Lecture 1   Set-constrained optimization and unconstrained optimization.

Monday, 3 January 2022   9:53 AM

Motivation: applies to any problem that involves decision making in engineering or economics or medical or other real-life applications.

- One needs to choose among several alternatives.
- The choice is governed by our desire to make the "best decision".

- The measure of how good is an alternative is described by an objective function.

Optimization theory and methods deal with selecting best alternative in the sense of an objective fn.

Unconstrained optimization / Constrained optimization

Optimization problem : $\min f(\underline{x})$ s.t. $\underline{x} \in \Omega \; (\subseteq \mathbb{R}^n)$

(Set constrained optimization)

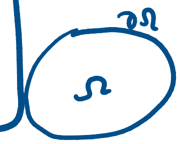
- $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is the objective fn / cost function
- $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$ $x_1, \dots, x_n$ (decision variables)
- $\Omega \subseteq \mathbb{R}^n$ Constraint set or feasible set.

Example $\Big[\min \; J(y, u) := \frac{1}{2} \| y - y_d \|^2 + \frac{\alpha}{2} \| u \|^2$

given (above y_d), control (above u), given (below α)

Example

$$\begin{bmatrix} \min \quad J(y,u) := \frac{1}{2}\|y - y_d\|^2 + \frac{\alpha}{2}\|u\|^2 \\ \text{s.t.} \quad \begin{bmatrix} \Delta y = u + \underset{\text{given}}{f} \text{ in } \Omega \\ y = 0 \text{ on } \partial\Omega. \end{bmatrix} \end{bmatrix}$$



$y(x)$

Decision problem

→ To choose the "best" vector $\underline{x}$ of the decision variable over all possible vectors in 'Ω'

→ The one that results in the smallest value of the objective fn

→ This vector is called a minimizer.

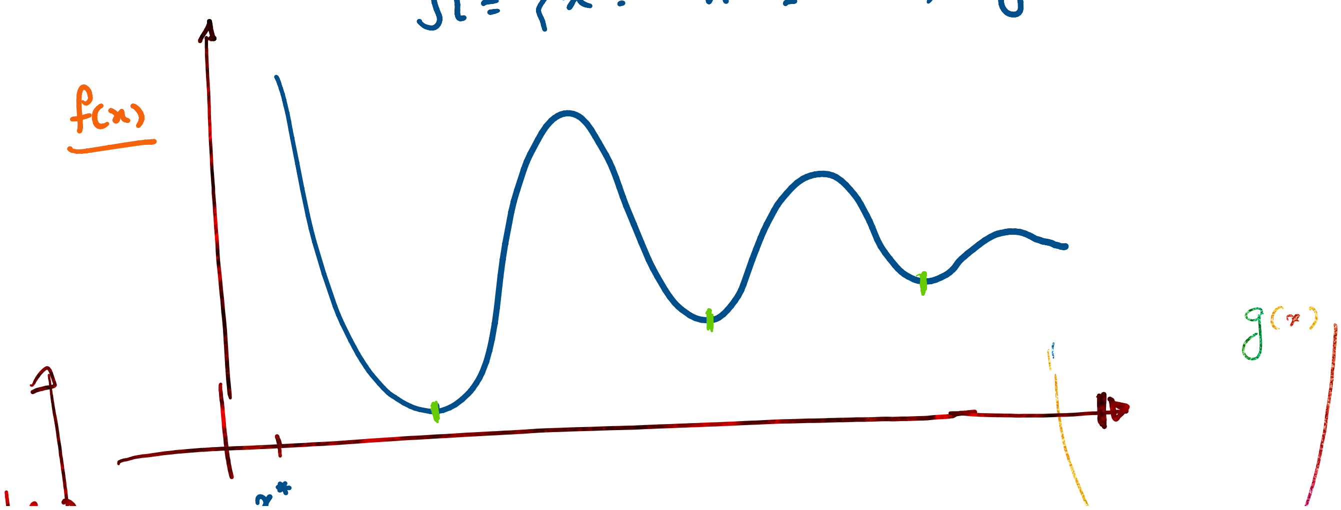
Maximizer → Minimizer for $(-f)$.

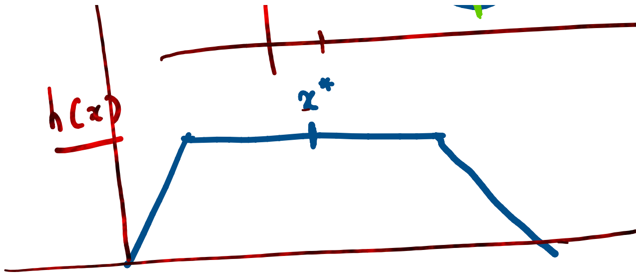
$$\min f(\underline{x}) \quad \text{s.t. } \underline{x} \in \Omega \subseteq \mathbb{R}^n$$

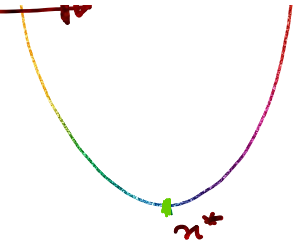
Constrained optimization	Unconstrained optimization
$\Omega \subset \mathbb{R}^n$	$\Omega = \mathbb{R}^n$

Example of a functional constraint

$$\Omega = \{x : h(\underline{x}) = 0, \; g(x) \leq 0\}.$$







Local Minimizer: Let $f : \mathbb{R}^n \to \mathbb{R}$ be a real-valued fn defined on some set $\Omega \subset \mathbb{R}^n$. The point $\underline{x}^* \in \Omega$ is a (strict) local minimizer of f over Ω if $\exists\ \varepsilon > 0$ such that $\boxed{f(\underline{x}) \geq f(\underline{x}^*)}$ $\forall\ \underline{x} \in \Omega \setminus \{\underline{x}^*\}$ and $\|\underline{x} - \underline{x}^*\| < \varepsilon$.

$f(x) > f(\underline{x}^*)$

Global minimizer ... $f(x) \geq f(\underline{x}^*)$ $\forall\ \underline{x} \in \Omega \setminus \{\underline{x}^*\}$.

$$f(\underline{x}^*) = \min_{x \in \Omega} f(\underline{x}) \qquad \underline{x}^* = \arg\min_{\underline{x} \in \Omega} f(\underline{x})$$

Unconstrained case: $\underline{x}^* = \arg\min_{\underline{x}} f(x)$

Examples: $f(x) = (x+1)^2 + 3$

$\Omega = \mathbb{R}$ (Unconstrained)

$$\min_x f(x) = 3$$

$$\arg\min_x f(x) = -1$$

$\Omega : x \geq 0$ (Constrained)

$$\min_{x \in \Omega} f(x) = 4$$

$$\arg\min_{x \in \Omega} f(x) = 0.$$

$\arg\min_x f(x)$ $\quad x \in \Omega$

Conditions for local minimizers

$\underline{x} = (x_1, \ldots, x_n)$; $f : \mathbb{R}^n \to \mathbb{R}$

Recall : $\boxed{1}$ $Df(\underline{x}) = \left[\frac{\partial f}{\partial x_1}(\underline{x}), \frac{\partial f}{\partial x_2}(\underline{x}), \ldots, \frac{\partial f}{\partial x_n}(\underline{x}) \right]$

I derivative

$$\nabla f(\underline{x}) = [Df(\underline{x})]^t = \begin{bmatrix} \frac{\partial f}{\partial x_1}(\underline{x}) \\ \vdots \\ \frac{\partial f}{\partial x_n}(\underline{x}) \end{bmatrix}$$

$\boxed{2}$ Second derivative

$$F(\underline{x}) = D^2 f(\underline{x})$$

$\boxed{\text{Hessian}}$

$$= \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2}(\underline{x}) & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_1}(\underline{x}) \\ \frac{\partial^2 f}{\partial x_1 \partial x_2}(\underline{x}) & \cdots & \vdots \\ \vdots & \cdots & \frac{\partial^2 f}{\partial x_n^2}(\underline{x}) \end{bmatrix}$$

Example $\quad f(x_1, x_2) = x_1^2 + 2x_2^2 - x_1 x_2 + 5x_1 - 8x_2$

Compute Df, $D^2 f$.

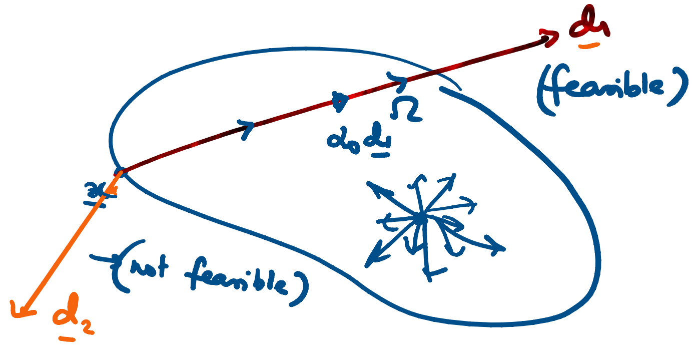
$$Df = \left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2} \right] = \left[\overline{2x_1 - x_2 + 5}, \ \overline{4x_2 - x_1 - 8} \right]$$

$$D^2 f = \begin{bmatrix} 2 & -1 \\ -1 & 4 \end{bmatrix}$$

$\boxed{3}$ Given an optimization problem with a constraint set Ω, minimizer may lie in the interior or boundary of Ω.
To study the case of finding minimizers, we introduce "<u>feasible directions</u>"

To study

we introduce "feasible directions"



Defn A vector $\underline{d} \in \mathbb{R}^n$, ($\underline{d} \neq 0$) is called a feasible direction at $\underline{x} \in \Omega$ if $\exists\ \alpha_0 > 0$ s.t.

$$\underline{x} + \alpha \underline{d} \in \Omega \quad \forall\ \alpha \in [0, \alpha_0].$$