Lecture 24

Monday, 28 March 2022 1:38 PM

Problem: [Tut Sheet 8] Let $f: \mathbb{R}^n \to \mathbb{R}$, $f \in \mathcal{C}^1$, f is a convex function on set of feasible points

$$\Omega = \{ \underline{x} \in \mathbb{R}^n : \underline{h}(\underline{x}) = 0 \}$$

$\left(\begin{array}{l} \underline{h}: \mathbb{R}^n \to \mathbb{R}^m \\ \Omega \text{ is convex} \end{array} \right)$.

Let $\exists$ $\underline{x}^* \in \Omega$ & $\underline{\lambda}^* \in \mathbb{R}^m$ such that

$$Df(x^*) + \underline{\lambda}^{*T} Dh(\underline{x}^*) = \underline{0}^T.$$

Then $\underline{x}^*$ is a global minimizer of f.

Proof:

(Thm 22.4) Convexity

f is convex on Ω

$\iff$ $\forall x, y \in \Omega$,

$$f(y) \geq f(x) + Df(\underline{x})(\underline{y} - x)$$

Let $\underline{x}, \underline{x}^* \in \Omega$. Since f is convex and $f \in C^{(1)}$,

$$f(\underline{x}) \geq f(\underline{x}^*) + Df(\underline{x}^*)(\underline{x} - \underline{x}^*)$$

Lagrange condtn

$$\Rightarrow f(\underline{x}) \geq f(\underline{x}^*) - \underbrace{\underline{\lambda}^{*T} D\underline{h}(\underline{x}^*)(\underline{x} - \underline{x}^*)}$$

$\underline{x}^*$; $\underline{x} \in \Omega$; Ω is convex

$\Rightarrow$ $(1-\alpha)\underline{x}^* + \alpha \underline{x} \in \Omega$ $\alpha \in [0, 1]$.

$\Rightarrow$ $h((1-\alpha)x^* + \alpha \underline{x}) = 0$

$\Rightarrow$ $\underline{h}(\underline{x}^* + \alpha(\underline{x} - x^*)) = 0$

$$\Rightarrow \quad \underline{h}(\underline{x}^* + \alpha(\underline{x} - x^*)) = 0$$

$$\Rightarrow \quad \frac{\underline{\lambda}^{*T} \underline{h}(\underline{x}^* + \overbrace{\alpha(x - x^*)})) - \underline{\lambda}^{*T} \underbrace{\underline{h}(\underline{x}^*)}_{=0}}{\alpha} = 0$$

Take limit as $\alpha \to 0$, $\underline{\lambda}^{*T} D\,\underline{h}(x^*)(x - \underline{x}^*) = 0.$

Hence $f(x) \geq f(x^*) \quad \forall\, \underline{x} \in \Omega$

$\Rightarrow \underline{x}^*$ is a global min.

__Problem 11__ Let $f : \mathbb{R}^n \to \mathbb{R}$, $f \in C^1$ is convex on

$\Omega = \{ x \in \mathbb{R}^n : \underline{h}(\underline{x}) = 0, \; g(\underline{x}) \leq 0 \}$; $h : \mathbb{R}^n \to \mathbb{R}^m$,

$\underline{g} : \mathbb{R}^n \to \mathbb{R}^p$, $h, g \in C^1$; Ω is convex.

Suppose $\exists\; \underline{x}^* \in \Omega$, $\underline{\lambda}^* \in \mathbb{R}^m$ and $\underline{\mu}^* \in \mathbb{R}^p$ s.t.

(i) $\underline{\mu}^* \geq 0$

(ii) $Df(\underline{x}^*) + \underline{\lambda}^{*T} D\underline{h}(\underline{x}^*) + \underline{\mu}^{*T} D\underline{g}(\underline{x}^*) = 0$

(iii) $\underline{\mu}^{*T} \underline{g}(\underline{x}^*) = 0.$

Then $\underline{x}^*$ is a global minimizer of f over Ω.

__Proof:__ Proceed as in the previous result.

$$Df(\underline{x}^*) = -\underline{\lambda}^{*T} Dh(\underline{x}^*) - \underline{\mu}^{*T} Dg(x^*).$$

$$f(\underline{x}) \geq f(\underline{x}^*) - \underbrace{\underline{\lambda}^{*T} D\underline{h}(x^*)(\underline{x} - \underline{x}^*)}_{=0} - \underline{\mu}^{*T} \underbrace{Dg(\underline{x}^*)(\underline{x} - \underline{x}^*)}$$

(as in the last result).

(as in$^{\circ}$ the last result).

$f(x) \geq f(x^*) + (+ve)$

$\overline{(1-\alpha) x^* + \alpha x} \in \Omega$ → Ω is convex.

$g((1-\alpha) x^* + \alpha x) \leq 0$

$g(x^* + \alpha(x - x^*)) \leq 0$

$\overbrace{\mu^{*T}}^{\geq 0 \text{ (by condtn (ii))}} g(x^* + \alpha(x - x^*)) - \overbrace{\mu^{*T} g(x^*)}^{=0 \text{ by condtn (iii)}}$

$$\lim_{\alpha \to 0} \frac{\mu^{*T} g(x^* + \alpha(x - x^*)) - \mu^{*T} g(x^*)}{\alpha} \leq 0$$

$$\mu^{*T} Dg(x^*)(x - x^*) \leq 0.$$

$$\Rightarrow f(x) \geq f(x^*) \quad \forall x \in \Omega.$$

Example: Linear programming problem in Standard form.

I. maximize $c^T x_{n\times 1}$

s.t. $A_{m\times n} x = b_{m\times 1}$

$x \geq 0$

II. maximize $c^T x$

s.t. $\boxed{\begin{matrix} Ax \leq b \\ x \geq 0 \end{matrix}}$

Write KKT Conditions.

[$m \leq n$ may not be true]

I. $\min \; -c^T x \longrightarrow f(x) = -c^T x = -c_1 x_1 - c_2 x_2 - \cdots - c_n x_n$

s.t. $Ax - b = 0 \longrightarrow h(x) = Ax - b = 0$ [m condtn].

$-x \leq 0$ $\quad g(x) = -x \leq 0.$ [n condition]

KKT conditions

[$\lambda^* \rightarrow$ no restriction on sign

KKT conditions

$$\left[\begin{array}{l} \underline{\lambda}^* \to \text{no restriction on sign} \\ \underline{\mu}^* \geq 0 \end{array}\right.$$

Primal Feasibility $\quad \left.\begin{array}{l} A\underline{x}^* = \underline{b} \\ \underline{x}^* \geq 0 \end{array}\right]$

$g_1(x) = -x_1 \qquad \nabla g_1 = \begin{pmatrix} -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

$\vdots$

$g_n(x) = -x_n \qquad \nabla g_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ -1 \end{pmatrix}$

Stationarity $\quad Df(\underline{x}^*) + \underline{\lambda}^{*T} Dh(\underline{x}^*) + \underline{\mu}^{*T} Dg(\underline{x}^*) = \underline{0}$

$$\boxed{\underline{\lambda}^{*T} A - \underline{\mu}^{*T} = \underline{c}^T}$$

$\underline{\mu}^{*T} = [\mu_1^*, \ldots, \mu_n^*]$

$Dg(\underline{x}^*) = \begin{bmatrix} -1 & 0 & \ldots & 0 \\ 0 & -1 & \ldots & 0 \\ 0 & \ldots & \ldots & -1 \end{bmatrix}$

Complementarity $\quad \underline{\mu}^{*T} \underline{x} = \underline{0}.$

II.

$$\max \ \underline{c}^T \underline{x} \quad \text{s.t.} \quad A\underline{x} \leq \underline{b}, \quad \underline{x} \geq 0$$

m + n

$$\min \ -\underline{c}^T \underline{x}$$

s.t.

	Dual variables
$g_1(\underline{x}) = A\underline{x} - \underline{b} \leq 0$	$\underline{w}^*$
$g_2(x) = -\underline{x} \leq 0$	$\underline{v}^*$

$\underline{\mu}^* = \begin{bmatrix} \underline{w}^* \\ \underline{v}^* \end{bmatrix}.$

- $\underline{w}^*, \underline{v}^* \geq 0.$
- $A\underline{x}^* \leq \underline{b}, \quad \underline{x}^* \geq 0 \quad$ [Primal feasibility].
- ✓ $\underline{w}^{*T} A - \underline{v}^{*T} = \underline{c}^T \quad$ [Stationarity] $\quad Df(\underline{x}^*) + \underline{w}^* Dg_1(x^*) + \underline{v}^* Dg_2(x^*) = 0$
- $\underline{w}^{*T}(A\underline{x}^* - \underline{b}) = 0; \quad \underline{v}^{*T} \underline{x}^* = 0.$

Example. $\quad \max \ f(x_1, x_2) = 7x_1 + 6x_2$

s.t.

	Dual variables
$3x_1 + x_2 \leq 120$	$\to w_1$
$x_1 + 2x_2 \leq 160$	$\to w_2$
$x_1 \leq 35$	$\to w_3$
$x_1 \geq 0,$	$\to v_1$

$x_1 \leq 35 \longrightarrow w_3$

$x_1 \geq 0, \longrightarrow v_1$

$x_2 \geq 0 \longrightarrow v_2$

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \\ 1 & 0 \end{bmatrix} \quad \underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \underline{b} = \begin{bmatrix} 120 \\ 160 \\ 35 \end{bmatrix}$$

$$\underline{c} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$$

<u>Primal feasibility</u>

$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} \leq \begin{bmatrix} 120 \\ 160 \\ 35 \end{bmatrix}$$

$$\begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

<u>Dual feasibility</u>

$$[w_1 \quad w_2 \quad w_3] \geq [0 \quad 0 \quad 0]$$

$$[v_1 \quad v_2] \geq 0.$$

<u>Stationarity</u>

$$[w_1 \quad w_2 \quad w_3] \begin{bmatrix} 3 & 1 \\ 1 & 2 \\ 1 & 0 \end{bmatrix} - [v_1 \quad v_2] = [7 \quad 6].$$

$\boxed{\text{Complementary Slack}}$

$$[w_1 \quad w_2 \quad w_3] \left(\begin{bmatrix} 3 & 1 \\ 1 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 120 \\ 160 \\ 35 \end{bmatrix} \right) = \underline{0}$$

$$[v_1 \quad v_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underline{0}$$

A Q M A √ F