Lecture 23 [Problems with inequality constraints]

Thursday, 24 March 2022 1:46 PM

$$\left[\begin{array}{ll} \text{minimize} & f(\underline{x}) \\ \text{s.t.} & \begin{cases} \underline{h}(\underline{x}) = 0 \\ \underline{g}(\underline{x}) \le 0 \end{cases} \end{array}\right] \qquad \begin{array}{l} f : \mathbb{R}^n \to \mathbb{R} \\ \underline{h} : \mathbb{R}^n \to \mathbb{R}^m \\ \underline{g} : \mathbb{R}^n \to \mathbb{R}^p \end{array}$$

$\boxed{\text{Defn}}$ An <u>inequality constraint</u> $g(x) \le 0$ is said to be <u>active</u> at $\underline{x}^*$ if $g_j(\underline{x}^*) = 0$. It is said to be <u>inactive</u> at $\underline{x}^*$ if $g_j(\underline{x}^*) < 0$.

---

min $f(x) := (x+5)^2$ s.t. $x \le 0$

$g_1(x) = x$. ↙ inactive constraint

↓

$\underline{x}^* = \underline{-5} < 0$

min $f(x) = (x-5)^2 \to \boxed{x=5}$

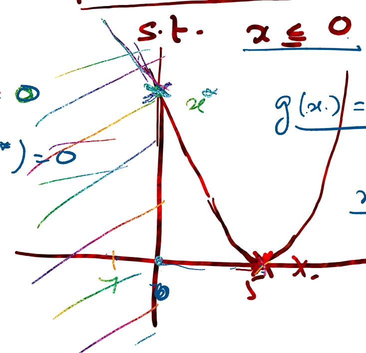
s.t. $x \le 0$

$g(x) = x$.

$x^* = 0$

$g(x^*) = 0$

$x^* = 0$



<u>Defn</u> $\boxed{\text{Regular point}}$ Let $\underline{x}^*$ satisfy $\left\{\begin{array}{l} \underline{h}(\underline{x}^*) = 0 \\ \underline{g}(\underline{x}^*) \le 0 \end{array}\right]$

and let $J(\underline{x}^*)$ be the index set of active inequality constraints

$$J(\underline{x}^*) = \{ j : g_j(\underline{x}^*) = 0 \}.$$

Then $\underline{x}^*$ is a <u>regular point</u> if the vectors

$$\nabla h_i(x^*) \quad (i = 1, \dots, m) \qquad \text{and} \qquad \nabla g_j(x^*) \quad j \in J(\underline{x}^*)$$

are <u>linearly independent</u>.

Karush Kuhn Tucker Theorem [KKT] FONC [KT condtn].

Karush-Kuhn-Tucker Theorem [KKT] (FONC) [KT condtn].

- Let $f, h, g \in \mathcal{C}^{(1)}$
- Let $\underline{x}^*$ be a regular point and a local minimizer for f s.t. $\underline{h}(\underline{x}) = 0$, $\underline{g}(x) \leq 0$.

Then $\exists$ $\underline{\lambda}^* \in \mathbb{R}^m$ and $\underline{\mu}^* \in \mathbb{R}^p$ such that

(i) $\underline{\mu}^* \geq 0$ → (dual feasibity)

(ii) $Df(x^*) + \underline{\lambda}^{*T} D\underline{h}(\underline{x}^*) + \underbrace{\underline{\mu}^{*T}}_{\text{KKT multiplier}} Dg(\underline{x}^*) = 0$ → Stationarity condition

Lagrange multiplier

(iii) $\underline{\mu}^{*T} \underline{g}(\underline{x}^*) = 0$ → (Complementary Slack Condition).

(iv) $\underline{h}(\underline{x}^*) = 0$

(v) $\underline{g}(\underline{x}^*) \leq 0$

(iv), (v) → primal feasibity condtns.

1. $$0 = \underbrace{\mu_1^*}_{\geq 0 \text{ from (i)}} \underbrace{g_1(x^*)}_{\leq 0 \text{ (from (v))}} + \mu_2^* g_2(x^*) + \dots + \mu_p^* g_p(x^*)$$

(iii) is ← $\left\{ \mu_j^* g_j(x^*) = 0 \quad j = 1, \dots p. \right.$ →

2.

	Inactive	Active
Constraints	$g_j(x^*) < 0$	$g_j(x^*) = 0$
KKT multiplies	$\mu_j^* = 0$	$\mu_j(x^*) \geq 0$
		$\mu_j > 0$ / $\mu_j = 0$

Second order Necessary Condition: $h_1, \dots, h_m$

$$\ell(\underline{x}, \underline{\lambda}, \underline{\mu}) = f(\underline{x}) + \underline{\lambda}^T \underline{h}(\underline{x}) + \underline{\mu}^T \underline{g}(\underline{x})$$

Define $$\mathcal{L}(\underline{x}, \underline{\lambda}, \underline{\mu}) = F(\underline{x}) + [\underline{\lambda} \quad H(\underline{x})] + [\underline{\mu} \quad G(\underline{x})]$$

[Hessian] μ_1(Hessian of g_1)

Hessian of f

λ_1 (Hessian of h_1) $+\cdots+\lambda_m$ (Hessian of h_m)

μ_1 (Hessian of g_1) $+\cdots+\mu_p$ (Hessian of g_p)

$$T(\underline{x}^*) = \{ \underline{y} \in \mathbb{R}^n : D\underline{h}(\underline{x}^*)\,\underline{y} = 0,\ Dg_j(\underline{x}^*)\,\underline{y} = 0;\ j \in J(\underline{x}^*) \}$$

$J(\underline{x}^*)$ → index set of active constraints.

SONC

Thm. Let $\underline{x}^*$ be a local minimizer of $f : \mathbb{R}^n \to \mathbb{R}$, s.t. $\boxed{\underline{h}(\underline{x}) = 0,\ \underline{g}(\underline{x}) \le 0,}$ $\underline{h} : \mathbb{R}^n \to \mathbb{R}^m$, $\underline{g} : \mathbb{R}^n \to \mathbb{R}^p$, $f, \underline{h}, \underline{g} \in \mathcal{C}^2$. Let $\underline{x}^*$ be regular. Then

$\exists\ \underline{\lambda}^* \in \mathbb{R}^m$, $\underline{\mu}^* \in \mathbb{R}^p$ s.t.

(i) $\underline{\mu}^* \ge 0$ (ii) $Df(\underline{x}^*) + \underline{\lambda}^{*T} D\underline{h}(\underline{x}^*) + \underline{\mu}^{*T} D\underline{g}(\underline{x}^*) = \underline{0}^T$

(iii) $\underline{\mu}^{*T} \underline{g}(\underline{x}^*) = 0$

(iv) $\forall\, \underline{y} \in T(\underline{x}^*)$, $\underline{y}^T \mathcal{L}(\underline{x}^*, \underline{\lambda}^*, \underline{\mu}^*)\,\underline{y} \ge 0$.

II order Suff. conditions (SOSC).

- $f, g, h \in \mathcal{C}^2$
- $\exists$ a feasible point $\underline{x}^* \in \mathbb{R}^n$ and vectors $\underline{\lambda}^* \in \mathbb{R}^m$ and $\underline{\mu}^* \in \mathbb{R}^p$ s.t.

$\boxed{\underline{h}(\underline{x}^*) = 0,\ \underline{g}(\underline{x}^*) \le 0.}$

(i) $\underline{\mu}^* \ge 0$ (ii) $Df(x^*) + \underline{\lambda}^{*T} Dh(\underline{x}^*) + \underline{\mu}^{*T} D\underline{g}(\underline{x}^*) = \underline{0}^T$

(iii) $\underline{\mu}^* g(\underline{x}^*) = 0$ (iv) $\forall\, y \in \tilde{T}(\underline{x}^*, \underline{\mu}^*)$; $\underline{y}^T \mathcal{L}(\underline{x}^*, \underline{\lambda}^*, \underline{\mu}^*)\,\underline{y} > 0$ $\boxed{\underline{y} \neq 0}$

$\tilde{T}(\underline{x}^*, \underline{\mu}^*) = \{ \underline{y} : Dh(\underline{x}^*)\,\underline{y} = 0;\ Dg_j(\underline{x}^*)\,\underline{y} = 0\ \ \forall j \in \tilde{J}(\underline{x}^*, \underline{\mu}^*) \}$

$\tilde{T}(\underline{x}^*, \underline{\mu}^*) = \{ \underline{y} : Dh(\underline{x}^*)\,\underline{y} = 0;\ Dg_j(\underline{x}^*)\,\underline{y} = 0 \quad \forall j \in \tilde{J}(\underline{x}, \underline{\mu}) \}$

Then $\underline{x}^*$ is a strict local minimizer of 'f' s.t. $\left.\begin{matrix} h(\underline{x}) = 0 \\ g(\underline{x}) \le 0 \end{matrix}\right]$

with $\tilde{J}(\underline{x}^*, \mu^*) = \{ j : g_j(\underline{x}^*) = 0,\ \mu_j^* > 0 \} \subset \underline{J(\underline{x}^*)}$

$= \{ j : \underline{g_j(x^*) = 0} \}$

Example $\quad \min \quad x_1 x_2$

$x_1 + x_2 \geq 2$

$x_2 \geq x_1$

$f(\underline{x}) = f(x_1, x_2) = x_1 x_2$

$g_1(\underline{x}) = 2 - x_1 - x_2 \le 0$

$g_2(\underline{x}) = x_1 - x_2 \le 0$

$\boxed{\text{KKT conditions}}$

$$[x_2 \quad x_1] + \mu_1 [-1 \quad -1] + \mu_2 [1 \quad -1] = 0$$

$x_1 - \mu_1 - \mu_2 = 0$

$x_2 - \mu_1 + \mu_2 = 0$

$\mu_1 (2 - x_1 - x_2) + \mu_2 (x_1 - x_2) = 0$

$\mu_1, \mu_2 \geq 0$

$2 - x_1 - x_2 \le 0$

$x_1 - x_2 \le 0$

$\left.\begin{matrix} x_1 = \mu_1 + \mu_2 \\ x_2 = \mu_1 - \mu_2 \end{matrix}\right] \qquad \mu_1 = \frac{x_1 + x_2}{2} \qquad \mu_2 = \frac{x_1 - x_2}{2}$

$\mu_1 (2 - 2\mu_1) + \mu_2 (2\mu_2) = 0$

$\mu_1 (1 - \mu_1) = 0 \qquad \underline{\mu_2 = 0}$

$\mu_1 = 0$ or $\mu_1 = 1$

$x_1 + x_2 \geq 2$

$\mu_1 \geq 1$

μ_1^*	~~0~~	1
μ_2^*	0	
x_1	1	
x_2	1	

1. $\boxed{\begin{matrix} x_1^* = 1 & \mu_1^* = 1 \\ x_2^* = 1 & \mu_2^* = 0 \end{matrix}}$ feasible pt.

(1) $\boxed{x_2^* = 1 \qquad \mu_2^* = 0}$ feasible pt.

(3) Is $\underline{x}^* = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ a regular point? Yes. [Check].

(3) $$\mathcal{L}(\underline{x}^*, \underline{\lambda}^*, \mu^*) = F(\underline{x}^*) + \mu_1 (\text{Hessian of } g_1) + \mu_2 (\text{Hessian of } g_2).$$

$f(x_1, x_2) = x_1 x_2$

$$\nabla f = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix} \qquad F = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$\nabla g_1 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

$\nabla g_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Hessian are 0

$$\mathcal{L}(\underline{x}^*, \lambda^*, \mu^*) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\tilde{T}(\underline{x}^*, \mu^*) = \left\{ \underline{y} \in \mathbb{R}^2 : \begin{bmatrix} -1 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 0 \right\}$$

$y_1 + y_2 = 0.$

$\alpha \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$

$$\underline{y}^T \mathcal{L}(\underline{x}^*, \underline{\lambda}^*, \underline{\mu}^*) \underline{y} = \alpha \begin{bmatrix} 1 \\ -1 \end{bmatrix}^T \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \alpha \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \alpha^2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}^T \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$= \alpha^2 \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -2\alpha^2 < 0.$$

$\boxed{\text{Local maximizer}}$