Simplex Method for LPP.

Thursday, 7 April 2022 1:33 PM

1. Start with an initial BFS.
2. Is the current BFS optimal?
3. If yes, Stop, else; if no move to a new and improved BFS and return to step 2.

Ex. maximise $x_1 + x_2$

s.t. $x_1 + 3x_2 \le 9$

$2x_1 + x_2 \le 8$

$x_1, x_2 \geq 0$

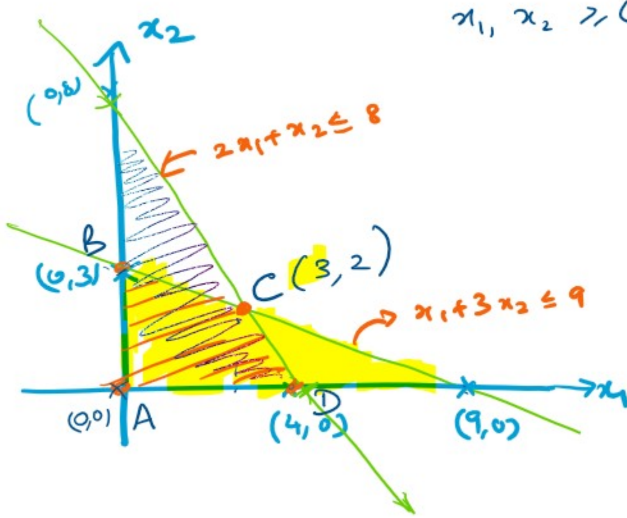
$f(x) = \boxed{x_1 + x_2}$ — ③

$x_1 + 3x_2 + \boxed{x_3} = 9$ — ①

$2x_1 + x_2 + \boxed{x_4} = 8$ — ②

$(x_3, x_4 \geq 0)$

→ slack variables.



Step I

$$\begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array}$$
$$\begin{bmatrix} 1 & 3 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \end{bmatrix}$$

x_3, x_4 basic

x_1, x_2 are free variables $x_1 = x_2 = 0$.

BFS $= [0, 0, 9, 8]^t$ $\quad f = 0$ $\quad (f = x_1 + x_2)$

At vertex A, we have $f = 0$.

From ③, increasing x_1 or x_2 will increase 'f'.

Let us increase x_1:

(i) From ①, increase x_1 to 9, decrease x_3 to 0

(ii) From ②, increase x_1 to 4, decrease x_4 to 0

(ii) From ③, increase x_1 to 4, decrease x_4 to 0

Choose the stricter restrictions, so that all variables remain positive.

$\uparrow x_1$ to 4, $x_2 = 0$, $\downarrow x_3 = 5$; $\downarrow x_4 = 0$

$[4, 0, 5, 0]$ $\qquad f = 4.$

free variables → x_2, x_4

basic variables → x_1, x_3

Write basic variables and f in terms of the free variables

$$\begin{bmatrix} 1 & 3 & 1 & 0 \\ 2 & 1 & 0 & 1 \\ \hline 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ f \end{bmatrix}$$

$x_1 + 3x_2 + x_3 = 9$

$$\begin{bmatrix} 1 & 1 & 3 & 0 \\ 2 & 0 & 1 & 1 \\ \hline 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \\ x_2 \\ x_4 \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ f \end{bmatrix}$$

$x_1 + x_3 + 3x_2 = 9$

$R_1 \to R_1 - \frac{1}{2} R_2$

$$\begin{bmatrix} 0 & 1 & 5/2 & -1/2 \\ 2 & 0 & 1 & 1 \\ \hline 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \\ x_2 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ f \end{bmatrix}$$

$R_2 \to \frac{R_2}{2}$

$$\begin{bmatrix} 0 & 1 & 5/2 & -1/2 \\ 1 & 0 & 1/2 & 1/2 \\ \hline 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \\ x_2 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ f \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 0 & 1 & 5/2 & -1/2 \\ 1 & 0 & 1/2 & 1/2 \\ \hline 0 & 0 & 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \\ x_2 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ \hline f-4 \end{bmatrix}$$

$$f = 4 \cancel{+} \frac{1}{2} x_2 \cancel{-} \frac{1}{2} x_4 \qquad \boxed{f = 4}$$

$$x_3 + \frac{5}{2} x_2 \cancel{-} \frac{1}{2} x_4 = 5$$
$$x_1 + \frac{1}{2} x_2 + \frac{1}{2} x_4 = 4$$

$x_2 = 0$
$x_4 = 0$

$$\boxed{[4, 0, 5, 0]^t \qquad f = 4.}$$

$f = x_1 + x_2$

$$\frac{5}{2} x_2 + x_3 - \frac{1}{2} x_4 = 5 \quad \text{— (4)}$$
$$x_1 + \frac{1}{2} x_2 + \frac{1}{2} x_4 = 4 \quad \text{— (5)}$$
$$\boxed{\frac{1}{2} x_2 - \frac{1}{2} x_4 + 4 = f \quad \text{— (6)}}$$

Increase x_2 to maximise f.

Go to (4), (5)
$$\begin{cases} x_2 = 2 & \text{from (4)} \\ x_2 = 8 & \text{from (5)} \end{cases}$$

$\min \left\{ \frac{5}{5/2}, \frac{4}{(1/2)} \right\}$

$x_2 = 2$, $x_3 = 0$, $x_4 = 0$

$x_1 = 3$

$\uparrow x_2$ to 2, decrease $(x_3 = 0)$

x_1

$$[3, 2, 0, 0]^t \qquad f = 5$$

$$\left[\begin{array}{cccc} 0 & 1 & 5/2 & -1/2 \\ 1 & 0 & 1/2 & 1/2 \\ \hline 0 & 0 & 1/2 & -1/2 \end{array}\right] \begin{bmatrix} \left.\begin{matrix} x_1 \\ x_3 \end{matrix}\right\} \text{basic} \\ x_2 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ f-4 \end{bmatrix}$$

x_3 free. $\qquad$ x_2 basic.

$$\left[\begin{array}{cccc} 0 & 5/2 & 1 & -1/2 \\ 1 & 1/2 & 0 & 1/2 \\ \hline 0 & 1/2 & 0 & -1/2 \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \left[\begin{array}{c} 5 \\ 4 \\ \hline f-4 \end{array}\right]$$

$R_1 \to \frac{2}{5} R_1$
$$\left[\begin{array}{cccc} 0 & 1 & 2/5 & -1/5 \\ 1 & 1/2 & 0 & 1/2 \\ \hline 0 & 1/2 & 0 & -1/2 \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ f-4 \end{bmatrix}$$

$R_2 \to R_2 - \frac{1}{2} R_1 \sim$
$$\left[\begin{array}{cccc} 0 & 1 & 2/5 & -1/5 \\ 1 & 0 & -1/5 & 3/5 \\ \hline 0 & 1/2 & 0 & -1/2 \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \left[\begin{array}{c} 2 \\ 3 \\ \hline f-4 \end{array}\right]$$

$R_3 \to R_3 - \frac{1}{2} R_1 \sim$
$$\left[\begin{array}{cccc} 0 & 1 & 2/5 & -1/5 \\ 1 & 0 & -1/5 & 3/5 \\ \hline 0 & 0 & -1/5 & -2/5 \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \left[\begin{array}{c} 2 \\ 3 \\ \hline f-5 \end{array}\right]$$

$x_3 = x_4 = 0$

$[3, 2, 0, 0] \qquad f = 5$

$$x_2 + \boxed{\frac{2}{5} x_3 - \frac{1}{5} x_4} = 2$$
$$x_1 - \underbrace{\frac{1}{5} x_3 + \frac{3}{5} x_4} = 3$$
$$f - 5 = -\frac{1}{5} x_3 - \frac{2}{5} x_4.$$

$$f = 5 - \frac{1}{5}x_3 - \frac{2}{5}x_4 \qquad f \leq 5.$$

- $B \rightarrow$ basic variables
- express x_i, $i \in B$ and f in terms of the free variables, x_i, $i \notin B$
- Set $x_i = 0$, $i \notin B$; and calculate $\underline{f, \& \; x_i}$ $(i \in B)$

(free = 0) x_1, x_2; basic x_3, x_4

	x_1	x_2	x_3	x_4		
0	1	3	1	0	9	ρ_1
1	$\boxed{2}$	1	0	1	8	ρ_2
0	1	1	0	0	0	ρ_3

$f = \underline{c}^T \underline{x}$

Initial

$\bar{a}_{ij} = \bar{A}$	$\bar{b}$
$\bar{\underline{c}}^T$	$\bar{f}$

$\bar{A} = A$
$\bar{b} = b$
$\bar{c}^T = c^T$
$\bar{f} = 0$

1. Choose a pivot column.

   Choose a j such that $\bar{c}_j > 0$ [Corresponds to the variable that we want to increase]

2. Choose a pivot row

   Among all i's with $\bar{a}_{ij} > 0$, choose i to minimize $\bar{b}_i / \bar{a}_{ij}$ [gives how much we can increase x_j].

1	3	1	0	9	ρ_1
2	1	0	1	8	ρ_2
[illegible]	1	0	0	0	ρ_3

0	$5/2 \to 1$	1	$-1/2$	5	$\rho_1' \to \rho_1 - \rho_2'$
1	$1/2 \to 0$	0	$1/2$	4	$\rho_2' \to \rho_2/2$
0	$1/2 \to 0$	0	$-1/2$	-4	$\rho_3' \to \rho_3 - \rho_2'$

$f - 4$

Pivot column = 2^{nd} column

Pivot row = $\min\left\{ 5/\tfrac{5}{2},\ 4/\tfrac{1}{2} \right\} = \min\{2, 8\}$

x_1	x_2	x_3	x_4		
0	1	$2/5$	$-1/5$	2	$\rho_1'' = \rho_1' \times \frac{2}{5}$
1	0	$-1/5$	$3/5$	3	$\rho_2'' = \rho_2' - \frac{1}{2}\rho_1''$
0	0	$-1/5$	$-2/5$	-5	

$f - 5$

$f - 5 = \frac{-1}{5} x_3 - \frac{2}{5} x_4.$