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2. Linear Differential Equations

An nth order linear differential equation on an interval I is, by definition, an operator equation of the form

$$Ly = h(x) \qquad (4.5)$$

in which h is continuous on I, and L is an nth order differential operator defined on I. Such an equation is said to be homogenous if h is identically equal to zero on I, non homogenous otherwise and normal whenever the leading coefficient $a_n(x)$ of the operator L does not vanish anywhere on I. Finally, a function $y = f(x)$ is said to be a solution of (4.5), if and only if $y(x)$ belongs to $\mathcal{C}^n(I)$ and satisfies the equation identically on I.

Thus, an nth order differential equation is simply an equation of the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = h(x). \qquad (4.6)$$

whose coefficients $a_0(x), a_1(x), \ldots, a_n(x)$ and the right hand side $h(x)$ are continuous on I and ~~h(x)~~ $a_n(x)$ is not identically 0. Typical examples are provided by the equations –

$$\frac{d^2y}{dx^2} + y = 0.$$

which is homogenous, normal, and of order 2 on $(-\infty,\infty)$ or any of its subintervals.

$$x^3\frac{d^3y}{dx^3} + x\frac{dy}{dx} = 3$$

which is non-homogenous normal and of order 3 on $(0,\infty)$ and $(-\infty,0)$, but is non normal on any interval containing the origin.

The primary objective in the study of linear differential equations is to find all solutions of any given equation on an interval I. As might be expected, this is a difficult problem, and a complete answer is known only for certain special types of equations. However, there exists a considerable body of knowledge concerning the general behavior of solutions of linear differential equations, and in this respect the theory of such equations stands in a refreshing contrast to that of non-linear equations. This, of course, is due to the fact, that the techniques of linear algebra can be used in this content. This, ~~is~~ of course, is due to the fact that the techniques of linear algebra can be used in this context, and the present chapter constitutes our first substantial application of these ideas to the study of a problem in analysis.

As an illustration of the way in which linear algebra intervenes in the study of differential equations, let

$$Ly = 0 \qquad (4.7)$$

be a normal, homogenous linear differential equation of order n on an interval I on the x-axis. In this case, the solution set of the equation is none other than the null space of the linear transformation L and hence is a subspace of $\mathcal{C}^n(I)$. Out of deference to the problem at hand this subspace is called the solution space of the equation and the task of solving (4.7) has been reduced to finding a basis for its solution space, provided, of course, the solution space is finite dimensional. It is; and later in this chapter, we shall in fact prove that the solution space of any normal nth order homogenous linear differential equation is an n-dimensional subspace of $\mathcal{C}^n(I)$. Thus, if L is normal, and if $y_1(x), \ldots, y_n(x)$ are n linearly independent solutions of (4.7), then every solution of that equation must be of the form

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \ldots + c_n y_n(x). \tag{4.8}$$

for suitable numbers c_i. Conversely, every function of this type is certainly a solution of (4.7) whenever $y_1(x), \ldots, y_n(x)$ are, and for this reason (4.8) with the arbitrary c_i is called the general solution of (4.7). Finally, any function obtained from the general solution by assigning definite values to the c_i is called a particular solution. We leave the reader to reflect upon the merits and shortcomings of this somewhat unfortunate choice of terminology.

By a familiar line of reasoning, these results are also pertinent to the study of non-homogenous equations. Indeed, we studied in chapter 3, that if $y_p(x)$ is a particular solution of the non-homogenous equation

$$Ly = h(x) \tag{4.9}$$

and if $y_h(x)$ is the general solution of the associated homogenous equation $Ly=0$, then the expression $y_p(x) + y_h(x)$ is the general solution of (4.9). In other words, the solution set of a non-homogenous linear differential equation can be found by adding all solutions of the associated homogenous equations to any particular solution of the given equation. Needless to say, this argument effectively reduces the problem of solving a non homogenous equation to that of finding the general solution of its associated homogenous equation. And in the next chapter we shall complete this reduction by giving a method whereby a particular solution $y_p(x)$ of (4.9) can always be found once the general solution of the associated homogenous equation is known.

Examples.

1) The functions $\sin(x)$ and $\cos(x)$ are easily seen to be solutions of the second-order equation

$$y'' + y = 0. \tag{4.10}$$

on the interval $(-\infty, \infty)$. Moreover, these functions are linearly independent in $\mathcal{C}^2(-\infty, \infty)$, since

$$c_1 \sin x + c_2 \cos x = 0$$

for all x implies, by setting $x = \pi/2$ and $x = 0$, $c_1 = 0$ and $c_2 = 0$. Thus, the general solution of (4.10) is

$$y = c_1 \sin x + c_2 \cos x. \tag{4.11}$$

where c_1 and c_2 are arbitrary constants. The reader should note that without a theorem such as the one cited above there would be no guarantee whatever that (4.11) includes every solution of the given equation.

2. The function $y_p(x) = x$ is obviously a solution of the non-homogeneous equation

$$y'' + y = x. \tag{4.12}$$

on $(-\infty, \infty)$. Hence, since $c_1 \sin x + c_2 \cos x$ is a general solution of the associated homogeneous equation $y'' + y = 0$, the general solution of (4.12) is

$$y = x + c_1 \sin x + c_2 \cos x.$$

Before leaving this section it may be instructive to compare the solution set of a non-linear differential equation with that of a linear equation. To this end, we consider

$$y' - 3y^{2/3} = 0. \tag{4.13}$$

which, as is easily seen has the family of cubic curves

$$y = (x+c)^3 \tag{4.14}$$

as its general solution on the interval $(-\infty, \infty)$. In particular, the functions x^3 and $(x+1)^3$ are solutions of (4.13). But, their sum is not, and hence, the solution set of this equation is not a subset of $C'(-\infty, \infty)$, even though the equation appears to be homogeneous. Moreover, all of the various solutions obtained from (4.14) by assigning different values to c are linearly independent in $C'(-\infty, \infty)$, and we conclude that a first-order non-linear differential equation can actually have infinitely many linearly independent solutions. Finally, (4.13) also admits an infinite number of solutions which cannot be obtained from $(x+c)^3$ by specializing the constant c. All of these somewhat peculiar solutions have the property that they are zero along an interval of the x-axis and are of the following three forms –

$$y = \begin{cases} (x-a)^3, & x \le a \\ 0, & x > a \end{cases} \qquad y = \begin{cases} 0, & x < b \\ (x-b)^3, & x \ge b \end{cases}$$

$$y = \begin{cases} (x-a)^3, & x \le a \\ 0, & a < x < b \\ (x-b)^3, & x \ge b. \end{cases}$$

Thus, to use the term "general solution" in reference to (4.14) is in this case a genuine misnomer. In short, every single one of the properties enjoyed by the solution set of a linear differential equation fails to hold here. – a fact which, if it does nothing else, should convince the student that linear differential equations are rather more pleasant to encounter than non-linear equations.

Problems

1. Determine the order of each of the following linear differential equations on the indicated intervals.

(a) $xy'' - (2x+1)y = 3$ on $(-\infty, \infty)$

This is a linear differential equation of order 2 and non-normal on any interval containing the origin.

(b) $(D+1)^3 y = 0$ on $(0,1)$

$$(D+1)^3 y = (D^3 + 3D^2 + 3D + 1)y$$
$$= \frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + y.$$

This will be a linear equation of order 3.

(c) $(x + |x|)y''' + (\sin x)y' = 2e^x$, on $(-1,1)$; on $(0,\infty)$.

This is a linear differential equation of order 3, but it is non-normal on any interval containing the origin.

(d) $\sqrt{x}\,y'' - 2y' + (\sin x)y = \ln x$, on $(1,\infty)$

This is a linear differential equation of order 2.

(e) $(x+1+|x+1|)y''' + (x+|x|)y' + 2y = 0$, on $(-\infty,\infty)$; on $(0,\infty)$; on $(-1,0)$

This is a linear differential equation of order 3.

2. In each of the following, show that the given function is a solution of the associated linear differential equation, and find the interval or intervals in which this is the case.

(a) $xy'' + y' = 0$; $\ln(1/x)$.

Suppose

$$f(x) = \ln\left(\frac{1}{x}\right)$$
$$f'(x) = x \cdot \left(-\frac{1}{x^2}\right) = -\frac{1}{x}.$$
$$f''(x) = \frac{1}{x^2}.$$

$$F[x, f(x), f'(x), f''(x)] = x f''(x) + f'(x)$$
$$= x \cdot \left(\frac{1}{x^2}\right) + \left(-\frac{1}{x}\right)$$
$$= 0.$$

The expression is identically equal to zero. So, $\ln(1/x)$ is a solution of the linear differential equation. The function $\ln\left(\frac{1}{x}\right)$ is defined for $x > 0$.

(b) $4x^2y'' + 4xy' + (4x^2-1)y = 0$; $\dfrac{\sqrt{2}}{\sqrt{\pi x}} \sin x$.

<u>solution</u>

$$f(x) = \sqrt{\frac{2}{\pi x}} \sin x$$

$$= \frac{2}{\sqrt{2\pi x}} \cdot \sin x$$

$$f'(x) = \sqrt{\frac{2}{\pi}} \left(\frac{\sin x}{\sqrt{x}}\right)'$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{\sqrt{x}\cos x - \sin x\,(1/2\sqrt{x})}{(\sqrt{x})^2}\right)$$

$$= \frac{1}{x}\sqrt{\frac{2}{\pi}} \left[\frac{2x\cos x - \sin x}{2\sqrt{x}}\right]$$

$$= \frac{1}{x}\frac{1}{\sqrt{2\pi x}} (2x\cos x - \sin x)$$

$$f''(x) = \frac{1}{\sqrt{2\pi}} \left[\frac{2x\cos x - \sin x}{x\sqrt{x}}\right]'$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{2\cos x}{\sqrt{x}} - \frac{\sin x}{(\sqrt{x})^3}\right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\left\{\frac{\sqrt{x}(-2\sin x) - (2\cos x)(1/2\sqrt{x})}{(\sqrt{x})^2}\right\} - \left\{\frac{(\sqrt{x})^3(\cos x) - (\sin x)\,3(\sqrt{x})^2 \cdot (1/2\sqrt{x})}{(\sqrt{x})^6}\right\}\right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\left\{\frac{-4x\sin x - 2\cos x}{2x\sqrt{x}}\right\} - \left\{\frac{2x^2\cos x - 3x\sin x}{2x^3\sqrt{x}}\right\}\right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\left\{\frac{-4x^2\sin x - 2x\cos x}{2x^2\sqrt{x}}\right\} - \left\{\frac{2x\cos x - 3\sin x}{2x^2\sqrt{x}}\right\}\right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{-4x^2\sin x - \cancel{2x\cos x} - 2x\cos x + 3\sin x}{2x^2\sqrt{x}}\right]$$

$$= \frac{1}{\sqrt{2\pi x}} \left[\frac{3\sin x - 4x^2\sin x - 4x\cos x}{2x^2}\right]$$

$$= \frac{1}{\sqrt{2\pi x}} \frac{(3-4x^2)\sin x - 4x\cos x}{2x^2}$$

$$\therefore 4x^2 f''(x) + 4x f'(x) + (4x^2-1) f(x)$$

$$= \frac{1}{\sqrt{2\pi x}} \left[(6-8x^2)\sin x - 8x\cos x + 8x\cos x - 4\sin x + (4x^2-1)\,2\sin x\right]$$

$$= \frac{1}{\sqrt{2\pi x}} \left[6\sin x - 8x^2\sin x - 8x\cos x + 8x\cos x - 4\sin x + 8x^2\sin x - 2\sin x\right]$$

$= 0$.

Thus, $f(x)$ is a solution of the ODE $4x^2y'' + 4xy' + (4x^2-1)y = 0$.
It is defined for all $x \neq 0$.

(c) $(1-x^2)y'' - 2xy' + 6y = 0$; $f(x) = 3x^2-1$.

Solution.

$$f(x) = 3x^2 - 1$$
$$f'(x) = 6x$$
$$f''(x) = 6$$

$$\begin{aligned}
&(1-x^2) f''(x) - 2x f'(x) + 6 f(x) \\
&= (1-x^2)\, 6 - 2x(6x) + 6(3x^2-1) \\
&= 6 - 6x^2 - 12x^2 + 18x^2 - 6 \\
&= 0.
\end{aligned}$$

Hence, $f(x) = 3x^2-1$ is a solution of the second order linear homogenous differential equation $(1-x^2)y'' - 2xy' + 6y = 0$.

(d) $x^2y'' - xy' + y = 1$; $f(x) = 1 + 2x \ln x$.

Solution.

$$\begin{aligned}
f(x) &= 1 + 2x \ln x \\
f'(x) &= 2(x)' \ln x + (2x)(\ln x)' \\
&= 2 \ln x + 2x \cdot \frac{1}{x} \\
&= 2 \ln x + 2
\end{aligned}$$

$$f''(x) = \frac{2}{x}$$

$$\begin{aligned}
&x^2 f''(x) - x f'(x) + f(x) - 1 \\
&= 2x - x(2\ln x + 2) + (1 + 2x \ln x) - 1 \\
&= 2x - 2x \ln x - 2x + 1 + 2x \ln x - 1 \\
&= 0.
\end{aligned}$$

Hence, the $f(x) = 1 + 2x \ln x$ is a solution of the ODE $x^2y'' - xy' + y = 1$ for all $x > 0$.

(e) $(1-x^2)y'' - 2xy' + 2y = 2$; $f(x) = x \tanh^{-1} x$.

3. (a) Show that $e^{ax}\cos bx$ and $e^{ax}\sin bx$ are linearly independent solutions of the equation

$$(D^2 - 2aD + a^2 + b^2)y = 0, \qquad b \neq 0.$$

on $(-\infty, \infty)$.

(b) What is the general solution of this equation?

(c) Find the particular solution of the equation in (a) which satisfies the initial conditions $y(0) = b$, $y'(0) = -a$.

Solution.

(a) Firstly, let $c_1 e^{ax}\cos bx + c_2 e^{ax}\sin bx = 0 \quad \forall x \in \mathbb{R}$.

If we let $x = 0$, $c_1 = 0$.

$x = \frac{\pi}{2}$, $c_2 = 0$.

Thus, $\{e^{ax}\cos bx, e^{ax}\sin bx\}$ form a linearly independent set.

Let, $f(x) = e^{ax}\cos bx$

$f'(x) = ae^{ax}\cos bx - be^{ax}\sin bx$

$f''(x) = a^2 e^{ax}\cos bx - abe^{ax}\sin bx - abe^{ax}\sin bx - b^2 e^{ax}\cos bx.$

$= (a^2 - b^2)e^{ax}\cos bx - 2abe^{ax}\sin bx$

$= e^{ax}[(a^2 - b^2)\cos bx - 2ab\sin bx]$

$f''(x) - 2af'(x) + (a^2 + b^2)f(x)$

$= e^{ax}[(a^2 - b^2)\cos bx - 2ab\sin bx - 2a^2\cos bx + 2ab\sin bx + (a^2 + b^2)\cos bx]$

$= 0.$

Hence, $f(x) = e^{ax}\cos bx$ is a solution of the ODE.

Further, let $g(x) = e^{ax}\sin bx$

$g'(x) = ae^{ax}\sin bx + be^{ax}\cos bx$

$= e^{ax}(a\sin bx + b\cos bx)$

$g''(x) = a^2 e^{ax}\sin bx + abe^{ax}\cos bx + bae^{ax}\cos bx - b^2 e^{ax}\sin bx$

$= (a^2 - b^2)e^{ax}\sin bx + 2abe^{ax}\cos bx$

$= e^{ax}[(a^2 - b^2)\sin bx + 2ab\cos bx].$

$g''(x) - 2ag'(x) + (a^2 + b^2)g(x)$

$= e^{ax}[(a^2 - b^2)\sin bx + 2ab\cos bx - 2a^2\sin bx - 2ab\cos bx + (a^2 + b^2)\sin bx]$

$= 0$

Hence, $g(x)$ is a solution of the ODE.

(b) The general solution of this equation is

$y = c_1 e^{ax}\cos bx + c_2 e^{ax}\sin bx$

(c) $y(0) = b$

$\Rightarrow c_1 = b$

$y'(0) = -a.$

$y'(x) = c_1(ae^{ax}\cos bx - be^{ax}\sin bx) + c_2(ae^{ax}\sin bx + be^{ax}\cos bx)$

$y'(0) = ac_1 + bc_2$
$-a = a(b) + bc_2$
$c_2 = -a\left(1+\frac{1}{b}\right)$.

The desired particular solution of the ODE is:-

$$y(x) = e^{ax}\left[b\cos bx - a\left(1+\frac{1}{b}\right)\sin bx\right]$$

4. (a) Show that e^{ax} and xe^{ax} are linearly independent solutions of the equation

$$(D-a)^2 y = 0.$$

(b) Find the particular solution of this equation which satisfies the initial conditions $y(0) = 1$, $y'(0) = 2$.

Solution.

(a) $(D-a)^2 y = 0$
$(D^2 - 2Da + a^2)y = 0$
$y'' - 2ay' + a^2 y = 0$

Let $f(x) = e^{ax}$
$f'(x) = ae^{ax}$
$f''(x) = a^2 e^{ax}$

$$f''(x) - 2af'(x) + a^2 f(x) = a^2 e^{ax} - 2a\cdot(ae^{ax}) + a^2 e^{ax}$$
$$= e^{ax}(a^2 - 2a^2 + a^2)$$
$$\equiv 0.$$

Hence, $f(x) = e^{ax}$ is a solution of the ODE.

Let $g(x) = xe^{ax}$.
$g'(x) = (x)'e^{ax} + x(e^{ax})'$
$= e^{ax} + axe^{ax}$
$g''(x) = ae^{ax} + a(e^{ax} + axe^{ax})$
$= 2ae^{ax} + a^2 x e^{ax}$.

$g''(x) - 2ag'(x) + a^2 g(x)$
$= 2ae^{ax} + a^2 xe^{ax} - 2a(e^{ax} + axe^{ax}) + a^2 xe^{ax}$
$= e^{ax}(2a + a^2 x - 2a - 2a^2 x + a^2 x)$
$= 0$

Hence, $g(x) = xe^{ax}$ is also a solution of the ODE.

(b) $y(0) = 1$, $y'(0) = 2$
$y(x) = c_1 e^{ax} + c_2 xe^{ax}$
$y(0) = c_1 = 1$
$\Rightarrow c_1 = 1$

And $y'(0) = 2$
$y'(x) = c_1 ae^{ax} + c_2 e^{ax} + c_2 axe^{ax}$
$y'(0) = 2 = c_1 + c_2$
$c_2 = 1$

$\Rightarrow y(x) = e^{ax} + xe^{ax}$

5. (a) Verify that $\sin^3 x$ and $\sin x - \frac{1}{3}\sin 3x$ are solutions of

$$y'' + (\tan x - 2\cot x)\, y' = 0$$

on any interval where $\tan x$ and $\cot x$ both are defined. Are these solutions linearly independent?

(b) Find the general solution of this equation.

Solution.

(a) Let $f(x) = \sin^3 x$

$$f'(x) = 3\sin^2 x \cos x$$

$$f''(x) = 3(2\sin x \cos x)\cos x + 3\sin^2 x(-\sin x)$$

$$= 6\sin x \cos^2 x - 3\sin^3 x$$

$$f''(x) + (\tan x - 2\cot x) f'(x)$$

$$= 6\sin x \cos^2 x - 3\sin^3 x + \left(\frac{\sin x}{\cos x} - \frac{2\cos x}{\sin x}\right)(3\sin^2 x \cos x)$$

$$= 6\sin x \cos^2 x - 3\sin^3 x + \left(\frac{\sin^2 x - 2\cos^2 x}{\sin x \cancel{\cos x}}\right)(3\sin^{\cancel{2}} x \cancel{\cos x})$$

$$= 6\sin x \cos^2 x - 3\sin^3 x + 3\sin^3 x - 6\sin x \cos^2 x$$

$$= 0.$$

Let $g(x) = \sin x - \frac{1}{3}\sin 3x$.

$$g'(x) = \cos x - \frac{1}{3}\cdot 3\cos 3x = \cos x - \cos 3x.$$

$$g''(x) = -\sin x + 3\sin 3x = 3\sin 3x - \sin x.$$

$$g''(x) + (\tan x - 2\cot x) g'(x)$$

$$= (-\sin x + 3\sin 3x) + \left(\frac{\sin x}{\cos x} - \frac{2\cos x}{\sin x}\right)(\cos x - \cos 3x)$$

$$= -\sin x + 3(3\sin x - 4\sin^3 x) + \left(\frac{\sin^2 x - 2\cos^2 x}{\sin x \cos x}\right)(\cos x - 4\cos^3 x + 3\cos x)$$

$$= -\sin x + 9\sin x - 12\sin^3 x + \left(\frac{\sin^2 x - 2\cos^2 x}{\sin x \cancel{\cos x}}\right)(4\cancel{\cos x} - 4\cos^{\cancel{3}2} x)$$

$$= 8\sin x(1 - \sin^2 x) - 4\sin^3 x + 4\sin x(\sin^2 x - 2\cos^2 x)$$

$$= \cancel{8\sin x\cos^2 x} - \cancel{4\sin^3 x} + \cancel{4\sin^3 x} - \cancel{8\sin x \cos^2 x}$$

$$= 0.$$

(b) The general solution of the ODE is –

$$y(x) = c_1\left(\sin x - \frac{1}{3}\sin 3x\right) + c_2 \sin^3 x.$$

3. Linear first order differential equations.

Let

$$a_1(x)\frac{dy}{dx} + a_0(x)\, y = h(x) \qquad (4.15)$$

be a normal first-order linear differential equation defined on an interval I of the x axis. Then, as we know, the general solution of this equation can be expressed in the form

$$y = y_p(x) + y_h(x) \qquad (4.16)$$