Method of the variation of parameters and Green's functions(contd.)

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These are concise notes on the method of the variation of parameters to find the particular solution of a non-homogeneous equation. We also look at, how to compute the Green's function for a linear differential operator L.

I. METHOD OF THE VARIATION OF PARAMETERS.

Recall the Cramer's rule for solving a system of equations. Consider the system of equations

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 &= b_1 \\
a_{21}x_1 + a_{22}x_2 &= b_2
\end{aligned}
$$

We can solve for $x = (x_1, x_2)$ as follows. Multiplying equation one by a_{22} and equation two by a_{12}, we obtain:

$$
\begin{aligned}
a_{22}a_{11}x_1 + a_{22}a_{12}x_2 &= a_{22}b_1 \\
a_{12}a_{21}x_1 + a_{12}a_{22}x_2 &= a_{12}b_2
\end{aligned}
$$

Subtracting equation two from equation one, we have -

$$
\begin{aligned}
(a_{11}a_{22} - a_{12}a_{21})x_1 &= (b_1 a_{22} - b_2 a_{12}) \\
x_1 &= \frac{b_1 a_{22} - b_2 a_{12}}{a_{11}a_{22} - a21 a_{12}}
\end{aligned}
$$

And likewise, if we multiply equation one by a_{21} and equation two by a_{11}, we find

$$x_2 = \frac{a_{11}b_2 - a_{12}b_1}{a_{11}a_{22} - a21 a_{12}}$$

If you prefer determinant notation,

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

$$x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

So, if the system of equations $Ax = b$ has a unique solution, and A_i denotes the determinant of A obtained by replacing the i-th column, x_i is given by the explicit formula -

$$x_i = \frac{det(A_i)}{det(A)}$$

A. The setup

The basic method of the variation of parameters can be easily extended to equations of arbitrary order. In this case, we begin with a normal equation

$$y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \ldots + a_1 y'(x) + a_0 y(x) = h(x) \tag{1}$$

defined on an interval I and again assume that the general solution

$$y_h = c_1 y_1(x) + c_2 y_2(x) + \ldots + c_n y_n(x) \tag{2}$$

of the associated homogeneous equation is known.

Then following the argument given in the second-order case, we seek a particular solution of the form :

$$y_p = c_1(x)y_1(x) + c_2(x)y_2(x) + \ldots + c_n(x)y_n(x) \tag{3}$$

B. Deriving an expression for the unknown functions $c_1(x), \ldots, c_n(x)$

In order to meet the requirement that y_p satisfies the given non-homogeneous equation, we impose the following $(n-1)$ conditions on the unknown functions $c_1(x), c_2(x), \ldots, c_n(x)$:

$$
\begin{aligned}
c_1'(x)y_1(x) + c_2'(x)y_2(x) + \ldots + c_n'(x)y_n(x) &= 0 \\
c_1'(x)y_1'(x) + c_2'(x)y_2'(x) + \ldots + c_n'(x)y_n'(x) &= 0 \\
&\vdots \\
c_1'(x)y_1^{(n-2)}(x) + c_2'(x)y_2^{(n-2)}(x) + \ldots + c_n'(x)y_n^{(n-2)}(x) &= 0
\end{aligned} \tag{4}
$$

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for all x in I. The function y_p can be differentiated as follows :

$$\begin{aligned}
y_p &= \sum_i c_i(x) y_i(x) \\
y_p' &= \sum_i c_i'(x) y_i(x) + \sum_i c_i(x) y_i'(x) \\
&= 0 + \sum_i c_i(x) y_i'(x) \text{ (using conditions above)} \qquad (5) \\
&\vdots \\
y_p^{(j)} &= \sum_i c_i(x) y_i^{(j)}(x)
\end{aligned}$$

The last differentiation gives -

$$y_p^{(n)} = \sum_i c_i'(x) y_i^{(n-1)}(x) + \sum_i c_i(x) y_i^{(n)}(x) \qquad (6)$$

Substituting the equations from (5) in the given ODE (1), we obtain:

$$\begin{aligned}
y_p^{(n)} + \sum_{j=0}^{n-1} a_j(x) y_p^{(j)} &= h(x) \\
y_p^{(n)} + \sum_{j=0}^{n-1} a_j(x) \left(\sum_i c_i(x) y_i^{(j)}(x) \right) &= h(x) \qquad (7) \\
y_p^{(n)} + \sum_i c_i(x) \sum_{j=0}^{n-1} a_j(x) y_i^{(j)}(x) &= h(x)
\end{aligned}$$

On substituting (6) in (7), we find that -

$$\begin{aligned}
&\sum_i c_i'(x) y_i^{(n-1)}(x) + \\
&\sum_i c_i(x) \left(y_i^{(n)}(x) + \sum_{j=0}^{n-1} a_j(x) y_i^{(j)}(x) \right) = h(x)
\end{aligned} \qquad (8)$$

But $y_1(x), y_2(x), \ldots, y_n(x)$ are solutions of the associated homogeneous equation, they form a basis of the solution space. Hence, the term in the brackets is identically equal to 0 for all x in the interval I. So, in order that y_p satisfies the given ODE, we must the left-over term equal $h(x)$:

$$c_1'(x) y_1^{(n-1)}(x) + \ldots + c_n'(x) y_n^{(n-1)}(x) = h(x) \qquad (9)$$

for each x in I.

The equations (4) together with equation (9) may be viewed as a system n linear equations in the unknowns $c_1', c_2', \ldots, c_n'$. Our earlier reasoning still applies and we can obtain a particular solution for the system (1) by solving for $c_1', \ldots, c_n'$, integrating and then substituting the resulting functions in (3).

In determinant notation,

$$c_k'(x) = \frac{W_k(x)}{W(x)} \qquad (10)$$

where W denotes the Wronskian of the functions $y_1(x), \ldots, y_n(x)$ and W_i is the determinant of W with its i-th column replaced by the right hand side column vector $(0, \ldots, 0, h(x))$.

In simple terms,

$$c_k'(x) = \frac{\begin{vmatrix} y_1 & y_2 & \ldots & y_{k-1} & 0 & y_{k+1} & \ldots & y_n \\ y_1' & y_2' & \ldots & y_{k-1}' & 0 & y_{k+1}' & \ldots & y_n' \\ \vdots & \vdots & & \vdots & & \vdots & & \\ y_1^{(n-1)} & y_2^{(n-1)} & \ldots & y_{k-1}^{(n-1)} & h(x) & y_{k+1}^{(n-1)} & \ldots & y_n^{(n-1)} \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 & \ldots & y_n y_1' & y_2' & \ldots & y_n' \\ \vdots & & & & & & \\ y_1^{(n-1)} & y_2^{(n-1)} & \ldots & y_n^{(n-1)} & & & \end{vmatrix}} \qquad (11)$$

II. THE GREEN'S FUNCTION FOR THE LINEAR DIFFERENTIAL OPERATOR L

The Green's operator G is the right inverse of the linear differential operator L, such that when applied to h, it yields y.

$$\begin{aligned}
Gh &= y \\
LGh &= Ly \qquad (12) \\
LGh &= h
\end{aligned}$$

The particular solution y_p of the non-homogeneous ODE $Ly = h$ may be written in its integral form as :

$$y_p(x) = y_1(x) \cdot \int_{x_0}^{x} c_1'(x) dx + \ldots + y_n(x) \cdot \int_{x_0}^{x} c_n'(x) dx \qquad (13)$$

Let $V_k(x)$ denote the determinant obtained from the $W[y_1(x), \ldots, y_n(x)]$ by replacing its k-th column with $(0, 0, \ldots, 0, 1)$.

Then,

$$c_k'(x) = \frac{W_k}{W[y_1(x), \ldots, y_n(x)]} = \frac{V_k(x) h(x)}{W[y_1(x), \ldots, y_n(x)]} \qquad (14)$$

Substituting (14) in the expression for the particular solution (13), we have :

$$y_p(x) = y_1(x)\int_{x_0}^{x} \frac{V_1(x)h(x)}{W(x)}dx + \ldots \\ + y_n(x)\int_{x_0}^{x} \frac{V_n(x)h(x)}{W(x)}dx \tag{15}$$

We would like to bring $y_i(x)$ inside. So, we change the variable inside the integral to a dummy variable t and write :

$$y_p(x) = \int_{x_0}^{x} \frac{y_1(x)V_1(t) + \ldots + y_n(x)V_n(t)}{W[y_1(t), \ldots, y_n(t)]} h(t)dt \tag{16}$$

or

$$y_p(x) = \int_{x_0}^{x} K(x,t)h(t)dt \tag{17}$$

where

$$K(x,t) = \int_{x_0}^{x} \frac{y_1(x)V_1(t) + \ldots + y_n(x)V_n(t)}{W[y_1(t), \ldots, y_n(t)]} \tag{18}$$

For the reader who prefers determinant notation, we can simplify the above.

$$\begin{aligned}
&\sum_k y_k(x)V_k(t) \\
&= \sum_k y_k(x) \begin{vmatrix} y_1(t) & \ldots & y_{k-1}(t) & 0 & y_{k+1}(t) & \ldots \\ y_1'(t) & \ldots & y_{k-1}' & 0 & y_{k+1}'(t) & \ldots \\ \vdots & & \vdots & \vdots & & \\ y_1^{(n-1)}(t) & \ldots & y_{k-1}^{(n-1)}(t) & 1 & y_{k+1}^{(n-1)}(t) & \ldots \end{vmatrix} \\
&= \begin{vmatrix} y_1(t) & y_2(t) & \ldots & y_n(t) \\ y_1'(t) & y_2'(t) & \ldots & y_n'(t) \\ \vdots & & & \\ y_1^{(n-2)}(t) & y_2^{(n-1)}(t) & \ldots & y_n^{(n-2)}(t) \\ y_1(x) & y_2(x) & \ldots & y_n(x) \end{vmatrix}
\end{aligned} \tag{19}$$

Therefore the Green's function $K(x,t)$ for the operator $L = D^n + a_{n-1}(x)D^{n-1} + a_1(x)D + a_0(x)$ is given by the expression :

$$K(x,t) = \frac{\begin{vmatrix} y_1(t) & y_2(t) & \ldots & y_n(t) \\ y_1'(t) & y_2'(t) & \ldots & y_n'(t) \\ \vdots & & & \\ y_1^{(n-2)}(t) & y_2^{(n-2)}(t) & \ldots & y_n^{(n-2)}(t) \\ y_1(x) & y_2(x) & \ldots & y_n(x) \end{vmatrix}}{\begin{vmatrix} y_1(t) & y_2(t) & \ldots & y_n(t) \\ y_1'(t) & y_2'(t) & \ldots & y_n'(t) \\ \vdots & & & \\ y_1^{(n-2)}(t) & y_2^{(n-2)}(t) & \ldots & y_n^{(n-2)}(t) \\ y_1^{(n-1)}(t) & y_2^{(n-1)}(t) & \ldots & y_n^{(n-1)}(t) \end{vmatrix}} \tag{20}$$

The expression

$$G[h] = \int_{x_0}^{x} K(x,t)h(t)dt \tag{21}$$

defines the right inverse $G : C(I) \to C^n(I)$ for the operator L. In fact, the integral operator G is the inverse of L that satisfies the initial conditions
$G(h)(x_0) = G(h)'(x_0) = \ldots = G(h)^{(n-1)}(x_0) = 0$