Lecture 22

Monday, 21 March 2022 1:35 PM

Recall SONC :

- $\underline{x}^*$ is a local minimizer of $f: \mathbb{R}^n \to \mathbb{R}$ s.t. $\underline{h}(\underline{x}) = \underline{0}$; $\underline{h}: \mathbb{R}^n \to \mathbb{R}^m$ $(m \leq n)$,
- $f, h \in C^2$
- $\underline{x}^*$ is regular rank$[Dh_1(x^*) \ldots Dh_m(x^*)] = m$

(A)

Then (B)

$\exists$ $\underline{\lambda}^* \in \mathbb{R}^m$ s.t.

(i) $Df(\underline{x}^*) + \underline{\lambda}^{*T} Dh(\underline{x}^*) = \underline{0}^T$

(ii) for all $\underline{y} \in T(\underline{x}^*)$, $y^T \mathcal{L}(\underline{x}^*, \underline{\lambda}^*)\, \underline{y} \geq 0$.

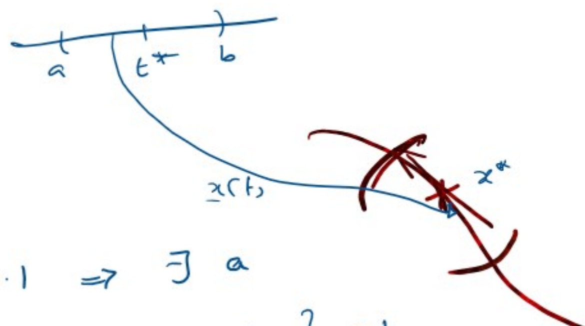
$\boxed{\text{SOSC}}$: Let (A) hold, B(i) hold.

For all $\underline{y} \in T(\underline{x}^*)$, $\underline{y} \neq \underline{0}$

$$\underline{y}^T \mathcal{L}(\underline{x}^*, \lambda^*)\, \underline{y} > 0$$

$\Rightarrow$ $\underline{x}^*$ is a strict local minimizer of f s.t. $\underline{h}(\underline{x}) = \underline{0}$.

Proof of SONC



(i) is I order condition.

(ii)

Let $y \in T(\underline{x}^*)$, Theorem 20.1 $\Rightarrow$ $\exists$ a twice differentiable curve $\{x(t),\ t \in (a,b)\}$ s.t.

$$\begin{cases} \underline{x}(t^*) = x^* \\ \dot{\underline{x}}(t^*) = y \end{cases} \quad \text{for some } t^* \in (a,b)$$

t^* is a local minimizer for $\phi(t) = \underline{f(\underline{x}(t))}$

$$\Leftarrow \begin{cases} \phi'(t^*) = 0 \\ \phi''(t^*) \geq 0 \end{cases}$$

$\leftarrow \left[\phi''(t^*) \geq 0 \right.$

$$\begin{cases} h(x(t)) = 0 \\ 0 = \frac{d^2}{dt^2}\left[\lambda_1^* h_1 + \cdots + \lambda_m^* h_m\right] \end{cases}$$

HW

$$\frac{d}{dt}\left[\underline{y}(t)^T \underline{z}(t)\right] = \underline{z}(t)^T \frac{d\underline{y}}{dt}(t) + \underline{y}(t)^T \frac{d\underline{z}}{dt}(t)$$

$$\boxed{\phi''(t^*) \geq 0}$$

$$\phi(t) = f(\underline{x}(t))$$

$$\frac{d^2\phi}{dt^2}(t^*) = \frac{d}{dt}\left[\frac{d\phi}{dt}(t^*)\right]$$

$$= \frac{d}{dt}\left[\overset{(\nabla f)^T}{Df}(\underline{x}(t^*))\ \dot{\underline{x}}(t^*)\right]$$

$$= \dot{\underline{x}}(t^*)^T \frac{d}{dt}\left[Df(\underline{x}(t^*))^T\right] + Df(\underline{x}(t^*))\ \ddot{\underline{x}}(t^*)$$

$$\phi''(t^*) = \dot{\underline{x}}(t^*)^T F(\underline{x}(t^*))\ \dot{\underline{x}}(t^*) + Df(\underline{x}(t^*))\ \ddot{\underline{x}}(t^*) \geq 0 \quad (1)$$

$$\underline{h}(\underline{x}(t)) = 0 \quad \forall\ t \in (a,b)$$

$$\Rightarrow \frac{d^2}{dt^2}\left[\lambda_1^* h_1(\underline{x}(t)) + \cdots \lambda_m^* h_m(\underline{x}(t))\right] = 0$$

$$\Rightarrow \frac{d^2}{dt^2}\left[\sum_{k=1}^{m} \lambda_k^* h_k(\underline{x}(t))\right] = 0$$

$$\Rightarrow \frac{d}{dt}\left[\sum_{k=1}^{m} \lambda_k^* \frac{d}{dt}\left(h_k(\underline{x}(t))\right)\right] = 0$$

$$\Rightarrow \frac{d}{dt}\left[\sum_{k=1}^{m} \lambda_k^* Dh_k(\underline{x}(t))\ \dot{\underline{x}}(t)\right] = 0$$

$$\Rightarrow \sum_{k=1}^{m} \lambda_k^* \frac{d}{dt}\left[Dh_k(x(t))\ \dot{\underline{x}}(t)\right] = 0$$

$$\Rightarrow \sum_{k=1} \quad \lambda_k \quad \frac{d}{dt}\left[\qquad\right]$$

$$\boxed{\begin{aligned}&\frac{d}{dt}\left[\underline{y}(t)^T \underline{z}(t)\right] \checkmark \\ &= \underline{z}(t)^T \frac{d\underline{y}}{dt} + \underline{y}(t)^T \frac{d\underline{z}}{dt}\end{aligned}}$$

$$= \sum_{k=1}^{m} \lambda_k^* \left(\begin{aligned} &(\dot{x}(t))^T H_k(x(t))\, \dot{x}(t) \\ &+ Dh_k(x(t))\, \ddot{x}(t) \end{aligned} \right) = 0$$

$$\Rightarrow \boxed{\begin{aligned}&\underbrace{\sum_{k=1}^{m} \lambda_k^* (\dot{x}(t^*))^T H_k(\underline{x}(t^*))}\, \dot{\underline{x}}(t^*) \\ &+ \sum_{k=1}^{m} \lambda_k^* Dh_k(x(t^*))\, \ddot{x}(t^*) = 0 \quad -③\end{aligned}}$$

$$① + ③ \geq 0$$

$$\boxed{\underline{y}^T F(\underline{x}^*)\, \underline{y} + \underline{y}^T \left[\underline{\lambda}^* H(\underline{x}^*)\right] \underline{y}} \quad \geq 0$$

$$+ \left[\underbrace{Df(x^*) + \underline{\lambda}^* D\underline{h}(\underline{x}^*)}_{=0}\right] \ddot{x}(t^*) \geq 0$$

(I order condition).

$\boxed{\underline{y} = \dot{x}(t^*)}$

SONC: $$\boxed{\underline{y}^T \mathcal{L}(\underline{x}^*, \underline{\lambda}^*)\, \underline{y} \geq 0 \quad \forall \underline{y} \in T(x^*)}$$

$l(x, \lambda) \rightarrow$ Lagrangian function.

Example

$(x \in \mathbb{R}^n)$

$$\boxed{\max \left(\frac{\underline{x}^T Q \underline{x}}{\underline{x}^T P \underline{x}} \right)} \qquad \left.\begin{aligned} Q = Q^T > 0 \\ P = P^T > 0 \end{aligned}\right]$$

→ If $\underline{x}$ is a solution, $t\underline{x}$ for $t \neq 0$ is also a solution.

→ To avoid multiplicity impose $\boxed{\underline{x}^T P \underline{x} = 1}$ ✓

$$\max \overbrace{\underline{x}^T Q \underline{x}}^{f} \quad \text{s.t.} \quad \overbrace{\underline{x}^T P \underline{x} = 1}^{h(\underline{x})} \qquad m = 1$$

$x^T Q x$

$$\max \sqrt{\underline{x}^T Q \underline{x}} \quad \text{s.t.} \quad \underline{x}^T P \underline{x} = 1$$

$$\ell(\underline{x}, \lambda) = \underbrace{\sqrt{\underline{x}^T Q \underline{x}}}_{f} + \lambda \big(\underbrace{1 - \underline{x}^T P \underline{x}}_{h}\big)$$

$$\underline{x}^T Q \underline{x} = a_{11} x_1^2 + 2a_{12} x_1 x_2 + \cdots + 2a_{1n} x_1 x_n + a_{22} x_2^2 + \cdots + a_{nn} x_n^2$$

I order condition

$$\begin{cases} \cancel{2}\, \underline{x}^T Q - \cancel{2} \lambda\, \underline{x}^T P = \underline{0}^T \\ 1 - \underline{x}^T P \underline{x} = 0 \end{cases}$$

$$Q \underline{x} - \lambda P \underline{x} = 0 \Rightarrow (\lambda P - Q)\underline{x} = 0$$

$$\Rightarrow \boxed{P^{-1} Q \underline{x} = \lambda \underline{x}}$$

Stationary points are eigenvectors of $P^{-1}Q$; Eigenvalues are Lagrange x.

$(\underline{\lambda}^*, \underline{x}^*)$ is optimal $\boxed{\underline{x}^{*T} P \underline{x}^* = 1}$

$$Q = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \qquad P = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

Verify: $$P^{-1} Q = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$(P^{-1}Q)\underline{x} = \lambda \underline{x}$$

$$\begin{vmatrix} 2-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} = 0 \Rightarrow \boxed{\lambda_1 = 2, \quad \lambda_2 = 1}$$

$\boxed{\lambda_1 = 2}$ Eigenvectors $$\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^* = \begin{pmatrix} \alpha \\ 0 \end{pmatrix} \qquad \begin{pmatrix} \alpha & 0 \end{pmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ 0 \end{bmatrix} = 1$$

$$\begin{pmatrix} \alpha & 0 \end{pmatrix} \begin{pmatrix} 2\alpha \\ 0 \end{pmatrix} = 1$$

$$2\alpha^2 = 1 \qquad \alpha = \pm \frac{1}{\sqrt{2}}$$

$\boxed{\lambda_1 = 2}$

Consider e.v. $\begin{pmatrix} 1/\sqrt{2} \\ 0 \end{pmatrix}, \begin{pmatrix} -1/\sqrt{2} \\ 0 \end{pmatrix}$ that satisfy $\underline{x}^* P \underline{x} = 1$

$\boxed{\lambda_2 = 1}$

e.v. $\rightarrow \begin{pmatrix} 0 \\ \alpha \end{pmatrix}$

$$\underline{x}^* P \underline{x} = 1$$

$$\begin{bmatrix} 0 & \alpha \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ \alpha \end{bmatrix} = 1$$

$$\alpha^2 = 1 \qquad \alpha = \pm 1$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$\left(2;\ \underbrace{\begin{pmatrix} 1/\sqrt{2} \\ 0 \end{pmatrix}}_{x_1^*}\right);\ \left(2,\ \underbrace{\begin{pmatrix} -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}}_{x_2^*}\right);\ \left(1,\ \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{x_3^*}\right);\ \left(1,\ \underbrace{\begin{pmatrix} 0 \\ -1 \end{pmatrix}}_{x_4^*}\right)$$

Tangent space for $1 - \underline{x}^T P \underline{x} = 0$ at x_1^*.

$$T(\underline{x}_1^*) = \{ \underline{y} \in \mathbb{R}^2 : Dh(x^*)\ \underline{y} = 0 \}$$

$$= \{ \underline{y} \in \mathbb{R}^2 : \underline{x}^{*T} \underline{P}\ \underline{y} = 0 \}$$

$$= \{ \underline{y} \in \mathbb{R}^2 : \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \underline{y} = 0 \}$$

$$= \{ \underline{y} \in \mathbb{R}^2 : \begin{bmatrix} \sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 0 \}$$

$$\sqrt{2} y_1 = 0 \Rightarrow y_1 = 0 \qquad y_2 = \text{free}.$$

$\sqrt{2}\, y_1 = 0 \Rightarrow y_1 = 0 \qquad y_2 = \text{free}.$

$$T(\underline{x}_1^*) = \left\{ \begin{pmatrix} 0 \\ \alpha \end{pmatrix} : \alpha \in \mathbb{R} \right\}$$

$\ell(x, \lambda) = \underline{x}^T Q x + \lambda (1 - x^T P x)$

$\mathcal{L}(\underline{x}_1^*, \lambda^*) = 2(Q - \lambda P)$

$= 2Q - 4P$

$= \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}$

$$\underline{y}^T \mathcal{L}\, \underline{y}$$

$$= \begin{bmatrix} 0 & \alpha \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ \alpha \end{bmatrix}$$

$$= -2\alpha^2 < 0$$

$\underline{x}_1^* = \begin{pmatrix} 1/\sqrt{2} \\ 0 \end{pmatrix}$ is a strict local maxima.

Problems with inequality constraints

$$\begin{cases} \min & f(\underline{x}) \\ \text{s.t} & \underline{h}(x) = 0 \\ & g(\underline{x}) \le 0 \end{cases}$$

$f : \mathbb{R}^n \to \mathbb{R}$

$\underline{h} : \mathbb{R}^n \to \mathbb{R}^m \; (m \le n)$

$\underline{g} : \mathbb{R}^n \to \mathbb{R}^p$

$\boxed{\text{KKT}}$