CS207 Graphs

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1 Definition

A simple graph $G := (V, E)$ represents a set of vertices V and edges E, where,

$$E \subseteq \{\{a, b\} | a, b \in V; a \neq b\}$$

A simple graph can be considered a symmetric(since undirected edges) and irreflexive(no self-loops).
A non-simple graph can have multiple edges between vertices, and can even have weights associated with edges.

2 Examples

2.1 Complete graph(K_n)

A graph with n vertices in which there is an edge between every possible pair of vertices. Mathematically,

$$E = \{\{a, b\} | a, b \in V; a \neq b\}$$

2.2 Cycle graph(C_n)

A graph with n vertices $V = \{v_1, \dots v_n\}$, in which the edge set is given as

$$E = \{\{v_i, v_{i+1}\} | i = [n - 1]\} \cup \{\{v_n, v_1\}\}$$

2.3 Bipartite graph

A graph is called bipartite graph if V can be partitioned into V_1 and V_2 such that $V_1 \cap V_2 = \Phi$ and there does not exist any edge between vertices of same set. Mathematically,

$$E \subseteq \{\{a, b\} | a \in V_1; b \in V_2\}$$

2.4 Complete Bipartite graph

A graph is called bipartite graph if V can be partitioned into V_1 and V_2(both non-empty) such that $V_1 \cap V_2 = \Phi$ and edge set is given by

$$E = \{\{a, b\} | a \in V_1; b \in V_2\}$$

2.5 Isomorphic graphs

Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are said to be isomorphic if there exists a bijection $f : V_1 \to V_2$ such that

$$\{u, v\} \in E_1 \iff \{f(u), f(v)\} \in E_2$$

Finding whether given two graphs are isomorphic or not is a computationally hard problem.

2.6 Subgraphs

A subgraph of a given graph $G = (V, E)$ is a graph $G' = (V', E')$ such that

$$V' \subseteq V; E' \subseteq E$$

An induced subgraph is the one in which all the possible edges between vertices i V' are present in G' (and are also present in E).

3 Walks and Path

3.1 Walk

A walk of length k from vertex a to vertex b is a ordered set of $k+1$ vertices $(v_0, \dots v_k)$ such that

- $v_0 = a$ and $v_k = b$
- $\{v_i, v_{i+1}\} \in E \forall i = 0, \dots k-1$

3.2 Path

A walk with **no repeating** vertices.

3.3 Cycle

A walk of length $k \geq 3$ with **only repeating** vertices being the first and the last element of the walk.
A graph G is acyclic if there does not exist a subgraph of G such that it is isomorphic to C_n

3.4 Connectivity

Vertex u is said to be connected to vertex v if there exists a walk from u to v.
Connected(u, v) is an equivalence relation, and the equivalence classes of this relation form connected components of the graph G.

3.5 Degree of a vertex($deg(v)$)

Number of edges incident on v. Mathematically,

$$deg(v) = |\{u | \{u, v\} \in E\}|$$

Important result: $\sum_{v \in V} deg(v) = 2|E|$
Degree sequence: Sorted list of degrees(invariant under isomorphism)
To check if a degree sequence is possible, distribute the highest degree among other degree(and hence subtracting from other degrees at top till the highest degree is exhausted), and then recursively solve the smaller problems.

3.6 Eulerian trail

A walk visiting every edge **exactly** once.
Property: Eulerian trail exists $\implies$ At most two odd degree vertices
Proof: There must be an edge to leave a particular vertex for every edge used for walking into the vertex, except for the extreme vertices.

Eulerian circuit: A **closed** walk visiting every edge **exactly** once.
Property: Eulerian trail exists $\implies$ No odd degree vertices Proof: There must be an edge to leave a particular vertex for every edge used for walking into the vertex.

Property: If a connected graph has no odd degree nodes then there exists an Eulerian circuit.
Proof: Given a graph G is connected and has no odd degree vertices, the graph must be cyclic(start from a walk from a vertex and keep visiting vertices until a repeating vertex is found). Them remove the edges of the cycle found to form multiple connected components, where each of the connected components would have no odd degree vertices. Then inductively carry out the above

process to find smaller Eulerian circuits and finally stitch them to get Eulerian circuit of original graph.

3.7 Hamiltonian cycle

A **cycle** that has all the vertices of a graph.
No efficient algorithm to compute Hamiltonian cycle exists.

3.8 Distance

Length of the shortest walk between two vertices.
A shortest walk between two vertices is always a path. (proof by contradiction)
Diameter is the largest distance over all the pairs of vertices of the graph.

4 Graph coloring

A coloring is a mapping from the vertices to a set of colors.(usually term coloring is used for proper coloring)
A perfect coloring(of k colors) is the one in which no two vertices sharing an edge has the same color. Mathematically,

$$u, v \in E \implies c(u) \neq c(v)$$

Chromatic number($\chi(G)$): Least k such that a proper coloring of k colors exist. Eg, $\chi(K_n) = n$

$$\text{G has k-coloring} \iff \chi(G) \leq k$$

Results:

- If H is a subgraph of G, then $\chi(G) \geq \chi(H)$
- Isomorphism preserves χ

Efficient algorithms for coloring a graph is known but finding chromatic number of a graph is believed to be NP-hard problem.
Claim: $\forall n \geq 1$, C_{2n+1} is not bipartite
Proof: By contradiction,
If bipartite, start coloring the vertices with two colors clockwise, leading to contradiction.

Claim: A graph is bipartite (with $|V| > 1$) $\iff$ it has no odd length cycles.
Proof: Forward implication is easy to prove. For reverse implication, that is, no odd length cycle implies bipartite,
Let's prove the contrapositive, if not bipartite then odd length cycle exists.
Now, not bipartite $\implies$ there exists a connected component C which is not bipartite
Consider a vertex v in C and partition C such that vertices with even distance are in one component and with odd distances are in another component. There must be an edge joining vertices within these component since the graph is not bipartite. But this would result in odd length cycle.

Claim: $\chi(G) = n \iff G$ **is isomorphic to** K_n Proof: Reverse implication is simple to prove. For forward implication we have to prove,
If $\chi(G) = n$, then G is isomorphic to K_n
Suppose there exists a pair $\{u, v\}$ which does not have an edge. Then give same color to the two vertices and different $n - 2$ colors for remaining vertices. This is a proper coloring with $n - 1$ colors, hence contradiction.

4.1 Clique number($\omega(G)$)

It is the largest natural number c such that G has a subgraph isomorphic to K_c

4.2 Independence number($\alpha(G)$)

Largest m such that G has a set of m vertices with no edges between them.
Given a proper coloring, the vertices with same color form a independent set.
Important Results:

- $n = \sum_c$number of vertices with color $c \leq \alpha(G)\chi(G)$
- $\chi(G) \leq maxdeg(G) + 1$
  Proof: Induction on number of vertices. Take any arbitrary graph and remove a vertices and corresponding edges. Color the graph using $\chi(G') \leq maxdeg(G') + 1 \leq maxdeg(G) + 1$. For the removed vertex, there would exists atleast one color different from the $deg(v) \leq maxdeg(G)$ vertices connected to it.
  Fact: Equality holds only for K_n and C_{2n+1}

5 Other families of graph

5.1 Path graph(P_n)

$V = [n]$ and $E = \{\{i, i+1\}|i \in [n-1]\}$
Bipartite graph

5.2 Wheel graph(W_n)

Cycle graph(C_n) with additional vertex and an edge between the vertex and every vertex of C_n

5.3 Ladder graph(L_n)

$V = \{0, 1\} \times [n]$
$E = \{\{(0, i), (1, i)\}|i \in [n]\} \cup \{\{(b, i), (b, i+1)\}|b \in \{0, 1\}; i \in [n-1]\}$
Bipartite graph

5.4 Cyclic Ladder(CL_n)

Two additional edges to L_n :{(0,1),(0,n)} and {(1,1),(1,n)}

5.5 Hypercube graph(Q_n)

$V =$ all possible n-bit strings (2^n vertices)
$E = \{\{u, v\}|$u and v differ in only one bit}
Diameter of the graph $= n$
Bipartite graph

5.6 $\overline{KG}_{n,k}$

Consider a graph Q_n and let the vertices(which are represented as n-bit strings) be represented as sets storing the positions at which 1 occurs. Let the vertices with same size,k , of such sets be V and let there be an edge between them if there is non-empty intersection of their corresponding sets.

The resulting graph is $\overline{KG}_{n,k}$.
A clique(completely connected component) in the above graph can have a maximum of ${}^{n-1}C_{k-1}$ vertices (having only one element common between each pair), if $k \leq n/2$(Erdos-ko-Rado Theorem).

5.7 Kneser Graph

Complement graph of $\overline{KG}_{n,k}$.(Complement of a graph: Interchange edges and non-edges)

6 Graph operations

Intersection/union of two graphs is component wise(for vertices and edges) intersection/union.

6.1 Powering

Given $G = (V, E)$. Then $G^2 = (V, E')$ is defined as

$$E' = \{\{u, v\} | \exists w \quad s.t. \ \{u, w\} \in E; \{v, w\} \in E\}$$

Moe generally, G^k has edge $\{x, y\} \iff$ There exists a path of length k in G between x and y.

6.2 Cross product

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. $G_1 \times G_2 = (V, E)$ where,

$$E = \{\{(u_1, u_2), (v_1, v_2)\} | \{u_1, v_1\} \in E_1 \wedge \{u_2, v_2\} \in E_2\}$$

6.3 Box product

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. $G_1 \Box G_2 = (V, E)$ where,

$$E = \{\{(u_1, u_2), (v_1, v_2)\} | (\{u_1, v_1\} \in E_1 \wedge u_2 = v_2) \vee (\{u_2, v_2\} \in E_2 \wedge u_1 = v_1)\}$$

Eg, $Q_m \Box Q_n = Q_{m+n}$

6.4 Hamming graph($H_{n,q}$)

Generalization of hypercubes.
$H_{n,q} = K_q \Box \ldots \Box K_q$ (n times K_q)
For $q = 2$, Hamming graphs are same as hypercube graphs.

7 Matching

A matching, M, is a subset of E having no common vertices between any two pairs of edges. Mathematically,

$$M \subseteq \ s.t. \quad \forall e_1, e_2 \in M, e_1 \neq e_2 \implies e_1 \cap e_2 = \Phi$$

Perfect matching: Matching with all vertices covered.
Efficient algorithms exist for finding maximum matching.

7.1 Matching in bipartite graphs

Given a bipartite graph, $G = (X, Y, E)$, a complete matching M from X to Y is a matching having $|M| = |X|$.
If$|X| = |Y|$, a complete matching from X to Y is also a complete matching from Y to X, and the matching is also a perfect matching.

7.2 Neighbourhood

Neighbourhood of a set of vertices is given by,

$$\Gamma(S) = \bigcup_{v \in S} \{u | \{u, v\} \in E\}$$

For a bipartite graph $G = (X, Y, E)$, let $S \subseteq X$

- S is shrinking if $|\Gamma(S)| < |S|$
- Let $B \subseteq Y$. Then S is shrinking in B if $|\Gamma(S) \cap B| < |S|$

7.3 Hall's Theorem

A bipartite graph $G = (X, Y, E)$ has a complete matching from X to Y **iff** no subset of X is shrinking.

Proof: Forward implication is simple to proof. For reverse implication, we need to prove
No shrinking $S \subseteq X \implies$ a complete matching from X to Y exists
Proof by induction. True for $|X| = 1$. Assume the claim for $|X| = k$
Induction step: Consider any arbitrary bipartite graph, with non-shrinking X, and $|X| = k + 1$. Consider any vertex,u , in X and find the its neighbour v in Y. If $\exists$ a complete matching from $X - \{u\}$ to $Y\{y\}$ then we have found a complete matching from X to Y(by including the removed vertices). However if not, then consider the shrinking subset,S of $X - \{u\}$ and complete matching into $\Gamma(S)$ can be found by induction hypothesis. Further it can be proved that there cannot be a shrinking subset of $X - S$ in $Y - \Gamma(S)$. This way a matching for the original graph is found.

Application: The edge set of a $d-$ regular bipartite graph can be partitioned into d complete matchings. (Proof by induction on d and using Hall's theorem after proving the non-shrinking property to remove a perfect matching from $d = k + 1$ regular graph).

8 Vertex cover

A vertex cover is a set of vertices, $C \subseteq V$ such that any edge has atleast one vertex in C, that is,

$$\forall e \in E \quad e \cup C \neq \Phi$$

Finding the smallest vertex cover for a arbitrary graph is a NP-hard problem.
Important result: For any graph G, $|C| \geq |M|$, $\forall$ vertex cover C and $\forall$ matchings M. (Proof: Any vertex in a vertex cover can correspond to atmost one edge in a matching, and all the edges of the matching must be used up).

8.1 Konig's Theorem

For a bipartite graph, the size of smallest vertex cover is equal to the size of the largest matching.
Proof: We need to prove that for the smallest vertex cover C there exists a matching M, such that $|M| \geq |C|$

Proof: Consider the sets $A = X \cup C$ and $B = Y \cup C$(where C is the smallest vertex set). Find a complete matching from A to $Y - B$ and from B to $X - A$. Hall's theorem can be used and it can be shown that there cannot be a shrinking subset of A, say S, in $Y - B$, because,

$$|C \cup \Gamma(S) - S| = |C| + |\Gamma(S) - B| - |S|(< |C| \text{ if shrinking})$$

(not possible since C is smallest)

8.2 Maximal matching

A maximal matching is a matching, having the property that adding any edge would violate the matching property.
If M is a maximal matching, $\exists$ a vertex cover of size $2|M|$.
This can be used to bound the size of smallest vertex cover C.

$$|M| \leq |C| \leq 2|M|$$

8.3 Independent set

I is an independent set(that is, set of vertices having no edges between them) of a graph $G \iff \overline{I}$ is vertex cover.

Therefore, **size of smallest vertex cover + size of largest independent set** $= n$

9 Trees

A connected acyclic graph is called a tree.
An acyclic graph is called a forest, and each connected component is a tree.
A leaf in a tree is a vertex with degree one.

Claim: Every tree with $|V| \geq 2$, has atleast 2 leaves.
Proof: Consider a maximal path. If the extreme vertices of the path are not leaves then the path would not be maximal(since they cannot be connected to any internal vertex of the path since trees are acyclic).

Claim: Deleting a leaf vertex of a tree and the edge incident on that vertex results in a tree.
Proof: Deleting a vertex cannot induce a cycle. And the vertices in the new tree are connected.

Claim: In a tree, for all pairs of vertices u, v, there is exactly one u-v path.
Proof: By contradiction,
If there exists two distinct paths, then there exists a cycle. But a tree is acyclic.

Claim: Number of edges in a tree $= |V| - 1$
Proof: Can be proved by induction on number of vertices and deleting a leaf in the induction step.

Claim: If a graph is connected and $|E| = |V| - 1$, then the graph is a tree.
Proof: By contradiction,
If a graph is connected and $|E| = |V| - 1$, has a cycle, delete any vertex of the cycle(the graph is still connected), and keep on doing so until no cycles are left. The remaining graph is a tree with $|E| < |V| - 1$(Contradiction)

Claim: In a forest the number of connected components $c = |V| - |E|$
Proof: Sum up edges in each connected component...

Claim: Deleting a degree d vertex from a tree results in d connected components.
Proof: From the above result.

10 Dilworth's Theorem

In a poset, size of any anti-chain $\leq$ size of any chain decomposition(partition of the poset into chains).
Dilworth's theorem states that equality is achieved.

Recall that, in a poset, size of any chain $\leq$ size of any anti-chain decomposition and **Mirsky's theorem** guarantees equality.

Proof sketch: We can prove that there exists an anti-chain atleast as large as a chin decomposition.
Consider a bipartite graph $G = (\{S\} \times \{0\}, \{S\} \times \{0\}, E)$, where

$$E = \{\{(u, 0), (v, 1)\} | u \leq v\}$$

Consider a vertex cover C and a matching M in G.
To get a large antichain, construct B such that it is the first element of all vertices in G(irrespective of the 0 or 1).
Then $A = S - B$ is an antichain, with property $|A| \geq |S| - |C|$
(since $|B| \leq |C|$)
Now using M construct $F = (S, \{\{u, v\} | \{(u, 0), (v, 1)\} \in M\})$.
It can be easily proved that F is a forest and each connected component of F is a **path**.
Number of connected components = Size of a chain decomposition = $|S|$-$|edges(F)|$
Now, using Konig's theorem $\exists$ C and M such that $|C| = |M|$. Hence proved.

10.1 Comparison graph

Given a poset $(S, \leq)$, its comparison graph $G = (S, E)$ is given as,

$$E = \{\{u, v\} | u \leq v; u \neq v\}$$

Important results:

- **Any induced subgraph of G is also a comparison graph.** (because a subset of poset is also a poset)
- **A chain corresponds to a clique(completely connected component) in G and an anti-chain corresponds to independent set in G(equivalently a clique $\overline{G}$).**
- **An anti-chain decomposition corresponds to a coloring in G and a chain decomposition corresponds to coloring of $\overline{G}$.**

10.2 Mirsky's Theorem and Dilworth's Theorem restated

Mirsky's Theorem: For a comparison graph, $\chi(G) = \omega(G)$
Dilworth's Theorem: For a comparison graph, $\chi(\overline{G}) = \omega(\overline{G})$

10.3 Perfect graph

A graph G is perfect if every induced sub graph, G' of G has property: $\chi(G') = \omega(G')$
Comparison graphs are perfect(using Mirsky's theorem).