Practice Problem Set 13 Released: Sunday November 3, 2019

1. Show that the following subsets of $\mathbb{R}$ are countably infinite:
   (a) $\{2^q : q \in \mathbb{Q}^+\}$
   (b) $\{x \in \mathbb{R} : x > 0 \wedge x^2 \in \mathbb{Q}\}$
2. Show that the following sets have the same cardinality as $\mathbb{R}$.
   (a) $(0, \infty)$
   (b) $[0, 1] \cup \mathbb{Z}$
   (c) $\mathbb{R} \times \mathbb{R}$ (*Hint: Flesh out the proof from the lecture.*)
3. Suppose for each $i \in \mathbb{N}$, S_i is a countable set. Using the result that $\mathbb{N} \times \mathbb{N}$ is countable, prove that the following sets are countable (but not necessarily infinite). Be sure to handle any special cases (e.g., some $S_i = \emptyset$). You may use the fact that S is countable iff $|S| \leq |\mathbb{N}|$.[1]
   (a) $\bigcup_{i \in \mathbb{N}} S_i$
   (b) For any $k \in \mathbb{N}$, $S_0 \times \cdots \times S_k$
4. Let A, B and C be sets such that $|A| < |B|$ and $B \subseteq C$. Prove that $|A| < |C|$.
5. $\mathbb{Z}[X]$ denotes the set of all polynomials in a variable X, with integer coefficients. Show that $\mathbb{Z}[X]$ is countable.

   Hint: Prove first that for every integer $n \geq 1$ the set P_n of all polynomials of degree $\leq n$ with integer coefficients is countable. Then use the fact that the union of countably many countable sets is countable.
6. Consider a machine which has n bits of memory (initialized to all 0s). Its state is determined by the contents of its memory. This machine takes inputs from the set $\Sigma = \{1, \ldots, n\}$: On input i, it toggles its i^{th} bit. (Toggling a bit changes its value from 0 to 1 or from 1 to 0.)
   (a) Write down the transition function for this machine for $n = 2$, as table, with columns "current state," "input" and "next state." (Note that each state is labeled with 2 bits.)
   (b) What graph does the state-diagram for this machine (for a general value of n) resemble? You can treat a pair of directed edges between two states, but pointing in opposite directions, as a single undirected edge. Justify your answer.

   Hint: For this part, it may be helpful to draw a diagram for the case $n = 2$ from the previous part.
7. Give deterministic finite-state acceptors for the following languages over the alphabet $\{0, 1\}$. In each case, write down its transition function (in the form of a table) and also draw the state diagram.
   (a) $\{x \mid x$ is non-empty and begins and ends with the same symbol$\}$
   (b) $\{x \mid x$ does not contain the substring 11$\}$
   (c) $\{x \mid x$ number of zeroes is even and number of ones is odd$\}$
8. Give non-deterministic finite-state acceptors for the following languages over the alphabet $\{0, 1\}$. In each case, write down its transition function (in the form of a table) and also draw the state diagram.
   (a) $\{x \mid x$ ends in 00 or 010 $\}$
   (b) $\{x \mid x$ contains the substring 11$\}$
9. It was mentioned in class that the language consisting of *all* palindromes does not have a finite-state acceptor. In this problem we consider palindromes of a fixed length.
   (a) Describe, as explicitly as you can, a deterministic finite-state acceptor for the language consisting of all palindromes of length exactly 100.

   Hint: The first 50 input bits need to be memorized. Arrange the states in a binary tree for this.

[1] In class, we defined a set S as countable if S is finite or $|S| = |\mathbb{N}|$. By the "axiom of countable choice" this is equivalent to $|S| \leq |\mathbb{N}|$.

(b) Using the pigeonhole principle, argue that any such acceptor should have at least 2^{50} states. Can you tighten this argument to show that your construction from above has the minimum possible number of states?

Hint: Consider the set of prefixes of strings in the language. Can two distinct prefixes take the machine to the same state?

10. Show that each of the following decision problems is in **NP**, by describing a "certificate" (for "yes" instances) that can be verified in time that is polynomial in the length of the instance.

(a) Is x the binary representation of a composite number?

(b) Is x the binary representation of an even number?

(c) Are the pair of graphs (G_1, G_2) (represented using their adjacency matrices) isomorphic to each other?

(d) Does the graph G have a Hamiltonian cycle?

(e) Does a polynomial $p(X)$ with integer coefficients have an integer root? *Hint: You may use the fact that all the real roots of a polynomial* $c_d X^d + \ldots + c_1 X + c_0$ *lie in the range* $[-s, s]$*, where* $s = \max\{1, |c_0| + \ldots + |c_n|\}$*. Why do you need to use this fact?*