Tutorial Sheet 7:

Q5. $f(x) = \frac{1}{2} x^T Q x - b^T x + c,$

$$Q = \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad c = \pi^2$$

Take: $H_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad x^{(0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$d^{(0)} = -g^{(0)} = -(Qx^{(0)} - b) = b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

$$x^{(1)} = x^{(0)} + \alpha_0 d^{(0)}, \quad \alpha_0 = \frac{-g^{(0)T} d^{(0)}}{d^{(0)T} Q d^{(0)}} = \frac{-\begin{bmatrix} 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}}{\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}} = \frac{1}{2}$$

<u>Next:</u> Compute B_1^{-1} from the formula in Lecture Slides 19.

Then update $d^{(1)}$. Compute $\alpha^{(1)}$.

Repeat it once more for getting B_2^{-1} etc...

Q6.b) Minimize $-4x_1 - x_2^2$

subject to $x_1^2 + x_2^2 = 9$

(Let's do this using Lagrange multipliers).

$$f(\underset{\sim}{x}) = -4x_1 - x_2^2$$

$$h(\underset{\sim}{x}) = x_1^2 + x_2^2 - 9$$

$$\left.\begin{aligned} \nabla f(\underset{\sim}{x}^*) + \lambda^* \nabla h(\underset{\sim}{x}^*) &= \underset{\sim}{0} \\ h(x^*) &= 0 \end{aligned}\right\} \longrightarrow \begin{aligned} &\begin{bmatrix} -4 \\ -2x_2 \end{bmatrix} + \begin{bmatrix} 2\lambda x_1 \\ 2\lambda x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ &x_1^2 + x_2^2 - 9 = 0 \end{aligned}$$

$$-4 + 2\lambda^* x_1 = 0$$

$$-2x_2 + 2\lambda^* x_2 = 0 \longrightarrow 2x_2(-1+\lambda^*) = 0$$

$$x_1^2 + x_2^2 = 9$$

<u>Case I</u>: $x_2 = 0$

Then $x_1 = \pm 3$, $\lambda^* = \pm \frac{2}{3}$, $x^* = \begin{bmatrix} \pm 3 \\ 0 \end{bmatrix}$

<u>Case II</u>: $\lambda^* = 1$

$x_1 = 2$, $x_2 = \pm\sqrt{5}$, $x^* = \begin{bmatrix} 2 \\ \pm\sqrt{5} \end{bmatrix}$

If x^* is a minimizer, then by SONC,

$$y^T L(x^*, \lambda^*) y \geq 0 \quad \text{for all} \quad y \in T(x^*),$$

where $L(x^*, \lambda^*) = F(x^*) + \lambda_1 H_1(x^*, \lambda^*) + \cdots + \lambda_k H_k(x^*, \lambda^*)$

In our case:

$$F(x) = \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}$$

$$H(x) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$\therefore$ SONC is satisfied at x^* whenever

$$y^T \begin{bmatrix} 2\lambda^* & 0 \\ 0 & -2+2\lambda^* \end{bmatrix} y \geq 0$$

for all $y \in T(x^*)$.

Consider the pair $(x^*, \lambda^*) = \left([3, 0]^T, \frac{2}{3} \right)$

$$L(x^*, \lambda^*) = \begin{bmatrix} 4/3 & 0 \\ 0 & -2/3 \end{bmatrix}$$

$$T(x^*) = \{ y : Dh(x^*) \cdot y = 0 \}.$$

$$= \left\{ y : \begin{bmatrix} 2 \cdot 3 & 2 \cdot 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 0 \right\}$$

$$= \left\{ y : \begin{bmatrix} 6 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 0 \right\}$$

$$= \{ y : y_1 = 0 \}.$$

So, an arbitrary vector $y \in T(x^*)$ ~~that~~ is of the form: $y = [0, y_2]^T$.

$$\therefore \quad y^T L(x^*, \lambda^*)\, y = -\frac{2}{3} y_2^2 \leq 0$$

In particular, $y = [0, 1]^T \in T(x^*)$ and we have $y^T L(x^*, \lambda^*)\, y < 0$.

$\therefore$ x^* is <u>not</u> a local minimizer.

Next, check for $(x^*, \lambda^*) = ([-3, 0]^T, -\frac{2}{3})$

$$L(x^*, \lambda^*) = \begin{bmatrix} -4/3 & 0 \\ 0 & -10/3 \end{bmatrix} < 0$$

$\therefore$ $x^* = [-3, 0]^T$ is NOT a local minimizer.

Next, check for $\lambda^* = 1$: $L(x^*, \lambda^*) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \geq 0$

$\therefore$ Potential minimizers are $x^* = \begin{bmatrix} 2 \\ \pm\sqrt{5} \end{bmatrix}$.

If x^* is a feasible point with Lagrange multiplier λ^*, such that

$$y^T L(x^*, \lambda^*) y > 0$$

for all $y \neq 0$ in $T(x^*)$, then by SOSC, x^* is a strict local minimizer for f.

$$T([2 \ \sqrt{5}]^T) = \{ y : Dh([2 \ \sqrt{5}]^T) \cdot y = 0 \}$$

$$= \{ y : [4 \ \ 2\sqrt{5}] \, y = 0 \}$$

$$= \{ y : 4y_1 + 2\sqrt{5} y_2 = 0 \}$$

$y^T \mathcal{L}(x^*, \lambda^*) y^* = 2y_1^2$ when $y = [y_1 \; y_2]^T$.

$x^* = [2 \; \sqrt{5}]^T$

Now, this is zero $\Leftrightarrow y_1 = 0$.

But if $y \in T(x^*)$ with $y_1 = 0$,

then we also must have that

$$4 \cdot 0 + 2\sqrt{5} \cdot y_2 = 0 \Rightarrow y_2 = 0.$$

$\therefore \; y^T \mathcal{L}(x^*, \lambda^*) y = 0$ for $y \in T(x^*) \Leftrightarrow y = 0$

Else it is > 0.

$\therefore$ SOSC is satisfied by $\begin{bmatrix} 2 & \pm\sqrt{5} \end{bmatrix}$.

So, both are strict local minimizers.

Q8. a)

minimize $2x_1x_2 + 2x_1x_3 + 2x_2x_3$

subject to $x_1x_2x_3 - V = 0$

$\underset{\sim}{h}(\underset{\sim}{x}) = \begin{bmatrix} h_1(x) \\ \vdots \\ h_m(x) \end{bmatrix}$

$\nabla f + \lambda \nabla h = 0$

$$\begin{bmatrix} 2x_2 + 2x_3 \\ 2x_1 + 2x_3 \\ 2x_1 + 2x_2 \end{bmatrix} + \lambda \begin{bmatrix} x_2x_3 \\ x_1x_3 \\ x_1x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

b) $h(\underset{\sim}{x}) = x_1x_2x_3 - V$

$\{\nabla h(x^*)\}$ is L.I. $\iff$ $\nabla h(x^*) \neq 0$.

Observe that in our case:

$$\left.\begin{aligned} \underline{x_1} x_2 &= 0 \\ x_2 \underline{x_3} &= 0 \\ x_1 x_3 &= 0 \end{aligned}\right\} \Leftrightarrow x_i = x_j = 0 \text{ for some } i \neq j$$

$\swarrow$ satisfying

However any ~~such point~~ $[x_1, x_2, x_3]$ does not satisfy the constraint $h(x) = x_1 x_2 x_3 - V$

$\therefore$ All feasible $\underset{\sim}{x}$ in this case are regular points.

What is a regular point?

A point $\underset{\sim}{x}^*$ satisfying $\underset{\sim}{h}(x^*) = 0$ is regular if $\nabla h_1(x^*), \ldots, \nabla h_m(x^*)$ are m linearly independent vectors in $\mathbb{R}^n$. [Here $h : \mathbb{R}^n \to \mathbb{R}^m$,

$$h(\underset{\sim}{x}) = \begin{bmatrix} h_1(x) \\ \vdots \\ h_m(\underset{\sim}{x}) \end{bmatrix}$$