Tutorial Sheet 2
Arithmetic Error Analysis

1. Let X be a sufficiently large number which result in an overflow of memory on a computing device. Let x be a sufficiently small number which result in underflow of memory on the same computing device. Then give the output of the following operations:
   (i) $x \times X$ (ii) $3 \times X$ (iii) $3 \times x$ (iv) x/X (v) X/x.

2. In this question, computations are done on a computer which uses 3-digit chopping arithmetic.
   i) Compute the mid-point of the interval $[0.982, 0.987]$. Show all the steps of the computation.
   ii) Give a different way of computing the mid-point of $[0.982, 0.987]$, so that the mid-point lies in the given interval $[0.982, 0.987]$. Show all the steps of the computation. Explain the differences between both these computations.

3. In the following problems, show all the steps involved in the computation.
   i) Using 5-digit rounding, compute $37654 + 25.874 - 37679$.
   ii) Let $a = 0.00456$, $b = 0.123$, $c = -0.128$. Using 3-digit rounding, compute $(a + b) + c$, and $a + (b + c)$. What is your conclusion?
   iii) Let $a = 2$, $b = -0.6$, $c = 0.602$. Using 3-digit rounding, compute $a \times (b + c)$, and $(a \times b) + (a \times c)$. What is your conclusion?

4. Consider a computing device having exponents e in the range $m \leq e \leq M$, $m, M \in \mathbb{Z}$. Let n be an integer such that $n \leq |m| + 1$.
   i) If the device uses n-digit rounding binary floating-point arithmetic, then show that $\delta = 2^{-n}$ is the machine epsilon.
   ii) What is the machine epsilon of the device if it uses n-digit rounding decimal floating-point arithmetic? Justify your answer.

5. Consider a computing device that uses n-digit chopping (decimal) arithmetic. Let $\mathrm{fl}(x)$ denote the floating-point approximation of a positive real number x in this device. Prove
$$\left|\frac{x - \mathrm{fl}(x)}{x}\right| \leq 10^{-n+1}.$$

6. The ideal gas law is given by $PV = nRT$ where R is the gas constant. We are interested in knowing the value of T for which $P = V = n = 1$. If R is known only approximately as $R_A = 8.3143$ with an absolute error at most 0.12×10^{-2}. Obtain an upper bound for the absolute relative error in the computation of T that results in using R_A instead of R?

7. Let $x_A = 3.14$ and $y_A = 2.651$ be obtained from x_T and y_T using 4-digit rounding. Find the smallest interval that contains
   (i) x_T (ii) y_T (iii) $x_T + y_T$ (iv) $x_T - y_T$ (v) $x_T \times y_T$ (vi) x_T/y_T.

8. Obtain the number of significant digits of $x_A = 0.025678$ present in $x = 0.025611$.

9. Instead of using the true values $x_T = 0.62457371$ and $y_T = 0.62457238$ in calculating $z_T = x_T - y_T$, if we use the approximate values $x_A = 0.62451251$ and $y_A = 0.62458125$, and calculate $z_A = x_A - y_A$, then find the loss of significant digits in the process of calculating z_A when compared to the significant digits in x_A.

10. Let $x_A = 0.04078$ has exactly 3 significant digits with respect to the real number x_T. Find the smallest interval in which x_T lies.

11. Given $x = 0.75371$ and $y = -0.49572$. Let the product $x * y$ be computed using 3-digit rounding floating-point arithmetic. What is the absolute value of the total error? [Give the final answer with at least 6-digits after decimal places]

12. For small values of x, the approximation $\sin x \approx x$ is often used. Obtain a range of values of x for which the approximation gives an absolute error of at most $\frac{1}{2} \times 10^{-6}$.

13. Is the process of computing the value of the function $f(x) = (e^x - 1)/x$ stable or unstable for $x \approx 0$? Justify your answer.