AIML Assignment 1 200050103

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a) (i) Let ~~α_y~~ α_y be the probability of Liam winning starting from y.
then, (p: prob. of winning a bet, $q = 1-p$, $\beta = \frac{q}{p}$)

$$\alpha_y = p\alpha_{y+1} + q\alpha_{y-1}$$

and we have boundary conditions

$$\left.\begin{aligned}\alpha_0 &= 0\\ \alpha_{X+Y} &= 1\end{aligned}\right\}$$ terminal states of ~~winni~~ losing and winning (by Lia-m) resp.

Solving recursion,

$$\boxed{\alpha_Y = \frac{1-\beta^Y}{1-\beta^{X+Y}}}$$ ← Prob. of Liam winning \$X starting with \$Y

(ii) the objective is to reach X from Y in above, but if we set objective to reach 0 and β to $\frac{1}{\beta}$ (the odds will be inverted), we find P(Liam loses) ($X = Y$ and $Y = X$)

$$P(\text{Loses}) = \frac{1 - \frac{1}{\beta^X}}{1 - \frac{1}{\beta^{X+Y}}} = \frac{\beta^{X+Y} - \beta^Y}{\beta^{X+Y} - 1}$$

(iii) $P(\text{win}) + P(\text{loses}) = \dfrac{\beta^{X+Y} - \beta^{Y} + \beta^{Y} - 1}{\beta^{X+Y} - 1} = 1$

$\therefore P(\text{playing } \infty\text{'ly}) = 0$

there are events where liam plays ∞'ly but chance/prob of these events $= 0$.

(iv) $\text{Expected gains} = \dfrac{1-\beta^{Y}}{1-\beta^{X+Y}} \cdot X + \dfrac{\beta^{Y} - \beta^{X+Y}}{1-\beta^{X+Y}}(-Y)$

$(X=1)\Big| \quad = \dfrac{1-\beta^{Y}}{1-\beta^{1+Y}} - \dfrac{\beta^{Y} - \beta^{1+Y}}{1-\beta^{1+Y}}(Y)$

$= \dfrac{1-\beta^{Y} - \beta^{Y}(1-\beta)Y}{1-\beta^{1+Y}}$

$= \dfrac{1-\beta^{Y}}{1-\beta^{Y+1}}(Y+1) - Y \leq \dfrac{Y+1}{\beta} - Y$

b)(i) if Liam wins in i^{th} toss,

$\Rightarrow$ he has lost $2^{i}-1$ till now

$\Rightarrow$ he wins 2^{i} now

$\Rightarrow$ total gain: \$1

$\Rightarrow$ he can stop playing.

Since he has $2^{Y}-1$, he can play Y tosses max

So liam wins if atleast one of the first Y tosses is head

$P(\text{liam wins}) = 1 - (1-p)^{Y}$

(ii) Expected gain $= (1-(1-p)^y) \times 1 - (1-p)^y (2^y - 1)$

$= 1 - (1-p)^y 2^y$

since $p < 0.5$, Exp. gain is -ve.

2) We win 1.4 times the betting amounts and fraction α of the current balance, then on winning with prob. 0.5 we increase current balance by ~~1+1.4α~~ $1+1.4\alpha$ and on losing with prob 0.5 we decreasing the current balance by $1-\alpha$.

Then the expected geometric growth rate r is (Kelly criterion)

$$r = (1+1.4\alpha)^{0.5}(1-\alpha)^{0.5}$$

$$E = \log r = \frac{1}{2}\left[\log(1+1.4\alpha) + \log(1-\alpha)\right]$$

$$\left.\frac{\partial E}{\partial \alpha}\right|_{\alpha^*} = \frac{1}{2}\left[\frac{1.4}{1+1.4\alpha^*} + \frac{-1}{1-\alpha^*}\right] = \frac{1}{2}\left[\frac{0.4 - 1.4\alpha^* - 1.4\alpha^*}{(1+1.4\alpha^*)(1-\alpha^*)}\right] = 0$$

$$0.4 = 2\times 1.4\alpha^*$$

$$\alpha^* = \frac{1}{7}$$

Ans: always bet $\frac{1}{7}$th of balance to maximise returns in longtime

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1(b) $\{A_n\}$ A_i is a $n \times n$ upper triangular matrix.

for A_n, let the ~~trace~~ diagnal elements be $X_{11}, X_{22}, \ldots, X_{nn}$ which are iid r.vs

$$X_{ii} \sim \text{Ber} \begin{cases} 4 & p = \frac{1}{5} \\ 5 & 1-p = \frac{4}{5} \end{cases}$$

$$E[X_{ii}] = \frac{4}{5} + 4 = \frac{24}{5}$$

(i) $$\frac{tr(A_n)}{n} = \frac{1}{n}\sum_{i=1}^{n} X_{ii}$$

$$\lim_{n\to\infty} \frac{tr(A_n)}{n} = \underbrace{\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} X_{ii} = \bar{X} = E[X]}_{\text{Law of large numbers.}}$$

$$E[X] = \frac{24}{5} = 4.8$$

$$\therefore \boxed{\lim_{n\to\infty} \frac{tr(A_n)}{n} = 4.8}$$

(ii) $$\det(A_n) = \prod_{i=1}^{n} X_{ii}$$

$$\frac{\log(\det(A_n))}{n} = \frac{1}{n}\sum_{i=1}^{n} \log X_i$$

$$\lim_{n\to\infty} \frac{\log(\det(A_n))}{n} = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \log X_i = E[\log(X)]$$

$$= \frac{4}{5}\log 5 + \frac{1}{5}\log 4$$

$$= \frac{1}{5}\left(4(\log 10 - \log 2) + 2\log 2\right)$$

$$= \frac{1}{5}\left(4\log 10 - 2\log 2\right)$$

$$= \frac{2}{5}\left(\log 100/2\right) = \boxed{\frac{2}{5}\log 50}$$

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4)

a) $\forall n \in V$, $\exists\, e(n,v)$, $\sum_{\forall v \in Neigh(n) \cup \{n\}} T(n,v) = 1$ (sum of all prob transitions from one node to neighbours $\sum_{v \in Neigh(n)} = 1$)

b) T

$i \downarrow$ $j \rightarrow$	V_1	V_2	V_3
V_1	1/2	1/2	0
V_2	1/3	1/3	1/3
V_3	0	1/2	1/2

c) $p^{(0)} = e_i$

Let $A = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/3 & 1/3 & 1/3 \\ 0 & 1/2 & 1/2 \end{bmatrix}$ and called the transition matrix.

then

$p^{(t)} = p^{(0)} A^t$, $t \in \mathbb{N}$

$|A| = \frac{1}{2}\left(\frac{1}{6} - \frac{1}{6}\right) - \frac{1}{2}\left(\frac{1}{6}\right) + 0 = \frac{-1}{12}$

As t increases, we must show that $p^{(t)}$ converges irrespective of $i \Rightarrow$ S.T A^t converges as t increases.

Eigenvalues of A

$A = \frac{1}{6}\begin{bmatrix} 3 & 3 & 0 \\ 2 & 2 & 2 \\ 0 & 3 & 3 \end{bmatrix}$

$|A - \lambda I| = 0$

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$\lambda' = 6\lambda$

$$\left|\frac{1}{6}\begin{bmatrix} 3-\lambda' & 3 & 0 \\ 2 & 2-\lambda' & 2 \\ 0 & 3 & 3-\lambda' \end{bmatrix}\right| = 0$$

$$\frac{1}{6^3}\left\{\left[(3-\lambda')\left[\cancel{(2-\lambda')}\ 6-5\lambda'+\lambda'^2-6\right] - 3\cdot\cancel{(6)}(6-2\lambda')\right]\right\} = 0$$

$\lambda' - 3$

$\lambda'(\cancel{3-\lambda'})(\lambda'-5) + 18 \cancel{\quad} - 6\lambda'$

1 2 3 3

$\lambda' = 6, 3, -1$

$\lambda = 1, \frac{1}{2}, -\frac{1}{6}$

6 3 $\cancel{5}$

A)

c) probalistic position of particle at t+1, $p^{(t+1)}$ can be found by multiplying the prob. pos matrix at t: $p^{(t)}$ by the transition matrix A

$$p^{(t+1)} = p^{(t)}T = p^{(0)}A^{t}$$

the eigenvalues of A are $1, \frac{1}{2}, -\frac{1}{6}$

we can write $A = Q\Lambda Q^{-1}$; also $A^{t} = Q\Lambda^{t}Q^{-1}$. at $t \gg 1$, only one term will survive in Λ, which is 1 which proves A^{t} converges to a fixed value A^{*} as $t \to \infty$ and thus

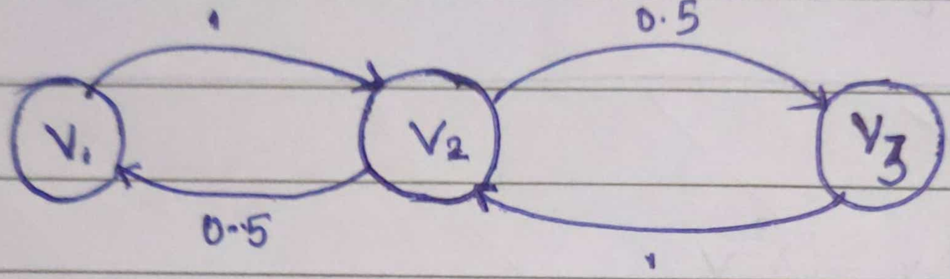
$$p^{(t)} \to p^{*} = p^{(0)}A^{*},$$

Using eigenvalue decomp,

$$A^{*} = \begin{bmatrix} 2/7 & 3/7 & 2/7 \\ 2/7 & 3/7 & 2/7 \\ 2/7 & 3/7 & 2/7 \end{bmatrix}$$

thus $p^{*} = [2/7 \quad 3/7 \quad 2/7]$ irrespective of $p^{(0)}$.

d)



After long time, the position oscillates btw V_2 and (V_1 or V_3) with equal probability.

5)

4) $y = ax + \epsilon \qquad \epsilon \sim \mathcal{N}(0, \sigma^2)$

$\rightarrow y \sim \mathcal{N}(ax, \sigma^2)$

a) y is a Gaussian, so the loss function used is the MSE loss

$$L = \frac{1}{n} \sum_{i=1}^{n} (y_i - ax_i)^2$$

(can be derived from MLE of Gaussian)

b) MLE estimate of a

$$\underset{a}{\arg\max} \left[P(y_1, y_2, \dots y_n \mid x_1, x_2, \dots x_n) \right]$$

$$= \underset{a}{\arg\max} \prod_{i=1}^{n} P(y_i \mid x_i) = \underset{a}{\arg\max} \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(y_i - ax_i)^2}{2\sigma^2}\right)$$

$$= \underset{a}{\arg\max} \left[\sum_{i=1}^{n} -\log(\sigma\sqrt{2\pi}) - \frac{(y_i - ax_i)^2}{2\sigma^2} \right]$$

$$= \underset{a}{\arg\min} \left[\sum_{i=1}^{n} (y_i - ax_i)^2 \right]$$

$$= \underset{a}{\arg\min} \left[\sum_{i=1}^{n} (y_i)^2 + a^2 x_i^2 - 2 y_i a x_i \right]$$

$$= \frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2}$$

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6) (r_i, t_i) $i = \{1, 2, \dots 52\}$

$\mu_r = \bar{r} = 1292.6$ mm

$\sigma_r = 201.0$

$\mu_t = \bar{t} = 57.1$

$\sigma_t = 10.9$

$\rho_{r,t} = 0.722$

a) we assume $\underset{t_{error}}{e} \sim \mathcal{N}(0, \sigma^2)$ is sampled independently. This makes minimizing our least-squares loss function equivalent to MLE estimate

b) $Loss = \frac{1}{N}\sum_{i=1}^{n} (t_i - a - br_i)^2$

$\frac{\partial Loss}{\partial a} = 0$, $\frac{\partial Loss}{\partial b} = 0$

$\frac{1}{N}\sum_{i=1}^{n} 2(t_i - a - br_i) = 0$

$\mu_t - a - b\mu_r = 0$

$\boxed{a + b\mu_r = \mu_t}$

$a = \mu_t - b\mu_r$ - a

$\frac{1}{N}\sum_{i=1}^{n} 2r_i(t_i - a - br_i) = 0$

$\frac{\sum_{i=1}^{n} r_i t_i}{N} - \frac{a\sum r_i}{N} - \frac{b\sum r_i^2}{N} = 0$

$\rho_{rt}\sigma_r\sigma_t + \mu_r\mu_t - a\mu_r - (\sigma_r^2 + \mu_r^2) b =$

$\sigma_r^2 = \frac{\sum r_i^2}{N} - \mu_r^2$ - b

$\rho_{rt} = \left(\frac{\sum r_i t_i}{N} - \mu_r\mu_t\right) / \sigma_r\sigma_t$ - c

using a,b,c

$\sigma_r\sigma_t\rho_{rt} + \cancel{\mu_r\mu_t} - \cancel{\mu_r\mu_t} \cancel{+ b\mu_r^2} - \cancel{\mu_r^2 b} - \sigma_r^2 b = 0$

$b = \frac{\rho_{rt}\sigma_t}{\sigma_r} = 0.0392$

$a = \mu_t - b\mu_r = 6.49$

c) Point farthest from the line changes the parameters the most.

Ans: ($r = 2000$, $t = 30$)

d) $a \sim N(30, 6)$

$b \sim N(0, 2)$

$e \sim \exp(\lambda = 2)$

$$P(a,b \mid (t_i, r_i)\ i \in 1 \ldots 32) = P(t_1, t_2, \ldots r_1, r_2, \ldots \mid a,b) \cdot P(a) P(b)$$

$$= \left[\prod_{i=1}^{N} \lambda e^{-\lambda(t_i - a - br_i)}\right] P(a) P(b)$$

$$= \left(\prod_{i=1}^{N} \lambda e^{-\lambda(t_i - a - br_i)}\right) \cdot \frac{1}{\sigma_a\sqrt{2\pi}} \exp\left(-\frac{(a-\mu_a)^2}{2\sigma_a^2}\right) \frac{1}{\sigma_b\sqrt{2\pi}} \exp\left(\frac{-(b-\mu_b)^2}{2\sigma_b^2}\right)$$

$$-\log(MLE) \approx \underbrace{\sum \lambda(t_i - a - br_i) + \frac{(a-\mu_a)^2}{2\sigma_a^2} + \frac{(b-\mu_b)^2}{2\sigma_b^2}}_{\text{Loss fxn.}}$$

(without constants)

to maximise MLE, we minimise neg. log(MLE) ~~to get~~

$\lambda = 2$, $\mu_a = 30$, $\sigma_a = 6$, $\mu_b = 0$, $\sigma_b = 2$.

Also need to ensure $t_i - a - br_i \geq 0$ cause error $(\epsilon) \sim \exp(\lambda) \geq 0$