

The following contains details statements of those results alluded to briefly in the lecture notes. It also contains solutions to each lectures exercises.

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# 1 Lecture 1 – Exercises

## 1.1 Exercise 1 (Normal Contractions)

(a) Show that the following maps from  $\mathbb{R}$  to  $\mathbb{R}$  are normal contractions:

- $C_+ : t \mapsto t \vee 0$
- $C_- : t \mapsto (-t) \vee 0$
- $C_{(-\infty, 1]} : t \mapsto t \wedge 1$
- $C_{[0, 1]} : t \mapsto 0 \vee (t \wedge 1)$

For which  $a \leq b$  is  $C_{[a, b]} : t \mapsto a \vee (t \wedge b)$  a normal contraction?

(b) Let  $X$  be a finite set and let  $Q$  be a symmetric bilinear form on  $C(X)$ . Show that  $Q$  is compatible with normal contractions if it is compatible with the map  $C_{(-\infty, 1]}$ .

**Solution:**

(a) –  $C_+(0) = 0$ .

$$\begin{aligned} |C_+(s) - C_+(t)| &= |(s \vee 0) - (t \vee 0)| \\ &= \begin{cases} 0 & \text{if } s, t \leq 0 \\ |s - t| & \text{if } s, t > 0 \\ |s| & \text{if } s \geq 0 > t \end{cases} \leq |s - t| \end{aligned}$$

–  $C_-(0) = 0$ .

$$\begin{aligned} |C_-(s) - C_-(t)| &= |(-s) \vee 0 - (-t) \vee 0| \\ &= \begin{cases} |t - s| & \text{if } s, t \leq 0 \\ 0 & \text{if } s, t > 0 \\ |t| & \text{if } s \geq 0 > t \end{cases} \leq |s - t| \end{aligned}$$

–  $C_{(-\infty, 1]}(0) = 0$ .

$$\begin{aligned} |C_{(-\infty, 1]}(s) - C_{(-\infty, 1]}(t)| &= |s \wedge 1 - t \wedge 1| \\ &= \begin{cases} |s - t| & \text{if } s, t \leq 1 \\ 0 & \text{if } s, t > 1 \\ |s - 1| & \text{if } s \leq 1 < t \end{cases} \leq |s - t| \end{aligned}$$

$$- C_{[0,1]}(0) = 0.$$

$$\begin{aligned} |C_{[0,1]}(s) - C_{[0,1]}(t)| &= |0 \vee (s \wedge 1) - 0 \vee (t \wedge 1)| \\ &= \begin{cases} |s - t| & \text{if } s, t \in [0, 1] \\ 0 & \text{if } s, t \notin [0, 1] \\ |s - 1| & \text{if } s \in [0, 1] \text{ and } t \notin [0, 1] \end{cases} \leq |s - t| \end{aligned}$$

To determine  $a, b$  notice if  $b < 0$  then

$$C_{[a,b]}(0) = a \vee (0 \wedge b) = a \wedge b \neq 0$$

So we must have  $b \geq 0$  in which case,

$$C_{[a,b]}(0) = a \vee 0 = 0 \iff a \leq 0$$

Now to ensure the contraction property,

$$\begin{aligned} |C_{[a,b]}(s) - C_{[a,b]}(t)| &= |a \vee (s \wedge b) - a \vee (t \wedge b)| \\ &= \begin{cases} |s - t| & \text{if } s, t \in [a, b] \\ 0 & \text{if } s, t \notin [a, b] \\ |s - b| & \text{if } s \in [a, b] \text{ and } t \notin [a, b] \end{cases} \leq |s - t| \end{aligned}$$

Therefore if  $a \leq 0 \leq b$  then  $C_{[a,b]}$  is a normal contraction.

## 1.2 Exercise 2 (First Beurling-Deny criterion)

Let  $X$  be a finite set and let  $Q$  be a symmetric bilinear form over  $X$ . For any  $f \in C(X)$ , let  $f_+ = f \vee 0$  be the *positive part* and let  $f_- = (-f) \vee 0$  be the *negative part* of  $f$ .

Show the following equivalence:

- (i)  $Q(|f|) \leq Q(f)$  for all  $f \in C(X)$ .
- (ii)  $Q(f_+, f_-) \leq 0$  for all  $f \in C(X)$ .
- (iii)  $Q(f \vee g) + Q(f \wedge g) \leq Q(f) + Q(g)$  for all  $f, g \in C(X)$ .

and for  $Q$  positive show that this is also equivalent to:

- (iv)  $Q(f_+) \leq Q(f)$  for all  $f \in C(X)$ .

**Solution:**

(i)  $\implies$  (ii) By properties of symmetric forms,

$$\begin{aligned} Q(f_-) + 2Q(f_+, f_-) + Q(f_-) &= Q(|f|) \\ &\leq Q(f) = Q(f_+) - 2Q(f_+, f_-) + Q(f_-) \end{aligned}$$

Then rearranging gives,

$$Q(f_+, f_-) \leq 0 \quad \forall f \in C(X)$$

(ii)  $\implies$  (i) By properties of symmetric forms,

$$\begin{aligned} Q(f) &= Q(f_+) - 2Q(f_+, f_-) + Q(f_-) \\ &\geq Q(f_+) + 2Q(f_+, f_-) + Q(f_-) = Q(|f|) \quad \forall f \in C(X) \end{aligned}$$

(i)  $\implies$  (iii) We can write  $f \vee g = \frac{1}{2}(f+g) + \frac{1}{2}|f-g|$  and  $f \wedge g = \frac{1}{2}(f+g) - \frac{1}{2}|f-g|$ . Then,

$$\begin{aligned} Q(f \vee g) &= \frac{1}{4}Q(f+g) + \frac{1}{2}Q(f+g, |f-g|) + \frac{1}{4}Q(|f-g|) \\ &\leq \frac{1}{4}Q(f+g) + \frac{1}{2}Q(f+g, |f-g|) + \frac{1}{4}Q(f-g) \\ &= \frac{1}{4}Q(f) + \frac{1}{2}Q(f, g) + \frac{1}{4}Q(g) + \frac{1}{2}Q(f+g, |f-g|) + \frac{1}{4}Q(f) - \frac{1}{2}Q(f, g) + \frac{1}{4}Q(g) \\ &= \frac{1}{2}Q(f) + \frac{1}{2}Q(g) + \frac{1}{2}Q(f+g, |f-g|) \end{aligned}$$

By the same method,

$$Q(f \wedge g) \leq \frac{1}{2}Q(f) + \frac{1}{2}Q(g) - \frac{1}{2}Q(f+g, |f-g|)$$

and so,

$$Q(f \vee g) + Q(f \wedge g) \leq Q(f) + Q(g)$$

(iii)  $\implies$  (i) By the same calculations as above,

$$\frac{1}{2}Q(f+g) + \frac{1}{2}Q(|f-g|) \leq Q(f) + Q(g)$$

rearranging and expanding,

$$\begin{aligned} \frac{1}{2}Q(|f-g|) &\leq Q(f) + Q(g) - \frac{1}{2}Q(f) - \frac{1}{2}Q(f, g) - \frac{1}{2}Q(g) \\ &= \frac{1}{2}Q(f-g) \end{aligned}$$

Choosing  $g = 0$  gives the result.

(i)  $\iff$  (iv) We work with the assumption that  $Q$  is positive now. If (ii) holds then,

$$Q(f^+) \leq Q(f^+) - 2Q(f^+, f^-) + Q(f^-) = Q(f)$$

If (iv) holds then,

$$0 \leq Q(f^-) - 2Q(f^+, f^-)$$

### 1.3 Exercise 3 (Harmonic functions and connected components)

Let  $(b, c)$  be a graph over a finite set measure space  $(X, m)$  with associated Laplacian  $L = L_{b, c, m}$  and let,

$$H = \{f \in C(X) : Lf = 0\}$$

be the subspace of harmonic functions. Show that  $\dim H$  is equal to the number of connected components of  $(b, c)$  on which  $c$  vanished.

**Solution:** On any connected component where  $c$  vanishes we have that any constant function (only on the connected component and zero everywhere else) is harmonic,

$$L\mathbb{1}(x) = \frac{1}{m(x)} \sum_{y \in X} b(x, y)(\mathbb{1}(x) - \mathbb{1}(y)) = 0$$

Let  $n \in \mathbb{N}$  denote the number of connected components of  $(b, c)$ . The constant functions  $\mathbb{1}$  on connected components where  $c$  vanishes form a linearly independent set in  $C(X)$  and so  $V$ . Hence  $\dim H \geq n$ .

If  $f \in H$  let us consider it over a connected component where  $c$  vanishes. Then,

$$(Lf)(x) = \frac{1}{m(x)} \sum_{y \in X} b(x, y)(f(x) - f(y)) = 0$$

for every  $x$  in the connected component. In particular, if we take the cut-off of  $f$  to any connected component where  $c$  vanishes then we have a collection of linearly independent functions. The number of such functions would clearly equal the number of connected components where  $c$  vanishes.

### 1.4 Exercise 4 (Poisson equation for $\alpha = 0$ )

Let  $b$  be a graph over a finite set measure space  $(X, m)$  (that is  $c = 0$ ) and let  $L = L_{b, 0, m}$  be the associated Laplacian. Furthermore let,

$$V := \left\{ f \in C(X) : \sum_{x \in X} f(x)m(x) = 0 \right\}$$

Show that for each  $f \in V$ , there is a unique functions  $u \in V$  such that,

$$Lu = f$$

*Hint:* Observe that for the scalar product in  $\ell^2(X, m)$ , we have for all  $f \in \ell^2(X, m)$ ,

$$\sum_{x \in X} f(x)m(x) = \langle f, 1 \rangle$$

## 2 Lecture 2 – Exercises

### 2.1 Exercise 1 (Positivity improvement of the inverse operator)

Let  $X$  be a finite set and let  $L$  be an injective operator on  $C(X)$ . Show that the following assertions are equivalent:

- (i) The inverse operator  $L^{-1}$  is positivity improving, i.e. for all  $f \in C(X)$  such that  $f \geq 0$  ( $\neq 0$ ) we have  $L^{-1}f > 0$ .
- (ii) For each function  $u \in C(X)$  satisfying the inequalities  $\max_X u(x) \geq 0$  and  $Lu \leq 0$ , we have  $u \equiv 0$ .

**Solution:**

(i)  $\implies$  (ii) Suppose  $u \in C(X)$  such that  $\max_X u(x) \geq 0$  and  $Lu \leq 0$ . Set  $v := Lu$  then by assumption  $L^{-1}v < 0 \implies u < 0$  if  $v \neq 0$ . Clearly this is not possible as  $\max_X u(x) \geq 0$ . Hence, we must have that  $v = 0 \implies u \equiv 0$ .

(ii)  $\implies$  (i) For any  $f \geq 0$  ( $\neq 0$ ) we have by the injectivity of  $L$  that there exists a unique  $g$  such that  $L(-g) = -f \leq 0$ . If  $\max_X -g(x) \geq 0$  then by assumption  $g \equiv 0$ . However this is not possible since  $f \neq 0$ . Therefore, we must have  $-g < 0 \implies g > 0$ . Applying  $L^{-1}$  we then have  $L^{-1}f = g > 0$ .

### 2.2 Exercise 2 (Cauchy problem / Heat equation)

Let  $(X, m)$  be a finite measure space and let  $L$  be a self-adjoint operator on  $\ell^2(X, m)$  and for  $t \geq 0$  let  $e^{-tL}$  be defined via the spectral calculus.

- (a) Show that for all  $t \geq 0$ ,

$$e^{-tL} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (tL)^n$$

In particular, show that the sum is absolutely convergent with respect to the operator norm.

- (b) Show that  $\{e^{-tL} : t \geq 0\}$ , equipped with the composition of operators, is an operator semigroup, i.e.,  $e^{0L} = I$  and  $e^{(t+s)L} = e^{tL}e^{sL}$  for all  $t, s \geq 0$  and  $t \mapsto e^{-tL}f$  is continuously differentiable at  $t = 0$  for all  $f \in \ell^2(X, m)$ . Moreover, show that (in this finite dimensional case),

$$\frac{d}{dt}e^{-tL} = -Le^{-tL} = -e^{tL}L$$



- (c) Show that for all  $f \in \ell^2(X, m)$ , the function  $t \mapsto \varphi_t := e^{-tL}f$  is the unique solution of the equation,

$$\frac{d}{dt}\varphi_t = -L\varphi_t, \quad \varphi_0 = f$$

for all  $t \geq 0$ .

**Solution:**

- (a) We first make sure the series given exists, that is, show that it is absolutely convergent. Treating the series as a power series we have coefficients  $a_n = \frac{(-1)^{n+1}L^n}{n!}$  so,

$$\frac{\left\| \frac{(-1)^{n+2}L^{n+1}}{(n+1)!} \right\|}{\left\| \frac{(-1)^{n+1}L^n}{n!} \right\|} = \frac{\|L\|}{n+1} \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

since  $\ell^2(X, m)$  is finite dimensional then  $\|L\| < \infty$ . Hence, the radius of convergence of the given series is  $\infty$  and it indeed exists.

Using the definition of  $e^{-tL}$  and a Taylor series of the real valued function  $e^{-\lambda t}$  we have,

$$\begin{aligned} e^{-tL} &= \sum_{\lambda \in \sigma(L)} e^{-\lambda t} E_\lambda = \sum_{\lambda \in \sigma(L)} \left( \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} t^n \right) E_\lambda \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} (-t)^n \left( \sum_{\lambda \in \sigma(L)} \lambda^n E_\lambda \right) = \sum_{n=0}^{\infty} \frac{1}{n!} (-tL)^n \end{aligned}$$

as,

$$L^2 = \sum_{\lambda \in \sigma(L)} \lambda^2 E_\lambda^2 = \sum_{\lambda \in \sigma(L)} \lambda^2 E_\lambda \implies L^n = \sum_{\lambda \in \sigma(L)} \lambda^n E_\lambda$$

by properties of projections.

- (b) Using the spectral calculus definition,

$$e^{0L} = \sum_{\lambda \in \sigma(L)} e^{\lambda 0} E_\lambda = \sum_{\lambda \in \sigma(L)} E_\lambda = I$$

and,

$$\begin{aligned} e^{(t+s)L} &= \sum_{\lambda \in \sigma(L)} e^{\lambda(t+s)} E_\lambda = \sum_{\lambda \in \sigma(L)} e^{\lambda t} e^{\lambda s} E_\lambda^2 \\ &= \sum_{\lambda \in \sigma(L)} e^{\lambda t} E_\lambda \sum_{\mu \in \sigma(L)} e^{\mu s} E_\mu = e^{tL} e^{sL} \end{aligned}$$

for all  $t, s \geq 0$ . By properties of power series we have that  $e^{-tL}$  is analytic and so it is continuously differentiable for all  $t \geq 0$ , in particular we have term-wise differentiate of the series which gives,

$$\frac{d}{dt}e^{-tL} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} n t^{n-1} L^n = -L \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{(n-1)!} (tL)^{n-1} = -L e^{-tL} = -e^{-tL} L$$

where the last equality follows since the operator is self-adjoint.

(c) This follows immediately from (b). The differential equation comes from,

$$\frac{d}{dt}\varphi_t = \left( \frac{d}{dt}e^{-tL} \right) f = -L\varphi_t$$

by the continuous differentiability of the map  $t \mapsto e^{-tL}f$ . The initial condition is immediate by the continuity of the map,  $\varphi_0 = e^{0L}f = f$ .

### 2.3 Exercise 3 (Stochastic incompleteness)

Let  $(b, c)$  be a connected graph over  $(X, m)$  and let  $L = L_{b,c,m}$  denote the associated Laplacian.

(a) Show that  $e^{-tL}1 < 1$  for all  $t > 0$  if and only if  $c \neq 0$ .

(b) Show that if  $e^{-tL}1 < 1$  for some  $t > 0$ , then  $e^{-tL}1 < 1$  for all  $t > 0$ .

**Solution:**

(a) By Theorem 1.20 in the notes we have that  $e^{-tL}$  has the Markov property, that is,

$$0 \leq e^{-tL}f \leq 1 \quad \text{for all } 0 \leq f \leq 1 \text{ and } t \geq 0$$

( $\implies$ ) If  $c = 0$  then  $L1 = 0$  and so  $u_t := 1$  is a solution of,

$$\frac{d}{dt}u_t = -Lu_t, \quad u_0 = 1$$

By uniqueness of solutions we then have  $e^{-tL}1 = 1$  for all  $t > 0$ . Hence, by contrapositive statements  $e^{-tL}1 < 1 \forall t > 0 \implies c \neq 0$ .

( $\Leftarrow$ ) Suppose  $c \neq 0$  then  $L1 = c \neq 0$ . So the problem,

$$\frac{d}{dt}u_t = Lu_t \quad u_0 = 1$$

cannot have a constant solution. As  $u_t := e^{-tL}1$  is a solution to the above problem we must have  $e^{-tL}1 \neq 1 \implies e^{-tL}1 < 1$  for all  $t > 0$ .

(b) Assume there exists  $t_0 > 0$  and some  $x_0 \in X$  such that,

$$(e^{-t_0L}1)(x_0) = 1$$

Then  $u_t(x_0) := (1 - e^{-tL}1)(x_0)$  has a minimum at  $t_0$ . So,

$$0 = L1(x_0) - Lu_{t_0}(x_0) = \frac{c(x_0)}{m(x_0)} + \sum_{y \sim x} b(x, y)u_{t_0}(y)$$

and we have  $c(x_0) = 0$ . By the connectedness of the graph we can iterate this argument and hence have  $c = 0$ . Now by (a) we must have  $e^{-tL}1 = 1$  for all  $t > 0$  however this is a contradiction to our assumption and so we must have that  $e^{-tL}1 < 1$  for all  $t > 0$ .

### 3 Lecture 3 – Exercises

#### 3.1 Exercise 1 ( $\mathcal{F} = C(X)$ )

Let  $\mathcal{F}$  be the domain of the formal Laplacian  $\mathcal{L}$  associated to a graph. Show that the following statements are equivalent.

- (i) The graph is locally finite.
- (ii)  $\mathcal{L}C_c(X) \subseteq C_c(X)$
- (iii)  $C(X) = \mathcal{F}$ .

**Solution:**

(i)  $\iff$  (ii) Any  $f \in C_c(X)$  has finite support so it is immediate that any  $f \in C_c(X)$  can be written as a linear combination of indicator functions. Consider now  $\mathbb{1}_x$  then,

$$\begin{aligned}\mathcal{L}\mathbb{1}_x(z) &= \frac{1}{m(z)} \left[ \sum_{y \in X} b(z, y)(\mathbb{1}_x(z) - \mathbb{1}_x(y)) + c(z)\mathbb{1}_x(z) \right] \\ &= \begin{cases} -\frac{b(x, z)}{m(z)} & \text{if } z \neq x \\ \text{Deg}(x) & \text{if } z = x \end{cases}\end{aligned}$$

So  $\mathcal{L}\mathbb{1}_x = \text{Deg}(x)\mathbb{1}_x - \frac{b(x, \cdot)}{m(\cdot)}$ . If  $\mathcal{L}C_c(X) \subseteq C_c(X)$  then  $\mathcal{L}\mathbb{1}_x$  has finite support. It is clear that  $\mathcal{L}\mathbb{1}_x(x) \neq 0$  however if  $\mathcal{L}\mathbb{1}_x(z) \neq 0$  for  $z \neq x$  then we must have that  $b(x, z) \neq 0 \implies z$  is a neighbour of  $x$ . Hence the set  $\{z: x \sim z\}$  is finite.

If we instead assume the graph is locally finite, then there are only finitely many elements which give  $b(x, z) \neq 0$  and so  $\mathcal{L}\mathbb{1}_x \in C_c(X)$  as it has finite support. Since any  $f \in C_c(X)$  can be written as a linear combination of indicator functions we have  $\mathcal{L}f \in C_c(X) \implies \mathcal{L}C_c(X) \subseteq C_c(X)$ .

(iii)  $\implies$  (i) Since  $C(X)$  contains all functions on  $X$ . For any  $x \in X$  we define the function,

$$f_x(y) := \begin{cases} \frac{1}{b(x, y)} & \text{if } x \sim y \\ 0 & \text{if } x \not\sim y \end{cases}$$

and have,

$$\sum_{y \in X} b(x, y)f_x(y) < \infty$$

Clearly we must have that  $\{y: x \sim y\}$  is a finite set otherwise the sum above is a series of 1 which diverges. Hence the graph is locally finite.

(i)  $\implies$  (iii) If the graph is locally finite then for any  $x \in X$ ,

$$\sum_{y \in X} b(x, y)$$

is a finite sum. So for any  $f \in C(X)$  we have that,

$$\sum_{y \in X} b(x, y) |f(y)| < \infty$$

as it is a finite sum. Hence,  $C(X) = \mathcal{F}$ .

### 3.2 Exercise 2 (Maximum principle)

Let  $\mathcal{A} : C_c(X) \rightarrow C_c(X)$  be a symmetric linear operator, i.e.,  $\mathcal{A}\mathbb{1}_x(y) = \mathcal{A}\mathbb{1}_y(x)$  for all  $x, y \in X$ . Show the following equivalence:

- (i)  $\mathcal{A} = \mathcal{L}_{b,c}$  on  $C_c(X)$  for a locally finite graph  $(b, c)$  over  $X$ .
- (ii)  $\mathcal{A}$  satisfies a maximum principle.

(i)  $\implies$  (ii) Suppose  $f \in C_c(X)$  has a non-negative local maximum at  $x$ . Then,

$$\mathcal{A}f(x) = \mathcal{L}_{b,c}f(x) = \frac{1}{m(x)} \left[ \sum_{y \in X} b(x, y)(f(x) - f(y)) + c(x)f(x) \right]$$

since  $f(x) \geq 0$  and  $f(x) \geq f(y)$  for all  $y \sim x$  then  $\mathcal{A}f(x) \geq 0$ .

(ii)  $\implies$  (i) Let  $a$  be the (infinite) matrix associated to  $\mathcal{A}$ . By the maximum principle if  $-\mathcal{A}\mathbb{1}_x(y) \geq 0$  for all  $y \neq x$ . Since  $\mathcal{A}\mathbb{1}_x(y) = a(x, y)$  then  $a(x, y) \leq 0$  for all  $y \neq x$ . Next for the one function we have  $\mathcal{A}\mathbb{1}(x) \geq 0$  for all  $x \in X$  and so  $\sum_{y \in X} a(x, y) \geq 0$ .

### 3.3 Exercise 3 (Uncountable graphs)

Let  $X$  be an arbitrary set and assume that  $b : X \times X \rightarrow [0, \infty)$  satisfies  $b(x, y) = b(y, x)$ ,  $b(x, x) = 0$  and,

$$\sum_{z \in X} b(x, z) = \sup_{U \subseteq X \text{ finite}} \sum_{y \in U} b(x, y) < \infty$$

for all  $x \in X$ . Call a subset  $Y$  and  $X$  connected if for arbitrary  $x, y \in Y$  there exists  $n \in \mathbb{N}$  and  $x_0, \dots, x_n \in Y$  with  $x_0 = x$ ,  $x_n = y$  and  $b(x_k, x_{k+1}) > 0$  for all  $k = 0, \dots, n-1$ . Show that any connected subset of  $X$  is connected.

### 3.4 Exercise 4 (Summability)

Let  $X$  be a countable set,  $b : X \times X \rightarrow [0, \infty)$  and  $Q : C(X) \rightarrow [0, \infty]$ ,

$$Q(f) = \frac{1}{2} \sum_{x,y \in X} b(x,y)(f(x) - f(y))^2$$

Show that,

$$Q(\varphi) < \infty$$

for all  $\varphi \in C_c(X)$  if and only if,

$$\sum_{y \in X} b(x,y) < \infty$$

for all  $x \in X$ .

## 4 Lecture 4 – Exercises

### 4.1 Exercise 1 (Resolvents are continuous)

Show that the resolvent map of an operator  $A$  on a Hilbert space  $H$ ,

$$\rho(A) \rightarrow B(H) \quad z \mapsto (A - z)^{-1}$$

is continuous.

### 4.2 Exercise 2 (Multiplication operators I)

Let  $(X, \mu)$  be a measure space and let  $u : X \rightarrow \mathbb{C}$  be measurable. The operator  $M_u$  of multiplication by  $u$  has domain,

$$D(M_u) = \{f \in L^2(X, \mu) : uf \in L^2(X, \mu)\}$$

and acts as,

$$M_u f = uf$$

for all  $f \in D(M_u)$ . Show the following statements.

- (a) The operator  $M_u$  is densely defined.
- (b) The operator  $M_u$  is closed.
- (c) The adjoint of  $M_u$  is given by  $(M_u)^* = M_{\bar{u}}$ . In particular,  $M_u$  is self-adjoint if  $u$  is real-valued.
- (d) The operator  $M_u$  is bounded if  $u \in L^\infty(X, \mu)$ .

**Solution:**

- (a) Consider the sets  $X_n := \{x \in X : |u(x)| \leq n\}$  for every  $n \in \mathbb{N}$ . For any  $f \in L^2(X, \mu)$  take the sequence  $\mathbb{1}_{X_n} f$ . Now,

$$|\mathbb{1}_{X_n} f(x)| \leq f(x) \quad \forall n \in \mathbb{N}$$

and  $\mathbb{1}_{X_n} f \rightarrow f$  a.e then by DCT  $\mathbb{1}_{X_n} f \rightarrow f$  in  $L^2(X, \mu)$ . We also have,

$$u \mathbb{1}_{X_n} f \leq n \mathbb{1}_{X_n} f \in L^2(X, \mu) \implies \mathbb{1}_{X_n} f \in D(M_u) \quad \forall n \in \mathbb{N}$$

Therefore,  $\overline{D(M_u)} = L^2(X, \mu)$ .

- (b) Suppose  $f_n \in D(M_u)$  such that  $f_n \rightarrow f$  and  $uf_n \rightarrow g$  in  $L^2(X, \mu)$ . We can take a subsequence  $f_{n_k}$  such that  $f_{n_k} \rightarrow f$  a.e and  $uf_{n_k} \rightarrow g$  in  $L^2(X, \mu)$ . Taking a further subsequence (which we relabel as  $\cdot_{n_k}$ ) we have  $uf_{n_k} \rightarrow uf$  a.e and so we must have that  $uf = g$  and  $f \in D(M_u)$ .
- (c) For any  $f \in D(M_u)$  and  $g \in D(M_u^*)$  we have the identity,

$$\langle g, M_u f \rangle = \langle M_u^* g, f \rangle$$

Since this is the  $L^2$  inner product and as it is linear in its second argument we have,

$$\langle g, M_u f \rangle = \int_X \bar{g} u f \, d\mu = \int_X \overline{\bar{u} g} f \, d\mu = \langle \bar{u} g, f \rangle$$

Hence,

$$\langle g, M_u f \rangle = \langle M_{\bar{u}} g, f \rangle$$

for all  $f \in D(M_u)$  and  $g \in D(M_u^*)$  so  $M_u^* = M_{\bar{u}}$ . Of course if  $u$  is real-valued we have  $\bar{u} = u$  and clearly  $M_u$  is self-adjoint.

- (d) If  $u \in L^\infty(X, \mu)$  we can write,

$$\|M_u f\|_2 = \|uf\|_2 \leq \|u\|_\infty \|f\|_2 \quad \forall f \in L^2(X, \mu)$$

### 4.3 Exercise 3 (Multiplication operators II)

Let  $(X, \mu)$  be a  $\sigma$ -finite measure space and  $M_u$  the multiplication operator for a measurable function  $u : X \rightarrow \mathbb{C}$ .

- (a) The operator  $M_u$  is self-adjoint if and only if the essential range of  $u$  is contained in  $\mathbb{R}$ , which, in turn, holds if and only if  $u$  is real-valued almost everywhere.
- (b) The operator  $M_u$  is bounded if and only if the essential range of  $u$  is bounded, which, in turn, holds if and only if  $u \in L^\infty(X, \mu)$ . In this case,

$$\|M_u\| = \|u\|_\infty = \sup \{|\lambda| : \lambda \text{ is in the essential range of } u\}$$

- (c)  $M_u = 0$  holds if and only if the essential range of  $u$  is  $\{0\}$  which, in turn, holds if and only if  $u = 0$  holds almost everywhere.



**Solution:**

(a)

$$\begin{aligned}
M_u \text{ self-adjoint} &\iff \int_X \overline{u} f f \, d\mu = \int_X \overline{f} u f \, d\mu \quad \forall f \in D(M_u) \\
&\iff \int_X (\overline{u} - u) |f|^2 \, d\mu = 0 \quad \forall f \in D(M_u) \\
&\iff u = \overline{u} \text{ a.e} \\
&\iff u \text{ is real-valued a.e} \\
&\iff \text{ess ran } u \subset \mathbb{R}
\end{aligned}$$

(b)

$$\begin{aligned}
M_u \text{ bounded} &\iff \int_X |u f|^2 \, d\mu \leq C \int_X |f|^2 \, d\mu \quad \forall f \in L^2(X, \mu) \\
&\iff \int_X (C - |u|^2) |f|^2 \, d\mu \geq 0 \quad \forall f \in L^2(X, \mu) \\
&\iff |u|^2 \leq C \\
&\iff \sup \{|\lambda| : \lambda \in \text{ess ran } u\} \leq K \\
&\iff \|u\|_\infty = \sup \{|\lambda| : \lambda \in \text{ess ran } u\} \text{ and } u \in L^\infty(X, \mu)
\end{aligned}$$

(c)

$$\begin{aligned}
M_u = 0 &\iff 0 = \|M_u f\|_2^2 = \int_X |u f|^2 \, d\mu \quad \forall (0 \neq) f \in D(M_u) \\
&\iff u = 0 \text{ a.e} \\
&\iff \text{ess ran } u = \{0\}
\end{aligned}$$

#### 4.4 Exercise 4 (Closure convergence)

Let  $(L_n)$  be a sequence of self-adjoint operators on a Hilbert space and let  $L$  be a self-adjoint operator. Assume that for a family  $(\Phi_\alpha)_{\alpha \in I}$  of measurable bounded functions from  $\mathbb{R}$  to  $\mathbb{R}$  and some index set  $I$  we have,

$$\lim_{n \rightarrow \infty} \Phi_\alpha(L_n) f = \Phi_\alpha(L) f$$

for all  $f$  in the Hilbert space and for all  $\alpha \in I$ . Let  $\mathcal{A}$  be the closure of  $\{\Phi_\alpha : \alpha \in I\}$  with respect to the supremum norm. Show that,

$$\lim_{n \rightarrow \infty} \Phi(L_n) f = \Phi(L) f$$

for all  $\Phi \in \mathcal{A}$  and  $f$  in the Hilbert space.

**Solution:** We perform a classical three epsilon argument. For any  $\Phi \in \mathcal{A}$  take  $\Phi_{\alpha_k}$  such that  $\|\Phi_{\alpha_k} - \Phi\|_{\infty} \rightarrow 0$  as  $k \rightarrow \infty$ . We can then write,

$$\begin{aligned} \|\Phi(L_n)f - \Phi(L)f\|_H &\leq \|\Phi(L_n)f - \Phi_{\alpha_k}(L_n)f\|_H + \|\Phi_{\alpha_k}(L_n)f - \Phi_{\alpha_k}(L)f\|_H \\ &\quad + \|\Phi_{\alpha_k}(L)f - \Phi(L)f\|_H \end{aligned}$$

and it is clear by assumption that  $\|\Phi_{\alpha_k}(L_n)f - \Phi_{\alpha_k}(L)f\|_H \rightarrow 0$ . Now by the Spectral theorem (Theorem 3.6 in the notes) we have,

$$\|\Phi_{\alpha_k}(L)f - \Phi(L)f\|_H = \|UM_{\Phi_{\alpha_k} \circ U}U^{-1}f - UM_{\Phi \circ U}U^{-1}f\|_H$$

Setting  $\psi := U^{-1}f$  and recalling that  $U$  is a unitary operator we can write,

$$\begin{aligned} \|\Phi_{\alpha_k}(L)f - \Phi(L)f\|_H &\leq \|M_{\Phi_{\alpha_k} \circ U}\psi - M_{\Phi \circ U}\psi\|_2 \\ &= \|M_{\Phi_{\alpha_k} \circ U - \Phi \circ U}\psi\|_2 \leq \|\Phi_{\alpha_k} - \Phi\|_{\infty} \|\psi\|_2 \rightarrow 0 \end{aligned}$$

where the last inequality comes from Exercise 3(b) (or Proposition 3.4 in the notes). By a similar argument we have,

$$\|\Phi(L_n)f - \Phi_{\alpha_k}(L_n)f\|_H \leq \|M_{\Phi \circ U_n - \Phi_{\alpha_k} \circ U_n}\psi_n\|_2$$

where  $\psi_n = U_n^{-1}f$ . Notice that  $\|\psi_n\|_2 = \|U_n^{-1}f\|_2 \leq \|f\|_2$  and so  $\psi_n$  is a bounded sequence. Hence,

$$\|\Phi(L_n)f - \Phi_{\alpha_k}(L_n)f\|_H \leq \|\Phi - \Phi_{\alpha_k}\|_{\infty} \|\psi_n\|_2 \rightarrow 0$$

Putting all this together gives,

$$\|\Phi(L_n)f - \Phi(L)f\|_H \rightarrow 0 \quad \text{as } n \rightarrow \infty$$