1. Vector spaces.

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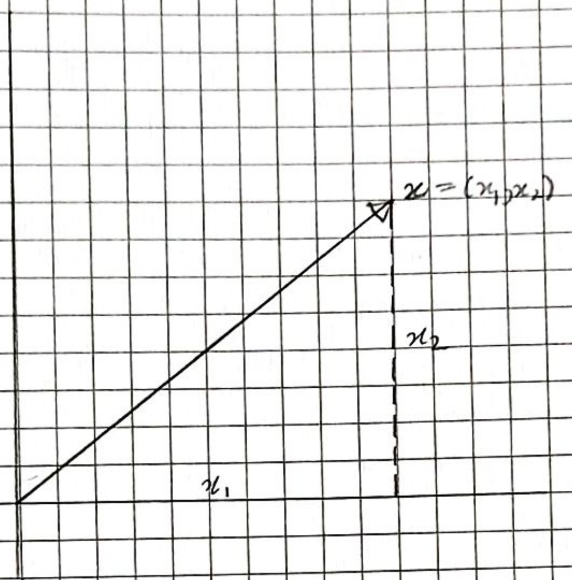
1. Introduction.

The Cartesian plane of analytic geometry, denoted by $\mathbb{R}^2$, is one of the most familiar examples of what is known in mathematics as a real vector space. Each of its points or vectors, is an ordered pair (x_1, x_2) of real numbers, whose individual entries are called the components of the vector. Geometrically, the vector $x = (x_1, x_2)$ may be represented by means of an arrow drawn from the origin of the coordinate system to the point (x_1, x_2).

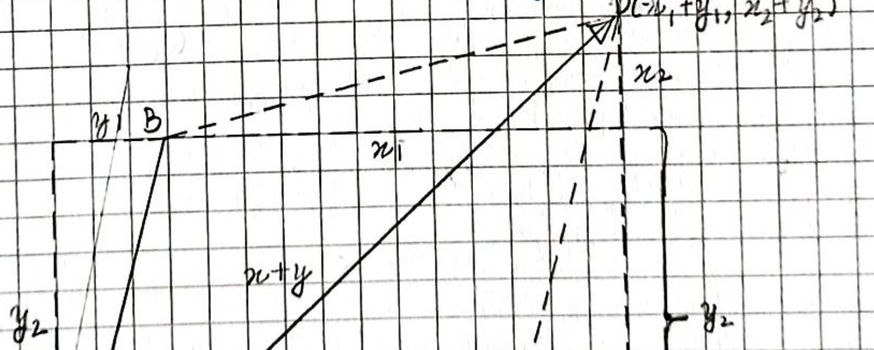


Geometric Vector.

If $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are any two vectors in $\mathbb{R}^2$, then by their vector sum is defined by -

$$x + y = (x_1 + x_2, y_1 + y_2). \tag{1.1}$$

obtained by adding the corresponding components of x and y. The graphical interpretation of this is the familiar parallelogram law, which states that, if two vectors $x = (x_1, x_2)$ and $y = (y_1, y_2)$ lie along the adjacent sides of a parallelogram, then the diagonal passing through the point of contact of the two vectors represents the vector sum $x + y$.



Parallelogram law of vector addition.

It follows from (1·1) that vector addition is both associative and commutative.

$$
\begin{aligned}
x + y &= (x_1, x_2) + (y_1, y_2) \\
&= (x_1 + y_1, x_2 + y_2) \\
&= (y_1 + x_1, y_2 + x_2) \\
&= (y_1, y_2) + (x_1, x_2) \\
&= y + x. \qquad (1\cdot2)
\end{aligned}
$$

And,

$$(x + y) + z = x + (y + z) \qquad (1\cdot3)$$

Moreover, if we let 0 denote the vector (0,0) and $-x$ the vector $(-x_1, -x_2)$ obtained by reflecting $x = (x_1, x_2)$ across the origin, then

$$
\begin{aligned}
x + 0 &= (x_1, x_2) + (0, 0) \\
&= (x_1 + 0, x_2 + 0) \\
&= (x_1, x_2) \\
&= x \qquad (1\cdot4)
\end{aligned}
$$

$$
\begin{aligned}
x + (-x) &= (x_1, x_2) + (-x_1, -x_2) \\
&= (x_1 - x_1, x_2 - x_2) \\
&= (0, 0) \\
&= 0. \qquad (1\cdot5)
\end{aligned}
$$

for every x. Taken together, the equations (1·2) – (1·5) simply mean that vector addition behaves very much like ordinary addition of arithmetic.

As well as being able to add vectors in $\mathbb{R}^2$, we can also form the product of a real number (scalar) α and a vector x. The result αx, is the vector ~~of~~, each of whose components are α times the corresponding component of x. Thus, if $x = (x_1, x_2)$, then

$$\alpha x = (\alpha x_1, \alpha x_2) \qquad (1\cdot6)$$

Geometrically, this vector can be viewed as a magnification of x by the factor α. The principal algebraic properties of this multiplication are the following.

$$
\begin{aligned}
(\alpha\beta) x &= \alpha(\beta x) \qquad (1\cdot7)\\
1x &= x \qquad (1\cdot8)\\
(\alpha + \beta) x &= \alpha x + \beta x \qquad (1\cdot9)\\
\alpha(x + y) &= \alpha x + \alpha y \qquad (1\cdot10)
\end{aligned}
$$

The validity of each of these equations can be deduced easily from the definition of the operations involved, and we prove by way of illustration equation (1·9).

$$
\begin{aligned}
(\alpha + \beta) x &= (\alpha + \beta)(x_1, x_2) \\
&= ((\alpha + \beta) x_1, (\alpha + \beta) x_2) \\
&= (\alpha x_1 + \beta x_1, \alpha x_2 + \beta x_2) \\
&= (\alpha x_1, \alpha x_2) + (\beta x_1, \beta x_2) \\
&= \alpha x + \beta x
\end{aligned}
$$

The reason for calling attention to the properties (1·7) – (1·10) is that they together with properties (1·2) through (1·5) for vector addition are precisely what make $\mathbb{R}^2$ a real vector space.

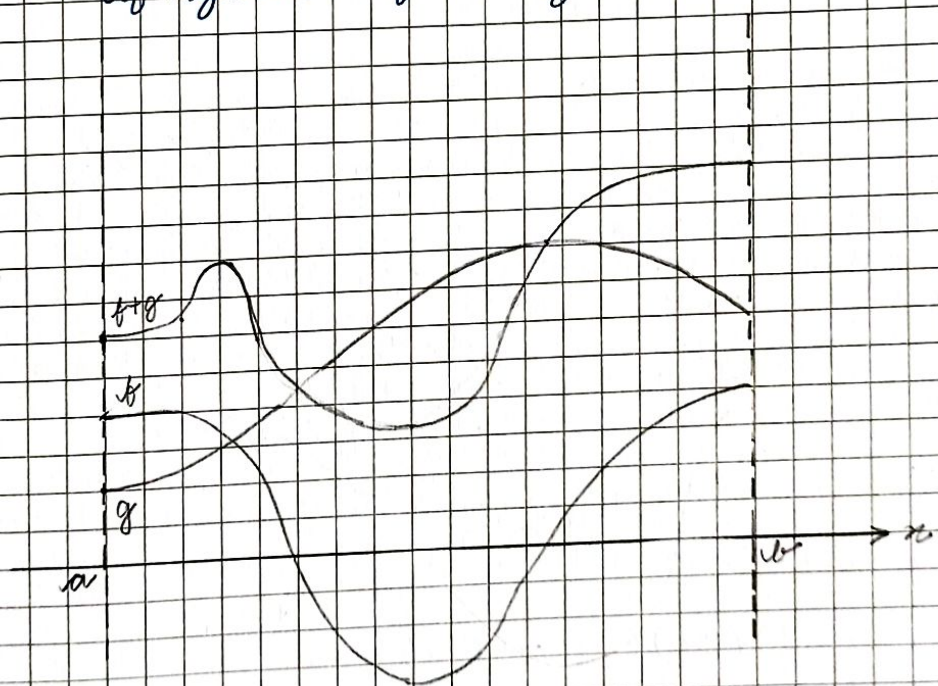
Indeed these results are none other than the general definition of such a space, and once this definition has been given, the above discussion constitutes a verification of the fact that $\mathbb{R}^2$ is a real vector space. But, before giving this definition, we pause to look at another example.

This time, we consider $C[a,b]$ - the set of all real-valued continous functions defined on a closed interval $[a,b]$ of the real number line. For reasons which will shortly become clear, we shall call any such function a vector. Thus, f is a vector in $C[a,b]$, if and only if, f is a continuous real valued function over the interval $[a,b]$.

At first sight, it may seem $C[a,b]$ and $\mathbb{R}^2$ have nothing in common, but the name real vector space. However, this is one of those instances in which first impressions are misleading, for as we shall see, these spaces are remarkably similar. This similarity arises from the fact that an addition and scalar multiplication can also be defined in $C[a,b]$ and that these operations enjoy the same properties as the corresponding operations in $\mathbb{R}^2$.

Turning first to addition, let f and g be any two vectors in $C[a,b]$. Then their sum, $f+g$ is defined to be the function (vector) whose value at any point x in $[a,b]$ is the sum of the values of f and g at x. In other words,

$$(f+g)(x) = f(x) + g(x). \tag{1.11}$$



At this point, it is important to observe that since the sum of two continuous functions is continuous, the definition is meaningful in the sense that $f+g$ is again a vector in $C[a,b]$.

It is now easy to verify that apart from the notation, equations (1.2) - (1.5) remain valid in $C[a,b]$.
In fact,

$$f+g = g+f \tag{1.12}$$

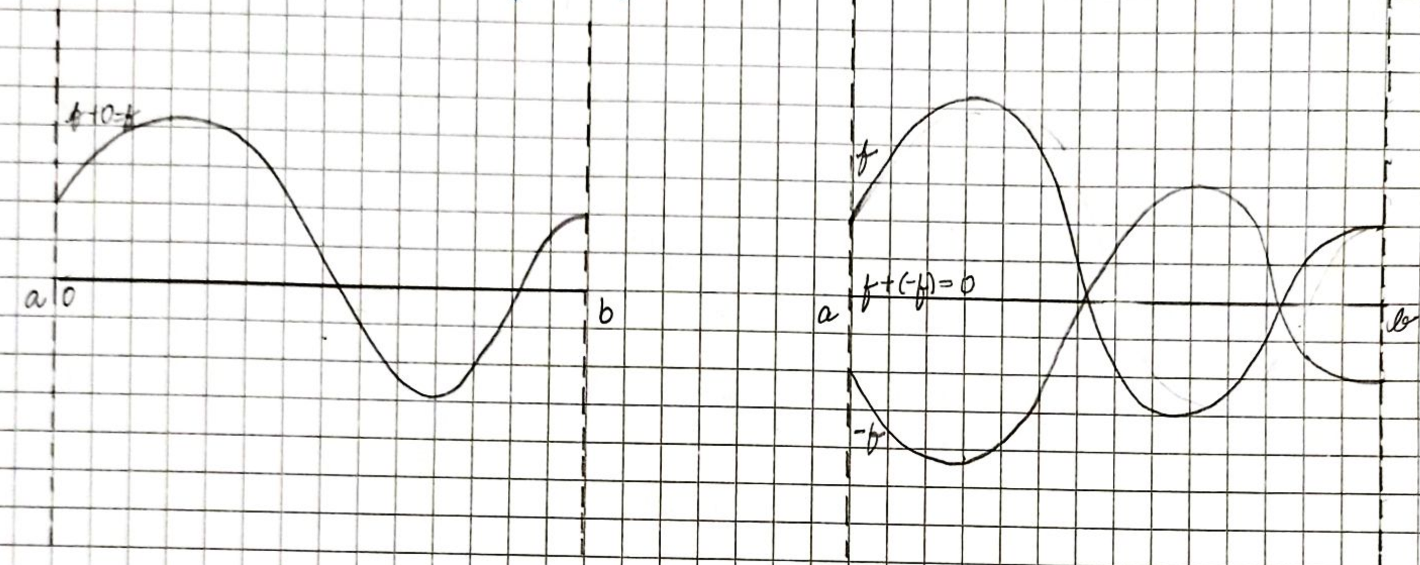
$$f+(g+h) = (f+g)+h \tag{1.13}$$

follow immediately from (1.11), while if 0 denotes the function whose value is, zero at each point of $[a,b]$ then

$$f + 0 = f \tag{1.14}$$

for every $f \in C[a,b]$. Finally, if $-f$ is the function whose value at x is $-f(x)$ (i.e. $-f$ is the reflection of f across O), then $f(x) + (-f(x))$ has the value zero at each point x of the interval $[a,b]$, and we have -

$$f + (-f) = 0. \tag{1.15}$$



We have seen that the sum of two vectors in $\mathbb{R}^2$ is found by adding their corresponding components - equation (1·1). A similar interpretation of vector addition is possible in the present example and may be achieved as follows. If f is any vector in $C[a,b]$, we agree to say that the component of f at the point x is its functional value at x. Of course, every vector in $C[a,b]$ then has infinitely many components, one for each x in the interval $[a,b]$, but once this fact has been accepted, it becomes clear that the equation (2·11) simply states that the sum of two vectors in $C[a,b]$ is obtained by adding the corresponding components just as in $\mathbb{R}^2$.

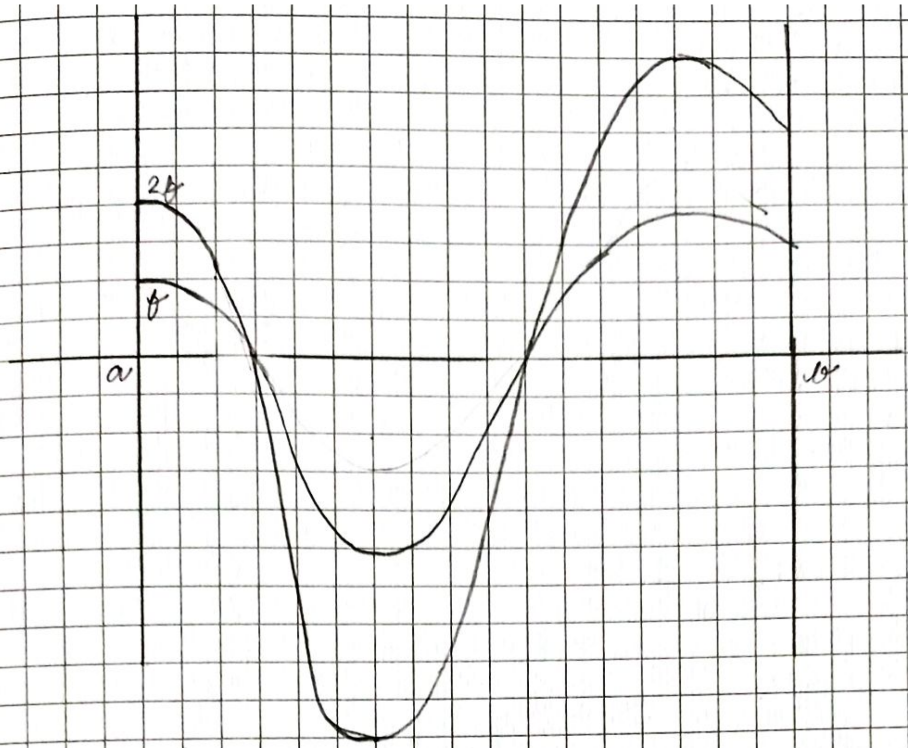
~~Next that~~, if f is any vector in $C[a,b]$ and α is any arbitrary real number, we define the product αf to be the vector in $C[a,b]$ whose defining equation is -

$$(\alpha f)(x) = \alpha f(x) \tag{1.16}$$

In other words, αf is the function whose value at x is the product of the real numbers α and $f(x)$. The similarity between this multiplication and the corresponding operation in $\mathbb{R}^2$ is clear since αf is also formed by multiplying each component of f by α. The next figure illustrates this multiplication when $\alpha = 2$.

The analogy with $\mathbb{R}^2$ is now complete, for equations (1·7) – (1·10) are also valid in $C[a,b]$. We restate them as -

$$\begin{aligned}
(\alpha\beta) f &= \alpha(\beta f) \\
1 \cdot f &= f \\
(\alpha + \beta) f &= \alpha f + \beta f \\
\alpha(f + g) &= \alpha f + \alpha g
\end{aligned}$$



scalar multiplication of f by a real number α

Problems.
For problems 1 through 5, compute the value $x+y$ and $\alpha(x+y)$ for the given vectors x and y in $\mathbb{R}^2$ and the real number α.

1. $x = (0,2),\ y = (-1,1),\ \alpha = 3$

solution.

$$x+y = (0,2) + (-1,1)$$
$$= (0-1,\ 2+1)$$
$$= (-1,3)$$
$$\alpha(x+y) = 3(-1,3)$$
$$= (-3,9)$$

2. $x = \left(\frac{1}{2}, 1\right),\ y = (1,-2),\ \alpha = -2.$

solution.

$$x+y = \left(\frac{1}{2}+1,\ 1-2\right)$$
$$= \left(\frac{3}{2}, -1\right)$$
$$\alpha(x+y) = -2\left(\frac{3}{2}, -1\right)$$
$$= (-3, 2).$$

3. $x = \left(-\frac{1}{2}, \frac{1}{3}\right),\ y = (-2,-1),\ \alpha = -1.$

solution.

$$x+y = \left(\frac{-1}{2} - 2,\ \frac{1}{3} - 1\right)$$
$$= \left(\frac{-5}{2}, \frac{-2}{3}\right).$$
$$\alpha(x+y) = (-1)\left(\frac{-5}{2}, \frac{-2}{3}\right)$$
$$= \left(\frac{5}{2}, \frac{2}{3}\right).$$

4. $x = (5,-2)$, $y = (-3,2)$, $\alpha = \frac{1}{2}$.

Solution.

$$x + y = (5,-2) + (-3,2) = (2,0)$$

$$\alpha(x+y) = \frac{1}{2}(2,0) = (1,0)$$

5. $x = (-5,-2)$, $y = (-1,-1)$, $\alpha = -3$.

$$x + y = (-5,-2) + (-1,-1) = (-6,-3)$$

$$\alpha(x+y) = -3(-6,-3) = (18, 9)$$

6. $f(x) = 2x$, $g(x) = x^2 - x + 1$, $\alpha = 2$.

Solution.

$$(f+g)(x) = f(x) + g(x) = x^2 + x + 1$$

$$(\alpha(f+g))(x) = \alpha f(x) + \alpha g(x) = 4x + 2x^2 - 2x + 2 = 2x^2 + 2x + 2.$$

7. $f(x) = \tan^2 x$, $g(x) = 1$, $\alpha = -1$.

Solution.

$$(f+g)(x) = 1 + \tan^2 x = \sec^2 x$$

$$(\alpha(f+g))(x) = -\sec^2 x.$$

8. $f(x) = e^x$, $g(x) = e^{-x}$, $\alpha = \frac{1}{2}$.

Solution

$$(f+g)(x) = e^x + e^{-x}$$

$$(\alpha(f+g))(x) = \frac{e^x + e^{-x}}{2}.$$

9. $f(x) = \frac{x+3}{x-2}$, $g(x) = -\frac{x-2}{x+3}$, $\alpha = \frac{1}{5}$.

Solution.

$$(f+g)(x) = \frac{x+3}{x-2} - \frac{x-2}{x+3}$$

$$= \frac{(x+3)^2 - (x-2)^2}{(x+3)(x-2)}$$

$$= \frac{(\cancel{x^2} + 6x + 9) - (\cancel{x^2} - 4x + 4)}{(x+3)(x-2)}$$

$$= \frac{10x + 5}{(x+3)(x-2)}$$

$$(\alpha(f+g))(x) = \frac{2x+1}{(x+3)(x-2)}$$

10. $f(x) = \cos^2 x$, $g(x) = \sin^2 x$, $\alpha = -3$.

Solution.

$$(f+g)(x) = f(x) + g(x) = \cos^2 x + \sin^2 x = 1$$

$$(\alpha(f+g))(x) = -3.$$

11. $\sin\left(\frac{x+1}{x-1}\right)$.

The function $f(x) = \sin\left(\frac{x+1}{x-1}\right)$ is not defined at $x=1$. Hence, it does not belong to $C[-1,1]$.

12. $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$

Solution.

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} |x| = \lim_{x\to 0} (x) = 0$

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} |x| = \lim_{x\to 0} (-x) = 0.$

$f(0) = 0.$

f is continuous at $x=0$. Hence f belongs to $C[a,b]$.

2. Vector spaces.

Before defining what a vector space is, let's establish the definition of a few important sets in all of mathematics.

The set of all real numbers in mathematics is denoted by $\mathbb{R}$.

$$\mathbb{R} := \{x : -\infty < x < \infty\}$$

The set of all ordered pairs of real numbers, which you can think of as a cartesian plane is denoted by $\mathbb{R}^2$.

$$\mathbb{R}^2 := \{(x_1, x_2) : x_1, x_2 \in \mathbb{R}\}$$

The set $\mathbb{R}^3$ which you can think of as ordinary space, consists of all ordered triples of real numbers:

$$\mathbb{R}^3 := \{(x_1, x_2, x_3) : x_1, x_2, x_3 \in \mathbb{R}\}.$$

To generalize $\mathbb{R}^2$ and $\mathbb{R}^3$ to higher dimensions, we first need to discuss the concept of tuples. Suppose n is a non-negative integer, $n \geq 0$. A list of length n, is an ordered collection of n objects (which might be numbers, other lists or more abstract entities). A list of length n looks like this

$$(x_1, x_2, \ldots, x_n).$$

Thus, a list of length 2 is an ordered pair, a list of length 3 is an ordered triple. For $j = \{1, 2, \ldots, n\}$, we say x_j is jth coordinate of the list above.

To define higher dimensional analogues of $\mathbb{R}^2$ and $\mathbb{R}^3$ we simply replace $\mathbb{R}$ with $\mathbb{F}$ (which equals $\mathbb{R}$ or $\mathbb{C}$) and replace 2 or 3 with n. We define $\mathbb{F}^n$ to be the set of all n-tuples, consisting of elements of $\mathbb{F}$:

$$\mathbb{F}^n = \{(x_1, x_2, \ldots, x_n) : x_j \in \mathbb{F} \ \forall j \in \{1, 2, 3, \ldots, n\}\}.$$

If $n \geq 4$, we cannot easily visualize $\mathbb{R}^n$ as a physical object. The same problem arises with complex numbers: $\mathbb{C}$ can be thought of as a plane, but for $n \geq 2$, the human brain cannot provide geometric models of $\mathbb{C}^n$. However, even when n is large, we can perform algebra on vectors in $\mathbb{F}^n$ as easily as in $\mathbb{R}^2$ or $\mathbb{R}^3$.