15. Prove that every non-trivial solution $u(x)$ of a normal second-order linear differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

has only simple zeros. [A point x_0 is said to be a zero of a function $u(x)$ if and only if $u(x_0) = 0$. A zero of $u(x)$ is simple if and only if $u'(x_0) \neq 0$.].

Proof.

4.5 Dimension of the solution space.

In this section, we shall use the existence and uniqueness theorem stated above to give a simple yet elegant proof of the fact that the dimension of the solution space of every normal homogenous linear differential equation is equal to the order of the equation.

The reader should note however, that this result fails in the case of an equation whose leading coefficient vanishes somewhere in the interval under consideration.

This said, we now prove –

Theorem 4.3.

If $a_0(x), a_1(x), \ldots, a_n(x)$ and $h(x)$ are each continuous functions ~~of~~ of x on a common interval I and $a_n(x) \neq 0$ when x is in I, then:

1. The homogenous linear differential equation

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots a_1(x)\frac{dy}{dx} + a_0(x)y = 0 \qquad (4.33)$$

defined on the interval I has n linearly independent solutions $y_1(x), y_2(x), \ldots, y_n(x)$. It is is an n-dimensional subspace of $\mathcal{C}(I)$.

2. The linear combination of these solutions

$$y_c(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x)$$

is also a solution of (4.33). It is an n-parameter family of solutions of (4.33).

3. The function

$$y(x) = y_c(x) + y_p(x)$$

where $y_c(x)$ is defined above and $y_p(x)$ is a particular solution of the non homogeneous linear differential equation corresponding to (4.33) namely

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}(y)}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = b(x) \tag{4.34}$$

is an n-parameter family of solutions of (4.34)

<u>Proof of 1.</u>

~~Let~~ Let x_0 be a fixed point in I.
Then by theorem (4.2) we know that this equation admits solutions $y_1(x), y_2(x), \ldots, y_n(x)$ which, respectively satisfy the initial conditions:

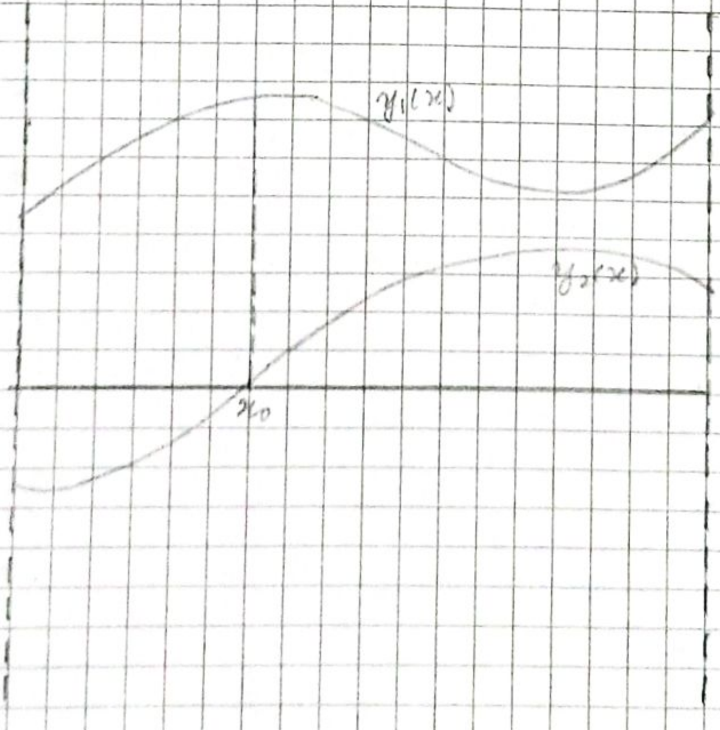
$$\left.\begin{array}{llll} y_1(x_0) = 1, & y_1'(x_0) = 0, & \ldots, & y_1^{n-1}(x_0) = 0, \\ y_2(x_0) = 0, & y_2'(x_0) = 1, & \ldots, & y_2^{(n-1)}(x_0) = 0, \\ \vdots & & & \\ y_n(x_0) = 0, & y_n'(x_0) = 0, & \ldots, & y_n^{(n-1)}(x_0) = 1. \end{array}\right\} \tag{4.35}$$

In other words, $y_1(x), \ldots, y_n(x)$ have the property that the vectors

$$(y_i(x_0), y_i'(x_0), \ldots, y_i^{(n-1)}(x_0)), \quad i = 1, 2, \ldots, n$$

are the standard basis vectors in $\mathbb{R}^n$.

We assert, that these ~~vectors~~ solutions form a ~~f~~ basis for the solution space of (4.33).



Indeed suppose that $c_1, \dots, c_n$ are real numbers such that

$$c_1 y_1(x) + c_2 y_2(x) + \dots + c_n y_n(x) \equiv 0 \qquad \forall x \in I.$$

Then this identity together with its $(n-1)$ derivatives, yields the system

$$\left.\begin{aligned} c_1 y_1(x) + c_2 y_2(x) + \dots + c_n y_n(x) &\equiv 0, \\ c_1 y_1'(x) + c_2 y_2'(x) + \dots + c_n y_n'(x) &\equiv 0, \\ &\vdots \\ c_1 y_1^{(n-1)}(x) + c_2 y_2^{(n-1)}(x) + \dots + c_n y_n^{(n-1)}(x) &\equiv 0. \end{aligned}\right\} \quad (4.36)$$

This implies setting $x = x_0$, we obtain

$$\left.\begin{aligned} c_1 y_1(x_0) + c_2 y_2(x_0) + \dots + c_n y_n(x_0) &\equiv 0 \\ c_1 y_1'(x_0) + c_2 y_2'(x_0) + \dots + c_n y_n'(x_0) &\equiv 0. \\ &\vdots \\ c_1 y_1^{(n-1)}(x_0) + c_2 y_2^{(n-1)}(x_0) + \dots + c_n y_n^{(n-1)}(x_0) &\equiv 0 \end{aligned}\right\}$$

and (4.35) now implies that $c_1 = c_2 = c_3 = \dots = c_n = 0$. Thus, $y_1(x), \dots, y_n(x)$ are linearly independent in $C(I)$.

Proof of 2.

It remains to prove that every solution of (4.33) can be expressed as a linear combination of the ~~t~~ functions $y_1(x), \dots, y_n(x)$.

~~For~~ since $y_1(x), y_2(x), \dots, y_n(x)$ are solutions of the differential equation, they satisfy the differential equation

$$Ly_1 = 0, \; Ly_2 = 0, \; Ly_3 = 0, \dots, Ly_n = 0$$

$\Leftrightarrow c_1 Ly_1 = 0, \; c_2 Ly_2 = 0, \dots, c_n Ly_n = 0,$ where not all $c_1, c_2, \dots c_n$ are zero.

$\Leftrightarrow c_1 Ly_1 + c_2 Ly_2 + \dots + c_n Ly_n = 0,$ not all $c_1, \dots, c_n$ are 0.

$\Leftrightarrow L(c_1 y_1 + c_2 y_2 + \dots + c_n y_n) = 0,$ not all $c_1, \dots, c_n$ are 0. (4.37)

$\Leftrightarrow y = c_1 y_1 + c_2 y_2 + \dots + c_n y_n$ is a solution of the homogenous differential equation of order n.

$\Leftrightarrow \{y_1, y_2, \dots, y_n\}$ form a basis of this n-dimensional subspace of $C(I)$.

Proof of 3.

Let y_p be a particular of the non-homogeneous linear differential equation $Ly = h$. Then,

$$Ly_p = h.$$

and y_c is the general solution of the associated homogeneous linear differential equation $Ly = 0$. So,

$$Ly_c = 0.$$

Let y_0 be an arbitrary solution of the equation $Ly = h$. We assert that $y_0 = y_p + y_c$. Indeed,

$$\begin{aligned} L(y_0 - y_p) &= L(y_0) - L(y_p) \\ &= h - h \\ &= 0. \end{aligned}$$

$\therefore$ ~~$y_0 - p_p$~~ $y_0 - y_p = y_c$

$\therefore y_0 = y_p + y_c$

Q.E.D.

Definition
The solution $y_c(x)$ of the homogenous equation $Ly = 0$ is called the complementary function of $Ly = h$. Hence, the use of the subscript c in $y_c(x)$.

Remark The subscript p in $y_p(x)$ of is used to distinguish from the y_c part of the n parameter family of solutions.

Examples.
1. The second order equation
$$\frac{d^2y}{dx^2} - y = 0 \tag{4.38}$$
is normal on the entire x-axis, and thus, its solution space is a 2-dimensional subspace of $C(-\infty, \infty)$. Moreover, it is easy to show that the functions
$$y_1(x) = \frac{1}{2}(e^x + e^{-x}) = \cosh(x)$$
$$y_2(x) = \frac{1}{2}(e^x - e^{-x}) = \sinh(x)$$
are solutions of (4.38) on $(-\infty, \infty)$, and since
$$y_1(0) = 1, \quad y_1'(0) = 0$$
$$y_2(0) = 0, \quad y_2'(0) = 1.$$
the argument implies that $\cosh x$ and $\sinh x$ are a basis for the solution space of this equation. Thus, the general solution of (4.38) is
$$y = c_1 \cosh(x) + c_2 \sinh(x).$$
where c_1, c_2 are arbitrary constants.

2. The functions
$$y_1(x) = e^x, \qquad y_2(x) = e^{-x}$$
provide a second pair of solutions of equation (4.38). In this case
$$y_1(0) = 1, \quad y_1'(0) = 1$$
$$y_2(0) = +1, \quad y_2'(0) = -1.$$
and since the vectors (1, 1) and (1, −1) are linearly independent in $\mathbb{R}^2$, e^x and e^{-x} also forms a basis for the solution space of the equation. It follows that the general solution of (4.38) may may also be written as
$$y(x) = c_1 e^x + c_2 e^{-x} \qquad (\cancel{4.3}$$
which is of course a variant of the solution obtained above.

3. The functions
$$y_1(x) = \sin 2x, \quad y_2(x) = \cos 2x$$
are solutions of the normal second-order equation
$$\frac{d^2y}{dx^2} + 4y = 0. \tag{4.39}$$
on $(-\infty, \infty)$.

Furthermore, $y_1(0) = 0, \; y_1'(0) = 2$
$y_2(0) = 1, \; y_2'(0) = 0.$

and the vectors $(0,2)$ and $(1,0)$ are linearly independent in $\mathbb{R}^2$, we conclude that $\sin 2x$ and $\cos 2x$ are linearly independent in $\mathcal{C}(-\infty,\infty)$. Hence, they are basis for the solution space of (4.31) and the general solution of the equation is

$$y(x) = c_1 \sin 2x + c_2 \cos 2x.$$

At this point it is impossible to escape the conclusion that in proving theorem (4.3), we also established a method for testing functions for linear independence. This fact is well worth bringing out into the open, since it will be used in the following sections to obtain a number of important results concerning linear differential equations. Specifically, we have:

Corollary 4.1. Let $y_1(x), y_2(x), \ldots, y_n(x)$ be functions in $\mathcal{C}(I)$, each of which possesses derivatives upto and including those of order $(n-1)$ every where in I, and suppose that at some point x_0 in I, the vectors

$$(y_i(x_0), y_i'(x_0), \ldots, y_i^{(n-1)}(x_0)), \quad i = 1, 2, 3, \ldots, n. \qquad (4.40)$$

are linearly independent in $\mathbb{R}^n$. Then, $y_1(x), \ldots, y_n(x)$ are linearly independent in $\mathcal{C}(I)$.

4. The functions

$$e^x, xe^x, x^2e^x$$

are linearly independent in $\mathcal{C}(-\infty,\infty)$, since the above test applied at $x=0$ yields the following vectors.

$$y_1(0) = 1, \quad y_1'(0) = 1, \quad y_1''(0) = 1$$
$$y_2(0) = 0, \quad y_2'(0) = 1, \quad y_2''(0) = 2$$
$$\begin{aligned} y_3(0)=0, \quad y_3'(0) = 0, \quad y_3''(0) &= (2xe^x + x^2e^x)'\big|_{x=0} \\ &= (2e^x + 2xe^x + 2xe^x + x^2e^x)\big|_{x=0} \\ &= (2e^x + 4xe^x + x^2e^x)\big|_{x=0} = 2. \end{aligned}$$

The vectors $(1,1,1)$, $(0,1,2)$ and $(0,0,2)$ are linearly independent in $\mathbb{R}^3$. For if:

$$c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} = 0$$

$$\left.\begin{aligned} c_1 &= 0 \\ c_1 + c_2 &= 0 \\ c_1 + 2c_2 + 2c_3 &= 0 \end{aligned}\right\} \Rightarrow c_1 = 0, c_2 = 0, c_3 = 0$$

Thus, the functions e^x, xe^x, x^2e^x are linearly independent in $\mathcal{C}(-\infty, \infty)$.

4.4 The Wronskian.

In the preceding section, we proved that $y_1, y_2, \ldots, y_n$ are linearly independent functions in $\mathcal{C}^{n-1}(I)$ whenever there exists a point x_0 in I, such that the vectors

$$(y_i(x_0), y_i'(x_0), \ldots, y_i^{(n-1)}(x_0)), \quad i=1,\ldots,n \qquad (4.41)$$

are linearly independent in $\mathbb{R}^n$.