Probability I (SI 427)

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Problem set 1

1. Let $\{A_i : i \in \mathbb{N}\}$ be a collection of sets. Prove (De Morgan's law)
$$(\cup_i A_i)^c = \cap_i A_i^c, \quad (\cap_i A_i)^c = \cup_i A_i^c.$$

2. Which of the following are identically true? For those that are not, say when they are true.
   (a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$,
   (b) $A \cap (B \cap C) = (A \cap B) \cap C$,
   (c) $(A \cup B) \cap C = A \cup (B \cap C)$,
   (d) $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$.

3. Let A, B belong to some sigma field $\mathcal{F}$. Show that $\mathcal{F}$ contains the sets $A \cap B, A \setminus B$ (difference) and $A \triangle B$ (symmetric difference).

4. Prove that if $\mathcal{F}_1$ and $\mathcal{F}_2$ are sigma fields of subsets of Ω, then $\mathcal{F}_1 \cap \mathcal{F}_2$ is also a sigma field. Is $\mathcal{F}_1 \cup \mathcal{F}_2$ also a sigma field? Justify your answer.

5. $\Omega = \{1, 2, 3\}$. Write down all possibles sigma fields of subsets of Ω.

6. For a family $\mathcal{S}$ of subsets of Ω, we define
$$\mathcal{F}_{\mathcal{S}} = \cap\{\mathcal{F} : \mathcal{F} \text{ is sigma field such that } \mathcal{S} \subset \mathcal{F}\}.$$
Show that $\mathcal{F}_{\mathcal{S}}$ is a sigma-filed and it is the smallest sigma field containing $\mathcal{S}$.

7. Let A, B be any two subsets of Ω. Write down the smallest sigma field, say $\mathcal{F}$ explicitly containing A and B.

8. Let Ω be a non-empty set and $A_1, A_2, \ldots, A_n \subseteq \Omega$. Also assume $A_i \neq \phi$ for all i and $A_i \cap A_j = \phi$ for $i \neq j$, and $\cup_{i=1}^n A_i = \Omega$, that is $\{A_1, A_2, \ldots, A_n\}$ is a partition of Ω. Describe the smallest sigma-algebra (sigma-field) $\mathcal{F}$ containing $A_1, A_2, \ldots, A_n$. Find cardinality of $\mathcal{F}$ and justify your answer.

9. Show that, $\mathrm{P}(A \cap B) \geq \mathrm{P}(A) + \mathrm{P}(B) - 1$ for $A, B \in \mathcal{F}$. (This is known as *Bonferroni inequality*) Suppose $A_1, A_2, \ldots, A_n \in \mathcal{F}$. Show that
$$\mathrm{P}(\cap_{i=1}^n A_i) \geq \sum_{i=1}^n \mathrm{P}(A_i) - (n-1).$$

10. Given n events $A_1, A_2, \ldots, A_n$, show that
$$\sum_{i=1}^n \mathrm{P}(A_i) - \sum_{i<j} \mathrm{P}(A_i \cap A_j) \leq \mathrm{P}(\cup_{i=1}^n A_i) \leq \sum_{i=1}^n \mathrm{P}(A_i).$$