b) $u_{xx} + u_{yy} = \cos(x+y)$,
$u_x v_y - u_y v_x = 1$

Here ~~This~~ This is a second-order system of differential equations.
x, y are the independent variables.
u, v are the dependent variables.

c) $\frac{\partial u}{\partial t} = \frac{\partial v}{\partial x}$, $\frac{\partial^2 v}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$.

This is a second-order system of differential equations.
u, v are the dependent variables.
x, t are the independent variables.

d) $u_t + u u_x + v u_y = p_x$
$v_t + u v_x + v v_y = p_y$,
$u_x + v_y = 0$;

This is a first-order system of differential equations.
u, v, p are the dependent variables.
t, x, y are the independent variables.

e) $u_t = v_{xxx} + v(1-v)$,
$v_t = u_{xxy} + vw$
$w_t = u_x + u_y$.

This is a third order system of differential equations.
Here, u, v, w are the dependent variables.
t, x, y are the independent variables.

3. Classical Solutions.

Let us now focus our attention on a single differential equation involving a single scalar-valued function u that depends on one or more independent variables. The function u is usually real-valued, although complex functions can and do, play a role in the analysis. Everything that we say in this section will, when suitably adapted, apply to systems of differential equations.

By a solution, we mean a sufficiently smooth function u of the independent variables that satisfies the differential equation at every point of its domain of definition. We do not necessarily require that the solution be defined for all possible values of the independent variable. Indeed, usually the differential equation is imposed on some domain D contained in the space of independent variables, and we seek a solution defined only on D. In general, the domain D will be an open subset usually connected, and in particular in equilibrium equations, often bounded, with a reasonably nice boundary, denoted by ∂D.

We will call a function smooth if it can be differentiated sufficiently often, atleast so that all of the derivatives appearing in the equation are well-defined on the domain of interest D. More specifically, if the differential equation has order n, then we require that the solution u must belong to function space $C^n(D)$, which means that it and all its derivatives of order $\leq n$ are continuous functions in D and such that the differential equation that relates the derivatives of u holds throughout D. However, on occasion when dealing with shock waves, we will consider the most general types of solutions. The most important such class consists of the so-called weak-solutions to be introduced in section 10.4. To emphasize the distinction, the smooth solutions described above are often referred to as classical solutions. In this course, the term solution without extra classification will usually mean classical solution.

Examples

1) A classical solution to the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad (1\cdot 6)$$

is a function $u(t,x)$ defined on a domain $D \subset \mathbb{R}^2$, such that all of the functions

$$u(t,x),\ \frac{\partial u}{\partial t}(t,x),\ \frac{\partial u}{\partial x}(t,x),\ \frac{\partial^2 u}{\partial t^2}(t,x),\ \frac{\partial^2 u}{\partial t \partial x}(t,x) = \frac{\partial^2 u}{\partial x \partial t}(t,x),\ \frac{\partial^2 u}{\partial x^2}(t,x)$$

are well-defined and continuous at every point $(t,x) \in D$, so that $u \in C^2(D)$ and moreover (1.5) holds for every $(t,x) \in D$. Observe that even though u_t and u_{xx} appear in the equation explicitly in the heat equation, we require continuity of all partial derivatives of order ≤ 2 in order that u qualifies as a classical solution. For example,

$$u(t,x) = t + \frac{x^2}{2}. \qquad (1\cdot 6)$$

is a solution to the heat equation that is defined on the full domain $D = \mathbb{R}^2$ because it is C^2 and moreover,

$$\frac{\partial u}{\partial t} = 1 = \frac{\partial^2 u}{\partial x^2}.$$

Another more complicated but extremely important solution is

$$u(t,x) = \frac{e^{-x^2/4t}}{2\sqrt{\pi t}}. \qquad (1\cdot 7)$$

One easily verifies that u belongs to C^2 and moreover solves the heat equation on the domain $D = \{(t,x) \mid t > 0\} \subset \mathbb{R}^2$. The reader is invited to verify this by computing $\frac{\partial u}{\partial t}$ and $\frac{\partial^2 u}{\partial x^2}$, and then checking that they are equal.

Verification

$$u = \frac{e^{-x^2/4t}}{2\sqrt{\pi t}}.$$

$$\frac{\partial u}{\partial t} = \frac{2\sqrt{\pi t}\,(e^{-x^2/4t})' - (e^{-x^2/4t})\cdot(2\sqrt{\pi t})'}{(2\sqrt{\pi t})^2}$$

$$= \frac{2\sqrt{\pi}\cdot \sqrt{t}\cdot(e^{-x^2/4t})\cdot(-x^2/4)\cdot(1/t)' - (\sqrt{t})'}{4\pi t}$$

$$= \frac{\cancel{2\sqrt{\pi}}}{2\cancel{4\pi t}}\left[\sqrt{t}\, e^{-x^2/4t}\cdot\left(-\frac{x^2}{4}\right)\left(-\frac{1}{t^2}\right) - \frac{e^{-x^2/4t}}{2\sqrt{t}}\right]$$

$$= \frac{e^{-x^2/4t}}{2\sqrt{\pi}\, t}\left[+\frac{x^2}{4t\sqrt{t}} - \frac{1}{2\sqrt{t}}\right]$$

$$= \frac{e^{-x^2/4t}}{4\sqrt{\pi}\, t\sqrt{t}}\left[\frac{x^2}{2t} - 1\right] = \frac{e^{-x^2/4t}\,(x^2 - 2t)}{8\sqrt{\pi}\, t^{5/2}}.$$

$$\frac{\partial u}{\partial x} = \frac{1}{2\sqrt{\pi t}}\,(e^{-x^2/4t})'_x = \frac{1}{2\sqrt{\pi t}}\, e^{-x^2/4t}\cdot\left(-\frac{x^2}{4t}\right)'_x = \frac{1}{2\sqrt{\pi t}}\, e^{-x^2/4t}\cdot\frac{(-2x)}{4t}$$

$$= -\frac{x\, e^{-x^2/4t}}{4\sqrt{\pi}\, t^{3/2}}.$$

$$\frac{\partial^2 u}{\partial x^2} = -\frac{1}{4\sqrt{\pi}\, t^{3/2}} \left(x e^{-x^2/4t}\right)'$$

$$= -\frac{1}{4\sqrt{\pi}\, t^{3/2}} \left[e^{-x^2/4t} + x\left(e^{-x^2/4t}\right)' \right]$$

$$= -\frac{1}{4\sqrt{\pi}\, t^{3/2}} \left[e^{-x^2/4t} + x e^{-x^2/4t} \left(\frac{-x^2}{4t}\right)'_x \right]$$

$$= -\frac{e^{-x^2/4t}}{4\sqrt{\pi}\, t^{3/2}} \left[1 + \frac{x}{4t} \cdot (-2x) \right]$$

$$= \frac{e^{-x^2/4t}}{4\sqrt{\pi}\, t^{3/2}} \left(\frac{x^2}{2t} - 1 \right)$$

$$= \frac{e^{-x^2/4t}\,(x^2 - 2t)}{8\sqrt{\pi}\, t^{3/2}}.$$

Thus, $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$.

Finally, with $i = \sqrt{-1}$ denoting the imaginary unit, we note that

$$u(t,x) = e^{-t+ix} = e^{-t}(\cos x + i \sin x)$$

the second expression following from Euler's formula defines a complex-valued solution to the heat equation. This can be verified directly, since the rules for differentiating complex exponentials are identical to those for their real counterparts.

$$\frac{\partial u}{\partial t} = -e^{-t+ix} \quad \frac{\partial u}{\partial x} = i e^{-t+ix} \quad \frac{\partial^2 u}{\partial x^2} = -e^{-t+ix}$$

So, $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$.

It is worth pointing out that both the real part, $e^{-t}\cos x$ and the imaginary part $e^{-t}\sin x$ of the complex solution are individual real solutions, which is indicative of a fairly general property.

Incidentally, most partial differential ~~a~~ equations arising in physical applications are real, although complex solutions often facilitate their analysis, at the end of the day we require real, physically meaningful solutions. A notable exception is quantum mechanics, which is an inherently complex-valued physical theory. For example, the one-dimensional Schrodinger equation

$$i\hbar \frac{\partial u}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 u}{\partial x^2} + V(x)\, u$$

where $\hbar$ denoting Planck's constant, which is real governs the dynamical evolution of the complex-valued wave function $u(t,x)$, describing the probabilistic distribution of a quantum particle of mass m, e.g. an electron, moving in the force field described by the potential function $V(x)$. While the solution u is complex valued, the independent variable t, x representing time and space, remain real.

4. Initial Conditions and Boundary Conditions.