Linear Programming Problems.

Monday, 4 April 2022 1:05 PM

Standard form

I. min $\underline{c}^T \underline{x}$ s.t. $A\underline{x} = \underline{b}$ $\underline{x} \geq 0$

$(A_{m\times n}\ \underline{x}_{n\times 1} = \underline{b}_{m\times 1})$

$m < n$

rank $A = m$.

$[\underline{b} \geq 0]$

II. min $\underline{c}^T \underline{x}$

s.t. $A\underline{x} \geq \underline{b}$

$\underline{x} \geq 0$

III. min $\underline{c}^T \underline{x}$

s.t. $A\underline{x} \leq \underline{b}$

$\underline{x} \geq 0$

Standard form for II. & III.

min $\underline{c}^T \underline{x}$

(Surplus variables)

s.t. $a_{i1}x_1 + \cdots + a_{in}x_n - y_i = b_i$

$i = 1, \ldots, m$

$x_1, x_2, \ldots, x_n \geq 0$

$y_1, y_2, \ldots y_m \geq 0$

$A\underline{x} - I_m \underline{y} = \underline{b}$

$A\underline{x} + I_m \underline{y} = \underline{b}$

(Slack variables)

$A_{m\times n}$ $m < n$

$A = [\underbrace{B} \mid D]$ (reordering)

s.t. $B_{m\times m}$ is non-singular

$D_{m\times(n-m)}$

$A\underline{x} = \underline{b}_{m\times 1}$

$\begin{bmatrix} x_1 \\ \end{bmatrix}$

$m \times (n-m)$

$$\underset{m\times n}{A}\,\underline{x} = \underline{b}_{m\times 1}$$

$$B\,\underline{x}_B = \underline{b}$$

$$\begin{array}{c} x_B \\ \\ \underline{x}_D \end{array} \begin{bmatrix} \left.\begin{matrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{matrix}\right\} m \\ \hline \left.\begin{matrix} x_{m+1} \\ \vdots \\ x_n \end{matrix}\right\} (n-m) \end{bmatrix}$$

$$\left[\underset{m\times m}{B} \;\middle|\; \underset{m\times(n-m)}{D}\right] \begin{bmatrix} \underline{x}_B \\ \hline \underline{\underline{x}}_D \end{bmatrix} = \underline{b}$$

$$\underline{x}_B = B^{-1}\underline{b}$$

$$\underbrace{B\,\underline{x}_B}_{\text{basic variables}} + \underbrace{D\,\underline{x}_D}_{\text{free variables.}} = \underline{b}$$

$$\boxed{\underline{x} = \left[\underline{\underline{x}}_B^T,\; \underline{0}^T\right]^T}$$

Ex.

$$A = \left[\underline{a}_1 \quad \underline{a}_2 \quad \underline{a}_3 \quad \underline{a}_4\right] = \begin{bmatrix} 1 & 1 & -1 & 4 \\ 1 & -2 & -1 & 1 \end{bmatrix} \qquad \underline{b} = \begin{bmatrix} 8 \\ 2 \end{bmatrix}.$$

$\underline{m = 2}$

Aim: To find <u>basic solutions</u>

$$B = [\underline{a}_1 \quad \underline{a}_2] \qquad D = [\underline{a}_3 \quad \underline{a}_4]$$

$$\left[\begin{array}{cc|cc} 1 & 1 & -1 & 4 \\ 1 & -2 & -1 & 1 \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \end{bmatrix}.$$

$R_2 \to R_2 - R_1$

$$\left[\begin{array}{cc|cc} 1 & 1 & -1 & 4 \\ 0 & -3 & 0 & -3 \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ \hline x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 8 \\ -6 \end{bmatrix}$$

$x_3 = x_4 = 0 \qquad -3x_2 = -6 \qquad x_2 = 2.$

$x_1 + x_2 = 8 \qquad x_1 = 6$

$$\underline{x}_B = \begin{bmatrix} 6 \\ 2 \end{bmatrix} \qquad \boxed{\underline{x} = \begin{bmatrix} 6 \\ 2 \\ 0 \\ 0 \end{bmatrix}} \rightarrow \boxed{\text{basic solution}}$$

$\underline{a}_1 x_1 + \underline{a}_2 x_2 + \underline{a}_3 x_3 + \underline{a}_4 x_4 = \underline{b}$

$$\begin{cases} x_1 + x_2 - x_3 + 4x_4 = b_1 \\ x_1 + 4x_4 + x_2 - x_3 = b_1 \end{cases}$$

Another choice:

$$B = [\underline{a}_1 \quad \underline{a}_4] \qquad \underline{D} = [\underline{a}_2 \quad \underline{a}_3]$$

$$\left[\begin{array}{cc|cc} 1 & 4 & 1 & -1 \\ 1 & 1 & -2 & -1 \end{array}\right] \begin{bmatrix} x_1 \\ x_4 \\ \hline x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \end{bmatrix}$$

$\boxed{R_2 \to R_2 - R_1}$

$$\left[\begin{array}{cc|cc} 1 & 4 & 1 & -1 \\ 0 & -3 & -3 & 0 \end{array}\right] \begin{bmatrix} x_1 \\ x_4 \\ \hline x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -6 \end{bmatrix}$$

$x_2 = x_3 = 0 \qquad -3x_4 = -6 \qquad x_4 = 2$

$x_1 + 4x_4 = 8 \qquad x_1 + 8 = 8 \qquad \boxed{x_1 = 0}$

Basic soln.

$$\underline{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix}$$ is the basic soln corresponding $\underline{a}_1$ and $\underline{a}_4$

(degenerate basic solution)

III choice

$B = [\underline{a}_2 \quad \underline{a}_3]$

$$
\begin{array}{cccc}
\underline{a}_2 & \underline{a}_3 & \underline{a}_1 & a_4 \\
\end{array}
$$

$$
\left[\begin{array}{cc|cc} 1 & -1 & 1 & 4 \\ -2 & -1 & 1 & 1 \end{array}\right]
\begin{bmatrix} x_2 \\ x_3 \\ x_1 \\ x_4 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \end{bmatrix}
$$

$R_2 \to R_2 + 2R_1$

$$
\left[\begin{array}{cc|cc} 1 & -1 & 1 & 4 \\ 0 & -3 & \cdots & \cdots \end{array}\right]
\begin{bmatrix} x_2 \\ x_3 \\ x_1 \\ x_4 \end{bmatrix} = \begin{bmatrix} 8 \\ 18 \end{bmatrix}
$$

$x_1 = x_4 = 0$ $\quad -3x_3 = 18$ $\quad \boxed{x_3 = -6}$

$x_2 - x_3 = 8$

$x_2 = 8 + -6 = 2.$

$$
\boxed{\underline{x} = \begin{bmatrix} 0 \\ 2 \\ -6 \\ 0 \end{bmatrix}} \longrightarrow \underline{x} \ngeq 0 \quad \left[\text{Basic but not feasible}\right].
$$

Def'ns:

- If some of the basic variables of the basic soln are zeroes, then the basic soln is called a degenerate basic soln.
- A vector $\underline{x}$ satisfying $A\underline{x} = \underline{b}$, $\underline{x} \geq 0$ is called a feasible soln.
- ... that is also basic is called a

a feasible soln.

- A feasible soln. that is also basic is called a basic feasible soln.

max # of basic solns $= {}^nC_m$.

Ex. 2 $A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 4 & 1 & 1 & -2 \end{bmatrix}$ $\underline{b} = \begin{bmatrix} -1 \\ 9 \end{bmatrix}$

$A\underline{x} = \underline{b}$

$\underline{x} = \underline{v} + \underline{h}$

g. soln. $A\underline{h} = 0$

$A\underline{v} = \underline{b}$

$\underline{v}$ particular soln

Gauss Jordan Elimination (Augmented matrix)

$$\left[\begin{array}{cccc|c} 2 & 3 & -1 & -1 & -1 \\ 4 & 1 & 1 & -2 & 9 \end{array}\right]$$

$\boxed{R_1 \to \frac{R_1}{2}}$ $\sim \left[\begin{array}{cccc|c} 1 & 3/2 & -1/2 & -1/2 & -1/2 \\ 4 & 1 & 1 & -2 & 9 \end{array}\right]$

$\boxed{R_2 \to R_2 - 4R_1}$ $\sim \left[\begin{array}{cccc|c} 1 & 3/2 & -1/2 & -1/2 & -1/2 \\ 0 & \boxed{-5} & 3 & 0 & 11 \end{array}\right]$

$\boxed{R_2 \to \frac{R_2}{-5}}$ $\sim \left[\begin{array}{cccc|c} 1 & 3/2 & -1/2 & -1/2 & -1/2 \\ 0 & 1 & -3/5 & 0 & -11/5 \end{array}\right]$

$$\begin{array}{cccc} x_1 & x_2 & & \\ \end{array}$$
$$\left[\begin{array}{cccc|c} 1 & 0 & 2/5 & -1/2 & 14/5 \end{array}\right.$$

$$\boxed{R_1 \to R_1 - 3/2\, R_2} \quad \sim \left[\begin{array}{cccc|c} 1 & 0 & 2/5 & -1/2 & 14/5 \\ 0 & 1 & -3/5 & 0 & -11/5 \end{array}\right]$$

$$x_1 + \underline{2/5\, x_3 - 1/2\, x_4} = 14/5$$

$$x_2 - 3/5\, x_3 = -11/5$$

$x_3 = s$

$x_4 = t$

$$x_1 = \frac{14}{5} - \frac{2}{5}s + \frac{1}{2}t$$

$$x_2 = -11/5 + 3/5\, s$$

$$x_3 = s$$

$$x_4 = t$$

$x_2 = x_4 = 0$

$\Rightarrow \boxed{s = 11/3} = x_3$

$x_1 = \frac{14}{5} - \frac{2}{5} \times \frac{11}{3}$

$= 4/3$

$$\begin{bmatrix} 4/3 \\ 0 \\ 11/3 \\ 0 \end{bmatrix}$$

$$\boxed{\underline{x} = \underbrace{\begin{pmatrix} 14/5 \\ -11/5 \\ 0 \\ 0 \end{pmatrix}}_{\underline{v}} + \underbrace{s\begin{pmatrix} -2/5 \\ 3/5 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1/2 \\ 0 \\ 0 \\ 1 \end{pmatrix}}_{\underline{h}}}$$

$$\underline{h} = s\begin{pmatrix} -2/5 \\ 3/5 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1/2 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\underline{v} = \begin{pmatrix} 14/5 \\ -11/5 \\ 0 \\ 0 \end{pmatrix}$$

$\underline{x} = \underline{v} + \underline{h}$

(general soln).

$\boxed{\text{Basic Solutions}}$?

$x_3 = x_4 = 0$ $\qquad x_B = \begin{pmatrix} 14/5 \\ -11/5 \end{pmatrix}$

$\begin{bmatrix} \frac{14}{5} & \boxed{\frac{-11}{5}} & 0 & 0 \end{bmatrix}^T$ solves $A\underline{x} = \underline{b}$

but not feasible.

$\boxed{x_2 = x_4 = 0}$

Importance of basic feasible solutions

While solving LPP, we need to consider only basic feasible solutions. The optimal solution (if it exists) is always achieved at the basic feasible soln.

$\boxed{\text{Convexity}}$ The set of points that satisfy

$A\underline{x} = \underline{b}, \quad \underline{x} \geq 0$ [feasible solns]

form a convex set.

$\Omega = \{ \underline{x} : A\underline{x} = \underline{b}, \ \underline{x} \geq 0 \}$

$A\underline{x}_1 = \underline{b} \qquad x_1 \geq 0$

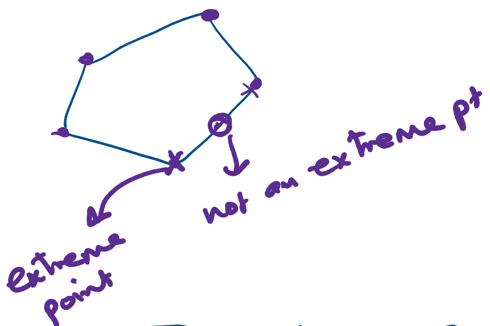
$A\underline{x}_2 = \underline{b} \qquad x_2 \geq 0$

$\underbrace{\alpha x_1 + (1-\alpha) x_2} \geq 0$

$A(\alpha \underline{x}_1 + (1-\alpha)\underline{x}_2)$

$= \alpha A x_1 + (1-\alpha) A x_2 = \underline{b}$

<u>Extreme point</u> $\underline{x}$ is an extreme point of Ω if there are no two distinct points $\underline{x}_1$ and $\underline{x}_2$ in Ω s.t. $\underline{x} = \alpha \underline{x}_1 + (1-\alpha)\underline{x}_2$, for $\alpha \in (0,1)$.



$\boxed{\text{Thm}}$ Let Ω be a convex set of all feasible solutions, that is, all the vectors satisfy

$$A\underline{x} = \underline{b}, \quad \underline{x} \geq 0. \qquad [A \in \mathbb{R}^{m \times n}, \ m < n]$$

Then $\underline{x}$ is <u>an extreme point</u> of Ω

$\iff$ $\underline{x}$ is a <u>basic feasible solution</u> of $A\underline{x} = \underline{b}$, $\underline{x} \geq 0$