Finding global minima is not easy in applications.

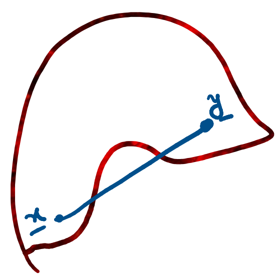
Convexity of feasible sets and objective functions provides information about minima: for example;

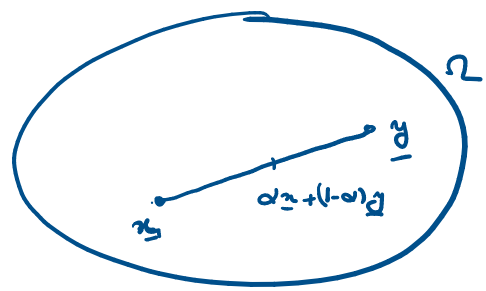
(i) do they exist ?

(ii) is it unique ?

(iii) how quickly can we find them using optimization algorithms ?

Convex sets.





A set $\Omega \subset \mathbb{R}^n$ is convex, if for every $\underline{x}, \underline{y} \in \Omega$ and $\alpha \in (0,1)$, $\alpha \underline{x} + (1-\alpha)\underline{y} \in \Omega$.

$\boxed{\Omega \subseteq \mathbb{R}^{(n)}}$

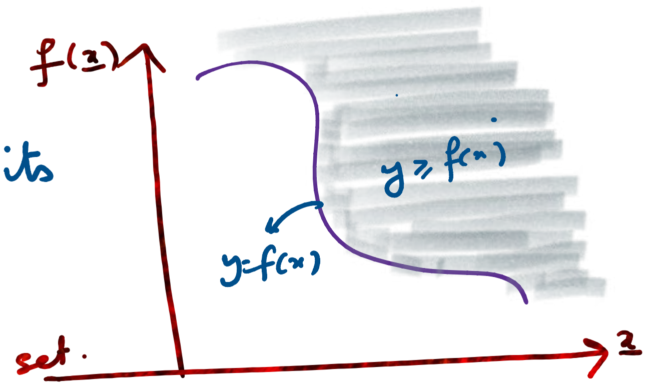
Exs $\rightarrow$ $\boxed{HW}$.

Defn [Graph]: For $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$, the graph of f is the set of points in $\Omega \times \mathbb{R} \subseteq \mathbb{R}^{n+1}$ and is given by

$$\left\{ \begin{pmatrix} \underline{x} \\ f(\underline{x}) \end{pmatrix} : \underline{x} \in \Omega \right\}.$$

Defn [Epigraph] The epigraph of a function $f : \Omega \to \mathbb{R}$, denoted by $epi(f) \subset \mathbb{R}^{n+1}$ is the set of points in $\Omega \times \mathbb{R}$ given by $epi(f) = \left\{ \begin{pmatrix} \underline{x} \\ \beta \end{pmatrix} : \underline{x} \in \Omega, \beta \in \mathbb{R}, \beta \geq f(x) \right\}$.

given by epi (f) ...



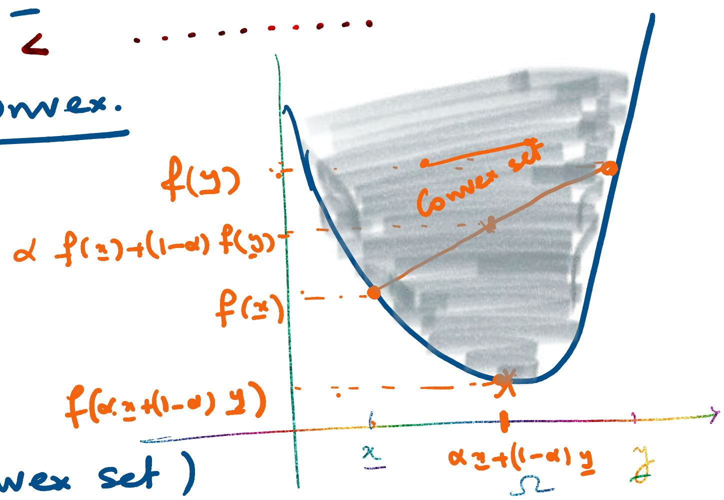
Defn [Convex fn] A function $f: \Omega \to \mathbb{R}$ is convex in Ω if its epigraph is a convex set.

Ex: If f is convex, Ω is a convex set.

Lemma A function $f: \Omega \to \mathbb{R}$ defined on a convex set $\Omega \subseteq \mathbb{R}^n$ is convex, if and only if, for all $\underline{x}, \underline{y} \in \Omega$ and $\alpha \in (0,1)$

$$f(\alpha \underline{x} + (1-\alpha)\underline{y}) \underset{<}{\leq} \alpha f(\underline{x}) + (1-\alpha) f(\underline{y})$$

[Strictly convex]



Proof: $\Rightarrow$ Let $f: \Omega \to \mathbb{R}$ be convex.

$\Rightarrow$ epi (f) is a convex set.

Let $f(\underline{x}) = a$, $f(\underline{y}) = b$ for $\underline{x}, \underline{y} \in \Omega$.

$\begin{bmatrix} \underline{x} \\ a \end{bmatrix}, \begin{bmatrix} \underline{y} \\ b \end{bmatrix} \in$ epi (f) (is a convex set)

$$\alpha \begin{bmatrix} \underline{x} \\ a \end{bmatrix} + (1-\alpha) \begin{bmatrix} \underline{y} \\ b \end{bmatrix} \in \text{epi}(f). \qquad \alpha \in (0,1)$$

$$\Rightarrow \begin{pmatrix} \alpha \underline{x} + (1-\alpha) \underline{y} \\ \alpha a + (1-\alpha) b \end{pmatrix} \in \text{epi}(f).$$

defn of epi (f).

That is, $\boxed{f(\alpha \underline{x} + (1-\alpha)\underline{y}) \leq \alpha f(\underline{x}) + (1-\alpha) f(\underline{y})}$

$$f(\alpha \underline{x} + (1-\alpha)\underline{y}) \leq \alpha f(\underline{x}) + (1-\alpha) f(\underline{y}) \text{ — } ⊛$$

"$\Leftarrow$": Let $f(\alpha \underline{x} + (1-\alpha)\underline{y}) \le \alpha f(\underline{x}) + (1-\alpha) f(\underline{y})$ — (*)

To show that epi(f) is a convex set.

Let $\begin{bmatrix}\underline{x}\\ a\end{bmatrix}$ and $\begin{bmatrix}\underline{y}\\ b\end{bmatrix}$ belong to epi(f) ; $\underline{x}, \underline{y} \in \Omega$, $a, b \in \mathbb{R}$.

$\Rightarrow f(\underline{x}) \le a$ and $f(\underline{y}) \le b$ [From defn of epi(f)]

Hence (*) gives $f(\overset{z}{\underbrace{\alpha\underline{x} + (1-\alpha)\underline{y}}_{\in\Omega}}) \le \overset{\beta}{\alpha a + (1-\alpha) b}$

as Ω is convex

$\Downarrow$

$f(z) \le \beta$

$\Rightarrow \begin{pmatrix} z \\ \beta \end{pmatrix} \in$ epi(f).

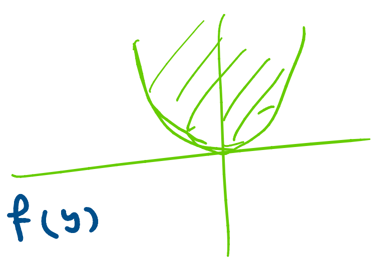
$\begin{pmatrix} z \\ \beta \end{pmatrix} \in$ epi(f)

That is, $\begin{pmatrix} \alpha\underline{x} + (1-\alpha)\underline{y} \\ \alpha a + (1-\alpha) b \end{pmatrix} \in$ epi(f).

$\Rightarrow \alpha \begin{pmatrix} \underline{x} \\ a \end{pmatrix} + (1-\alpha)\begin{pmatrix} \underline{y} \\ b \end{pmatrix} \in$ epi(f)

$\Rightarrow$ epi(f) is a convex set

$\Rightarrow$ f is convex.



Ex. $f(x) = x^2$ is convex.

if $f(\alpha x + (1-\alpha) y) \le \alpha f(x) + (1-\alpha) f(y)$

if $(\alpha x + (1-\alpha) y)^2 \le \alpha x^2 + (1-\alpha) y^2$

if $\alpha^2 x^2 + \underbrace{(1-\alpha)^2 y^2}_{(1+\alpha^2-2\alpha)y^2} + 2\alpha(1-\alpha) y x \le \alpha x^2 + y^2 - \alpha y^2$

$\ldots y^2 + \alpha^2 y^2 - 2\alpha y^2 + 2(\alpha - \alpha^2) x y$

if $\alpha^2x^2 + \cancel{y^2} + \underbrace{\alpha^2y^2 - \cancel{2}\alpha y^2} + 2(\alpha-\alpha^2)xy$
$\le \alpha x^2 + \cancel{y^2} - \cancel{\alpha y^2}$

if $(\alpha^2-\alpha)x^2 + (\alpha^2-\alpha)y^2 + 2(\alpha-\alpha^2)xy \le 0$

$(\alpha^2-\alpha)\left[x^2+y^2-2xy\right] \le 0$

if $(\alpha^2-\alpha)\left[x-y\right]^2 \le 0$ True for $\alpha \in (0,1)$

Convex Optimization Problems

Objective function and constraint set are convex.

→ Local minimizers are global minimizers.

→ Qns.! If $f \in C^1, C^2$ do we have additional characterisation?

Feasible sets affect optimization problem (Exs).

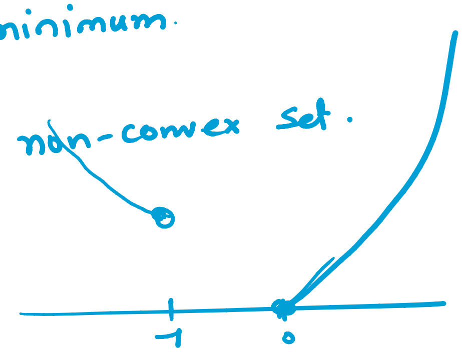
$\min_{x \in \Omega} f(x)$

$f(x) = x^2$ convex. $f: \mathbb{R} \to \mathbb{R}$ unique global minimum at $x=0$. min value 0.

- $\Omega = \{1\}$. minimum is 1.
- $\Omega = \mathbb{R}\setminus\{0\}$ non-convex, no minimum.
- $\Omega = (-\infty,-1] \cup [0,\infty)$ non-convex set.

  local minimum at $x=-1$

  global minimum at $x=0$



- $\Omega = (-\infty,-1] \cup [1,\infty)$ non-convex set

- $\Omega = (-\infty, -1] \cup [1, \infty)$ non-convex set

Global minimum

attained at $x = -1$ and $x = 1$

min value 1.