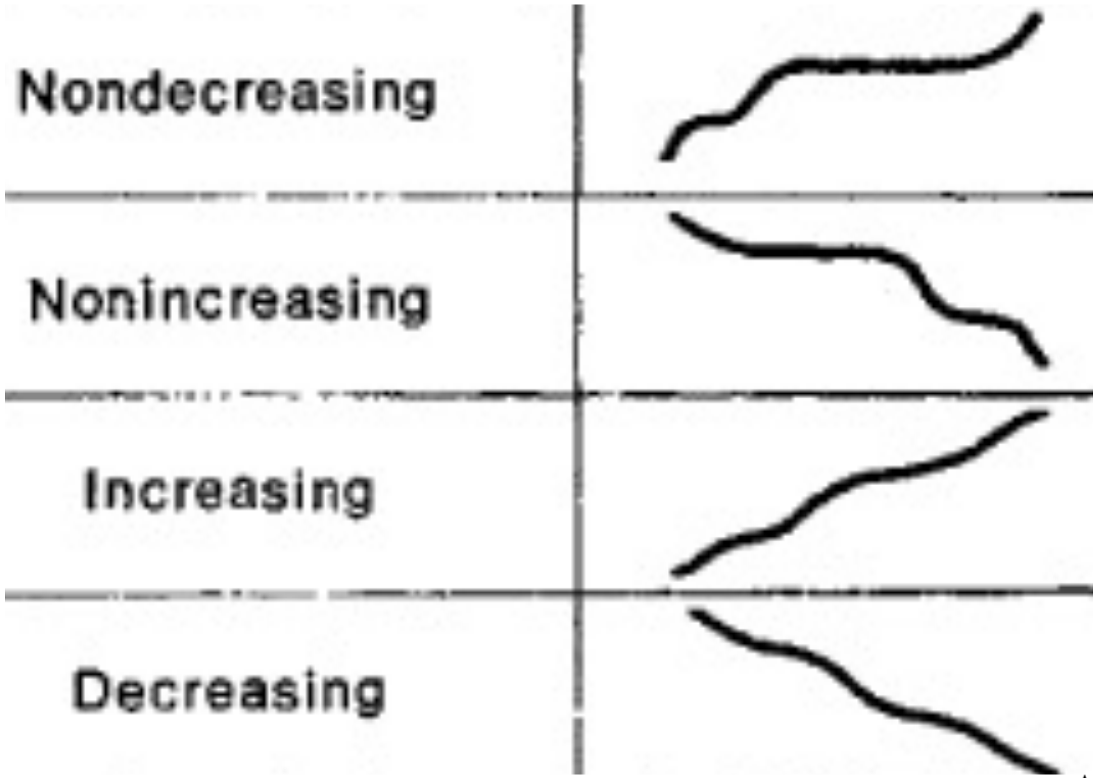
CS 215
Data Analysis and Interpretation

Transformation of Random Variables

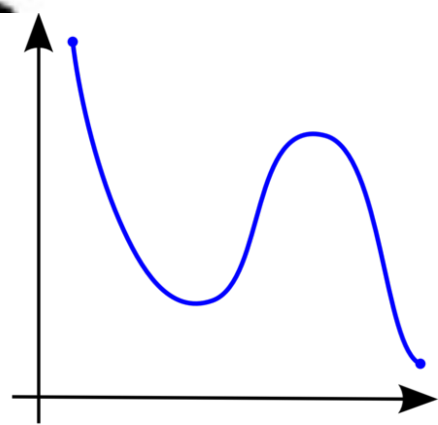
Suyash P. Awate

Transformation of Random Variables

- Let X be a (continuous) random variable (RV) with probability density function (PDF) p(X)
- Let function g(.) be strictly monotonically **increasing**
  - If a < b, then g(a) < g(b)



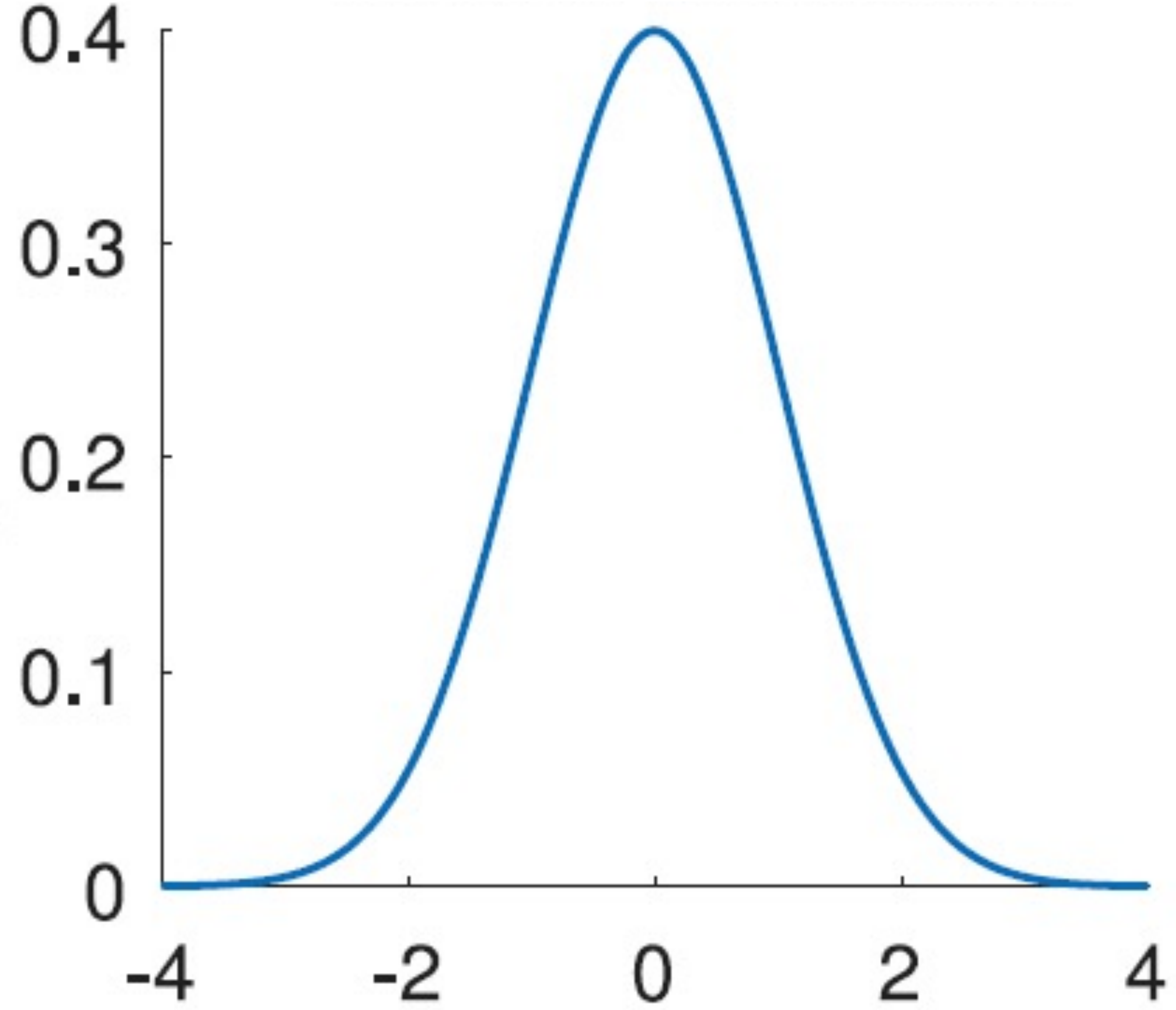
  - We will generalize/extend the class of functions later

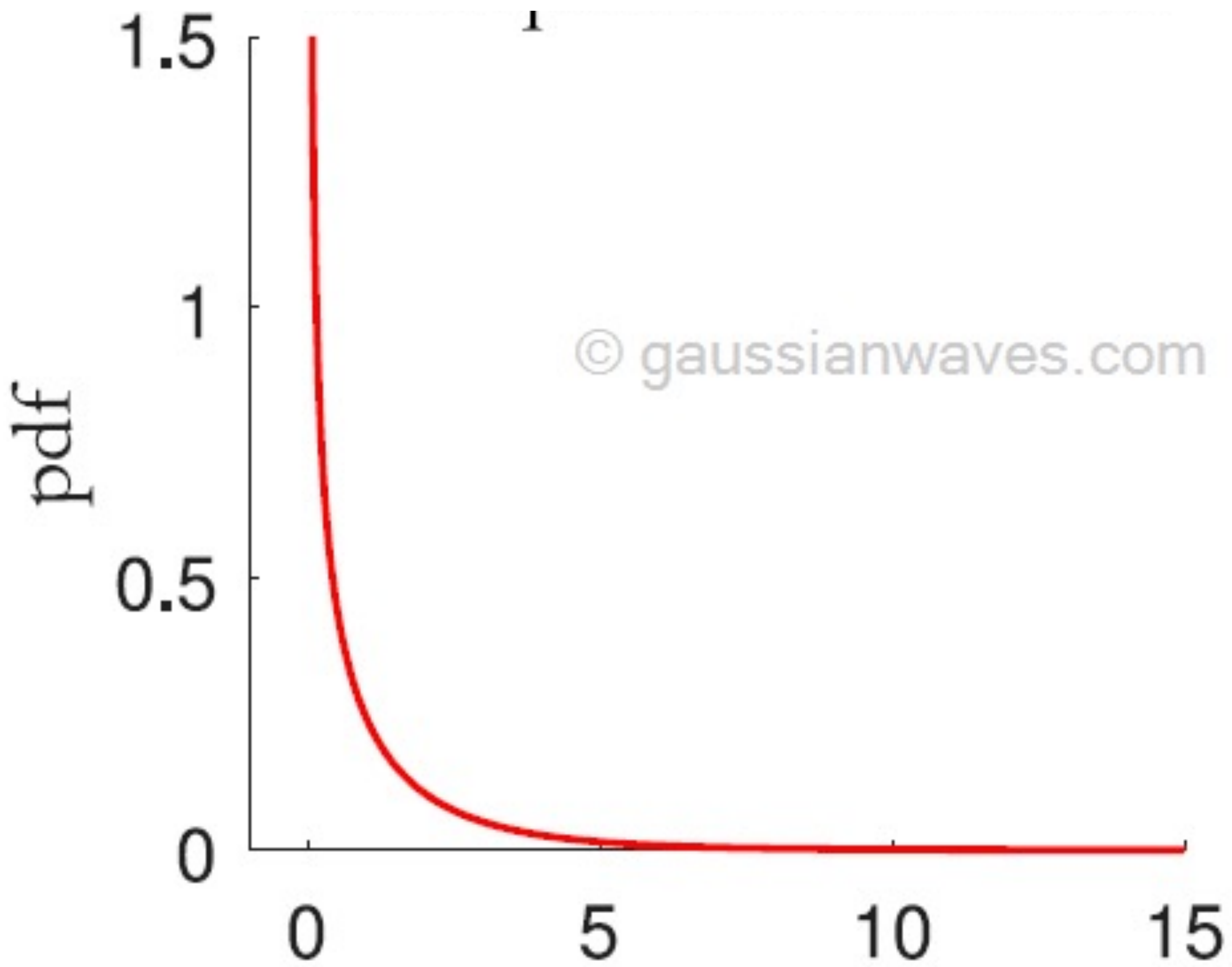


- Consider the transformed variable Y := g(X)
- What is the PDF q(Y) of RV Y ?

Transformation of Random Variables

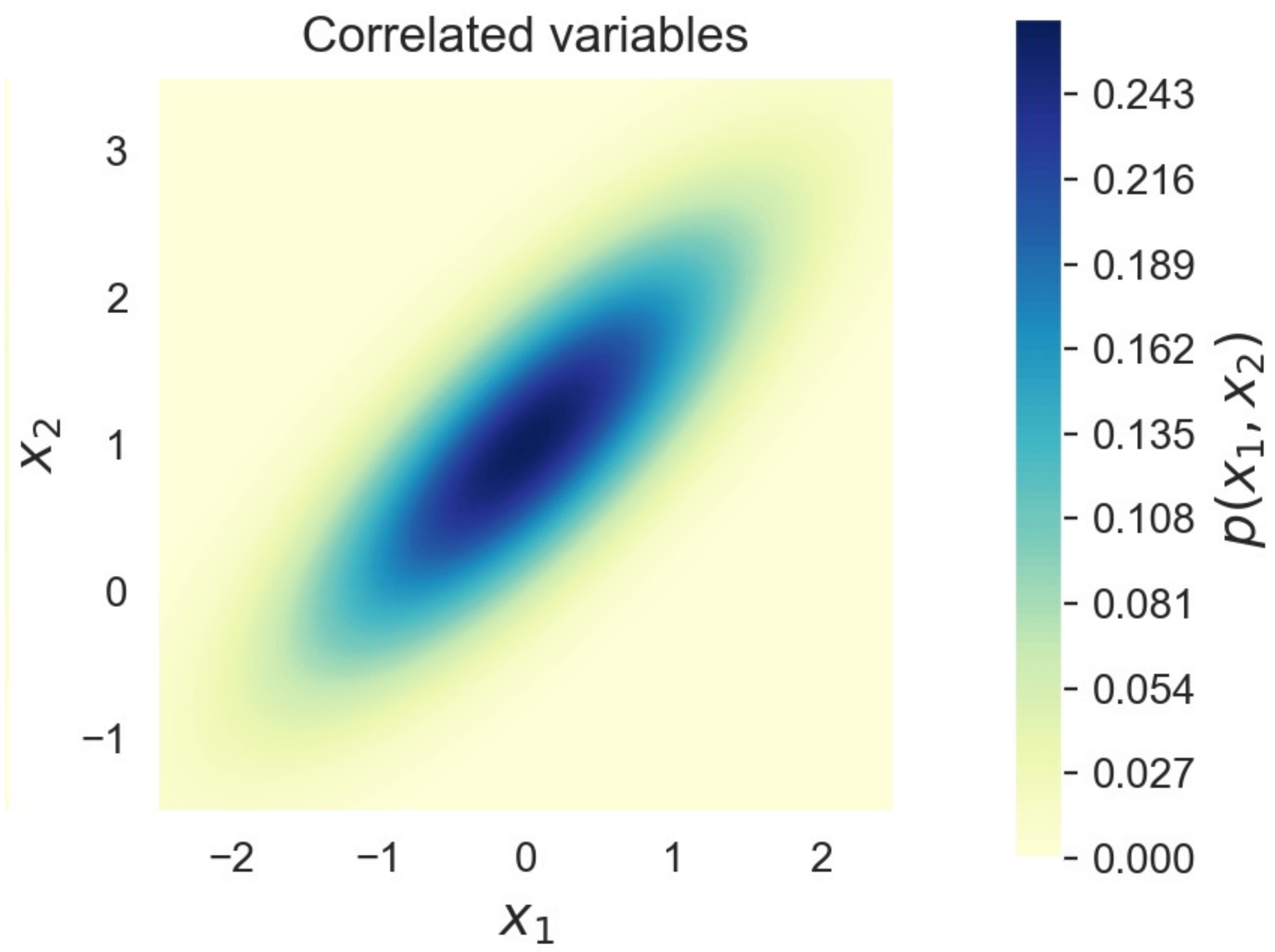
- Example
  - If X has a Normal PDF, then what is the PDF for Y := X^2 ?





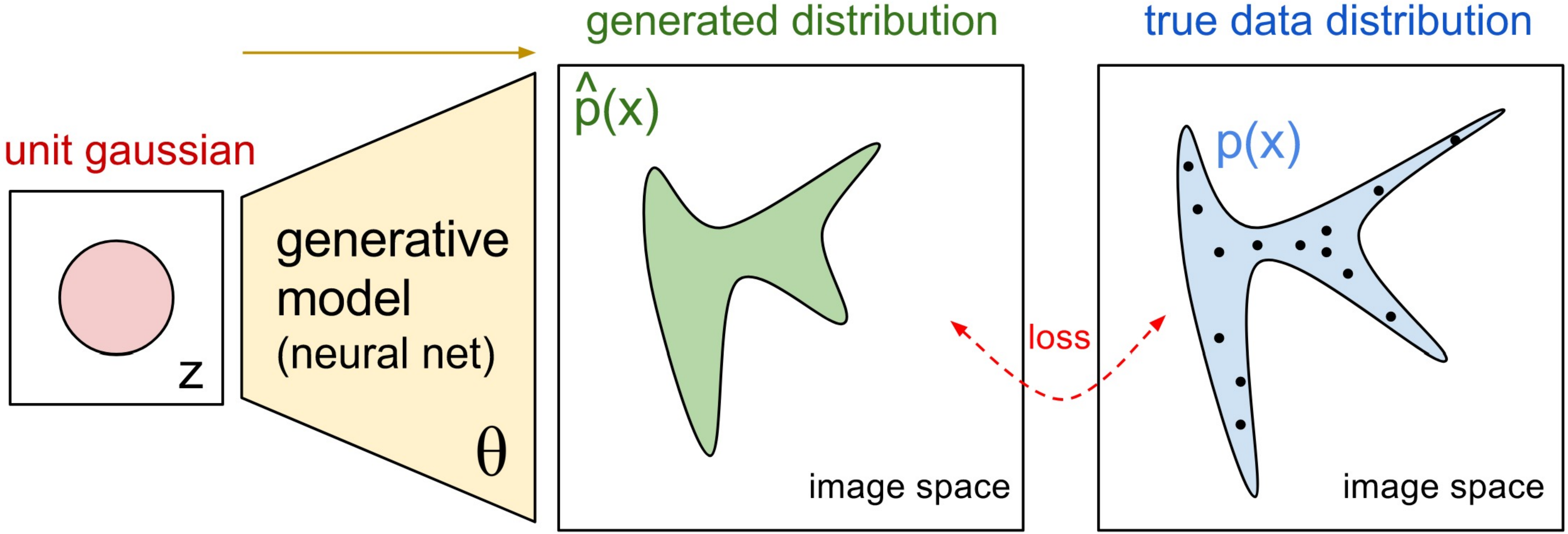
Transformation of Random Variables

- Example
  - If
    - RVs U,V are independent Gaussian
    - X1 = aU + bV
    - X2 = cU + dV
  - Then
    - What is P(X1,X2) ?



Transformation of Random Variables

- Example



Transformation of Random Variables

- Principle of probability mass conservation
  - Consider the events $\{x : x \in (a, b)\}$ and $\{y : y \in (g(a), g(b))\}$
  - Because we assumed that g(.) was increasing, $P(g(a) < Y < g(b)) = P(a < X < b)$

  - So, the probability "mass" of X in interval (a, b)
    gets mapped to the mass of Y in interval (g(a), g(b))

Now, $P(g(a) < Y < g(b)) := \int_{g(a)}^{g(b)} q(y)dy$

Also, $P(a < X < b) := \int_{a}^{b} p(x)dx$

  - Write the second integral in terms of y, using the known relationship y = g(x)

Transformation of Random Variables

- We found that these probabilities → are equal

$$P(g(a) < Y < g(b)) := \int_{g(a)}^{g(b)} q(y)dy$$

$$P(a < X < b) := \int_a^b p(x)dx$$

- We have, $x = g^{-1}(y)$

$$dx = \left(\frac{d}{dy} g^{-1}(y)\right) dy$$

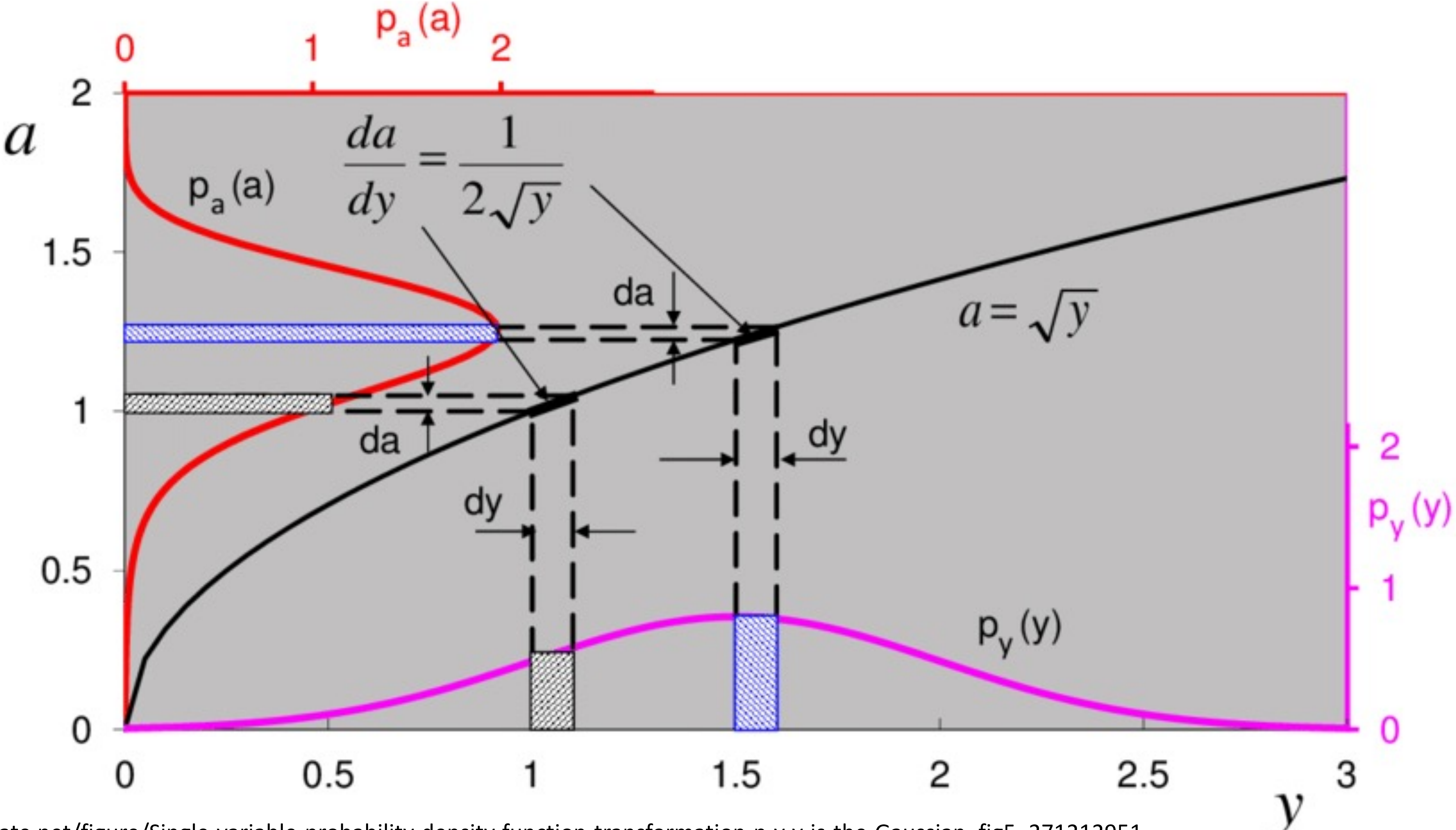
Then, $P(a < X < b) = \int_{g(a)}^{g(b)} p(g^{-1}(y)) \left(\frac{d}{dy} g^{-1}(y)\right) dy$

- This mass conservation holds for **every** interval (a,b), however small

Thus, $q(y) = p(g^{-1}(y)) \frac{d}{dy} g^{-1}(y)$, for all y

Transformation of Random Variables

- Example
  - P(Y)
  - A := sqrt(Y)
  - g(.) is sqrt(.)
  - To find P(A)



https://www.researchgate.net/figure/Single-variable-probability-density-function-transformation-p-y-y-is-the-Gaussian_fig5_271212951

Transformation of Random Variables

- We found the relationship between PDF q(.) of Y and PDF p(.) of X, in terms of the strictly-increasing transformation function g(.)

$$q(y) = p(g^{-1}(y))\frac{d}{dy}g^{-1}(y), \text{ for all } y$$

- If g(·) is strictly increasing, then:
  - a < b ⇒ g(a) < g(b)
  - Derivative of g-inverse(.) is positive
  - So, the above formula holds good
- If g(·) is strictly decreasing, then:
  - a < b ⇒ g(a) > g(b)
  - Derivative of g-inverse(.) is negative
  - What to do then ?

Transformation of Random Variables

- For convenience, to handle both cases above, we:
  - Write $$q(y) = p(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|$$
  - Taking the absolute value ensures that PDF q(.) is always non-negative
  - Take the integral limits to go from a smaller number to a larger number

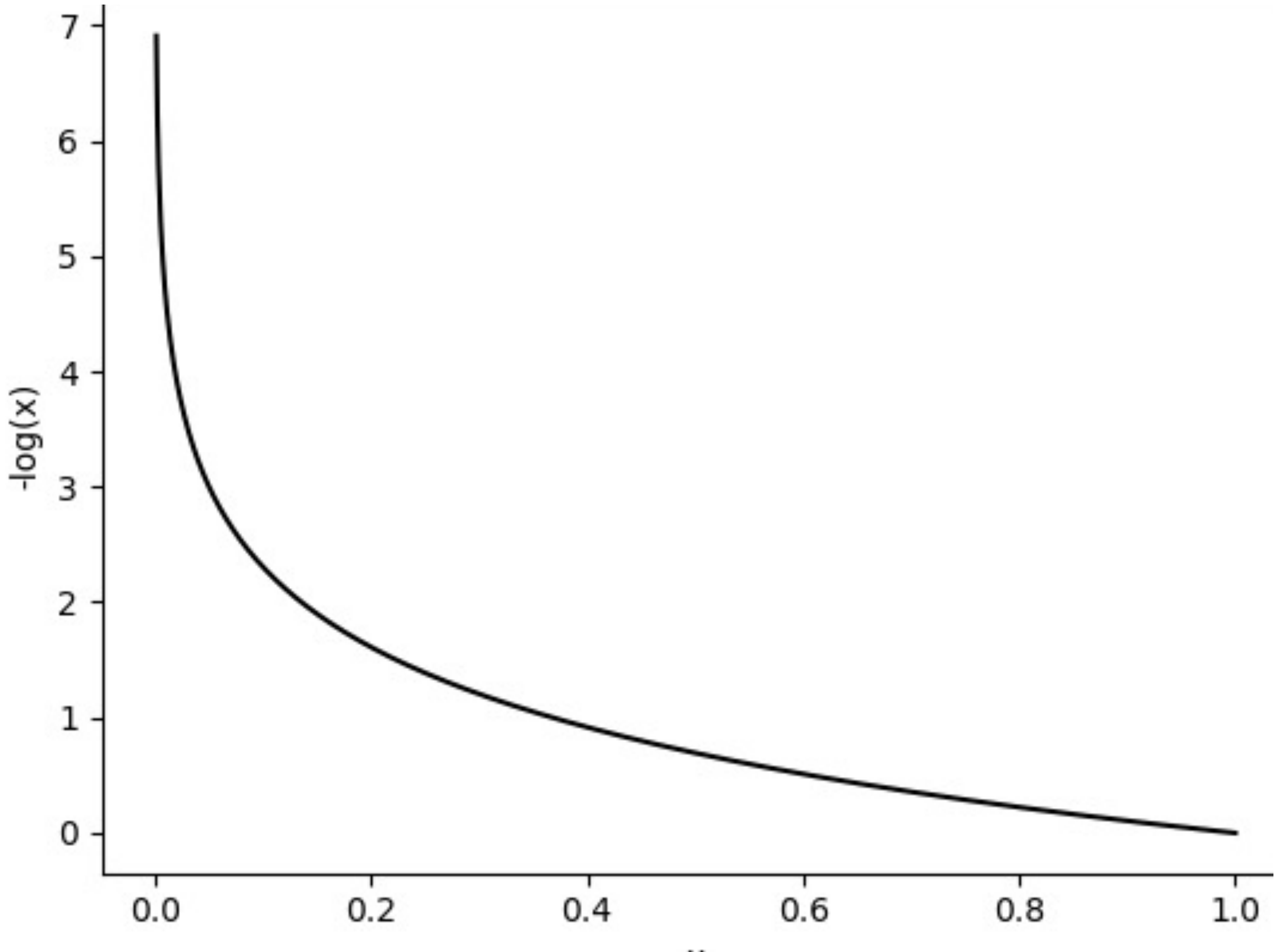
We have, $x = g^{-1}(y)$

$$dx = \left(\frac{d}{dy} g^{-1}(y) \right) dy$$

Then, $P(a < X < b) = \int_{g(a)}^{g(b)} p(g^{-1}(y)) \left(\frac{d}{dy} g^{-1}(y) \right) dy$

Transformation of Random Variables

- Consider a RV $X \sim U(0,1)$ (generated by the C/C++ rand() function)
- Consider the transformation $Y := (-1/\lambda)\log(X)$, where $\lambda > 0$
- What is $q(Y)$?



Transformation of Random Variables

- Consider a RV X ~ U(0,1) (generated by the C/C++ rand() function)
- Consider the transformation Y := (−1/λ) log(X), where λ > 0
- What is q(Y) ?

$y = -(1/\lambda)\log(x) \implies x = \exp(-\lambda y)$. This is the $g^{-1}(\cdot)$ function.

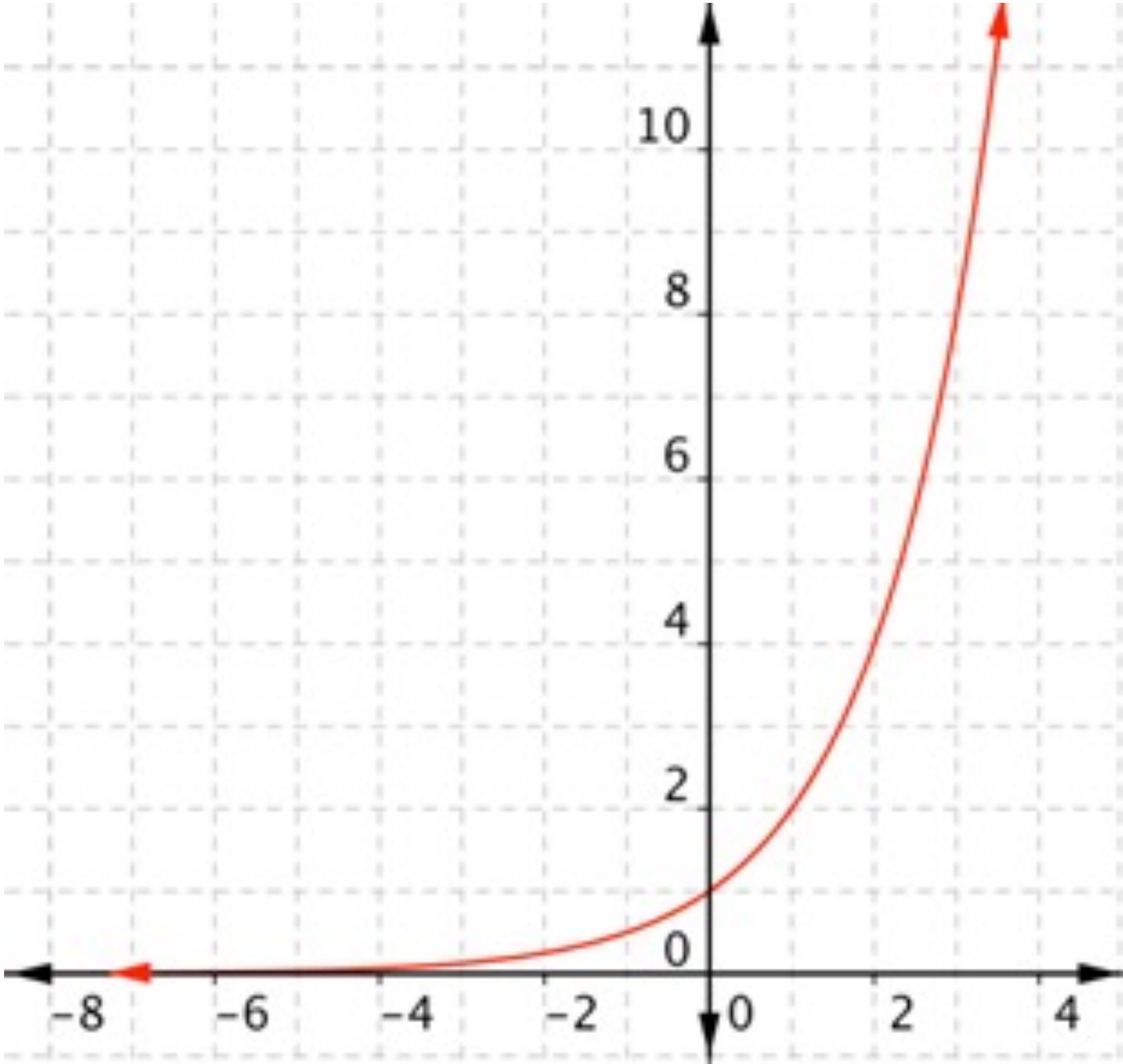
$$\left|\frac{d}{dy}g^{-1}(y)\right| = \lambda\exp(-\lambda y)$$

So, $q(y) = p(g^{-1}(y))\left|\frac{d}{dy}g^{-1}(y)\right| = \lambda\exp(-\lambda y)$

- Thus, Y has the exponential PDF with parameter λ, i.e., mean = 1/λ

Transformation of Random Variables

- Consider a RV $X \sim U(-a/2, +a/2)$
- Consider $Y := \exp(X)$
- What is $q(Y)$?



Transformation of Random Variables

- Consider a RV X ~ U(−a/2, +a/2)
- Consider Y := exp(X)
- What is q(Y) ?

$y = \exp(x) \implies x = \log(y)$. This is the $g^{-1}(\cdot)$ function.

$$\left|\frac{d}{dy} g^{-1}(y)\right| = 1/y$$

So, $q(y) = p(g^{-1}(y)) \left|\frac{d}{dy} g^{-1}(y)\right| = (1/a)(1/y)$

- Thus, Y has PDF q(y) = 1/(ay) for y ∈ (exp(−a/2), exp(a/2))

Transformation of Random Variables

- Consider a RV X ~ G(0,1) (standard Normal PDF)
- Consider Y := aX, with ‘a’ non-zero
- What is q(Y) ?

$$y := ax \implies x = y/a \implies g^{-1}(y) = y/a$$

$$\left|\frac{d}{dy} g^{-1}(y)\right| = 1/a$$

$$q(y) := p(g^{-1}(y)) \left|\frac{d}{dy} g^{-1}(y)\right| = p\left(\frac{y}{a}\right)\frac{1}{a} = \frac{1}{a\sqrt{2\pi}} \exp\left(-\frac{y^2}{2a^2}\right)$$

- Thus, p(Y) is also a Gaussian with variance σ^2 scaled by a factor of a^2

Transformation of Random Variables

- Consider a RV $X \sim G(0,a^2)$
- Consider $Y := X + b$
- What is $q(Y)$?

$$y := b + x \implies x = y - b \implies g^{-1}(y) = y - b$$

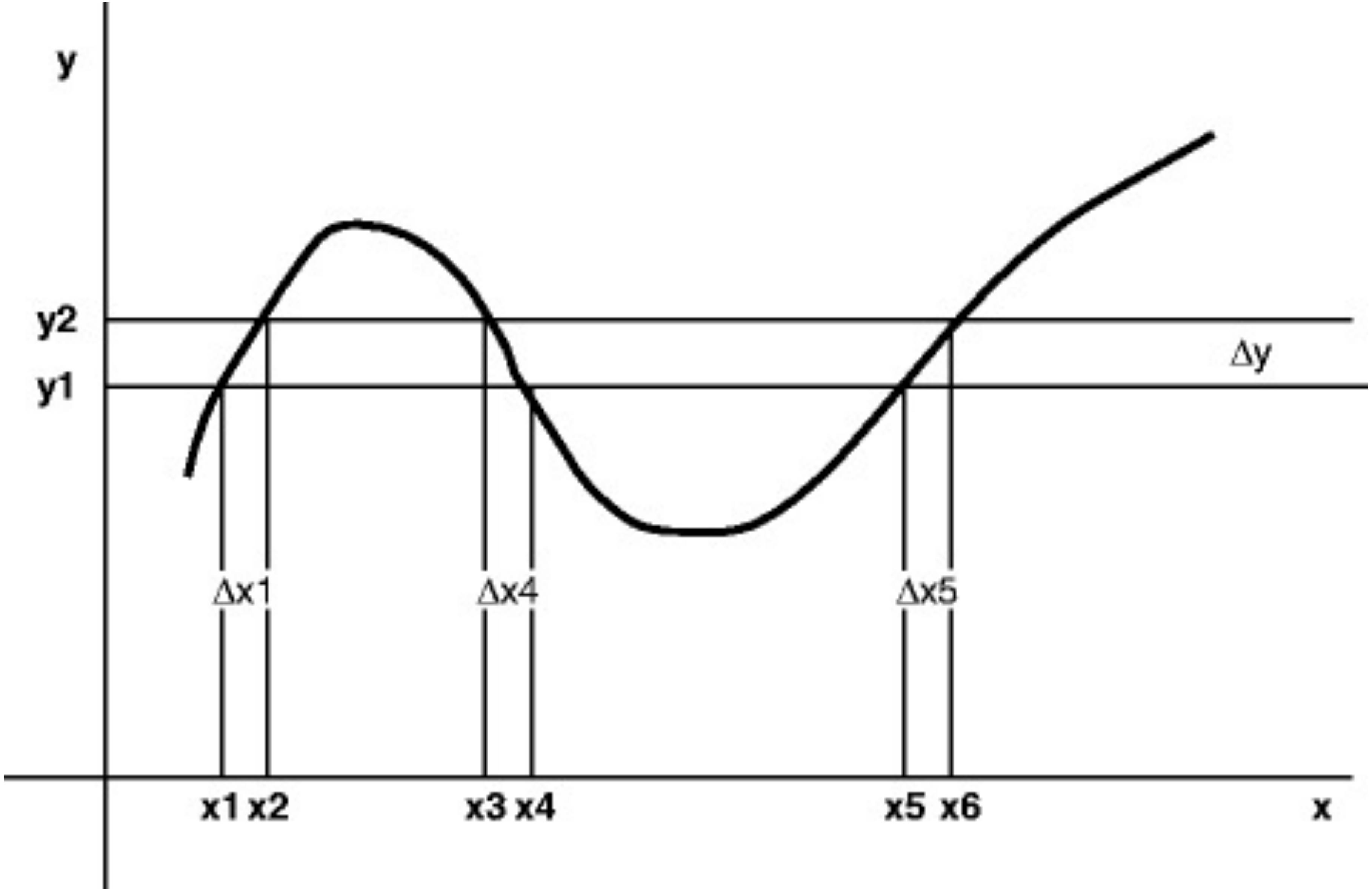
$$\left|\frac{d}{dy} g^{-1}(y)\right| = 1$$

$$q(y) := p(g^{-1}(y)) \left|\frac{d}{dy} g^{-1}(y)\right| = p(y-b) \cdot 1 = \frac{1}{a\sqrt{2\pi}} \exp\left(-\frac{(y-b)^2}{2a^2}\right)$$

- Thus, $p(Y)$ is also a Gaussian with μ translated by b

Transformation of Random Variables

- General non-monotonic functions

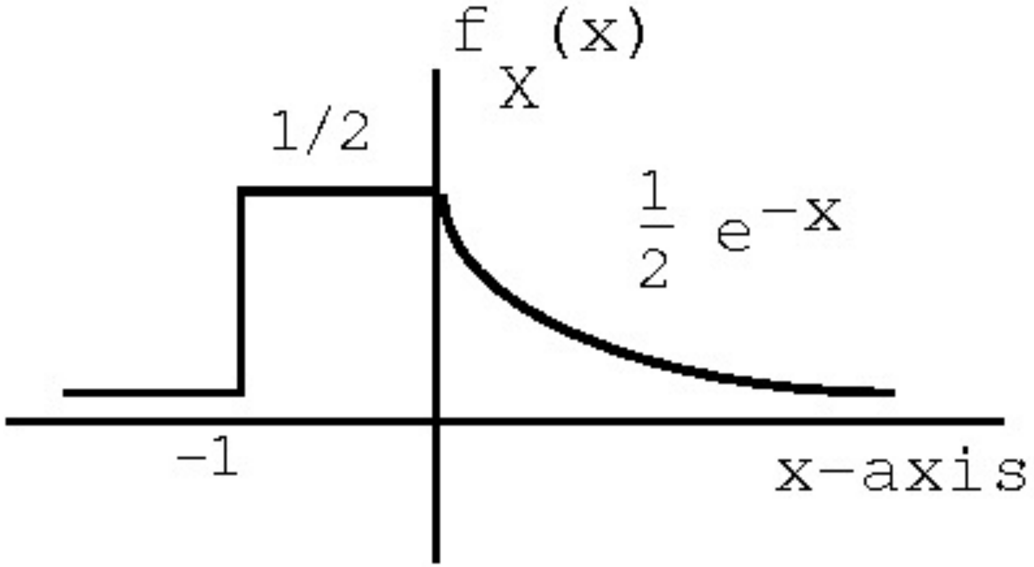


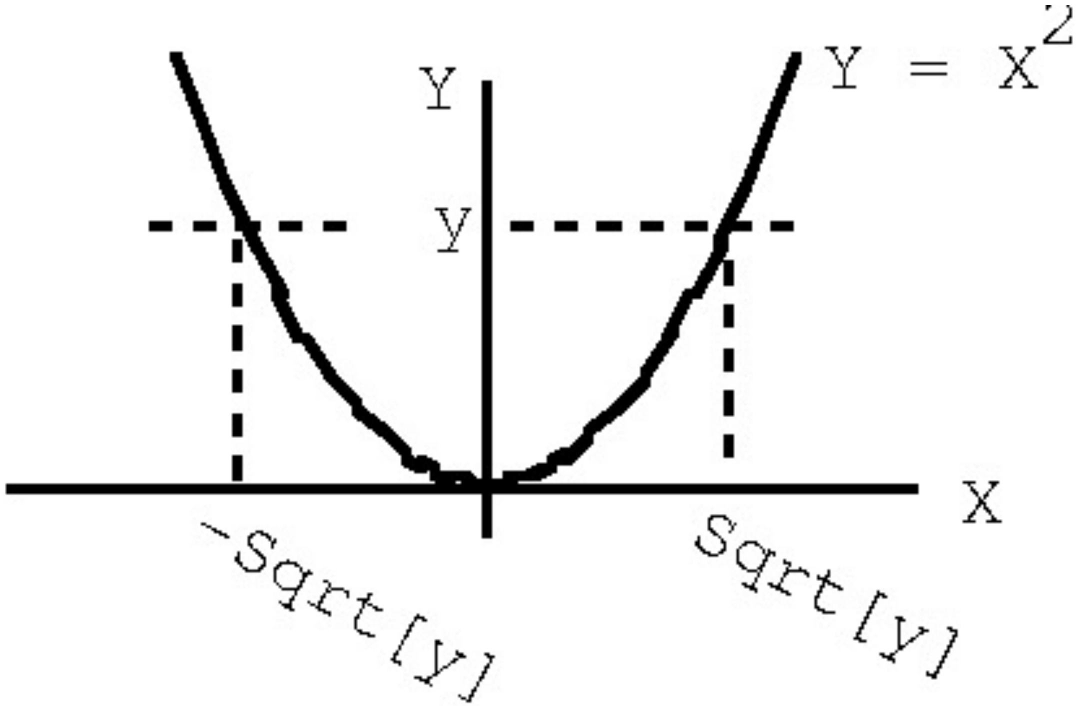
Transformation of Random Variables

- Consider a PDF P(X) as follows:

$$P(x) := 0 \text{ for } x \leq -1$$
$$P(x) := 0.5 \text{ for } x \in (-1, 0)$$
$$P(x) := 0.5 \exp(-x) \text{ for } x \geq 0$$

- Consider a transformation function Y := g(X) := X^2
- What is PDF q(y) of Y ?





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- Consider a transformation function Y := g(X) := X^2
- What is PDF q(y) of Y ?

$$y := x^2 \implies x = \pm\sqrt{y} \implies g^{-1}(y) = \pm\sqrt{y}$$

$$\left|\frac{d}{dy} g^{-1}(y)\right| = \frac{1}{2\sqrt{y}}$$

Transformation of Random Variables

- Consider a PDF P(X) as follows: $P(x) := 0$ for $x \leq -1$
  $P(x) := 0.5$ for $x \in (-1, 0)$
  $P(x) := 0.5 \exp(-x)$ for $x \geq 0$
- Consider a transformation function Y := g(X) := X^2
- What is PDF q(y) of Y ?

Case 1: $x \in (-1, 0)$. In this case, $g(\cdot)$ is a *decreasing* function. Mass conservation applies.

$$\text{For } y \in (0,1) : q_1(y) := p(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| = (0.5)\frac{1}{2\sqrt{y}} = \frac{1}{4\sqrt{y}}$$

Case 2: $x \geq 0$. In this case, $g(\cdot)$ is a *increasing* function. Mass conservation applies.

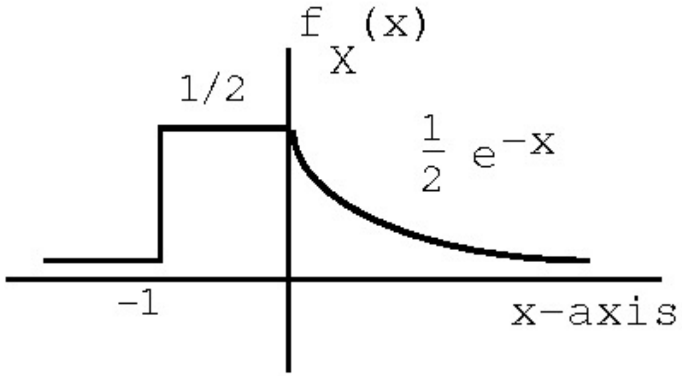
$$\text{For } y \geq 0 : q_2(y) := p(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| = (0.5 \exp(-\sqrt{y}))\frac{1}{2\sqrt{y}} = \frac{\exp(-\sqrt{y})}{4\sqrt{y}}$$

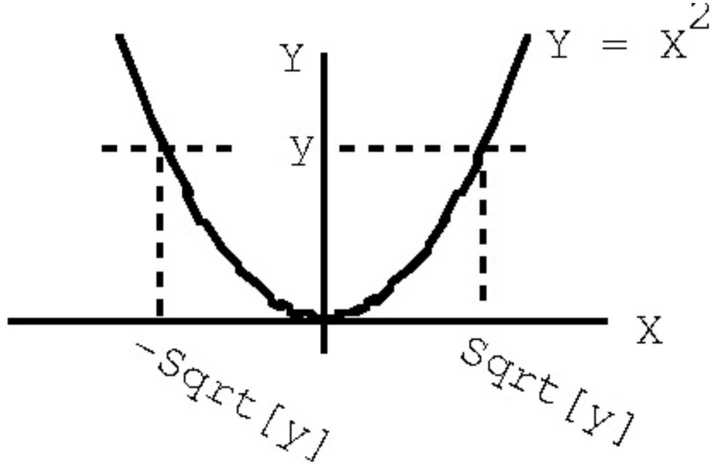
Transformation of Random Variables

- Consider a PDF P(X) as follows:

$$P(x) := 0 \text{ for } x \leq -1$$
$$P(x) := 0.5 \text{ for } x \in (-1, 0)$$
$$P(x) := 0.5 \exp(-x) \text{ for } x \geq 0$$

- Consider a transformation function Y := g(X) := X^2
- What is PDF q(y) of Y ?
- Desired PDF q(y) = $q_1(y) + q_2(y)$
- In the region y ∈ (0,1), probability mass comes from Case 1 & Case 2





Transformation of Random Variables

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$$P(x) := 0 \text{ for } x \leq -1$$
$$P(x) := 0.5 \text{ for } x \in (-1, 0)$$
$$P(x) := 0.5 \exp(-x) \text{ for } x \geq 0$$

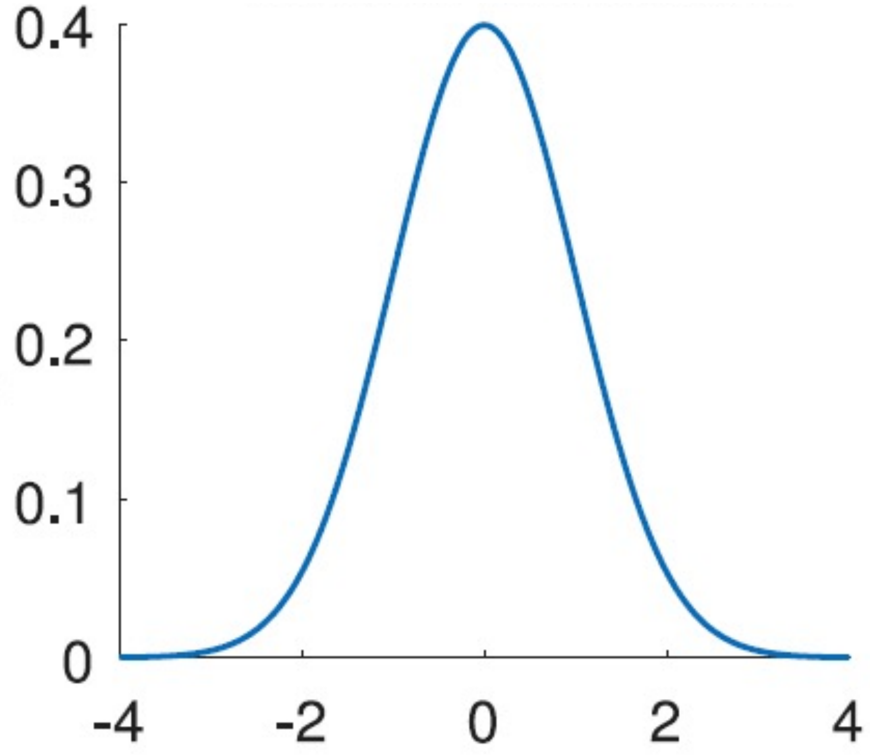
- Consider a transformation function Y := g(X) := X^2
- What is PDF q(y) of Y ?

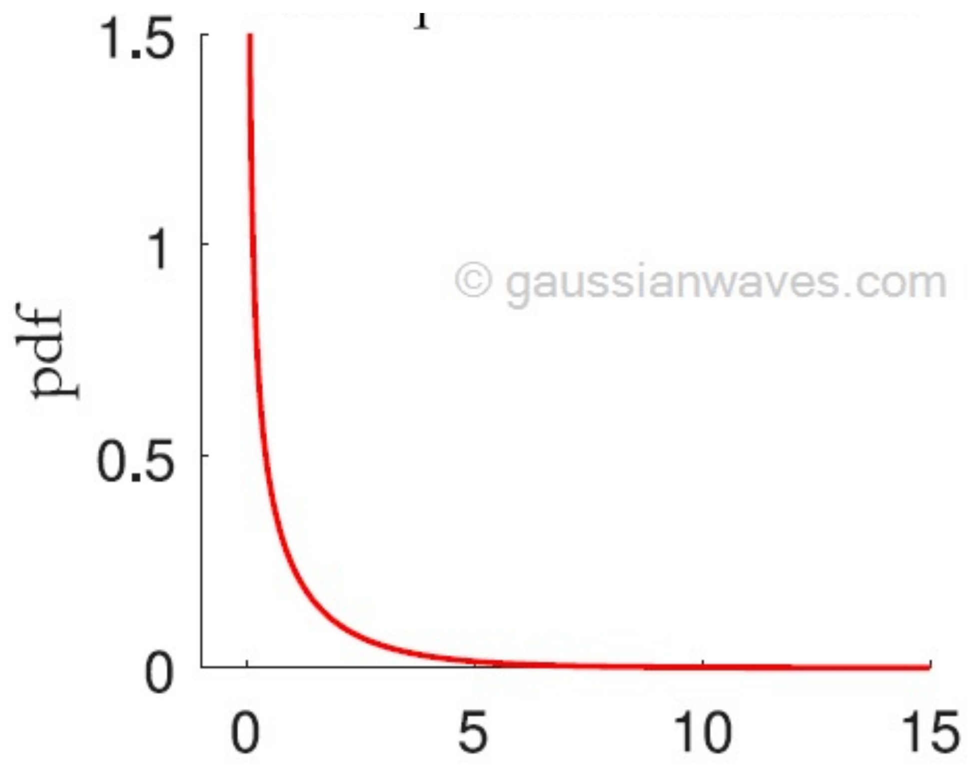
Thus,
(i) for $y \in (0,1)$, PDF $q(y) = \frac{1}{4\sqrt{y}}(1 + \exp(-\sqrt{y}))$
(ii) for $y \geq 1$, PDF $q(y) = \frac{\exp(-\sqrt{y})}{4\sqrt{y}}$

- There will be a jump discontinuity at y = 1,
  where left limit = (1+exp(-1))/4 and right limit = exp(-1)/4

Transformation of Random Variables

- Let X ~ G(0,1)
- Let Y := X^2
- What is P(Y), defined as the chi-square PDF ?





Transformation of Random Variables

- Let X ~ G(0,1)
- Let Y := X^2
- What is P(Y) ?

$$y := x^2 \implies x = \pm\sqrt{y} \implies g^{-1}(y) = \pm\sqrt{y}$$

$$\left|\frac{d}{dy}g^{-1}(y)\right| = \frac{1}{2\sqrt{y}}$$

- Case 1: x ≤ 0. Here, g(.) is a decreasing function

For $y \geq 0 : q_1(y) := p(g^{-1}(y))\left|\frac{d}{dy}g^{-1}(y)\right| = \frac{\exp(-0.5(\sqrt{y})^2)}{\sqrt{2\pi}}\frac{1}{2\sqrt{y}} = \frac{\exp(-0.5y)}{2\sqrt{y2\pi}}$

- Case 2: x > 0. Here, g(·) is a increasing function

For $y > 0 : q_2(y) := \frac{\exp(-0.5y)}{2\sqrt{y2\pi}}$

- Desired chi-square PDF: $q(y) = q_1(y) + q_2(y) = (1/\mathrm{sqrt}(y2\pi))\exp(-0.5y)$

Transformation of Random Variables

- Let X have a Gamma PDF,

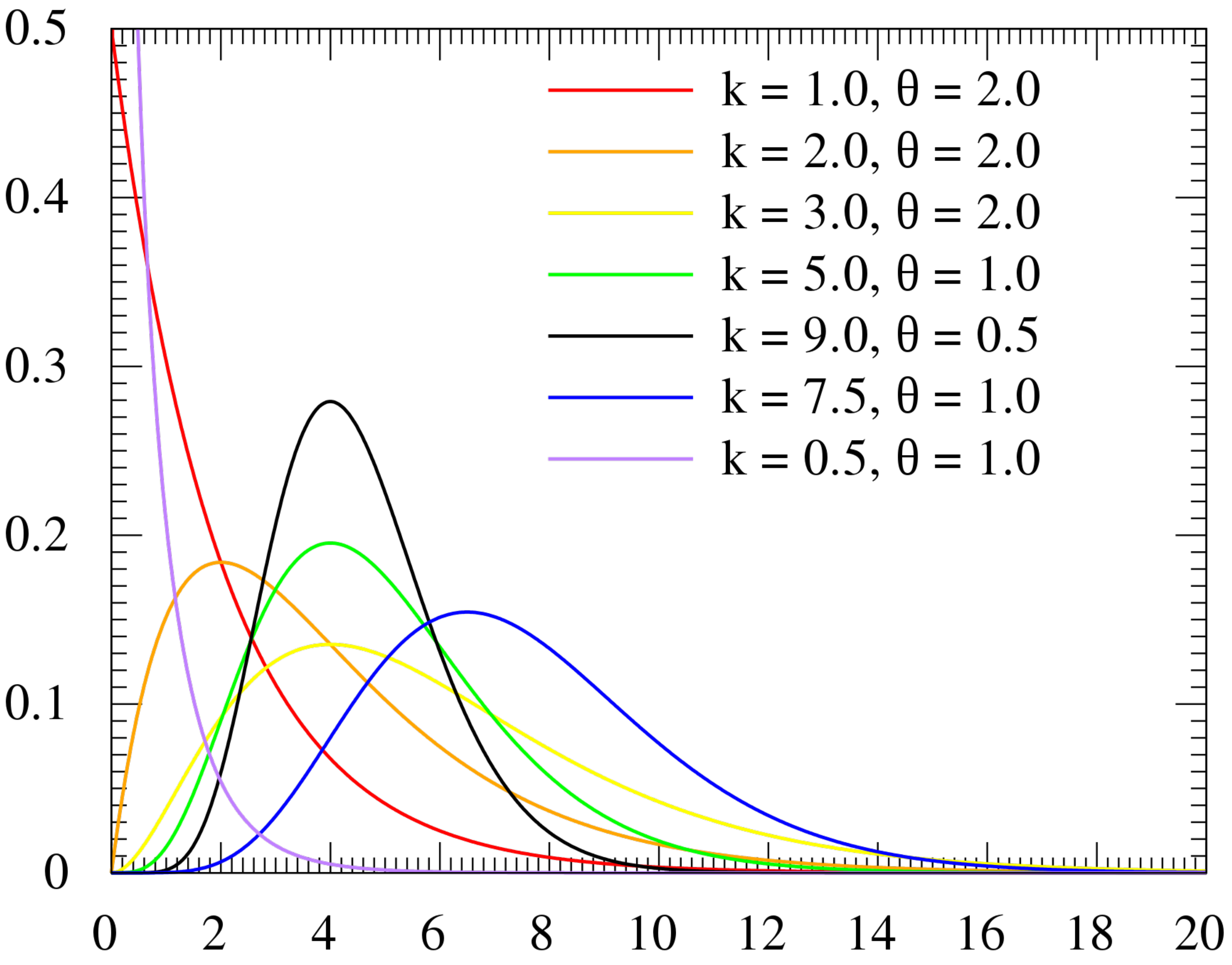
$$P(x) \;=\; \mathsf{Gamma}(x|\alpha,\beta) \;=\; (\beta^{\alpha}/\Gamma(\alpha))x^{\alpha-1}\exp(-\beta x)$$

where α (shape) > 0, β (rate) > 0, x > 0, Γ(·) = gamma function defined for all complex numbers with real part positive

  - $\Gamma(z) = \int_0^\infty x^{z-1} e^{-x}\, dx, \qquad \Re(z) > 0$
  - For positive integer n, gamma(n) = factorial(n-1)

Transformation of Random Variables

- Gamma PDF
  - k = shape = α
  - theta = scale = $1/\text{rate} = 1/\beta$
  - [Link]()



Transformation of Random Variables

- Let X have a Gamma PDF,

$$P(x) = \text{Gamma}(x|\alpha, \beta) = (\beta^\alpha / \Gamma(\alpha)) x^{\alpha-1} \exp(-\beta x)$$

where α (shape) > 0, β (rate) > 0, x > 0, Γ(·) = gamma function defined for all complex numbers with real part positive

  - $\Gamma(z) = \int_0^\infty x^{z-1} e^{-x}\, dx, \qquad \Re(z) > 0$
  - For positive integer n, gamma(n) = factorial(n-1)
- Consider the transformation Y := 1/X
- What is the PDF of Y ?

$$y := 1/x \implies x = 1/y \implies g^{-1}(y) = 1/y$$

$$\left|\frac{d}{dy} g^{-1}(y)\right| = \frac{1}{y^2} \text{for } y > 0$$

Transformation of Random Variables

- Let X have a Gamma PDF,

$$P(x) = \text{Gamma}(x|\alpha, \beta) = (\beta^{\alpha}/\Gamma(\alpha))x^{\alpha-1}\exp(-\beta x)$$

where α > 0, β > 0, x > 0, and Γ(·) is the well-known gamma function defined for all complex numbers with real part positive.

- Consider the transformation Y := 1/X
- What is the PDF of Y ?

$$y := 1/x \implies x = 1/y \implies g^{-1}(y) = 1/y$$

$$\left|\frac{d}{dy}g^{-1}(y)\right| = \frac{1}{y^2} \text{ for } y > 0$$

$$q_1(y) := p(g^{-1}(y))\left|\frac{d}{dy}g^{-1}(y)\right| = (\beta^{\alpha}/\Gamma(\alpha))y^{1-\alpha}\exp(-\beta/y)\frac{1}{y^2} = (\beta^{\alpha}/\Gamma(\alpha))y^{-\alpha-1}\exp(-\beta/y)$$

  - This is called the inverse-Gamma PDF

Transformation of Random Variables

- Inverse-Gamma PDF