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5.4 Non-homogeneous Equations — Variation of Parameters and Green's Functions

In section (4.2), we observed that the general solution of a non-homogenous linear differential equation

$$a_n(x)\frac{d^n y}{dx^n} + \dots + a_0(x)y = h(x) \tag{5.16}$$

may be written in the form

$$y = y_p + y_h \tag{5.17}$$

where y_p is the particular solution of (5.16) and y_h is the general solution of the associated homogenous equation

$$a_n(x)\frac{d^n y}{dx^n} + \dots + a_0(x)y = 0. \tag{5.18}$$

Using the language of linear operators, the problem of finding a particular solution of (5.16) — which we assume defined and normal on an interval I, consists of finding exactly one solution in $\mathscr{C}^n(I)$ which satisfies the equation

$$Ly = h \tag{5.19}$$

where L is the linear differential operator $a_n(x)D^n + a_{n-1}(x)D^{n-1} + \dots + a_0(x)$. And this as we know is equivalent to the problem of constructing a right inverse for L; meaning of course a linear transformation $G : \mathscr{C}(I) \to \mathscr{C}^n(I)$ such that:

$$Gh = y$$
$$LGh = Ly$$
$$LGh = h.$$

The existence of such inverses is guaranteed by the fact that L is surjective; the equation (5.19) has a solution y in $\mathscr{C}^n(I)$ for every h in $\mathscr{C}(I)$, and the only open question is how to go about selecting a particular inverse for L from the infinitely many that exist. In other words how do we impose conditions on equation (5.19) to ensure that it has unique solution for each h in $\mathscr{C}(I)$? When asked in these terms, the answer is obvious: We simply require that the solution satisfy a "complete" set of initial conditions at some point x_0 in the interval I. This requirement can be viewed as restricting the domain of L in such a way that L becomes one-to-one, hence it is both injective and surjective, it is a bijection, it has an inverse.

~~(The reader should note~~ since the particular solution obtained is quite immaterial we choose the simplest of all possible initial conditions, namely

$$y(x_0) = 0,\ y'(x_0) = 0,\ y''(x_0) = 0, \dots,\ y^{(n-1)}(x_0) = 0.$$

And with this in fact we have defined the right inverse G for the operator L. Specifically, G can be described as the linear mapping from $\mathscr{C}(I)$ to $\mathscr{C}^n(I)$ which sends each function h in $\mathscr{C}(I)$ onto the solution of (5.19) which satisfies the initial conditions given above. In this section, we shall obtain an explicit formula for G in terms of a basis for the solution space of the ~~eqn~~ homogenous equation $Ly = 0$ where L is an operator of order two. In the next section, these results will be generalized to operators of arbitrary order, and once this has been done, the study of linear differential equations will have reduced to the homogenous case.

The reader should note that this portion of our discussion is not restricted to constant coefficient operators. Thus, we begin by considering a normal second-order linear differential equation

$$\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)\,y = h(x), \qquad (5.21)$$

defined on an interval I of the axis, and the general solution

$$y_h = c_1 y_1(x) + c_2 y_2(x). \qquad (5.22)$$

of its associated homogeneous equation. We seek a particular solution y_p of (5.21) such that, it satisfies the initial condition

$$y_p(x_0) = 0, \quad y_p'(x_0) = 0 \qquad (5.23)$$

where x_0 is a fixed, but otherwise arbitrary point in ~~(I)~~ I.

The construction of y_p is begun by making the unjustified but not unreasonable assumption that any particular solution of (5.21) ought to be related ~~to~~ to the expression for y_h, and we therefore attempt to alter the latter in such a way that it becomes a solution of the given equation. One way of doing this is to allow the parameters c_1 and c_2 in (5.22) to vary with x in the hope of finding a solution of (5.21) of the form

$$y_p = c_1(x)\,y_1(x) + c_2(x)\,y_2(x). \qquad (5.24)$$

If (5.24) is substituted in (5.21), and the notation simplified by suppressing mention of the variable x, we obtain—

$$(c_1y_1 + c_2y_2)'' + a_1(c_1y_1 + c_2y_2)' + a_0(c_1y_1 + c_2y_2) = h$$
$$(c_1y_1)'' + (c_2y_2)'' + a_1(c_1y_1)' + a_1(c_2y_2)' + a_0(c_1y_1 + c_2y_2) = h.$$
$$c_1''y_1 + 2c_1'y_1' + c_1y_1'' + c_2''y_2 + 2c_2'y_2' + c_2y_2'' + a_1(c_1'y_1 + c_1y_1')$$
$$+ a_1(c_2'y_2 + c_2y_2') + a_0(c_1y_1 + c_2y_2) = h$$
$$\therefore c_1(y_1'' + a_1y_1' + a_0y_1) + c_2(y_2'' + a_1y_2' + a_0y_2) + (c_1'y_1 + c_2'y_2)'$$
$$+ a_1(c_1'y_1 + c_2'y_2) + (c_1'y_1' + c_2'y_2') = h \qquad (5.25).$$

Moreover, since y_1 and y_2 are solutions of the homogeneous equation $y'' + a_1y' + a_0y = 0$, the first two terms in (5.25) are zero, and we have—

$$(c_1'y_1 + c_2'y_2)' + a_1(c_1'y_1 + c_2'y_2) + (c_1'y_1' + c_2'y_2') = h.$$

This identity which must hold if (5.24) is to be a solution of (5.21) will obviously be satisfied if c_1 and c_2 can be chosen so that

$$c_1'(x)\,y_1(x) + c_2'(x)\,y_2(x) = 0$$
$$c_1'(x)\,y_1'(x) + c_2'(x)\,y_2'(x) = h(x) \qquad (5.26)$$

for all x in I. Thus, it remains to be shown that these equations serve to determine $c_1(x)$ and $c_2(x)$, and that this can be done in such a way that the function

$$y_p = c_1(x)\,y_1(x) + c_2(x)\,y_2(x)$$

satisfies the initial conditions given in (5.23).

But, for each x in I, (5.26) may be viewed as a pair of linear equations in the unknowns $c_1'(x)$ and $c_2'(x)$. As such the determinant of its coefficients is—

$$\begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$$

which we recognize as the Wronskian of the linearly independent solutions $y_1(x)$ and $y_2(x)$ of the homogenous equation associated with (5.21). We now recall theorem (4.6) that this determinant is a continuous function of x which never vanishes on I. Hence, (5.26) has a unique solution for $c_1'(x)$ and $c_2'(x)$ and once this solution is known, $c_1(x)$ and $c_2(x)$ can be found by integration. Moreover, by suitably choosing the limits of integration, the required initial conditions can also satisfied, and the argument is complete. For obvious reasons, this method of constructing a particular solution for a non-homogenous linear differential equation out of the general solution of its associated homogenous equation is known as the method of the variation of parameters.

Starting with (5.26), an easy calculation gives—

$$c_1'(x) = \frac{-h(x)\,y_2(x)}{W[y_1(x), y_2(x)]} \qquad c_2'(x) = \frac{h(x)\,y_1(x)}{W[y_1(x), y_2(x)]}.$$

Thus,

$$c_1(x) = -\int_{x_0}^{x} \frac{h(t)\,y_2(t)}{W[y_1(t), y_2(t)]}\,dt, \qquad c_2(x) = \int_{x_0}^{x} \frac{h(t)\,y_1(t)}{W[y_1(t), y_2(t)]}\,dt \tag{5.27}$$

and if these values are substituted in (5.24), and the terms combined, we find that y_p can be written in the integral form as

$$y_p(x) = \int_{x_0}^{x} \frac{y_2(x)\,y_1(t) - y_1(x)\,y_2(t)}{W[y_1(t), y_2(t)]}\,h(t)\,dt. \tag{5.28}$$

One reason for calling attention to this expression is that it can be read as the definition of a right-inverse of a linear differential operator $L = D^2 + a_1(x)D + a_0(x)$, and in fact is the particular right inverse discussed earlier in this section. For if h is any function in $\mathcal{C}(I)$ and we set

$$G(h) = \int_{x_0}^{x} K(x,t)\,h(t)\,dt \tag{5.29}$$

where

$$K(x,t) = \frac{y_2(x)\,y_1(t) - y_1(x)\,y_2(t)}{W[y_1(t), y_2(t)]} \tag{5.30}$$

then G maps $\mathcal{C}(I)$ to $\mathcal{C}^2(I)$, acts as a right inverse for L, and has further the property that $G(h)$ satisfies the initial conditions $G(h)(x_0) = G(h)'(x_0) = 0$. It should be pointed out that the function $K(x,t)$ defined by (5.30) is independent of the particular choice of x_0 in the interval I and is completely determined by the operator L. As such, it is referred to as the Green's function for L for initial value problems on the interval I, or more simply as the Green's function for L.

Examples.

1) Find the general solution of the second-order equation

$$y'' + y = \tan x \tag{5.31}$$

Solution.

In this case, the associated homogenous equation

$$(D^2+1)\,y = 0$$

has the general solution

$$y_h = (c_1 \cos x + c_2 \sin x).$$

Thus, we seek a particular solution of (5.31) of the form

$$y_p = c_1(x)\cos x + c_2(x)\sin x.$$

where $c_1(x)$ and $c_2(x)$ are determined from the pair of equations –

$$y_1(x)\, c_1'(x) + y_2(x)\, c_2'(x) = 0$$
$$y_1'(x)\, c_1'(x) + y_2'(x)\, c_2'(x) = h(x)$$

That is:

$$\cos x \cdot c_1'(x) + \sin x \cdot c_2'(x) = 0$$
$$-\sin x \; c_1'(x) + \cos x \cdot c_2'(x) = \tan x.$$

$$c_1'(x) = \frac{\sin x \cdot \tan x}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = -\sin x \cdot \tan x.$$

$$c_2'(x) = \frac{\cos x \cdot \tan x}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = +\cos x \cdot \tan x = \sin x.$$

Thus, it follows that:

$$c_1(x) = -\int \sin x \tan x \, dx$$
$$= -\int \frac{\sin^2 x}{\cos x}\, dx$$
$$= -\int \frac{1-\cos^2 x}{\cos x}\, dx$$
$$= -\int (\sec x - \cos x)\, dx$$
$$= -\ln|\sec x - \tan x| + \sin x.$$

$$c_2(x) = \int \sin x \, dx = -\cos x.$$

Thus,

$$y_p = (-\ln|\sec x \mp \tan x| + \sin x)\cos x - \cos x \sin x$$
$$= -\cos x \cdot \ln|\sec x - \tan x|$$

And the general solution of (5.31) is

$$y = -\cos x \ln|\sec x - \tan x| + c_1 \cos x + c_2 \sin x.$$

An alternate method of solving (5.31) relies upon determining the Green's function $K(x,t)$ for the operator $L = D^2+1$. According to (5.30)

$$K(x,t) = \frac{y_2(x)\, y_1(t) - y_1(x)\, y_2(t)}{W[y_1(t), y_2(t)]}$$

$$= \frac{\sin x \cos t - \cos x \sin t}{\begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix}}$$

$$= \sin(x-t).$$

and if we set $x_0 = 0$ in (5.29) the expression

$$G(h) = \int_0^x \sin(x-t)\, h(t)\, dt$$

defines a right inverse for L. The reader should note at this point that we are in fact in a position to solve all linear differential equation involving D^2+1, a fact which vividly illustrates the economy of using Green's functions. It now follows that a particular solution of (5.31) can be obtained by applying G to the function $\tan x$; ie by computing

$$y_p(x) = \int_0^x \sin(x-t)\tan t \, dt \tag{5.32}$$

$$= \int_0^x (\sin x \cos t - \cos x \sin t)\tan t \, dt$$

$$= \int_0^x \sin x \sin t \, dt - \cos x \int_0^x \frac{\sin^2 t}{\cos t} dt$$

$$= \sin x \int_0^x \sin t \, dt - \cos x \int_0^x \frac{\sin^2 t}{\cos t} dt$$

$$= \sin x \cdot [\cos t]_0^x - \cos x \int_0^x \frac{1-\cos^2 t}{\cos t} dt$$

$$= \sin x[-\cos x + \cos 0] - \cos x \int_0^x (\sec t - \cos t)\, dt$$

$$= -\sin x \cos x + \sin x - \cos x\,[\ln(\sec t - \tan t) - \sin t]_0^x$$

$$= \sin x \cos x + \sin x - \cos x\,[\ln(\sec x - \tan x - \sin x]$$

$$= \cancel{-\sin x \cos x} + \sin x - \cos x \ln(\sec x - \tan x) + \cancel{\sin x \cos x}$$

$$= \sin x - \cos x \ln(\sec x - \tan x).$$

2) In section (5.8), we will show that the general solution of

$$xy'' + y' = 0$$

on $(0,\infty)$ and $(-\infty, 0)$ is $c_1 + c_2 \ln|x|$. Hence, the non-homogeneous equation

$$xy'' + y' = x+1 \tag{5.33}$$

has a particular solution of the form $y_p = c_1(x) + c_2(x)\ln(x)$, where $c_1(x)$ and $c_2(x)$ are determined from the equations:

$$y(x)c_1'(x) + y_2(x)c_2'(x) = 0$$
$$y_1'(x)c_1'(x) + y_2'(x)c_2'(x) = h(x)$$

That is:

$$c_1'(x) + \ln(x)\, c_2'(x) = 0$$
$$\frac{1}{x} c_2'(x) = \frac{x+1}{x}.$$

[Here $h(x) = (x+1)/x$, since we must divide by x to (5.33) in the normal form.]

Thus,

$$c_1'(x) = -(x+1)\ln|x|,$$
$$c_2'(x) = x+1.$$

and

$$c_1(x) = -\int (x+1)\ln(x)\, dx$$

Let $\quad u = \ln(x) \qquad dv = x+1$

$\quad du = \frac{1}{x} \qquad v = \frac{x^2}{2} + x$

$$c_1(x) = -\left[\left(\frac{x^2}{2}+x\right)\ln x - \int \frac{1}{x}\left(\frac{x^2}{2}+x\right)dx\right.$$

$$= -\left(\frac{x^2}{2}+x\right)\ln x + \frac{1}{2}\int x\,dx + \int dx$$

$$= -\left(\frac{x^2}{2}+x\right)\ln x + \frac{x^2}{4} + x.$$

$$c_2(x) = \frac{x^2}{2} + x.$$

$$\therefore y_p(x) = -\left(\frac{x^2}{2}+x\right)\ln x + \left(\frac{x^2}{4}+x\right) + \left(\frac{x^2}{2}+x\right)\ln(x) = \frac{x^2}{4} + x.$$

and the general solution of (5.33) is -

$$y = \left(\frac{x^2}{4} + x\right) + c_1 + c_2 \cdot \ln|x|.$$

Problems

Find the general solution of each of the following differential equations.

1) $(D^2+1)\,y = \dfrac{1}{\cos x}$. (a)

solution

The general solution of the associated homogeneous equation

$$(D^2+1)\,y = 0$$

is

$$y_h = c_1 \cos x + c_2 \sin x.$$

Hence, we can construct a particular solution of the form

$$y_p = c_1(x) \cos x + c_2(x) \sin x.$$

where $c_1(x)$ and $c_2(x)$ are determined from the equations -

$$y_1(x)\,c_1'(x) + y_2(x)\,c_2'(x) = 0$$
$$y_1'(x)\,c_1'(x) + y_2'(x)\,c_2'(x) = h(x).$$

That is:

$$\cos x\; c_1'(x) + \sin x\; c_2'(x) = 0$$
$$-\sin x\; c_1'(x) + \cos x\; c_2'(x) = \frac{1}{\cos x}.$$

$$c_1'(x) = \frac{-\sin x\,(1/\cos x)}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}}, \quad c_2'(x) = 1$$

$$= -\tan x$$

$$c_1(x) = -\int \tan x\, dx = \ln(\cos x)$$
$$c_2(x) = x.$$

The particular solution of the differential equation (a) is:

$$y_p = \ln(\cos x) \cdot \cos x + x \sin x.$$

The general solution of the non-homogeneous linear differential equation is -

$$y_n = \ln(\cos x)\cos x + x \sin x + c_1 \cos x + c_2 \sin x.$$

2) $(D^2 - D - 2)\,y = e^{-x} \sin x$. (a)

Solution.

The associated homogeneous linear differential equation is -

$$(D^2 - D - 2)\,y = 0 \quad (b)$$

The characteristic equation is:

$$D^2 - D - 2 = 0$$
$$D^2 - 2D + D - 2 = 0$$
$$D(D-2) + 1(D-2) = 0$$
$$(D+1)(D-2) = 0.$$

The roots of the characteristic equation are -

$\alpha_1 = -1$ and $\alpha_2 = 2$

The general solution of the homogeneous linear differential equation (b) is:

$$y_h = c_1 e^{-x} + c_2 e^{2x}.$$

Hence, the non-homogeneous equation $(D^2 - D - 2)\,y = e^{-x} \sin x$ has a particular solution of the form:

$$y_p = c_1(x)\, e^{-x} + c_2(x)\, e^{2x}.$$

where $c_1(x)$ and $c_2(x)$ are determined from the equations:

$$y_1(x)\,c_1'(x) + y_2(x)\,c_2'(x) = 0$$
$$y_1'(x)\,c_1'(x) + y_2'(x)\,c_2'(x) = h(x)$$

That is -

$$e^{-x} c_1'(x) + e^{2x} c_2'(x) = 0$$
$$-e^{-x} c_1'(x) + 2e^{2x} c_2'(x) = e^{-x}\sin x.$$

$$c_1'(x) = \frac{-e^{x}\sin x}{\begin{vmatrix} e^{-x} & e^{2x} \\ -e^{-x} & 2e^{2x} \end{vmatrix}} = \frac{e^{x}\sin x}{2e^{x} + e^{x}} = \frac{e^{x}\sin x}{3e^{x}} = \frac{\sin x}{3}.$$

$$c_2'(x) = \frac{e^{-2x}\sin x}{3e^{x}} = \frac{e^{-3x}\sin x}{3}.$$

$$c_1(x) = -\frac{1}{3}\int \sin x \, dx = +\frac{1}{3}\cos x.$$

$$c_2(x) = \frac{1}{3}\int e^{-3x}\sin x \, dx$$

u	dv
e^{-3x} +	$\sin x$
$-3e^{-3x}$	$-\cos x$
$9e^{-3x}$	$-\sin x$

$$I(x) = -e^{-3x}\cos x - 3e^{-3x}\sin x - 9\int e^{-3x}\sin x \, dx$$
$$10 I(x) = -e^{-3x}(\cos x + 3\sin x)$$
$$I(x) = -\frac{1}{10}e^{-3x}(\cos x + 3\sin x)$$
$$c_2(x) = \frac{1}{3}\left[-\frac{1}{10}e^{-3x}(\cos x + 3\sin x)\right]$$

The particular solution of the non-homogeneous differential equation (a) is:-

$$y_p = \frac{1}{3}\left[\cos x \cdot e^{-x} + \left(-\frac{1}{10}e^{-3x}(\cos x + 3\sin x)\cdot e^{2x}\right)\right]$$

$$= \frac{e^{-x}}{3}\left[\cos x - \frac{1}{10}\cos x - \frac{3}{10}\sin x\right]$$
$$= \frac{e^{-x}}{3}\left[\frac{9}{10}\cos x - \frac{3}{10}\sin x\right]$$
$$= e^{-x}\left[\frac{3}{10}\cos x - \frac{1}{10}\sin x\right]$$

The general solution of the non-homogeneous equation is

$$y = y_p + y_h$$
$$= e^{-x}\left[\frac{3}{10}\cos x - \frac{1}{10}\sin x\right] + c_1e^{-x} + c_2e^{2x}.$$

3) $(D^2+4D+4)y = xe^{2x}$. (a)

<u>Solution.</u>

The associated homogeneous equation of (a) is

$$(D^2+4D+4)y = 0 \qquad (b)$$

The characteristic equation of (b) is:-

$$D^2+4D+4 = 0$$
$$(D+2)^2 = 0$$

The roots of the characteristic equation are:-

$\alpha_1 = -2$ with multiplicity 2

The general solution of the homogeneous linear differential equation (b) is:-

$$y_h = (c_1 + c_2 x)e^{-2x}$$

Hence, the non-homogeneous linear differential equation (a) has a particular solution of the form:

$$y_p = (c_1(x) + x\, c_2(x))e^{-2x}$$
$$= c_1(x)e^{-2x} + c_2(x)\, x e^{-2x}$$

where $c_1(x)$ and $c_2(x)$ satisfy the equations-

$$y_1(x)c_1'(x) + y_2(x)c_2'(x) = 0$$
$$y_1'(x)c_1'(x) + y_2'(x)c_2'(x) = r(x)$$

That is:

$$e^{-2x}c_1'(x) + xe^{-2x}c_2'(x) = 0$$
$$-2e^{-2x}c_1'(x) + (e^{-2x} - 2xe^{-2x})c_2'(x) = xe^{2x}$$

$$W[e^{-2x}, xe^{-2x}] = \begin{vmatrix} e^{-2x} & xe^{-2x} \\ -2e^{-2x} & e^{-2x} - 2xe^{-2x} \end{vmatrix} = e^{-4x}\begin{vmatrix} 1 & x \\ -2 & 1-2x \end{vmatrix} = e^{-4x}$$

$$c_1'(x) = -\frac{x^2}{e^{-4x}}, \quad c_2'(x) = \frac{x}{e^{-4x}}$$
$$= -x^2e^{4x} \qquad\qquad = xe^{4x}$$

$$\begin{array}{ll} x^2 & + \; e^{4x} \\ 2x & - \; e^{4x}/4 \\ 2 & \; e^{4x}/16 \end{array}$$

$$c_1(x) = \int c_1'(x)\,dx = -\int x^2 e^{4x}\,dx$$
$$= -\left\{\frac{x^2}{4}e^{4x} - \frac{x}{8}e^{4x} + \frac{1}{32}e^{4x}\right\}$$

$$\begin{array}{ll} x & + \; e^{4x} \\ 1 & \; e^{4x}/4 \end{array}$$

$$c_2(x) = \int c_2'(x)\,dx = \int xe^{4x}\,dx$$
$$= \frac{x}{4}e^{4x} - \frac{1}{16}e^{4x}$$

The particular solution of the non-homogeneous linear differential equation is-

$$y_p = \left(-\frac{x^2}{4}e^{4x} + \frac{x}{8}e^{4x} - \frac{1}{32}e^{4x}\right)e^{-2x} + \left(\frac{x}{4}e^{4x} - \frac{1}{16}e^{4x}\right)\cdot xe^{-2x}$$
$$= \frac{x}{16}e^{2x} - \frac{1}{32}e^{2x}$$

The general solution of the non-homogeneous linear differential equation $(D^2+4D+4)y = xe^{2x}$ is:

$$y = y_p + y_h$$
$$= \frac{1}{32}(2x-1)e^{2x} + (c_1 + c_2 x)e^{-2x}.$$

(a)

4) $(D^2+3D-4)y = x^2e^x$.

solution

The associated homogeneous equation is:

(a)

$$(D^2+3D-4)y = 0$$

The characteristic equation is:

$$(D^2+3D-4) = 0$$

The roots of the characteristic equation are-

$$D^2+3D-4 = 0$$
$$D^2+4D-D-4 = 0$$
$$D(D+4) - 1(D+4) = 0$$
$$(D-1)(D+4) = 0.$$
$$\alpha_1 = 1, \alpha_2 = -4$$

The general solution of the homogeneous linear differential equation

$$y_h = c_1 e^{x} + c_2 e^{-4x}.$$

Hence, the non-homogenous linear differential equation $(D^2+3D-4)y = x^2 e^{x}$ has a particular solution of the form

$$y_p = c_1(x)\, e^{x} + c_2(x)\, e^{-4x}$$

where $c_1(x)$ and $c_2(x)$ are given by the equations:-

$$y_1(x)\, c_1'(x) + y_2(x)\, c_2'(x) = 0$$
$$y_1'(x)\, c_1'(x) + y_2'(x)\, c_2'(x) = h(x).$$

That is:

$$e^{x}\, c_1'(x) + e^{-4x}\, c_2'(x) = 0$$
$$e^{x}\, c_1'(x) - 4e^{-4x}\, c_2'(x) = x^2 e^{x}$$

$$c_1'(x) = \frac{-x^2 e^{-3x}}{\begin{vmatrix} e^{x} & e^{-4x} \\ e^{x} & -4e^{-4x} \end{vmatrix}} = \frac{-x^2 e^{-3x}}{e^{-3x}\begin{vmatrix} 1 & 1 \\ 1 & -4 \end{vmatrix}} = \frac{-x^2 e^{-3x}}{-5e^{-3x}} = \frac{x^2}{5}.$$

$$c_2'(x) = \frac{x^2 e^{2x}}{-5e^{-3x}} = -\frac{1}{5} x^2 e^{5x}.$$

$$c_1(x) = \frac{1}{5}\int x^2\, dx = \frac{1}{15} x^3.$$

$$c_2(x) = -\frac{1}{5}\int x^2 e^{5x}\, dx$$

Let $I(x) = \int x^2 e^{-5x}\, dx$

u	dv
x^2	$+\ e^{-5x}$
$2x$	$-\ e^{+5x}/5$
2	$+\ e^{+5x}/25$
0	$e^{+5x}/125$

$$I(x) = \frac{1}{5} x^2 e^{5x} - \frac{2}{25} x e^{5x} + \frac{2}{125} e^{5x}$$

$$c_2(x) = -\frac{1}{25} x^2 e^{5x} + \frac{2}{125} x e^{5x} - \frac{2}{625} e^{5x}$$

$$= e^{5x}\left[-\frac{x^2}{25} + \frac{2x}{125} - \frac{2}{625}\right].$$

The particular solution of the non-homogenous linear differential equation $(D^2+3D-4)y = x^2 e^{x}$ is:

$$y_p = c_1(x) e^{x} + c_2(x) e^{-4x}$$

$$= \frac{x^3}{15} e^{x} + \left[-\frac{x^2}{25} + \frac{2x}{125} - \frac{2}{625}\right] e^{5x} \cdot e^{-4x}$$

$$= e^{x}\left(\frac{x^3}{15} - \frac{x^2}{25} + \frac{2x}{125} - \frac{2}{625}\right).$$

The general solution of the non-homogenous linear differential equation $(D^2+3D-4)y = x^2 e^{x}$ is:

$$y = y_p + y_h$$
$$= e^{x}\left(\frac{x^3}{15} - \frac{x^2}{25} + \frac{2x}{125} - \frac{2}{625}\right) + c_1 e^{x} + c_2 e^{-4x}.$$

5) $(4D^2+4D+1)y = xe^{-x/2}\sin x.$ (a)

Solution.

The associated homogeneous equation is:-

$$(4D^2+4D+1)y = 0.$$ (b)

The characteristic equation is:

$$(4D^2+4D+1) = 0$$

$$(2D+1)^2 = 0$$

The roots of the characteristic equation are:

$$\alpha_1 = -\tfrac{1}{2} \text{ with multiplicity } 2.$$

The general solution of the homogeneous linear differential equation (b) is given by:-

$$y_h = c_1 e^{-x/2} + c_2 x e^{-x/2}.$$ (c)

Hence, the non-homogeneous linear differential equation $(4D^2+4D+1)y = xe^{-x/2}\sin x$ has a particular solution of the form

$$y_p = c_1(x)\, e^{-x/2} + c_2(x)\, x e^{-x/2}.$$

where $c_1(x)$ and $c_2(x)$ are given by the equations:

$$y_1(x)\, c_1'(x) + y_2(x)\, c_2'(x) = 0$$

$$y_1'(x)\, c_1'(x) + y_2'(x)\, c_2'(x) = h(x).$$

That is:

$$e^{-x/2} c_1'(x) + xe^{-x/2} c_2'(x) = 0$$

$$-\frac{1}{2} e^{-x/2} c_1'(x) + \left(e^{-x/2} - \frac{1}{2} x e^{-x/2}\right) c_2'(x) = \frac{xe^{-x/2}\sin x}{4}.$$

The Wronskian of $y_1(x), y_2(x)$ is:-

$$W[y_1(x), y_2(x)] = \begin{vmatrix} e^{-x/2} & xe^{-x/2} \\ -\frac{1}{2} e^{-x/2} & \left(1-\frac{x}{2}\right) e^{-x/2} \end{vmatrix}$$

$$= e^{-x} \begin{vmatrix} 1 & x \\ -1/2 & (1-x/2) \end{vmatrix}$$

$$= e^{-x}\left[1 - \frac{x}{2} + \frac{x}{2}\right] = e^{-x}.$$

$$c_1'(x) = \frac{-x\, e^{-x/2} \cdot x e^{-x/2} \cdot \sin x}{4e^{-x}} = -\frac{x^2 \sin x}{4}.$$

$$c_2'(x) = \frac{e^{-x/2} \cdot x e^{-x/2} \cdot \sin x}{e^{-x}} = x \sin x.$$

$$c_1(x) = -\int x^2 \sin x \, dx.$$

u		dv
x^2	+	$\sin x$
$2x$	−	$-\cos x$
2	+	$-\sin x$
0		$\cos x$

$$= -\left[-x^2\cos x + 2x\sin x + 2\cos x\right]/4$$

$$= (x^2\cos x - 2x\sin x - 2\cos x)/4$$

$$c_2(x) = \int x \sin x \, dx.$$

u		dv
x	+	$\sin x$
1	−	$-\cos x$
0		$-\sin x$

$$c_2(x) = (-x\cos x + \sin x)/4$$

The particular solution of the non-homogeneous linear differential equation is:-

$$y_p = \left[(x^2 \cos x - 2x \sin x - 2 \cos x)\, e^{-x/2} + (-x \cos x + \sin x)\, x e^{-x/2} \right] / 4$$

$$= \left[(x^2 \cos x - 2x \sin x - 2 \cos x - x^2 \cos x + x \sin x)\, e^{-x/2} \right] / 4$$

$$= -\left(\frac{1}{4} x \sin x - \frac{1}{2} \cos x \right) e^{-x/2}$$

The general solution of the non-homogeneous equation is:

$$y = y_p + y_h$$

$$= c_1 e^{-x/2} + c_2 x e^{-x/2} - \frac{1}{4} e^{-x/2} \cdot x \sin x - \frac{1}{2} e^{x/2} \cos x.$$

6) $(D^2+4)\, y = \frac{e^{2x}}{2}$ (a)

Solution.

The associated homogeneous equation is:

$$(D^2+4)\, y = 0.$$

The characteristic equation is:

$$D^2 + 4 = 0$$

The roots of the characteristic equation are:

$$\alpha_1 = 2i, \quad \alpha_2 = -2i$$

The general solution of the equation is:

$$y_h = c_1 \cos 2x + c_2 \sin 2x.$$

Hence, the non-homogeneous linear differential equation $(D^2+4)y = \frac{e^{2x}}{2}$ has a particular solution of the form -

$$y_p = c_1(x) \cos 2x + c_2(x) \sin 2x.$$

where $c_1(x)$ and $c_2(x)$ are given by the equations -

$$y_1(x)\, c_1'(x) + y_2(x)\, c_2'(x) = 0$$

$$y_1'(x)\, c_1'(x) + y_2'(x)\, c_2'(x) = h(x).$$

That is -

$$\cos 2x\, c_1'(x) + \sin 2x\, c_2'(x) = 0$$

$$-2 \sin 2x\, c_1'(x) + 2 \cos 2x\, c_2'(x) = \frac{e^{2x}}{2}$$

$$c_1'(x) = \frac{-e^{2x} \sin 2x / 2}{\begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix}} = \frac{-e^{2x} \sin 2x / 2}{2\cos^2 2x + 2\sin^2 2x} = -\frac{e^{2x} \sin 2x}{4}.$$

$$c_2'(x) = \frac{e^{2x} \cos 2x / 2}{2} = \frac{e^{2x} \cos 2x}{4}$$

$$c_1(x) = -\frac{1}{4} \int e^{2x} \sin 2x\, dx \qquad c_2(x) = \frac{1}{4} \int e^{2x} \cos 2x\, dx$$

u		dv
e^{2x}	$+$	$\sin 2x$
$2e^{2x}$	$-$	$-\frac{1}{2} \cos 2x$
$4e^{2x}$		$-\frac{1}{4} \sin 2x$

u		dv
e^{2x}	$+$	$\cos 2x$
$2e^{2x}$	$-$	$\frac{1}{2} \sin 2x$
$4e^{2x}$		$-\frac{1}{4} \cos 2x$

$$I_1(x) = -\frac{1}{2} e^{2x} \cos 2x + \frac{1}{2} e^{2x} \sin 2x - \int e^{2x} \sin 2x\, dx$$

$$2I_1(x) = \frac{1}{2} e^{2x} (-\cos 2x + \sin 2x)$$

$$I_1(x) = \frac{1}{4} e^{2x} (-\cos 2x + \sin 2x)$$

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$I_2(x) = \frac{1}{2} e^{2x}\sin 2x + \frac{1}{2} e^{2x}\cos 2x - \int e^{2x}\cos 2x\,dx$.

$2I_2(x) = \frac{1}{2} e^{2x}(\cos 2x + \sin 2x)$.

$I_2(x) = \frac{1}{4} e^{2x}(\cos 2x + \sin 2x)$.

$c_1(x) = -\frac{1}{16} e^{2x}(-\cos 2x + \sin 2x)$

$c_2(x) = \frac{1}{16} e^{2x}(\cos 2x + \sin 2x)$.

The particular solution of the non-homogeneous differential equation is:

$$y_p(x) = -\frac{1}{16} e^{2x}(\sin 2x - \cos 2x)\cos 2x + \frac{1}{16} e^{2x}(\sin 2x + \cos 2x)\sin 2x$$

$$= -\frac{1}{16} e^{2x}\sin 2x\cos 2x + \frac{1}{16} e^{2x}\cos^2 2x + \frac{1}{16} e^{2x}\sin^2 2x + \frac{1}{16} e^{2x}\sin 2x\cos 2x$$

$$= \frac{1}{16} e^{2x}(\sin^2 2x + \cos^2 2x) = \frac{1}{16} e^{2x}$$

Hence, the general solution of the homogeneous linear differential equation (1) is:

$$y = y_p + y_h$$
$$= \frac{1}{16} e^{2x} + c_1\cos 2x + c_2\sin 2x$$

7) $(D^2+10D-12)\,y = \frac{(e^{2x}+1)^2}{e^{2x}}$. (a)

solution.

The associated homogeneous equation is:

$$(D^2+10D-12)\,y = 0.$$

The characteristic equation is:

$$D^2+10D-12=0$$
$$D^2+2(D)(5)+(5)^2-37=0$$
$$(D+5)^2-(\sqrt{37})^2=0$$

The roots of the characteristic equation are:

$$\alpha_1 = -(5+\sqrt{37}),\quad \alpha_2 = -(5-\sqrt{37}).$$

The general solution of the equation is:

$$y_h = c_1 e^{-(5+\sqrt{37})x} + c_2 e^{-(5-\sqrt{37})x}$$

Hence, the non-homogeneous differential equation has a particular solution of the form:

$$y_p = c_1(x)\, e^{-(5+\sqrt{37})x} + c_2(x)\, e^{-(5-\sqrt{37})x}$$

where $c_1(x)$ and $c_2(x)$ are given by the equations:

$$y_1(x)\,c_1'(x) + y_2(x)\,c_2'(x) = 0$$
$$y_1'(x)\,c_1'(x) + y_2'(x)\,c_2'(x) = h(x)$$

That is:

$$e^{-(5+\sqrt{37})x}\,c_1'(x) + e^{-(5-\sqrt{37})x}\,c_2'(x) = 0$$
$$-(5+\sqrt{37})\,e^{-(5+\sqrt{37})x}\,c_1'(x) - (5-\sqrt{37})\,e^{-(5-\sqrt{37})x}\,c_2'(x) = \frac{(e^{2x}+1)^2}{e^{2x}}$$

The wronskian of the functions $e^{-(5+\sqrt{37})x}$ and $e^{-(5-\sqrt{37})x}$ is:

$$W[e^{-(5+\sqrt{37})x}, e^{-(5-\sqrt{37})x}] = \begin{vmatrix} e^{-(5+\sqrt{37})x} & e^{-(5-\sqrt{37})x} \\ -(5+\sqrt{37})e^{-(5+\sqrt{37})x} & -(5-\sqrt{37})e^{-(5-\sqrt{37})x} \end{vmatrix}$$

$$= e^{-(5+\sqrt{37})x}\cdot e^{-(5-\sqrt{37})x}\begin{vmatrix} 1 & 1 \\ -(5+\sqrt{37}) & -(5-\sqrt{37}) \end{vmatrix}$$

$$= e^{-10x}\cdot[-5+\sqrt{37}+5+\sqrt{37}]$$
$$= 2\sqrt{37}\,e^{-10x}$$

$$c_1'(x) = \frac{e^{-(5-\sqrt{37})x} \cdot (e^{2x}+1)^2 / e^{2x}}{2\sqrt{37}\, e^{-10x}}$$

$$= \frac{-1}{2\sqrt{37}} \; e^{-(5-\sqrt{37})x} \cdot e^{10x} \cdot e^{-2x} \cdot (e^{4x} + 2e^{2x} + 1)$$

$$= \frac{-1}{2\sqrt{37}} \; e^{(3+\sqrt{37})x} \cdot (e^{4x} + 2e^{2x} + 1)$$

$$= \frac{-1}{2\sqrt{37}} \left\{ e^{(7+\sqrt{37})x} + 2e^{(5+\sqrt{37})x} + e^{(3+\sqrt{37})x} \right\}$$

$$c_1(x) = \frac{-1}{2\sqrt{37}} \left[\frac{e^{(7+\sqrt{37})x}}{7+\sqrt{37}} + \frac{2e^{(5+\sqrt{37})x}}{5+\sqrt{37}} + \frac{e^{(3+\sqrt{37})x}}{3+\sqrt{37}} \right]$$

$$c_2'(x) = \frac{e^{-(5+\sqrt{37})} \cdot (e^{2x}+1)^2 / e^{2x}}{2\sqrt{37}\, e^{-10x}}$$

$$c_2(x) = \frac{1}{2\sqrt{37}} \left[\frac{e^{(7-\sqrt{37})x}}{7-\sqrt{37}} + \frac{2e^{(5-\sqrt{37})x}}{5-\sqrt{37}} + \frac{e^{(3-\sqrt{37})x}}{3-\sqrt{37}} \right]$$

$$y_p = \frac{1}{2\sqrt{37}} \left[e^{(-5-\sqrt{37})x} \left\{ \frac{e^{(7+\sqrt{37})x}}{7+\sqrt{37}} + \frac{2e^{(5+\sqrt{37})x}}{5+\sqrt{37}} + \frac{e^{(3+\sqrt{37})x}}{3+\sqrt{37}} \right\} \right.$$

$$\left. + e^{(-5+\sqrt{37})x} \left\{ \frac{e^{(7-\sqrt{37})x}}{7-\sqrt{37}} + \frac{2e^{(5-\sqrt{37})x}}{5-\sqrt{37}} + \frac{e^{3-\sqrt{37}}}{3-\sqrt{37}} \right\} \right]$$

$$= \frac{1}{2\sqrt{37}} \left[e^{2x} \left(\frac{-1}{7+\sqrt{37}} + \frac{1}{7-\sqrt{37}} \right) + 2 \cdot \left(\frac{-1}{5+\sqrt{37}} + \frac{1}{5-\sqrt{37}} \right) \right.$$

$$\left. + e^{-2x} \left(\frac{-1}{3+\sqrt{37}} + \frac{1}{3-\sqrt{37}} \right) \right]$$

$$= \frac{1}{2\sqrt{37}} \left[\frac{e^{2x} \cdot 2\sqrt{37}}{(49-37)} + \frac{2 \cdot (2\sqrt{37})}{(25-37)} + \frac{e^{-2x} \cdot 2\sqrt{37}}{9-37} \right]$$

$$= \frac{e^{2x}}{12} + \frac{2}{(-12)} + \frac{e^{-2x}}{(-28)}$$

The general solution of the non-homogeneous differential equation is:

$$y = y_p + y_h$$
$$= c_1 e^{(-5-\sqrt{37})x} + c_2 e^{(-5+\sqrt{37})x} + \frac{e^{2x}}{12} - \frac{e^{-2x}}{28} - \frac{1}{6}$$

8) $(D+3)^2 y = (x+1)e^x$.

Solution.

The associated homogeneous differential equation is –

$$(D+3)^2 y = 0 \qquad (a)$$

The roots of the characteristic equation are

$$\alpha_1 = -3 \text{ with multiplicity } 2. \qquad (b)$$

The general solution of the homogeneous differential equation is –

$$y = c_1 e^{-3x} + c_2 x e^{-3x}.$$

Hence, the particular solution of the non-homogeneous differential equation has the form –

$$y_p = c_1(x) e^{-3x} + c_2(x) x e^{-3x}$$

where $c_1(x)$ and $c_2(x)$ are given by the equations –

$$y_1(x) c_1'(x) + y_2(x) c_2'(x) = 0$$
$$y_1'(x) c_1'(x) + y_2'(x) c_2'(x) = h(x)$$

That is —

$$e^{-3x}\cdot c_1'(x) + xe^{-3x}\cdot c_2'(x) = 0$$
$$-3e^{-3x}\, c_1'(x) + (e^{-3x} - 3xe^{-3x})\, c_2'(x) = (x+1)e^{x}.$$

The Wronskian of the functions e^{-3x}, xe^{-3x} is

$$W[e^{-3x}, xe^{-3x}] = \begin{vmatrix} e^{-3x} & xe^{-3x} \\ -3e^{-3x} & (e^{-3x} - 3xe^{-3x}) \end{vmatrix} = e^{-6x}\begin{vmatrix} 1 & x \\ -3 & (1-3x) \end{vmatrix}$$
$$= e^{-6x}\left[(1-3x)+3x\right] = e^{-6x}.$$

$$c_1'(x) = \frac{-xe^{-3x}\cdot(x+1)e^{x}}{e^{-6x}}$$
$$= -x(x+1)e^{4x}$$
$$= -(x^2e^{4x} + xe^{4x})$$

x^2	$+$	e^{4x}
$2x$	$-$	$e^{4x}/4$
2	$+$	$e^{4x}/16$
0		$e^{4x}/64$

x	$+$	e^{4x}
1	$-$	$e^{4x}/4$
0		$e^{4x}/16$

$$c_1(x) = -\left(\int x^2e^{4x}\,dx + \int x\, e^{4x}\,dx\right)$$
$$= -\left[\left(\frac{1}{4}x^2 - \frac{1}{8}x + \frac{1}{32}\right)e^{4x} + \left(\frac{1}{4}x - \frac{1}{16}\right)e^{4x}\right]$$
$$= -\left[\frac{1}{4}x^2 + \frac{1}{8}x - \frac{1}{16}\right]e^{4x}$$
$$= \left(-\frac{1}{4}x^2 - \frac{1}{8}x + \frac{1}{16}\right)e^{4x}.$$

$$c_2'(x) = \frac{e^{-3x}\cdot(x+1)e^{x}}{e^{-6x}}$$
$$= (x+1)e^{4x}$$
$$= xe^{4x} + e^{4x}$$

$$c_2(x) = \int xe^{4x}\,dx + \int e^{4x}\,dx$$
$$= \left(\frac{1}{4}x - \frac{1}{16}\right)e^{4x} + \frac{e^{4x}}{4}$$

The particular solution of the non-homogeneous differential equation is —

$$y_p = c_1(x)e^{-3x} + c_2(x)\, xe^{-3x}$$
$$= e^{x}\left[\left(-\frac{1}{4}x^2 - \frac{1}{8}x + \frac{1}{16}\right) + \left(\frac{1}{4}x^2 - \frac{1}{16}x + \frac{1}{4}x\right)\right]$$
$$= e^{x}\left[\frac{1}{16}x + \frac{1}{16}\right].$$

The general solution of the non-homogeneous differential equation is

$$y = y_p + y_h$$
$$= \frac{1}{16}xe^{x} + \frac{1}{16}e^{x} + c_1e^{-3x} + c_2xe^{-3x}.$$

9) $(D^2 - 2D + 2)y = e^{2x}\sin x.$

Solution.

The associated homogeneous differential equation is —

$$(D^2 - 2D + 2)y = 0$$

The characteristic equation is —

$$D^2 - 2D + 2 = 0$$
$$(D-1)^2 + 1^2 = 0$$

The roots of the characteristic equation are:
$$\alpha_1 = 1+i \qquad \alpha_2 = 1-i$$

The general solution of the homogeneous differential equation is –
$$y = e^x (c_1 \cos x + c_2 \sin x)$$

The non-homogeneous equation has a particular solution of the form:
$$y = c_1(x) e^x \cos x + c_2(x) e^x \sin x$$
where $c_1(x)$ and $c_2(x)$ is determined by the equations –
$$y_1(x) c_1'(x) + y_2(x) c_2'(x) = 0$$
$$y_1'(x) c_1'(x) + y_2'(x) c_2'(x) = h(x)$$

That is:
$$e^x \cos x \, c_1'(x) + e^x \sin x \, c_2'(x) = 0$$
$$e^x (\cos x - \sin x) c_1'(x) + e^x (\sin x + \cos x) c_2'(x) = e^{2x} \sin x$$

The wronskian of the functions $e^x \cos x$, $e^x \sin x$ is given by –
$$W[e^x \cos x, e^x \sin x] = \begin{vmatrix} e^x \cos x & e^x \sin x \\ e^x(\cos x - \sin x) & e^x(\sin x + \cos x) \end{vmatrix}$$
$$= e^{2x}[(\sin x + \cos x)\cos x - \sin x (\cos x - \sin x)]$$
$$= e^{2x}[\sin x \cos x + \cos^2 x - \sin x \cos x + \sin^2 x]$$
$$= e^{2x}.$$

$$c_1'(x) = -\frac{e^{3x} \cdot \sin^2 x}{e^{2x}} = -e^x \sin^2 x = -e^x\left(\frac{1-\cos 2x}{2}\right) = \left(\frac{e^x \cos 2x - e^x}{2}\right)$$
$$c_2'(x) = \frac{e^{3x} \sin x \cdot \cos x}{e^{2x}} = e^x \sin x \cdot \cos x = \frac{1}{2} e^x \sin 2x$$

$$c_1(x) = \frac{1}{2}\left[\int e^x \cos 2x \, dx - \int e^x \, dx\right]$$

Let $I_1(x) = \int e^x \cos 2x \, dx$

u	dv
e^x +	$\cos 2x$
e^x −	$+\frac{\sin 2x}{2}$
e^x	$-\frac{\cos 2x}{4}$

$$I_1(x) = \frac{1}{2} e^x \sin 2x + \frac{1}{4} e^x \cos 2x - \frac{1}{4}\int e^x \cos 2x \, dx$$
$$\frac{5}{4} I_1(x) = \frac{e^x}{4}(2\sin 2x + \cos 2x)$$
$$I_1(x) = \frac{e^x}{5}(2\sin 2x + \cos 2x)$$

Let $I_2(x) = \int e^x \sin 2x$

e^x +	$\sin 2x$
e^x	$-\frac{\cos 2x}{2}$
e^x	$-\frac{\sin 2x}{4}$

$$I_2(x) = -\frac{1}{2} e^x \cos 2x + \frac{1}{4} e^x \sin 2x - \frac{1}{4}\int e^x \sin 2x \, dx$$
$$\frac{5}{4} I_2(x) = \frac{e^x}{4}(-2\cos 2x + \sin 2x)$$

$I_2(x) = \frac{e^x}{5}(-2\cos 2x + \sin 2x).$

$c_1(x)y_1(x) = \frac{1}{10}(2\sin 2x + \cos 2x)\, e^{2x}\cos x - \frac{1}{2}e^{2x}\cos x$

$c_2(x)y_2(x) = \frac{1}{10}(\sin 2x - 2\cos 2x)\, e^{2x}\sin x$

$y_p = 1/10\, \{2\sin 2x\cos x + \cos 2x\cos x - 2\cos 2x\sin x + \sin 2x\sin x\}e^{2x} = \frac{1}{10}(2\sin x + \cos x)e^{2x} - \frac{1}{2}e^{2x}\cos x$

$y(x) = \frac{1}{5}e^{2x}\sin x - \frac{2}{5}e^{2x}\cos x + c_1e^x\cos x + c_2e^x\sin x.$

10) $(4D^2 - 8D + 5)y = e^x\tan^2\left(\frac{x}{2}\right).$

Solution

The associated homogeneous differential equation is –

$$(4D^2 - 8D + 5)y = 0$$

The characteristic equation is

$$4D^2 - 8D + 5 = 0$$
$$(2D)^2 - 2\cdot(2D)(2) + (2)^2 + 1 = 0$$
$$(2D - 2)^2 + 1 = 0$$

The roots of the characteristic equation are:

$$\alpha_1 = \frac{2+i}{2} = 1 + \frac{1}{2}i$$

$$\alpha_2 = \frac{2-i}{2} = 1 - \frac{1}{2}i$$

The general solution of the equation is –

$$y = e^x\left(c_1\cos\frac{x}{2} + c_2\sin\frac{x}{2}\right)$$

The particular solution of the non-homogeneous differential equation can be written in the form:

$$y_p = c_1(x)e^x\cos\frac{x}{2} + c_2(x)e^x\sin\frac{x}{2}.$$

where $c_1(x)$ and $c_2(x)$ are given by the equations –

$$c_1'(x)e^x\cos x/2 + c_2'(x)e^x\sin x/2 = 0$$

$$c_1'(x)\left(e^x\cos x/2 - \frac{1}{2}e^x\sin x/2\right) + c_2'(x)\left(e^x\sin x/2 + \frac{1}{2}e^x\cos x/2\right) = \frac{1}{4}e^x\tan^2\frac{x}{2}$$

$W(y_1, y_2) = e^{2x}\sin x/2\cos x/2 + 1/2\, e^{2x}\cos^2 x/2 - e^{2x}\sin x/2\cos x/2 + 1/2\, e^{2x}\sin^2 x/2 = \frac{1}{2}e^{2x}$

so, $c_1'(x) = -\frac{e^{2x}\sin x/2\cdot\tan^2 x/2}{4(1/2\, e^{2x})}$

$$= -\frac{1}{2}\cdot\frac{\sin^3 x/2}{\cos^2 x/2}$$

$$= -\frac{1}{2}\,\frac{\sin x/2\cdot\sin^2 x/2}{\cos^2 x/2} = -\frac{1}{2}\,\frac{\sin x/2\,(1-\cos^2 x/2)}{\cos^2 x/2}$$

$$= -\frac{1}{2}\left\{\sec x/2\cdot\tan x/2 - \sin x/2\right\}$$

$$c_1(x) = \frac{1}{2}\int\sin x/2\,dx - \frac{1}{2}\int\sec x/2\tan x/2\,dx$$

$$= -\frac{1}{2}\,\frac{\cos x/2}{(1/2)} - \frac{1}{2}\,\frac{\sec x/2}{(1/2)}$$

$$= -\cos(x/2) - \sec(x/2)$$

$$c_2'(x) = \frac{e^{2x} \cdot \cos x/2 \cdot \tan^2 x/2}{4(1/2)e^{2x}}$$

$$= \frac{1}{2}\frac{\sin^2 x/2}{\cos x/2} = \frac{1}{2}\frac{1-\cos^2 x/2}{\cos x/2}$$

$$= \frac{1}{2}\left\{\sec(x/2) - \cos(x/2)\right\}$$

$$c_2(x) = \frac{1}{2}\left\{\int \sec x/2\, dx - \int \cos x/2\, dx\right\}$$

$$= \frac{1}{2}\left\{\int \frac{\sec^2 x/2 + \sec x/2 \tan x/2}{\sec x/2 + \tan x/2}\, dx - \int \cos\frac{x}{2}\, dx\right\}$$

$$= \frac{1}{2}\left\{\frac{\log(\sec x/2 + \tan x/2)}{(1/2)} - \frac{\sin x/2}{(1/2)}\right\}$$

$$= \log(\sec x/2 + \tan x/2) - \sin x/2.$$

$$y_p = c_1(x)e^x \cos x/2 + c_2(x)e^x \sin x/2$$

$$= -e^{2x}\cos^2 x/2 - e^{2x} + \log(\sec x/2 + \tan x/2)e^{2x}\sin x/2 - e^{2x}\sin^2 x/2$$

$$= -2e^{2x} + \log(\sec x/2 + \tan x/2)\cdot e^{2x}\sin x/2.$$

We can simplify $\sec x/2 + \tan x/2$ as follows -

$$\sec x/2 + \tan x/2 = \frac{1+\sin x/2}{\cos x/2} = \frac{1 + 2\sin x/4 \cos x/4}{\cos^2 x/4 - \sin^2 x/4}$$

$$= \frac{(\cos x/4 + \sin x/4)^2}{\cos^2 x/4 - \sin^2 x/4} = \frac{\cos x/4 + \sin x/4}{\cos x/4 - \sin x/4}$$

$$\therefore \log(\sec x/2 + \tan x/2) = \log(\cos x/4 + \sin x/4) - \log(\cos x/4 - \sin x/4)$$

so, $$y_p = -2e^{2x} + \log(\cos x/4 + \sin x/4)\, e^x \sin x/2 - \log(\cos x/4 - \sin x/4)\, e^x \sin x/2$$

$$y = c_1 e^x \cos\frac{x}{2} + c_2 e^x \sin\frac{x}{2} - 2e^{2x} + \log\left(\cos\frac{x}{4} + \sin\frac{x}{2}\right)e^x \sin\frac{x}{2} - \log\left(\cos\frac{x}{4} - \sin\frac{x}{4}\right)e^x \sin\frac{x}{2}.$$

For each of the following differential equations verify that the given expression is the general solution of the associated homogenous equation and then find a particular solution of the equation.

11) $x^2y'' - 2xy' + 2y = x^3 \ln x$, $x > 0$; $y_h = c_1x + c_2x^2$.

solution

$y = c_1x + c_2x^2$
$y' = c_1 + 2c_2x$
$y'' = 2c_2$.

$x^2y'' - 2xy' + 2y$
$= x^2(2c_2) - 2x(c_1 + 2c_2x) + 2(c_1x + c_2x^2)$
$= 2c_2x^2 - 2c_1x - 4c_2x^2 + 2c_1x + 2c_2x^2$
$= 0$.

Thus, the non-homogenous differential equation has a particular solution of the form:

$y_p = c_1(x)\,x + c_2(x)\,x^2$.

where $c_1(x)$ and $c_2(x)$ are given by the equations:

$y_1(x)\,c_1'(x) + y_2(x)\,c_2'(x) = 0$
$y_1'(x)\,c_1'(x) + y_2'(x)\,c_2'(x) = h(x)$

That is -

$x\,c_1'(x) + x^2\,c_2'(x) = 0$
$c_1'(x) + 2x\,c_2'(x) = x\ln(x)$

$$c_1'(x) = -\frac{x^3\ln(x)}{\begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix}} = -\frac{x^3\ln(x)}{x^2} = -x\ln(x)$$

$c_1(x) = -\int x\ln(x)\,dx$

$\ln(x) \quad + \quad x$
$\frac{1}{x} \quad \frac{x^2}{2}$

$$= -\left[\frac{x^2}{2}\ln(x) - \frac{1}{2}\int x\,dx\right]$$
$$= -\left[\frac{x^2}{2}\ln(x) - \frac{x^2}{4}\right].$$

$$c_2'(x) = \frac{x^2\ln(x)}{x^2} = \ln(x)$$

$c_2(x) = \int \ln(x)\,dx$

$\ln x \quad 1$
$\frac{1}{x} \quad x$

$= x\ln x - \int dx$
$= x\ln x - x$

The particular solution of the non homogenous differential equation is

$y_p = c_1(x)\,x + c_2(x)\cdot x^2$
$= \left(-\frac{x^2}{2}\ln(x) + \frac{x^2}{4}\right)x + (x\ln x - x)\,x^2$
$= -\frac{x^3}{2}\ln(x) + \frac{x^3}{4} + x^3\ln x - x^3$
$= \frac{x^3}{2}\ln x - 3\frac{x^3}{4}$.

12) $x^2y'' - xy' + y = x(x+1)$; $y_h = (c_1 + c_2\ln(x))\,x$.

solution.

$y_h = (c_1 + c_2 \ln(x))x = c_1 x + c_2 x \ln(x)$

$y' = c_1 + c_2 \ln x + c_2 x \left(\frac{1}{x}\right)$

$= c_1 + c_2 + c_2 \ln x.$

$y'' = \frac{c_2}{x}.$

$x^2 y'' - xy' + y$

$= x^2 \left(\frac{c_2}{x}\right) - x(c_1 + c_2 + c_2 \ln x) + c_1 x + c_2 x \ln x$

$= c_2 x - c_1 x + - c_2 x - c_2 x \ln x + c_1 x + c_2 x \ln x$

$= 0.$

The non-homogenous differential equation has a particular solution of the form:

$$y = c_1(x)x + c_2(x) \cdot x \ln x$$

where $c_1(x)$ and $c_2(x)$ are given by the equations:

$$y_1(x)c_1'(x) + y_2(x)c_2'(x) = 0$$
$$y_1'(x)c_1'(x) + y_2'(x)c_2'(x) = h(x)$$

that is -

$$x c_1'(x) + x \ln(x) c_2'(x) = 0$$
$$c_1'(x) + (\ln x - 1) c_2'(x) = (x^2 + x)/x^2 = \left(\frac{x+1}{x}\right)$$

$W[y_1(x), y_2(x)] = \begin{vmatrix} x & x \ln x \\ 1 & \ln x + 1 \end{vmatrix} = x \ln x + x - x \ln x$

$c_1'(x) = -\frac{x \ln x (x+1)/x}{x} = \frac{(x+1)\ln x}{x} = -\left(\ln x + \frac{\ln x}{x}\right)$

$c_1(x) = -\left(\int \ln x \, dx + \int \frac{\ln x}{x} dx\right) = -\left\{x \ln x - x + \frac{1}{2}(\ln x)^2\right)$

$= -x \ln x + x - \frac{1}{2} \ln(x)^2$

$c_2'(x) = \frac{x+1}{x}$; $c_2(x) = \int dx + \int \frac{dx}{x} = x + \ln x$

$y_p = c_1(x) \cdot x + c_2(x) \cdot x \ln x$

$= -x^2 \cancel{\ln x} + x^2 - \frac{1}{2} x(\ln x)^2 + \cancel{x^2 \ln x} + x(\ln x)^2$

$= \frac{1}{2} x(\ln x)^2 + x^2.$

14) $xy'' - (1+2x^2)y' = x^5 e^{x^2}$; $y_h = c_1 + c_2 e^{x^2}$

Solution.

$$y = c_1 + c_2 e^{x^2}$$
$$y' = c_2 2x \cdot e^{x^2} = 2c_2 x e^{x^2}$$
$$y'' = 2c_2 (e^{x^2} + 2x^2 e^{x^2})$$
$$= 2c_2 e^{x^2} + 4c_2 x^2 e^{x^2}.$$

$$xy'' - (1+2x^2)y'$$
$$= 2c_2 x e^{x^2} + 4c_2 x^3 e^{x^2} - (1+2x^2)(2c_2 x e^{x^2})$$
$$= 2c_2 x e^{x^2} + 4c_2 x^3 e^{x^2} - 2c_2 x e^{x^2} - 4x^3 c_2 e^{x^2}$$
$$= 0.$$

The non-homogeneous differential equation has a particular solution of the form –

$$y_p = c_1(x) + c_2(x) e^{x^2}$$

where $c_1(x)$ and $c_2(x)$ are given by the equations –

$$y_1(x) c_1'(x) + y_2(x) c_2'(x) = 0$$
$$y_1'(x) c_1'(x) + y_2'(x) c_2'(x) = (x^5 e^{x^2})/x.$$

That is,

$$c_1'(x) + e^{x^2} c_2'(x) = 0$$
$$2x e^{x^2} c_2'(x) = x^4 e^{x^2}.$$

$$\therefore \quad c_2'(x) = \frac{1}{2} x^3.$$

$$c_1'(x) = -\frac{1}{2} x^3 e^{x^2}.$$

$$c_1(x) = -\frac{1}{2} \int x^3 e^{x^2} dx$$

$$= -\frac{1}{4} \int x^2 (2x) e^{x^2} dx = -\frac{1}{4} \int t e^t dt.$$

t	e^t
1	e^t
0	e^t

$$= -\frac{1}{4} \left[t e^t - \int e^t dt \right]$$
$$= -\frac{1}{4} (t e^t - e^t)$$
$$= -\frac{1}{4} (x^2 e^{x^2} - e^{x^2})$$

$$c_2(x) = \frac{1}{8} x^4.$$

$$y_p = \frac{1}{8} x^4 e^{x^2} - \frac{1}{4} x^2 e^{x^2} + \frac{1}{4} e^{x^2}.$$

15) $(1-x^2) y'' - 2xy' = 2x$, $-1 < x < 1$; $y_h = c_1 + c_2 \ln\left(\frac{1+x}{1-x}\right)$.

Solution.

$$y = c_1 + c_2 \ln\left(\frac{1+x}{1-x}\right)$$
$$y' = c_2 \left[\frac{1}{1+x} + \frac{1}{1-x}\right] = \frac{2c_2}{1-x^2}$$
$$y'' = \frac{-2c_2}{(1-x^2)^2} \cdot (-2x) = \frac{4c_2 x}{(1-x^2)^2}$$

$$(1-x^2) y'' - 2xy' = (1-x^2) \frac{4c_2 x}{(1-x^2)^2} - 2x\left(\frac{2c_2}{1-x^2}\right)$$
$$= \frac{4c_2 x}{(1-x^2)} - \frac{4c_2 x}{1-x^2} = 0.$$

The non-homogenous differential equation has a particular solution of the form

$$y_p = c_1(x) + c_2(x)\ln\left(\frac{1+x}{1-x}\right)$$

where $c_1(x)$ and $c_2(x)$ are given by the equations:-

$$y_1(x)c_1'(x) + y_2(x)c_2'(x) = 0$$
$$y_1'(x)c_1'(x) + y_2'(x)c_2'(x) = h(x)$$

that is:

$$c_1'(x) + c_2'(x)\ln\left(\frac{1+x}{1-x}\right) = 0$$
$$2c_2'(x)\frac{1}{1-x^2} = \frac{2x}{1-x^2}$$

$c_2'(x) = x$

$c_2(x) = \int x\,dx = \frac{x^2}{2}$

$c_1'(x) = -x\ln\left(\frac{1+x}{1-x}\right)$

$\ln\left(\frac{1+x}{1-x}\right)$ $\quad x$

$\frac{2}{1-x^2}$ $\quad \frac{x^2}{2}$

$$c_1(x) = -\left\{\frac{x^2}{2}\ln\left(\frac{1+x}{1-x}\right) - \int\frac{x^2}{1-x^2}dx\right\}$$
$$= -\left\{\frac{x^2}{2}\ln\left(\frac{1+x}{1-x}\right) + \int\left(\frac{x^2}{x^2-1}\right)dx\right\}$$
$$= -\left\{\frac{x^2}{2}\ln\left(\frac{1+x}{1-x}\right) + \int\frac{x^2-1+1}{x^2-1}dx\right\}$$
$$= -\left\{\frac{x^2}{2}\ln\left(\frac{1+x}{1-x}\right) + \int\left(1+\frac{1}{x^2-1}\right)dx\right\}$$
$$= -\left\{\frac{x^2}{2}\ln\left(\frac{1+x}{1-x}\right) + x + \int\frac{dx}{x^2-1}\right\} = \left\{\frac{x^2}{2}\ln\left(\frac{1+x}{1-x}\right) + x + \frac{1}{2}\left(\int\frac{dx}{x-1} - \frac{dx}{x+1}\right)\right.$$
$$= -\left(\frac{x^2}{2}\ln\left(\frac{1+x}{1-x}\right) + x + \frac{1}{2}\ln\left(\frac{x-1}{x+1}\right)\right) =$$
$$= -\left(\frac{x^2}{2}\ln\left(\frac{1+x}{1-x}\right) + x + \frac{1}{2}\ln\left(\frac{1-x}{1+x}\right)\right) =$$
$$= -\frac{x^2}{2}\ln\left(\frac{1+x}{1-x}\right) - x + \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

$$y_p = -\frac{x^2}{2}\ln\left(\frac{1+x}{1-x}\right) - x + \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) + \frac{x^2}{2}\ln\left(\frac{1+x}{1-x}\right) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) - x.$$

In each of the following exercises find a Green's function for the given linear differential operator.

16) D^2+3.

solution.

The Green's function $K(x,t)$ for the linear differential operator $L = D^2+3$ is

$$K(x,t) = \frac{y_2(x)y_1(t) - y_1(x)y_2(t)}{W[y_1(t), y_2(t)]}.$$

The characteristic equation is

$$D^2+3=0$$

The roots of the characteristic equation are,

$$\alpha_1 = \sqrt{3}i, \quad \alpha_2 = -\sqrt{3}i$$

The general solution of the homogeneous linear differential equation $(D^2+3)y = 0$ is,

$$y = c_1\cos\sqrt{3}x + c_2\sin\sqrt{3}x.$$

$$W[y_1(t), y_2(t)] = \begin{vmatrix} \cos\sqrt{3}t & \sin\sqrt{3}t \\ -\sqrt{3}\sin\sqrt{3}t & \sqrt{3}\cos\sqrt{3}t \end{vmatrix}$$

$$= \sqrt{3}(\cos^2\sqrt{3}t + \sin^2\sqrt{3}t)$$
$$= \sqrt{3}.$$

$$K(x,t) = \frac{\sin\sqrt{3}x \cdot \cos\sqrt{3}t - \cos\sqrt{3}x \cdot \sin\sqrt{3}t}{\sqrt{3}}$$
$$= \frac{\sin\sqrt{3}(x-t)}{\sqrt{3}}$$

17) $D^2 - D - 2$.

Solution

The characteristic equation is

$$D^2 - D - 2 = 0$$
$$D^2 - 2D + D - 2 = 0$$
$$D(D-2) + 1(D-2) = 0$$
$$(D+1)(D-2) = 0$$

The roots of the characteristic equation are –

$$\alpha_1 = -1, \alpha_2 = 2.$$

The general solution of the equation is –

$$y = c_1 e^{-x} + c_2 e^{2x}.$$

The Wronskian of the functions e^{-x}, e^{2x} is:

$$W[e^{-x}, e^{2x}] = \begin{vmatrix} e^{-x} & e^{2x} \\ -e^{-x} & 2e^{2x} \end{vmatrix}$$

$$= e^{-x}\begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} = 3e^{x}.$$

The Green's function $K(x,t)$ for the linear differential operator $L = D^2 - D - 2$ is given by,

$$K(x,t) = \frac{y_1(x)y_2(t) - y_2(x)y_1(t)}{W[y_1(t), y_2(t)]}$$

$$= \frac{e^{2x} \cdot e^{-t} - e^{-x} \cdot e^{2t}}{3e^{t}}$$

$$= \frac{1}{3}[e^{2(x-t)} - e^{-(x-t)}].$$

18) $D^2 + 4D + 4$.

Solution.

The characteristic equation is

$$D^2 + 4D + 4 = 0$$
$$(D+2)^2 = 0$$

The roots of the characteristic equation are

$$\alpha = -2 \text{ with multiplicity } 2.$$

Thus, the general solution of the homogeneous differential equation is:

$$y = c_1 e^{-2x} + c_2 x e^{-2x}.$$

The Wronskian of the functions e^{-2x}, xe^{-2x} is –

$$W[e^{-2x}, xe^{-2x}] = \begin{vmatrix} e^{-2x} & xe^{-2x} \\ -2e^{-2x} & e^{-2x} - 2xe^{-2x} \end{vmatrix} = e^{-4x}\begin{vmatrix} 1 & x \\ -2 & 1-2x \end{vmatrix}$$

$= e^{-4x}(1-2x+2x) = e^{4x}.$

The Green's function $K(x,t)$ for the linear differential operator $L = D^2 - D - 2$ is given by,

$$K(x,t) = \frac{y_2(x)\,y_1(t) - y_1(x)\cdot y_2(t)}{W[y_1(t), y_2(t)]}$$

$$= \frac{xe^{-2x}\cdot e^{-2t} - e^{-2x}\cdot te^{-2t}}{e^{-4t}}$$

$$= e^{2t}(xe^{-2x} - te^{-2x})$$

$$= e^{2t-2x}(x-t)$$

$$= e^{-2(x-t)}(x-t).$$

19) $4D^2 - 8D + 5.$

<u>solution.</u>

The characteristic equation is-

$$4D^2 - 8D + 5 = 0$$

$$(2D)^2 - 2(2D)(2) + 2^2 + 1 = 0$$

$$(2D-2)^2 + 1^2 = 0$$

The roots of the characteristic equation are-

$$\alpha_1 = 1 + \frac{1}{2}i,\ \alpha_2 = 1 - \frac{1}{2}i.$$

The general solution of the equation is-

$$y = e^{x}\left(c_1 \cos\frac{x}{2} + c_2 \sin\frac{x}{2}\right).$$

The Wronskian of the functions $e^{x}\cos\frac{x}{2}$, $e^{x}\sin\frac{x}{2}$ is

$$W\left[e^{x/2}\cos\frac{x}{2}, e^{x/2}\sin\frac{x}{2}\right] = \begin{vmatrix} e^{x}\cos\frac{x}{2} & e^{x}\sin\frac{x}{2} \\ e^{x}\left(\cos\frac{x}{2} - \frac{1}{2}\sin\frac{x}{2}\right) & e^{x}\left(\sin\frac{x}{2} + \frac{1}{2}\cos\frac{x}{2}\right) \end{vmatrix}$$

$$= e^{2x}\begin{vmatrix} \cos x/2 & \sin x/2 \\ \cos x/2 - 1/2\sin x/2 & \sin x/2 + 1/2\cos x/2 \end{vmatrix}$$

$$= e^{x}(\sin x/2 + \tfrac{1}{2}\cos x/2)\cdot \cos x/2 - \sin x/2\,(\cos x/2 - \tfrac{1}{2}\sin x/2)$$

$$= e^{x}(\sin x/2\cos x/2 + \tfrac{1}{2}\cos^2 x/2 - \sin x/2\cos x/2 + \tfrac{1}{2}\sin^2 x/2)$$

$$= \frac{e^{x}}{2}$$

The Green's function $K(x,t)$ for the linear differential operator $L = 4D^2 - 8D + 5$ is:

$$K(x,t) = \frac{y_2(x)\,y_1(t) - y_1(x)\,y_2(t)}{W[y_1(t), y_2(t)]}$$

$$= \frac{e^{x}\sin\frac{x}{2}\,e^{t}\cos\frac{t}{2} - e^{x}\cos\frac{x}{2}\cdot e^{t}\sin\frac{t}{2}}{(e^{2t}/2)}$$

$$= 2e^{(x-t)}\,\sin x/2\cdot\cos t/2 - \cos x/2\cdot\sin t/2$$

$$= 2e^{(x-t)}\sin(x-t)/2.$$

20) D^2+3D-4

solution

The characteristic equation is–

$$D^2+3D-4=0$$
$$D^2+4D-D-4=0$$
$$D(D+4)-1(D+4)=0$$
$$(D-1)(D+4)=0$$

The roots of the characteristic equation are–

$$\alpha_1=1,\ \alpha_2=-4.$$

The general solution of the equation is:

$$y=c_1e^{x}+c_2e^{-4x}$$

The Wronskian of the functions e^{x}, e^{-4x} is–

$$W[e^{x}, e^{-4x}] = \begin{vmatrix} e^{x} & e^{-4x} \\ e^{x} & -4e^{-4x} \end{vmatrix}$$

$$= e^{-3x}\begin{vmatrix} 1 & 1 \\ 1 & -4 \end{vmatrix}$$

$$= -5e^{-3x}$$

The Green's function $K(x,t)$ for the linear differential operator $L=D^2+3D-4$ is given by,

$$K(x,t) = \frac{y_2(x)\,y_1(t)-y_1(x)\,y_2(t)}{W[y_1(t),y_2(t)]}$$

$$= \frac{e^{-4x}\cdot e^{t}-e^{x}\cdot e^{-4t}}{-5e^{-3t}}$$

$$= \frac{e^{-4x}\cdot e^{4t}-e^{x}\cdot e^{-t}}{-5}$$

$$= \frac{e^{(x-t)}-e^{-4(x-t)}}{5}.$$

21) $x^2D^2-2xD+2.$

solution.

The characteristic equation is–

$$x^2D^2-2xD+2=0$$
$$(xD-1)^2+1^2=0$$

The roots of this equation are

$$xD=1+i,\ xD=1-i$$

22) Solve the initial value problem

$$(D^2 + 2aD + b^2)y = \sin \omega t, \quad y(0) = y'(0) = 0. \qquad (a)$$

where a, b, ω are real constants, $a < b$. Consider ~~the~~ separately the cases $\omega \neq \sqrt{b^2-a^2}$ and $\omega = \sqrt{b^2-a^2}$ and sketch the solution curve in each case.

Solution.

The characteristic equation is

$$m^2 + 2am + b^2 = 0, \quad a < b.$$

$$(m+a)^2 + b^2 - a^2 = 0$$

$$(m+a)^2 + (\sqrt{b^2-a^2})^2 = 0$$

$$(m + a + i\sqrt{b^2-a^2})(m + a - i\sqrt{b^2-a^2}) = 0$$

$$\alpha_1 = -a + i\sqrt{b^2-a^2}$$

$$\alpha_2 = -a - i\sqrt{b^2-a^2}$$

$$y_h = e^{-at}(c_1 \cos\sqrt{b^2-a^2}\,t + c_2 \sin\sqrt{b^2-a^2}\,t)$$

For the economy of space, denote $a = \alpha$

$$\sqrt{b^2-a^2} = \beta.$$

$$y_h = e^{-\alpha t}(c_1 \cos\beta t + c_2 \sin\beta t)$$

Let $y_p = c_1(t)e^{-\alpha t}\cos\beta t + c_2(t)e^{-\alpha t}\sin\beta t$

The Wronskian $W[y_1(t), y_2(t)] = \begin{vmatrix} e^{-\alpha t}\cos\beta t & e^{-\alpha t}\sin\beta t \\ -\alpha e^{-\alpha t}\cos\beta t - \beta e^{-\alpha t}\sin\beta t & -\alpha e^{-\alpha t}\sin\beta t + \beta e^{-\alpha t}\cos\beta t \end{vmatrix}$

$$= e^{-2\alpha t}\begin{vmatrix} \cos\beta t & \sin\beta t \\ -\alpha\cos\beta t - \beta\sin\beta t & -\alpha\sin\beta t + \beta\cos\beta t \end{vmatrix}$$

$$= e^{-2\alpha t}\{\cos\beta t(-\alpha\sin\beta t) + \beta\cos^2\beta t + \alpha\sin\beta t\cos\beta t + \beta\sin^2\beta t\}$$

$$= \beta e^{-2\alpha t}.$$

We have

$$c_1'(t)e^{-\alpha t}\cos\beta t + c_2'(t)e^{-\alpha t}\sin\beta t = 0$$

$$c_1'(t)(-\alpha e^{-\alpha t}\cos\beta t - \beta e^{-\alpha t}\sin\beta t) + c_2'(t)(-\alpha e^{-\alpha t}\sin\beta t + \beta e^{-\alpha t}\cos\beta t) = \sin\omega t$$

$$c_1'(t) = -\frac{e^{-\alpha t}\sin\beta t\cdot\sin\omega t}{\beta e^{-2\alpha t}} = -\frac{e^{\alpha t}}{\beta}\sin\beta t\cdot\sin\omega t.$$

$$= -\frac{1}{2\beta}\cdot e^{\alpha t}[-\cos(\omega+\beta)t + \cos(\omega-\beta)t]$$

$$c_2'(t) = \frac{e^{-\alpha t}\cos\beta t\cdot\sin\omega t}{\beta e^{-2\alpha t}} = \frac{e^{\alpha t}\cdot\sin\omega t\cos\beta t}{\beta}$$

$$= \frac{1}{2\beta}\cdot e^{\alpha t}[\sin(\omega+\beta)t + \sin(\omega-\beta)t].$$

Let $I_1(t) = \int e^{\alpha t}\cdot\cos(\omega\pm\beta)t$

$$\begin{array}{ll} e^{\alpha t} & +\cos(\omega\pm\beta)t \\ \alpha e^{\alpha t} & -\sin(\omega\pm\beta)t/(\omega\pm\beta) \\ \alpha^2 e^{\alpha t} & -\cos(\omega\pm\beta)t/(\omega\pm\beta)^2 \end{array}$$

$$I_1(t) = \frac{1}{(\omega\pm\beta)}e^{\alpha t}\sin(\omega\pm\beta t) + \frac{\alpha}{(\omega\pm\beta)^2}\cos(\omega\pm\beta)t$$

$$-\frac{\alpha^2}{(\omega\pm\beta)^2}I_1(t).$$

$$\left[1 + \frac{\alpha^2}{(\omega\pm\beta)^2}\right]I_1(t) = \frac{e^{\alpha t}}{(\omega\pm\beta)^2}\{(\omega\pm\beta)\sin(\omega\pm\beta)t + \alpha\cos(\omega\pm\beta)t\}$$

$$I_1(t) = \frac{e^{\alpha t}}{\alpha^2 + (\omega\pm\beta)^2}\{(\omega\pm\beta)\sin(\omega\pm\beta)t + \alpha\cos(\omega\pm\beta)t\}.$$

$$I_2(t) = \int e^{\alpha t} \sin(\omega \pm \beta)t$$

$$\begin{array}{lcl} e^{\alpha t} & + & \sin(\omega\pm\beta)t \\ \alpha e^{\alpha t} & - & -\cos(\omega\pm\beta)t/(\omega\pm\beta) \\ \alpha^2 e^{\alpha t} & & -\sin(\omega\pm\beta)t/(\omega\pm\beta)^2 \end{array}$$

$$I_2(t) = \frac{e^{\alpha t}}{(\omega\pm\beta)} \cdot -\cos(\omega\pm\beta)t + \frac{\alpha}{(\omega\pm\beta)^2}\sin(\omega\pm\beta)t - \frac{\alpha^2}{(\omega\pm\beta)^2} I_1(t)$$

$$\left[1+\frac{\alpha^2}{(\omega\pm\beta)^2}\right] I_2(t) = \frac{e^{\alpha t}}{(\omega\pm\beta)^2}\{\alpha \sin(\omega\pm\beta)t - (\omega\pm\beta)\cos(\omega\pm\beta)t\}$$

$$I_2(t) = \frac{e^{\alpha t}}{\alpha^2+(\omega\pm\beta)^2}\{\alpha\sin(\omega\pm\beta)t - (\omega\pm\beta)\cos(\omega\pm\beta)t\}.$$

$$c_1(t)y_1(t) = -\frac{1}{2\beta}\left\{-\int e^{\alpha t}\cdot\cos(\omega+\beta)t\,dt + \int e^{\alpha t}\cos(\omega-\beta)t\,dt\right\} y_1(t)$$

$$= -\frac{1}{2\beta}\left[\frac{e^{\alpha t}}{\alpha^2+(\omega+\beta)^2}\{\alpha\cos(\omega+\beta)t - (\omega+\beta)\sin(\omega+\beta)t\} + \frac{e^{\alpha t}}{\alpha^2+(\omega-\beta)^2}\{\alpha\cos(\omega-\beta)t + (\omega-\beta)\sin(\omega-\beta)t\}\right] e^{-\alpha t}\cdot\cos\beta t$$

$$c_2(t)y_2(t) = \frac{1}{2\beta}\left\{\int e^{\alpha t}\sin(\omega+\beta)t\,dt + \int e^{\alpha t}\sin(\omega-\beta)t\,dt\right\} y_2(t)$$

$$= \frac{1}{2\beta}\left[\frac{e^{\alpha t}}{\alpha^2+(\omega+\beta)^2}\{\alpha\sin(\omega+\beta)t - (\omega+\beta)\cos(\omega+\beta)t\} + \frac{e^{\alpha t}}{\alpha^2+(\omega-\beta)^2}\{\alpha\sin(\omega-\beta)t - (\omega-\beta)\cos(\omega-\beta)t\}\right] e^{-\alpha t}\sin\beta t.$$

$$y_p(t) = c_1(t)y_1(t) + c_2(t)y_2(t)$$

$$= \frac{1}{2\beta}\left[\frac{1}{\alpha^2+(\omega+\beta)^2}\{\alpha\sin(\omega+\beta)t\cdot\sin\beta t + \alpha\cos(\omega+\beta)t\cdot\cos\omega t + (\omega+\beta)\sin(\omega+\beta)t\cdot\cos\beta t - (\omega+\beta)\cos(\omega+\beta)\cdot\sin\beta t\}\right.$$
$$\left. + \frac{1}{\alpha^2+(\omega-\beta)^2}\{-\alpha\cos(\omega-\beta)t\cos\beta t + \alpha\sin(\omega-\beta)t\sin\omega t - (\omega-\beta)\sin(\omega-\beta)t\cdot\cos\beta t - (\omega-\beta)\cos(\omega-\beta)t\cdot\sin\beta t\}\right]$$

$$= \frac{1}{2\beta}\left[\frac{1}{\alpha^2+(\omega+\beta)^2}\{\alpha\cos\omega t + (\omega+\beta)\sin\omega t\} + \frac{1}{\alpha^2+(\omega-\beta)^2}\{-\alpha\cos\omega t - (\omega-\beta)\sin\omega t\}\right]$$

$$= \frac{1}{2\beta}\left[\alpha\cos\omega t\left(\frac{1}{\alpha^2+(\omega+\beta)^2} - \frac{1}{\alpha^2+(\omega-\beta)^2}\right) + \omega\sin\omega t\left(\frac{1}{\alpha^2+(\omega+\beta)^2} - \frac{1}{\alpha^2+(\omega-\beta)^2}\right) + \beta\sin\omega t\left(\frac{1}{\alpha^2+(\omega+\beta)^2} + \frac{1}{\alpha^2+(\omega-\beta)^2}\right)\right]$$

Now, $\frac{1}{\alpha^2+(\omega+\beta)^2} - \frac{1}{\alpha^2+(\omega-\beta)^2} = \frac{-4\omega\beta}{\{\alpha^2+(\omega+\beta)^2\}\{\alpha^2+(\omega-\beta)^2\}}$

$$\frac{1}{\alpha^2+(\omega+\beta)^2} + \frac{1}{\alpha^2+(\omega-\beta)^2} = \frac{2(\alpha^2+\omega^2+\beta^2)}{\{\alpha^2+(\omega+\beta)^2\}\{\alpha^2+(\omega-\beta)^2\}}$$

so, $y_p(t) = \dfrac{-4\alpha\omega\beta\cos\omega t - 4\omega^2\beta\sin\omega t + 2(\alpha^2+\beta^2+\omega^2)\beta\sin\omega t}{\{\alpha^2+(\omega+\beta)^2\}\{\alpha^2+(\omega-\beta)^2\}} \times \dfrac{1}{2\beta}$

$= \dfrac{-2\alpha\omega\cos\omega t - 2\omega^2\sin\omega t + (\alpha^2+\beta^2+\omega^2)\sin\omega t}{\{\alpha^2+(\omega+\beta)^2\}\{\alpha^2+(\omega-\beta)^2\}}$

$= \dfrac{(\alpha^2+\beta^2-\omega^2)\sin\omega t - 2\alpha\omega\cos\omega t}{\{\alpha^2+(\omega+\beta)^2\}\{\alpha^2+(\omega-\beta)^2\}}$

Re-writing $\alpha = a$, $\beta = \sqrt{b^2-a^2}$,

$y_p(t) = \dfrac{(a^2+b^2-a^2-\omega^2)\sin\omega t - 2a\omega\cos\omega t}{(a^2+\omega^2+b^2-a^2+2\omega\sqrt{b^2-a^2})(a^2+\omega^2+b^2-a^2-2\omega\sqrt{b^2-a^2})}$

$= \dfrac{(b^2-\omega^2)\sin\omega t - 2a\omega\cos\omega t}{(\omega^2+b^2)^2 - 4\omega^2(b^2-a^2)}$

24) Let $K(x,t)$ denote the Green's function for initial value problems involving the operator $L = D^2 + a_1(x)D + a_0(x)$. and assume that L is defined on an interval I of the x-axis.

(a) What is the domain of $K(x,t)$ in the xt-plane?

(b) Prove that $K(x,x) = 0$ and $K_x(x,x) = 1$ for all x in I. [Note: K_x denotes the partial derivatives of $K(x,t)$ with respect to x.]

(c) Show that for each fixed t in I, the function $\phi(x) = K(x,t)$ is a solution on I of the initial value problem $Ly = 0$; $\phi(t) = 0$, $\phi'(t) = 1$.

(d) Use the results of (b) and (c) to deduce that $K(x,t)$ is independent of the particular basis $y_1(x)$ and $y_2(x)$ chosen for the solution space of the homogeneous equation $Ly = 0$.

Solution.

(a) Domain of $K(x,t)$

The Green's function $K(x,t)$ is defined for all x in I, and for all $t \in \mathbb{R}$, as $y_1(t)$ and $y_2(t)$ are linearly independent in $\mathcal{C}^2(I)$, the Wronskian $W[y_1(t), y_2(t)]$ is never identically equal to zero. Further, $y_1(t)$ and $y_2(t)$ are defined for all $t \in I$.

Hence, $K(x,t) = \dfrac{y_2(x)\,y_1(t) - y_1(x)\,y_2(t)}{W[y_1(t), y_2(t)]}$

is defined for all $x \in I$, $t \in I$.

(b) Prove that $K(x,x) = 0$ and $K_x(x,x) = 1$.

$$K(x,x) = \frac{y_2(x)\cdot y_1(x) - y_1(x)\cdot y_2(x)}{W[y_1(x), y_2(x)]}$$

$$= 0.$$

$$K_x(x,t) = \frac{1}{W(t)}\frac{\partial}{\partial x}\left(y_2(x)y_1(t) - y_1(x)y_2(t)\right)$$

$$= \frac{y_2'(x)\,y_1(t) - y_1'(x)\,y_2(t)}{W[y_1(t), y_2(t)]}$$

$$K_x(x,x) = \frac{y_2'(x)\,y_1(x) - y_1'(x)\,y_2(x)}{\begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}}$$

$$= \frac{y_2'(x)\,y_1(x) - y_1'(x)\,y_2(x)}{y_1(x)\cdot y_2'(x) - y_1'(x)\,y_2(x)}$$

$$= 1.$$

(c) Show that for each fixed t in I, the function $\phi(x) = K(x,t)$ is a solution of I of the initial value problem $Ly = 0$, $\phi(t) = 0$, $\phi'(t) = 1$.

Solution

$$\phi(x) = K(x,t) = \frac{1}{W[y_1(t), y_2(t)]}\left[y_1(t)\cdot y_2(x) - y_2(t)\,y_1(x)\right]$$

$$\phi'(x) = \frac{1}{W[y_1(t), y_2(t)]}\left\{y_1(t)\,y_2'(x) - y_2(t)\cdot y_1'(x)\right\}$$

$$\phi''(x) = \frac{1}{W[y_1(t), y_2(t)]}\left\{y_1(t)\,y_2''(x) - y_2(t)\,y_1''(x)\right\}$$

$$L\phi(x) = (D^2 + a_1(x)D + a_0(x))\,\phi(x)$$
$$= \phi''(x) + a_1(x)\phi'(x) + a_0(x)\phi(x)$$
$$= \frac{1}{W[y_1(t), y_2(t)]}\Big\{ y_1(t)\left(y_2''(x) + a_1(x)y_2'(x) + a_0(x)y_2(x)\right) - y_2(t)\left(y_1''(x) + a_1(x)y_1'(x) + a_0(x)y_1(x)\right)\Big\}$$

Since, y_1 and y_2 are solutions of homogeneous linear differential equation $Ly = 0$, the terms $(y_2'' + a_1 y_2' + a_0 y_2)$ and $(y_1'' + a_1 y_1' + a_0)$ are zero.

Hence, $L\phi = 0$.
Thus, $\phi(x)$ is a solution of the homogeneous linear differential equation $Ly = 0$. Further,

$$\phi(t) = 0$$
$$\phi'(t) = 1.$$

Hence, $\phi(x)$ is the solution of the initial value problem

$$Ly = 0$$
$$\phi(t) = 0$$
$$\phi'(t) = 1$$

(d) Irrespective of the basis chosen

$$\phi(x) = K(x,t)$$

will always satisfy the initial value problem

$$Ly = 0$$
$$\phi(t) = 0$$
$$\phi'(t) = 1$$

By the existence and uniqueness theorem, every initial value problem with 'n' initial conditions has a unique solution. So, $\phi(x) = K(x,t)$ is unique and does not depend on the basis $y_1(x), y_2(x)$.

25) With $K(x,t)$ as in the preceding exercise, show that the function y_p defined by

$$y_p = \int_{x_0}^{x} K(x,t)\,h(t)\,dt$$

satisfies the initial conditions $y_p(x_0) = y_p'(x_0) = 0$ for all h in $\mathcal{C}(I)$.
[Hint: Use Leibnitz formula for differentiating integrals, namely -

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,t)\,dx = \int_{a(x)}^{b(x)} f_x(x,t)\,dx + \left[f(x, b(x))\,b'(x) - f(x, a(x))\,a'(x)\right].$$

Solution.