Tutorial Sheet 7

Polynomial Interpolation (Week 2)
Nonlinear Equations (Week 1)

1. If $f \in C^{n+1}[a,b]$ and if $x_0, x_1, \cdots, x_n$ are distinct nodes in $[a,b]$, then for $x \in (a,b)$ with $x \neq x_i$, $i = 1, 2, \ldots, n$, show that there exists a point $\xi_x \in (a,b)$ such that
$$f[x_0, x_1, \cdots, x_n, x] = \frac{f^{(n+1)}(\xi_x)}{(n+1)\,!}$$
2. Prove that if we take any set of 23 nodes in the interval $[-1,1]$ and interpolate the function $f(x) = \cosh x$ with a polynomial p_{22} of degree less than or equal to 22, then at each $x \in [-1,1]$ the relative error satisfies the bound
$$\frac{|f(x) - p_{22}(x)|}{|f(x)|} < 0.38134 \times 10^{-15}.$$
3. Derive the piecewise quadratic interpolating function for
$$f(x) = \cos x$$
on the interval $[0, 2\pi]$ with the partition $\left\{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi\right\}$.
4. Let bisection method be used to solve the nonlinear equation
$$x^5 - 4x^4 + 2 = 0$$
starting with the initial interval $[0,1]$. In order to approximate a solution of the nonlinear equation with an absolute error less than or equal to 10^{-4}, what is the minimum number of iterations required as per the error estimate of the bisection method? Using bisection method compute x_3.
5. Consider the equation $x^2 - 6x + 5 = 0$.
   i) Taking $x_0 = 0$ and $x_1 = 4.5$, obtain the secant method iterates x_2, x_3, $\ldots$, x_8.
   ii) Take the initial interval as $[a_0, b_0] = [0, 4.5]$, obtain the regula-falsi method iterates x_2, x_3, $\ldots$, x_7.

   Observe to which roots of the given equation does the above two sequences converge.
6. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ such that the equation $f(x) = 0$ has an isolated real root with the condition that the Newton-Raphson method converges but does not have quadratic convergence.
7. Find the iterative method based on Newton-Raphson method for finding $N^{5/2}$, where N is a positive real number. Take $N = 2$ and $x_0 = 6$ and obtain x_i, for $i = 1, 2, 3, 4$ using the Newton-Raphson method. Assume that the sequence converges and show that it converges qudratically.