$\cdots$, x_n in I. How well does $p_n(x)$ approximate f on the interval I? This question leads to the analysis of *interpolation error*.

As we discussed at the beginning of Chapter 3 (Error Analysis), the error given by

$$\mathrm{ME}_n(x) = f(x) - p_n(x). \tag{8.17}$$

is the *mathematical error* involved in the interpolating polynomial. But, when we perform the calculations using finite precision floating-point arithmetic, then the polynomial obtained, denoted by $\tilde{p}_n(x)$, need not be the same as the interpolating polynomial $p_n(x)$. The error

$$\mathrm{AE}_n(x) = p_n(x) - \tilde{p}_n(x). \tag{8.18}$$

is the *arithmetic error* involved in the interpolating polynomial.

The *total error*, denoted by $\mathrm{TE}_n(x)$, involved in the polynomial interpolation is therefore given by

$$\mathrm{TE}_n(x) = f(x) - \tilde{p}_n(x) = (f(x) - p_n(x)) + (p_n(x) - \tilde{p}_n(x)) = \mathrm{ME}_n(x) + \mathrm{AE}_n(x).$$

In Subsection 8.3.1, we derive the mathematical error involved in polynomial interpolation. We analyze the effect of arithmetic error in Subsection 8.3.2.

8.3.1 Mathematical Error

The following theorem provides a formula for the interpolation error when we assume that the necessary data are given exactly without any floating-point approximation.

Theorem 8.3.1 [Mathematical Error in Interpolation].

Let $p_n(x)$ be the polynomial interpolating a function $f \in C^{n+1}[a,b]$ at the nodes $x_0, x_1, \cdots, x_n$ lying in $I = [a,b]$. Then for each $x \in I$, there exists a $\xi_x \in (a,b)$ such that

$$\mathrm{ME}_n(x) = \frac{f^{(n+1)}(\xi_x)}{(n+1)!} \prod_{i=0}^{n} (x - x_i) \tag{8.19}$$

Proof.

If x is one of the nodes, then (8.19) holds trivially for any choice of $\xi_x \in (a,b)$. Therefore, it is enough to prove the theorem when x is not a node. The idea is to obtain a function having at least $n+2$ distinct zeros; and then apply Rolle's theorem

$n+1$ times to get the desired conclusion.

For a given $x \in I$ with $x \neq x_i$ $(i = 0, 1, \cdots, n)$, define a new function ψ on the interval I by

$$\psi(t) = f(t) - p_n(t) - \lambda \prod_{i=0}^{n} (t - x_i), \quad t \in I, \tag{8.20}$$

where λ is chosen so that $\psi(x) = 0$. This gives

$$\lambda = \frac{f(x) - p_n(x)}{\displaystyle\prod_{i=0}^{n} (x - x_i)}.$$

Note that $\psi(x_i) = 0$ for each $i = 0, 1, \cdots, n$. Thus, ψ has at least $n+2$ distinct zeros. By Rolle's theorem, the function ψ' has at least $n+1$ distinct zeros in (a, b). A repeated application of Rolle's theorem $n+1$ times to the function ψ gives that $\psi^{(n+1)}$ has at least one zero in (a, b); call it ξ_x. Differentiating (8.20) $(n+1)$-times and noting that

$$p_n^{(n+1)}(t) = 0, \quad \left(\prod_{i=0}^{n} (t - x_i) \right)^{(n+1)} = (n+1)!, \quad \psi^{(n+1)}(\xi_x) = 0,$$

we see that ξ_x satisfies

$$0 = \psi^{(n+1)}(\xi_x) = f^{(n+1)}(\xi_x) - \frac{f(x) - p_n(x)}{\displaystyle\prod_{i=0}^{n} (x - x_i)} (n+1)\,!.$$

Thus,

$$f^{(n+1)}(\xi_x) - \frac{\mathrm{ME}_n(x)}{\displaystyle\prod_{i=0}^{n} (x - x_i)} (n+1)\,! = 0.$$

The last equation yields (8.19).

The following theorem plays an important role in the error analysis of numerical integration. The idea behind the proof of this theorem is similar to the idea used in the above theorem and is left as an exercise.

Theorem 8.3.2.

If $f \in C^{n+1}[a,b]$ and if $x_0, x_1, \cdots, x_n$ are distinct nodes in $[a,b]$, then for any $x \neq x_i$, $i = 0, 1, \cdots, n$, there exists a point $\xi_x \in (a,b)$ such that

$$f[x_0, x_1, \cdots, x_n, x] = \frac{f^{(n+1)}(\xi_x)}{(n+1)\,!}. \tag{8.21}$$

Remark 8.3.3.

It is interesting to note that when all the nodes coincide, then (8.21) reduces to (**??**).

Definition 8.3.4 [Infinity Norm].

If f is continuous on a closed interval $I = [a,b]$, then the ***infinity norm*** of f, denoted by $\|f\|_{\infty,I}$, is defined as

$$\|f\|_{\infty,I} = \max_{x\in I}|f(x)|. \tag{8.22}$$

Example 8.3.5.

Let us obtain an upper bound of the mathematical error for the linear interpolating polynomial with respect to the infinity norm. As in Example 8.1.11, the linear interpolating polynomial for f at x_0 and x_1 ($x_0 < x_1$) is given by

$$p_1(x) = f(x_0) + f[x_0, x_1](x - x_0),$$

where $f[x_0, x_1]$ is given by (8.10). For each $x \in I := [x_0, x_1]$, using the formula (8.19), the error $\mathrm{ME}_1(x)$ is given by

$$\mathrm{ME}_1(x) = \frac{(x - x_0)(x - x_1)}{2}\, f''(\xi_x),$$

where $\xi_x \in (x_0, x_1)$ depends on x. Therefore,

$$|\mathrm{ME}_1(x)| \leq |(x - x_0)(x - x_1)|\, \frac{\|f''\|_{\infty,I}}{2}.$$

Note that the maximum value of $|(x - x_0)(x - x_1)|$ as x varies in the interval $[x_0, x_1]$, occurs at $x = (x_0 + x_1)/2$. Therefore, we have

$$|(x - x_0)(x - x_1)| \leq \frac{(x_1 - x_0)^2}{4}.$$

Using this inequality, we get an upper bound

$$|\mathrm{ME}_1(x)| \leq (x_1 - x_0)^2 \, \frac{\|f''\|_{\infty,I}}{8}, \text{ for all } x \in [x_0, x_1].$$

Note that the above inequality is true for all $x \in [x_0, x_1]$. In particular, this inequality is true for an x at with the function $|\mathrm{ME}_1|$ attains its maximum. Thus, we have

$$\|\mathrm{ME}_1\|_{\infty,I} \leq (x_1 - x_0)^2 \, \frac{\|f''\|_{\infty,I}}{8}.$$

The right hand side quantity, which is a real number, is an upper bound for the mathematical error in linear interpolation with respect to the infinity norm.

Example 8.3.6.

Let the function

$$f(x) = \sin x$$

be approximated by an interpolating polynomial $p_9(x)$ of degree less than or equal to 9 for f at the nodes x_0, x_1, $\cdots$, x_9 in the interval $I := [0, 1]$. Let us obtain an upper bound for $\|\mathrm{ME}_9\|_{\infty,I}$, where (from (8.19))

$$\mathrm{ME}_9(x) = \frac{f^{(10)}(\xi_x)}{10!} \prod_{i=0}^{9} (x - x_i).$$

Since $|f^{(10)}(\xi_x)| \leq 1$ and $\prod_{i=0}^{9} |x - x_i| \leq 1$, we have

$$|\sin x - p_9(x)| = |\mathrm{ME}_9(x)| \leq \frac{1}{10!} < 2.8 \times 10^{-7}, \text{ for all } x \in [0, 1].$$

Since this holds for all $x \in [0, 1]$, we have

$$\|\mathrm{ME}_9\|_{\infty,I} < 2.8 \times 10^{-7}.$$

The right hand side number is the upper bound for the mathematical error ME_9 with respect to the infinity norm on the interval I.

8.3.2 Arithmetic Error

Quite often, the polynomial interpolation that we compute is based on the function data subjected to floating-point approximation. In this subsection, we analyze the arithmetic error arising due to the floating-point approximation $\mathrm{fl}(f(x_i))$ of $f(x_i)$ for each node point x_i, $i = 0, 1, \cdots, n$ in the interval $I = [a, b]$. All other computations are assumed to be performed with infinite precision, for the sake of simplicity.

The Lagrange form of interpolating polynomial that uses the values $\mathrm{fl}(f(x_i))$ instead of $f(x_i)$, $i = 0, 1, \cdots, n$ is given by

$$\tilde{p}_n(x) = \sum_{i=0}^{n} \mathrm{fl}(f(x_i))\ l_i(x).$$

We now analyze the arithmetic error. Denoting

$$\epsilon_i := f(x_i) - \mathrm{fl}(f(x_i); \quad ||\epsilon||_\infty = \max\{|\epsilon_i| \,:\, i = 0, 1, \cdots, n\},$$

we have

$$|\mathrm{AE}_n(x)| = |p_n(x) - \tilde{p}_n(x)| = \left| \sum_{i=0}^{n} \Big(f(x_i) - \mathrm{fl}(f(x_i)) \Big)\ l_i(x) \right| \leq ||\epsilon||_\infty \sum_{i=0}^{n} ||l_i||_{\infty,I},$$

for all $x \in I$. Therefore,

$$\|\mathrm{AE}_n\|_{\infty,I} = \|p_n - \tilde{p}_n\|_{\infty,I} \leq ||\epsilon||_\infty \sum_{i=0}^{n} ||l_i||_{\infty,I}. \tag{8.23}$$

The upper bound in (8.23) might grow quite large as n increases, especially when the nodes are equally spaced as we will study now.

Assume that the nodes are equally spaced in the interval $[a, b]$, with $x_0 = a$ and $x_n = b$, and $x_{i+1} - x_i = h$ for all $i = 0, 1, \cdots, n-1$. Note that $h = (b-a)/n$. We write

$$x_i = a + ih, \quad i = 0, 1, \cdots, n.$$

Any $x \in I$ can be written as

$$x = a + \eta h,$$

where $0 \leq \eta \leq n$. Here η is not necessarily an integer. Therefore, for any $x \in I$, we have

$$l_k(x) = \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{(x - x_i)}{(x_k - x_i)} = \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{(\eta - i)}{(k - i)}, \quad k = 0, \cdots, n.$$

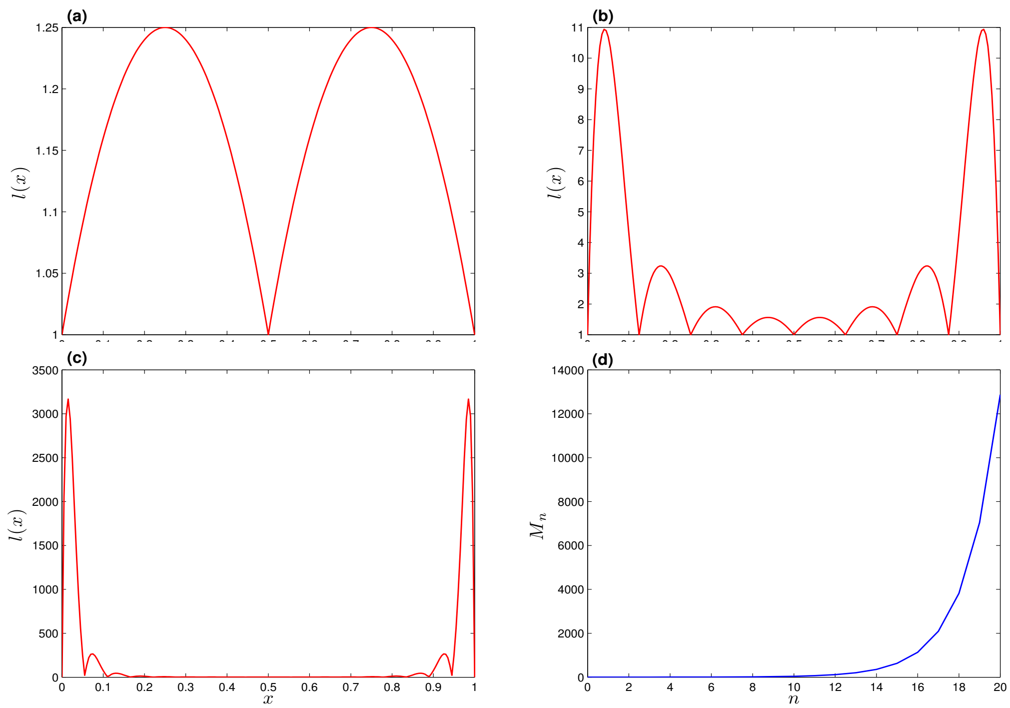

Figure 8.2: (a) to (c) depicts the graph of function l given by (8.24) for $x \in [0,1]$ when $n = 2, 8$, and 18. (d) depicts the function n in the x-axis and M_n given by (8.25) in the y-axis.

Clearly, the Lagrange polynomials are not dependent on the choice of a, b, and h. They depend entirely on n and η (which depends on x). The Figure 8.2 (a), (b) and (c) shows the graph of the function

$$l(x) = \sum_{i=0}^{n} |l_i(x)| \tag{8.24}$$

for $n = 2, 8$ and 18. It is observed that as n increases, the maximum of the function l increases. In fact, as $n \to \infty$, the maximum of l tends to infinity and it is observed in Figure 8.2 (d) which depicts n in the x-axis and the function

$$M_n = \sum_{i=0}^{n} ||l_i||_{\infty,I} \tag{8.25}$$

in the y-axis. This shows that the upper bound of the arithmetic error AE_n given in (8.23) tends to infinity as $n \to \infty$. This gives the possibility that the arithmetic error may tend to increase as n increases. Thus, as we increase the degree of the interpolating polynomial, the approximation may go worser due to the presence of floating-point approximation. In fact, this behavior of the arithmetic error in polynomial interpolation can also be analyzed theoretically, but this is outside the scope of the present course.

8.3.3 Total Error

Let us now estimate the total error, which is given by

$$\mathrm{TE}_n(x) = f(x) - \tilde{p}_n(x) = (f(x) - p_n(x)) + (p_n(x) - \tilde{p}_n(x)). \tag{8.26}$$

Taking infinity norm on both sides of the equation (8.26) and using triangle inequality, we get

$$\|\mathrm{TE}_n(x)\|_{\infty,I} = ||f - \tilde{p}||_{\infty,I} \leq ||f - p_n||_{\infty,I} + ||p_n - \tilde{p}||_{\infty,I} \leq ||f - p_n||_{\infty,I} + ||\epsilon||_\infty M_n.$$

It is clear from the Figure 8.2 (d) that M_n increases exponentially with respect to n. This implies that even if $||\epsilon||_\infty$ is very small, a large enough n can bring in a significantly large error in the interpolating polynomial.

Thus, for a given function and a set of equally spaced nodes, even if the mathematical error is bounded, the presence of floating-point approximation in the given data can lead to significantly large arithmetic error for larger values of n.

8.3.4 Runge Phenomenon

In the previous section, we have seen that even a small arithmetic error may lead to a drastic magnification of total error as we go on increasing the degree of the polynomial. This gives us a feeling that if the calculation is done with infinite precision (that is, without any finite digit floating point arithmetic) and the function f is smooth, then we always have a better approximation for a larger value of n. In other words, we expect

$$\lim_{n\to\infty} \|f - p_n\|_{\infty,I} = 0.$$

But this is not true in the case of equally spaced nodes. This was first shown by Carl Runge, where he discovered that there are certain functions for which, as we go on increasing the degree of interpolating polynomial, the total error increases drastically and the corresponding interpolating polynomial oscillates near the boundary of the interval in which the interpolation is done. Such a phenomenon is called the *Runge Phenomenon*. This phenomenon is well understood by the following example given by Carl Runge.

Example 8.3.7 [Runge's Function].

Consider the *Runge's function* defined on the interval $[-1, 1]$ given by

$$f(x) = \frac{1}{1 + 25x^2}. \tag{8.27}$$

The interpolating polynomials with $n = 2$, $n = 8$ and $n = 18$ are depicted in Figure 8.3. This figure clearly shows that as we increase the degree of the polynomial, the

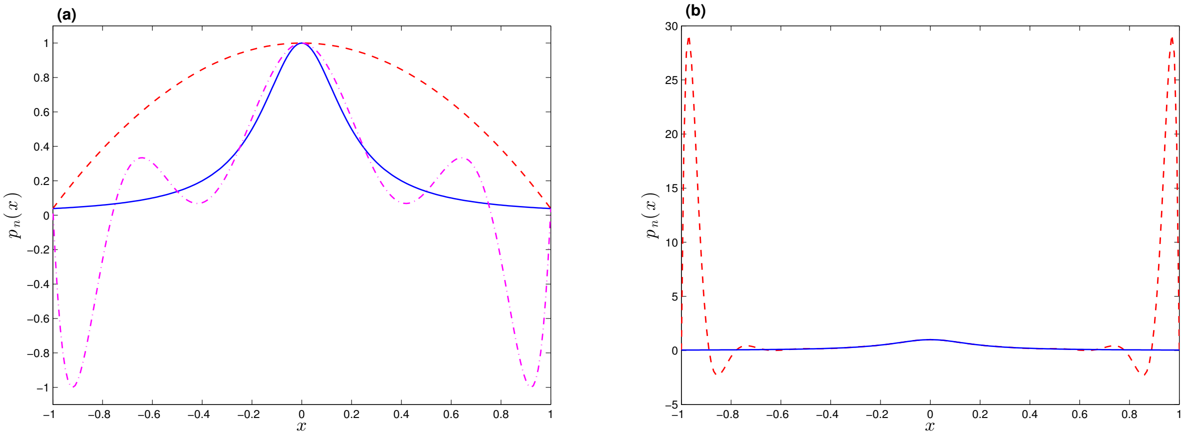

Figure 8.3: Runge Phenomenon is illustrated. Figure (a) depicts the graph of the function f given by (8.27) (blue solid line) along with the interpolating polynomial of degree 2 (red dash line) and 8 (magenta dash dot line) with equally spaced nodes. Figure (b) shows the graph of f (blue solid line) along with the interpolating polynomial of degree 18 (red dash line) with equally spaced nodes.

interpolating polynomial differs significantly from the actual function especially, near the end points of the interval.

In the light of the discussion made in Subsection 8.3.2, we may think that the Runge phenomenon is due to the amplification of the arithmetic error. But, even if the calculation is done with infinite precision (that is, without any finite digit floating point arithmetic), we may still have the Runge phenomenon due to the amplification in mathematical error. This can be observed by taking infinity norm on both sides of the formula (8.19). This gives an upper bound of the infinity norm of $\mathrm{ME}_n(x)$ as

$$\|\mathrm{ME}_n\|_{\infty,I} \leq \frac{(b-a)^{n+1}}{(n+1)!}\|f^{(n+1)}\|_{\infty,I}.$$

Although the first part, $(b-a)^{n+1}/(n+1)!$ in the upper bound tends to zero as $n \to \infty$, if the second part, $\|f^{(n+1)}\|_{\infty,I}$ increases significantly as n increases, then the upper bound can still increase and makes it possible for the mathematical error to be quite high.

A more deeper analysis is required to understand the Runge phenomenon more rigorously.

8.3.5 Convergence of Sequence of Interpolating Polynomials

We end this section by stating without proof a negative result and a positive result concerning the convergence of sequence of interpolating polynomials.

Theorem 8.3.8 [Faber].

For $n \in \mathbb{N}$, let the sequence of nodes

$$a \leq x_0^{(n)} < x_1^{(n)} < \cdots < x_n^{(n)} \leq b$$

be given. Then there exists a continuous function f defined on the interval $[a, b]$ such that the polynomials $p_n(x)$ that interpolate the function f at these nodes have the property that $\|p_n - f\|_{\infty,[a,b]}$ **does not** tend to zero as $n \to \infty$.

Example 8.3.9.

In fact, the interpolating polynomial $p_n(x)$ for the Runge's function goes worser and worser as shown in Figure 8.3 for increasing values of n with equally spaced nodes. That is, $\|f - p_n\|_{\infty,[-1,1]} \to \infty$ as $n \to \infty$ for any sequence of equally spaced nodes.

Let us now state a positive result concerning the convergence of sequence of interpolating polynomials to a given continuous function.

Theorem 8.3.10.

Let f be a continuous function on the interval $[a, b]$. Then there exists a sequence of nodes

$$a \leq x_0^{(n)} < x_1^{(n)} < \cdots < x_n^{(n)} \leq b, \text{ for } n \in \mathbb{N},$$

such that the polynomials $p_n(x)$ that interpolate the function f at these nodes satisfy $\|p_n - f\|_{\infty,[a,b]} \to 0$ as $n \to \infty$.

The Theorem 8.3.10 is very interesting because it implies that for the Runge's function, we can find a sequence of nodes for which the corresponding interpolating polynomial yields a good approximation even for a large value of n.

Example 8.3.11.

For instance, define a sequence of nodes

$$x_i^{(n)} = \cos\left(\frac{(2i+1)\pi}{2(n+1)}\right), i = 0, 1, \cdots, n \tag{8.28}$$

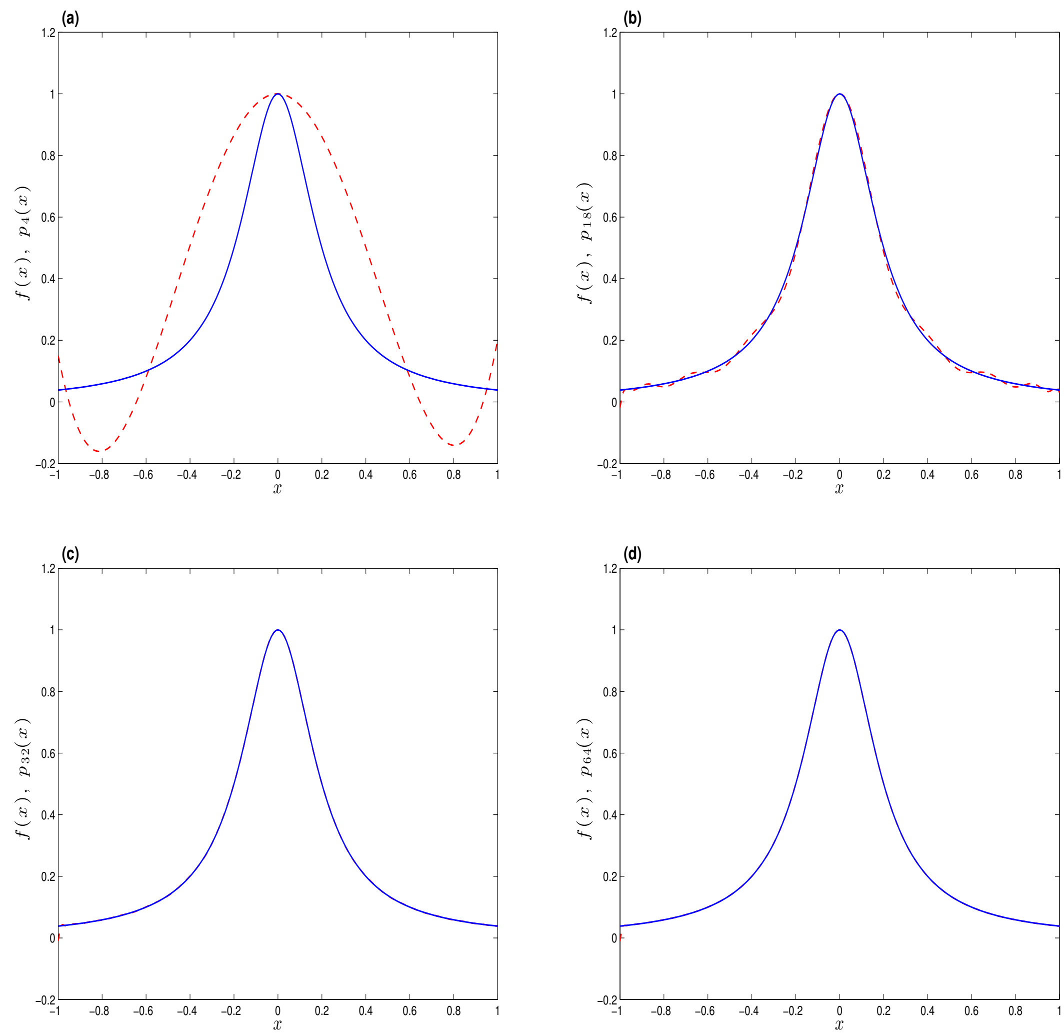

Figure 8.4: Runge Phenomenon is illustrated with Chebyshev nodes. Figure (a) to (d) shows the graph of the Runge function (blue solid line) and the interpolating polynomial with Chebyshev nodes (red dash line) for $n = 4$, 18, 32 and 64 respectively. Note that the two graphs in Figure (c) and (d) are indistinguishable.

for each $n = 0, 1, 2, \cdots$. The nodes $x_i^{(n)}$ defined by (8.28) are called ***Chebyshev nodes***.

In particular, if $n = 4$, the nodes are

$$x_0^{(4)} = \cos(\pi/10), \quad x_1^{(4)} = \cos(3\pi/10),$$
$$x_2^{(4)} = \cos(5\pi/10), \quad x_3^{(4)} = \cos(7\pi/10), \quad x_4^4 = \cos(9\pi/10).$$

Figure 8.4 depicts $p_n(x)$ for $n = 4$, 18, 32, and 64 along with the Runge's function. From these figures, we observe that the interpolating polynomial $p_n(x)$ agrees well with the Runge's function.

8.4 Piecewise Polynomial Interpolation

Quite often polynomial interpolation will be unsatisfactory as an approximation tool. This is true if we insist on letting the order of the polynomial get larger and larger. However, if we keep the order of the polynomial fixed, and use different polynomials over different intervals, with the length of the intervals getting smaller and smaller, then the resulting interpolating function approximates the given function more accurately.

Let us start with linear interpolation over an interval $I = [a, b]$ which leads to

$$p_1(x) = f(a) + f[a, b](x - a) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a) = \frac{x - b}{a - b}f(a) + \frac{x - a}{b - a}f(b),$$

where the nodes are $x_0 = a$, $x_2 = b$. In addition to these two nodes, we now choose one more point x_1 such that $x_0 < x_1 < x_2$. With these three nodes, can obtain a quadratic interpolation polynomial. Instead, we can interpolate the function $f(x)$ in $[x_0, x_1]$ by a linear polynomial with nodes x_0 and x_1, and in $[x_1, x_2]$ by another linear polynomial with nodes x_1 and x_2. Such polynomials are given by

$$p_{1,1}(x) := \frac{x - x_1}{x_0 - x_1}f(x_0) + \frac{x - x_0}{x_1 - x_0}f(x_1), \quad p_{1,2}(x) := \frac{x - x_2}{x_1 - x_2}f(x_1) + \frac{x - x_1}{x_2 - x_1}f(x_2)$$

and the interpolating function is given by

$$s(x) = \begin{cases} p_{1,1}(x) & , x \in [x_0, x_1] \\ p_{1,2}(x) & , x \in [x_1, x_2] \end{cases}.$$

Note that $s(x)$ is a continuous function on $[x_0, x_2]$, which interpolates $f(x)$ and is linear in $[x_0, x_1]$ and $[x_1, x_2]$. Such an interpolating function is called *piecewise linear interpolating function*.

In a similar way as done above, we can also obtain piecewise quadratic, cubic interpolating functions and so on.

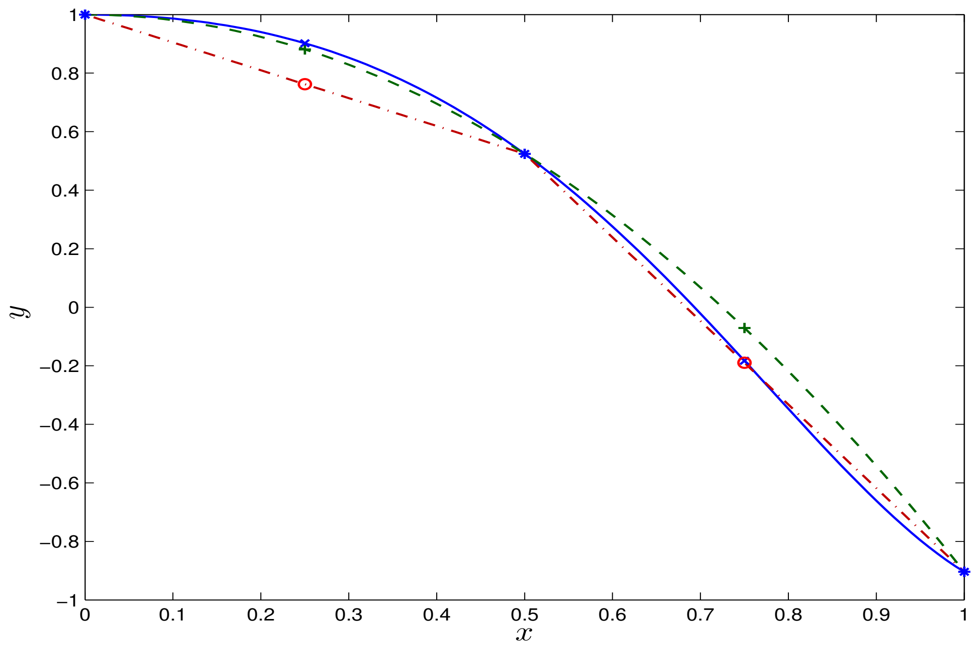

Figure 8.5: The function $f(x) = \sin\left(\frac{\pi}{2}e^x\right)$ (blue line), $p_2(x)$ (green bash line) and the piecewise linear interpolation $s(x)$ (red dash and dot line) are shown. Blue dots represent the given data, blue 'x' symbol indicates the value of $f(0.25)$ and $f(0.75)$, green '+' symbol indicates the value of $p_2(0.25)$ and $p_2(0.75)$, and the red 'O' symbol represents the value of $s(0.25)$ and $s(0.75)$.

Example 8.4.1.

Consider the Example 8.1, where we have obtained the quadratic interpolating polynomial for the function

$$f(x) = \sin\left(\frac{\pi}{2}e^x\right).$$

The piecewise linear polynomial interpolating function for the data

x	0	0.5	1
$f(x)$	1.0000	0.5242	−0.9037

is given by

$$s(x) = \begin{cases} 1 - 0.9516\,x & , x \in [0, 0.5] \\ 1.9521 - 2.8558\,x & , x \in [0.5, 1]. \end{cases}$$

The following table shows the value of the function f at $x = 0.25$ and $x = 0.75$ along with the values of $p_2(x)$ and $s(x)$ with relative errors.

x	$f(x)$	$p_2(x)$	$s(x)$	$E_r(p_2(x))$	$E_r(s(x))$
0.25	0.902117	0.881117	0.762105	0.023278	0.155203
0.75	-0.182750	-0.070720	-0.189732	0.613022	0.038204

Figure 8.5 depicts the graph of f, $p_2(x)$ and $s(x)$. In this figure, we observe that the quadratic polynomial $p_2(x)$ agrees well with $f(x)$ than $s(x)$ for $x \in [0, 0.5]$, whereas $s(x)$ agrees well with $f(x)$ than $p_2(x)$ for $x \in [0.5, 1]$.

8.5 Exercises

Polynomial Interpolation

1. Let $x_0, x_1, \cdots, x_n$ be distinct nodes. If $p(x)$ is a polynomial of degree less than or equal to n, then show that

$$p(x) = \sum_{i=0}^{n} p(x_i) l_i(x),$$

where $l_i(x)$ is the i^{th} Lagrange polynomial.

2. Show that the polynomial $1 + x + 2x^2$ is an interpolating polynomial for the data

x	0	1	2
y	1	4	11

Find an interpolating polynomial for the new data

x	0	1	2	3
y	1	4	11	-2

Does there exist a quadratic polynomial that satisfies the new data? Justify your answer.

3. The quadratic polynomial $p_2(x) = \frac{3}{4}x^2 + \frac{1}{4}x + \frac{1}{2}$ interpolates the data

x	-1	0	1
y	1	$\frac{1}{2}$	$\frac{3}{2}$

Find a node x_3 ($x_3 \notin \{-1, 0, 1\}$), and a real number y_3 such that the polynomial $p_3(x)$ interpolating the data

x	-1	0	1	x_3
y	1	1/2	3/2	y_3

is a polynomial of degree less than or equal to 2.

4. Let $p(x)$, $q(x)$, and $r(x)$ be interpolating polynomials for the three sets of data

x	0	1
y	y_0	y_1

,

x	0	2
y	y_0	y_2

, and

x	1	2	3
y	y_1	y_2	y_3

respectively. Let $s(x)$ be the the interpolating polynomial for the data

x	0	1	2	3
y	y_0	y_1	y_2	y_3

If

$$p(x) = 1 + 2x, \; q(x) = 1 + x, \text{ and } r(2.5) = 3,$$

then find the value of $s(2.5)$.

5. Obtain Lagrange form of interpolating polynomial for equally spaced nodes.
6. Find the Largrange form of interpolating polynomial for the data:

x	-2	-1	1	3
y	-1	3	-1	19

7. Find the Lagrange form of interpolating polynomial $p_2(x)$ that interpolates the function $f(x) = e^{-x^2}$ at the nodes $x_0 = -1$, $x_1 = 0$ and $x_2 = 1$. Further, find the value of $p_2(-0.9)$ (use 6-digit rounding). Compare the value with the true value $f(-0.9)$ (use 6-digit rounding). Find the percentage error in this calculation.

Newton's Divided Difference Formula

8. For the particular function $f(x) = x^m$ $(m \in \mathbb{N})$, show that
$$f[x_0, x_1, \cdots, x_n] = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n > m \end{cases}$$
9. Let $x_0, x_1, \cdots, x_n$ be nodes, and f be a given function. Define $w(x) = \prod_{i=0}^{n}(x - x_i)$. Prove that
$$f[x_0, x_1, \cdots, x_n] = \sum_{i=0}^{n} \frac{f(x_i)}{w'(x_i)}.$$
10. The following data correspond to a polynomial $P(x)$ of unknown degree

x	0	1	2
$P(x)$	2	-1	4

Determine the coefficient of x in the polynomial $P(x)$ if all the third order divided differences are 1.

Error in Polynomial Interpolation

11. Let $p_n(x)$ be a polynomial of degree less than or equal to n that interpolates a function f at a set of distinct nodes $x_0, x_1, \cdots, x_n$. If $x \notin \{\, x_0, x_1, \cdots, x_n \,\}$, then show that the error is given by
$$f(x) - p_n(x) = f[x_0, x_1, \cdots, x_n, x] \prod_{i=0}^{n}(x - x_i).$$
12. If $f \in C^{n+1}[a, b]$ and if $x_0, x_1, \cdots, x_n$ are distinct nodes in $[a, b]$, then show that there exists a point $\xi_x \in (a, b)$ such that
$$f[x_0, x_1, \cdots, x_n, x] = \frac{f^{(n+1)}(\xi_x)}{(n+1)\,!}$$

13. Let N be a natural number. Let $p_1(x)$ denote the linear interpolating polynomial on the interval $[N, N+1]$ interpolating the function $f(x) = x^2$ at the nodes N and $N+1$. Find an upper bound for the mathematical error ME_1 using the infinity norm on the interval $[N, N+1]$ (*i.e.*, $\|\mathrm{ME}_1\|_{\infty,[N,N+1]}$).

14. Let $p_3(x)$ denote a polynomial of degree less than or equal to 3 that interpolates the function $f(x) = \ln x$ at the nodes $x_0 = 1$, $x_1 = \frac{4}{3}$, $x_2 = \frac{5}{3}$, $x_3 = 2$. Find a lower bound on the absolute value of mathematical error $|\mathrm{ME}_3(x)|$ at the point $x = \frac{3}{2}$, using the formula for mathematical error in interpolation.

15. Let $f : [0, \frac{\pi}{6}] \to \mathbb{R}$ be a given function. The following is the meaning for the **Cubic interpolation in a table of function values**

x	x_0	x_1	$\cdots$	x_N
$f(x)$	$f(x_0)$	$f(x_1)$	$\cdots$	$f(x_N)$

The values of $f(x)$ are tabulated for a set of equally spaced points in $[a, b]$, say x_i for $i = 0, 1, \cdots, N$ with $x_0 = 0, x_N = \frac{\pi}{6}$, and $h = x_{i+1} - x_i > 0$ for every $i = 0, 1, \cdots, N-1$. For an $\bar{x} \in [0, \frac{\pi}{6}]$ at which the function value $f(\bar{x})$ is not tabulated, the value of $f(\bar{x})$ is taken to be the value of $p_3(\bar{x})$, where $p_3(x)$ is the polynomial of degree less than or equal to 3 that interpolates f at the nodes $x_i, x_{i+1}, x_{i+2}, x_{i+3}$ where i is the least index such that $x \in [x_i, x_{i+3}]$.
Take $f(x) = \sin x$ for $x \in [0, \frac{\pi}{6}]$; and answer the following questions.
   i) When $\bar{x}$ and p_3 are as described above, then show that $|f(\bar{x}) - p_3(\bar{x})| \leq \frac{h^4}{48}$.
   ii) If $h = 0.005$, then show that cubic interpolation in the table of function values yields the value of $f(\bar{x})$ with at least 10 decimal-place accuracy.

16. Let x_0, x_1, $\cdots$, x_n be $n+1$ distinct nodes, and f be a function. For each $i = 0, 1, \cdots, n$, let $\mathrm{fl}\,(f(x_i))$ denote the floating point approximation of $f(x_i)$ obtained by rounding to 5 decimal places (note this is different from using 5-digit rounding). Assume that $0.1 \leq f(x_i) < 1$ for all $i = 0, 1, \cdots, n$. Let $p_n(x)$ denote the Lagrange form of interpolating polynomial corresponding to the data $\{(x_i, f(x_i)) : i = 0, 1, \cdots, n\}$. Let $\tilde{p}_n(x)$ denote the Lagrange form of interpolating polynomial corresponding to the data $\{(x_i, \mathrm{fl}\,(f(x_i))) : i = 0, 1, \cdots, n\}$. Show that the arithmetic error at a point $\tilde{x}$ satisfies the inequality

$$|p_n(\tilde{x}) - \tilde{p}_n(\tilde{x})| \leq \frac{1}{2} 10^{-5} \sum_{k=0}^{n} |l_k(\tilde{x})|.$$