Tutorial Sheet 1
Mathematical Preliminaries

1. Let L be a real number and let $\{a_n\}$ be a sequence of real numbers. If there exists a positive integer N and a $\mu \in (0,1)$ such that
$$|a_n - L| \le \mu |a_{n-1} - L|$$
holds for all $n \ge N$, then show that $a_n \to L$ as $n \to \infty$.

2. Show that the equation $\sin x + x^2 = 1$ has at least one solution in the interval $[0,1]$.

3. Let $f(x)$ be a continuous function on $[a,b]$, let $x_1, \ldots, x_n$ be points in $[a,b]$, and let $g_1, \ldots, g_n$ be non-positive real numbers. Then show that
$$\sum_{i=1}^{n} f(x_i) g_i = f(\xi) \sum_{i=1}^{n} g_i, \quad \text{for some } \xi \in [a,b].$$

4. Let $f : [a,b] \to [a,b]$ be a continuous function. Prove that the equation $f(x) = x$ has at least one solution lying in the interval $[a,b]$ (Note: A solution of this equation is called a *fixed point* of the function f). Further if $\max\limits_{x \in [a,b]} |f'(x)| < 1$, then show that the equation $f(x) = x$ has a unique solution in $[a,b]$.

5. Let g be a continuously differentiable function (C^1 function) such that the equation $g(x) = 0$ has at least n distinct roots. Show that the equation $g'(x) = 0$ has at least $n-1$ distinct roots.

6. Evaluate an approximate value of the function $f(x) = e^{x^2}$ at $x = 1$ using $T_2(x)$ about the point $a = 0$. Obtain the remainder $R_2(1)$ in terms of some unknown real number ξ. Compute (approximately) a possible value of ξ.

7. For every $x \in \mathbb{R}$, show that there exists a $\xi_x \in \mathbb{R}$
$$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{\cos(\xi_x)}{6!} x^6.$$

8. Determine the best value of $\alpha \in \mathbb{R}$ in the equation
$$\tan^{-1} x = x + O(x^\alpha) \text{ as } x \to 0$$

9. Let $F : [0,1] \to [0,1]$ be a differentiable function. Let a sequence $\{x_n\}$ defined inductively by $x_{n+1} = F(x_n)$ be such that $\lim_{n\to\infty} x_n$ exists and denote the limit by x. Further assume that $F'(x) = 0$. Show that
$$x_{n+2} - x_{n+1} = o(x_{n+1} - x_n) \text{ as } n \to \infty.$$

10. Prove or disprove:
(i) $\dfrac{n+1}{n^2} = O\left(\dfrac{1}{n}\right)$ as $n \to \infty$ (ii) $\dfrac{1}{\ln n} = o\left(\dfrac{1}{n}\right)$ as $n \to \infty$