Q1. WTS: $f(x^{(k)} + \alpha d^{(k)}) < f(x^{(k)})$ for all $\alpha \in (0, \bar{\alpha}]$, for some $\bar{\alpha} > 0$.

$$\phi(\alpha) = f(x^{(k)} + \alpha d^{(k)})$$

$$\phi'(\alpha) = \nabla f(x^{(k)} + \alpha d^{(k)})^T d^{(k)} \quad \text{[Chain Rule]}$$

$$\therefore \phi'(0) = \nabla f(x^{(k)})^T d^{(k)} = -\underbrace{g^{(k)T} F(x^{(k)})^{-1} g^{(k)}} < 0$$

because $F(x^{(k)}) > 0$ and inverse of a p.d. matrix is p.d.

$\therefore \exists\ \bar{\alpha} > 0$ s.t. $\phi(\alpha) < \phi(0)$ for all $\alpha \in (0, \bar{\alpha}]$.

Q2. WTS: In the conjugate gradient algorithm,

$$g^{(k+1)^T} d^{(i)} = 0$$

for all $0 \leq k \leq n-1$, $0 \leq i \leq k$. Let's assume that Q is SPD.

Hint: Use induction.

Let's show this for $k=0$, $i=0$.

$$g^{(1)^T} d^{(0)} \overset{?}{=} 0$$

$$g^{(1)T} = (Qx^{(1)} - \underset{\sim}{b})^T$$

$$f(x) = \frac{1}{2} x^T Q x - b^T x + c$$

$$Q = Q^T > 0$$

$$g^{(1)T} d^{(0)} = (Qx^{(1)} - b)^T d^{(0)}$$

$$= x^{(1)T} Q d^{(0)} - b^T d^{(0)}$$

$$\left[x^{(1)} = x^{(0)} + \alpha_0 d^{(0)}, \quad \text{where } \alpha_0 = - \frac{g^{(0)T} d^{(0)}}{d^{(0)T} Q d^{(0)}} \right]$$

$$= \underbrace{x^{(0)T} Q d^{(0)}} - \frac{g^{(0)T} d^{(0)}}{\cancel{d^{(0)T} Q d^{(0)}}} \cancel{d^{(0)T} Q d^{(0)}} \underbrace{- b^T d^{(0)}}$$

$$= (Qx^{(0)} - b)^T d^{(0)} - g^{(0)T} d^{(0)}$$

$$= g^{(0)T} d^{(0)} - g^{(0)T} d^{(0)} = 0.$$

Consider $Q(x^{(k+1)} - x^{(k)}) = Qx^{(k+1)} - b - (Qx^{(k)} - b)$

$$= g^{(k+1)} - g^{(k)}.$$

$$g^{(k+1)} = g^{(k)} + \alpha_k Q d^{(k)}. \qquad 0 \le i < k$$

$$g^{(k+1)^T} d^{(k)} = (Qx^{(k+1)} - b)^T d^{(k)}$$

$$= \left(x^{(k)} - \left(\frac{g^{(k)T} d^{(k)}}{d^{(k)T} Q d^{(k)}}\right) d^{(k)}\right)^T Q d^{(k)}$$

$$- b^T d^{(k)}$$

$$= \underbrace{x^{(k)T} Q d^{(k)}} - g^{(k)T} d^{(k)} \underbrace{- b^T d^{(k)}}$$

$$= (Qx^{(k)} - b)^T d^{(k)} - g^{(k)T} d^{(k)} = 0$$

Q3. $f(x_1, x_2) = \frac{5}{2}x_1^2 + x_2^2 - 3x_1x_2 - x_2 - 7$

$$= \frac{1}{2}\left[5x_1^2 + 2x_2^2 - 6x_1x_2\right] - x_2 - 7$$

$$Q = \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}$$

$Q = Q^T > 0$ (the Sylvester's criterion)

$$b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$c = -7$$

$$\boxed{x^{(0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}}$$

$$\nabla f(x^{(0)}) = Qx^{(0)} - b = \begin{bmatrix} 0 \\ -1 \end{bmatrix} = d^{(0)}$$

$$\alpha_0 = -\frac{g^{(0)T} d^{(0)}}{d^{(0)T} Q d^{(0)}} = \frac{-1}{2}$$

$$x^{(1)} = x^{(0)} + \alpha_0 d^{(0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \left(-\frac{1}{2}\right) \cdot \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}.$$

$$g^{(1)} = Qx^{(1)} - b = \begin{bmatrix} -3/2 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -3/2 \\ 0 \end{bmatrix}$$

$$\beta_1 = \frac{g^{(1)T} g^{(1)}}{g^{(0)T} g_0^{(0)}} = \frac{9/4}{1} = \frac{9}{4}$$

or β_0

$$d^{(1)} = g^{(1)} + \beta^{(1)} d^{(0)} = \begin{bmatrix} -3/2 \\ 0 \end{bmatrix} + \frac{9}{4}\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -3/2 \\ -9/4 \end{bmatrix}$$

Verify:

$$d^{(0)T} Q \, d^{(1)} = \begin{bmatrix} 0 & -1 \end{bmatrix} \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} -3/2 \\ -9/4 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -2 \end{bmatrix} \begin{bmatrix} -3/2 \\ -9/4 \end{bmatrix}$$

$$= \frac{-9}{2} + \frac{9}{2} = 0. \checkmark$$