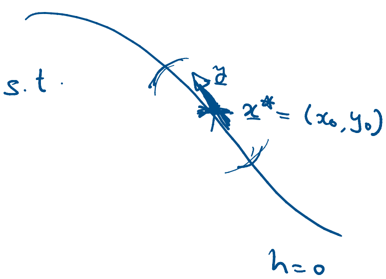
Lecture 21

Thursday, 17 March 2022 1:40 PM

[Revisit - Implicit function Thm and Lagrange Thm].

Implicit fn Thm ($h : \mathbb{R}^2 \to \mathbb{R}$)



Assume
- (i) $h(x,y) \in C^1$ near (x_0, y_0) s.t.
- (ii) $h(x_0, y_0) = 0$
- (iii) $\frac{\partial h}{\partial y}(x_0, y_0) \neq 0$

$h(x,y) = 0$

$h(x, f(x)) = 0$

Then, $\exists$ a unique function $y = f(x) \in C^1$ in the neighbourhood of x_0 s.t.

(a) $y_0 = f(x_0)$

(b) $h(x, f(x)) = 0 \quad \forall x$ near x_0

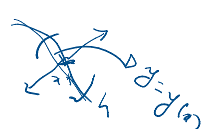
(c) $$f'(x) = -\frac{h_x(x, f(x))}{h_y(x, f(x))}$$

$$\left[\begin{matrix} x_1(t) = t \\ x_2(t) = f(t) \end{matrix}\right] \qquad \dot{x}(t) = \begin{bmatrix} 1 \\ f'(t) \end{bmatrix} \neq \underline{0}$$

Lagrange multiplier theorem

[• min $f(\underline{x})$ s.t. $h(\underline{x}) = 0$, $f : \mathbb{R}^2 \to \mathbb{R}$.]

- $f \in C^{(1)}$, $h \in C^{(1)}$, $\nabla h \neq 0$
- $\underline{x}^* = (x_0, y_0)$ is an extremum such that $h(x_0, y_0) = 0$.



Conclusion: $\exists\, \lambda \in \mathbb{R}$ s.t. $\nabla(f + \lambda h)(x_0, y_0) = 0$.

Proof: Note that the conditions of implicit fn thm are satisfied.

$\Rightarrow \exists$ a function $y = y(x)$ that satisfies

$h(x, y(x)) = 0$ [in nbd of $\underline{x}^*$]

$h_x + h_y \cdot y'(x) = 0$

$$h_x + h_y \, y'(x) = 0$$

$$\Rightarrow \boxed{y'(x) = -\frac{h_x}{h_y}}$$

Given $f(x, y(x))$ has a <u>local extrema</u> at (x_0, y_0).

$$f_x + f_y \, y'(x) = 0 \longrightarrow [\text{At } (x_0, y_0)]$$

$$f_x(x_0, y_0) - f_y(x_0, y_0) \frac{h_x(x_0, y_0)}{h_y(x_0, y_0)} = 0$$

$$\begin{cases} f_x(x_0, y_0) + \lambda h_x(x_0, y_0) = 0 \\ f_y(x_0, y_0) + \lambda h_y(x_0, y_0) = 0 \\ h(x_0, y_0) = 0 \end{cases} \qquad \left[\frac{f_y(x_0, y_0)}{h_y(x_0, y_0)} = -\lambda \right]$$

$$\boxed{\nabla (f + \lambda h)(x_0, y_0) = 0}$$

<u>Lagrange Theorem</u>: $f : \mathbb{R}^n \to \mathbb{R}$,
$\underline{h} : \mathbb{R}^n \to \mathbb{R}^m$ $\quad m \le n$.

<u>Necessary condtn</u>

(i) Let $\underline{x}^*$ be a <u>local minimizer</u> of $f : \mathbb{R}^n \to \mathbb{R}$ s.t.
$\underline{h}(\underline{x}) = 0$, $\underline{h} : \mathbb{R}^n \to \mathbb{R}^m$, $m \le n$.

(ii) Let $\underline{x}^*$ be a <u>regular point</u>.

(iii) $f, h \in \mathcal{C}^{(1)}$

Then $\exists \, \underline{\lambda} \in \mathbb{R}^m$ s.t.

$$\boxed{Df(\underline{x}^*) + \underline{\lambda}^T D\underline{h}(\underline{x}^*) = \underline{0}^T} \quad \text{(A)}$$

$$\nabla f = \begin{bmatrix} \\ \\ \end{bmatrix}_{n \times 1}$$

$$Df = \begin{bmatrix} & \end{bmatrix}_{1 \times n}$$

$$\underline{h} = (h_1, \ldots h_m)$$

$$D\underline{h} = \begin{bmatrix} & \end{bmatrix}_{m \times n}$$

$$\underline{\lambda} = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{bmatrix}_{m \times 1}$$

<u>Second Order necessary condition</u>

Let $\underline{x}^*$ be a local minimizer of $f : \mathbb{R}^n \to \mathbb{R}$

$\mathcal{C}^{(2)}$ [... other assumptions

Let $\underline{x}^*$ be a local minimizer of $f : \mathbb{R}^n \to \mathbb{R}$ s.t. $h(\underline{x}) = 0$, $f, h \in C^{(2)}$ [with all other assumptions as above.]

Then $\exists\ \lambda^* \in \mathbb{R}^m$ s.t.

(i) (A) holds [from I order necessary condtn]

(ii) $\forall\ \underline{y} \in T(\underline{x}^*)$, $\underline{y}^T\ \mathcal{L}(\underline{x}^*, \underline{\lambda}^*)\ \underline{y} \geq 0.$

$$\mathcal{L}(\underline{x}^*, \underline{\lambda}^*) = F(\underline{x}^*, \lambda^*) + \lambda_1^* H_1(\underline{x}^*, \underline{\lambda}^*) + \cdots + \lambda_m^* H_m(\underline{x}^*, \lambda^*)$$

Recall

[$l(\underline{x}, \underline{\lambda}) = f(\underline{x}) + \underline{\lambda}^T \underline{h}(\underline{x})$
$= f(\underline{x}) + \lambda_1 h_1(\underline{x}) + \cdots + \lambda_m h_m(\underline{x})$.

Example Min $x_1^2 + 2x_1x_2 + 3x_2^2 + 4x_1 + 5x_2 + 6x_3$

s.t. $\begin{bmatrix} x_1 + 2x_2 = 3 \\ 4x_1 + 5x_3 = 6 \end{bmatrix}$ $\boxed{\underline{h}(\underline{x}) = 0}$

$$l(\underline{x}, \underline{\lambda}) = \overbrace{x_1^2 + 2x_1x_2 + 3x_2^2 + 4x_1 + 5x_2 + 6x_3}^{f} + \lambda_1 (\underbrace{x_1 + 2x_2 - 3}_{h_1}) + \lambda_2 (\underbrace{4x_1 + 5x_3 - 6}_{h_2}).$$

$\underline{x} \to x_1, x_2, x_3$, $\underline{\lambda} \to \lambda_1, \lambda_2$

$\underline{x}^* = \underline{x}$

$$\nabla(\underline{f} + \lambda \underline{h})(x^*) = 0$$

$$\begin{bmatrix} 2x_1 + 2x_2 + 4 \\ 2x_1 + 6x_2 + 5 \\ 6 \end{bmatrix} + \lambda_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Constraints. $\begin{cases} x_1 + 2x_2 - 3 = 0 \\ 4x_1 + 5x_3 - 6 = 0. \end{cases}$

Constraints. $\left\{ \underline{4x_1 + 5x_3 - 6 = 0} \right.$

$$\begin{bmatrix} 2 & 2 & 0 & 1 & 4 \\ 2 & 6 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 5 \\ 1 & 2 & 0 & 0 & 0 \\ 4 & 0 & 5 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} -4 \\ -5 \\ -6 \\ 3 \\ 6 \end{bmatrix}$$

$$\boxed{\underline{x}^* = \begin{bmatrix} 16/5 \\ -1/10 \\ -34/25 \end{bmatrix}} \qquad \underline{\lambda}^* = \begin{bmatrix} -27/5 \\ -6/5 \end{bmatrix} \rightarrow \boxed{\text{check}}$$

II order

$$\underline{y}^T \mathcal{L}(\underline{x}^*, \underline{\lambda}^*)\, \underline{y} \geq 0$$

$$\forall\, \underline{y} \in T(\underline{x}^*).$$

$$T(x^*) = \left\{ \underline{y} : \begin{bmatrix} 1 & 2 & 0 \\ 4 & 0 & 5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \underline{0} \right\}.$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 4 & 0 & 5 \end{bmatrix} \xrightarrow{R_2 \to R_2/4} \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 5/4 \end{bmatrix}$$

$$\xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 2 & 0 \\ 0 & \boxed{-2} & 5/4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} =$$

y_2 is basic.

$y_1 \rightarrow$ basic variable.
$y_2 \rightarrow$ basic.
y_3 = free.

$$-2y_2 + \frac{5}{4} y_3 = 0 \Rightarrow \underline{y_2 = \frac{5}{8} y_3}$$

$$y_1 + 2y_2 = 0$$

$$y_1 = -2 \times y_2$$

$$= -2 \times \frac{5}{8} y_3$$

$$= -5/4\, y_3.$$

$$T(x^*) = \left\{ \underline{a} \begin{bmatrix} -5/4 \\ 5/8 \\ 1 \end{bmatrix} : a \in \mathbb{R} \right\}$$

$T(x^*)$

Check: $\quad \underline{y}^T \mathcal{L}(\underline{x}^*, \lambda^*)\, \underline{y} = \frac{75}{32} a^2 > 0$ ✓ $\quad a = 0$

$\Rightarrow$ $\boxed{\underline{x}^* \text{ is a local minimizer}}$

Proof of Lagrange Theorem

$f(x) = x^3$

$f'(0) = 0$

$f''(0) = 0$

$x = 0$

$$Df(\underline{x}^*) + \underline{\lambda}^{*T} Dh(\underline{x}^*) = \underline{0}$$

for some $\underline{\lambda}^* \in \mathbb{R}^m$.

$$\underline{\nabla f(\underline{x}^*)} = -\left(D\underline{h}(\underline{x}^*)^T \underline{\lambda}^*\right)$$

$$\underline{\nabla f(\underline{x}^*)} \in \mathcal{R}\left(D\underline{h}(\underline{x}^*)^T\right) = \underline{N(x^*)}$$ normal space

$= \{ \underline{x} \in \mathbb{R}^n : \underline{x} = Dh(x^*)^T \underline{z},\ \underline{z} \in \mathbb{R}^m \}$

S.T. $\nabla f(\underline{x}^*) \in T(\underline{x}^*)^{\perp}$ $= T(x^*)^{\perp}$ (Tangent space)

(Thm 20.1)

Let $\underline{y} \in T(\underline{x}^*) \Rightarrow \exists$ a differentiable curve $\{\underline{x}(t) : t \in (a,b)\}$ such that

$\forall t \in (a,b)$, $\underline{h}(\underline{x}(t)) = 0$

$\underline{x}(t^*) = x^*$

$\dot{\underline{x}}(t^*) = \underline{y}$ in tangent space

Let $\phi(t) = f(\underline{x}(t))$

f is attaining an extrema at t^*.

$\phi'(t) = 0$,

$$\phi'(t) = 0$$

$$\Rightarrow \quad Df(\underline{x}^*) \cdot \underbrace{\dot{\underline{x}}(t^*)} = 0 \qquad [\text{Chain Rule}]$$

$$\Rightarrow \quad \underbrace{Df(\underline{x}^*)}_{\text{row}} \; \underline{y} = 0 \qquad \forall \underline{y} \in T(\underline{x}^*)$$

$$\Rightarrow \quad \boxed{Df(\underline{x}^*) \in T(\underline{x}^*)^{\perp}.}$$

Concludes the proof.

$$\boxed{\text{VS ZSH R}}$$