and the vectors $(0,2)$ and $(1,0)$ are linearly independent in $\mathbb{R}^2$, we conclude that $\sin 2x$, and $\cos 2x$ are linearly independent in $\mathcal{C}(-\infty,\infty)$. Hence, they are basis for the solution space of (4.39) and the general solution of the equation is

$$y(x) = c_1 \sin 2x + c_2 \cos 2x.$$

At this point it is impossible to escape the conclusion that in proving theorem (4.3), we also established a method for testing functions for linear independence. This fact is well worth bringing out into the open, since it will be used in the following sections to obtain a number of important results concerning linear differential equations. Specifically, we have:

Corollary 4.1. Let $y_1(x), y_2(x), \ldots, y_n(x)$ be functions in $\mathcal{C}(I)$, each of which possesses derivatives upto and including those of order $(n-1)$ everywhere in I, and suppose that at some point x_0 in I, the vectors

$$(y_i(x_0), y_i'(x_0), \ldots, y_i^{(n-1)}(x_0)), \quad i = 1, 2, 3, \ldots, n. \qquad (4.40)$$

are linearly independent in $\mathbb{R}^n$. Then, $y_1(x), \ldots, y_n(x)$ are linearly independent in $\mathcal{C}(I)$.

4. The functions

$$e^x, xe^x, x^2e^x$$

are linearly independent in $\mathcal{C}(-\infty,\infty)$, since the above test applied at $x=0$ yields the following vectors:

$$\begin{aligned}
&y_1(0) = 1, \quad y_1'(0) = 1, \quad y_1''(0) = 1\\
&y_2(0) = 0, \quad y_2'(0) = 1, \quad y_2''(0) = 2\\
&y_3(0) = 0, \quad y_3'(0) = 0, \quad y_3''(0) = (2xe^x + x^2e^x)'|_{x=0}\\
&\qquad = (2e^x + 2xe^x + 2xe^x + x^2e^x)|_{x=0}\\
&\qquad = (2e^x + 4xe^x + x^2e^x)|_{x=0} = 2.
\end{aligned}$$

The vectors $(1,1,1)$, $(0,1,2)$ and $(0,0,2)$ are linearly independent in $\mathbb{R}^3$. For if:

$$c_1\begin{pmatrix}1\\1\\1\end{pmatrix} + c_2\begin{pmatrix}0\\1\\2\end{pmatrix} + c_3\begin{pmatrix}0\\0\\2\end{pmatrix} = 0$$

$$\left.\begin{aligned} c_1 &= 0\\ c_1 + c_2 &= 0\\ c_1 + 2c_2 + 2c_3 &= 0\end{aligned}\right\} \Rightarrow c_1 = 0, c_2 = 0, c_3 = 0$$

Thus, the functions e^x, xe^x, x^2e^x are linearly independent in $\mathcal{C}(-\infty,\infty)$.

4.6 The Wronskian.

In the preceding section, we proved that $y_1, y_2, \ldots, y_n$ are linearly independent functions in $\mathcal{C}^{n-1}(I)$ whenever there exists a point x_0 in I, such that the vectors

$$(y_i(x_0), y_i'(x_0), \ldots, y_i^{(n-1)}(x_0)), \quad i = 1, \ldots, n \qquad (4.41)$$

are linearly independent in $\mathbb{R}^n$.

For our present purposes, this result can be stated more conveniently in terms of the determinant of a certain matrix, as follows:

Let $y_1, \dots, y_n$ be arbitrary functions in $\mathcal{C}^{n-1}(I)$, and for each x in I, consider the matrix

$$\begin{bmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y_1'(x) & y_2'(x) & \cdots & y_n'(x) \\ y_1''(x) & y_2''(x) & \cdots & y_n''(x) \\ y_1'''(x) & y_2'''(x) & \cdots & y_n'''(x) \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & & y_n^{(n-1)}(x) \end{bmatrix} \qquad (4.42)$$

Then (4.42) defines a function on the interval I, whose value at x is indicated ~~by the~~ matrix, and by forming the determinant of this matrix, we obtain a real valued function on I known as the Wronskian of $y_1, \dots, y_n$. This function will be denoted by $W[y_1, \dots, y_n]$ to indicate its dependence on $y_1, \dots, y_n$ and its value at x by $W[y_1(x), y_2(x), \dots, y_n(x)]$. In short, the Wronskian of $y_1, \dots, y_n$ is the real-valued function whose defining equation is

$$W[y_1(x), y_2(x), \dots, y_n(x)] = \begin{vmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y_1'(x) & y_2'(x) & & y_n'(x) \\ y_1''(x) & y_2''(x) & & y_n''(x) \\ y_1'''(x) & y_2'''(x) & & y_n'''(x) \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{vmatrix} \qquad (4.43)$$

For example,

$$W[x, \sin x] = \begin{vmatrix} x & \sin x \\ 1 & \cos x \end{vmatrix} = x\cos x - \sin x.$$

$$W[x, 2x] = \begin{vmatrix} x & 2x \\ 1 & 2 \end{vmatrix} = 2x - 2x = 0.$$

We now recall that the determinant of an $n \times n$ matrix is non-zero if and only if the columns of the matrix are linearly independent vectors in $\mathbb{R}^n$. Thus, the Wronskian of $y_1, \dots, y_n$ is different from zero at x_0, if and only if the columns of (4.42) are linearly independent when $x = x_0$. But, for each x_0 in I, the columns of (4.42) are none other than the vectors in (4.40), and therefore we have the following theorem.

<u>Theorem 4.4</u>. The functions $y_1, y_2, \dots, y_n$ are linearly independent in $\mathcal{C}^{n-1}(I)$, and hence also in $\mathcal{C}(I)$, whenever their Wronskian is not identically zero on I.

<u>Examples.</u>

1) since $W[e^x, e^{-x}] = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -1 - 1 = -2.$

The functions e^x, e^{-x} are linearly independent in $\mathcal{C}(I)$ for any interval I.

2) The functions $x, x^{1/2}, x^{3/2}$ are linearly independent in $C(I)$ for any sub-interval I of the positive x-axis since

$$W[x, x^{1/2}, x^{3/2}] = \begin{vmatrix} x & x^{1/2} & x^{3/2} \\ 1 & \frac{1}{2}x^{-1/2} & \frac{3}{2}x^{1/2} \\ 0 & -\frac{1}{4}x^{-3/2} & +\frac{3}{4}x^{-1/2} \end{vmatrix}$$

$$= x \begin{vmatrix} \frac{1}{2}x^{-1/2} & \frac{3}{2}x^{1/2} \\ -\frac{1}{4}x^{-3/2} & \frac{3}{4}x^{-1/2} \end{vmatrix} - 1 \begin{vmatrix} x^{1/2} & x^{3/2} \\ -\frac{1}{4}x^{-3/2} & \frac{3}{4}x^{-1/2} \end{vmatrix}$$

$$= x\left(\frac{3}{8} + \frac{3}{8}\cdot\frac{1}{x}\right) - 1\left(\frac{3}{4} + \frac{1}{4}\right)$$

$$= \frac{3x}{8} + \frac{3}{8} - \frac{1}{2}$$

$$= \frac{3x}{8} - \frac{1}{4}.$$

The Wronskian will be different from 0 for all positive x. Hence, the functions $x, x^{1/2}, x^{3/2}$ are linearly independent.

3) Verify ~~that~~ if the below functions
(a) $y_1(x) = 9\cos(2x)$, $y_2(x) = 2\cos^2 x - 2\sin^2 x$
(b) $f(t) = 2t^2$, $g(t) = t^4$
are linearly independent sets.

Solution.

a)

$$W(x) = \begin{vmatrix} 9\cos 2x & 2\cos^2 x - 2\sin^2 x \\ -18\sin 2x & -4\sin x\cos x - 4\sin x\cos x \end{vmatrix}$$

$$= \begin{vmatrix} 9\cos 2x & 2\cos 2x \\ -18\sin 2x & -4\sin 2x \end{vmatrix}$$

$$= -36\sin 2x\cos 2x + 36\sin 2x\cos 2x$$
$$= 0.$$

The Wronskian is identically equal to zero. Hence, the functions are

b)

$$W[f,g] = \begin{vmatrix} f(t) & g(t) \\ f'(t) & g'(t) \end{vmatrix} = \begin{vmatrix} 2t^2 & t^4 \\ 4t & 4t^3 \end{vmatrix}$$

$$= 8t^5 - 4t^5$$
$$= 4t^5.$$

The Wronskian is non-zero for all $t \neq 0$. This is not a problem. As long as the Wronskian is not identically equal to zero for all $t \in (-\infty, \infty)$, we are okay.

4) Determine if the following functions are linearly independent.
(a) $f(t) = \cos t$, $g(t) = \sin t$
(b) $y_1(x) = 6^x$, $y_2(x) = 6^{x+2}$
~~(c) $\{\sin t, \cos t\}$~~

Solution.

a) $$W[f,g] = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = \cos^2 t + \sin^2 t = 1.$$

The Wronskian is not identically equal to zero for all $t \in (-\infty, \infty)$. Hence, $\{\cos t, \sin t\}$ is a linearly independent set.

b) $\{6^x, 6^{x+2}\}$

$$W[y_1, y_2] = \begin{vmatrix} 6^x & 6^{x+2} \\ 6^x \log 6 & 6^{x+2} \cdot \log 6 \end{vmatrix}$$
$$= 6^{2x+2} \log 6 - 6^{2x+2} \log 6$$
$$= 0.$$

The Wronskian is identically equal to zero for all x in $\mathbb{R}$. Hence, the functions are not linearly independent.

5) The functions x^3 and $|x|^3$ are linearly independent on $\mathcal{C}(-\infty, \infty)$, for if $c_1 x^3 + c_2 |x|^3 \equiv 0$, then

$$c_1 (1)^3 + c_2 |+1|^3 = 0$$
$$c_1 (-1)^3 + c_2 |-1|^3 = 0$$

which yields $\left.\begin{array}{l} c_1 + c_2 = 0 \\ -c_1 + c_2 = 0 \end{array}\right\} \cdot \; c_1 = 0, \; c_2 = 0.$

On the other hand, the Wronskian of x^3 and $|x|^3$ is identically zero ~~for~~ on $(-\infty, \infty)$, since:

If $x \geq 0$,

$$W[x^3, |x|^3] = \begin{vmatrix} x^3 & x^3 \\ 3x^2 & 3x^2 \end{vmatrix} = 0$$

If $x < 0$,

$$W[x^3, |x|^3] = \begin{vmatrix} x^3 & -x^3 \\ 3x^2 & -3x^2 \end{vmatrix} = -3x^5 + 3x^5 = 0.$$

Thus, the converse of the theorem (4.4), if $y_1, \ldots, y_n$ are linearly independent $\rightarrow W[y_1, \ldots, y_n] \not\equiv 0$ is false, and one cannot linear dependence of a set of functions in $\mathcal{C}(I)$ from the fact that their Wronskian vanishes identically on I.

This example notwithstanding, it is true that the Wronskian of a linearly dependent set of functions in $\mathcal{C}(I)$ vanishes identically on I, provided of course that the Wronskian exists in the first place. Hence, rather than abandon the search for a converse to the theorem (4.4), we weaken our requirements and ask, whether it is possible to impose additional conditions on a set of functions which together with the vanishing of the Wronskian will imply linear ~~in~~dependence.

This can in fact be done, simply by requiring that the functions be solutions of the homogeneous linear differential equation. We prove this assertion as:

<u>Theorem (4.5</u>. Let $y_1, y_2, \ldots, y_n$ be the solutions of a nth order homogeneous linear differential equation.

$$a_n(x) \frac{d^n y}{dx^n} + \cdots + a_0(x) y = 0 \qquad (4.44)$$

on an interval I and suppose that $W[y_1, \ldots, y_n]$ is identically zero on I. Then, $y_1, \ldots, y_n$ are linearly dependent in $\mathcal{C}(I)$.

Proof.

Let x_0 be any point ~~on~~ in I, and consider the system of equations.

$$\left.\begin{aligned} c_1 y_1(x_0) + c_2 y_2(x_0) + \dots + c_n y_n(x_0) &= 0 \\ c_1 y_1'(x_0) + c_2 y_2'(x_0) + \dots + c_n y_n'(x_0) &= 0 \\ &\vdots \\ c_1 y_1^{(n-1)}(x_0) + c_2 y_2^{(n-1)}(x_0) + \dots + c_n y_n^{(n-1)}(x_0) &= 0 \end{aligned}\right\} \qquad (4.45)$$

in the unknowns $c_1, c_2, \dots, c_n$. Since the Wronskian of $y_1, \dots, y_n$ vanishes identically on I, the determinant of (4.45), ~~the determinants of (4.45)~~ is zero, and the system has a nontrivial solution. $(\bar{c}_1, \bar{c}_2, \dots, \bar{c}_n)$. Thus, the function

$$y(x) = \sum \bar{c}_i\, y_i(x).$$

is a solution of the initial-value problem consisting of (4.44) and the initial conditions

$$y(x_0) = 0,\ y'(x_0) = 0, \dots, y^{(n-1)}(x_0) = 0.$$

But, the zero function is also a solution of this problem and hence theorem (4.2) implies that

$$\bar{c}_1 y_1(x) + \bar{c}_2 y_2(x) + \bar{c}_3 y_3(x) + \dots + \bar{c}_n y_n(x) = 0$$

for all x in I. The linear dependence of $y_1, \dots, y_n$ follows from the fact that not all c_i's are zero.
This closes the proof.

Once again we have established a result which is stronger than the one advertised. For the above proof, we only made use of the fact that the Wronskian $y_1, \dots, y_n$ vanished at a single point in I, and hence the conclusion remains true under this more restrictive hypothesis. Combined with theorem (4.4) this observation yields:

Theorem 4.6 A set of solutions of the nth order homogenous linear differential equation is linearly independent in $C(I)$ and hence is a basis for the solution space of the equation, if and only if its Wronskian never vanishes on I.

Examples.

1) By direct substitution, the student can verify that $\sin^3 x$, $1/\sin^2 x$ are solutions of

$$\frac{d^2y}{dx^2} + \tan x \frac{dy}{dx} - 6(\cot^2 x)\, y = 0$$

on any interval I on which $\tan x$ and $\cot x$ are both defined. ~~Moreover,~~

Verification:

$$y_1 = \sin^3 x$$

$$y_1' = 3\sin^2 x \cos x$$

$$y_1'' = (6\sin x \cos x)\cos x + (3\sin^2 x)(-\sin x)$$

$$= 6\sin x \cos^2 x - 3\sin^3 x.$$

$$y_1'' + \tan x\, y_1' - 6\cot^2 x\,(y_1)$$

$$= (6\sin x\cos^2 x - 3\sin^3 x) + \tan x \cdot 3\sin^2 x\cos x - 6\cot^2 x \cdot \sin^3 x$$

$$= 6\sin x\cos^2 x - 3\sin^3 x + 3\sin^3 x - 6\cos^2 x \sin x$$

$$= 0.$$

Also, let $y_2(x) = \frac{1}{\sin^2 x}$.

$$y_2'(x) = +\frac{1}{\sin^4 x} \cdot 2\sin x \cdot \cos x$$
$$= \frac{2\cos x}{\sin^3 x}.$$

$$y_2''(x) = 2\left[\frac{\sin^3 x \cdot (-\sin x) - \cos x \cdot (3\sin^2 x \cdot \cos x)}{\sin^6 x}\right]$$
$$= -2\left[\frac{\sin^2 x - 3\sin^2 x \cdot \cos^2 x}{\sin^6 x}\right]$$
$$= -2\left[\frac{\sin^2 x - 3\cos^2 x}{\sin^4 x}.\right]$$

$$y_2'' + \tan x \, y_2' - 6\cot^2 x (y_2)$$
$$= \frac{-2\sin^2 x - 6\cos^2 x}{\sin^4 x} + \left(\frac{2\cos x \cdot \sin x}{\sin^4 x}\right)\tan x - 6\cot^2 x \cdot (-1/\sin^2 x)$$
$$= \frac{-2\sin^2 x - 6\cos^2 x + 2\sin^2 x + 6\cos^2 x}{\sin^4 x} = 0.$$

$$W\left[\sin^3 x, \frac{1}{\sin^2 x}\right] = \begin{vmatrix} \sin^3 x & -1/\sin^2 x \\ 3\sin^2 x \cos x & 2\cos x/\sin^3 x \end{vmatrix}$$
$$= 2\cos x + 3\cos x$$
$$= 5\cos x.$$

Since $\cos x$ is never zero on $I = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, the above theorem implies that $\sin^3 x$ and $\frac{1}{\sin^2 x}$ are linearly independent in $C(I)$ and the general solution of (4.46) therefore is

$$y = c_1 \sin^3 x + \frac{c_2}{\sin^2 x}.$$

2) The functions
$$y_1(x) = \sin^3 x, \quad y_2(x) = \sin x - \frac{1}{3}\sin 3x$$
are solutions of
$$\frac{d^2y}{dx^2} + (\tan x - 2\cot x)\frac{dy}{dx} = 0.$$
on any interval I in which $\tan x$ and $\cot x$ are defined. But,
$$W[y_1(x), y_2(x)] = \begin{vmatrix} \sin^3 x & \sin x - 1/3\sin 3x \\ 3\sin^2 x \cdot \cos x & \cos x - \cos 3x \end{vmatrix}$$
$$= \sin^3 x \cdot \cos x - \sin^3 x \cdot \cos 3x - 3\sin^3 x \cos x + \sin 3x \sin^2 x \cos x.$$
$$= -2\sin^3 x \cos x + \sin^2 x (\sin 3x \cos x - \cos 3x \sin x)$$
$$= -2\sin^3 x \cos x + \sin^2 x \sin 2x$$
$$= \sin^2 x (\sin 2x - 2\sin x \cos x)$$
$$= \sin^2 x (\sin 2x - \sin 2x)$$
$$= 0.$$

3] Hence, y_1 and y_2 are linearly dependent in $C(I)$ and do not form a basis for the solution space of (4.47). In this case, it is clear that any constant c is a solution of (4.47), and since c and $\sin^3 x$ are obviously linearly independent in $C(I)$, the general solution of the equation is:

$$y = c_1 + c_2 \sin^3 x.$$

Of course, this expression may also be written

$$y = c_1 + c_2 (\sin x - 1/3 \sin 3x).$$

<u>Problems.</u>

By computing Wronskians, show that each of the following sets of functions is linearly independent in $\mathcal{C}(I)$ for the indicated interval I.

1. $1, e^{-x}, 2e^{2x}$. on any interval I.

<u>Solution</u>

$$W[1, e^{-x}, e^{2x}] = \begin{vmatrix} 1 & e^{-x} & e^{2x} \\ 0 & -e^{-x} & 2e^{2x} \\ 0 & e^{-x} & 4e^{2x} \end{vmatrix}$$

$$= 1 \begin{vmatrix} -e^{-x} & 2e^{2x} \\ e^{-x} & 4e^{2x} \end{vmatrix} = -4e^{x} - 2e^{x} = -6e^{x}.$$

$W(x)$ is not identically zero for all x in $(-\infty, \infty)$. Hence, the functions $1, e^{-x}, 2e^{2x}$ are linearly independent in $\mathcal{C}(-\infty, \infty)$.

2. $e^{x}, \sin 2x$ on any interval I.

<u>Solution</u>

$$W[e^{x}, \sin 2x] = \begin{vmatrix} e^{x} & \sin 2x \\ e^{x} & 2\cos 2x \end{vmatrix}$$

$$= e^{x} (2\cos 2x - \sin 2x)$$

$-(2\cos 2x - \sin 2x) \neq 0$

$W(x)$ is equal to zero if and only if $2\cos 2x - \sin 2x = 0$. so, $W(x)$ is not identically zero. $e^{x}, \sin 2x$ are linearly independent in $\mathcal{C}(-\infty, \infty)$.

3. $1, x, x^2, \ldots, x^n$ on any interval I.

<u>Solution</u>

$$W[1, x, x^2, \ldots, x^n] = \begin{vmatrix} 1 & x & x^2 & x^3 & \ldots & x^n \\ 0 & 1 & 2x & 3x^2 & \ldots & nx^{n-1} \\ 0 & 0 & 2 & 6x & \ldots & n(n-1)x^{n-2} \\ & & & & \ddots & \\ & & & & & n! \end{vmatrix}$$

The Wronskian of this upper triangular matrix is therefore the product of all elements on the principal diagonal $1 \times 1 \times 2! \times 3! \times 4! \times \ldots \times n! = n!!$.

Thus The Wronskian is not identically equal to zero for all x in I.

Hence, $1, x, x^2, \ldots, x^n$ are linearly independent functions $\mathcal{C}(I)$.

4. $\ln x, x \ln x$.

$$W[\ln x, x \ln x] = \begin{vmatrix} \ln x & x \ln x \\ 1/x & \ln x + 1 \end{vmatrix}$$

$$= \ln^2 x + \ln x - \ln x$$
$$= \ln^2 x.$$

The Wronskian is not identically zero for all x in I. Therefore, the functions $\ln x, x \ln x$ are linearly independent in $\mathcal{C}(I)$.

5. $x^{1/2}, x^{1/3}$

$$W[x^{1/2}, x^{1/3}] = \begin{vmatrix} x^{1/2} & x^{1/3} \\ \frac{1}{2}x^{-1/2} & \frac{1}{3}x^{-2/3} \end{vmatrix}$$

$$= \frac{1}{3} x^{1/2 - 2/3} - \frac{1}{2} x^{\frac{1}{3} - \frac{1}{2}}$$

$$= \frac{1}{3} x^{-1/6} - \frac{1}{2} x^{-1/6}$$

$$= -\frac{1}{6} x^{-1/6} = \frac{-1}{6x^{1/6}}$$

The Wronskian is not identically equal to zero for all x in the interval $(0, \infty)$. So, the given functions $x^{1/2}, x^{1/3}$ are linearly independent in $C(0, \infty)$.

6. $e^{ax}\sin bx, e^{ax}\cos bx$ $(b \neq 0)$ on any interval I.

Solution.

$$W[e^{ax}\sin bx, e^{ax}\cos bx] = \begin{vmatrix} e^{ax}\sin bx & e^{ax}\cos bx \\ (ae^{ax}\sin bx + be^{ax}\cos bx) & (ae^{ax}\cos bx - be^{ax}\sin bx) \end{vmatrix}$$

$$= (e^{ax})^2 \sin bx \cdot (a\cos bx - b\sin bx)$$
$$- (e^{ax})^2 \cos bx\,(a\sin bx + b\cos bx)$$
$$= e^{2ax}[a\cos bx \cdot \sin bx - b\sin^2 bx - a\sin bx\cos bx - b\cos^2 bx]$$
$$= -e^{2ax}(b\cos^2 bx + \sin^2 bx)$$
$$= -b\,e^{2ax}.$$

As $b \neq 0$, the Wronskian is not identically equally to 0, for all x in $(-\infty, \infty)$. Hence, $\{e^{ax}\sin bx, e^{ax}\cos bx\}$ is a linearly independent set in $C(-\infty, \infty)$.

7. $e^x, e^x \sin x$.

Solution

$$W[e^x, e^x \sin x] = \begin{vmatrix} e^x & e^x \sin x \\ e^x & e^x \sin x + e^x \cos x \end{vmatrix}$$

$$= e^{2x} \begin{vmatrix} 1 & \sin x \\ 1 & \sin x + \cos x \end{vmatrix}$$

$$= e^{2x}\{(\sin x + \cos x) - \sin x\}$$
$$= e^{2x}\cos x.$$

8. $e^{-x}, xe^{-x}, x^2 e^{-x}$. on any interval. I.

Solution.

$$W[e^{-x}, xe^{-x}, x^2e^{-x}] = \begin{vmatrix} e^{-x} & xe^{-x} & x^2e^{-x} \\ -e^{-x} & e^{-x} - xe^{-x} & 2xe^{-x} - x^2e^{-x} \\ e^{-x} & -2e^{-x} + xe^{-x} & 2e^{-x} - 4xe^{-x} + x^2e^{-x} \end{vmatrix}$$

$(xe^{-x})' = e^{-x} - xe^{-x}$

$(xe^{-x})'' = -e^{-x} - (e^{-x} - xe^{-x})$

$= -2e^{-x} + xe^{-x}$

$$= e^{-3x} \begin{vmatrix} 1 & x & x^2 \\ -1 & 1 - x & 2x - x^2 \\ 1 & -2 + x & 2 - 4x + x^2 \end{vmatrix}$$

$(x^2e^{-x})' = 2xe^{-x} - x^2e^{-x}$

$(x^2e^{-x})'' = 2(e^{-x} - xe^{-x}) - (2xe^{-x} - x^2e^{-x})$

$= 2e^{-x} - 4xe^{-x} + x^2e^{-x}$

$$= e^{-3x} \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & -2 & 2-4x \end{vmatrix}$$

$$= e^{-3x}(2-4x+4x)$$
$$= 2e^{-3x}.$$

Thus, the Wronskian of the functions $e^{-x}, xe^{-x}, x^2e^{-x}$ not identically equal to zero for all x in $(-\infty,\infty)$. Hence, the functions $\{e^{-x}, xe^{-x}, x^2e^{-x}\}$ are linearly independent in $\mathcal{C}(-\infty,\infty)$.

9. $1, \sin^2 x, 1-\cos x$ on any interval I.

Solution.

$$W[1, \sin^2 x, 1-\cos x] = \begin{vmatrix} 1 & \sin^2 x & 1-\cos x \\ 0 & 2\sin x\cos x & \sin x \\ 0 & 2\cos 2x & \cos x \end{vmatrix}$$

$$= 1 \begin{vmatrix} \sin 2x & \sin x \\ 2\cos 2x & \cos x \end{vmatrix}$$

$$= \sin 2x \cos x - 2\cos 2x \sin x$$
$$= \sin 2x \cos x - \cos 2x \sin x - \cos 2x \sin x$$
$$= \sin x - \cos 2x \sin x$$
$$= \sin x(1-\cos 2x) = \sin x \cdot 2\sin^2 x$$
$$= 2\sin^3 x.$$

The Wronskian of $\{1, \sin^2 x, 1-\cos x\}$ is not identically equal to zero for all $x \neq n\pi$. Hence, the functions $\{1, \sin^2 x, 1-\cos x\}$ are linearly independent in $\mathcal{C}(-\infty,\infty)$.

10. $\ln\left(\frac{x-1}{x+1}\right), 1$ on $(-\infty,-1)$.

Solution.

$$W\left[\ln\left(\frac{x-1}{x+1}\right), 1\right] = \begin{vmatrix} \ln\left(\frac{x-1}{x+1}\right) & 1 \\ \frac{x+1}{x-1}\cdot\frac{(x+1)-(x-1)}{(x+1)^2} & 0 \end{vmatrix} = \begin{vmatrix} \ln\left(\frac{x-1}{x+1}\right) & 1 \\ \frac{2}{x^2-1} & 0 \end{vmatrix}$$

$$= \frac{2}{1-x^2}.$$

The Wronskian of $\{\ln(x-1)/(x+1), 1\}$ is not identically equal to 0 for all x in $(-\infty,-1)$. Hence, the functions $\{\ln(x-1)/(x+1), 1\}$ are linearly independent in $\mathcal{C}(-\infty,-1)$.

11. $\sqrt{1-x^2}, x$ on $(-1,1)$

Solution.

$$W[\sqrt{1-x^2}, x] = \begin{vmatrix} \sqrt{1-x^2} & x \\ \frac{-x}{\sqrt{1-x^2}} & 1 \end{vmatrix}$$

$$= \sqrt{1-x^2} + \frac{x^2}{\sqrt{1-x^2}}$$

$$= \frac{1-x^2+x^2}{\sqrt{1-x^2}} = \frac{1}{\sqrt{1-x^2}}.$$

This function is

The Wronskian is not identically equal to zero for all x in the interval $(-1,1)$. Hence, $\{\sqrt{1-x^2}, x^3\}$ form a linearly independent set in $\mathcal{C}(-1,1)$.

12. $\sin\frac{x}{2}$, $\cos^2 x$ on any interval I.

Solution.

$$W\left[\sin\frac{x}{2}, \cos^2 x\right] = \begin{vmatrix} \sin\left(\frac{x}{2}\right) & \cos^2 x \\ \frac{1}{2}\cos\left(\frac{x}{2}\right) & 2\cos x\,(-\sin x) \end{vmatrix}$$

$$= \sin\frac{x}{2}\left(-\sin 2x\right) - \left(\cos^2 x\right)\left(\frac{1}{2}\cos\frac{x}{2}\right)$$

$= -$ [illegible]

13. Show that $x^\alpha, x^\beta, x^\gamma$ are linearly independent in $\mathcal{C}(0,\infty)$ if and only if α, β, γ are distinct real numbers. [Hint: If α, β, γ are distinct and $c_1x^\alpha + c_2x^\beta + c_3x^\gamma \equiv 0$ on $(0,\infty)$, show that $c_1 = c_2 = c_3$ be consider what happens as x tends to infinity].

Solution

Consider the linear combination of the functions $\{x^\alpha, x^\beta, x^\gamma\}$. Suppose

$$c_1x^\alpha + c_2x^\beta + c_3x^\gamma \equiv 0 \quad \forall x \in \mathbb{R}.$$

Then, [illegible] passing to the limit as $x \to \infty$,

$$\lim_{x\to\infty}\left(c_1x^\alpha + c_2x^\beta + c_3x^\gamma\right) = \infty.$$

Thus, the expressing is identically equal to zero $\forall x$ for all x in $\mathbb{R}$, if and only if $c_1 = c_2 = c_3 = 0$. Thus, $\{x^\alpha, x^\beta, x^\gamma\}$ form a linearly independent set in $(0,\infty)$.

14. Show that $x^\alpha, x^\beta, x^\gamma$ are linearly independent in $\mathcal{C}(0,\infty)$ if and only if they are linearly independent in $\mathcal{C}(I)$ for every subinterval of $(0,\infty)$. [Hint: First establish the following assertions and then use [illegible] 4.6]:

(a) $x^\alpha, x^\beta, x^\gamma$ satisfy the linear differential equation

$$x^3y''' + a_2x^2y'' + a_1xy' + a_0y = 0.$$

where $a_2 = 3-\alpha-\beta-\gamma$, $a_1 = 1-\alpha-\beta-\gamma+\alpha\beta+\beta\gamma+\gamma\alpha$, $a_0 = -\alpha\beta\gamma$.

(b) $$W[x^\alpha, x^\beta, x^\gamma] = x^{\alpha+\beta+\gamma-3}\begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha(\alpha-1) & \beta(\beta-1) & \gamma(\gamma-1) \end{vmatrix}$$

and hence $W(x^\alpha, x^\beta, x^\gamma)$ either vanishes nowhere in $(0,\infty)$ or vanishes identically.

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Solution.

(a) Let $y = x^\alpha$

$y' = \alpha \cdot x^{\alpha-1}$

$y'' = \alpha \cdot (\alpha-1)\, x^{\alpha-2}$

$y''' = \alpha(\alpha-1)(\alpha-2)\, x^{\alpha-3}$.

$$
\begin{aligned}
& x^3 y''' + a_2 x^2 y'' + a_1 x y' + a_0 y \\
&= x^3 \cdot x^{\alpha-3} \cdot \alpha(\alpha-1)(\alpha-2) + (3-\alpha-\beta-\gamma)\cdot x^2 \cdot \alpha(\alpha-1)\, x^{\alpha-2} \\
&\quad + (1-\alpha-\beta-\gamma+\alpha\beta+\beta\gamma+\gamma\alpha)\, x \cdot \alpha \cdot x^{\alpha-1} - \alpha\beta\gamma \cdot x^\alpha \\
&= x^\alpha \left[\alpha(\alpha-1)(\alpha-2) + \alpha(\alpha-1)(3-(\alpha+\beta+\gamma)) + \alpha(1-\alpha-\beta-\gamma+\alpha\beta+\beta\gamma+\gamma\alpha) - \alpha\beta\gamma \right]. \\
&= \alpha \cdot x^\alpha \left[(\alpha-1)(\alpha-2) + (\alpha-1)(3-(\alpha+\beta+\gamma)) + 1-\alpha-\beta-\gamma+\alpha\beta+\beta\gamma+\gamma\alpha - \beta\gamma \right] \\
&= \alpha \cdot x^\alpha \left[\alpha^2 - 3\alpha + 2 + 3\alpha - 3 - (\alpha-1)(\alpha+\beta+\gamma) + 1-\alpha-\beta-\gamma+\alpha\beta+\gamma\alpha \right] \\
&= \alpha \cdot x^\alpha \left[\alpha^2 - 1 - (\alpha-1)(\alpha+\beta+\gamma) + 1-\alpha-\beta-\gamma+\alpha\beta+\gamma\alpha \right] \\
&= \alpha \cdot x^\alpha \left[\cancel{\alpha^2} - 1 - \cancel{\alpha^2} - \cancel{\alpha\beta} - \cancel{\alpha\gamma} + \cancel{\alpha} + \cancel{\beta} + \cancel{\gamma} + 1 - \cancel{\alpha} - \cancel{\beta} - \cancel{\gamma} + \cancel{\alpha\beta} + \cancel{\gamma\alpha} \right] \\
&= 0.
\end{aligned}
$$

Thus, x^α, x^β and x^γ satisfy the linear differential equation.

(b)

$$
W[x^\alpha, x^\beta, x^\gamma] = \begin{vmatrix} x^\alpha & x^\beta & x^\gamma \\ \alpha \cdot x^{\alpha-1} & \beta \cdot x^{\beta-1} & \gamma \cdot x^{\gamma-1} \\ \alpha(\alpha-1)x^{\alpha-2} & \beta(\beta-1)x^{\beta-2} & \gamma\cdot(\gamma-1)x^{\gamma-2} \end{vmatrix}
$$

$$
= x^{\alpha-1} \cdot x^{\beta-1} \cdot x^{\gamma-1} \begin{vmatrix} x & x & x \\ \alpha & \beta & \gamma \\ \frac{\alpha(\alpha-1)}{x} & \frac{\beta(\beta-1)}{x} & \frac{\gamma(\gamma-1)}{x} \end{vmatrix}
$$

$$
= x^{\alpha+\beta+\gamma-3} \cdot x \left(\frac{1}{x}\right) \begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha(\alpha-1) & \beta(\beta-1) & \gamma(\gamma-1) \end{vmatrix}
$$

$$
= x^{\alpha+\beta+\gamma-3} \begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha(\alpha-1) & \beta(\beta-1) & \gamma(\gamma-1) \end{vmatrix}
$$

Hence, the Wronskian $W[x^\alpha, x^\beta, x^\gamma]$ either vanishes nowhere in $(0,\infty)$, if the determinant $\neq 0$, or is identically equal to 0.

So it follows that if $x^\alpha, x^\beta, x^\gamma$ are linearly independent in $\mathcal{C}(I)$ for every subinterval I of $(0,\infty)$, they are linearly independent. The converse is obviously true.

15. Generalize the results of problems 13 and 14(b) to show that $x^{\alpha_1}, \ldots, x^{\alpha_n}$ are linearly independent in $\mathcal{C}(I)$ for any subinterval I of $(0,\infty)$ if and only if $\alpha_1, \ldots, \alpha_n$ are distinct real numbers.

Solution

$$
W[x^{\alpha_1}, \ldots, x^{\alpha_n}] = \begin{vmatrix} x^{\alpha_1} & x^{\alpha_2} & \cdots & x^{\alpha_n} \\ \alpha_1 x^{\alpha_1-1} & \alpha_2 x^{\alpha_2-1} & \cdots & \alpha_n x^{\alpha_n-1} \\ \alpha_1(\alpha_1-1)x^{\alpha_1-2} & \alpha_2(\alpha_2-1)x^{\alpha_2-2} & \cdots & \alpha_n(\alpha_n-1)x^{\alpha_n-2} \\ \vdots & & & \\ \alpha_1(\alpha_1-1)\ldots(\alpha_1-n)\, x^{\alpha_1-n+1} & & & \alpha_n(\alpha_n-1)\ldots(\alpha_n-n+1)\, x^{\alpha_n-n+1} \end{vmatrix}
$$

$$= x^{\alpha_1 - n+1} \cdot x^{\alpha_2 - n+1} \cdots x^{\alpha_n - n+1} \begin{vmatrix} x^{n-1} & x^{n-1} & \cdots & x^{n-1} \\ \alpha_1 x^{n-2} & \alpha_2 x^{n-2} & \cdots & \alpha_n x^{n-2} \\ \alpha_1(\alpha_1-1)x^{n-3} & \alpha_2(\alpha_2-1)x^{n-3} & \cdots & \alpha_n(\alpha_n-1)x^{n-3} \\ \vdots & & & \\ \alpha_1(\alpha_1-1)\cdots(\alpha_1-n+1) & \alpha_2(\alpha_2-1)\cdots(\alpha_2-n+1) & & \alpha_n(\alpha_n-1)\cdots(\alpha_n-n+1) \end{vmatrix}$$

$$= x^{\alpha_1+\alpha_2+\cdots\alpha_n - n(n-1)} \cdot x^{1+2+3+\cdots+(n-1)} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ \alpha_1(\alpha_1-1) & \alpha_2(\alpha_2-1) & \cdots & \alpha_n(\alpha_n-1) \\ \vdots & & & \\ \alpha_1(\alpha_1-1)\cdots(\alpha_1-n+1) & \alpha_2(\alpha_2-1)\cdots(\alpha_2-n+1) & & \alpha_n(\alpha_n-1)\cdots(\alpha_n-n+1) \end{vmatrix}$$

$$= x^{\alpha_1+\alpha_2+\cdots+\alpha_n - n(n-1) + n(n-1)/2} \cdot D$$
$$= x^{\alpha_1+\alpha_2+\cdots+\alpha_n - n(n-1)/2} \cdot D.$$

If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are distinct real numbers, the determinant $D \neq 0$. The Wronskian $W[x^{\alpha_1}, \ldots, x^{\alpha_n}]$ vanishes nowhere in $(0,\infty)$. Hence, the set of functions $\{x^{\alpha_1}, \ldots, x^{\alpha_n}\}$ are linearly independent in $\mathcal{C}(I)$ for every subinterval I in $(0,\infty)$.

16. Let f belong to $\mathcal{C}'[a,b]$ and suppose that f is not the zero function. By computing the Wronskian, show that $f(x)$ and $x f(x)$ are linearly independent in $\mathcal{C}'[a,b]$.

Solution.

$$W[f(x), x f(x)] = \begin{vmatrix} f(x) & x f(x) \\ f'(x) & f(x) + x f'(x) \end{vmatrix}$$

$$= (f(x))^2 + x f(x) \cdot f'(x) - x f(x) \cdot f'(x)$$
$$= (f(x))^2.$$

As f is not the zero function, the Wronskian $W[f(x), x f(x)] = (f(x))^2$ is not identically equal to zero for all x. Hence, $f(x), x f(x)$ are linearly independent in $\mathcal{C}[a,b]$.

17. Let f, g be any two functions in $\mathcal{C}'(I)$, and suppose that g never vanishes in I. Prove that if $W[f(x), g(x)] \equiv 0$ on I, then f and g are linearly dependent in $\mathcal{C}'(I)$. [Hint: Calculate $d/dx\,(f(x)/g(x))$.]

Solution

$$W[f(x), g(x)] = \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix} = f(x) g'(x) - f'(x) g(x)$$

Now, $$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x) f'(x) - f(x) g'(x)}{(g(x))^2}$$

As the numerator is identically equal to zero, which implies

$$\left(\frac{f(x)}{g(x)}\right)' \equiv 0$$

$$\Rightarrow \frac{f(x)}{g(x)} = k$$

$f(x) = k_1 g(x)$
Hence, $f(x)$ is a scalar multiple of $g(x)$ for all x. Thus, $f(x)$ and $g(x)$ are linearly dependent.

18. Let f and g be any two functions in $\mathcal{C}^1(I)$ which have only finitely many zeros in I and have no common zeros. Prove that if $W[f(x), g(x)] \equiv 0$ on I, then f, g are linearly dependent in $\mathcal{C}(I)$.
[Hint: Apply the result of problem 17 to the finite number of subintervals of I on which ~~f an~~ or g never vanishes.].

Solution.
Let $I = [a, b]$ and suppose $x_1, x_2, \ldots, x_n$ are the roots of the equation $f(x) = 0$ and $g(x) = 0$. Consider the subintervals $[a, x_1), (x_1, x_2), (x_2, x_3), \ldots, (x_n, b]$. In each of these subintervals the Wronskian $W[f(x), g(x)]$ is identically equal to zero. $g(x)$ never vanishes in any of the subintervals. By the result obtained earlier,

19. (a) Show that

$$W[e^{a_1 x}, e^{a_2 x}, \ldots, e^{a_n x}] = e^{(a_1+a_2+\cdots+a_n)x} \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ a_1 & a_2 & a_3 & \cdots & a_n \\ a_1^2 & a_2^2 & a_3^2 & \cdots & a_n^2 \\ \vdots & & & & \vdots \\ a_1^{n-1} & a_2^{n-1} & a_3^{n-1} & \cdots & a_n^{n-1} \end{vmatrix}$$

(b) The determinant appearing in (a) is known as a Vandermonde determinant. Show that it is zero if and only if $a_i = a_j$ for some pair of indices i, j, $i \neq j$. [Hint: Expand the determinant by cofactors of 1st column to obtain a polynomial in a_1. Is a_2 a root of this polynomial?].

Solution.

$$W[e^{a_1 x}, e^{a_2 x}, \ldots, e^{a_n x}] = \begin{vmatrix} e^{a_1 x} & e^{a_2 x} & \cdots & e^{a_n x} \\ a_1 e^{a_1 x} & a_2 e^{a_2 x} & \cdots & a_n e^{a_n x} \\ a_1^2 e^{a_1 x} & a_2^2 e^{a_2 x} & \cdots & a_n^2 e^{a_n x} \\ \vdots & \vdots & & \vdots \\ a_1^{n-1} e^{a_1 x} & a_2^{n-1} e^{a_2 x} & \cdots & a_n^{n-1} e^{a_n x} \end{vmatrix}$$

$$= e^{(a_1+a_2+\cdots+a_n)x} \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ a_1 & a_2 & a_3 & \cdots & a_n \\ a_1^2 & a_2^2 & a_3^2 & \cdots & a_n^2 \\ \vdots & & & & \\ a_1^{n-1} & a_2^{n-1} & a_3^{n-1} & & a_n^{n-1} \end{vmatrix}$$

(b) If $a_i = a_j$, the
We can simplify the determinant as follows –

$$= e^{(a_1+a_2+\cdots+a_n)x}\begin{vmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ a_1 & a_2-a_1 & a_3-a_1 & a_4-a_1 & \cdots & a_n-a_1 \\ a_1^2 & a_2^2-a_1^2 & a_3^2-a_1^2 & a_4^2-a_1^2 & \cdots & a_n^2-a_1^2 \\ \vdots & & & & & \vdots \\ a_1^{n-1} & a_2^{n-1}-a_1^{n-1} & a_3^{n-1}-a_1^{n-1} & a_4^{n-1}-a_1^{n-1} & \cdots & a_n^{n-1}-a_1^{n-1} \end{vmatrix}$$

$$= e^{(a_1+a_2+\cdots+a_n)x}\begin{vmatrix} a_2-a_1 & a_3-a_1 & a_4-a_1 & \cdots & a_n-a_1 \\ a_2^2-a_1^2 & a_3^2-a_1^2 & a_4^2-a_1^2 & \cdots & a_n^2-a_1^2 \\ \vdots & & & & \\ a_2^{n-1}-a_1^{n-1} & a_3^{n-1}-a_1^{n-1} & a_4^{n-1}-a_1^{n-1} & \cdots & a_n^{n-1}-a_1^{n-1} \end{vmatrix}$$

$$= e^{(a_1+\cdots+a_n)x}(a_2-a_1)(a_3-a_1)\cdots(a_n-a_1)\begin{vmatrix} 1 & 1 & \cdots \\ a_2+a_1 & a_3+a_1 & \cdots \\ a_2^2+a_1a_2+a_1^2 & a_3^2+a_1a_3+a_1^2 & \cdots \\ \vdots & \vdots & \end{vmatrix}$$

$$= e^{(a_1+a_2+\cdots+a_n)x}(a_2-a_1)(a_3-a_1)\cdots(a_n-a_1)\begin{vmatrix} 1 & 0 & 0 \\ a_2+a_1 & a_3-a_2 & a_4-a_2 \\ & a_3^2-a_2^2+a_1(a_3-a_2) & \end{vmatrix}$$

$$= e^{(a_1+a_2+\cdots+a_n)x}(a_2-a_1)\cdots(a_n-a_1)(a_3-a_2)(a_4-a_2)(a_5-a_2)\cdots$$

$$= e^{(a_1+a_2+\cdots+a_n)x}\prod_{i>j}(a_i-a_j)$$

If $a_i = a_j$, the Wronskian $W[e^{a_1x}, e^{a_2x}, \ldots, e^{a_nx}]$ is identically equal to zero.

4.7 Abel's Formula.

According to theorem 4.6, the Wronskian of a set of solutions of a linear homogenous differential equation either vanishes identically or not at all. This fact can also be deduced from the following theorem, which gives an explicit formula for the Wronskian in this case.

Theorem 4.7. Let $y_1, y_2, \ldots, y_n$ be solutions on an interval I of a linear, homogenous n-th order differential equation –

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_0(x)y = 0. \qquad (4.48)$$

and suppose $a_n(x) \neq 0$ everywhere in I. Then,