1. Consider $X \sim \text{Poisson}(\lambda)$ and $P(Y | X = l) = \text{Binomial}(l, p)$ where $0 \le p \le 1$ and $\lambda > 0$. Show that $Y \sim \text{Poisson}(\xi)$ where ξ is a function of λ and p. This process is called as the thinning of a poisson distribution by a binomial distribution. It has realistic applications in image processing.

Solution:

$$P(Y=k) = \sum_{l=k}^{\infty} P(Y=k | X=l)\, P(X=l)$$

$$= \sum_{l=k}^{\infty} \binom{l}{k} p^k (1-p)^{l-k} \frac{\lambda^l}{l!} e^{-\lambda}$$

$$= \frac{e^{-\lambda}}{k!} p^k \lambda^k \sum_{l=k}^{\infty} \frac{l!}{(l-k)!} (1-p)^{l-k} \lambda^{l-k}$$

$$= \frac{e^{-\lambda}}{k!} (\lambda p)^k \sum_{l=k}^{\infty} \frac{(\lambda(1-p))^{l-k}}{(l-k)!}$$

$$= \frac{e^{-\lambda}}{k!} (\lambda p)^k \sum_{l=0}^{\infty} \frac{(\lambda(1-p))^{l}}{l!}$$

$$= \frac{(\lambda p)^k}{k!} e^{-\lambda} e^{\lambda(1-p)} = \frac{(\lambda p)^k}{k!} e^{-\lambda p}$$

$$= \text{Poisson}(\lambda p)$$

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