Probability I

1. Introduction to probability theory – Hoel, Port and stone.
2. A first course in probability – Sheldon Ross

After Midsem

3. An introduction to probability and statistics. - Rohatgi and Saleh
4. Probability - Alan F Karr

Quiz 1 : August 26 (8:15–9:25)

Quiz 2 : October 17 (11:35– 12:50)

(216+ Ramanujan)

Quiz 3 : Surprise

30% Quiz

30% mid

40% End

P-Probability

$(\Omega, \mathcal{F}, P) \rightarrow$ Probability space

Ω – non empty set (Sample space)

$\mathcal{F}$ – σ field. (Events space)

$\mathcal{F}$ - Collection of some subsets of Ω, which satisfy some conditions.

P – Probability measure.

It is a map $\mathcal{F} \longrightarrow [0,1]$

Eg: Toss a coin

Set of outcomes is sample space. (Ω)

$\Omega = \{H, T\}$

Events are outcome is H | ~~or~~ outcome is T or | outcome is H or T | outcome is H and T.

~~All these~~ Some events together is called event space. (Set of ~~all~~ some events).

Event is a subset of Ω.

If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$

If $A, B \in \mathcal{F}$, $A \cup B$, $A \cap B \in \mathcal{F}$

$\phi \in \mathcal{F}$

Def: Let Ω be a non empty set. A collection $\mathcal{A}$ of subsets of Ω is called field if the following holds:

(i) $\phi \in \mathcal{A}$

(ii) If $A \in \mathcal{A}$ then $A^c \in \mathcal{A}$

(iii) If $A, B \in \mathcal{A}$ then $\underbrace{A \cup B \in \mathcal{A} \text{ and } A \cap B \in \mathcal{A}}$

one condition among these is enough

Assume $A, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A}$

TST: $A, B \in \mathcal{A} \Rightarrow A \cap B \in \mathcal{A}$

$A, B \in \mathcal{A} \Rightarrow A^c, B^c \in \mathcal{A}$

$A^c \cup B^c \in \mathcal{A}$

$(A \cap B)^c \in \mathcal{A}$

$\Rightarrow ((A \cap B)^c)^c \in \mathcal{A}$

$\Rightarrow A \cap B \in \mathcal{A}$

Example ① $\Omega = \{H, T\}$

$\mathcal{A}_1 = \{\phi, \Omega, \{H\}, \{T\}\}$

$\mathcal{A}_2 = \{\phi, \Omega\}$

② $\Omega = \{1, 2, \dots 5, 6\}$

$\mathcal{A}_1 = \{\phi, \Omega\}$

$\mathcal{A}_2 =$ Power set of Ω

$\mathcal{A}_3 = \{\phi, \Omega, \{1\}, \{2,3,4,5,6\}\}$

$\mathcal{A}_4 = \{\phi, \Omega, \{2,3\}, \{1,4,5,6\}\}$

Ω, $A \subset \Omega$

$\mathcal{A} = \{\phi, \Omega, A, A^c\} \rightarrow$ field ✓

Ex. $\mathcal{A}$ is a field. Suppose $A_1, A_2, \dots A_n \in \mathcal{A}$, then show that $\bigcup_{i=1}^{n} A_i$, $\bigcap_{i=1}^{n} A_i \in \mathcal{A}$. (PMI) we need these happer

Note: 1) If contral space (Ω) is finite, then event space ~~could be~~ is a field.

2) If Ω is infinite, (countable) then field is not enough to be considered as an event space.

<u>Example:</u> Tossing a coin repeatedly until we get a H. We are interested in number of trials required to get an H.

$$\Omega = \{H, TH, TTH, TTTH, \dots\}$$

$$\Omega = \{\omega_1, \omega_2, \omega_3, \dots\}$$

A = H occurs after an odd number of tosses.

$A = \{\omega_2, \omega_4, \omega_6, \dots\}$ - infinite countable set.

Sigma ~~Algebra~~ field

A collection $\mathcal{F}$ of subsets of Ω is called σ-field if the following holds.

(i) $\phi \in \mathcal{F}$

(ii) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$

(iii) $A_1, A_2, A_3, \dots \in \mathcal{F}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$
(sequence of events)

Remark

1. A σ-field is a field.
2. If Ω is finite, then any field $\mathcal{A}$ of subsets of Ω is a σ-field.
3. Suppose Ω is infinite and $\mathcal{A}$ is a finite field, the $\mathcal{A}$ is a σ-field.

Examples of σ-field

1. $\mathcal{F} = \{\phi, \Omega\}$
2. $\mathcal{F} = \{\phi, A, A^c, \Omega\}$ $\quad A \subset \Omega$
3. $\Omega = \mathbb{N}$
$\mathcal{F} = P(\mathbb{N})$ (Power set of $\mathbb{N}$)

Example.. of field, but not a σ- field.

$\Omega = \mathbb{N}$

$\mathcal{F} = \{A : A \text{ is a finite or } A^c \text{ is finite}\}$

Exercise - Show that $\mathcal{F}$ is a field. $\rightarrow$ by taking different case $\rightarrow$ (3 cases. ($\because$ 1 symmetric)) & ~~doing~~ using notations carefully.

$A_i = \{2i\}$

$\bigcup_{i=1}^{\infty} A_i = \{2n : n \in \mathbb{N}\} \notin \mathcal{F}$

$P : \mathcal{F} \longrightarrow [0,1]$, a map.

Example: 1) Toss a coin.

$\Omega = \{H, T\}$

$\mathcal{F} = \{\phi, \{H\}, \{T\}, \{T, H\}\}$

(i) Suppose it is a fair coin.

$P(\{H\}) = \frac{1}{2} = P(\{T\})$

$P(\Omega) = 1 = P(\{H\}) + P(\{T\})$

(ii) Coin is not fair

Toss it N number of times.

N_1 = No. of H

N_2 = No. of T.

$P(\{H\}) = \frac{N_1}{N}$ $\quad$ $P(\{T\}) = \frac{N_2}{N}$.

2) Throw a dice.

$\Omega = [6]$. $\quad$ $\mathcal{F} = P(\Omega)$ (power set)

(i) Dice is fair

$P(\{i\}) = \frac{1}{\ }$ for $i \in [6]$

(ii) Dice is not fair.

Repeat experiment N times.

N_i = no. of times i occurs.

$$P(\{i\}) = \frac{N_i}{N}, \quad i \in [6]$$

Conditions on probability:

(i) $P(\Omega) = 1$, $P(\phi) = 0$

(ii) If $A_1, A_2, A_3 \dots \in \mathcal{F}$ and $A_i \cap A_j = \phi$ for $i \neq j$, then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

$(\Omega, \mathcal{F}, P) \longrightarrow$ probability space.

$\mathcal{F}$ ↓ sigma field

$P(A_1 \cup A_2 \cup \phi \cup \phi \cup \phi) = \sum P(A_i) = \dots + 0 + 0 + \dots$

$P(A_1 \cup A_2) = P(A_1) + P(A_2)$ if $A_1 \cap A_2 = \phi$

$A_1 = A \quad A_2 = A^c$

$\therefore$ (1).

<u>Exercise</u>:- 1) $P(A^c) = 1 - P(A)$

2) $A \subseteq B$, $A, B \in \mathcal{F}$

then $P(A) \leq P(B)$

$\boxed{?} \rightarrow$ 3) $A, B \in \mathcal{F}$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

3) $B = \Omega \cap B = (A \cup A^c) \cap B = (A \cap B) \cup (A^c \cap B)$

$(A \cap B) \cap (A^c \cap B) = \phi$

$\therefore P((A \cap B) \cup (A^c \cap B)) = P(A \cap B) + P(A^c \cap B)$ —— (i)

$P(B) = P(A \cap B) + P(A^c \cap B)$

2) $A \subseteq B \Rightarrow A \cap B = A$

$P(B) = P(A) + P(A^c \cap B)$

Since $P(A^c \cap B) \geq 0$, $\quad P(B) - P(A) \geq 0$

$P(B) \geq P(A)$

3. $A \cup B = A \cup (B - A)$

$= A \cup (B - A \cap B)$

$P(A \cup B) = P(A) + P(B - A \cap B)$

$B = (B - A \cap B) \cup A \cap B$

$P(B) = P(B - A \cap B) + P(A \cap B)$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

4. Inclusion Exclusion Principle:

Let $A_1, A_2, \dots A_n \in \mathcal{F}$

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k) - \cdots + (-1)^{n+1} P\left(\bigcap_{i=1}^{n} A_i\right)$$

We use induction

Suppose result is true for $k = n-1$ ($\because$ $k=2$ is already done)

We will show that result is true for $k=n$

$$P\left(\bigcup_{i=1}^{n} A_i\right) = P(B \cup A_n) \quad \Bigg| \quad B = \bigcup_{i=1}^{n-1} A_i$$

$$P\left(\bigcup_{i=1}^{n} A_i\right) = P(B) + P(A_n) - P(B \cap A_n)$$

$$B \cap A_n = \left(\bigcup_{i=1}^{n-1} A_i\right) \cap A_n$$

$$= \bigcup_{i=1}^{n-1} (A_i \cap A_n) = \bigcup_{i=1}^{n-1} A_i'$$

$$P(B \cap A_n) = P\left(\bigcup_{i=1}^{n-1} (A_i \cap A_n)\right) = \cancel{\sum_{i=1}^{n-1} P(A_1 A_2 \cap \dots A_n)}$$

$$= \sum_{i=1}^{n-1} P(A_i \cap A_n) - \sum_{i<j} P(A_i \cap A_j \cap A_n) + \dots + (-1)^n P\left(\bigcap_{i=1}^{n-1} A_i \cap A_n\right)$$

$$= \underbrace{\sum_{i=1}^{n-1} P(A_i \cap A_n) - \sum_{i<j} P(A_i \cap A_j \cap A_n) + \dots + (-1)^n P\left(\bigcap_{i=1}^{n} A_i\right)}_{n-1 \text{ terms}}$$

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n-1} P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k) \dots + (-1)^n P\left(\bigcap_{i=1}^{n-1} A_i\right) +$$

$$\underbrace{P(A_n) - \left(\sum_{i=1}^{n-1} P(A_i \cap A_n) - \sum_{i<j} P(A_i \cap A_j A_n) + \dots + (-1)^n P\left(\bigcap_{i=1}^{n} A_i\right)\right)}_{n \text{ terms}}$$

$$= \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k) \dots - (-1)^n P\left(\bigcap_{i=1}^{n} A_i\right)$$

$+ (-1)^{n+1}$

Example:

Suppose n people have come to a meeting with umbrellas (distinct). They keep the umbrella in one place and at the end of the meeting, everyone an umbrella at random. What is the probability that no one gets his own umbrella?

Ans: A_i = i-th person get his own.

$$P\left(\bigcap_{i=1}^{n} A_i^c\right) = 1 - P\left(\left(\bigcap_{i=1}^{n} A_i^c\right)^c\right)$$

$$= 1 - P\left(\bigcup_{i=1}^{n} A_i\right) \quad \text{(Demorgan's law)}$$

$$= 1 - \left[\sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \dots + (-1)^{n+1} P\left(\bigcap_{i=1}^{n} A_i\right)\right]$$

$$\cancel{= 1 - [\;]}$$

$$P(A_1) = \frac{(n-1)!}{n!}$$

Result: (Continuity of Probability Measure)

Let $A_1, A_2, \ldots$ be an increasing sequence of events such that $A_1 \subseteq A_2 \subseteq A_3 \ldots$. Then $P\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} P(A_n)$

Similarly, if $B_1, B_2 \ldots$ be decreasing sequence of events, then

Exercise $\to$ $P\left(\bigcap_{n=1}^{\infty} B_n\right) = \lim_{n\to\infty} P(B_n)$

$$\lim_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} A_n \quad (\text{property of } \uparrow \text{ sequence of sets})$$

$$P\left(\lim_{n\to\infty} A_n\right) = \lim_{n\to\infty} P(A_n)$$

$P\left(\bigcup_{n=1}^{\infty} A_n\right)$ $\qquad A_1 \subseteq A_2 \subseteq A_3$

$$\bigcup_{n=1}^{\infty} A_n = A_1 \cup (A_2 \setminus A_1) \cup (A_3 \setminus A_2) \cup \ldots\ldots$$

$$= \bigcup_{n=1}^{\infty} A_n'$$

$A_1' = A_1$

$A_2' = A_2 \setminus A_1$

$A_3' = A_3 \setminus A_2$

$\vdots$

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n') \quad [\because A_n' \text{s are disjoint}]$$

finite

So sum has to converge.

finite $= P(A_1) + \sum_{i=1}^{\infty} \left(P(A_{i+1}) - P(A_i)\right)$ implies sum of partial sums

$$= P(A_1) + \lim_{n\to\infty} \sum_{i=1}^{n} P(A_{i+1}) - P(A_i)$$

$$= \lim_{n\to\infty} P(A_1) + \sum_{i=1}^{n} P(A_{i+1}) - P(A_i)$$

$$= \lim_{n\to\infty} P(A_{n+1}) = \lim_{n\to\infty} P(A_n)$$

$P(A_2') = P(A_2 \cap A_1^c)$

$= P(A_2) - P(A_1 \cap A_2)$

$= P(A_2) - P(A_1)$

$P(A \cup B) = P(A \cap B) + P(A \cap B^c) + P(A^c \cap B)$

$A = A_1, B = A_2$ $\qquad \phi$

$P(A_2) = P(A_1) + P(A_1 - A_2) + P(A_2 - A_1)$

Conditional Probability

$(\Omega, \mathcal{F}, P)$

Let $B \in \mathcal{F}$ and $P(B) > 0$,

Then define conditional probability of an event A given B has

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Example

Throw a fair dice. What is the probability that {2 occurs given that the outcome is an even number} ↳ C

$P(C) = \frac{1}{3}$

$$P(C) = \frac{1}{3} = \frac{1/6}{3/6} = \frac{P(A)}{P(B)} = \frac{P(A \cap B)}{P(B)}$$

Observation

1. $B \in \mathcal{F}$, $0 < P(B) < 1$

$$P(A) = P(A|B)P(B) + P(A|B^c)\, P(B^c)$$

2. $A_1, A_2, \ldots, A_n \in \mathcal{F}$, $P(A_i) > 0$

$A_i \cap A_j = \phi$ for $i \neq j$

$\bigcup_{i=1}^{n} A_i = \Omega$ (even $\bigcup_{i=1}^{n} A_i = B$ also works) (check with RHS of eq^n. Disjoint set P's sum is P(Union)).

Then $P(B) = \sum_{i=1}^{n} P(B|A_i)\, P(A_i)$

Baye's Formula

A_i's as given in observation (2).

Let $P(B) > 0$.

Then $P(A_j|B) = P(B|A_j)\, P(A_j) \Big/ \sum_{i=1}^{n} P(B|A_i)\, P(A_i)$

Check: $\tilde{P}(A) = P(A|B)$

$\tilde{P}: \mathcal{F} \to [0,1]$

$\tilde{P}(\Omega) = P(\Omega|B) = \frac{P(\Omega \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$

$\tilde{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = P\left(\bigcup_{i=1}^{\infty} A_i | B\right) \ldots = \sum_{i=1}^{\infty} \tilde{P}(A_i)$

Check it satisfy all properties of probability measure.

Example: We have urn.

Urn 1: 2 black balls & 5 red balls

Urn 2: 3 black & 2 red.

Draw a ball from urn 1 at random and put it in Urn 2. Then draw a ball from urn 2 at random. What is the probability that colour of second ball is red?

Sol^n: A = 2^{nd} ball is red.

B = 1^{st} ball is red.

$$P(A) = P(A|B)\, P(B) + P(A|B^c)\, P(B^c)$$

$$= \frac{3}{6} \times \frac{5}{7} + \frac{2}{6} \times \frac{2}{7} = \frac{19}{42}$$

?

Exercise

Suppose there are b-black balls and r red balls in a urn. We draw a ball at random and note its colour. and then add c more balls of same colour along with the drawn ball to the urn. ~~Let B_n = No. of black balls~~

Let B_n = Colour of the n^{th} ball picked, is black.

R_n = Colour of n^{th} ball picked is red.

$$P(B_1) = \frac{b}{r+b} \qquad P(R_1) = \frac{r}{r+b}$$

$$P(B_2) = P(B_2|B_1) \cdot P(B_1) + P(B_2|R_1) \cdot P(R_1)$$

$$= \frac{2b}{r+2b} \cdot \frac{b}{r+b} + \frac{b}{2r+b} \times \frac{r}{r+b} \Rightarrow$$

$$= \frac{b}{r+b}\left[\frac{2b}{r+2b} + \frac{r}{2r+b}\right]$$

$$= \frac{b}{r+b}\left[\frac{4br + 2b^2 + r^2 + 2br}{(r+2b)(2r+b)}\right]$$

$$\cancel{P(B_2) = \frac{b}{r+b}\left[\frac{r^2+6br+2b^2}{2r^2+br+4br+2b^2}\right]}$$

Exercise: Show that $P(B_n) = \dfrac{b}{b+r}$ $\forall\, n \in \mathbb{N}$.

$$P(B_2) = P(B_2|B_1)\cdot P(B_1) + P(B_2|R_1)\, P(R_1) = \frac{b+c}{r+b+c} \times \frac{b}{r+b} + \frac{b}{r+b+c} \times \frac{r}{r+b}$$

$$= \frac{b}{(r+b+c)(r+b)}\left[b+c+r\right] = \frac{b}{r+b};$$

Assume $P(B_{n-1}) = \dfrac{b}{b+r}$

$$P(B_n) = P(B_n|B_{n-1})\, P(B_{n-1}) + P(B_n|R_{n-1})\cdot P(R_{n-1}) = \frac{\cancel{b+ic}}{\cancel{r+b+(n-1)c}} \cdot \frac{\cancel{b}}{\cancel{b+r}} +$$

Independence

Suppose probability of occurence of an event A does not depend on the occurence of an event B, then we expect $P(A|B) = P(A)$.

ie, $P(A \cap B) = P(B)P(A)$.

Define two events A and B are independent if $P(A\cap B) = P(A)P(B)$

A collection of events $\{A_i : i \in I\}$ is called independent if

$$P\left(\bigcap_{i\in J} A_i\right) = \prod_{i \in J} P(A_i) \quad \text{for any finite subset of } J \text{ of } I.$$

Remark: A common mistake:

A and B are independent if $A \cap B = \phi$ (Think meaningwise with A & A^c. Their occurrence are related).

Pairwise Independence

A collection of events $\{A_i : i \in I\}$ is called pairwise independent if $P(A_i \cap A_j) = P(A_i)\,P(A_j)$ for all $i, j \in I$ & $i \neq j$

$\{A_1, A_2, A_3\}$

Pairwise $\Rightarrow$
$$P(A_1 \cap A_2) = P(A_1)\,P(A_2)$$
$$P(A_1 \cap A_3) = P(A_1)\,P(A_3)$$
$$P(A_2 \cap A_3) = P(A_2)\,P(A_3)$$

Independence $\Rightarrow$ $P(A_1 \cap A_2 \cap A_3) = P(A_1)\,P(A_2)\,P(A_3)$ &

Independence $\Rightarrow$ pairwise independence. (from definition)

$\Omega = \{1,2,3,4\}$

$A = \{1,2\}$

$B = \{1,3\}$

$C = \{1,4\}$

Exercise:

Find $(\Omega, \mathcal{F}, P)$ and some collection of events such that collection is pairwise independent but not independent.

Definition: Suppose $C \in \mathcal{F}$ and $P(C) > 0$. We say events A and B conditionally independent given C if
$$P(A \cap B \mid C) = P(A \mid C)\,P(B \mid C)$$

Random Variable

We have a random experiment. But we are not interested in (detailed) outcomes, rather we are interested to a particular property of the outcomes.

Example: Toss a fair coin n-many times. We are interested in number of H.

$$\Omega = \{(a_1, a_2, \ldots a_n) : a_i = H \text{ or } T,\ 1 \leq i \leq n\}$$

$$X\big((a_1, a_2, \dots a_n)\big) = \sum_{i=1}^{n} 1_{\{H\}}(a_i)$$

where $1_{\{H\}}(a_i) = \begin{cases} 1 & \text{if } a_i = H \\ 0 & \text{otherwise.} \end{cases}$

<u>Def</u> Let $(\Omega, \mathcal{F}, P)$ be a probability space.

A function $X : \Omega \to \mathbb{R}$ is called random variable if

why needed $\leq$?

$$\boxed{\{\omega \in \Omega : X(\omega) \leq x\}} \in \mathcal{F} \quad \forall x \in \mathbb{R}$$

$$\| $$

$$\{\omega : X(\omega) \leq x\}$$

$$\| $$

$$\{X \leq x\}$$

$P(X=1)$

$P(X=k) \quad k \in \mathbb{R}$

$P(X \leq x) \quad x \in \mathbb{R}$

Questions to ask.

If Ω is uncountable, just $=$ condition here is not enough.

For countable Ω, $\leq$ not necessary. $=$ is enough.

Distribution Function

$(\Omega, \mathcal{F}, P)$

X

Distribution function F of X is a function $F : \mathbb{R} \to [0,1]$ defined as.

$$F(x) = P(X \leq x)$$

$(\Omega, \mathcal{F}, P)$

$X(\omega) = 10 \quad \forall \omega \in \Omega.$

$\{X \leq x\} = \begin{cases} \phi & \text{if } x < 10 \\ \Omega & \text{if } x \geq 10 \end{cases}$ each belong to $\mathcal{F}$. So it's a random variable.

$$F(x) = \begin{cases} 0 & \text{if } x < 10 \\ 1 & \text{if } x \geq 10 \end{cases}$$

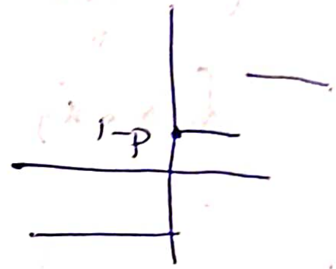
2. $\Omega = \{H, T\}$ $\quad \mathcal{F} = P(\Omega)$ $\quad P(H) = p$ where $0 < p < 1$.

$X : \Omega \to \mathbb{R}$.

$X(H) = 1 \qquad X(T) = 0$.

$$\{X \leq x\} = \begin{cases} \phi & \text{if } x < 0 \\ \{T\} & \text{if } 0 \leq x < 1 \\ \Omega & \text{if } x \geq 1 \end{cases}$$

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1-p & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$



This is known as Bernoulli distribution denoted by $Ber(p)$.

3. $(\Omega, \mathcal{F}, P)$

$A \in \mathcal{F}$, $A \neq \phi, \Omega$.

$$I_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{otherwise} \end{cases}$$

Check it's a random variable.

$$F(x) = \begin{cases} 0 & \text{if } x < 0. \\ P(A^c) & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$

4. $\Omega = \{(a_1, a_2, \cdots a_n) : a_i \in \{H, T\}\}$

$\mathcal{F} = P(\Omega)$

$P(\omega) = \frac{1}{2^n}$

$X : \Omega \to \mathbb{R}$.

$X(\omega) =$ no. of H in ω.

$= \sum_{i=1}^{n} 1_{\{H\}}(a_i)$ where $\omega = (a_1, a_2, \dots a_n)$

$P(X = k) = \frac{^nC_k}{2^n} = P_k$, say.

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ P_0 & \text{if } 0 \leq x < 1 \\ P_0 + P_1 & \text{if if } 1 \leq x < 2 \\ \vdots & \\ 1 & \text{if } x \geq n. \end{cases}$$

This is called binomial distribution.

In short, $\text{Bino}(n, \frac{1}{2})$

In general, $\text{bino}(n, p)$

Discrete Random Variable

A randome variable X is called discrete random variable if it takes values in some countable subset.

ie, $\{x_1, x_2, x_3 \dots\}$ of $\mathbb{R}$.

Probability Mass function

... of a discrete random variable X is

$$p_i = P(X = x_i)$$

Q. Suppose $f: \mathbb{R} \to [0,1]$ s.t $f=0$ outside a countable set $\{x_1, x_2, ...\}$

$f(x_i) \geq 0$, $\sum_{i=1}^{\infty} f(x_i) = 1$

Can f be a P.m.F?

Ans: $\Omega = \mathbb{N}$, $\mathcal{F} = P(\mathbb{N})$

$$P(\{k\}) = f(x_k)$$

$X: \Omega \to \mathbb{R}$

$X(k) = x_k$

PMF of X is f.

$\Omega = \{x_1, x_2, ...\}$

$\mathcal{F} = P(\Omega)$

$P(\{x_k\}) = f(x_k)$

$X(\omega) = \omega \quad \forall \omega \in \Omega$.

Properties of Distribution Functions

1. $P(X > x) = 1 - F(x)$
2. $P(a < X \leq b) = F(b) - F(a)$
3. F is non decreasing. If $x \leq y$, then $F(x) \leq F(y)$.
4. F is right continuous.

   ie, $\lim_{h \to 0^+} F(x+h) = F(x)$

$F(x) = P(X \leq x)$

Let $\{h_n\}$ be a decreasing ~~d~~ sequence and $\{h_n\} \to 0$.

$\lim_{n\to\infty} F(x+h_n) = F(x)$ (To Show)

$\lim_{n\to\infty} F(x+h_n) = \lim_{n\to\infty} P(X \leq x+h_n)$

$A_n = \{\omega : X(\omega) \leq x+h_n\}$

$A_1 \supseteq A_2 \supseteq A_3 \supseteq \dots$

$\bigcup_{n\geq 1} [-1, -1/n) = [-1, 0)$

$\boxed{?}$

why not otherwise

$\lim_{n\to\infty} F(x+h_n) = \lim_{n\to\infty} P(A_n)$

$= P\left(\bigcap_{n=1}^{\infty} A_n\right)$

$\bigcap_{n=1}^{\infty} A_n = \{\omega : X(\omega) \leq x\}$

$\lim_{n\to\infty} F(x+h_n) = P(A) = F(x)$

5. $\lim_{x\to\infty} F(x) = 1$ $\lim_{x\to-\infty} F(x) = 0$ (Exercise)

6. $P(X=x) = P\left(\{X \leq x\} \setminus \{X < x\}\right)$

$= P(X \leq x) - P(X < x)$ $\boxed{?}$

$= F(x) - P(X < x) = F(x) - \lim_{h \to 0^+} F(x-h)$

$P(X<x) = \lim_{h \to 0^+} F(x-h)$

$= \lim_{y \to x^-} F(y)$

Test (1)

4. $\mathcal{F}_1, \mathcal{F}_2$ are sigma fields.

$\Omega \in \mathcal{F}_1$ & $\Omega \in \mathcal{F}_2$

$\therefore \Omega \in \mathcal{F}_1 \cap \mathcal{F}_2$

Let $A \in \mathcal{F}_1 \cap \mathcal{F}_2 \Rightarrow A \in \mathcal{F}_1$ & $A \in \mathcal{F}_2$

$\Rightarrow A^c \in \mathcal{F}_1$ & $A^c \in \mathcal{F}_2$.

$\Rightarrow A^c \in \mathcal{F}_1 \cap \mathcal{F}_2$

Let $A_1, A_2, \ldots \in \mathcal{F}_1 \cap \mathcal{F}_2 \Rightarrow A_1, A_2, \ldots \in \mathcal{F}_1$ & $A_1, A_2, \ldots \in \mathcal{F}_2$

$\Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}_1$ & $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}_2$

$\Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}_1 \cap \mathcal{F}_2$

$\therefore \mathcal{F}_1 \cap \mathcal{F}_2$ is also a sigma field.

$\mathcal{F}_1 \cup \mathcal{F}_2$ is not necessarily a sigma field.

$A \neq B$, $A, B \neq \phi$, $A \cup B \subset \Omega$ (strict subset)

eg: $\mathcal{F}_1 = \{\phi, A, A^c, \Omega\}$

$\mathcal{F}_2 = \{\phi, B, B^c, \Omega\}$

$\mathcal{F}_1 \cup \mathcal{F}_2 = \{\phi, A, B, A^c, B^c, \Omega\} \longrightarrow$ is not a σ-field

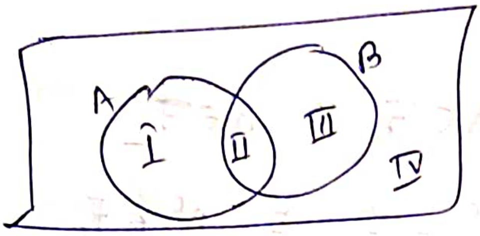
$\because A \cup B \notin \mathcal{F}_1 \cup \mathcal{F}_2$.

5. $\{\Omega, \phi\}$

6. $\mathcal{F}_S = \cap \{ \mathcal{F} : \mathcal{F}$ is σ field such that $S \subset \mathcal{F} \}$.

7. $\mathcal{F} = \{ \phi, A, B, A\cup B, A\cap B, A^c, B^c, A\cup B^c, A\cap B^c, A^c \cap B^c, A^c \cup B^c, B \cup A^c, B$

Find partition. Take their all possible unions.



$A_1 = \text{I} \rightarrow A\cap B^c$

$A_2 = \text{II} \rightarrow A\cap B$

$A_3 = \text{III} \rightarrow B\cap A^c$

$A_4 = \text{IV} \rightarrow (A\cup B)^c$

{All possible unions}

8. $\updownarrow$

2^n number. (Using bijection from → to binary string of length n.

$1 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_n = 2^n$

To show small

Take any sigma field. (K) containing $A_1, A_2, \ldots A_n$.

Show it $\subset K$.

$\therefore$

9. $0 \leq P(A\cup B) \leq 1$

$-1 \leq -P(A\cup B) \leq 0$

$$P(A)+P(B)-1 \leq P(A)+P(B)-P(A\cup B) \leq P(A)+P(B)$$

$$P(A)+P(B)-P(A\cup B) = P(A\cap B)$$

$$P(A\cap B) \geq P(A)+P(B)-1$$

next problem.

$$P\left(\bigcap_{i=1}^{n} A_i\right) = 1 - P\left(\bigcup_{i=1}^{n} A_i^c\right)$$

$$\cancel{= 1 - \sum P\left(\bigcup A_i^c\right)}$$

$$\cancel{1 - \sum (1 - P(A_i))}$$

$$\geq 1 - \sum_{i=1}^{n} P(A_i^c)$$

$$\geq 1 - \sum_{i=1}^{n} 1 - P(A_i)$$

$$\geq 1 - \left(n - \sum_{i=1}^{n} P(A_i)\right)$$

$$\geq \sum_{i=1}^{n} P(A_i) - (n-1)$$

10. Using induction..

Try proving for countable also.(Q'n is actually for finite only).

Random Vector

$\bar{X}: \Omega \to \mathbb{R}^k$ map.

$\pi : \mathbb{R}^k \longrightarrow \mathbb{R}$

$(x_1, \ldots, x_k) \longrightarrow x_1$

$\{\omega : \bar{X} \leq \bar{x}\} \in \mathcal{F} \quad \forall \bar{x} \in \mathbb{R}^k$

$\bar{X} = (X_1, X_2, \ldots X_k) \qquad \bar{x} = (x_1, x_2, \ldots x_k)$

$\pi \circ X : \Omega \to \mathbb{R}$

$(-\infty, t]$

$\bar{X} \leq \bar{x}$ mean $X_i \leq x_i \quad \forall i \in [k]$

$(\pi \circ X)^{-1}((-\infty, t])$

Note: 1) If $\bar{X}$ is a random vector, then each X_i is a random variable.

$(a.b)^{-1} = b^{-1}a^{-1}$

check [?] $X_1 : \Omega \to \mathbb{R}$

$X^{-1}(\pi^{-1}((-\infty, t]))$

$\{\omega : X_1(\omega) \leq x\} \in \mathcal{F} \quad \forall x \in \mathbb{R}$

2) Suppose $X_1, X_2, \ldots X_k$ are random variables defined on $(\Omega, \mathcal{F}, P)$. Then $\bar{X} = (X_1, X_2, \ldots X_k)$ is a random vector.

$$\{\omega : \bar{X}(\omega) \leq \bar{x}\}$$

$$= \bigcap_{i=1}^{k} \underbrace{\{\omega : X_i(\omega) \leq x_i\}}_{= Y_j} \in \mathcal{F}$$

($\because$ Each X_i is R.Variable. So, $Y_j \; \forall j \in [k] \in \mathcal{F}$. Then, $\bigcap_{i=1}^{k} Y_j \in \mathcal{F}$ ($\because \mathcal{F}$ is sigmafield))

Proof of ① $\rightarrow$

$$\{\omega : X_1(\omega) \leq x\} \overset{?}{=} \bigcup_{n=1}^{\infty} \{\omega : (X_1, X_2, \ldots X_k) \leq (x, n, n, \ldots n)\} \in \mathcal{F}$$

check.

Discrete Random Vector

If $\bar{X}$ takes values in a countable set of $\mathbb{R}^k$.

We ~~will~~ can talk about pmf of $\bar{X}$.

Distribution Function

$$F : \mathbb{R}^k \longrightarrow [0,1]$$

$$F(x_1, x_2, \dots x_k) = P(\bar{X} \leq (x_1, x_2, \dots x_k))$$

Properties (for $k = 2$)

1. If $(x_1, y_1) \leq (x_2, y_2)$, then
$$F(x_1, y_1) \leq F(x_2, y_2)$$

2. F is right continuous)

3. $\lim\limits_{x,y \to \infty} F(x,y) = 1$ $\quad \lim\limits_{x,y \to -\infty} F(x,y) = 0$

Proof: Exercise.

Probability Mass Function of Discrete Random vector. $(k=2)$

Suppose (X, Y) is a random vector which takes values on

$\{(x_i, y_j),\ i,j \in \mathbb{N}\}$

Countable set.

$$p(x,y) = P(X = x, Y = y)$$

$p(x,y) = 0$ if $x \neq x_i$ or $y \neq y_j$ $\forall\, i,j \in \mathbb{N}$

$X = x_i$ & $y \in$ anything

$$p_X(x_i) = \sum_{j=1}^{\infty} p(x_i, y_j) \quad \rightarrow \text{pmf of } X$$

$$p_Y(y_j) = \sum_{i=1}^{\infty} p(x_i, y_j) \quad \rightarrow \text{pmf of } Y$$

Qn. X, Y are random variables defined on $(\Omega, \mathcal{F}, P)$. P_X, P_Y are PMF of X, Y respectively.

(X, Y) is a random vector.

Calculate PMF of (X,Y) in terms of P_X and P_Y.

$(\Omega, \mathcal{F}, P)$

A & B are independent if $P(A \cap B) = P(A)P(B)$

we can't do that in every case, but in some cases including independence.

<u>Definition</u>: Two discrete rv's X, Y are called independent if

$$P(X=x, Y=y) = P(X=x)\, P(Y=y)$$

$\boxed{?}$

<u>Exercise</u>: X discrete r.v taking values in $\{x_1, x_2, \dots\}$

$$X = \sum_{i=1}^{\infty} x_i I_{A_i} \quad \text{where} \quad A_i = \{\omega : X(\omega) = x_i\}$$

$$Y = \sum_{j=1}^{\infty} y_j I_{B_j} \quad \text{where} \quad B_j = \{\omega : Y(\omega) = y_j\}$$

X, Y are independent iff A_i, B_j are independent $\forall\, i,j$.

<u>Example</u>: Tossing a coin once.

$$P(\{H\}) = p$$

$$P(\{T\}) = 1-p = q$$

X: denote no. of H

Y: denote no. of T

$$P(X=0, Y=0) = 0$$

$$P(X=0)\, P(Y=0) = (1-p)\, p = qp$$

Toss the coin random number of times, denote it by N.

X: denotes no. of H

Y: denotes no. of T.

N takes values in $\mathbb{N} \cup \{0\} = \{0, 1, 2, \ldots\}$

Result: Let X, Y be discrete RVs defined on same probability space. Suppose X, Y are independent. Let $g, h: \mathbb{R} \to \mathbb{R}$ be functions. Then $g(X)$ and $h(Y)$ are independent.

- First we showed that $g(X)$ and $h(Y)$ are random variables.
- $g(X)$ and $h(Y)$ are independent.
  - The theorem holds even if $g=h$.
- Similarly if $X_1, X_2, \dots X_n$ are independent and $g_i: \mathbb{R} \to \mathbb{R}$ for $1 \leq i \leq n$. Then $g_1(X_1), g_2(X_2) \dots g_n(X_n)$ are independent.

Suppose we are playing a game (throwing a dice) in a casino. Casino charges k Rs for each game. You will get x Rs back from casino after each game if outcome is x. (biased or unbiased dice unknown).

Qn. Whether you will play the game or not?

$$\frac{x_1 + x_2 + \dots + x_n}{n} \geq k.$$

$$= \sum_{i=1}^{6} \frac{n_i}{n} \times i \qquad n_i \to \text{no. of times of } i \text{ occurs in } n \text{ trials}$$

$$= \sum_{i=1}^{6} p_i \times i \quad \text{(Mean value / expected value)}$$

Def:

Let X be a finite discrete random variable with pmf p.

Then $E(X) = \sum_{i=1}^{n} x_i \, p_X(x_i)$.

Example: ① Bernoulli

$X \sim Ber(p)$

$E(X) = p$

(2) $X \sim Bino(n,p)$

$E(X)$ $P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$

$$E(X) = np$$

$Bino(n,p)$: $X = \sum_{i=1}^{n} X_i$, where X_i are independent $Ber(p)$.

$$\boxed{E(X) = \sum_{i=1}^{n} E(X_i)} \rightarrow \text{prove this first.}$$

$$= np$$

Def

Let X be a (infinite) discrete random variable with pmf p.
Then we say X has finite expectation / expectation of X exists

if $\sum_{i=1}^{\infty} |x_i| p(x_i) < \infty$ —(*)

If (*) holds, then $E(X) = \sum_{i=1}^{\infty} x_i p(x_i)$

Example: (3) $X \sim Poi(\lambda)$

$$E(X) = \sum_{k=0}^{\infty} \frac{k e^{-\lambda} \lambda^k}{k!} = \lambda$$

Exercise: Calculate expected value of X $(E(X))$ where X is geometric, negative binomial.

Result: Let X be discrete rv with pmf p and $E(X)$ exists. Let $g: R \to R$ be a f^n.
[$g(X)$ is a rv]. already seen.

Then $E(g(X))$ exist $\iff \sum |g(x_i)| p(x_i) < \infty$ ← (*)

If (*) holds, then $E(g(X)) = \sum_{i=1}^{\infty} g(x_i) p(x_i)$

Notation

$$E(X) = \sum_{i=1}^{\infty} x_i\, p(x_i) = \sum_{x} x\, p(x)$$

$$\overset{g(X)}{\overset{\parallel}{Y}}$$

Proof: X is discrete $\Rightarrow Y$ is discrete

Suppose Y takes values in $\{y_1, y_2, y_3, \cdots\} = T$

Suppose X takes values in $\{x_1, x_2, x_3, \cdots\} = S$

For $y_j \in T$, there is at least one $x_i \in S$ such that $g(x_i) = y_j$

There can be more than one x_i also.

$$A_j = \{x_i : g(x_i) = y_j\}$$

Pmf of $Y = p_Y(y_j)$

$$= P(Y = y_j)$$

$$= P(\{\omega : Y(\omega) = y_j\})$$

$$= \sum_{i : x_i \in A_j} P(\{\omega : X(\omega) = x_i\}) \quad (\because \text{countable additivity})$$

$$\{\omega : Y(\omega) = y_j\}$$
$$= \{\omega : X(\omega) \in A_j\}$$
$$= \bigcup_{i : x_i \in A_j} \{\omega : X(\omega) = x_i\}$$
$\in \mathcal{F}$ as X is a r.v.

$$= \sum_{x_i \in A_j} \underset{p(x_i)}{P(X = x_i)}$$

$$\sum_j |y_j|\, p_Y(y_j) = \sum_j |y_j| \sum_{x_i \in A_j} p(x_i)$$

(checking existence)

$$= \sum_j \sum_{x_i \in A_j} |g(x_i)|\, p(x_i)$$

$$= \sum_i |g(x_i)|\, p(x_i)$$

Result: Let $\bar{X}$ be a discrete random vector in $\mathbb{R}^d$ with pmf p. Suppose $E($

Let $g: \mathbb{R}^d \to \mathbb{R}$ be a f^n.

(then $g(\bar{X})$ is a random vector)

$Eg(\bar{X})$ exist $\iff \sum_i |g(\bar{x}_i)| \, p(\bar{x}_i) < \infty$. — $(*)$

If $(*)$ holds, then

$$Eg(\bar{x}) = \sum_{i=1}^{\infty} g(\bar{x}_i) p(\bar{x}_i)$$

- Main point is the we do not have to calculate the pmf of Y. Using pmf of X, we can calculate $E(Y)$.

Result: X, Y are discrete random variables with finite expectation,

(i) $c_1X + c_2Y$ has finite expectation. where $c_1, c_2 \in \mathbb{R}$,

$E(c_1X + c_2Y) = c_1E(X) + c_2E(Y)$.

(ii) If $X \leq Y$, then $E(X) \leq E(Y)$

(iii) $P(X \leq Y) = 1$; then $E(X) \leq E(Y)$.

(iv) $|E(X)| \leq E(|X|)$

Proof: (i) Let $Z = c_1X + c_2Y$, is a random variable.

$X, Y : (\Omega, \mathcal{F}, P) \to \mathbb{R}$ are RVs. Then

$(X, Y) : (\Omega, \mathcal{F}, P) \to \mathbb{R}^2$ is RVector.

$f(x,y) : \mathbb{R}^2 \to \mathbb{R}$ $\quad f(x,y) = c_1x + c_2y$

Then, $c_1X + c_2Y = f(X, Y)$

To check existence of $E(c_1X + c_2Y)$, we have to check

$\sum f(\bar{x}) \, p(\bar{x}) < \infty$.

For that we need pmf of (X, Y). Let $\{p(x_i, y_j)\}$ is pmf of (X, Y).

Tut 2

1. (a) $\liminf_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k$

$x \in \limsup_{n\to\infty} A_n$

$\Rightarrow \cancel{x \in \bigcup_{n=1}^{\infty} \bigcap_{k=}}$

$\Rightarrow x \in \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k.$

$\Rightarrow x \in \bigcup_{k=i}^{\infty} A_k. \quad \forall i$

$\Rightarrow x \in A_k$ ~~for some $k = 0, 1, 2, \ldots$~~ for infinitely many k.

$S_n = \bigcap_{k=n}^{\infty} A_k$

$x \in \liminf_{n\to\infty} A_n \Rightarrow x \in \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k.$

$\Rightarrow x \in \bigcup_{n=1}^{\infty} S_n$

$\Rightarrow \omega \in S_j$ for some j

$\Rightarrow \omega \in A_k \quad \forall k \geq j$

ie, all but finitely many.

(b) next)

$A_i \in \mathcal{F} \quad \bigcup_{i=1}^{\infty} A_i \in \mathcal{F} \quad \bigcap_{n=1}^{\infty} A_i \in \mathcal{F}$

$C_j = \bigcup_{i=j}^{\infty} A_i \in \mathcal{F}$

$\Rightarrow \bigcap_{j=1}^{\infty} C_j \in \mathcal{F}$

$\limsup_{n\to\infty} A_n \in \mathcal{F}$

(e) $A_n \to A \Leftrightarrow \limsup_{n\to\infty} A_n = \liminf_{n\to\infty} A_n$.

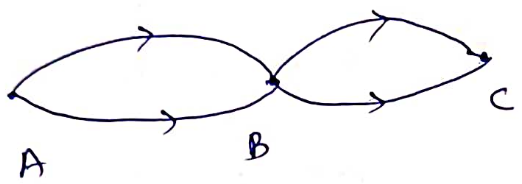
To show: $P(A_n) \to P(A)$

Continuity.

2. $A =$ sum(=)7

$B =$

3.



$A \to B$

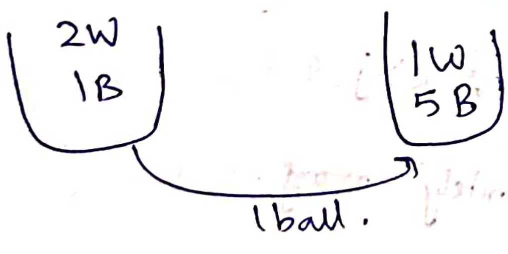
$P(\text{both road blocked}) = p^2$

$P(\text{reaching B from A}) = 1-p^2$

$P(\text{reaching C from B}) = 1-p^2$ (Similarly)

Req. probability $= (1-p^2)(1-p^2) = (1-p^2)^2$

4.



$P(\text{transfered white} \mid \text{got white})$

B

A: transfered white

B: Got white from Urn2.

$$P(A|B) = ? = \frac{\cancel{P(A \cap B)}}{\cancel{P(B)}} = \frac{P(B|A)\,P(A)}{P(B)} = \frac{\frac{2}{7} \times \frac{2}{3}}{\frac{5}{21}} = \frac{4}{5}$$

$$P(A) = \frac{2}{3}$$

$$P(B) = P(B|A) \cdot P(A) + P(B|A^c) \cdot P(A^c)$$

$$P(B) = \frac{2}{7} \times \frac{2}{3} + \frac{1}{7} \times \frac{1}{3} = \frac{5}{21}$$

Consult.

7) $P(X_1 = x_1, X_2 = x_2, \ldots X_r = x_r)$

$$= \binom{n}{x_1} p_1^{x_1} \binom{n - x_1}{x_2} p_2^{x_2} \cdots\cdots \binom{n - x_1 - x_2 - \cdots - x_{r-1}}{x_r} p_r^{x_r}$$

$$= \frac{n! \quad p_1^{x_1} p_2^{x_2} \cdots p_r^{x_r}}{x_1!\, x_2! \cdots x_r!\; 0!}$$

8) $A \rightarrow$ same

$$P(A_1^c \cap A_2^c \cap \ldots A_n^c) = 1 - P\left(\bigcup_{i=1}^{n} A_i\right)$$

$$= 1 - \left[\sum_i P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} \cdots\right]$$

$$= 1 - \left[\frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} \cdots + \frac{(-1)^n}{n!} \ldots\right]$$

Total ways of pairing $= n!$

$E_1, E_2, \ldots E_N$ get their own.

$\Rightarrow (n-N)!$

$$\sum P(E_1 E_2 \ldots E_N) = \binom{n}{N} \frac{(n-N)!}{n!}$$

2nd part

$$P(E_1 \cap E_2^c \cap E_3^c \ldots) = P\left[\frac{E_2^c \cap E_3^c \ldots}{E_1}\right] \cdot P(E_1)$$

$\Rightarrow$

$$\sum_{i,j} |f(x_i, y_j)| p(x_i, y_j) = \sum_{i,j} |c_1 x_i + c_2 y_j| p(x_i, y_j)$$

$$\leq \sum_{i,j} \left[|c_1||x_i| + |c_2||y_j| \right] p(x_i, y_j)$$

$$= |c_1| \sum_{i,j} |x_i| p(x_i, y_j) + |c_2| \sum_{i,j} |y_j| p(x_i, y_j)$$

$$= |c_1| \sum_i |x_i| \sum_j p(x_i, y_j) + |c_2| \sum_j |y_j| \sum_i p(x_i, y_j)$$

$$= |c_1| \sum_i |x_i| p_X(x_i) + |c_2| \sum_j |y_j| p_Y(y_j)$$

$< \infty$ (as $E(X)$ & $E(Y)$ exist)

$$E(c_1 X + c_2 Y) = E(f(X, Y))$$

$$= \sum_{i,j} f(x_i, y_j) p(x_i, y_j)$$

have to be done as without modulus.

$$= c_1 E(X) + c_2 E(Y)$$

<u>Remark:</u> $X_1, X_2, \ldots X_k$ random variables with finite expectation. Then $\sum_{i=1}^{k} c_i X_i$ has finite expectation & $E\left(\sum_{i=1}^{k} c_i X_i\right) = \sum_{i=1}^{k} c_i E(X_i)$

(ii, iii) $X \leq Y \Rightarrow X(\omega) \leq Y(\omega) \quad \forall \omega \in \Omega$

$P(X \leq Y) = 1$

$$P(\underbrace{\{\omega : X(\omega) \leq Y(\omega)\}}_{=A}) = 1.$$

$\Rightarrow P(A) = 1$

$A \subseteq \Omega$. If $A = \Omega$, then both are same. (Conditions on (ii) & (iii)).

$A \subsetneq \Omega$

example $\Omega = \{1, 2, \ldots, n\}$

$\mathcal{F} = P(\Omega)$

$P(\{1\}) = 0$

$P(\{i\}) = \frac{1}{n-1}, \quad i = 2, 3, \ldots, n$

$(\Omega, \mathcal{F}, P)$ is a probability space.

$P(\{1, 2, 3\}) = \frac{2}{n-1}$

$X, Y : \Omega \to \mathbb{R}$

$$X(\omega) = \begin{cases} \omega & \text{if } \omega = 2, \ldots, n \\ n+1 & \text{if } \omega = 1 \end{cases}$$

$$Y(\omega) = \begin{cases} 0 & \text{if } \omega = 1 \\ \omega + 1 & \text{if } \omega = 2, 3, \ldots n. \end{cases}$$

$P(X \leq Y) = P(\{2, 3, \ldots n\}) = 1$

but $X \not\leq Y$ as $X(1) > Y(1)$

Proof (ii) If $X \leq Y$

Define $Z = Y - X$

$$\left. \begin{array}{l} \text{Then } E(Z) \text{ exists} \\ E(Z) = E(Y) - E(X) \end{array} \right\} \text{by part (i)}$$

$Z \geq 0 \; (\because Y \geq X)$

$E(Z) = \sum z_i P_Z(z_i) \geq 0 \qquad (\because z_i \geq 0 \;\&\; P_Z(z_i) \geq 0)$

$\Rightarrow E(Y) \geq E(X)$

~~(iii) If $P(X \leq Y) = 1$~~

(iv) $\cancel{|E(X)| = \left| \sum x_i p(x_i) \right|}$

$$-|X| \leq X \leq |X|$$

$$-E(|X|) \leq E(X) \leq E(|X|) \quad \text{from part (ii)}$$

$$|E(X)| \leq E(|X|)$$

part (iii) $\rightarrow$ exercise.

Result: Let X be a rv and $|X| \leq m$ where $m > 0$; Then $E(X)$ exist. and $|E(X)| \leq m$.

$-m \leq X \leq m$ $\qquad X \leq Y$

Let $Y = m$ be constant rv.

Show that $E(X)$ exist. $\sum |x_i| p(x_i) \leq \sum m\, p(x_i)$

$\leq m. \sum p(x_i)$

$\leq m.1 < \infty$.

$\therefore \quad E(X) \leq E(Y)$

// Prove, easy:

$E(X) \leq m$.

Result: Let X be a rv and $P(|X| \leq m) = 1$, where $m > 0$.

Then $E(X)$ exist and $|E(X)| \leq m$.

Proof: exercise.

Proof (iii): $Z = Y - X$

Then $E(Z)$ exists ~~as~~ $\left(\sum_i |z_i| \, p_Z(z_i) < \infty \right)$

$$E(Z) = E(Y) - E(X)$$

$$E(Z) = \sum_i z_i \, p_Z(z_i)$$

$$= \sum_{i: z_i \geq 0} z_i \, p_Z(z_i) + \sum_{i: z_i < 0} z_i \, p_Z(z_i)$$

Note: if $z_i < 0$, then $0 \leq p_Z(z_i) = P(Z = z_i) \leq P(X > Y) = 0$

$Y - X < 0$

$\Rightarrow$

$$E(Z) = \sum_{i\, z_i \geq 0} z_i \, p(z_i) \geq 0$$

$$\Rightarrow \quad E(Y) \geq E(X)$$

<u>Result:</u> X discrete r.v.

$P(|X| \leq M) = 1$ for some $M > 0$.

Then $|E(X)| \leq M$

<u>k^{th} order moment of a discrete R.V X</u>, where $k \in \mathbb{N}$

$$E(X^k) = \sum_i x_i^k \, p(x_i) \quad \text{provided} \quad \sum_i |x_i|^k \, p(x_i) < \infty$$

k-th order central moment is

$$E\left[(X - \mu)^k\right] = \sum_i (x_i - \mu)^k \, p(x_i) \quad \text{provided} \quad \sum_i |x_i - \mu|^k \, p(x_i)$$

where $\mu = E(X)$

Variance of X

$Var(X) = E[(X-\mu)^2]$ provided it exists.

X, Y are two discrete random variables. Covariance of X and Y is defined as

$$Cov(X,Y) = E[(X-E(X))(Y-E(Y))] = E(XY) - E(X)E(Y)$$

__Result:__

X, Y discrete and independent and E(X), E(Y) exist. Then

$$E(XY) = E(X)E(Y)$$

Hence $Cov(X,Y) = 0$

$$E(XY) = \sum_i \sum_j x_i y_j \underbrace{p(x_i, y_j)}_{}$$ ⟶ joint pmf of X & Y or pmf of (X, Y)

$$= \sum_i \sum_j x_i y_j \, p_X(x_i) \, p_Y(y_j)$$

$$= \sum_i x_i p_X(x_i) \sum_j y_j p_Y(y_j)$$

$$= E(X)E(Y)$$

__Remark:__ $Cov(X,Y) = 0 \not\Rightarrow$ X, Y are independent

$\Leftarrow$ ✓

__Results:__

Suppose k-th moment exist for a discrete R.V X.

Then m-th moment of X exists $\forall \; 1 \le m \le k$, $m \in \mathbb{N}$.

__Proof:__ k-th order moment exist means $\sum_i |x_i|^k p(x_i) < \infty$

TST: $\sum_i |x_i|^m p(x_i) < \infty \quad \forall \; 1 \leq m \leq k.$

$$\sum_i |x_i|^m p(x_i) = \sum_{i:|x_i|\geq 1} |x_i|^m p(x_i) + \sum_{|x_i|<1} |x_i|^m p(x_i)$$

$$\leq \sum_{i|x_i|\geq 1} |x_i|^k p(x_i) + \sum_{|x_i|<1} p(x_i) \quad \left(\because |x_i|^m \leq 1 \; |x_i| \leq 1\right)$$

$$\leq \sum_{i|x_i|\geq 1} |x_i|^k p(x_i) + 1$$

$$\leq \sum_i |x_i|^k p(x_i) + 1 < \infty$$

Result :

X, Y discrete ~~with finite~~ and k-th moment exist. Then $(X+Y)$ has finite k-th moment.

Proof:- $\sum_i |x_i|^k p_X(x_i) < \infty \qquad \sum_j |y_j|^k p_Y(y_j) < \infty$

TST: $\sum_{i,j} |x_i+y_j|^k p(x_i,y_j) \leq \sum_{i,j} (|x_i|+|y_j|)^k p(x_i,y_j)$

$$\boxed{\begin{aligned} &\sum_j \sum_i |x_i|^k p(x_i,y_j) \\ &= \sum_i |x_i|^k \sum_j p(x_i,y_j) \\ &= \sum_i |x_i|^k p_X(x_i) \end{aligned}}$$

$$(|x|+|y|)^k \leq \left(2\max\{|x|,|y|\}\right)^k$$

$$\leq 2^k \max\{|x|^k, |y|^k\}$$

$$\leq 2^k \left(|x|^k + |y|^k\right)$$

$$\sum_{i,j} |x_i+y_j|^k p(x_i,y_j) \leq 2^k \sum_{i,j} \left(|x_i|^k + |y_j|^k\right) p(x_i,y_j)$$

by separating

$$\leq 2^k \left[\sum_i |x_i|^k p_X(x_i) + \sum_j |y_j|^k p_Y(y_j) \right] < \infty$$

<u>Exercise</u>

X - has k^{th} order moment iff X has k-th order central moment.

<u>Correlation Coefficient of X, Y</u>

$$\text{Corr}(X,Y) = \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}\,X}\,\sqrt{\text{Var}\,Y}}$$, provided $\text{Var}\,X, \text{Var}\,Y \neq 0$

Q. $\text{Var}(X) = 0 \iff P(X=c) = 1$

not necessarily $X(\omega) = c \quad \forall \omega \in \Omega$.

not necessarily probability zero aakam. So not

($\because$ non constant point probability zero aakam. So not necessary X is constant function).

<u>Cauchy Schwatz Inequality</u>

Suppose X, Y discrete r.v with finite second moment. Then

$(E(XY))^2 \leq E(X^2)E(Y^2)$

and equality holds if $P(Y=0) = 1$ or $P(X = cY) = 1$. for some constant c.

$-1 \leq \text{Corr}(X,Y) \leq 1$ using $\tilde{X} = X - E(X)$, $\tilde{Y} = Y - E(Y)$ in cauchy schwartz.

Y like $\subseteq XY$

Proof: If $\boxed{P(Y=0) = 1}$ then $P(XY = 0) = 1$.

$\Downarrow$

equality holds.

Assume $P(Y=0) < 1$. Then, $E(Y^2) > 0$.

$0 \leq E\left[(X - \alpha Y)^2\right] = E(X^2) - 2\alpha E(XY) + \alpha^2 E(Y^2) = \phi(\alpha)$ say.

has 2nd moment

X, Y has 2nd moment

$\phi(\alpha)$ has a minimum at $\alpha_0 = \dfrac{E(XY)}{E(Y^2)}$ (by calculus)

$$\phi(\alpha_0) \geq 0$$

$$\Rightarrow (E(XY))^2 \leq E(X^2)E(Y^2)$$

∴ Equality holds if $E(X-\alpha Y)^2 = 0$

$$\Rightarrow P(X-\alpha Y = 0) = 1$$

$$P(X = \alpha Y) = 1$$

Quiz syllabus

Tut

$\{A_n\}$ | Same last but question.

$A_n \to A$

$$A = \limsup_{n\to\infty} A_n = \liminf_{n\to\infty} A_n \quad \text{---(i)}$$

To Show: $P(A_n) \to P(A)$

$A_n \uparrow$ $\qquad\qquad\qquad\qquad$ $A_n \downarrow$

$$P(A) = P\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} P(A_n) \qquad P\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} P(A_n)$$

$$\limsup A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$$

$\wedge$

$$\liminf_n A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k$$

$$P(A) = P\left(\limsup_n A_n\right) = P\left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k\right) = \lim_{n\to\infty} P\left(\bigcup_{k=n}^{\infty} A_k\right) = \lim_{n\to\infty} P(B_n)$$

$$P(A) = P\left(\liminf_n A_n\right) = P\left(\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k\right) = \lim_{n\to\infty} P\Big(\underbrace{\bigcap_{k=n}^{\infty} A_k}_{C_n}\Big) = \lim_{n\to\infty} P\Big(\underbrace{\bigcap_{k=n}^{\infty} A_k}_{C_n}\Big)$$

from (i), $\quad \lim_{n\to\infty} P(B_n) = \lim_{n\to\infty} P(C_n)$

$$C_n \subseteq A_n \subseteq B_n$$

$$P(C_n) \leq P(A_n) \leq P(B_n)$$

as $n \to \infty$.

$$P(A) \leq \lim_{n\to\infty} P(A_n) \leq P(A)$$

$$\therefore \lim_{n\to\infty} P(A_n) = P(A)$$

$$P(A_n) \longrightarrow P(A)$$

last tat

8. A_i = i^{th} boy paired with i^{th} girl.

$$P(\underbrace{A_1 \cap A_2^c \cap A_3^c \cdots A_n^c}_{B}) = P(A_1 \cap B)$$

$$P(B) = P(\Omega \cap B) = P((A_1 \cup A_1^c) \cap B) = P((A_1 \cap B) \cup (A_1^c \cap B))$$

$$\underbrace{}_{\text{we know}} = \underset{\substack{\downarrow \\ \text{find this}}}{P(A_1 \cap B)} + \underbrace{P(A_1^c \cap B)}_{\text{we know}}$$

$$P(A_1 \cap B) = \underset{\substack{\downarrow \\ n-1 \text{ derangement}}}{P(B)} - P(A_1^c \cap B) \quad \text{complete derangement}$$

$$= P\left(\bigcap_{i=2}^{n} A_i^c\right) - P\left(\bigcap_{i=1}^{k} A_i^c\right)$$

9. $X, Y \sim Geo(p)$

$$P(X=k) = (1-p)^{k-1} p \qquad k = 1, 2, 3, \ldots$$

(a) $$P(X=Y) = \sum_k P(X=Y=k)$$

$$= \sum_k P(X=k, Y=k)$$

$$= \sum_k P(X=k) P(Y=k) \qquad [\because X, Y \text{ are independent}]$$

$$= \sum_k \left[(1-p)^{k-1} p\right]\left[(1-p)^{k-1} p\right]$$

$$= \sum_k (1-p)^{2k-2} p^2 \qquad k = 1, 2, 3 \ldots$$

$$= p^2 \frac{1}{1-(1-p)^2} = \cancel{p} = \frac{p^2}{2p - p^2} = \frac{p}{2-p}$$

$$P(X>Y) + P(X=Y) + P(X<Y) = 1$$

By symmetry, ($\because$ both follow same distribution)

$$P(X>Y) = P(X<Y)$$

$$2P(X>Y) = 1 - \frac{p}{2-p}$$

$$P(X>Y) = \frac{1}{2}\left[1 - \frac{p}{2-p}\right]$$

$$P(X<Y) = \frac{1}{2}\left[1 - \frac{p}{2-p}\right]$$

10. $U = \max\{X, Y\}$

IMP Standard Idea for Discrete R.V.

$$P(U=u) = P(U \le u) - P(U<u)$$

$$= P(\max\{X,Y\} \le u) - P(\max\{X,Y\} < u)$$

$$= P(X \le u \cap Y \le u) - P(X \le u-1, Y \le u-1)$$

X, Y are independent

$$P(X \le u) = \sum_{i=0}^{u} \frac{1}{n+1} = \frac{u+1}{n+1} = P(Y \le u)$$

$$P(X<u) = P(X \le u-1) = \sum_{i=0}^{u-1} \frac{1}{n+1} = \frac{u}{n+1} = P(Y<u)$$

$$P(U=u) = \frac{(u+1)^2 - u^2}{(n+1)^2} = \frac{2u+1}{(n+1)^2}$$

$V = \min\{X, Y\}$

$$P(V=v) = P(V \le v) - P(V<v)$$

$$= P(\min\{X,Y\} \le v) - P(\min\{X,Y\} < v)$$

use independence $\rightarrow$ $= \left[1 - P(\min\{X,Y\} > v\right] - \left[1 - P(\min\{X,Y\} \geq v\right]$

12. Let $\Omega = \{1,2,3,4\}$

$\mathcal{F} = P(\Omega)$

$P(i) = \frac{1}{4} \quad \forall\, i = 1,2,3,4$

$A = \{1,2\} \quad B = \{2,3\} \quad C = \{1,3\}$

$P(A \cap B) = \frac{1}{4}$

$P(A) = \frac{2}{4} = \frac{1}{2} = P(B)$

A, B pair wise independent.

$P(A \cap B \cap C) = \underline{\underline{0}} \quad P(A) \neq 0 \;\; P(B) \neq 0 \;\; P(C) \neq 0.$

$\therefore$ Not independent.

11. Possible values of X $\{m, m+1, m+2, \ldots\}$

$P(X=n) = P(X \geq n) - P(X > n) \quad n \geq m$

$= P(X > n-1) - P(X > n)$

~~$P(X > n) =$~~

$P(X=n) = P(X > n-1) - P(X > n)$

$P(X > n) = 1 \quad$ if $1 \leq n \leq m-1$

$P(X > n) = P($In 1^{st} n trial, at least one of the outcome hasn't appeared$)$

$= P\left(\bigcup_{i=1}^{m} A_i\right)$ where A_i = i^{th} outcome does not app[ear] in 1^{st} n trial

Conditional Probability Mass Function

$$P(A|B) = \frac{P(A\cap B)}{P(B)} \quad \text{provided } P(B) > 0$$

Suppose X, Y are two discrete random variables on $(\Omega, \mathcal{F}, P)$.

Joint pmf of (X,Y) is $p(x,y)$.

pmf of X is denoted by $p_X(x)$

" " Y is " " " $p_Y(y)$.

Conditional pmf of Y given $X = x$ is defined as

$$p_{Y|X}(y|x) = P(Y=y \mid X=x) \quad \text{for all } x \text{ such that } P(X=x)>0.$$

$$= \frac{P(X=x, Y=y)}{P(X=x)} = \frac{p(x,y)}{p_X(x)}$$

Conditional Distribution Function of Y given $X=x$:

$$F_{Y|X}(y|x) = P(Y \le y \mid X=x)$$

$$= \sum_{i:\, y_i \le y} P(Y=y_i \mid X=x) = \sum_{i:\, y_i \le y} p_{Y|X}(y_i|x)$$

provided x is such that $P(x)$

Denote $p_{Y|X}$ by $\tilde{p}$.

$$\tilde{p}(y) = p_{Y|X}(y|x)$$

$\tilde{p}(y)$ is a probability mass function.

$$\tilde{p}(y) \ge 0$$

$$\sum_y \tilde{p}(y) = 1.$$

$p_{X|Y}$ is a pmf of Y under the condition that $X=x$.

Expected value of Y under the condition $X=x$ is

$$\sum_y y\, p_{Y|X}(y|x) = E_{Y|X}(Y|X=x) = E(Y|X=x)$$

This is conditional expectation of Y given $X=x$.

Given $X=x$, we get $E(Y|X=x)$, a number.

Let $\varphi(x) = E(Y|X=x)$

Then, $\varphi(X)$ can be thought as a random variable on $(\Omega, \mathcal{F}, P)$

$$(\Omega, \mathcal{F}, P) \xrightarrow{X} \{x_1, x_2, \dots\} \xrightarrow{\varphi} \varphi(x) = E(Y|X=x)$$

$$E(\varphi(X)) = \sum_i \varphi(x_i)\, p_X(x_i)$$ (using earlier theorem: push over thing)

$$= \sum_i E(Y|X=x_i)\, p_X(x_i)$$

$$= \sum_i \sum_j y_j\, p_{Y|X}(y_j|x_i)\, p_X(x_i)$$

$$= \sum_i \sum_j y_j \frac{p(y_j, x_i)}{p_X(x_i)} p_X(x_i)$$

$$= \sum_i \sum_j y_j\, p(y_j, x_i)$$

$$= \sum_j y_j\, p_Y(y_j)$$

$$= E(Y)$$

$$\boxed{E\big(\underbrace{E(Y|X)}_{\text{Conditional expectation of } Y \text{ given } X.}\big) = E(Y)} \longrightarrow \text{Crazy useful formula.}$$

Summary

X, Y two rvs on $(\Omega, \mathcal{F}, P)$.

def^n $P_{Y|X}(y|x)$

def^n $E(Y|X=x)$

fund $E(Y|X) \rightarrow$ random variable.

$$E(E(Y|X)) = E(Y)$$

Exercise $(\Omega, \mathcal{F}, P)$

Let $\{A_i : i \in \mathbb{N}\}$ be a partition of Ω such that $A_i \in \mathcal{F}$ and $P(A_i) > 0$.

Let Y be a random variable on $(\Omega, \mathcal{F}, P)$.

$$E(Y) = \sum_i E(Y|A_i)\, P(A_i)$$

Define : $X : \Omega \longrightarrow \mathbb{R}$.

$X(\omega) = i$ if $\omega \in A_i$

Then X is a random variable.

$$E(Y) = E(E(Y|X))$$

$$= \sum_i E(Y|X=i)\, P(X=i)$$

$$\boxed{E(Y) = \sum_i E(Y|A_i)\, P(A_i)}$$

Example

A bird lays N number of eggs per day: where N follows $Poi(\lambda)$. Each egg hatches with probability p $(0<p<1)$ independent of other eggs. Find $E(Y)$. Y denotes the number of hatched eggs.

$N \sim Poi(\lambda)$

$$P(N=n) = \frac{e^{-\lambda}\lambda^{n}}{n!}$$

Ans: Conditional pmf of Y given $N=n$

$\cancel{p(Y \mid N=n)}$

$$X_i = \begin{cases} 1 & \text{if } i^{th} \text{ egg hatches} \\ 0 & \text{if otherwise} \end{cases}$$

$P(X_i=1) = p$

$P(X_i=0) = 1-p$

$Y \mid N=n = \sum_{i=1}^{n} X_i$ where X_i are independent and $X_i \sim Ber(p)$

Conditional probability mass function of Y given $N=n$, is

$$P_{Y|N}(k|n) = \binom{n}{k} p^k (1-p)^{n-k}$$

$E(Y) = E(E(Y|N))$

$E(Y|N=n) = np$ $\quad$ ($\because$ bino has expectation np)

$E(Y|N) = Np$

$E(E(Y|N)) = E(Np)$

$= pE(N) = p\lambda$ $\quad$ ($\because$ N is poisson & $E(\text{poisson}) = \lambda$)

$E(N|Y=k) = ?$

$$P_{N|Y}(N=n|Y=k) = \frac{P(N=n, Y=k)}{P(Y=k)} \qquad \boxed{n \geq k}$$

sort of Baye's theorem

$$= \frac{P(Y=k|N=n)\,P(N=n)}{\sum_{m \geq k} P(Y=k|N=n)\,P(N=m)}$$

Then calculate $E(N|Y=k)$ and get $E(N|Y)$.

Exercise

$$\psi(X) = E(Y|X)$$

Suppose g is a f^n $g: \mathbb{R} \to \mathbb{R}$.

Show that $E\left(\psi(X)g(X)\right) = E\left(Yg(X)\right)$

what should be $Var(Y|X=x)$

$Var(Y) = E\left[(Y-\mu)^2\right]$

1. $E(X) = \sum_{k=1}^{\infty} P(X \geq k)$

RHS $= \sum_{k=1}^{\infty} P(X=k) + P(X=k+1) + \cdots$

$= \sum_{k=1}^{\infty} \sum_{x=k}^{\infty} P(X=x)$

$1 \leq k \leq x < \infty$

$= \sum_{x=1}^{\infty} \sum_{k=1}^{x} P(X=x)$

$= \sum_{x=1}^{\infty} x P(X=x)$

$= E(x) =$ LHS

2. $P_Y(y) = P(Y=y)$

$= P(Y \leq y) - P(Y<y)$

$= P(Y \leq y) - P(Y \leq y-1)$

$P(Y \leq y) = P(X \leq y, M \leq y)$

$= P(X \leq y)\, I_{M \leq y}$ where $I_{M \leq y} = \begin{cases} 1 & \text{if } y \geq M \\ 0 & \text{otherwise} \end{cases}$

$= \sum_{x=1}^{y} p q^{x-1} I_{M \leq y}$

$= \frac{p}{q} \cdot \frac{q(q^y - 1)}{q-1} I_{M \leq y}$

$= p\left(\frac{1-q^y}{1-q}\right) I_{M \leq y} = (1-q^y) I_{M \leq y}$

$$P_Y(y) = (1-q^y)\, 1_{y \geq m} - (1-q^{y-1})\, 1_{y \geq m+1}$$

$$= \begin{cases} 1-q^y & \text{if } y = m \\ q^{y-1} - q^y & \text{if } y \geq m+1 \\ 0 & \text{otherwise} \end{cases}$$

$$E(Y) = m(1-q^m) + \sum_{m+1}^{\infty} y(q^{y-1} - q^y)$$

Note that

$$\sum_{m+1}^{\infty} q^y = \frac{q^{m+1}}{1-q}$$

Diff. wrt q,

$$\sum_{m+1}^{\infty} q^y = \frac{(1-q)(m+1)q^m + q^{m+1}}{(1-q)^2}$$

$$E(Y) = m(1-q^m) + \frac{(1-q)(m+1) + q^{m+1}}{(1-q)^2} + \frac{((1-q) + q^{m-1})q}{(1-q)^2}$$

3. Similar to 1

4. Using 1

6. X, Y, Z: Number appeared on 1st, 2nd, 3rd dice respectively.

$$P(Y) = P(Z) = P(X=i) = \begin{cases} \frac{1}{6} & i \in \{1, 2, \ldots 6\} \\ 0 & \text{otherwise} \end{cases}$$

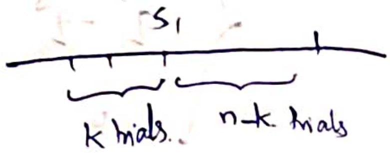
$$E(X+Y+Z) = E(X) + E(Y) + E(Z)$$
$$= 3E(X)$$
$$= 3 \cdot \left(\frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 2 + \cdots + \frac{1}{6} \cdot 6\right) = 3 \times \frac{1}{6} \times \frac{6 \times 7}{2} = 10.5$$

7. N_n: number of success in n trials

T_i : counts trials upto & including i^{th} success.

let p = probability of success $0<p<1$

(a) $$P(T_1 = k \mid N_n = 1) = \frac{P(T_1 = k, N_n = 1)}{P(N_n = 1)}$$



$$= \frac{p q^{k-1} q^{n-k}}{\binom{n}{1} p q^{n-1}} = \frac{1}{n}$$

(b) $$P(T_1 = k_1, T_2 = k_2, \dots T_r = k_r \mid N_n = r) = \frac{P(T_1 = k_1, \dots, T_r = k_r, N_r = r)}{P(N_n = r)}$$

$$= \frac{\left(p q^{k_1 - 1}\right)\left(p q^{k_2 - k_1 - 1}\right) \dots \left(p q^{k_r - k_{r-1} - 1}\right)\left(q^{n-k_r}\right)}{\binom{n}{r} p^r q^{n-r}}$$

$$= \frac{1}{\binom{n}{r}}$$

8. $$X_i = \begin{cases} 1 & \text{if } i^{th} \text{ box is empty} \\ 0 & \text{otherwise.} \end{cases}$$

(a) $$E(X_i) = 1 \cdot P(i^{th} \text{ box is empty}) + 0 \cdot P(\text{else})$$

$$= P(i^{th} \text{ box is empty})$$

$$= (r-1)^n$$

Since there are r boxes & n balls, no. of ways ball doesn't goes to i^{th} box $= (r-1)^n$.

Total no. of ways $= r^n$

(b) $E(X_i X_j) = 1 \times P(i^{th}$ & j^{th} box is empty$) + 0 + 0 + 0$
$i \neq j$

No. of way such that i^{th} & j^{th} box is empty $= (r-2)^n$.

$$E(X_i, X_j) = \frac{(r-2)^n}{r^n}.$$

(c) ~~Var(N) =~~

X_i : i^{th} box is empty.

Let $N = \sum_{i=1}^{r} X_i$ ($\sum X_i$: total no. of empty box)

$$V(N) = V(\sum X_i)$$

$$\boxed{V(\sum X_i) = \sum V(X_i) + \sum_{i<j} 2\,Cov(X_i, X_j)}$$ → formula

$$V(X_i) = E(X_i^2) - [E(X_i)]^2.$$

$$= \left(\frac{r-1}{r}\right)^n - \left(\frac{r-1}{r}\right)^{2n}.$$

$$Cov(X_i, X_j) = E(X_i, X_j) - E(X_i)\,E(X_j)$$

$$= \left(\frac{r-2}{r}\right)^n - \left(\left(\frac{r-1}{r}\right)^n\right)^2.$$

$$V(\sum X_i) = \sum V(X_i) + 2 \sum_{i<j} Cov(X_i, X_j)$$

10. application of def^n.

X, Y discrete rvs on $(\Omega, \mathcal{F}, P)$.

$$p_{Y|X}(y|x) = \frac{P(Y=y, X=x)}{P(X=x)}$$

$$\psi(x) = E(Y|X=x)$$

$E(Y) = \mu$

$\psi(X)$ is a rv.

$Var(Y) = E(Y-\mu)^2$

$$Var(Y|X=x) = \sum_i (y_i - \mu_x)^2 p_{Y|X}(y_i|x)$$

$= \sum_i (y_i - \mu)^2 p_Y(y_i)$

where $\mu_x = E(Y|X=x)$

$\sum p_i = 1$

$$= \sum_i y_i^2 p_{Y|X}(y_i|x) - 2\mu_x \sum_i y_i p_{Y|X}(y_i|x) + [\mu(x)]^2 \times 1$$

$$= E(Y^2|X=x) - \mu_x^2$$

$$Var(Y|X=x) = E(Y^2|X=x) - (E(Y|X=x))^2$$

Exercise

$$Var(Y) = E(Var(Y|X)) + Var(E(Y|X))$$

$$Var(Y|X) = E(Y^2|X) - (E(Y|X))^2$$

Exercise

$\{A_n\}$ sequence of events such that $\sum_{n=1}^{\infty} P(A_n) < \infty$.

Show that $P(\limsup A_n) = 0$

Proof: $Var(Y|X) = E(Y^2|X) - (E(Y|X))^2$

Note that $Var(Y|X)$ is a function of X and a random variable.

Denote $\phi(X) = Var(Y|X)$

Also observe, $E(Y^2|X)$, $E(Y|X)$ is also a random-variable.

let $\psi(X) = E(Y|X)$

Take expectation.

$$E(Var(Y|X)) = E(E(Y^2|X)) - E(\psi(X)^2)$$

we know that $\boxed{E(E(Y|X)) = E(Y)}$

$$E(Var(Y|X)) = E(Y^2) - E((\psi(X))^2)$$

$[\because Var(Y) = E(Y^2) - E(Y)^2]$

$$= Var(Y) + (E(Y))^2 - E((\psi(X))^2)$$

$$= Var(Y) + [E(E(Y|X))]^2 - E((\psi(X))^2)$$

$$= Var(Y) - \left[E[(\psi(X))^2] - [E(\psi(X))]^2\right]$$

$$= Var(Y) - \left[E[Z^2] - [E(Z)]^2\right] \qquad Z = \psi(X) = E(Y|X)$$

$$= Var(Y) - Va\,Z$$

$$= Var(Y) - Var(E(Y|X))$$

$$\Rightarrow Var(Y) = E(Var(Y|X)) + Var(E(Y|X))$$

$$\boxed{E(Yg(X)|X) = ? \quad = g(X)\,E(Y|X)}$$

where g is some function.

$$E(Yg(X)|X=x) = \sum_y y\,g(x)\,p_{Y|X}(y|x)$$

$$= g(x) \sum_y y\,p_{Y|X}(y|x) = g(x)\,E(Y|X=x)$$

Inequalities

(i) Markov Inequality: Let X be a (discrete) non negative random variable. For $a>0$,

$$P(X \geq a) \leq \frac{E(X)}{a}$$

Proof: Define $I(\omega) = \begin{cases} 1 & \text{if } X(\omega) \geq a \\ 0 & \text{otherwise} \end{cases}$

observe, $\forall \omega \in \Omega$

$$I(\omega) \leq \frac{X(\omega)}{a}$$

Taking expectation.

$$P(X \geq a) \leq \frac{E(X)}{a}.$$

(ii) Chebyshev's Inequality

X (discrete) random variable with mean μ and variance σ^2. Then,

for $a>0$,

$$P(|X-\mu| \geq a) \leq \frac{\sigma^2}{a^2}$$

Proof: $P(|X-\mu| \geq a) = P((X-\mu)^2 \geq a^2)$

$$\leq \frac{E((X-\mu)^2)}{a^2} = \frac{\sigma^2}{a^2}$$

Markov's Inequality

(iii) One sided Chebyshev's Inequality

X discrete rv with mean 0, variance σ^2. Then for $a>0$,

$$P(X \geq a) \leq \frac{\sigma^2}{\sigma^2 + a^2}$$

Proof: $P(X > a) \leq P(|X| > a)$

$$\leq \frac{\sigma^2}{a^2}$$

$P(X > a) = \underline{P(X + b > a + b)}$ $\rightarrow$ let $b > 0$

also (verify that this is not a true if $b < 0$)

$$\leq \frac{E(X+b)^2}{(a+b)^2} \quad \text{(by Markov)}$$

$$\leq \frac{E(X^2) + b^2}{(a+b)^2} = g(b), \text{ say} \quad (\because E(X) = 0, \text{ so is middle term})$$

$$\leq \frac{\sigma^2 + b^2}{(a+b)^2}$$

$$g'(b) = \frac{(a+b)^2 \cdot 2b - (\sigma^2 + b^2)\, 2(a+b)}{(a+b)^4}$$

$g'(b) = 0 \Rightarrow$ $\qquad b = \dfrac{\sigma^2 + b^2}{a+b}$

$$\Rightarrow b = \frac{\sigma^2}{a}$$

$$g''\left(\frac{\sigma^2}{a}\right) > 0$$

$$P(X > a) \leq \frac{\sigma^2 + \frac{\sigma^4}{a^2}}{\left(a + \frac{\sigma^2}{a}\right)^2} \leq \frac{\sigma^2}{\sigma^2 + a^2}.$$

Corollary also $\rightarrow$ Similarly $\quad P(X < -a) \leq \dfrac{\sigma^2}{\sigma^2 + a^2}$

$P(X < -a) = P(-X > a)$

Denote $Y = -X$

$Var(Y) = Var(X) = \sigma^2$

$= P(Y > a)$

$\leq \dfrac{\sigma^2}{\sigma^2 + a^2}$

Indicator Function

$(\Omega, \mathcal{F}, P)$

$A \in \mathcal{F}$

$$I_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{otherwise} \end{cases}$$

I_A is a Random variable.

$I_A + I_{A^c} = 1$

$A, B \in \mathcal{F}$

$I_{A \cap B} = I_A \times I_B$

$$\begin{aligned} I_{A \cup B} &= 1 - I_{(A \cup B)^c} \\ &= 1 - I_{A^c \cap B^c} \\ &= 1 - I_{A^c} I_{B^c} \\ &= 1 - (1 - I_A)(1 - I_B) \end{aligned}$$

$I_{A \cup B} = I_A + I_B - I_A I_B \Rightarrow I_A + I_B - I_{A \cap B}$

Taking expectation.

$$P(A\cup B) = P(A)+P(B) - P(A\cap B)$$

Inclusion Exclusion Principle

$A_1, A_2, \dots A_n \in \mathcal{F}$

$$B = \bigcup_{i=1}^{n} A_i$$

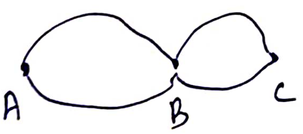
$$I_B = 1 - I_{B^c}$$

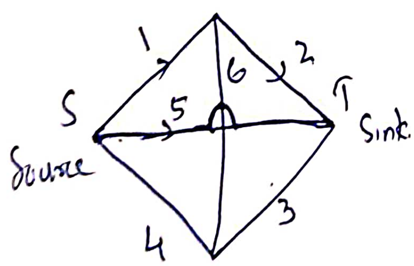
$$= 1 - I_{\bigcap_{i=1}^{n} A_i^c}$$

$$= 1 - I_{A_1^c} I_{A_2^c} \cdots I_{A_n^c}$$

$$= 1 - (1 - I_{A_1})(1 - I_{A_2}) \cdots (1 - I_{A_n})$$

which will give Inclusion Exclusion Principle.

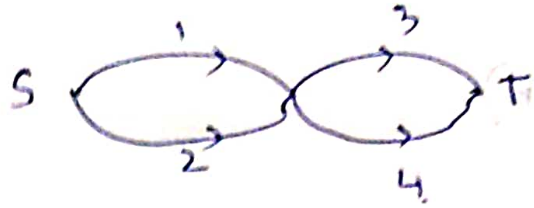




Information is sent from S to T. It travels via one of the path.

$$P(i\text{th edge not working}) = p_i$$

$P($Information will go from S to T$)$



P(info to go from S to T)

$\quad = A$

$$P(A) = E(I_A)$$

$$= 1 - E(I_{A^c})$$