CS207 Functions

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1 Basics

- A function maps each element in domain to co-domain
- All elements of co-domain may not be used. The elements in co-domain which are used form image set. Mathematically,

$$Im(f) = \{x \in \text{co-domain}(f) | \exists y \in \text{domain}(f)\ f(y) = x\}$$

- Functions can be considered as a heterogeneous relation between domain and co-domain.

$$R_f = \{(x, f(x)) | x \in domain(f)\}$$

- Composition of two functions ($f \circ g$) is a function mapping domain of f to co-domain of g

$$f \circ g(x) \equiv f(g(x))$$

  Defined only if $Im(f) \subseteq domain(g)$

2 Types of functions

- Onto(surjection): Im(f)=co-domain(f) i.e., $\forall x \in$ co-domain$(f) \exists y \in$ domain(f) s.t. $f(y) = x$
- One-to-one(injection): f(x)=f(y) $\implies$ x=y
- Bijection: Surjection+Injection

Invertible function: A function is said to be invertible if $\exists g$ such that $g \circ f =$ Identity
Claim: **Injective functions are invertible.**
Proof:$\forall y \in Im(f)$ let g(y)=x, where f(x)=y
$\forall y \in$ co-domain$(f) - Im(f)$ let g(y)=x (arbitrary) where $x \in$ domain(f)
Note that g may not be invertible in such a mapping.

Claim: **Invertible functions are one-to-one**
Proof: Suppose invertible functions are not one-to-one.
Apply mapping g to $f(x_1) = f(x_2)$, then $x_1 = x_2$. Contradiction.

Inverse of a bijective function is also invertible.

3 Important Properties

Suppose $f : A \to B$ Some properties(if A and B are finite sets):-

- $|A| \leq |Im(f)|$, equality iff f is injective
- $|B| \leq |Im(f)|$, equality iff f is surjective
- If f is onto, then $|A| \geq |B|$
- If f is injective, then $|A| \leq |B|$
- If f is bijective, then $|A| = |B|$

Properties of composition of functions f and g (assume **co-domain(f)=domain(g)**):

- If f and g are onto, then $g \circ f$ is also onto. Converse does not hold. However, if $g \circ f$ is onto, then g is onto.
- If f and g are injective, then $g \circ f$ is injective. Agsin converse does not hold. However, if $g \circ f$ is injective, then f is injective.

Above results may change if the assumption **co-domain(f)=domain(g)** is modified.