4. If $X \sim \mathcal{N}(0,1)$, then prove that $P(|X| \geq u) \leq \sqrt{2/\pi}\frac{e^{-u^2/2}}{u}$ for all $u > 0$. How does this bound compare with that given by Chebyshev's inequality? [10+5 = 15 points]

5. Consider n values $\{x_i\}_{i=1}^n$ drawn independently from a Laplacian distribution with mean 0 and parameter σ. The probability density for a Laplacian random variable X is given by $f_X(x) = \frac{1}{2\sigma}e^{-|x|/\sigma}$ (note the absolute value in the exponent). Given $\{x_i\}_{i=1}^n$, derive the maximum likelihood estimate for σ, as well as its bias, variance, MSE. [15 points]

6. In this problem, we will derive higher order moments of specific random variables in a new way.

   (a) Consider $X \sim \mathcal{N}(\mu, \sigma^2)$. Then prove that $E[g(X)(X-\mu)] = \sigma^2 E[g'(X)]$ where g is a differentiable function such that $E[|g'(X)|] < \infty, |g(x)| < \infty$. Use this to derive an expression for $E[X^3]$ in terms of μ and σ^2. Do not use any other method (eg: MGFs) to derive $E[X^3]$. [5+5=10 points]

   (b) Consider $X \sim \text{Poisson}(\lambda)$. Then prove that $E[\lambda g(X)] = E[Xg(X-1)]$ where g is a function such that $-\infty < E[g(X)] < \infty, -\infty < g(-1) < \infty$. Use this to derive an expression for $E[X^3]$ assuming known expressions for $E[X], E[X^2]$. Do not use any other method (eg: MGFs) to derive $E[X^3]$. [5+5=10 points]

7. (a) A student is trying to design a procedure to generate a sample from a distribution function F, where F is invertible. For this, (s)he generates a sample u_i from a $[0,1]$ uniform distribution using the 'rand' function of MATLAB, and computes $v_i = F^{-1}(u_i)$. This is repeated n times for $i = 1...n$. Prove that the values $\{v_i\}_{i=1}^n$ follow the distribution F. [6 points]

   (b) Let $Y_1, Y_2, ..., Y_n$ represent data from a continuous distribution F. The empirical distribution function F_e of these data is defined as $F_e(x) = \frac{\sum_{i=1}^n \mathbf{1}(Y_i \leq x)}{n}$ where $\mathbf{1}(z) = 1$ if the predicate z is true and 0 otherwise. Now define $D = \max_x |F_e(x) - F(x)|$. Also define $E = \max_{0 \leq y \leq 1} \left|\frac{\sum_{i=1}^n \mathbf{1}(U_i \leq y)}{n} - y\right|$ where $U_1, U_2, ..., U_n$ represent data from a $[0,1]$ uniform distribution. Now prove that $P(E \geq d) = P(D \geq d)$. Briefly explain what you think is the practical significance of this result in statistics. [8+6=14 points]