Elementary_Matrices

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Elementary Matrices

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Supplementary Material

1. Elementary matrices and properties
2. Gauss Jordan method

Elementary Matrices

An $m \times m$ elementary matrix is a matrix obtained from the $m \times m$ identity matrix I_m by one of the elementary operations; namely,

1. interchange of two rows,
2. multiplying a row by a non-zero constant,
3. adding a constant multiple of a row to another row.

That is, an elementary matrix is a matrix which differs from the identity matrix by one single elementary row operation.

1. Row Switching Operation

Interchange of two rows - $\boxed{R_i \leftrightarrow R_j}$.

The elementary matrix P_{ij} corresponding to this operation on I_m is obtained by swapping row i and row j of the identity matrix.

$$P_{ij} = \begin{array}{cc} & \begin{array}{ccccccc} C_1 & \cdots & C_i & \cdots & C_j & \cdots & C_m \end{array} \\ \begin{array}{c} R_1 \\ \vdots \\ R_i \\ \vdots \\ R_j \\ \vdots \\ R_m \end{array} & \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 0 & \cdots & 1 & & \\ & & & \ddots & & & \\ & & 1 & \cdots & 0 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \end{array}$$

Example : Consider I_3. $R_1 \leftrightarrow R_2$ gives $P_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

2. Row Multiplying Transformation

Multiplying a row by a non-zero constant - $\boxed{R_i \longrightarrow kR_i}$. The elementary matrix $M_i(k)$ corresponding to this operation on I_m is obtained by multiplying row i of the identity matrix by a non-zero constant k.

$$M_i(k) = \begin{array}{cc} & \begin{array}{cccccc} C_1 & C_2 & \cdots & C_i & \cdots & C_m \end{array} \\ \begin{array}{c} R_1 \\ R_2 \\ \vdots \\ R_i \\ \vdots \\ R_m \end{array} & \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \ddots & & & \\ & & & k & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix} \end{array}$$

Example : Consider I_3. $R_3 \to 7R_3$ gives $M_3(7) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{bmatrix}$.

3. Row Addition Transformation

$\boxed{R_i \longrightarrow R_i + kR_j}$: The elementary matrix $E_{ij}(k)$ corresponding to this operation on I_m is obtained by multiplying row j of the identity matrix by a non-zero constant k and adding with row i.

$$E_{ij}(k) = \begin{array}{c} \\ R_1 \\ \vdots \\ R_i \\ \vdots \\ R_j \\ \vdots \\ R_m \end{array} \begin{array}{c} \begin{array}{ccccccc} C_1 & \cdots & C_i & \cdots & C_j & \cdots & C_m \end{array} \\ \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & k & & \\ & & & \ddots & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \end{array}$$

Example : $R_2 \to R_2 + (-3)R_1$ for I_3 gives $E_{21}(-3) = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Proposition 1

Let A be an $m \times n$ matrix. If $\widetilde{A}$ is obtained from A by an elementary row operation, and $\mathscr{E}$ is the corresponding $m \times m$ elementary matrix, then $\mathscr{E}A = \widetilde{A}$.

Proof :

Let $I_m = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = \begin{bmatrix} e_1^T \\ e_2^T \\ \vdots \\ e_m^T \end{bmatrix}$

where $e_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$, with 1 is in the i^{th} position, $1 \le i \le m$.

Also, $e_i^T A = A_{(i)}$, the i^{th} row of A.

Recall that row i of AB is row i of A times B.

$R_i \leftrightarrow R_j (i < j)$

$$P_{ij}A = \begin{bmatrix} \vdots \\ e_j^T \\ \vdots \\ e_i^T \\ \vdots \end{bmatrix} A = \begin{bmatrix} \vdots \\ e_j^T A \\ \vdots \\ e_i^T A \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ A_{(j)} \\ \vdots \\ A_{(i)} \\ \vdots \end{bmatrix} = \widetilde{A}.$$

$R_i \longrightarrow kR_i$

$$M_i(k)A = \begin{bmatrix} \vdots \\ ke_i^T \\ \vdots \end{bmatrix} A = \begin{bmatrix} \vdots \\ kA_{(i)} \\ \vdots \end{bmatrix} = \widetilde{A}.$$

$R_i \longrightarrow R_i + kR_j$

$$E_{ij}(k)A = \begin{bmatrix} \vdots \\ e_i^T + ke_j^T \\ \vdots \end{bmatrix} A = \begin{bmatrix} \vdots \\ (e_i^T + ke_j^T)A \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ A_{(i)} + kA_{(j)} \\ \vdots \end{bmatrix} = \widetilde{A}.$$

Reduced Row Echelon form contd.

Exercise: Let A be an $m \times n$ matrix. There exist elementary matrices $E_1, E_2, ..., E_N$ of order m such that the product $\underline{E_N \cdots E_2 E_1 A}$ is a row echelon form of A.

Proposition 2 :

Elementary matrices are invertible and the inverses are also elementary matrices.

In fact,

- If $\mathscr{E}$ corresponds to $R_i \leftrightarrow R_j$, then $\mathscr{E}^{-1} = P_{ij}$;
- if $\mathscr{E}$ corresponds to $R_i \longrightarrow kR_i$, then $\mathscr{E}^{-1}$ is the elementary matrix corresponding to $R_i \to \frac{1}{k}R_i$; that is, $M_i(1/k)$.
- if $\mathscr{E}$ corresponds to $R_i \longrightarrow R_i + kR_j$, then $\mathscr{E}^{-1}$ is the elementary matrix corresponding to $R_i \to R_i - kR_j$, that is, $E_{ij}(-k)$.

Exercise : In each case, check $\mathscr{E}\mathscr{E}^{-1} = I$.

Since row operations are reversible, elementary matrices are invertible, for if $\mathscr{E}$ is produced by a row operation on I_m, there is another row operation of the same type, that changes $\mathscr{E}$ back to I_m. Hence, there is an elementary operation $\mathscr{F}$ such that $\mathscr{F}\mathscr{E} = I_m$. Since $\mathscr{E}$ and $\mathscr{F}$ correspond to reverse operations, $\mathscr{E}\mathscr{F} = I_m$ too. $\square$

Example

Find the inverses of

$$E_{31}(-4) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix},\ M_2(7) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 1 \end{bmatrix}\ \&\ P_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Soln: $E_{31}(-4)^{-1} = E_{31}(4) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}, M_2(7)^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/7 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\&\ P_{13}^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$

Proposition 3 :

If A and B are row equivalent square matrices and A is invertible, then B is invertible.

Proof :

- If $A_1, \dots, A_k$ are invertible matrices of same size, then the product $A_1 \dots A_k$ is also invertible; and $(A_1 \dots A_k)^{-1} = A_k^{-1} \dots A_1^{-1}$.
  (Why?
  Hint : Use the associative law for multiplication to derive

  $$(AB)(B^{-1}A^{-1}) = \dots = A(BB^{-1})A^{-1} = \dots = I.$$

  Generalize.)
- If $\mathscr{E}_1, \dots, \mathscr{E}_k$ are elementary matrices such that

  $$\boxed{B = \mathscr{E}_k \dots \mathscr{E}_1 A,}$$

  then B is a product of invertible matrices; and hence is invertible.

Theorem

A square matrix A is invertible if and only if A is row equivalent to the identity matrix.

Proof :

$(\Longleftarrow)$ Let A be row equivalent to the identity matrix. From Proposition 3, as I is invertible, A is also invertible.

$(\Longrightarrow)$ If A is invertible, then its row echelon form

$$B = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ 0 & b_{22} & \dots & b_{2n} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & b_{nn} \end{pmatrix}$$

is an upper triangular matrix with $b_{ii} \neq 0$ for all i. (Why?)
(B is obtained from A via $B = \mathscr{E}_k \dots \mathscr{E}_1 A$. Since A is invertible, B is invertible (Proposition 3). Now, if $b_{ii} = 0$ for any i, this would imply $det(B) = 0$ and this contradicts invertibility of B, as ($since\ |B||B^{-1}| = 1$)).

Proof continued...

Applying $R_i \to b_{ii}^{-1} R_i$ (for $i = 1, \ldots, n$) gives

$$B \sim \begin{pmatrix} 1 & b_{11}^{-1} b_{12} & \ldots & b_{11}^{-1} b_{1n} \\ 0 & 1 & \ldots & b_{22}^{-1} b_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1 \end{pmatrix}.$$

Now applying some more row operations of the type $R_i \to R_i - cR_j$, we get the required result, that is,

$$A \sim B \sim I.$$

Result: Let A be a square matrix, say of order $n \times n$. There exist elementary matrices $E_1, E_2, ..., E_N$ of order n such that the product $E_N \cdots E_2 E_1 A$ is either the $n \times n$ identity matrix I or its last row is 0.

Proof:

Consider the *reduced* row echelon form of A. Recall that there must be $p \leq n$ pivots in all. If there are $p = n$ pivots then the *reduced* REF must be I. If there are $p < n$ pivots,then the last $n - p$ rows must vanish.

Result: If A is an invertible matrix, then A can be written as a product of elementary matrices.

Proof : If A is invertible, then A is row equivalent to identity matrix, that is, there exists elementary matrices $\mathscr{E}_1, \ldots, \mathscr{E}_k$ such that $\mathscr{E}_k \ldots \mathscr{E}_1 A = I$.

This gives $A = (\mathscr{E}_k \ldots \mathscr{E}_1)^{-1} = \mathscr{E}_1^{-1} \ldots \mathscr{E}_k^{-1}$.

Gauss-Jordan method for finding A^{-1}

Let A be invertible, and $\mathscr{E}_k \ldots \mathscr{E}_1 A = I$.

Then, $\mathscr{E}_k \ldots \mathscr{E}_1 I = A^{-1}$.

That is, the elementary row operations performed on A to reduce it to identity matrix, performed in the same order on I reduces I to A^{-1}.

Inverse of A : an example

$$[A|I] = \left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 4 & -6 & 0 & 0 & 1 & 0 \\ -2 & 7 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \to R_2 + (-2)R_1} \xrightarrow{R_3 \to R_3 + (1)R_1} \left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & -8 & -2 & -2 & 1 & 0 \\ 0 & 8 & 3 & 1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 + (1)R_1} \left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & -8 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] = [B|L]$$

This completes the forward elimination. The first half of elimination has taken A to echelon form B, and now the second half will take B to I. That is, we create $0's$ above the pivots in the last matrix.

Example continued...

$$[B|L] \xrightarrow{R_2 \to R_2 + (2)R_3} \xrightarrow{R_1 \to R_1 + (-1)R_3} \left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 2 & -1 & -1 \\ 0 & -8 & 0 & -4 & 3 & 2 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 + (1/8)R_3} \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & \frac{12}{8} & -\frac{5}{8} & -\frac{6}{8} \\ 0 & -8 & 0 & -4 & 3 & 2 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{(1) \times R_1/2} \xrightarrow{(2) \times R_2/-8} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{12}{16} & -\frac{5}{16} & -\frac{6}{16} \\ 0 & 1 & 0 & \frac{4}{8} & -\frac{3}{8} & -\frac{2}{8} \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] = [I|A^{-1}]$$

Example

Write $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ as a product of 4 elementary row matrices if $D = ad - bc \neq 0$.

Case 1 ($a \neq 0$): $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} a & b \\ 0 & D/a \end{bmatrix} \mapsto \begin{bmatrix} 1 & b/a \\ 0 & 1 \end{bmatrix} \mapsto \mathrm{I}_2$.
where $D = ad - bc$ (see next slide for details).
Hence

$$A = \begin{bmatrix} 1 & 0 \\ c/a & 1 \end{bmatrix} \begin{bmatrix} 1/a & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & D/a \end{bmatrix} \begin{bmatrix} 1 & b/a \\ 0 & 1 \end{bmatrix}$$

$$= E_{21}(c/a)M_1(a)M_2(D/a)E_{12}(b/a).$$

Case 2: $(a = 0) \Longrightarrow -D = bc \neq 0$: Then

$$A = P_{12}M_2(b)M_1(c)E_{12}(d/c).$$

Detailing the previous example (Case 1)

Start with $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, let $D = ad - bc$.

Operation	Matrix	Product with elementary matrix
$R_2 \to R_2 - (c/a)R_1$	$\begin{bmatrix} a & b \\ 0 & D/a \end{bmatrix}$	$E_{21}(-c/a)A$
$R_2 \to R_2/(D/a)$ $R_1 \to R_1/a$	$\begin{bmatrix} 1 & b/a \\ 0 & 1 \end{bmatrix}$	$M_1(1/a)M_2(a/D)E_{21}(-c/a)A$
$R_1 \to R_1 - (b/a)R_2$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$E_{12}(-b/a)M_1(1/a)M_2(a/D)$ $\times E_{21}(-c/a)A$

Hence, I can be obtained from A by performing $E_{12}(-b/a)M_1(1/a)M_2(a/D)E_{21}(-c/a)$ on A; that is,

$$\boxed{I = E_{12}(-b/a)M_1(1/a)M_2(a/D)E_{21}(-c/a)A.}$$

Arguing in a similar manner,

$$A = E_{21}(c/a)M_2(D/a)M_1(a)E_{12}(b/a)I$$
$$= \begin{bmatrix} 1 & 0 \\ c/a & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & D/a \end{bmatrix} \begin{bmatrix} 1/a & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & b/a \\ 0 & 1 \end{bmatrix} (\mathit{Verify!})$$

Finding inverse by Gauss Jordan Method (Exercise)

Example: Find the inverse of $A = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$.

Solution:

$$\begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{E_{21}(3), E_{31}(-1)} \begin{bmatrix} -1 & 1 & 2 \\ 0 & 2 & 7 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{E_{32}(-1)} \begin{bmatrix} -1 & 1 & 2 \\ 0 & 2 & 7 \\ 0 & 0 & -5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -4 & -1 & 1 \end{bmatrix}$$

Example contd.

$$\xrightarrow{M_1(-1), M_2(1/2), M_3(-1/5)} \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 7/2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 3/2 & 1/2 & 0 \\ 4/5 & 1/5 & -1/5 \end{bmatrix}$$

$$\xrightarrow{E_{13}(2), E_{23}(-7/2)} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3/5 & 2/5 & -2/5 \\ -13/10 & -1/5 & 7/10 \\ 4/5 & 1/5 & -1/5 \end{bmatrix}$$

$$\xrightarrow{E_{12}(1)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -7/10 & 1/5 & 3/10 \\ -13/10 & -1/5 & 7/10 \\ 4/5 & 1/5 & -1/5 \end{bmatrix}$$

Example contd.

It follows that

$$A^{-1} = \frac{1}{10}\begin{bmatrix} -7 & 2 & 3 \\ -13 & -2 & 7 \\ 8 & 2 & -2 \end{bmatrix}.$$

Also putting all the row ops together,

$$\begin{aligned} A^{-1} = {} & E_{12}(1)E_{13}(2)E_{23}(-7/2)M_1(-1) \\ & \times M_2(1/2)M_3(-1/5)E_{32}(-1)E_{21}(3)E_{31}(-1) \end{aligned}$$

as a product of ERM's.