The following contains details statements of those results alluded to briefly in the lecture notes. It also contains solutions to each lectures exercises.

Contents

1 Lecture 1 – Exercises

1.1 Exercise 1 (Normal Contractions)

(a) Show that the following maps from $\mathbb{R}$ to $\mathbb{R}$ are normal contractions:

(a) $C_+ : t \mapsto t \vee 0$

(b) $C_- : t \mapsto (-t) \vee 0$

(c) $C_{(-\infty,1]} : t \mapsto t \wedge 1$

(d) $C_{[0,1]} : t \mapsto 0 \vee (t \wedge 1)$

For which $a \leq b$ is $C_{[a,b]} : t \mapsto a \vee (t \wedge b)$ a normal contraction?

(b) Let X be a finite set and let Q be a symmetric bilinear form on $C(X)$. Show that Q is compatible with normal contractions if it is compatible with the map $C_{(-\infty,1]}$.

Solution:

(a) (a) $C_+(0) = 0$.

$$|C_+(s) - C_+(t)| = |(s \vee 0) - (t \vee 0)| = \begin{cases} 0 & \text{if } s,t \leq 0 \\ |s-t| & \text{if } s,t > 0 \\ |s| & \text{if } s \geq 0 > t \end{cases} \leq |s-t|$$

(b) $C_-(0) = 0$.

$$|C_-(s) - C_-(t)| = |(-s) \vee 0 - (-t) \vee 0| = \begin{cases} |t-s| & \text{if } s,t \leq 0 \\ 0 & \text{if } s,t > 0 \\ |t| & \text{if } s \geq 0 > t \end{cases} \leq |s-t|$$

(c) $C_{(-\infty,1]}(0) = 0$.

$$|C_{(\infty,1]}(s) - C_{(\infty,1]}(t)| = |s \wedge 1 - t \wedge 1| = \begin{cases} |s-t| & \text{if } s,t \leq 1 \\ 0 & \text{if } s,t > 1 \\ |s-1| & \text{if } s \leq 1 < t \end{cases} \leq |s-t|$$

(d) $C_{[0,1]}(0) = 0$.

$$\begin{aligned}|C_{[0,1]}(s) - C_{[0,1]}(t)| &= |0 \vee (s \wedge 1) - 0 \vee (t \wedge 1)| \\ &= \begin{cases} |s-t| & \text{if } s,t \in [0,1] \\ 0 & \text{if } s,t \notin [0,1] \\ |s-1| & \text{if } s \in [0,1] \text{ and } t \notin [0,1] \end{cases} \qquad \leq |s-t|\end{aligned}$$

To determine a, b notice if $b < 0$ then

$$C_{[a,b]}(0) = a \vee (0 \wedge b) = a \wedge b \neq 0$$

So we must have $b \geq 0$ in which case,

$$C_{[a,b]}(0) = a \vee 0 = 0 \iff a \leq 0$$

Now to ensure the contraction property,

$$\begin{aligned}|C_{[a,b]}(s) - C_{[a,b]}(t)| &= |a \vee (s \wedge b) - a \vee (t \wedge b)| \\ &= \begin{cases} |s-t| & \text{if } s,t \in [a,b] \\ 0 & \text{if } s,t \notin [a,b] \\ |s-b| & \text{if } s \in [a,b] \text{ and } t \notin [a,b] \end{cases} \qquad \leq |s-t|\end{aligned}$$

Therefore if $a \leq 0 \leq b$ then $C_{[a,b]}$ is a normal contraction.

1.2 Exercise 2 (First Beurling-Deny criterion)

Let X be a finite set and let Q be a symmetric bilinear form over X. For any $f \in C(X)$, let $f_+ = f \vee 0$ bet the *positive part* and let $f_- = (-f) \vee 0$ be the *negative part* of f.

Show the following equivalence:

(i) $Q(|f|) \leq Q(f)$ for all $f \in C(X)$.

(ii) $Q(f_+, f_-) \leq 0$ for all $f \in C(X)$.

(iii) $Q(f \vee g) + Q(f \wedge g) \leq Q(f) + Q(g)$ for all $f, g \in C(X)$.

and for Q positive show that this is also equivalent to:

(iv) $Q(f_+) \leq Q(f)$ for all $f \in C(X)$.

Solution:

(i) $\Longrightarrow$ (ii) By properties of symmetric forms,

$$\begin{aligned}Q(f_=) + 2Q(f_+, f_-) + Q(f_-) &= Q(|f|)\\ &\le Q(f) = Q(f_+) - 2Q(f_+, f_-) + Q(f_-)\end{aligned}$$

Then rearranging gives,

$$Q(f_+, f_-) \le 0 \quad \forall f \in C(X)$$

(ii) $\Longrightarrow$ (i) By properties of symmetric forms,

$$\begin{aligned}Q(f) &= Q(f_+) - 2Q(f_+, f_-) + Q(f_-)\\ &\ge Q(f_+) + 2Q(f_+, f_-) + Q(f_-) = Q(|f|) \quad \forall f \in C(X)\end{aligned}$$

(i) $\Longrightarrow$ (iii) We can write $f \vee g = \frac{1}{2}(f+g) + \frac{1}{2}|f-g|$ and $f \wedge g = \frac{1}{2}(f+g) - \frac{1}{2}|f-g|$. Then,

$$\begin{aligned}Q(f \vee g) &= \frac{1}{4}Q(f+g) + \frac{1}{2}Q(f+g, |f-g|) + \frac{1}{4}Q(|f-g|)\\ &\le \frac{1}{4}Q(f+g) + \frac{1}{2}Q(f+g, |f-g|) + \frac{1}{4}Q(f-g)\\ &= \frac{1}{4}Q(f) + \frac{1}{2}Q(f,g) + \frac{1}{4}Q(g) + \frac{1}{2}Q(f+g, |f-g|) + \frac{1}{4}Q(f) - \frac{1}{2}Q(f,g) + \frac{1}{2}Q(g)\\ &= \frac{1}{2}Q(f) + \frac{1}{2}Q(g) + \frac{1}{2}Q(f+g, |f-g|)\end{aligned}$$

By the same method,

$$Q(f \wedge g) \le \frac{1}{2}Q(f) + \frac{1}{2}Q(g) - \frac{1}{2}Q(f+g, |f-g|)$$

and so,

$$Q(f \vee g) + Q(f \wedge g) \le Q(f) + Q(g)$$

(iii) $\Longrightarrow$ (i) By the same calculations as above,

$$\frac{1}{2}Q(f+g) + \frac{1}{2}Q(|f-g|) \le Q(f) + Q(g)$$

rearranging and expanding,

$$\begin{aligned}\frac{1}{2}Q(|f-g|) &\le Q(f) + Q(g) - \frac{1}{2}Q(f) - \frac{1}{2}Q(f,g) - \frac{1}{2}Q(g)\\ &= \frac{1}{2}Q(f-g)\end{aligned}$$

Choosing $g = 0$ gives the result.

(i) $\Longleftrightarrow$ (iv) We work with the assumption that Q is positive now. If (ii) holds then,

$$Q(f^+) \le Q(f^+) - 2Q(f^+, f^-) + Q(f^-) = Q(f)$$

If (iv) holds then,

$$0 \le Q(f^-) - 2Q(f^+, f^-)$$

1.3 Exercise 3 (Harmonic functions and connected components)

Let (b, c) be a graph over a finite set measure space (X, m) with associated Laplacian $L = L_{b,c,m}$ and let,

$$H = (f \in C(X) \colon Lf = 0)$$

be the subspace of harmonic functions. Show that $\dim H$ is equal to the number of connected components of (b, c) on which c vanished.

Solution: On any connected component where c vanishes we have that any constant function (only on the connected component and zero everywhere else) is harmonic,

$$L\mathbb{1}(x) = \frac{1}{m(x)} \sum_{y \in X} b(x, y)(\mathbb{1}(x) - \mathbb{1}(y)) = 0$$

Let $n \in \mathbb{N}$ denote the number of connected components of (b, c). The constant functions $\mathbb{1}$ on connected components where c vanishes form a linearly independent set in $C(X)$ and so V. Hence $\dim H \ge n$.

If $f \in H$ let us consider it over a connected component where c vanishes. Then,

$$(Lf)(x) = \frac{1}{m(x)} \sum_{y \in X} b(x, y)(f(x) - f(y)) = 0$$

for every x in the connected component. In particular, if we take the cut-off of f to any connected component where c vanishes then we have a collection of linearly independent functions. The number of such functions would clearly equal the number of connected components where c vanishes.

1.4 Exercise 4 (Poisson equation for $\alpha = 0$)

Let b be a graph over a finite set measure space (X, m) (that is $c = 0$) and let $L = L_{b,0,m}$ be the associated Laplacian. Furthermore let,

$$V := \left\{ f \in C(X) \colon \sum_{x \in X} f(x)m(x) = 0 \right\}$$

Show that for each $f \in V$, there is a unique functions $u \in V$ such that,

$$Lu = f$$

Hint: Observe that for the scalar product in $\ell^2(X, m)$, we have for all $f \in \ell^2(X, m)$,

$$\sum_{x \in X} f(x)m(x) = \langle f, 1 \rangle$$

2 Lecture 2 – Exercises

2.1 Exercise 1 (Positivity improvement of the inverse operator)

Let X be a finite set and let L be an injective operator on $C(X)$. Show that the following assertions are equivalent:

(i) The inverse operator L^{-1} is positivity improving, i.e. for all $f \in C(X)$ such that $f \geq 0 (\neq 0)$ we have $L^{-1} f > 0$.

(ii) For each function $u \in C(X)$ satisfying the inequalities $\max_X u(x) \geq 0$ and $Lu \leq 0$, we have $u \equiv 0$.

Solution:

(i) $\Longrightarrow$ (ii) Suppose $u \in C(X)$ such that $\max_X u(x) \geq 0$ and $Lu \leq 0$. Set $v := Lu$ then by assumption $L^{-1} v < 0 \implies u < 0$ if $v \neq 0$. Clearly this is not possible as $\max_X u(x) \geq 0$. Hence, we must have that $v = 0 \implies u \equiv 0$.

(ii) $\Longrightarrow$ (i) For any $f \geq 0 (\neq 0)$ we have by the injectivity of L that there exists a unique g such that $L(-g) = -f \leq 0$. If $\max_X -g(x) \geq 0$ then by assumption $g \equiv 0$. However this is not possible since $f \neq 0$. Therefore, we must have $-g < 0 \implies g > 0$. Applying L^{-1} we then have $L^{-1} f = g > 0$.

2.2 Exercise 2 (Cauchy problem / Heat equation)

Let (X, m) be a finite measure space and let L be a self-adjoint operator on $\ell^2(X, m)$ and for $t \geq 0$ let e^{-tL} be defined via the spectral calulus.

(a) Show that for all $t \geq 0$,

$$e^{-tL} = \sum_{n=0}^{\infty} \frac{-1}{n!} (-tL)^n$$

In particular, show that the sum is absolutely convergent with respect to the operator norm.

(b) Show that $\{e^{-tL} : t \geq 0\}$, equipped with the composition of operators, is an operator semigroup, i.e., $e^{0L} = I$ and $e^{(t+s)L} = e^{tL} e^{sL}$ for all $t, s \geq 0$ and $t \mapsto e^{-tL} f$ is continuously differentiable at $t = 0$ for all $f \in \ell^2(X, m)$. Moreover, show that (in this finite dimensional case),

$$\frac{d}{dt} e^{-tL} = -L e^{-tL} = -e^{tL} L$$

(c) Show that for all $f \in \ell^2(X,m)$, the function $t \mapsto \varphi_t := e^{-tL}f$ is the unique solution of the equation,

$$\frac{d}{dt}\varphi_t = -L\varphi_t, \quad \varphi_0 = f$$

for all $t \geq 0$.

Solution:

(a) We first make sure the series given exists, that is, show that it is absolutely convergent. Treating the series as a power series we have coefficients $a_n = \frac{(-1)^{n+1}L^n}{n!}$ so,

$$\frac{\left\|\frac{(-1)^{n+2}L^{n+1}}{(n+1)!}\right\|}{\left\|\frac{(-1)^{n+1}L^n}{n!}\right\|} = \frac{\|L\|}{n+1} \to 0 \quad \text{as} \quad n \to \infty$$

since $\ell^2(X,m)$ is finite dimensional then $\|L\| < \infty$. Hence, the radius of convergence of the given series is ∞ and it indeed exists.

Using the definition of e^{-tL} and a Taylor series of the real valued function $e^{-\lambda t}$ we have,

$$\begin{aligned}e^{-tL} &= \sum_{\lambda\in\sigma(L)} e^{-\lambda t}E_\lambda = \sum_{\lambda\in\sigma(L)}\left(\sum_{n=0}^{\infty}\frac{(-\lambda)^n}{n!}t^n\right)E_\lambda \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}(-t)^n\left(\sum_{\lambda\in\sigma(L)}\lambda^n E_\lambda\right) = \sum_{n=0}^{\infty}\frac{1}{n!}(-tL)^n\end{aligned}$$

as,

$$L^2 = \sum_{\lambda\in\sigma(L)}\lambda^2 E_\lambda^2 = \sum_{\lambda\in\sigma(L)}\lambda^2 E_\lambda \implies L^n = \sum_{\lambda\in\sigma(L)}\lambda^n E_\lambda$$

by properties of projections.

(b) Using the spectral calculus definition,

$$e^{0L} = \sum_{\lambda\in\sigma(L)} e^{\lambda 0}E_\lambda = \sum_{\lambda\in\sigma(L)} E_\lambda = I$$

and,

$$\begin{aligned}e^{(t+s)L} &= \sum_{\lambda\in\sigma(L)} e^{\lambda(t+s)}E_\lambda = \sum_{\lambda\in\sigma(L)} e^{\lambda t}e^{\lambda s}E_\lambda^2 \\ &= \sum_{\lambda\in\sigma(L)} e^{\lambda t}E_\lambda \sum_{\mu\in\sigma(L)} e^{\mu s}E_\mu = e^{tL}e^{sL}\end{aligned}$$

for all $t, s \geq 0$. By properties of power series we have that e^{-tL} is analytic and so it is continuously differentiable for all $t \geq 0$, in particular we have term-wise differentiate of the series which gives,

$$\frac{d}{dt}e^{-tL} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} nt^{n-1} L^n = -L \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{(n-1)!} (tL)^{n-1} = -Le^{-tL} = -e^{-tL} L$$

where the last equality follows since the operator is self-adjoint.

(c) This follows immediately from (b). The differential equation comes from,

$$\frac{d}{dt}\varphi_t = \left(\frac{d}{dt}e^{-tL}\right) f = -L\varphi_t$$

by the continuous differentiability of the map $t \mapsto e^{-tL} f$. The initial condition is immediate by the continuity of the map, $\varphi_0 = e^{0L} f = f$.

2.3 Exercise 3 (Stochastic incompleteness)

Let (b, c) be a connected graph over (X, m) and let $L = L_{b,c,m}$ denote the associated Laplacian.

(a) Show that $e^{-tL}1 < 1$ for all $t > 0$ if and only if $c \neq 0$.

(b) Show that if $e^{-tL}1 < 1$ for some $t > 0$, then $e^{-tL}1 < 1$ for all $t > 0$.

Solution:

(a) By Theorem 1.20 in the notes we have that e^{-tL} has the Markov property, that is,

$$0 \leq e^{-tL} f \leq 1 \quad \text{for all} \quad 0 \leq f \leq 1 \text{ and } t \geq 0$$

$(\implies)$ If $c = 0$ then $L1 = 0$ and so $u_t := 1$ is a solution of,

$$\frac{d}{dt}u_t = -Lu_t, \quad u_0 = 1$$

By uniqueness of solutions we then have $e^{-tL}1 = 1$ for all $t > 0$. Hence, by contrapositive statements $e^{-tL}1 < 1 \forall t > 0 \implies c \neq 0$.

($\Longleftarrow$) Suppose $c \neq 0$ then $L1 = c \neq 0$. So the problem,

$$\frac{d}{dt} u_t = L u_t \qquad u_0 = 1$$

cannot have a constant solution. As $u_t := e^{-tL}1$ is a solution to the above problem we must have $e^{-tL}1 \neq 1 \implies e^{-tL}1 < 1$ for all $t > 0$.

(b) Assume there exists $t_0 > 0$ and some $x_0 \in X$ such that,

$$(e^{-t_0 L}1)(x_0) = 1$$

Then $u_t(x_0) := \left(1 - e^{-tL}1\right)(x_0)$ has a minimum at t_0. So,

$$0 = L1(x_0) - Lu_{t_0}(x_0) = \frac{c(x_0)}{m(x_0)} + \sum_{y \sim x} b(x, y) u_{t_0}(y)$$

and we have $c(x_0) = 0$. By the connectedness of the graph we can iterate this argument and hence have $c = 0$. Now by (a) we must have $e^{-tL}1 = 1$ for all $t > 0$ however this is a contradiction to our assumption and so we must have that $e^{-tL}1 < 1$ for all $t > 0$.

3 Lecture 3 – Exercises

3.1 Exercise 1 ($\mathcal{F} = C(X)$)

Let $\mathcal{F}$ be the domain of the formal Lapalacian $\mathcal{L}$ associated to a graph. Show that the following statements are equivalent.

(i) The graph is locally finite.

(ii) $\mathcal{L}C_c(X) \subseteq C_c(X)$

(iii) $C(X) = \mathcal{F}$.

Solution:

(i) $\iff$ (ii) Any $f \in C_c(X)$ has finite support so it is immediate that any $f \in C_c(X)$ can be written as a linear combination of indicator functions. Consider now $\mathbb{1}_x$ then,

$$\begin{aligned}\mathcal{L}\mathbb{1}_x(z) &= \frac{1}{m(z)}\left[\sum_{y\in X} b(z,y)(\mathbb{1}_x(z) - \mathbb{1}_x(y)) + c(z)\mathbb{1}_x(z)\right] \\ &= \begin{cases} -\frac{b(x,z)}{m(z)} & \text{if } z \neq x \\ \mathrm{Deg}(x) & \text{if } z = x \end{cases}\end{aligned}$$

So $\mathcal{L}\mathbb{1}_x = \mathrm{Deg}(x)\mathbb{1}_x - \frac{b(x,\cdot)}{m(\cdot)}$. If $\mathcal{L}C_c(X) \subseteq C_c(X)$ then $\mathcal{L}\mathbb{1}_x$ has finite support. It is clear that $\mathcal{L}\mathbb{1}_x(x) \neq 0$ however if $\mathcal{L}\mathbb{1}_x(z) \neq 0$ for $z \neq x$ then we must have that $b(x,z) \neq 0 \implies z$ is a neighbour of x. Hence the set $\{z : x \sim z\}$ is finite.

If we instead assume the graph is locally finite, then there are only finitely many elements which give $b(x,z) \neq 0$ and so $\mathcal{L}\mathbb{1}_x \in C_c(X)$ as it has finite support. Since any $f \in C_c(X)$ can be written as a linear combination of indicator functions we have $\mathcal{L}f \in C_c(X) \implies \mathcal{L}C_c(X) \subseteq C_c(X)$.

(iii) $\implies$ (i) Since $C(X)$ contains all functions on X. For any $x \in X$ we define the function,

$$f_x(y) := \begin{cases} \frac{1}{b(x,y)} & \text{if } x \sim y \\ 0 & \text{if } x \nsim y \end{cases}$$

and have,

$$\sum_{y\in X} b(x,y) f_x(y) < \infty$$

Clearly we must have that $\{y : x \sim y\}$ is a finite set otherwise the sum above is a series of 1 which diverges. Hence the graph is locally finite.

(i) $\implies$ (iii) If the graph is locally finite then for any $x \in X$,

$$\sum_{y \in X} b(x, y)$$

is a finite sum. So for any $f \in C(X)$ we have that,

$$\sum_{y \in X} b(x, y)|f(y)| < \infty$$

as it is a finite sum. Hence, $C(X) = \mathcal{F}$.

3.2 Exercise 2 (Maximum principle)

Let $\mathcal{A} : C_c(X) \to C_c(X)$ be a symmetric linear operator, i.e., $\mathcal{A}1\!\!1_x(y) = \mathcal{A}1\!\!1_y(x)$ for all $x, y \in X$. Show the following equivalence:

(i) $\mathcal{A} = \mathcal{L}_{b,c}$ on $C_c(X)$ for a locally finite graph (b, c) over X.

(ii) $\mathcal{A}$ satisfies a maximum principle.

(i) $\implies$ (ii) Suppose $f \in C_c(X)$ has a non-negative local maximum at x. Then,

$$\mathcal{A}f(x) = \mathcal{L}_{b,c}f(x) = \frac{1}{m(x)}\left[\sum_{y \in X} b(x, y)(f(x) - f(y)) + c(x)f(x)\right]$$

since $f(x) \geq 0$ and $f(x) \geq f(y)$ for all $y \sim x$ then $\mathcal{A}f(x) \geq 0$.

(ii) $\implies$ (i) Let a be the (infinite) matrix associated to $\mathcal{A}$. By the maximum principle if $-\mathcal{A}1\!\!1_x(y) \geq 0$ for all $y \neq x$. Since $\mathcal{A}1\!\!1_x(y) = a(x, y)$ then $a(x, y) \leq 0$ for all $y \neq x$. Next for the one function we have $\mathcal{A}1\!\!1(x) \geq 0$ for all $x \in X$ and so $\sum_{y \in X} a(x, y) \geq 0$.

3.3 Exercise 3 (Uncountable graphs)

Let X be an arbitrary set and assume that $b : X \times X \to [0, \infty)$ satisfies $b(x, y) = b(y, x)$, $b(x, x) = 0$ and,

$$\sum_{z \in X} b(x, z) = \sup_{U \subseteq X \text{ finite}} \sum_{y \in U} b(x, y) < \infty$$

for all $x \in X$. Call a subset Y and X connected if for arbitrary $x, y \in Y$ there exists $n \in \mathbb{N}$ and $x_0, \ldots, x_n \in Y$ with $x_0 = x$, $x_n = y$ and $b(x_k, x_{k+1}) > 0$ for all $k = 0, \ldots, n-1$. Show that any connected subset of X is connected.

3.4 Exercise 4 (Summability)

Let X be a countable set, $b : X \times X \to [0, \infty)$ and $Q : C(X) \to [0, \infty]$,

$$Q(f) = \frac{1}{2} \sum_{x,y \in X} b(x, y)(f(x) - f(y))^2$$

Show that,

$$Q(\varphi) < \infty$$

for all $\varphi \in C_c(X)$ if and only if,

$$\sum_{y \in X} b(x, y) < \infty$$

for all $x \in X$.

4 Lecture 4 – Exercises

4.1 Exercise 1 (Resolvents are continuous)

Show that the resolvent map of an operator A on a Hilbert space H,

$$\rho(A) \to B(H) \quad z \mapsto (A - z)^{-1}$$

is continuous.

4.2 Exercise 2 (Multiplication operators I)

Let (X, μ) be a measure space and let $u : X \to \mathbb{C}$ be measurable. The operator M_u of multiplcation by u has domain,

$$D(M_u) = \left\{ f \in L^2(X, \mu) \colon uf \in L^2(X, \mu) \right\}$$

and acts as,

$$M_u f = uf$$

for all $f \in D(M_u)$. Show the following statements.

(a) The operator M_u is densely defined.

(b) The operator M_u is closed.

(c) The adjoint of M_u is given by $(M_u)^* = M_{\overline{u}}$. In particular, M_u is self-adjoint if u is real-valued.

(d) The operator M_u is bounded if $u \in L^\infty(X, \mu)$.

Solution:

(a) Consider the sets $X_n := \{x \in X \colon |u(x)| \leq n\}$ for every $n \in \mathbb{N}$. For any $f \in L^2(X, \mu)$ take the sequence $\mathbb{1}_{X_n} f$. Now,

$$|\mathbb{1}_{X_n} f(x)| \leq f(x) \quad \forall n \in \mathbb{N}$$

and $\mathbb{1}_{X_n} f \to f$ a.e then by DCT $\mathbb{1}_{X_n} f \to f$ in $L^2(X, \mu)$. We also have,

$$u \mathbb{1}_{X_n} f \leq n \mathbb{1}_{X_n} f \in L^2(X, \mu) \implies \mathbb{1}_{X_n} f \in D(M_u) \quad \forall n \in \mathbb{N}$$

Therefore, $\overline{D(M_u)} = L^2(X, \mu)$.

(b) Suppose $f_n \in D(M_u)$ such that $f_n \to f$ and $uf_n \to g$ in $L^2(X, \mu)$. We can take a subsequence f_{n_k} such that $f_{n_k} \to f$ a.e and $uf_{n_k} \to g$ in $L^2(X, \mu)$. Taking a further subsequence (which we relabel as $\cdot_{n_k}$) we have $uf_{n_k} \to uf$ a.e and so we must have that $uf = g$ and $f \in D(M_u)$.

(c) For any $f \in D(M_u)$ and $g \in D(M_u^*)$ we have the identity,

$$\langle g, M_u f\rangle = \langle M_u^* g, f\rangle$$

Since this is the L^2 inner product and as it is linear in its second argument we have,

$$\langle g, M_u f\rangle = \int_X \overline{g} u f \, d\mu = \int_X \overline{\overline{u} g} f \, d\mu = \langle \overline{u} g, f\rangle$$

Hence,

$$\langle g, M_u f\rangle = \langle M_{\overline{u}} g, f\rangle$$

for all $f \in D(M_u)$ and $g \in D(M_u^*)$ so $M_u^* = M_{\overline{u}}$. Of course if u is real-valued we have $\overline{u} = u$ and clearly M_u is self-adjoint.

(d) If $u \in L^\infty(X, \mu)$ we can write,

$$\|M_u f\|_2 = \|uf\|_2 \leq \|u\|_\infty \|f\|_2 \quad \forall f \in L^2(X, \mu)$$

4.3 Exercise 3 (Multiplication operators II)

Let (X, μ) be a σ-finite measure space and M_u the multiplication operator for a measurable function $u : X \to \mathbb{C}$.

(a) The operator M_u is self-adjoint if and only if the essential range of u is contained in $\mathbb{R}$, which, in turn, holds if and only if u is real-valued almost everywhere.

(b) The operator M_u is bounded if and only if the essential range of u is bounded, which, in turn, holds if and only if $u \in L^\infty(X, \mu)$. In this case,

$$\|M_u\| = \|u\|_\infty = \sup\{|\lambda| : \lambda \text{ is in the essential range of } u\}$$

(c) $M_u = 0$ holds if and only if the essential range of u is $\{0\}$ which, in turn, holds if and only if $u = 0$ holds almost everywhere.

Solution:

(a)

$$\begin{aligned}
M_u \text{ self-adjoint } &\iff \int_X \overline{uf} f \, d\mu = \int_X \overline{f} uf \, d\mu \quad \forall f \in D(M_u) \\
&\iff \int_X (\overline{u} - u)|f|^2 \, d\mu = 0 \quad \forall f \in D(M_u) \\
&\iff u = \overline{u} \text{ a.e} \\
&\iff u \text{ is real-valued a.e} \\
&\iff \text{ess ran } u \subset \mathbb{R}
\end{aligned}$$

(b)

$$\begin{aligned}
M_u \text{ bounded } &\iff \int_X |uf|^2 \, d\mu \leq C \int_X |f|^2 \, d\mu \quad \forall f \in L^2(X, \mu) \\
&\iff \int_X (C - |u|^2)|f|^2 \, d\mu \geq 0 \quad \forall f \in L^2(X, \mu) \\
&\iff |u|^2 \leq C \\
&\iff \sup\{|\lambda| \colon \lambda \in \text{ess ran } u\} \leq K \\
&\iff \|u\|_\infty = \sup\{|\lambda| \colon \lambda \in \text{ess ran } u\} \text{ and } u \in L^\infty(X, \mu)
\end{aligned}$$

(c)

$$\begin{aligned}
M_u = 0 &\iff 0 = \|M_u f\|_2^2 = \int_X |uf|^2 \, d\mu \quad \forall (0 \neq) f \in D(M_u) \\
&\iff u = 0 \text{ a.e} \\
&\iff \text{ess ran } u = \{0\}
\end{aligned}$$

4.4 Exercise 4 (Closure convergence)

Let (L_n) be a sequence of self-adjoint operators on a Hilbert space and let L be a self-adjoint operator. Assume that for a family $(\Phi_\alpha)_{\alpha \in I}$ of measurable bounded functions from $\mathbb{R}$ to $\mathbb{R}$ and some index set I we have,

$$\lim_{n \to \infty} \Phi_\alpha(L_n) f = \Phi_\alpha(L) f$$

for all f in the Hilbert space and for all $\alpha \in I$. Let $\mathcal{A}$ be the closure of $\{\Phi_\alpha \colon \alpha \in I\}$ with respect to the supremum norm. Show that,

$$\lim_{n \to \infty} \Phi(L_n) f = \Phi(L) f$$

for all $\Phi \in \mathcal{A}$ and f in the Hilbert space.

Solution: We perform a classical three epsilon argument. For any $\Phi \in \mathcal{A}$ take Φ_{α_k} such that $\left\|\Phi_{\alpha_k} - \Phi\right\|_\infty \to 0$ as $k \to \infty$. We can then write,

$$\begin{aligned}\|\Phi(L_n)f - \Phi(L)f\|_H \leq \left\|\Phi(L_n)f - \Phi_{\alpha_k}(L_n)f\right\|_H + \left\|\Phi_{\alpha_k}(L_n)f - \Phi_{\alpha_k}(L)f\right\|_H \\ + \left\|\Phi_{\alpha_k}(L)f - \Phi(L)f\right\|_H\end{aligned}$$

and it is clear by assumption that $\left\|\Phi_{\alpha_k}(L_n)f - \Phi_{\alpha_k}(L)f\right\|_H \to 0$. Now by the Spectral theorem (Theorem 3.6 in the notes) we have,

$$\left\|\Phi_{\alpha_k}(L)f - \Phi(L)f\right\|_H = \left\|UM_{\Phi_{\alpha_k}\circ u}U^{-1}f - UM_{\Phi\circ u}U^{-1}f\right\|_H$$

Setting $\psi := U^{-1}f$ and recalling that U is a unitary operator we can write,

$$\begin{aligned}\left\|\Phi_{\alpha_k}(L)f - \Phi(L)f\right\|_H &\leq \left\|M_{\Phi_{\alpha_k}\circ u}\psi - M_{\Phi\circ u}\psi\right\|_2 \\ &= \left\|M_{\Phi_{\alpha_k}\circ u - \Phi\circ u}\psi\right\|_2 \leq \left\|\Phi_{\alpha_k} - \Phi\right\|_\infty \|\psi\|_2 \to 0\end{aligned}$$

where the last inequality comes from Exercise 3(b) (or Proposition 3.4 in the notes). By a similar argument we have,

$$\left\|\Phi(L_n)f - \Phi_{\alpha_k}(L_n)f\right\|_H \leq \left\|M_{\Phi\circ u_n - \Phi_{\alpha_k}\circ u_n}\psi_n\right\|_2$$

where $\psi_n = U_n^{-1}f$. Notice that $\|\psi_n\|_2 = \left\|U_n^{-1}f\right\|_2 \leq \|f\|_2$ and so ψ_n is a bounded sequence. Hence,

$$\left\|\Phi(L_n)f - \Phi_{\alpha_k}(L_n)f\right\|_H \leq \left\|\Phi - \Phi_{\alpha_k}\right\|_\infty \|\psi_n\|_2 \to 0$$

Putting all this together gives,

$$\|\Phi(L_n)f - \Phi(L)f\|_H \to 0 \quad \text{as} \quad n \to \infty$$

5 Lecture 5 – Exercises

6 Lecture 6 – Exercises

6.1 Exercise 1 (Density of C_c)

Let (X, m) be an infinite discrete measure space and $p \in [1, \infty]$. Show that $C_c(X)$ is dense in $\ell^p(X, m)$ if and only if $p \in [1, \infty)$.

Solution: We first recall the supremum norm definition,

$$\|f\|_\infty := \inf \left\{ c \geq 0 \colon \sum_{|f(x)|>c} m(x) = 0 \right\}$$

($\Longrightarrow$): Suppose that $\overline{C_c(X)}^{\|\cdot\|_\infty} = \ell^\infty(X, m)$. Since $\mathbb{1} \in \ell^\infty(X, m)$ then there exists $(f_n) \in C_c(X)$ such that $\|f_n - \mathbb{1}\|_\infty \to 0$. By definition, f_n has finite support for each $n \in \mathbb{N}$ and so for every f_n there exists $x_n \in X$ such that $f(x_n) = 0$ and hence $\|f_n - \mathbb{1}\|_\infty = 1$ for all n. This is clearly a contradiction and so we must have $p \in [1, \infty)$.

($\Longleftarrow$): For any $f \in \ell^p(X, m)$ we can assume $\mathrm{supp}(f)$ is countable. Setting $K_n = \{x_1, \ldots, x_n \colon x_i \in \mathrm{supp}(f),\ i = 1, \ldots, n\}$ then K_n is finite, $K_n \subset K_{n+1}$ and $\bigcup_n K_n = \mathrm{supp}(f)$. Taking the sequence $(\mathbb{1}_{K_n} f)$ it is clear that $\mathbb{1}_{K_n} f \to f$ pointwise and $|\mathbb{1}_{K_n} f| \leq f$ a.e, hence by dominated convergence $\|\mathbb{1}_{K_n} f - f\|_p \to 0$ for $p \in [1, \infty)$. So $\overline{C_c(X)}^{\|\cdot\|_p} = \ell^p(X, m)$ for $p \in [1, \infty)$.

6.2 Exercise 2 (Inclusion of ℓ^p spaces)

Let (X, m) be a discrete measure space.

(a) Show the equivalence of the following statements:

- (i) $\ell^1(X, m) \subseteq \ell^\infty(X, m)$
- (ii) $\ell^1(X, m) \subseteq C_0(X) := \overline{C_c(X)}^{\|\cdot\|_\infty}$
- (iii) There exists $\alpha > 0$ such that $m \geq \alpha$

(b) Show the equivalence of the following statements:

- (i) $\ell^1(X, m) \supseteq \ell^\infty(X, m)$
- (ii) $m(X) < \infty$

Solution:

(a) ((i) $\iff$ (ii)): The inclusion in (i) is equivalent to the existence of $c > 0$ such that $\|f\|_\infty \le c\|f\|_1$ for all $f \in \ell^1(X,m)$. From exercise 1 we have $\overline{C_c(X)}^{\|\cdot\|_1} = \ell^1(X,m)$ so it is sufficient to show $\overline{C_c(X)}^{\|\cdot\|_1} \subseteq \overline{C_c(X)}^{\|\cdot\|_\infty}$. Let $(f_n) \in C_c(X)$ be a Cauchy sequence with respect to $\|\cdot\|_1$, then there exists $f \in \overline{C_c(X)}^{\|\cdot\|_1}$ such that $\|f_n - f\|_1 \to 0$. By the earlier inequality we also have that (f_n) is Cauchy with respect to $\|\cdot\|_\infty$, $f \in \ell^\infty(X,m)$ and $\|f_n - f\|_\infty \to 0$. So $f \in \overline{C_c(X)}^{\|\cdot\|_\infty}$. Therefore, $\overline{C_c(X)}^{\|\cdot\|_1} \subseteq \overline{C_c(X)}^{\|\cdot\|_\infty}$.

The reverse implication is trivial since $C_c(X) \subset \ell^\infty(X,m)$.

((i) $\iff$ (iii)): If (i) is satisfied then from above we have,

$$\frac{1}{c}\|f\|_\infty \le \|f\|_1 \quad \forall f \in \ell^1(X,m)$$

For any $x \in X$ taking $f = \mathbb{1}_x$ we have,

$$\frac{1}{c} \le m(x)$$

Setting $\alpha = \frac{1}{c}$ gives (iii). Now if (iii) is satisfied for any $f \in \ell^1(X,m)$ we have,

$$\alpha|f(x)| \le |f(x)|m(x) \le \|f\|_1$$

Taking an essential supremum gives $\|f\|_\infty \le \frac{1}{\alpha}\|f\|_1$ and so (i) holds.

(b) First suppose $\ell^1(X,m) \supseteq \ell^\infty(X,m)$. Since $\mathbb{1} \in \ell^\infty(X,m)$ we have,

$$m(X) = \sum_{x\in X} m(x) = \|\mathbb{1}\|_1 < \infty$$

Conversely if $m(X) < \infty$ then for any $f \in \ell^\infty(X,m)$ we have,

$$\sum_{x\in X} |f(x)|m(x) \le \|f\|_\infty \sum_{x\in X} m(x) = \|f\|_\infty m(X) < \infty$$

Hence, $\ell^1(X,m) \supseteq \ell^\infty(X,m)$.

6.3 Exercise 3 (Boundedness)

Let (b,c) be a graph over (X,m). Show that $\mathcal{L}$ is bounded on $\ell^2(X,m)$ if and only if it is bounded on $\ell^p(X,m)$ for some $p \in [1,\infty]$.

Solution: We recall the following facts from Lecture 3: the *weighted degree function* of a graph is defined by

$$\operatorname{Deg}(x) := \frac{1}{m(x)}\left[\sum_{y\in X} b(x,y) + c(x)\right],$$

and the Laplacian $\mathcal{L}$ is bounded on $\ell^2(X,m)$ if and only if $\operatorname{Deg}(\cdot)$ is bounded on X (see Theorem 2.18). Thus it suffices to prove that if $\mathcal{L}$ is bounded on $\ell^p(X,m)$ for some $p \in [1,\infty]$, then the weighted degree function is bounded on X.

We write $\|\cdot\|_p$ for the norm on $\ell^p(X,m)$. Let κ_p be the operator norm of $\mathcal{L}$ on $\ell^p(X,m)$, i.e. $\kappa_p = \sup_{\|f\|_p\le 1}\|\mathcal{L}f\|_p$. We now compute

$$\mathcal{L}\mathbb{1}_x(y) = \begin{cases}\operatorname{Deg}(x) & y = x\\ -\frac{b(x,y)}{m(y)} & y \neq x.\end{cases} \tag{6.1}$$

The above calculation shows that $\mathcal{L}\mathbb{1}_x(y) \le 0$ if $y \neq x$. We treat the case $p = \infty$ first. Fix an arbitrary $x \in X$, and observe that

$$\|\mathcal{L}\mathbb{1}_x\|_\infty = \sup_{y\in X}|\mathcal{L}\mathbb{1}_x(y)| = \mathcal{L}\mathbb{1}_x(x) = \operatorname{Deg}(x).$$

Clearly $\|\mathbb{1}_x\|_\infty = 1$ for all $x \in X$, so it follows that

$$\sup_{x\in X}\operatorname{Deg}(x) = \sup_{x\in X}\|\mathcal{L}\mathbb{1}_x\|_\infty \le \kappa_\infty \sup_{x\in X}\|\mathbb{1}_x\|_\infty = \kappa_\infty.$$

Hence $\operatorname{Deg}(\cdot)$ is bounded on X.

We have that $\|\mathbb{1}_x\|_p = m(x)^{1/p}$ for each $x \in X$, where $1/\infty := 0$. Thus if $1 \le p < \infty$, it follows that

$$\sum_{y\in X} b(x,y) + c(x) = m(x)\mathcal{L}\mathbb{1}_x(x) = \langle \mathbb{1}_x, \mathcal{L}\mathbb{1}_x\rangle \le \|\mathcal{L}\mathbb{1}_x\|_p\|\mathbb{1}\|_{p'},$$

where $\langle\cdot,\cdot\rangle$ is the duality pairing between $\ell^p(X,m)$ and $\ell^{p'}(X,m)$ given by

$$\langle f, g\rangle = \sum_{x\in X} f(x)g(x)m(x), \qquad f \in \ell^{p'}(X,m), g \in \ell^p(X,m).$$

Therefore

$$m(x)\operatorname{Deg}(x) \le \|\mathcal{L}\mathbb{1}_x\|_p\|\mathbb{1}\|_{p'} \le \kappa_p m(x)^{1/p}m(x)^{1/p'} = \kappa_p m(x), \tag{6.2}$$

which yields the bound $\sup_{x\in X}\operatorname{Deg}(x) \le \kappa_p$.