CHAPTER 8

Interpolation

Let a physical experiment be conducted and the outcome is recorded only at some finite number of times. If we want to know the outcome at some intermediate time where the data is not available, then we may have to repeat the whole experiment once again to get this data. In the mathematical language, suppose that the finite set of values

$$\{f(x_i) \,:\, i = 0, 1, \cdots, n\}$$

of a function f at a given set of points

$$\{x_i \,:\, i = 0, 1, \cdots, n\}$$

is known and we want to find the value of $f(x)$, for some $x \in (x_m, x_M)$, where $x_m = \min\limits_{j=0,1,\cdots,n} x_j$ and $x_M = \max_{j=0,1,\cdots,n} x_j$. One way of obtaining the value of $f(x)$ is to compute this value directly from the expression of the function f. Often, we may not know the expression of the function explicitly and only the *data*

$$\{(x_i, y_i) \,:\, i = 0, 1, \cdots, n\}$$

is known, where $y_i = f(x_i)$. In terms of the physical experiments, repeating an experiment will quite often be very expensive. Therefore, one would like to get at least an approximate value of $f(x)$ (in terms of experiment, an approximate value of the outcome at the desired time). This is achieved by first constructing a function whose value at x_i coincides exactly with the value $f(x_i)$ for $i = 0, 1, \cdots, n$ and then finding the value of this constructed function at the desired points. Such a process is called *interpolation* and the constructed function is called the *interpolating function* for the given data.

In certain circumstances, the function f may be known explicitly, but still too difficult to perform certain operations like differentiation and integration. Thus, it is often preferred to restrict the class of interpolating functions to polynomials, where the differentiation and the integration can be done more easily.

In Section 8.1, we introduce the basic problem of polynomial interpolation and prove the existence and uniqueness of polynomial interpolating the given data. There are at least two ways to obtain the unique polynomial interpolating a given data, one is the *Lagrange* and another one is the *Newton*. In Section 8.1.2, we introduce Lagrange form of interpolating polynomial, whereas Section 8.1.3 introduces the notion of divided differences and Newton form of interpolating polynomial. The error analysis of the polynomial interpolation is studied in Section 8.3. In certain cases, the interpolating polynomial can differ significantly from the exact function. This is illustrated by Carl Runge and is called the *Runge Phenomenon*. In Section 8.3.4, we present the example due to Runge and state a few results on convergence of the interpolating polynomials. The concept of *piecewise polynomial interpolation* and *Spline interpolation* are discussed in Section 8.5.

8.1 Polynomial Interpolation

Polynomial interpolation is a concept of fitting a polynomial to a given data. Thus, to construct an interpolating polynomial, we first need a set of points at which the data values are known.

Definition 8.1.1.

Any collection of distinct real numbers $x_0, x_1, \cdots, x_n$ (not necessarily in increasing order) is called ***nodes***.

Definition 8.1.2 [Interpolating Polynomial].

Let $x_0, x_1, \cdots, x_n$ be the given nodes and $y_0, y_1, \cdots, y_n$ be real numbers. A polynomial $p_n(x)$ of degree less than or equal to n is said to be a ***polynomial interpolating the given data*** or an ***interpolating polynomial for the given data*** if

$$p_n(x_i) = y_i, \quad i = 0, 1, \cdots n. \tag{8.1}$$

The condition (8.1) is called the ***interpolation condition***.

Remark 8.1.3.

Let $x_0, x_1, \cdots, x_n$ be given nodes, and $y_0, y_1, \cdots, y_n$ be real numbers. Let $p_n(x)$ be a polynomial interpolating the given data. Then the graph of $p_n(x)$ passes through the set of $(n+1)$ distinct points in the xy-plane given by the table

x	x_0	x_1	x_2	x_3	$\cdots$	x_n
y	y_0	y_1	y_2	y_3	$\cdots$	y_n

.

We call the set $\{(x_i, y_i), i = 0, 1, \cdots, n\}$ as *data* and quite often we represent this set in the above form of a table.

8.1.1 Existence and Uniqueness of Interpolating Polynomial

The following result asserts that an interpolating polynomial exists and is unique.

Theorem 8.1.4 [Existence and Uniqueness of Interpolating polynomial].

Let $x_0, x_1, \cdots, x_n$ be given nodes, and $y_0, y_1, \cdots, y_n$ be real numbers.

1. Then there exists a polynomial $p_n(x)$ of degree less than or equal to n such that
$$p_n(x_i) = y_i, \quad i = 0, 1, \cdots, n.$$
That is, there exists an interpolating polynomial $p_n(x)$ for the given data $\{(x_i, y_i), i = 0, 1, \cdots, n\}$.
2. Such a polynomial is unique.

Proof.

The given data may be represented as

x	x_0	x_1	x_2	x_3	$\cdots$	x_n
y	y_0	y_1	y_2	y_3	$\cdots$	y_n

.

We look for an interpolating polynomial having the form

$$p_n(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n,$$

where a_0, a_1, …, a_n need to be determined using the interpolation conditions

$$p_n(x_k) = y_k, \quad k = 0, 1, \ldots, n.$$

This leads to the system of linear equations for a_is given by

$$\begin{aligned} a_0 + x_0 a_1 + x_0^2 a_2 + \ldots + x_0^n a_n &= y_0, \\ a_0 + x_1 a_1 + x_1^2 a_2 + \ldots + x_1^n a_n &= y_1, \\ \cdots\cdots & \\ a_0 + x_n a_1 + x_n^2 a_2 + \ldots + x_n^n a_n &= y_n. \end{aligned}$$

The above system may be written as

$$V\boldsymbol{a} = \boldsymbol{y},$$

where V is the *Vandermonde matrix*

$$V = \begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ & & \cdots & \cdots & \\ & & \cdots & \cdots & \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{pmatrix},$$

$\boldsymbol{a} = (a_0, a_1, \ldots, a_n)^T$, and $\boldsymbol{y} = (y_0, y_1, \ldots, y_n)^T$.

Let the j^{th} column of V be denoted as $\boldsymbol{v}_j$. Then we claim that the vectors $\boldsymbol{v}_0$, $\boldsymbol{v}_2$, $\ldots$, $\boldsymbol{v}_n$ are linearly independent.

Let the constants c_0, c_1, $\ldots$, c_n be such that

$$c_0\boldsymbol{v}_0 + c_1\boldsymbol{v}_1 + \ldots + c_n\boldsymbol{v}_n = 0.$$

The k^{th} equation of the above system is

$$c_0 + c_1 x_k + c_2 x_k^2 + \ldots + c_n x_k^n = 0, \quad k = 0, 1, \ldots, n.$$

Thus, we obtained a polynomial $q(x) = c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n$ of degree less than or equal to n, and having $n+1$ distinct roots, namely, x_0, x_1, $\ldots$, x_n. This shows that the polynomial $q(x)$ is the zero polynomial. That is, $c_0 = c_1 = \ldots = c_n = 0$. This shows that the vectors $\boldsymbol{v}_0$, $\boldsymbol{v}_2$, $\ldots$, $\boldsymbol{v}_n$ are linearly independent. This implies that the Vandermonde matrix is invertible and hence the above system has a unique solution $\boldsymbol{a} = (a_0, a_1, \ldots, a_n)^T$. Thus, we obtained all the coefficients of $p_n(x)$ and are unique. This proves the existence and uniqueness of the interpolating polynomial of a given data set.

Remark 8.1.5.

A special case is when the data values y_i, $i = 0, 1, \cdots, n$, are the values of a function f at given nodes x_i, $i = 0, 1, \cdots, n$. In such a case, a polynomial interpolating the given data

x	x_0	x_1	x_2	x_3	$\cdots$	x_n
y	$f(x_0)$	$f(x_1)$	$f(x_2)$	$f(x_3)$	$\cdots$	$f(x_n)$

is said to be the *polynomial interpolating the given function* or the *interpolating polynomial for the given function* and has a special significance in applications of Numerical Analysis for computing approximate solutions of differential equations and numerically computing complicated integrals.

Example 8.1.6.

Let the following data represent the values of f:

x	0	0.5	1
$f(x)$	1.0000	0.5242	-0.9037

The questions are the following:

1. What is the exact expression for the function f?
2. What is the value of $f(0.75)$?

We cannot get the exact expression for the function f just from the given data, because there are infinitely many functions having same value at the given set of points. Due to this, we cannot expect an exact value for $f(0.75)$, in fact, it can be any real number. On the other hand, if we look for f in the class of polynomials of degree less than or equal to 2, then Theorem 8.1.4 tells us that there is exactly one such polynomial and hence we can obtain a unique value for $f(0.75)$.

The interpolating polynomial happens to be

$$p_2(x) = -1.9042x^2 + 0.0005x + 1$$

and we have

$$p_2(0.75) = -0.0707380.$$

The function used to generate the above table of data is

$$f(x) = \sin\left(\frac{\pi}{2}e^x\right).$$

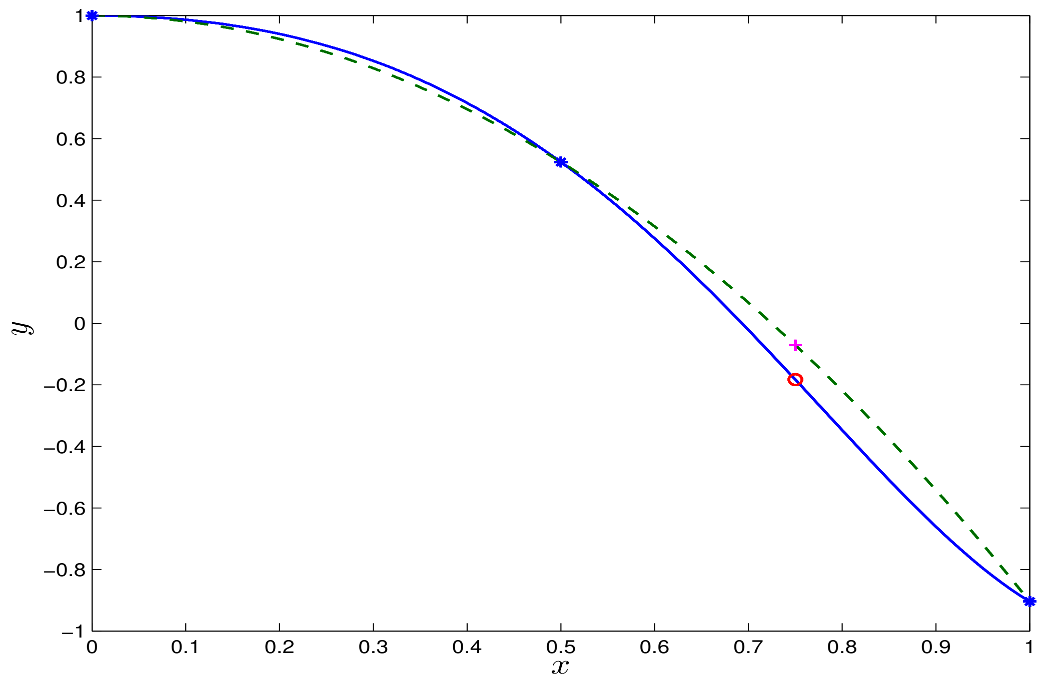

Figure 8.1: The function $f(x) = \sin\left(\frac{\pi}{2}e^x\right)$ (blue solid line) and $p_2(x)$ (green dash line). Blue dots represent the given data, magenta '+' symbol indicates the value of $p_2(0.75)$ and the red 'O' symbol represents the value of $f(0.75)$.

With this expression of f, we have (using 7-digit rounding)

$$f(0.75) \approx -0.1827495.$$

The relative error is given by

$$E_r(p_2(0.75)) = \frac{f(0.75) - p_2(0.75)}{f(0.75)} \approx 0.6129237.$$

That is, at the point $x = 0.75$ the polynomial approximation to the given function f has more than 61% error. The graph of the function f (blue solid line) and p_2 (green dash line) are depicted in Figure 8.1. The blue dots denote the given data, magenta '+' symbol indicates the value of $p_2(0.75)$ and the red 'O' symbol represents the value of $f(0.75)$. It is also observed from the graph that if we approximate the function f for $x \in [0, 0.5]$, then we obtain a better accuracy than approximating f in the interval $(0.5, 1)$.

8.1.2 Lagrange's Form of Interpolating Polynomial

Definition 8.1.7 [Lagrange's Polynomial].

Let $x_0, x_1, \cdots, x_n$ be the given nodes. For each $k = 0, 1, \cdots, n$, the polynomial $l_k(x)$ defined by

$$l_k(x) = \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{(x - x_i)}{(x_k - x_i)} \tag{8.2}$$

is called the k^{th} ***Lagrange polynomial*** or the k^{th} ***Lagrange cardinal function***.

Remark 8.1.8.

Note that the k^{th} Lagrange polynomial depends on all the $n+1$ nodes $x_0, x_1, \cdots, x_n$.

The Lagrange polynomials $l_0,\ l_1,\ \cdots,\ l_n$ form a basis for the space of polynomials of degree $\leq n$.

Theorem 8.1.9 [Lagrange's form of Interpolating Polynomial].

Hypothesis:

1. Let $x_0, x_1, \cdots, x_n$ be given nodes.
2. Let the values of a function f be given at these nodes.
3. For each $k = 0, 1, \cdots, n$, let $l_k(x)$ be the k^{th} Lagrange polynomial.
4. Let $p_n(x)$ (of degree $\leq n$) be the polynomial interpolating the function f at the nodes $x_0, x_1, \cdots, x_n$.

Conclusion: Then, $p_n(x)$ can be written as

$$p_n(x) = \sum_{i=0}^{n} f(x_i) l_i(x). \tag{8.3}$$

This form of the interpolating polynomial is called the ***Lagrange's form of Interpolating Polynomial***.

Proof.

Firstly, we will prove that $q(x) := \sum_{i=0}^{n} f(x_i) l_i(x)$ is an interpolating polynomial for the function f at the nodes $x_0, x_1, \cdots, x_n$. Since

$$l_i(x_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases},$$

we get $q(x_j) = f(x_j)$ for each $j = 0, 1, \cdots, n$. Thus, $q(x)$ is an interpolating polynomial. Since interpolating polynomial is unique by Theorem 8.1.4, the polynomial $q(x)$ must be the same as $p_n(x)$. This completes the proof of the theorem.

Remark 8.1.10.

The set of Lagrange polynomials $\{l_0(x),\ l_1(x),\ \ldots,\ l_n(x)\}$ forms a basis for the space of all polynomials of degree less than or equal to n.

Example 8.1.11.

Consider the case $n = 1$ in which we have two distinct points x_0 and x_1. Thus, we have

$$l_0(x) = \frac{x - x_1}{x_0 - x_1}, \quad l_1(x) = \frac{x - x_0}{x_1 - x_0}$$

and therefore,

$$\begin{aligned} p_1(x) &= f(x_0)l_0(x) + f(x_1)l_1(x) \\ &= f(x_0)\frac{x - x_1}{x_0 - x_1} + f(x_1)\frac{x - x_0}{x_1 - x_0} \\ &= \frac{f(x_0)(x - x_1) - f(x_1)(x - x_0)}{x_0 - x_1} \end{aligned}$$

After a rearrangement of terms, we arrive at

$$p_1(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0). \tag{8.4}$$

This is the *linear interpolating polynomial* of the function f. Similarly, if we are given three nodes with corresponding values, then we can generate the *quadratic interpolating polynomial* and so on..

Example 8.1.12.

Let the values of the function $f(x) = e^x$ be given at $x_0 = 0.82$ and $x_1 = 0.83$ by

$$e^{0.82} \approx 2.270500, \quad e^{0.83} \approx 2.293319.$$

In this example, we would like to obtain an approximate value of $e^{0.826}$ using the polynomial $p_1(x)$ that interpolates f at the nodes x_0, x_1. The polynomial $p_1(x)$ is given by

$$p_1(x) \approx 2.270500 + \frac{2.293319 - 2.270500}{0.83 - 0.82}(x - 0.82) = 2.2819x + 0.399342.$$

The approximate value of $e^{0.826}$ is taken to be $p_1(0.826)$, which is given by

$$p_1(0.826) \approx 2.2841914.$$

The true value of $e^{0.826}$ is

$$e^{0.826} \approx 2.2841638.$$

Note that the approximation to $e^{0.826}$ obtained using the interpolating polynomial $p_1(x)$, namely 2.2841914, approximates the exact value to at least five significant digits.

If we are given an additional node $x_2 = 0.84$ and the value of f at x_2 is taken to be $f(x_2) \approx 2.316367$, then we would like to use the quadratic interpolating polynomial p_2 to obtain an approximate value of $e^{0.826}$. In fact,

$$p_2(0.826) \approx 2.2841639.$$

Note that the approximation to $e^{0.826}$ obtained using the interpolating polynomial $p_2(x)$, namely 2.2841639, approximates the exact value to at least eight significant digits.

Remark 8.1.13.

The above example gives us a feeling that if we increase the number of nodes, and thereby increasing the degree of the interpolating polynomial, the polynomial approximates the orginal function more accurately. But this is not true in general, and we will discuss this further in Section 8.3.4.

Remark 8.1.14.

Let $x_0, x_1, \cdots, x_n$ be nodes, and f be a function. Recall that computing an interpolating polynomial in Lagrange's form requires us to compute for each $k = 0, 1, \cdots, n$, the k^{th} Lagrange's polynomial $l_k(x)$ which depends on the given nodes $x_0, x_1, \cdots, x_n$. Suppose that we have found the corresponding interpolating polynomial $p_n(x)$ of f in the Lagrange's form for the given data. Now if we add one more node x_{n+1}, the computation of the interpolating polynomial $p_{n+1}(x)$ in the Lagrange's form requires us to compute a new set of Lagrange's polynomials corresponding to the set of $(n+1)$ nodes, and no advantage can be taken of the fact that p_n is already available.

An alternative form of the interpolating polynomial, namely, *Newton's form of interpolating polynomial*, avoids this problem, and will be discussed in the next section.

8.1.3 Newton's Form of Interpolating Polynomial

We saw in the last section that it is easy to write the Lagrange form of the interpolating polynomial once the Lagrange polynomials associated to a given set of nodes have been written. However we observed in Remark 8.1.14 that the knowledge of p_n (in Lagrange form) cannot be utilized to construct p_{n+1} in the Lagrange form. In this section we describe *Newton's form of interpolating polynomial*, which uses the knowledge of p_n in constructing p_{n+1}.

Theorem 8.1.15 [Newton's form of Interpolating Polynomial].

Hypothesis:

1. Let $x_0, x_1, \cdots, x_n$ be given nodes.
2. Let the values of a function f be given at these nodes.
3. Let $p_n(x)$ (of degree $\leq n$) be the polynomial interpolating the function f at the nodes $x_0, x_1, \cdots, x_n$.

Conclusion: Then, $p_n(x)$ can be written as

$$p_n(x) = A_0 + A_1(x-x_0) + A_2(x-x_0)(x-x_1) + A_3\prod_{i=0}^{2}(x-x_i) + \cdots + A_n\prod_{i=0}^{n-1}(x-x_i) \quad (8.5)$$

where $A_0, A_1, \cdots, A_n$ are constants.

This form of the interpolating polynomial is called the ***Newton's form of interpolating polynomial***.

Proof.

We show that the interpolating polynomial can be written in the form (8.5) using mathematical induction.

If $n = 0$, then the constant polynomial

$$p_0(x) = y_0$$

is the required polynomial and its degree is less than or equal to 0. Thus, by taking $A_0 = y_0$, we see that $p_0(x)$ is in the form (8.5) as required.

Assume that the result is true for $n = k$. We will now prove that the result is true for $n = k + 1$.

Let the data be represented by

x	x_0	x_1	x_2	x_3	$\cdots$	x_k	x_{k+1}
y	y_0	y_1	y_2	y_3	$\cdots$	y_k	y_{k+1}

By the assumption, there exists a polynomial $p_k(x)$ of degree less than or equal to k such that the first k interpolating conditions

$$p_k(x_i) = y_i, \quad i = 0, 1, \cdots, k$$

hold. Define a polynomial $p_{k+1}(x)$ of degree less than or equal to $k + 1$ by

$$p_{k+1}(x) = p_k(x) + c(x - x_0)(x - x_1) \cdots (x - x_k), \tag{8.6}$$

where the constant c is such that the $(k + 1)^{\text{th}}$ interpolation condition $p_{k+1}(x_{k+1}) = y_{k+1}$ holds. This is achieved by choosing

$$c = \frac{y_{k+1} - p_k(x_{k+1})}{(x_{k+1} - x_0)(x_{k+1} - x_1) \cdots (x_{k+1} - x_k)}.$$

Note that $p_{k+1}(x_i) = y_i$ for $i = 0, 1, \cdots, k$ and therefore $p_{k+1}(x)$ is an interpolating polynomial for the given data. This proves the result for $n = k + 1$ with $A_{k+1} = c$. By the principle of mathematical induction, the result is true for any natural number n.

Remark 8.1.16.

Let us recall the equation (8.6) from the proof of Theorem 8.1.4 now.

1. It says that for each $n \in \mathbb{N}$, we have
$$p_n(x) = p_{n-1}(x) + A_n \prod_{i=0}^{n-1}(x - x_i) \tag{8.7}$$
for some constant A_n. This shows the recursive nature of computing Newton's form of interpolating polynomial.
2. Indeed A_n is given by
$$A_n = \frac{f(x_n) - p_{n-1}(x_n)}{(x_n - x_0)(x_n - x_1)\cdots(x_n - x_{n-1})}. \tag{8.8}$$
From the last equality, note that A_0 depends only on $f(x_0)$. A_1 depends on the values of f at x_0 and x_1 only. In general, A_n depends on the values of f at $x_0, x_1, x_2, \cdots, x_n$ only.
3. To compute Newton's form of interpolating polynomial $p_n(x)$, it is enough to compute A_k for $k = 0, 1, \cdots, n$. However note that the formula (8.8) is not well-suited to compute A_k because we need to evaluate all the successive interpolating polynomials $p_k(x)$ for $k = 0, 1, \cdots, n-1$ and then evaluate them at the node x_k which is computationally costly. It then appears that we are in a similar situation to that of Lagrange's form of interpolating polynomial as far as computational costs are concerned. But this is not the case, as we shall see shortly that we can compute A_n directly using the given data (that is, the given values of the function at the nodes), and this will be done in Section 8.2.

8.2 Newton's Divided Differences

8.2.1 Divided Differences having Distinct Arguments

Definition 8.2.1 [Divided Differences].

Let $x_0, x_1, \cdots, x_n$ be distinct nodes. Let $p_n(x)$ be the polynomial interpolating a function f at the nodes $x_0, x_1, \cdots, x_n$. The coefficient of x^n in the polynomial $p_n(x)$ is denoted by $f[x_0, x_1, \cdots, x_n]$, and is called an n^{th} ***divided difference of*** f.

Remark 8.2.2.

1. Since the interpolating polynomial for the function f at the nodes $x_0, x_1, \cdots, x_n$ is unique, there is one and only one coefficient of x^n; even though interpolation polynomial may have many forms like Lagrange's form and Newton's

form. Thus the quantity $f[x_0, x_1, \cdots, x_n]$ is well-defined.

2. More generally, the divided difference $f[x_i, x_{i+1}, \cdots, x_{i+k}]$ is the coefficient of x^k in the polynomial interpolating f at the nodes $x_i, x_{i+1}, \cdots, x_{i+k}$.
3. The Newton's form of interpolating polynomial may be written, using divided differences, as

$$p_n(x) = f[x_0] + \sum_{k=1}^{n} f[x_0, x_1, \cdots, x_k] \prod_{i=0}^{k-1} (x - x_i) \tag{8.9}$$

Example 8.2.3.

As a continuation of Example 8.1.11, let us construct the linear interpolating polynomial of a function f in the Newton's form. In this case, the interpolating polynomial is given by

$$p_1(x) = f[x_0] + f[x_0, x_1](x - x_0),$$

where

$$f[x_0] = f(x_0), \quad f[x_0, x_1] = \frac{f(x_0) - f(x_1)}{x_0 - x_1} \tag{8.10}$$

are *zeroth* and *first order divided differences*, respectively. Observe that this polynomial is exactly the same as the interpolating polynomial obtained using Lagrange's form in Example 8.1.11.

The following result is concerning the symmetry properties of divided differences.

Theorem 8.2.4 [Symmetry].

The divided difference is a symmetric function of its arguments. That is, if $z_0, z_1, \cdots, z_n$ is a permutation of $x_0, x_1, \cdots, x_n$, then

$$f[x_0, x_1, \cdots, x_n] = f[z_0, z_1, \cdots, z_n] \tag{8.11}$$

Proof.

Since $z_0, z_1, \cdots, z_n$ is a permutation of $x_0, x_1, \cdots, x_n$, which means that the nodes $x_0, x_1, \cdots, x_n$ have only been re-labelled as $z_0, z_1, \cdots, z_n$, and hence the polynomial interpolating the function f at both these sets of nodes is the same. By definition $f[x_0, x_1, \cdots, x_n]$ is the coefficient of x^n in the polynomial interpolating the function f at the nodes $x_0, x_1, \cdots, x_n$, and $f[z_0, z_1, \cdots, z_n]$ is the coefficient of x^n in the polynomial interpolating the function f at the nodes $z_0, z_1, \cdots, z_n$. Since both the interpolating polynomials are equal, so are the coefficients of x^n in them. Thus, we get

$$f[x_0, x_1, \cdots, x_n] = f[z_0, z_1, \cdots, z_n].$$

This completes the proof.

The following result helps us in computing recursively the divided differences of higher order.

Theorem 8.2.5 [Higher-order divided differences].

Divided differences satisfy the equation

$$f[x_0, x_1, \cdots, x_n] = \frac{f[x_1, x_2, \cdots, x_n] - f[x_0, x_1, \cdots, x_{n-1}]}{x_n - x_0} \tag{8.12}$$

Proof.

Let us start the proof by setting up the following notations.

- Let $p_n(x)$ be the polynomial interpolating f at the nodes x_0, x_1, $\cdots$, x_n.
- Let $p_{n-1}(x)$ be the polynomial interpolating f at the nodes x_0, x_1, $\cdots$, x_{n-1}.
- Let $q(x)$ be the polynomial interpolating f at the nodes x_1, x_2, $\cdots$, x_n.

Claim: We will prove the following relation between p_{n-1}, p_n, and q:

$$p_n(x) = p_{n-1}(x) + \frac{x - x_0}{x_n - x_0}\big(q(x) - p_{n-1}(x)\big) \tag{8.13}$$

Since both sides of the equality in (8.13) are polynomials of degree less than or equal to n, and $p_n(x)$ is the polynomial interpolating f at the nodes $x_0, x_1, \cdots, x_n$, the equality in (8.13) holds for all x if and only if it holds for $x \in \{\, x_0, x_1, \cdots, x_n \,\}$ and both sides of the equality reduce to $f(x)$ for $x \in \{\, x_0, x_1, \cdots, x_n \,\}$. Let us now verify the equation (8.13) for $x \in \{\, x_0, x_1, \cdots, x_n \,\}$.

1. When $x = x_0$,

$$p_{n-1}(x_0) + \frac{x_0 - x_0}{x_n - x_0}\big(q(x_0) - p_{n-1}(x_0)\big) = p_{n-1}(x_0) = f(x_0) = p_n(x_0).$$

2. When $x = x_k$ for $1 \le k \le n-1$, $q(x_k) = p_{n-1}(x_k)$ and thus we have

$$p_{n-1}(x_k) + \frac{x_k - x_0}{x_n - x_0}\big(q(x_k) - p_{n-1}(x_k)\big) = p_{n-1}(x_k) = f(x_k) = p_n(x_k).$$

3. When $x = x_n$, we have

$$p_{n-1}(x_n)+\frac{x_n - x_0}{x_n - x_0}\big(q(x_n)-p_{n-1}(x_n)\big) = p_{n-1}(x_n)+\big(f(x_n)-p_{n-1}(x_n)\big) = f(x_n) = p_n(x_n).$$

This finishes the proof of the Claim.

The coefficient of x^n in the polynomial $p_n(x)$ is $f[x_0, x_1, \cdots, x_n]$. The coefficient of x^n using the right hand side of the equation (8.13) is given by

$$\Big(\text{coefficient of } x^n \text{ in } p_{n-1}(x)\Big)+\frac{1}{x_n - x_0}\Big(\text{coefficient of } x^n \text{ in } (x-x_0)\big(q(x)-p_{n-1}(x)\big)\Big).$$

On noting that the coefficient of x^{n-1} in the polynomial p_{n-1} is $f[x_0, x_1, \cdots, x_{n-1}]$, the coefficient of x^{n-1} in the polynomial q is $f[x_1, x_2, \cdots, x_n]$, and the coefficient of x^n in the polynomial p_{n-1} is zero, we get that the coefficient of x^n using the right hand side of the equation (8.13) becomes

$$\frac{f[x_1, x_2, \cdots, x_n] - f[x_0, x_1, \cdots, x_{n-1}]}{x_n - x_0}.$$

Comparing the coefficients of x^n in the left and right hand sides of the equation (8.13) yields

$$f[x_0, x_1, \cdots, x_n] = \frac{f[x_1, x_2, \cdots, x_n] - f[x_0, x_1, \cdots, x_{n-1}]}{x_n - x_0}.$$

Remark 8.2.6.

Let $i, j \in \mathbb{N}$. Applying Theorem 8.2.5 to a set of nodes $x_i, x_{i+1}, \cdots, x_{i+j}$, we conclude

$$f[x_i, x_{i+1}, \cdots, x_{i+j}] = \frac{f[x_{i+1}, x_{i+2}, \cdots, x_{i+j}] - f[x_i, x_{i+1}, \cdots, x_{i+j-1}]}{x_{i+j} - x_i} \tag{8.14}$$

Note that the divided differences $f[x_0, x_1, \cdots, x_n]$ are defined only for distinct nodes $x_0, x_1, \cdots, x_n$.

8.2.2 Divided Differences Table

Given a collection of $(n+1)$ nodes $x_0, x_1, \cdots, x_n$ and the values of the function f at these nodes, we can construct the Newton's form of interpolating polynomial $p_n(x)$ using divided differences. As observed earlier, the Newton's form of interpolation polynomial has the formula

$$p_n(x) = f[x_0] + \sum_{k=1}^{n} f[x_0, x_1, \cdots, x_k] \prod_{i=0}^{k-1} (x - x_i) \tag{8.15}$$

One can explicitly write the formula (8.15) for $n = 1, 2, 3, 4, 5, \cdots$. For instance, when $n = 5$, the formula (8.15) reads

$$\left.\begin{array}{rl} p_5(x) = & \boldsymbol{f[x_0]} \\ + & \boldsymbol{f[x_0, x_1]}(x-x_0) \\ + & \boldsymbol{f[x_0, x_1, x_2]}(x-x_0)(x-x_1) \\ + & \boldsymbol{f[x_0, x_1, x_2, x_3]}(x-x_0)(x-x_1)(x-x_2) \\ + & \boldsymbol{f[x_0, x_1, x_2, x_3, x_4]}(x-x_0)(x-x_1)(x-x_2)(x-x_3) \\ + & \boldsymbol{f[x_0, x_1, x_2, x_3, x_4, x_5]}(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4) \end{array}\right\}. \tag{8.16}$$

For easy computation of the divided differences in view of the formula (8.12), it is convenient to write the divided differences in a table form. For $n = 5$, the divided difference table is given by

$$\begin{array}{c|c||ccccc}
x_0 & \boldsymbol{f(x_0)} & & & & & \\
 & & \boldsymbol{f[x_0,x_1]} & & & & \\
x_1 & f(x_1) & & \boldsymbol{f[x_0,x_1,x_2]} & & & \\
 & & f[x_1,x_2] & & \boldsymbol{f[x_0,x_1,x_2,x_3]} & & \\
x_2 & f(x_2) & & f[x_1,x_2,x_3] & & \boldsymbol{f[x_0,x_1,x_2,x_3,x_4]} & \\
 & & f[x_2,x_3] & & f[x_1,x_2,x_3,x_4] & & \boldsymbol{f[x_0,x_1,x_2,x_3,x_4,x_5]} \\
x_3 & f(x_3) & & f[x_2,x_3,x_4] & & f[x_1,x_2,x_3,x_4,x_5] & \\
 & & f[x_3,x_4] & & f[x_2,x_3,x_4,x_5] & & \\
x_4 & f(x_4) & & f[x_3,x_4,x_5] & & & \\
 & & f[x_4,x_5] & & & & \\
x_5 & f(x_5) & & & & &
\end{array}$$

Comparing the above divided differences table and the interpolating polynomial p_5 given by (8.16), we see that the leading members of each column (denoted in bold font) are the required divided differences used in $p_5(x)$.

8.3 Error in Polynomial Interpolation

Let f be a function defined on an interval $I = [a, b]$. Let $p_n(x)$ be a polynomial of degree less than or equal to n that interpolates the function f at $n+1$ nodes x_0, x_1,

8.6 Exercises

Polynomial Interpolation

1. Let $x_0, x_1, \cdots, x_n$ be distinct nodes. If $p(x)$ is a polynomial of degree less than or equal to n, then show that

$$p(x) = \sum_{i=0}^{n} p(x_i) l_i(x),$$

where $l_i(x)$ is the i^{th} Lagrange polynomial.

2. Show that the polynomial $1 + x + 2x^2$ is an interpolating polynomial for the data

x	0	1	2
y	1	4	11

Find an interpolating polynomial for the new data

x	0	1	2	3
y	1	4	11	-2

Does there exist a quadratic polynomial that satisfies the new data? Justify your answer.

3. The quadratic polynomial $p_2(x) = \frac{3}{4}x^2 + \frac{1}{4}x + \frac{1}{2}$ interpolates the data

x	-1	0	1
y	1	$\frac{1}{2}$	$\frac{3}{2}$

Find a node x_3 ($x_3 \notin \{-1, 0, 1\}$), and a real number y_3 such that the polynomial $p_3(x)$ interpolating the data

x	-1	0	1	x_3
y	1	1/2	3/2	y_3

is a polynomial of degree less than or equal to 2.

4. Let $p(x)$, $q(x)$, and $r(x)$ be interpolating polynomials for the three sets of data

x	0	1
y	y_0	y_1

,

x	0	2
y	y_0	y_2

, and

x	1	2	3
y	y_1	y_2	y_3

respectively. Let $s(x)$ be the the interpolating polynomial for the data

x	0	1	2	3
y	y_0	y_1	y_2	y_3

If

$$p(x) = 1 + 2x, \; q(x) = 1 + x, \text{ and } r(2.5) = 3,$$

then find the value of $s(2.5)$.

5. Obtain Lagrange form of interpolating polynomial for equally spaced nodes.

6. Find the Largrange form of interpolating polynomial for the data:

x	-2	-1	1	3
y	-1	3	-1	19

7. Find the Lagrange form of interpolating polynomial $p_2(x)$ that interpolates the function $f(x) = e^{-x^2}$ at the nodes $x_0 = -1$, $x_1 = 0$ and $x_2 = 1$. Further, find the value of $p_2(-0.9)$ (use 6-digit rounding). Compare the value with the true value $f(-0.9)$ (use 6-digit rounding). Find the percentage error in this calculation.

Newton's Divided Difference Formula

8. For the particular function $f(x) = x^m$ $(m \in \mathbb{N})$, show that
$$f[x_0, x_1, \cdots, x_n] = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n > m \end{cases}$$

9. Let $x_0, x_1, \cdots, x_n$ be nodes, and f be a given function. Define $w(x) = \prod_{i=0}^{n}(x - x_i)$. Prove that
$$f[x_0, x_1, \cdots, x_n] = \sum_{i=0}^{n} \frac{f(x_i)}{w'(x_i)}.$$

10. The following data correspond to a polynomial $P(x)$ of unknown degree

x	0	1	2
$P(x)$	2	-1	4

Determine the coefficient of x in the polynomial $P(x)$ if all the third order divided differences are 1.