Probability I (SI 427)
Department of Mathematics, IIT Bombay
July, 2022–December, 2022
Problem set 4

1. X negative binomial. $P(X = n) = \binom{n-1}{r-1} p^r (1-p)^{n-r}$ for $n = r, r+1, \ldots$ and $0 < p < 1$. Find expected value of X.

2. Suppose n couples have arrived in a party. The boys and girls are then paired off at random to dance. Find the expected number of boys who get his own girl to dance.

3. Suppose there are N different type of coupons and each time one obtains a coupon it is equally likely to be any one of the N types.

   (a) Find the expected number of different types of coupons that are contained in a set of n coupons.

   (b) Find the expected number of coupons one needs to collect to get atleast one coupon of each type.

4. Suppose X, Y, Z are discrete random variables on $(\Omega, \mathcal{F}, \mathrm{P})$. Show the following:

   (a) $E(aY + bZ|X) = aE(Y|X) + bE(Z|X)$ for $a, b \in \mathbb{R}$.

   (b) $E(Y|X) \geq 0$ if $Y \geq 0$.

   (c) $E(1|X) = 1$.

   (d) $E(E(Y|X, Z)|X) = E(Y|X) = E(E(Y|X)|X, Z)$.

5. Let $\{X_n\}$ be sequence of independent random variables with common mean μ and variance σ^2. Let N be a positive integer valued random variable with finite mean and variance, and suppose that N is independent of all X_n. Define $S_N = \sum_{i=1}^{N} X_i$. Show that $E(S_N^2) = \sigma^2 E(N) + \mu^2 E(N^2)$ and $\mathrm{Var}(S_N) = \sigma^2 E(N) + \mu^2 \mathrm{Var}(N)$.

6. Suppose that the number of people entering a departmental store on a given day is a random variable with mean 50. Suppose that the amounts of money spent by the customers are independent random variables having a common mean of \$10. Assume also that the amount of money spent by a customer is independent of the total number of customers to enter the store. What is the expected amount of money spent in the store on a given day?

7. Give an appropriate definition of the conditional covariance $\mathrm{Cov}(X, Y|Z)$. Show that

$$\mathrm{Cov}(X, Y) = E(\mathrm{Cov}(X, Y|Z)) + \mathrm{Cov}(E(X|Z), E(Y|Z)).$$

8. Suppose that conditional on $Y = y$, the random variables X_1 and X_2 are independent with mean y. Show that $\mathrm{Cov}(X_1, X_2) = Var(Y)$.

9. Suppose we know that the number of items produced in a factory during a week is a random variable with mean 500.

   (a) What can be said about the probability that this week's production will be atleast 1000?

(b) If the variance of the week's production is known to equal 100, then what can be said about the probability that this week's production will be between 400 and 600?

Remark: Importance of Markov and Chebyshev's inequality is that they enable us to derive bounds on probabilities when only the mean, or both the mean and the variance of the probability distribution are known. Of course, if the actual distribution is known, then the desired probabilities could be exactly computed, and we would not need to resort to bounds.

10. If the number of items produced in a factory during a weak, X is a random variable with mean 100 and variance 400. Show that $\mathrm{P}(X \geq 120) \leq \frac{1}{2}$.