Thus, A is invertible.

One-sided invertibility for a square matrix implies invertibility.

1.4 Homogenous system of equations

Definition

Any system of the form

$$Ax = 0$$

is called a homogenous system of linear algebraic equations.

Let R be the row-reduced echelon form of the matrix A. Let $i = 1, 2, 3, \ldots, r$ be the non-zero rows of R. Let $c_1, c_2, \ldots, c_r$ be the column numbers in which the first leading non-zero entry of the rows $i = 1, 2, \ldots, r$ occurs. As R is row-reduced echelon matrix of A, $c_1 < c_2 < \ldots < c_r$.

We label the above variables as $x_{c_1}, x_{c_2}, \ldots, x_{c_r}$. Let $J := \{1, 2, 3, \ldots n\} \setminus \{c_1, c_2, \ldots, c_r\}$. The remaining variables are labelled x_j, $j \in J$. The system $Rx = 0$ can be expanded as —

$$\left.\begin{aligned} x_{c_1} + \sum_{j \in J} \alpha_{1j} x_j &= 0 \\ x_{c_2} + \sum_{j \in J} \alpha_{2j} x_j &= 0 \\ \vdots \quad & \quad \vdots \\ x_{c_r} + \sum_{j \in J} \alpha_{rj} x_j &= 0 \end{aligned}\right]$$

These are the equations corresponding to the first r rows of R.

The other equations don't appear, because they correspond to zero rows. We can ~~be In~~ assign arbitrary values to the x_j's, $j \in J$. These ~~are~~ are called free-variables (parameters). ~~In particular, if we a~~ Assigning arbitrary values to x_j, $j \in J$, the values of $x_{c_1}, x_{c_2}, \ldots, x_{c_r}$ can be obtained. In particular, if $r < n$, there is atleast one free variable.

Thus, if $r < n$, by assigning one of the x_j's, the value unity, while setting all other x_j's ~~zero~~ $(n-r-1)$ free variables zero, we get a non-zero (non-trivial) solution for $Ax = 0$.

Therefore, if $r < n$ for a homogenous system of equations, the system has atleast one non-trivial solution. Further, if $R = R^{n \times n}$ and rank(A) = size$(A) = n$, then $R = I_n$.

$$R = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & \\ \vdots & & & \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Off-diagonal entries are zero, principal diagonal entries are 1.

It is easy to infer, that the only row-reduced echelon matrix, whose non-zero rows equal the order of the matrix is the identity matrix.

Theorem 19 The number of linearly independent solutions of a homogenous system of equations $Ax = 0$ is $(n-r)$. The dimension of the solution space of the homogenous equa system $Ax = 0$ is $(n-r)$.

Proof.

Define $J := \{1, 2, 3, \ldots, n\} \setminus \{c_1, c_2, \ldots, c_r\}$. $x_{c_1}, x_{c_2}, \ldots, x_{c_r}$ are the unknowns that correspond to the columns $c_1, c_2, \ldots, c_r$.

Remember, $c_1, c_2, \ldots, c_r$ represent the columns in which the first non-zero element is in row $1, 2, \ldots, r$.

The system of equations $Rx=0$ can be expanded in the full form as—

$$x_{c_1} + \sum_{j\in J} \alpha_{1j} x_j = 0$$
$$x_{c_2} + \sum_{j\in J} \alpha_{2j} x_j = 0$$
$$\vdots$$
$$x_{c_r} + \sum_{j\in J} \alpha_{rj} x_j = 0.$$

The cardinality of J is $(n-r)$.

We look at the variables $x_j, j \in J$. We assign arbitrary values to them. These are the free variables. If we substitute them in the above equation, we get $x_{c_1}, x_{c_2}, \ldots, x_{c_r}$. This is one set of values $x_j, j \in J$. You can compute another set of values for $x_{c_1}, x_{c_2}, \ldots, x_{c_r}$ by choosing ~~are~~ some other arbitrary values for the free variables.

Let us denote a solution vector as s^j; $j \in J$. s^j is the solution vector, which has its jth coordinate $x_j = 1$, and all other free variables set to zero. We are looking at one specific assignment for the free variables

$$x_j = 1, \; j \in J$$
$$x_i = 0, \; \forall i \in J, i \neq j.$$

I take the first $j \in J$, set $x_j = 1$. I set all other entries $x_i = 0$, $i \in J$, $i \neq j$. We then substitute these values in the system of equations and find the values of $x_{c_1}, x_{c_2}, \ldots, x_{c_r}$. We fill these values up in s_j.

We repeat this procedure for each $j \in J$. We get $(n-r)$ solution vectors, s_j's. Each s_j is a solution of $R x = 0$ and ~~here~~ since $A \sim R$, it is a solution of $Ax=0$. So, each s^j belongs to the solution space S.

Much like the standard basis vectors $e_1, e_2, \ldots, e_n$ in $\mathbb{R}^n$, ~~each~~ if we, for a minute forget about the variables $x_{c_1}, x_{c_2}, \ldots, x_{c_r}$, each s^j could be of the form:

$$s^1 = \begin{bmatrix} x_{c_1} \\ x_{c_2} \\ \vdots \\ x_{c_r} \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad s^2 = \begin{bmatrix} x_{c_1} \\ x_{c_2} \\ \vdots \\ x_{c_r} \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \cdots \quad s^{n-r} = \begin{bmatrix} x_{c_1} \\ x_{c_2} \\ \vdots \\ x_{c_r} \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Just like the standard basis vectors, each s^j will have its jth coordinate 1 and all other free variables 0. s^{j+1} will have a 1 in the $(j+1)$th coordinate and all other free variables zero.

Assume
$$\sum_{j\in J} \beta_j s^j = 0 \text{ vector}.$$

The jth coordinate of this vector must be β_j. This is true for all $j \in J$.
This is true by the fact that, s_j has 1 in column j and all other s^{j-1}, s^{j+1}, ... etc. have 0 there. ~~He~~

Since the resulting vector is identical to the zero vector, each $\beta_j = 0$.
Therefore,

$$\sum_{j \in J} \beta_j s^j = 0 \Rightarrow \beta_j = 0. \ \forall j$$

Clearly, the s^j's are linearly independent. There are $(n-r)$ s_j^j's. Further, we can verify that other solution in the set S can be expressed as a linear combination of these basis vectors.

Thus, a homogenous system of linear algebraic equations $Ax = 0$ has $(n-r)$ linearly independent solutions.

<u>Example</u>
Consider the system of 3 equations

$$\begin{aligned} 3x - 2y + z &= 0 \\ x + y \quad &= 0 \\ x \quad - 3z &= 0. \end{aligned}$$

How many solutions does it have which are linearly independent over $\mathbb{R}^3$?
<u>Solution</u>
The above system of equations can be written in the matrix form as

$$Ax = 0$$

$$\begin{bmatrix} 3 & -2 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$A = \begin{bmatrix} 3 & -2 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & -3 \end{bmatrix}$$

Applying R_{13},

$$\sim \begin{bmatrix} 1 & 0 & -3 \\ 1 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix}$$

Applying $R_{21}(-1)$, $R_{31}(-3)$

$$\sim \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 3 \\ 0 & -2 & 10 \end{bmatrix}$$

Applying $R_{32}(2)$,

$$\sim \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 3 \\ 0 & 0 & 16 \end{bmatrix}$$

Applying $R_3\left(\frac{1}{16}\right)$

$$\sim \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

Applying $R_{23}(-3)$, $R_{13}(3)$,

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\sim I_3$$

Thus, rank $(A) = r = 3$. Since, $r = n$, the system has $n - r = 0$ linearly independent solutions. $Ax = 0$ has only the trivial solution $x = 0$.

Corollary. A homogenous system of 'm' equations in 'n' unknowns has atleast one non-trivial solution if $m <$ 'n'.

Proof.
Let $A \in \mathbb{R}^{m \times n}$ with $m < n$. We are required to prove that ~~that~~ $Ax = 0$ has atleast one non-trivial solution. Let R be the row-reduced echelon matrix, row-equivalent to A. Then, $Ax = 0$ and $Rx = 0$ have the same solution space. Let r be the number of non-zero rows in R.

Then, $\quad r \leq m < n$.
So, $\quad r < n$.

r is strictly less than n. Hence, $Rx = 0$ has a non-trivial solution. Since, $A \sim R$, $Ax = 0$ has atleast one non-trivial solution.

Example.
Give a set of linearly independent solutions for the system of equations

$$x + 2y + 3z = 0$$
$$2x + 4y + z = 0$$

Solution
The system of equations is

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So, $$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \end{bmatrix}$$

Applying $R_{21}(-2)$,

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & -5 \end{bmatrix}$$

Applying $R_2\left(-\frac{1}{5}\right)$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

Applying $R_{12}(-3)$ $$\sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

rank $(A) = r = 2$.
The number of linearly independent solutions of $Ax = 0$ are $(n - r) = 3 - 2 = 1$.

The equivalent system of equations is—

$$\begin{aligned} x_1 + 2x_2 \quad &= 0 \\ x_3 &= 0. \end{aligned}$$

x_2 is a free-variable. Assigning $x_2 = -1$, $x_1 = 2$. Thus, $(2, -1, 0)$ is a linearly independent solution of the given system of equations.

1.5 Non-homogenous system of linear equations.

A system of equations $Ax = b$, where the right side $b \neq 0$ is called a non-homogenous system of linear algebraic equations.

The essential difference between a homogenous system and a non-homogenous system is that a non-homogenous system need not have a solution.
A homogenous system of equations always has a solution.

Consider $Ax = b$, $A = \mathbb{R}^{m\times n}$, $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, b is requirement vector, $b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$.

Let $A' = (A, b)$. We adjoin the matrix b. We apply elementary row operations on A' to get R', the row-reduced echelon matrix.

$$A' \sim R'$$
$$\text{where } R' = (R, d).$$

We know that, if (R, d) is obtained by applying elementary row operations to (A, b), then any solution of $Ax = b$ is a solution of $Rx = d$. and vice versa. Let us write $Rx = d$ in the expanded form

$$x_{c_1} + \sum_{j\in J} \alpha_{1j} x_j = d_1$$

$$x_{c_2} + \sum_{j\in J} \alpha_{2j} x_j = d_2$$

$$\vdots$$

$$x_{c_r} + \sum_{j\in J} \alpha_{rj} x_j = d_r$$

What about the other $(m-r)$ equations? In the homogenous case, these $(m-r)$ equations don't contribute anything or pose any constraint. But, for a non-homogenous system, the last $(m-r)$ zero rows of R, determines if the system has a solution or does not have a solution.

The last $(m-r)$ equations are:

$$\begin{aligned} 0 &= d_{r+1} \\ 0 &= d_{r+2} \\ &\vdots \\ 0 &= d_m \end{aligned}$$

The d_i's are zero in the homogenous case. Thus, $Ax = b$ has a solution if and only if $d_i = 0$, for all $i > r$.

$$d_i = 0, \quad r+1 \leq i \leq m.$$