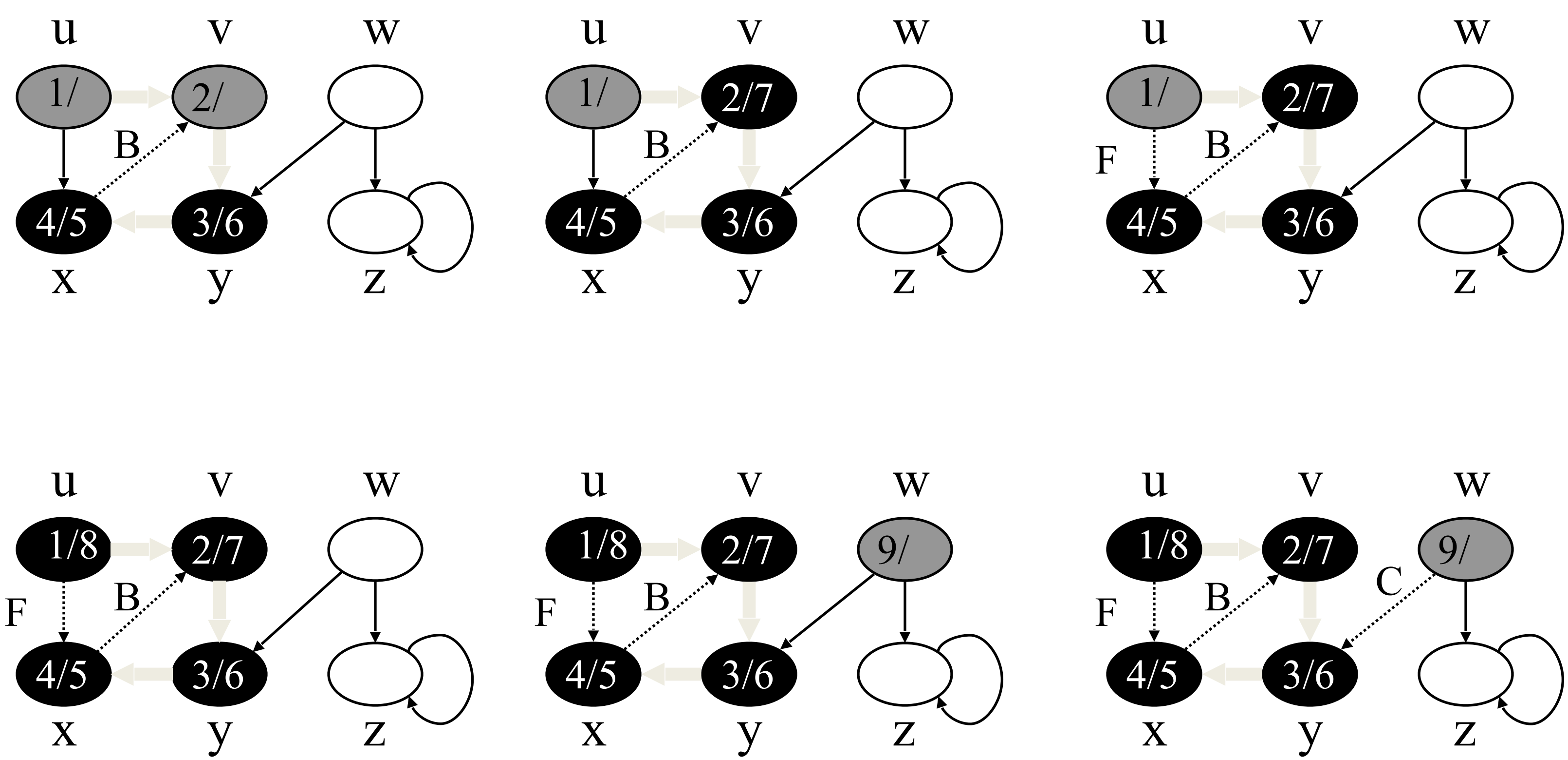
```
color[u] ← GRAY                  ▷ White vertex u discovered.
d[u] ← time                      ▷ Mark with discovery time.
time ← time + 1                  ▷ Tick global time.
for each v ∈ Adj[u]              ▷ Explore all edges (u, v).
    do if color[v] = WHITE
        then DFS-VISIT(v)
color[u] ← BLACK                 ▷ Blacken u; it is finished.
f[u] ← time                      ▷ Mark with finishing time.
time ← time + 1                  ▷ Tick global time.
```

Visit all children recursively

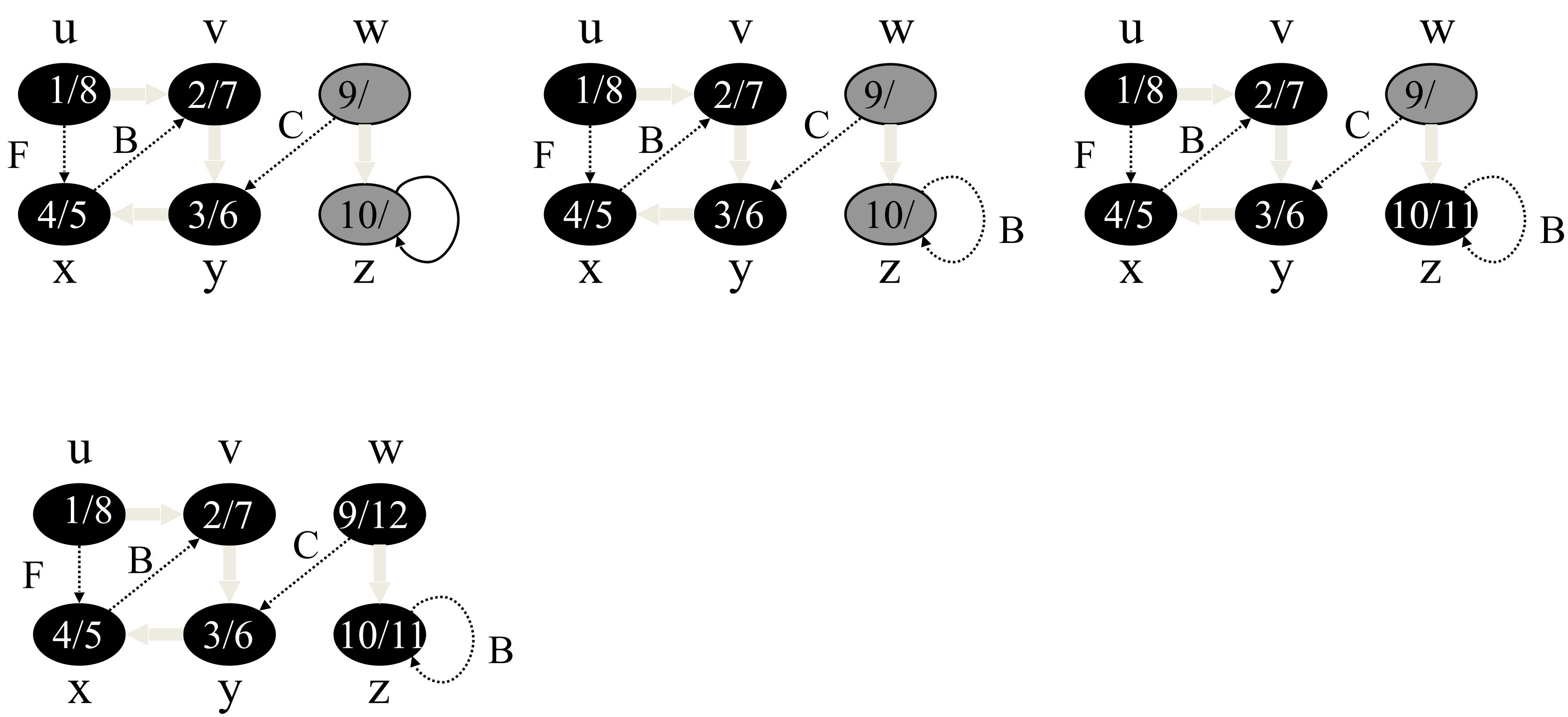
DFS Example


u v w
1/
x y z
u v w
1/ 2/
x y z
u v w
1/ 2/
3/
x y z
u v w
1/ 2/
4/ 3/
x y z
u v w
1/ 2/
B
4/ 3/
x y z
u v w
1/ 2/
B
4/5 3/
x y z

DFS Example (2)

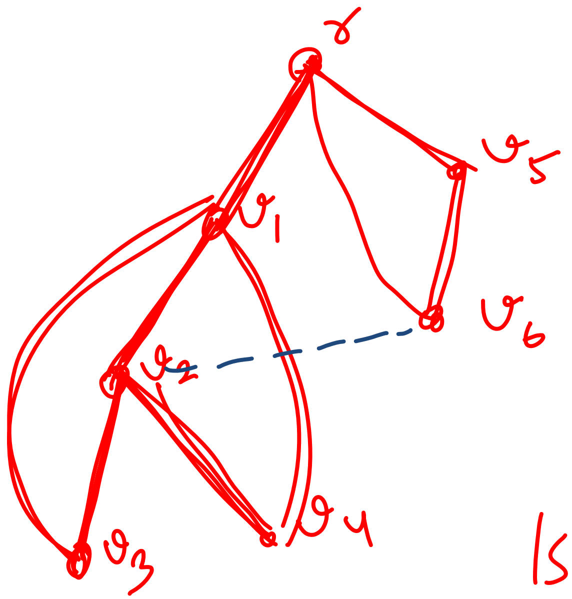


DFS Example (3)



Assume $G = (V, E)$ is connected

undirected DFS

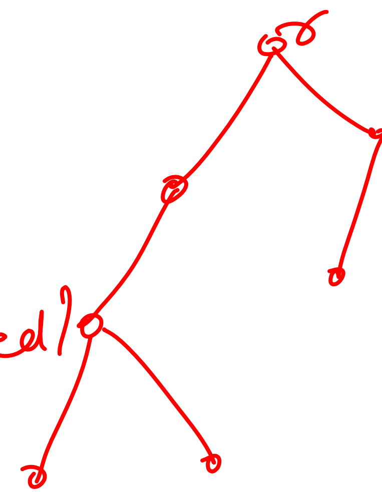


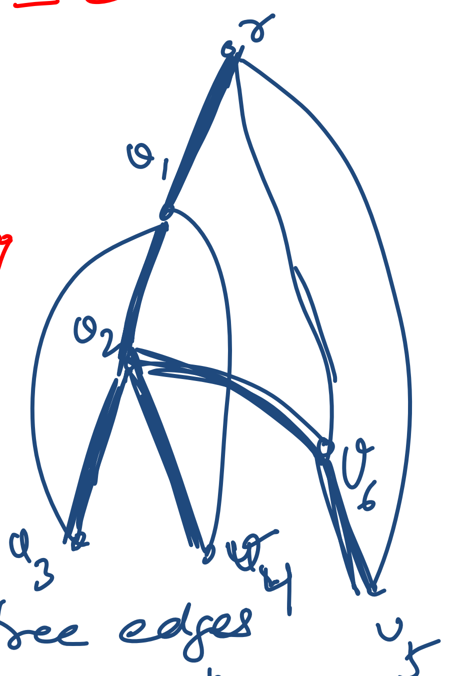
T: tree edges $\qquad T \subseteq E$

(V, T)

$|T| = n-1$

Is (V,T) connected?





$G = (V, E)$

all edges of G which are not tree edges have an ancestor-descendant relationship between their end-points

a => b

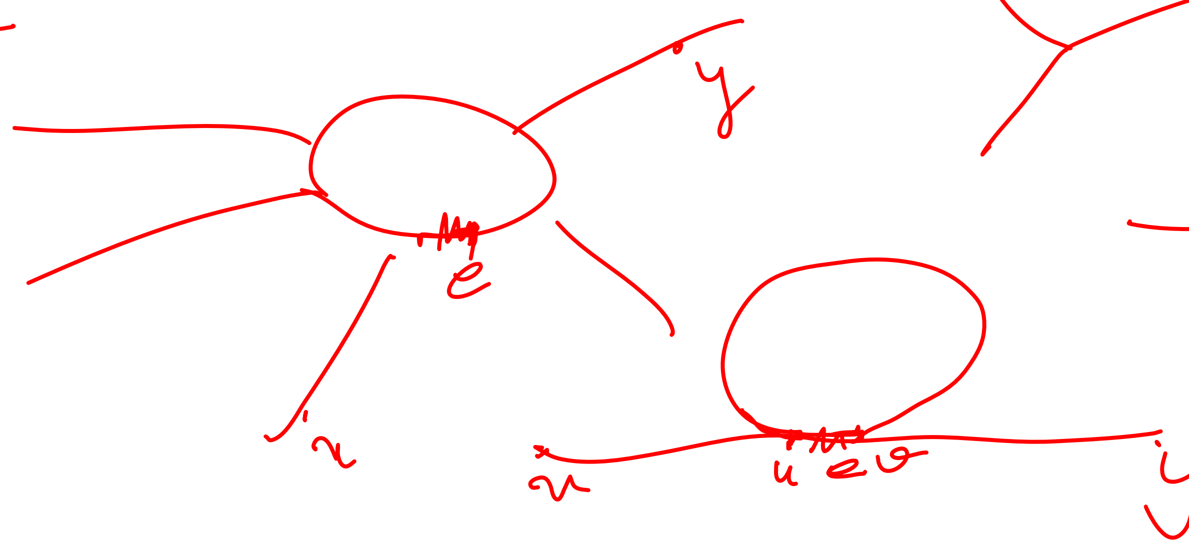
Equivalent definitions of a tree

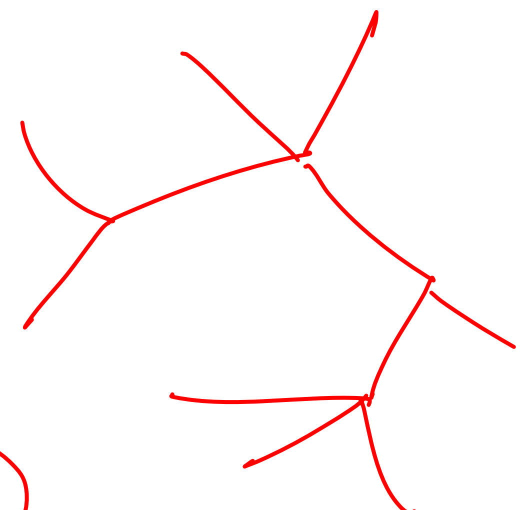
a) Connected graph containing no cycles

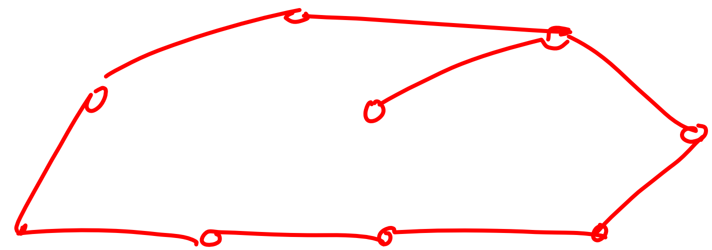
b) connected graph containing $|V| - 1$ edges

c) minimal connected graph

C => a





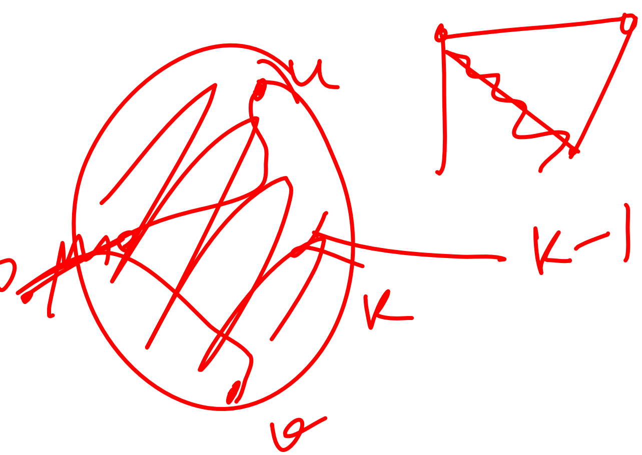


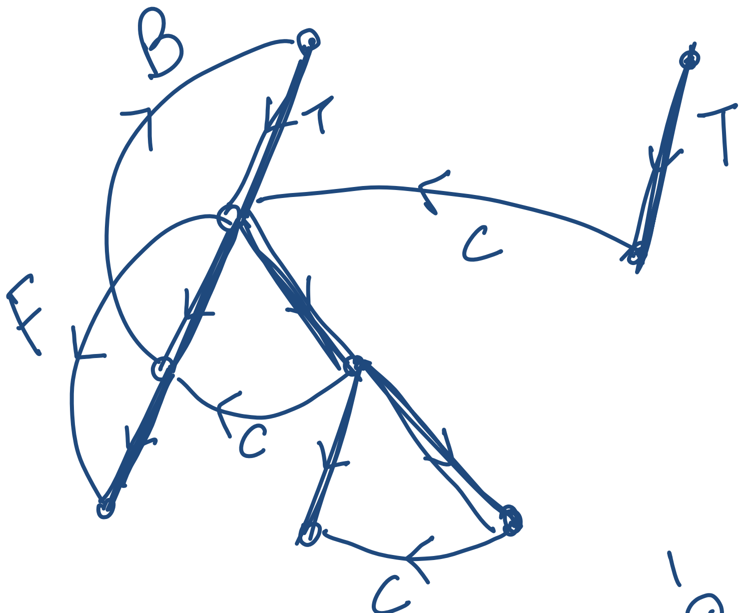
Induction on no. of vertices

every connected graph on k vertices which does not have a cycle has k-1 edges

Ind step: Consider a graph on k+1 vertices which does not have a cycle. Such a graph has at least one vertex of degree 1 (say w)

~~total~~ ~~complete~~ graph has k edges w

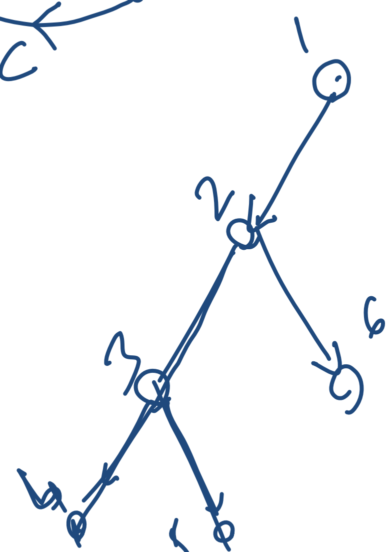




B: back edges from node to ancestor

F: forward edges: from node to descendant

C: cross edges: from node to another node which is not an ancestor/descendant



DFS Algorithm (3)

- When DFS returns, every vertex *u* is assigned
  - a discovery time *d*[*u*], and a finishing time *f*[*u*]
- Running time
  - the loops in DFS take time Θ(V) each, excluding the time to execute DFS-Visit
  - DFS-Visit is called once for every vertex
    - its only invoked on white vertices, and
    - paints the vertex gray immediately
  - for each DFS-visit a loop interates over all Adj[*v*]
  - the total cost for DFS-Visit is Θ(E)

$$\sum_{v \in V} \left| Adj[v] \right| = \Theta(E)$$

  - **the running time of DFS is Θ(V+E)**

Predecessor Subgraph

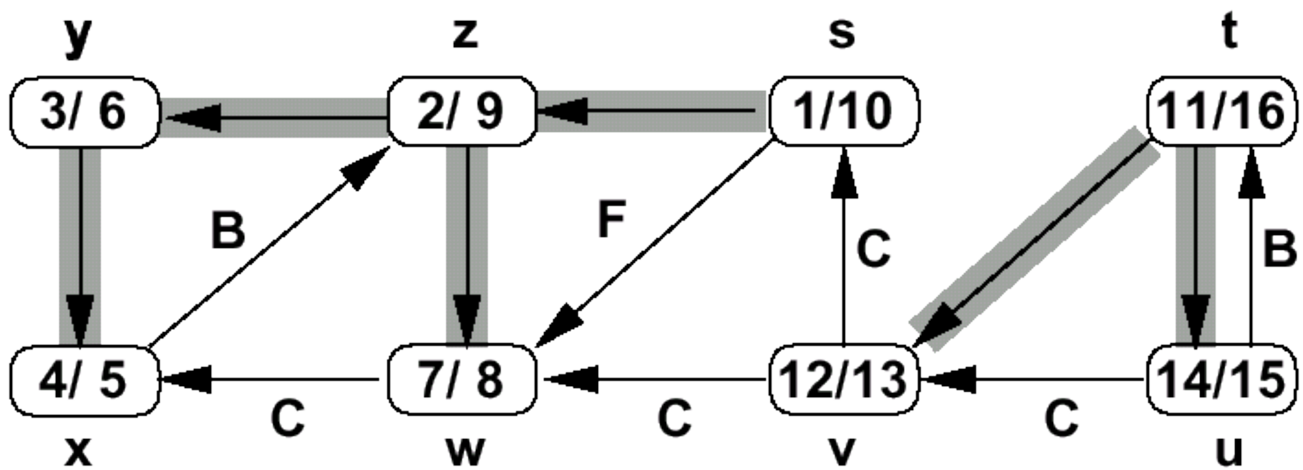
- Define slightly different from BFS

$$G_\pi = (V, E_\pi)$$
$$E_\pi = \left\{(\pi[v], v) \in E : v \in V \text{ and } \pi[v] \neq \text{NIL}\right\}$$

- The PD subgraph of a depth-first search forms a **depth-first forest** composed of several depth-first trees
- The edges in G_π are called tree edges

DFS Timestamping

- The DFS algorithm maintains a monotonically increasing global clock
  - discovery time *d*[*u*] and finishing time *f*[*u*]
- For every vertex *u*, the inequality *d*[*u*] < *f*[*u*] must hold



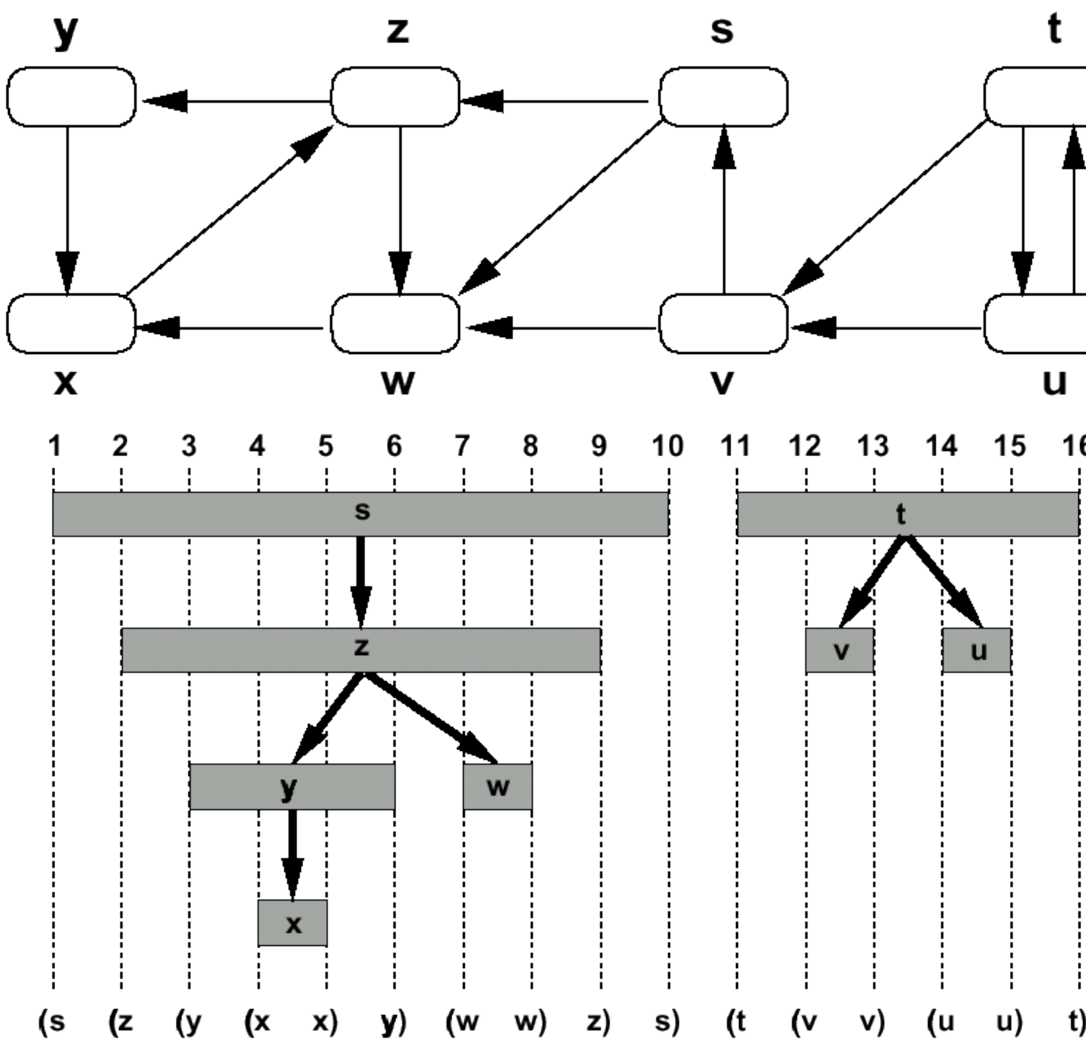
DFS Timestamping

- Vertex *u* is
  - white before time *d*[*u*]
  - gray between time *d*[*u*] and time *f*[*u*], and
  - black thereafter
- Notice the structure througout the algorithm.
  - gray vertices form a linear chain
  - correponds to a stack of vertices that have not been exhaustively explored (DFS-Visit started but not yet finished)

DFS Parenthesis Theorem

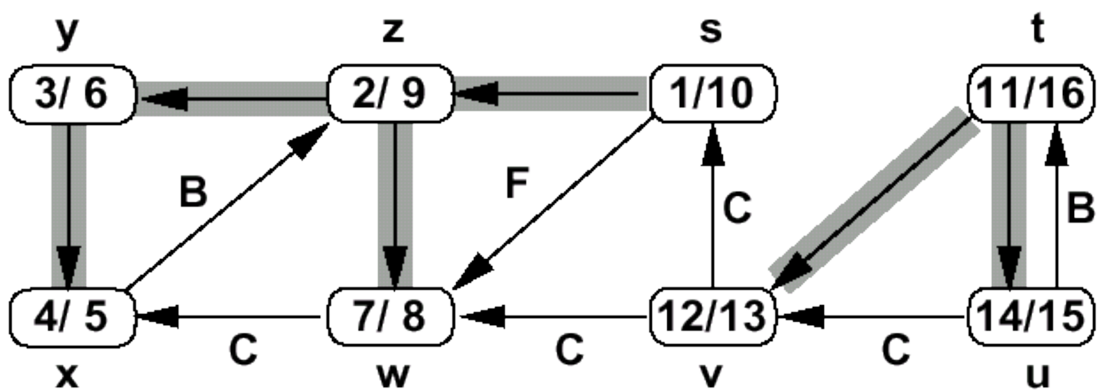
- Discovery and finish times have parenthesis structure
  - represent discovery of *u* with left parenthesis "(u"
  - represent finishin of *u* with right parenthesis "u)"
  - history of discoveries and finishings makes a well-formed expression (parenthesis are properly nested)
- Intuition for proof: any two intervals are either disjoint or enclosed
  - Overlaping intervals would mean finishing ancestor, before finishing descendant or starting descendant without starting ancestor

DFS Parenthesis Theorem (2)



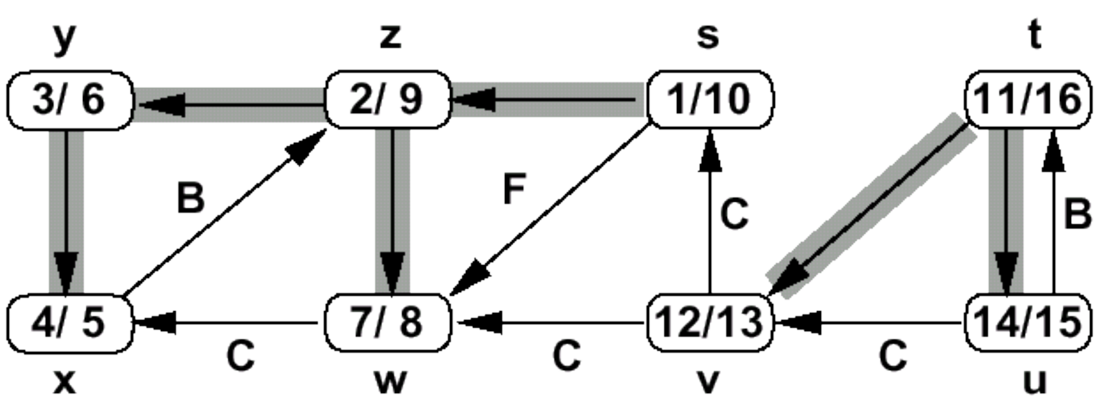
DFS Edge Classification

- Tree edge (gray to white)
  - encounter new vertices (white)
- Back edge (gray to gray)
  - from descendant to ancestor



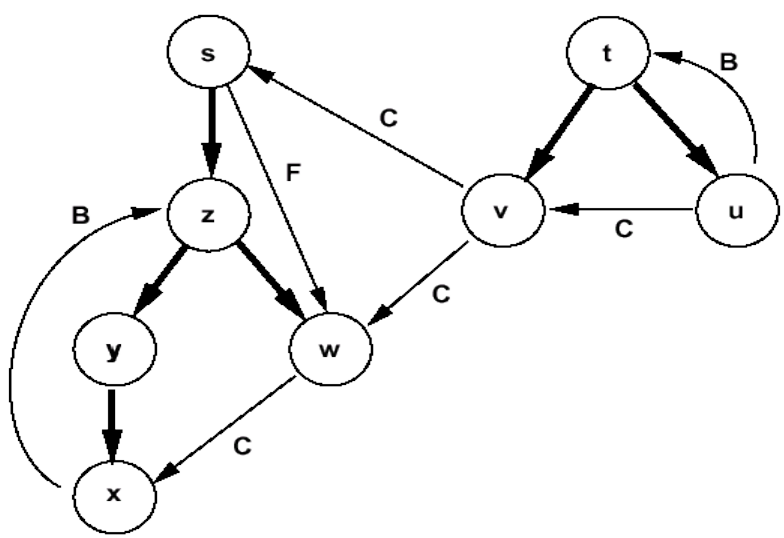
DFS Edge Classification (2)

- Forward edge (gray to black)
  - from ancestor to descendant
- Cross edge (gray to black)
  - remainder – between trees or subtrees



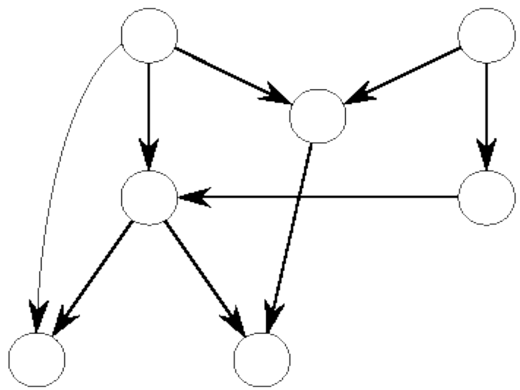
DFS Edge Classification (3)

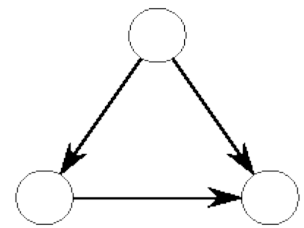
- Tree and back edges are important
- Most algorithms do not distinguish between forward and cross edges



Directed Acyclic Graphs

- A DAG is a directed graph with no cycles





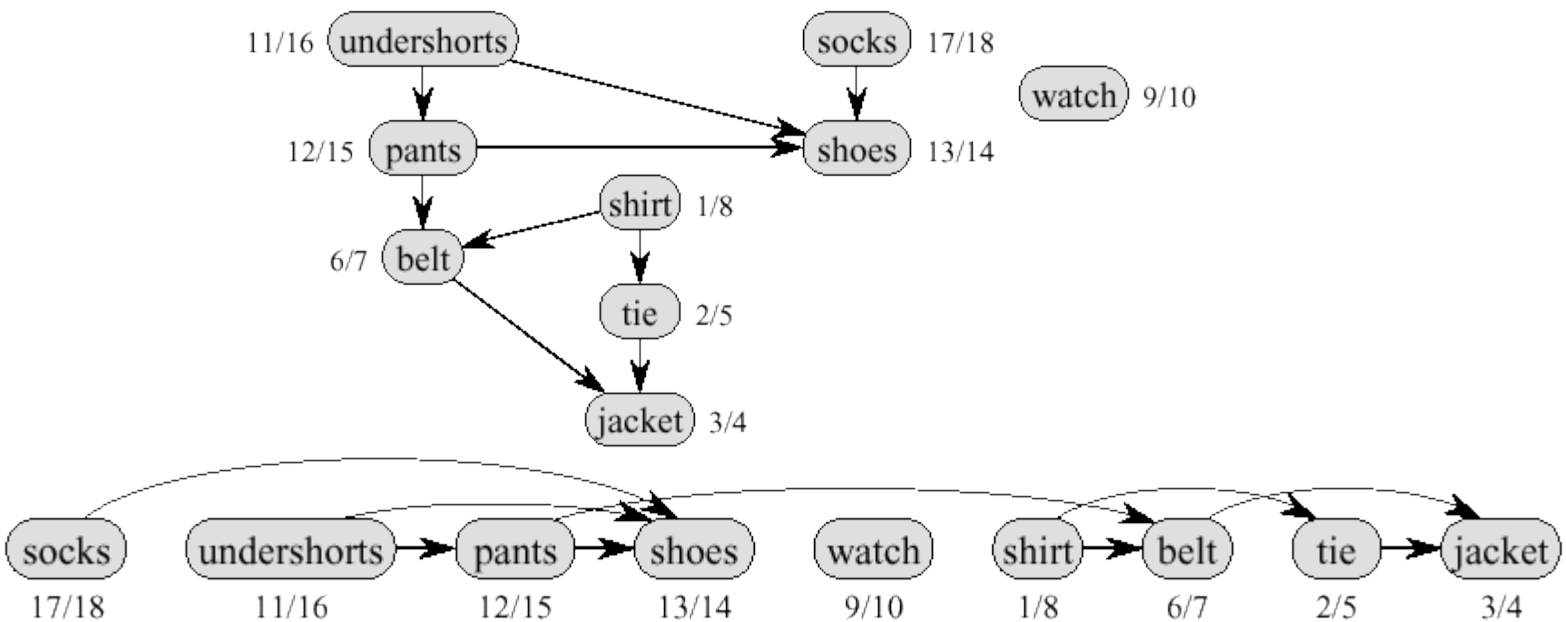
- Often used to indicate precedences among events, i.e., event *a* must happen before *b*
- An example would be a parallel code execution
- Total order can be introduced using **Topological Sorting**

DAG Theorem

- A directed graph *G* is acyclic if and only if a DFS of *G* yields no back edges. Proof:
  - **suppose there is a back edge (*u*,*v*);** *v* is an ancestor of *u* in DFS forest. Thus, there is a path from *v* to *u* in G and (*u*,*v*) completes the cycle
  - **suppose there is a cycle *c*;** let *v* be the first vertex in *c* to be discovered and *u* is a predecessor of *v* in *c*.
    - Upon discovering *v* the whole cycle from *v* to *u* is white
    - We must visit all nodes reachable on this white path before return DFS-Visit(*v*), i.e., vertex *u* becomes a descendant of *v*
    - Thus, (*u,v*) is a back edge
- Thus, we can verify a DAG using DFS!

Topological Sort Example

- Precedence relations: an edge from *x* to *y* means one must be done with *x* before one can do *y*
- Intuition: can schedule task only when all of its subtasks have been scheduled



Topological Sort

- Sorting of a directed acyclic graph (DAG)
- A topological sort of a DAG is a linear ordering of all its vertices such that for any edge (*u*,*v*) in the DAG, *u* appears before *v* in the ordering
- The following algorithm topologically sorts a DAG

Topological-Sort(G)
1) call DFS(G) to compute finishing times *f*[*v*] for each vertex *v*
2) as each vertex is finished, insert it onto the front of a linked list
3) return the linked list of vertices

- The linked lists comprises a total ordering

Topological Sort

- Running time
  - depth-first search: $O(V+E)$ time
  - insert each of the $|V|$ vertices to the front of the linked list: $O(1)$ per insertion
- Thus the total running time is $O(V+E)$

Topological Sort Correctness

- Claim: for a DAG, an edge $(u,v) \in E \Rightarrow f[u] > f[v]$
- When (u,v) explored, u is gray. We can distinguish three cases
  - v = gray
    $\Rightarrow$ (u,v) = back edge (cycle, contradiction)
  - v = white
    $\Rightarrow$ v becomes descendant of u
    $\Rightarrow$ v will be finished before u
    $\Rightarrow$ $f[v] < f[u]$
  - v = black
    $\Rightarrow$ v is already finished
    $\Rightarrow$ $f[v] < f[u]$
- The definition of topological sort is satisfied