Probability I (SI 427)
Department of Mathematics, IIT Bombay
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Problem set 3

1. Let X be a non-negative integer valued random variable. Then X has finite expectation if and only if $\sum_{k=1}^{\infty} \mathrm{P}(X \geq k) < \infty$. And in this case
$$E(X) = \sum_{k=1}^{\infty} \mathrm{P}(X \geq k).$$

2. Let X be a geometrically distributed random variable with parameter p and let $M > 0$ be an integer. Let $Y = \max\{X, M\}$. Find probability mass function of Y and $E(Y)$.

3. Let X, Y are both non-negative integer valued random variables. If $E(XY)$ exists, then show that
$$E(XY) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \mathrm{P}(X \geq n, Y \geq m).$$

4. We say that X is *stochastically* larger than Y, written $X \geq_{st} Y$, if for all $t \in \mathbb{R}$,
$$\mathrm{P}(X > t) \geq \mathrm{P}(Y > t).$$
If X, Y are non-negative integer valued random variable, then show that $E(X) \geq E(Y)$.

5. Construct a discrete random variable X such that k-th order $(k > 1, k \in \mathbb{N})$ moment exists but $(k+1)$-th doesn't exist. Hence no higher order moment exists than order k.

6. Calculate the expected sum obtained when three fair dice are rolled.

7. Let N_n denote the number of success in n trials and let T_i be the number of trials up to and including i-th success. Show that
   (a) $\mathrm{P}(T_1 = k | N_n = 1) = \frac{1}{n}$ for $k = 1, 2, \ldots, n$.
   (b) $\mathrm{P}(T_1 = k_1, T_2 = k_2, \ldots, T_r = k_r | N_n = r) = \binom{n}{r}^{-1}, \quad 0 < k_1 < k_2 < \ldots < k_r \leq n.$

8. Suppose n distinct balls are distributed at random into r distinct boxes. For $i = 1, 2, \ldots, n$, define
$$X_i = \begin{cases} 1 & \text{if } i\text{-th box is empty} \\ 0 & \text{otherwise} \end{cases}$$
   (a) Compute $E(X_i)$.
   (b) Find $E(X_i X_j)$ for $i \neq j$.
   (c) Let N be the number of empty boxes. Find variance of N.

9. Suppose X, Y are two discrete random variables such that
$$\mathrm{P}(|X - Y| \leq M) = 1$$
for some constant M. Show that if Y has finite expectation then X has finite expectation and $|E(X) - E(Y)| \leq M$.

10. If X and Y are independent binomial random variables with identical parameter n and p, calculate the conditional expected value of X, given that $X + Y = m$.