5.2 Linear Homogeneous equations of order two.

We have already emphasized that the technique for solving constant coefficient linear differential equations depends upon the commutativity of the operator multiplication involved. To make this dependence explicit and at the same time phrase it in a form best suited to our immediate needs, we begin by establishing –

Lemma 5.1 If $L_1, \ldots, L_n$ constant coefficient linear differential operators, then the null space of each of them is contained in the null space of their product.

Proof.

To prove this assertion, we must show that

$$(L_1 \ldots L_n) y = 0$$

whenever

$$L_i y = 0.$$

But, this is a triviality since:

$$\begin{aligned}(L_1 \ldots L_n) y &= (L_1 \ldots L_{i-1} L_i L_{i+1} \ldots L_n) y \\ &= (L_1 \ldots L_{i-1} L_{i+1} \ldots L_n L_i) y \\ &= (L_1 \ldots L_{i-1} L_{i+1} \ldots L_n)(L_i(y)) \\ &= (L_1 \ldots L_{i-1} L_{i+1} \ldots L_n)(0) \\ &= 0.\end{aligned}$$

(since product of constant coefficient linear operators is commutative).

Examples.

1) The second-order equation

$$(D^2 - 4) y = 0 \qquad (5.4)$$

may be re-written as

$$(D+2)(D-2) y = 0$$

Hence, e^{2x} and e^{-2x} are solutions of (5.4), since they are respectively solutions of the first order equations $(D-2) y = 0$ and $(D+2) y = 0$. Furthermore, these functions are linearly independent in $\mathcal{C}(-\infty, \infty)$, that is –

$$W[e^{2x}, e^{-2x}] = \begin{vmatrix} e^{2x} & e^{-2x} \\ 2e^{2x} & -2e^{-2x} \end{vmatrix} = -2-2 = -4,$$

the Wronskian is not identically equal to zero. And it therefore follows that the general solution of (5.4) is

$$y = c_1 e^{2x} + c_2 e^{-2x}.$$

where c_1 and c_2 are arbitrary constants.

This simple example suggests that we attempt to solve the general second order equation

$$(D^2 + a_1 D + a_0) y = 0.$$

by decomposing the operator $D^2 + a_1 D + a_0$ into linear factors. To this end we first find the roots α_1, α_2 of the quadratic equation

$$m^2 + a_1 m + a_0 = 0$$

known as the auxiliary or characteristic equation of (5.5), and then rewrite (5.5) as

$$(D - \alpha)(D - \beta) y = 0$$

This done, the argument falls into cases depending on the nature of α_1 and α_2 as follows:

Case 1. α and β are real and unequal.
Here the reasoning used in the above example carries over without change: the functions $e^{\alpha_1 x}$ and $e^{\alpha_2 x}$ are linearly independent solutions of (5.7) and

$$y = c_1 e^{\alpha_1 x} + c_2 e^{\alpha_2 x}$$

is the general solution.

Note: $(D-a)y = 0$ has the solution:

$$\frac{dy}{dx} = ay$$

$$\frac{dy}{y} = a\,dx$$

$$\int \frac{dy}{y} = \int a\,dx + \ln c_1$$

$$\ln y = ax + \ln c_1$$

$$y = c_1 e^{ax}.$$

Case 2 Real and equal roots. $\alpha_1 = \alpha_2 = \alpha$. In this case, (5.7) becomes

$$(D-\alpha)^2 y = 0$$

and our earlier argument yields but one solution of the equation namely $e^{\alpha x}$. Using it however, we can apply the method developed in section (4.7) to find the second linearly independent solution by solving the first order equation:

$$W[y_1(x), y_2(x)] = c e^{-\int [a_1(x)/a_2(x)]\,dx}$$

We have: $D^2 y - 2\alpha D y + \alpha^2 y = 0.$

$a_1(x) = -2\alpha$

$a_2(x) = 1.$

$$\therefore \quad W[e^{\alpha x}, y_2(x)] = c_1 e^{-\int(-2\alpha)\,dx}$$

$$W[e^{\alpha x}, y_2(x)] = c_1 e^{2\alpha x}.$$

$$\begin{vmatrix} e^{\alpha x} & y_2(x) \\ \alpha e^{\alpha x} & y_2'(x) \end{vmatrix}$$

$$e^{\alpha x} \cdot y' - \alpha e^{\alpha x} \cdot y = c_1 e^{2\alpha x}$$

$$\frac{e^{\alpha x} y' - \alpha e^{\alpha x} y}{(e^{\alpha x})^2} = c_1$$

$$\frac{e^{\alpha x}\,dy - y\,\alpha e^{\alpha x}\,dx}{(e^{\alpha x})^2} = c_1$$

$$d\left(\frac{y}{e^{\alpha x}}\right) = dx$$

$$\int d\left(\frac{y}{e^{\alpha x}}\right) = c_1 \int dx + c_2$$

$$\frac{y}{e^{\alpha x}} = c_1 x + c_2$$

$$y = (c_1 x + c_2) e^{\alpha x}.$$

Case 3. α_1 and α_2 are complex.
Here $\alpha_1 = a + bi$, $\alpha_2 = a - bi$, a and b real, $b > 0$, and the above method apparently breaks down. Nevertheless, if we pretend that $e^{\alpha_1 x}$ and $e^{\alpha_2 x}$ continue to make sense when α_1 and α_2 are complex

the discussion under case I would imply that the general solution of (5·7) is —

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$$y = c_1 e^{\alpha_1 x} + c_2 e^{\alpha_2 x}$$
$$= c_1 e^{(a+ib)x} + c_2 e^{(a-ib)x}$$
$$= e^{a}\left(c_1 e^{ibx} + c_2 e^{-ibx}\right)$$

At this point, we invoke Euler's famous formula:
$$e^{ix} = \cos x + i \sin x.$$

Thus,
$$y = e^{ax}[c_1(\cos bx + i\sin bx) + c_2(\cos bx - i\sin bx)]$$
$$= e^{ax}[(c_1+c_2)\cos bx + i(c_1-c_2)\sin bx]$$
$$= c_3 e^{ax}\cos bx + c_4 e^{ax}\sin bx.$$

Thus, on purely formal grounds we are led to $e^{ax}\cos bx$ and $e^{ax}\sin bx$ as a basis for the solution space of (5·7) when $\alpha_1 = a+bi$ and $\alpha_2 = a-bi$. Of course, we must now verify that these functions are actually solutions of the given differential equation, and that they are linearly independent in $\mathcal{C}(-\infty,\infty)$. But this is routine and has been left as an exercise to the reader.

Verification.

Let the differential equation be
$$(D-\alpha_1)(D-\alpha_2)y = 0$$
$$\therefore \{D^2 - (\alpha_1+\alpha_2)D + \alpha_1\alpha_2\}y = 0.$$

$\alpha_1 = a+bi,\ \alpha_2 = a-bi$

$\alpha_1+\alpha_2 = 2a,\ \alpha_1\alpha_2 = a^2+b^2$
$$(D^2 - 2aD + a^2+b^2)y = 0.$$

We are interested to show that,
$$y = e^{ax}\cos bx + e^{ax}\sin bx$$
satisfies the above second-order differential equation.

$$y = e^{ax}\cos bx + e^{ax}\sin bx = e^{ax}(\cos bx + \sin bx)$$
$$y' = ae^{ax}(\cos bx + \sin bx) + e^{ax}(-b\sin bx + b\cos bx)$$
$$= e^{ax}[(a+b)\cos bx + (a-b)\sin bx]$$
$$y'' = ae^{ax}[(a+b)\cos bx + (a-b)\sin bx] + e^{ax}[-b(a+b)\sin bx + b(a-b)\cos bx]$$
$$= e^{ax}[(a^2+ab+ab-b^2)\cos bx + (a^2-ab-ab-b^2)\sin bx]$$
$$= e^{ax}[(a^2+2ab-b^2)\cos bx + (a^2-2ab-b^2)\sin bx].$$

$$y'' - 2ay' + (a^2+b^2)y$$
$$= e^{ax}[(a^2+2ab-b^2)\cos bx + (a^2-2ab-b^2)\sin bx - 2a\{(a+b)\cos bx + (a-b)\sin bx\} + (a^2+b^2)(\cos bx + \sin bx)]$$
$$= e^{ax}[(a^2+2ab-b^2-2a^2-2ab+a^2+b^2)\cos bx + (a^2-2ab-b^2-2a^2+2ab+a^2+b^2)\sin bx]$$
$$= 0.$$

Thus, $y = e^{ax}(\cos bx + \sin bx)$ is a solution of the linear homogeneous second order ODE $y'' - 2ay' + (a^2+b^2)y = 0$.

$$W[e^{ax}\cos bx, e^{ax}\sin bx] = \begin{vmatrix} e^{ax}\cos bx & e^{ax}\sin bx \\ ae^{ax}\cos bx - be^{ax}\sin bx & ae^{ax}\sin bx + be^{ax}\cos bx \end{vmatrix}$$
$$= e^{2ax}[a\sin bx\cdot\cos bx + b\cos^2 bx - a\sin bx\cos bx + b\sin^2 bx]$$
$$= be^{2ax}$$

The Wronskian is not identically equal to zero for all x in $(-\infty,\infty)$.

Hence, $e^{ax}\cos bx$ & and $e^{ax}\sin bx$ are linearly independent in the function space $\mathcal{C}(-\infty,\infty)$.

Since the above three cases include all possible combinations of α_1 and α_2, we have completed the task of solving the general second order ~~linear~~ homogenous linear differential equation with constant ~~co~~ coefficients. For convenience of reference, we conclude by summarizing our results:

To solve a second-order homogenous linear differential equation of the form

$$(D^2 + a_1 D + a_0)\, y = 0,$$

first find the roots α_1 and α_2 of the characteristic equation

$$m^2 + a_1 m + a_0 = 0$$

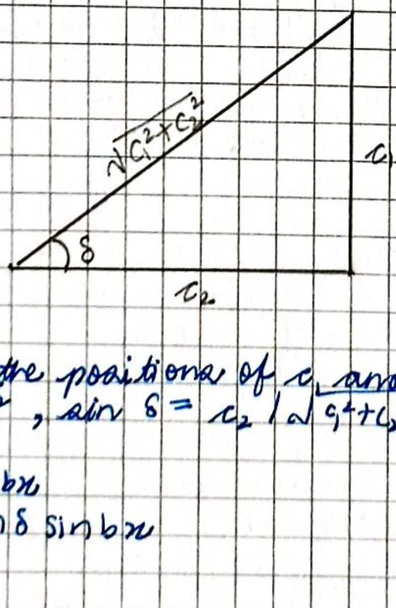
Then, the general solution of the equation can be expressed in terms of α_1 and α_2 as follows -

α_1, α_2	General solution
Real, $\alpha_1 \neq \alpha_2$	$c_1 e^{\alpha_1 x} + c_2 e^{\alpha_2 x}$
Real and equal roots $\alpha_1 = \alpha_2 = \alpha$	$(c_1 + c_2 x)\, e^{\alpha x}$.
Complex roots $\alpha_1 = a+ib,\ \alpha_2 = a-ib$.	$e^{ax}(c_1\cos bx + c_2\sin bx)$

~~Problems.~~

Another form of writing the general solution, ~~of~~ when the roots are complex, more useful for practical purposes, is obtained as follows. Write the equality

$$c_1\cos bx + c_2\sin bx$$
$$= \sqrt{c_1^2+c_2^2}\left(\frac{c_1}{\sqrt{c_1^2+c_2^2}}\cos bx + \frac{c_2}{\sqrt{c_1^2+c_2^2}}\sin bx\right)$$
$$= \sqrt{c_1^2+c_2^2}\,(\sin\delta\cdot\cos bx + \cos\delta\,\sin bx)$$
$$= \sqrt{c_1^2+c_2^2}\,\sin(bx+\delta).$$



If we interchange the positions of c_1 and c_2 in the above figure, then $\cos\delta = c_1/\sqrt{c_1^2+c_2^2}$, $\sin\delta = c_2/\sqrt{c_1^2+c_2^2}$ and we have-

$$c_1\cos bx + c_2\sin bx$$
$$= \sqrt{c_1^2+c_2^2}\,(\cos\delta\cdot\cos bx + \sin\delta\,\sin bx)$$
$$= \sqrt{c_1^2+c_2^2}\,\cos(bx-\delta).$$

Hence, in the case of complex roots, the general solution of a linear ~~ho~~ homogenous differential equation differential equation of order 2, whose characteristic equation has the conjugate roots $(a+bi)$ and $(a-bi)$ can be written in any of the following forms -

(a) $y = c_1 e^{(a+ib)x} + c_2 e^{(a-ib)x}$
(b) $y = e^{ax}(c_1 \cos bx + c_2 \sin bx)$
(c) $y = c\, e^{ax} \cos(bx - \delta)$
(d) $y = c\, e^{ax} \sin(bx + \delta)$.

The significance of the two arbitrary constants c and δ which appear in (c) and (d) will be discussed at length, when learning about ~~appli-~~ the applications of second-order homogeneous linear differential equations.

Examples.

1) Find the general solution of

$$y''' + 2y'' - y' - 2y = 0. \qquad (a)$$

Solution

$(D^3 + 2D^2 - D - 2)\, y = 0$

~~D^3~~ The characteristic equation of (a) is:

$$\begin{aligned} & D^3 + 2D^2 - D - 2 \\ =\; & D^3 - D^2 + 3D^2 - D - 2 \\ =\; & D^3 - D^2 + 3D^2 - 3D + 2D - 2 \\ =\; & D^2(D-1) + 3D(D-1) + 2(D-1) \\ =\; & (D-1)(D^2 + 3D + 2) \\ =\; & (D-1)(D+1)(D+2). \end{aligned}$$

Thus, the general solution of (a) is –

$$y = c_1 e^{x} + c_2 e^{-x} + c_3 e^{-2x}.$$

2) Find the particular solution $y(x)$ of

$$y'' - 3y' + 2y = 0. \qquad (a)$$

Solution.

The characteristic equation of (a) is:

$$\begin{aligned} & m^2 - 3m + 2 \\ =\; & m^2 - 2m - m + 2 \\ =\; & m(m-2) - 1(m-2) \\ =\; & (m-1)(m-2) \end{aligned}$$

Thus, the general solution of (a) is –

$$y = c_1 e^{x} + c_2 e^{2x}.$$

3) Find the particular solution of

$$y'' - 2y' + y = 0. \qquad (a)$$

for which $y(0) = 1$, $y'(0) = 0$.

Solution.

The characteristic equation of (a) is –

$$D^2 - 2D + 1 = 0$$
$$(D-1)^2 = 0$$

~~$D = 1$ is a~~ Thus, the roots of this equation are $D = 1$, with multiplicity 2.

Hence, the general solution of (a) is:

$$y = (c_1 + c_2 x)e^{x}. \qquad (b)$$

The differentiation of (b) gives

$$y' = c_2 e^{x} + (c_1 + c_2 x)e^{x} \qquad (c)$$

To find the particular solution for which $x = 0$, $y = 1$, $y' = 0$, we substitute this values in (b) and (c).

$$1 = c_1.$$
$$0 = c_2 + c_1 \quad \Rightarrow c_2 = -1.$$

∴ The desired particular solution is –

$$y = (1-x)e^{x}.$$

Problems.

Find the general solution of each of the following differential equations.

1) $y'' + y' - 2y = 0.$

Solution

The characteristic equation is

$m^2 + m - 2 = 0$

$m^2 + 2m - m - 2 = 0$

$m(m+2) - 1(m+2) = 0$

$(m-1)(m+2) = 0.$

The general solution of the equation is –

$y = c_1 e^{x} + c_2 e^{-2x}.$

2) $3y'' - 5y' + 2y = 0$

Solution

The characteristic equation is

$3m^2 - 5m + 2 = 0$

$\cancel{3m^2 - 6m +}$ $\Delta = b^2 - 4ac = (5)^2 - 4(3)(2)$

$= 25 - 24 = 1$

$3m^2 - 3m - 2m + 2 = 0$

$3m(m-1) - 2(m-1) = 0$

$(3m-2)(m-1) = 0$

The general solution of the equation is –

$y = c_1 e^{2x/3} + c_2 e^{x}.$

3) $8y'' + 14y' - 15y = 0.$

Solution

The characteristic equation is:

$8m^2 + 14m - 15 = 0$

$8m^2 + 20m - 6m - 15 = 0$

$4m(2m+5) - 3(2m+5) = 0$

$(2m+5)(4m-3) = 0$

The roots of the equation are $\alpha_1 = -\frac{5}{2}$ and $\alpha_2 = \frac{3}{4}$.

The general solution of the equation is –

$y = c_1 e^{-5x/2} + c_2 e^{3x/4}.$

4) $y'' - 2y' = 0.$

Solution

The characteristic equation is

$m^2 - 2m = 0$

$m(m-2) = 0$

The general solution of the equation is

$y = c_1 + c_2 e^{2x}.$

5) $y'' + 4y = 0.$

Solution

The characteristic equation is

$m^2 + 4 = 0$

$m = \pm 2i$

The general solution of the equation is

$y = c_1 \cos 2x + c_2 \sin 2x.$

6) $3y'' + 2y = 0$.

solution

The characteristic equation is

$$3m^2 + 2 = 0$$

$$m = \pm \sqrt{\frac{2}{3}}\, i$$

The general solution of the equation is

$$y = c_1 \cos\left(\sqrt{\frac{2}{3}}\, x\right) + c_2 \sin\left(\sqrt{\frac{2}{3}}\, x\right).$$

7) $y'' + 4y' + 8y = 0$

solution

The characteristic equation is -

$$m^2 + 4m + 8 = 0$$

$$(m+2)^2 + 2^2 = 0$$

$$m + 2 = \pm 2i$$

$$m = -2 + 2i \quad \text{or } m = -2 - 2i.$$

The roots of the equation are $-2+2i$ and $-2-2i$.

The general solution of the equation is

$$y = e^{-2x}(c_1 \cos 2x + c_2 \sin 2x).$$

8) $4y'' - 4y' + 3y = 0$.

solution

The characteristic equation is -

$$4m^2 - 4m + 3 = 0.$$

$$\cancel{4m^2 - 6m + 2m + 3 = 0}$$

$$\cancel{2m}\ (2m)^2 - 2\cdot(2m)(1) + (1)^2 + 2 = 0.$$

$$(2m-1)^2 + 2 = 0$$

$$2m = 1 \pm \sqrt{2}\, i$$

$$m = \frac{1 \pm \sqrt{2}\, i}{2}.$$

The roots of the equation are $\alpha_1 = \frac{1+\sqrt{2}i}{2}$ and $\alpha_2 = \frac{1-\sqrt{2}i}{2}$.

The general solution of the equation is

$$y = e^{x/2}\left(c_1 \cos\left(\frac{x}{\sqrt{2}}\right) + c_2 \sin\left(\frac{x}{\sqrt{2}}\right)\right).$$

9) $y'' - 2y' + 2y = 0$.

solution

The characteristic equation is -

$$m^2 - 2m + 2 = 0$$

$$(m-1)^2 + 1 = 0$$

~~m~~ The roots of the equation are:

$$\alpha_1 = 1 + i, \quad \alpha_2 = 1 - i$$

The general solution of the equation is,

$$y = e^{x}(c_1 \cos x + c_2 \sin x).$$

10) $9y'' - 12y' + 4y = 0$.

solution.

The characteristic equation is

$$9m^2 - 12m + 4 = 0.$$

$$9m^2 - 6m - 6m + 4 = 0$$

$$3m(3m-2) - 2(3m-2) = 0$$

$$(3m-2)^2 = 0.$$

The equation has roots $\alpha = \frac{2}{3}$ with multiplicity 2.

The general solution of the equation is
$$y = (c_1 + c_2 x) e^{2x/3}.$$

11) $y'' + 2y' + 4 = 0.$

Solution

The characteristic equation is -

$m^2 + 2m + 4 = 0$

$(m+1)^2 + 3 = 0$

The roots of this equation are: $\alpha_1 = -1 + \sqrt{3}i$ and $\alpha_2 = -1 - \sqrt{3}i$.

The general solution of the equation is
$$y = e^{-x}(c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x).$$

12) $2y'' - 2\sqrt{2}y' + y = 0.$

Solution

The characteristic equation is -

$2m^2 - 2\sqrt{2}m + 1 = 0.$

$(\sqrt{2}m)^2 - 2(\sqrt{2}m)(1) + (1)^2 = 0$

$(\sqrt{2}m - 1)^2 = 0$

The roots of this equation are: $\alpha = \frac{1}{\sqrt{2}}$ with multiplicity 2.

The general solution of the equation is,
$$y = (c_1 + c_2 x) e^{x/\sqrt{2}}.$$

13) $2y'' - 5\sqrt{3}y' + 6y = 0.$

Solution

The characteristic equation is -

$2m^2 - 5\sqrt{3}m + 6 = 0$

The discriminant is

$\Delta = b^2 - 4ac$

$= (5\sqrt{3})^2 - 4(2)(6)$

$= 75 - 48$

$= 27$

$\sqrt{\Delta} = 3\sqrt{3}.$

The roots of the equation by the quadratic formula are

$\alpha_1 = \frac{5\sqrt{3} + 3\sqrt{3}}{4} = 2\sqrt{3}$

$\alpha_2 = \frac{5\sqrt{3} - 3\sqrt{3}}{4} = \frac{\sqrt{3}}{2}.$

The general solution of the equation is
$$y = c_1 e^{2\sqrt{3}x} + c_2 e^{\sqrt{3}x/2}.$$

14) $9y'' + 6y' + y = 0$

Solution

The characteristic equation is:

$9m^2 + 6m + 1 = 0$

$(3m+1)^2 = 0$

The general solution of the equation is
$$y = (c_1 + c_2 x) e^{-x/3}.$$

15) $64y'' - 48y' + 17y = 0$

Solution.

The characteristic equation is:

$64m^2 - 48m + 17 = 0.$

$(8m)^2 - 2\cdot(8m)(4) + (4)^2 + 1 = 0$

$(8m - 4)^2 + 1 = 0.$

The roots of the equation are $\alpha_1 = \frac{4+i}{8}$, $\alpha = \frac{4-i}{8}.$

The general solution of this equation is

$$y = e^{x/2}\left(c_1 \cos\left(\frac{x}{8}\right) + c_2 \sin\left(\frac{x}{8}\right)\right). \quad (a)$$

In problems 16-25, find the solutions of the given initial-value problem.

16) $2y'' - y' - 3y = 0$; $y(0) = 0$, $y'(0) = \frac{-7}{2}$.

Solution:

The characteristic equation is –

$2m^2 - m - 3 = 0$

$2m^2 - 3m + 2m - 3 = 0$

$m(2m-3) + 1(2m-3) = 0$

$(m+1)(2m-3) = 0$

$m = -1$ or $m = \frac{3}{2}$.

The general solution of the equation is –

$y = c_1 e^{-x} + c_2 e$

The differentiation of $y(x) = c_1 e^{-x} + c_2 e^{3x/2}$ gives (b)

$y'(x) = -c_1 e^{-x} + \frac{3}{2} c_2 e^{3x/2}$ (c)

To find the particular solution for which $x = 0$, $y = 0$, $y' = -\frac{7}{2}$, we substitute these values in (b) and (c).

$$0 = c_1 + c_2$$

$$-\frac{7}{2} = -c_1 + \frac{3}{2} c_2$$

as $c_2 = -c_1$,

$$-\frac{7}{2} = -c_1 + \frac{5}{2} c_2$$

$$-\frac{7}{5} = - c_2, \quad c_1 = \frac{7}{5}.$$

The desired particular solution is

$$y = \frac{7}{5} e^{-x} + \left(-\frac{7}{5}\right) e^{3x/2}$$

17) $y'' - 8y' + 16y = 0$; $y(0) = \frac{1}{2}$, $y'(0) = -\frac{1}{3}$.

Solution:

The characteristic equation is

$m^2 - 8m + 16 = 0$

$(m-4)^2 = 0$

The root of this equation is $\alpha = 4$ which is repeated twice.

The general solution of the equation is:

$y = (c_1 + c_2 x) e^{4x}$.

The differentiation of $y = (c_1 + c_2 x) e^{4x}$ gives

$y' = c_2 e^{4x} + 4(c_1 + c_2 x) e^{4x}$.

To find the particular solution for which $x = 0$, $y = \frac{1}{2}$, $y' = -\frac{1}{3}$ we substitute these values in (b) and (c).

$$\frac{1}{2} = c_1$$

$$-\frac{1}{3} = c_2 + 4c_1.$$

$$\therefore c_2 = -\frac{1}{3} - 4\left(\frac{1}{2}\right) = -\frac{1}{3} - 2 = -\frac{7}{3}.$$

The required particular solution is

$$y = \left(\frac{1}{2} - \frac{7}{3} x\right) e^{4x}.$$

18) $4y'' - 4y' + 5y = 0$; $y(0) = \frac{1}{2}$, $y'(0) = 1$

solution

The characteristic equation is —

$4m^2 - 4m + 5 = 0$

$(2m-1)^2 + 4 = 0$

The roots of the equation are $\alpha_1 = \frac{1+2i}{2}$, $\alpha_2 = \frac{1-2i}{2}$. (a)

The general solution of the equation is

$y = e^{x/2}(c_1 \cos x + c_2 \sin x)$. (b)

$y' = \frac{1}{2} e^{x/2}(c_1 \cos x + c_2 \sin x) + e^{x/2}(-c_1 \sin x + c_2 \cos x)$. (c)

To find the particular solution for which $x = 0$, $y = \frac{1}{2}$, $y' = \frac{7}{2}$, we substitute these values in (b) and (c).

$\frac{1}{2} = c_1$.

$1 = \frac{1}{2} c_1 + c_2$. $\therefore 1 = \frac{1}{4} + c_2$

$\therefore c_2 = \frac{3}{4}$.

The desired particular solution is

$y = e^{x/2}\left[\frac{1}{2} \cos x + \frac{3}{4} \sin x\right]$.

19) $y'' + 2y = 0$; $y(0) = 2$, $y'(0) = 2\sqrt{2}$.

solution.

The characteristic equation is —

$m^2 + 2 = 0$

$m = \sqrt{2}i$ or $m = -\sqrt{2}i$

The general solution of the equation is

$y = c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x$

$y' = -\sqrt{2}c_1 \sin\sqrt{2}x + \sqrt{2}c_2 \cos\sqrt{2}x$

To find the particular solution for which $x = 0$, $y = 2$, $y' = 2\sqrt{2}$ we substitute these values in (b) and (c).

$y = 2 = c_1$

$2\sqrt{2} = \sqrt{2} c_2$

$c_2 = 2$.

The required particular solution is

$y = 2\cos\sqrt{2}x + 2\sin\sqrt{2}x$.

20) $4y'' - 12y' + 9y = 0$; $y(0) = \frac{1}{2}$, $y'(0) = \frac{7}{2}$

solution

The characteristic equation is —

$4m^2 - 12m + 9 = 0$

$4m^2 - 6m - 6m + 9 = 0$

$2m(2m-3) - 3(2m-3) = 0$

$(2m-3)^2 = 0$.

The general solution of the equation is

$y = (c_1 + c_2 x)e^{3x/2}$.

$y' = c_2 e^{3x/2} + \frac{3}{2}(c_1 + c_2 x)e^{3x/2}$

To find the particular solution for which $x = 0$, $y = \frac{1}{2}$, $y' = \frac{7}{2}$ we substitute these values in (b) and (c).

$\frac{1}{2} = c_1$

$\frac{7}{2} = c_2 + \frac{3}{2} c_1 = c_2 + \frac{3}{4}$

$c_2 = \frac{11}{4}$.

The required particular solution is –

$y = \left(\frac{1}{2} + \frac{4x}{4}\right) e^{3x/2}$

21) $y'' + 4y' + 13y = 0$; $y(0) = 0$, $y'(0) = -2$.

Solution.

The characteristic equation is

$m^2 + 4m + 13 = 0$

$(m+2)^2 + 9 = 0$

$m = -2+3i$ or $m = -2-3i$.

~~The~~ The general solution of the equation is –

$y = e^{-2x}(c_1 \cos 3x + c_2 \sin 3x)$.

$y' = -2e^{-2x}(c_1 \cos 3x + c_2 \sin 3x) + e^{-2x}(-3c_1 \sin 3x + 3c_2 \cos 3x)$

To find the particular solution for which $x=0$, $y=0$, $y'=2$, we substitute these values in (b) and (c).

$0 = c_1$

$2 = 2c_1 + 3c_2$

$c_2 = \frac{2}{3}$.

The required particular solution is

$y(x) = \cancel{2}e^{-2x} \cdot \left(\frac{-2}{3} \sin 3x\right)$

22) $9y'' - 3y' - 2y = 0$. , $y(0) = 3$, $y'(0) = 1$.

Solution

The characteristic equation is :

$\cancel{y} = 9m^2 - 3m - 2 = 0$

$9m^2 - 6m + 3m - 2 = 0$

$3m(3m-2) + 1(3m-2) = 0$

$(3m+1)(3m-2) = 0$

The roots of the equation are $\alpha_1 = -\frac{1}{3}$, $\alpha_2 = \frac{2}{3}$.

The general solution of the equation is :

$y = c_1 e^{-x/3} + c_2 e^{2x/3}$.

$y' = -\frac{1}{3} c_1 e^{-x/3} + \frac{2}{3} c_2 e^{2x/3}$.

To find the particular solution for which $x=0$, $y=3$, $y'=1$, we substitute these values in (b) and (c)

$3 = c_1 + c_2$

$1 = -\frac{1}{3}c_1 + \frac{2}{3}c_2$. $\Rightarrow$ $3 = -c_1 + 2c_2$

$3 = c_1 + c_2$

$6 = 3c_2$ $c_2 = 2$, $c_1 = 1$.

The required particular solution is –

$y = e^{-x/3} + 2e^{2x/3}$.

23) $y'' - 2\sqrt{5}\, y' + 5y = 0$. ; $y(0) = 0$, $y'(0) = 3$

Solution:

The characteristic equation is :

$m^2 - 2\sqrt{5}\, m + 5y = 0$

$(m - \sqrt{5})^2 = 0$.

The roots of the equation are $\alpha = \sqrt{5}$ with multiplicity 2.

The general solution of the equation is:

$y = (c_1 + c_2 x) e^{\sqrt{5}x}$

$y' = c_2 e^{\sqrt{5}x} + \sqrt{5}(c_1 + c_2 x) e^{\sqrt{5}x}$

To find the particular solution for which $x=0$, $y=0$, $y'=3$ we substitute these values in (b) and (c).

$0 = c_1$

$3 = c_2 + \sqrt{3} c_1.$

$\Rightarrow c_1 = 0, c_2 = 3$

The desired particular solution is —

$y = 3x e^{\sqrt{3}x}.$

24) $16y'' + 8y' + 5y = 0, y(0) = 4, y'(0) = -1.$

Solution.

The characteristic equation is

$16m^2 + 8m + 5 = 0$

$(4m)^2 + 2\cdot(4m) + 1 + 4 = 0$

$(4m+1)^2 + 4 = 0$

The roots of the equation are $\alpha_1 = -1+2i$, $\alpha_2 = -1-2i$.

The general solution of the equation is

$y = e^{-x}(c_1 \cos 2x + c_2 \sin 2x)$

To find the particular solution for which $x=0, y=4, y'=1$ we substitute these values in the equations:

$y = e^{-x}(c_1 \cos 2x + c_2 \sin 2x)$

$y' = -e^{-x}(c_1 \cos 2x + c_2 \sin 2x) + e^{-x}(-2c_1 \sin 2x + 2c_2 \cos 2x)$

$$\left.\begin{aligned} 4 &= c_1 \\ 1 &= -c_1 + 2c_2 \end{aligned}\right\} \quad \begin{aligned} c_1 &= 4 \\ c_2 &= 5/2. \end{aligned}$$

The desired particular solution is

$y = e^{-x}\left[4\cos 2x + \frac{5}{2}\sin 2x\right].$

25) $y'' - \sqrt{2}y' + y = 0.\ ; y(0) = \sqrt{2}, y'(0) = 0.$

Solution.

The characteristic equation is:

$m^2 - \sqrt{2}m + 1 = 0$

$m^2 - 2(\sqrt{2}m)\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}}\right)^2 + \frac{1}{2} = 0.$

$\left(m - \frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = 0$

$m = \frac{1+i}{\sqrt{2}}$ or $m = \frac{1-i}{\sqrt{2}}.$

The general solution of the equation is

$y = e^{x/\sqrt{2}}(c_1 \cos(x/\sqrt{2}) + c_2 \sin(x/\sqrt{2})).$

Differentiating y, we have —

$y' = \frac{1}{\sqrt{2}} e^{x/\sqrt{2}}\left[c_1 \cos\frac{x}{\sqrt{2}} + c_2 \sin\frac{x}{\sqrt{2}}\right] + \frac{e^{x/\sqrt{2}}}{\sqrt{2}}\left[-c_1 \sin\frac{x}{\sqrt{2}} + c_2 \cos\frac{x}{\sqrt{2}}\right]$

To find the particular solution for which $x=0, y=\sqrt{2}, y'=0$, we substitute these values in (b) and (c).

$\sqrt{2} = c_1$

$0 = \frac{c_1}{\sqrt{2}} + \frac{c_2}{\sqrt{2}}$

$\therefore c_2 = -c_1 = -\sqrt{2}.$

The desired particular solution is:

$y = e^{x/\sqrt{2}}\left[\sqrt{2}\cos\left(\frac{x}{\sqrt{2}}\right) - \sqrt{2}\sin\left(\frac{x}{\sqrt{2}}\right)\right].$

14/12/2019.

26) Prove that $e^{\alpha_1 x}$ and $e^{\alpha_2 x}$ are linearly independent in $\mathcal{C}(-\infty,\infty)$, whenever α_1 and α_2 are distinct real numbers.

Solution.

$$W[e^{\alpha_1 x}, e^{\alpha_2 x}] = \begin{vmatrix} e^{\alpha_1 x} & e^{\alpha_2 x} \\ \alpha_1 e^{\alpha_1 x} & \alpha_2 e^{\alpha_2 x} \end{vmatrix}$$

$$= e^{(\alpha_1+\alpha_2)x}[\alpha_2-\alpha_1].$$

If α_1 and α_2 are distinct real numbers, $\alpha_1 \neq \alpha_2$. The Wronskian $W[e^{\alpha_1 x}, e^{\alpha_2 x}]$ is not identically equal to zero in the interval $(-\infty,\infty)$. Hence, $e^{\alpha_1 x}$ and $e^{\alpha_2 x}$ are linearly independent in $\mathcal{C}(-\infty,\infty)$.

27) Verify that $xe^{\alpha x}$ is a solution of the second order differential equation $(D-\alpha)^2 y = 0$. Prove that this solution and $e^{\alpha x}$ are linearly independent in $\mathcal{C}(-\infty,\infty)$.

Solution.

Let $y = x e^{\alpha x}$

$y' = e^{\alpha x} + \alpha x e^{\alpha x}$

$y'' = \alpha e^{\alpha x} + \alpha(e^{\alpha x} + \alpha x e^{\alpha x})$

$= 2\alpha e^{\alpha x} + \alpha^2 x e^{\alpha x}$

$y'' - 2\alpha y' + \alpha^2 y$

$= \alpha^2 x e^{\alpha x} + 2\alpha e^{\alpha x} - 2\alpha e^{\alpha x} - 2\alpha^2 x e^{\alpha x} + \alpha^2 x e^{\alpha x}$

$= 0.$

Hence, $y = xe^{\alpha x}$ satisfies the differential equation.

$$W[e^{\alpha x}, xe^{\alpha x}] = \begin{vmatrix} e^{\alpha x} & xe^{\alpha x} \\ \alpha e^{\alpha x} & e^{\alpha x} + \alpha x e^{\alpha x} \end{vmatrix}$$

$$= e^{2\alpha x}[1+\alpha x - \alpha x]$$

$$= e^{2\alpha x}.$$

The Wronskian $W[e^{\alpha x}, xe^{\alpha x}]$ is not identically equal to zero in the interval $(-\infty,\infty)$. Hence, $e^{\alpha x}$ and $xe^{\alpha x}$ are linearly independent in $\mathcal{C}(-\infty,\infty)$.

28) Verify that $e^{ax}\cos bx$ and $e^{ax}\sin bx$ are linearly independent solutions of the equation