# 1. Differentiation in several variables

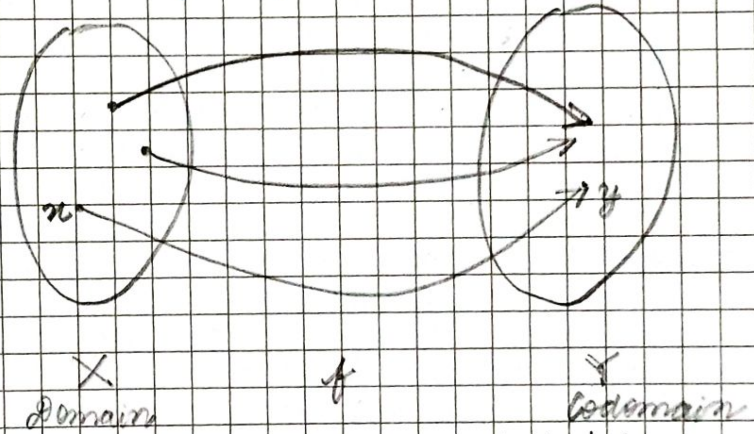
## 1.1 Functions of several variables

The volume and the surface area of a sphere depend on its radius, the formulas describing their relationships being $V = \frac{4}{3}\pi r^3$ and $S = 4\pi r^2$. (Here, V and S are, respectively, the volume and the surface area of the sphere and r its radius.) The equations define the volume and surface area as functions of the radius. The essential characteristic of a function is that the so called independent variable determines a unique value of the dependent variable. (V or S). Every element in the domain has one and only one image.

You can also think of many quantities that are determined uniquely ~~so~~ not only by one variable (as the volume of a sphere being determined by its radius) but by several: the area of a rectange $A = lb$, the volume of cylinder ~~or~~ or a cone, the average annual rainfall in Cleveland, or the national debt. Realistic modeling of the world requires that we understand the concept of a function of more than one variable and find meaningful ways to visualise such functions.

### Definitions, Notations and examples.

A function, any function, has three features: (1) a domain set X, (2) a codomain set Y (3) a rule of assignment that associates to each element $x$ in domain X a unique element, usually denoted $f(x)$, in the codomain Y. We will frequently use the notation $f: X \to Y$ for a function. Such notation indicates all the ingredients of a particular function, although it does not make the nature of the rule of assignment explicit. This notation also suggests the mapping nature of a function.




Example 1. Abstract definitions are necessary, but it is just as important to understand functions as they actually occur. Consider the act of assigning to each US citizen his or her social security number. This pairing defines a function: each citizen is assigned one social security number. The domain is the set of US citizens and the codomain is set of all nine-digit strings of numbers.

On the other hand, when a university assigns students to dormitory rooms, it is unlikely that it is creating a function from the set of available rooms to the set of students. This is because some rooms may have more than one student assigned to them, so that a particular room does not determine a unique student occupant.

Definition 1 (Range) The range of a function $f: X \to Y$ is the set of all elements of Y that are the actual values of Y. That is, the range of $f$ is the set image of the set X under $f$. It consists of those $y \in Y$ such $y = f(x)$ for some $x \in X$.

$$\text{Range } f = \{ y \in Y \mid y = f(x) \text{ for some } x \in X \}.$$

In the social security ~~example~~ function of example 1, the range consists of those nine-digit numbers actually used as active social security numbers. For example, the number 000-00-0000 is not in the range, since no one is actually assigned this number.

Definition 1.2 (surjective function).
A function $f: X \to Y$ is said to be onto (or surjective) if every element of $Y$ has a pre-image in $X$; if every $y \in Y$ is the element of some element of $X$, that is range $f = Y$.

The social security function is not surjective, since 000-00-000 is in the codomain but not in the range. Pictorially, an ~~onto function is sugg~~

Definition 1.3 (One-to-one function). A function $f: X \to Y$ is called one-to-one (injective) if no two distinct elements of the domain has the same image under $f$. For one-to-one functions, distinct elements have distinct images. That is, $f$ is one-to-one if whenever $x_1, x_2 \in X$ and $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.

When you studied single-variable Calculus, the functions of interest were those whose domains and codomains were subsets of $\mathbb{R}$ (the real numbers). It was probably the case that only the rule of assignment was made explicit; it is generally assumed that domain is the largest possible subset in $\mathbb{R}$ for which the function makes sense. The codomain is generally taken to be all of $\mathbb{R}$.

Example 2. Suppose $f: \mathbb{R} \to \mathbb{R}$ is given by $f(x) = x^2$. Then the domain and codomain are, explicitly, all of $\mathbb{R}$, but the range of $f$ is the subinterval $[0, \infty)$. Thus, $f$ is not onto, since the codomain is strictly larger than the range. Note that $f$ is not one-to-one since $f(2) = f(-2) = 4$, but $2 \neq -2$.

Example 3. Suppose $g$ is a function such that $g(x) = \sqrt{x-1}$. Then, if we take the codomain to be all of $\mathbb{R}$, the domain cannot be larger than $[1, \infty)$. If the domain included any values less than one, the radicand would be negative and hence $g$ would not be real-valued.

Now we're ready to think about functions of more than one variable. In the most general terms, these are functions whose domains are subsets $X$ of $\mathbb{R}^n$ and whose codomains are subsets of $\mathbb{R}^m$ for some positive integers $n$ and $m$. (For simplicity of notation, we'll take the codomains to be all of $\mathbb{R}^m$, except when specified otherwise.) That is, such a function is a mapping $f: X \subseteq \mathbb{R}^n \to \mathbb{R}^m$ that associates a vector (or point) $x$ in $X$, a unique vector (point) $f(x)$ in $\mathbb{R}^m$.

Example 4. Let $T: \mathbb{R}^3 \to \mathbb{R}$ be defined by $T(x, y, z) = xy + yz + zx$. We can think of $T$ as a sort of a temperature function. Given a point $x = (x, y, z)$ in $\mathbb{R}^3$, $T(x)$ calculates the temperature at that point.

Example 5. Let $L: \mathbb{R}^n \to \mathbb{R}$ be given by $L(x) = \|x\|$. This is a length function in that it computes the length of any vector $x$ in $\mathbb{R}^n$. Note that $L$ is not one-to-one, since $L(e_i) = L(e_j) = 1$, where $e_i$ and $e_j$ are any of the two standard basis vectors of $\mathbb{R}^n$. $L$ fails to be onto, since the length of a vector is non-negative.

Example 6. Consider the function given by $N(x) = x / \|x\|$, where $x$ is a vector in $\mathbb{R}^3$. Note that, $N$ is not defined if $x = 0$, so the largest possible domain for $N$ is $\mathbb{R}^3 - \{0\}$. The range of $N$ consists of all unit vectors in $\mathbb{R}^3$. The function $N$ is the normalization function, that is, the function that takes a non-zero vector in $\mathbb{R}^3$ and returns the unit vector that points in the same direction.

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Example 7.

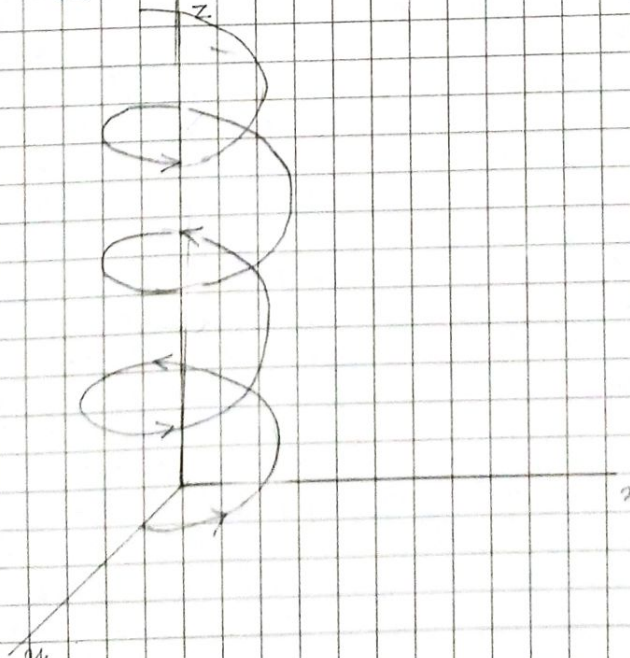
Sometimes a function may be given numerically by a table. One such example is the notion of a windchill - the apparent temperature one feels when taking into account both the actual air temperature and the speed of the wind. A standard table of wind chill values is shown below. From it we see that if the air temperature is 20°F and the wind speed is 25 mph, the windchill temperature (how "cold" it feels) is 3°F. Similarly, if the air temperature is 35°F and the windspeed is 10 mph, then the windchill is 27°F. In other words, if $s$ denotes the windspeed and $t$, the air temperature, then the windchill is a function $W(s,t)$.

The functions described in examples 4, 5 and 7 are scalar-valued functions, that is functions whose codomains are $\mathbb{R}$ or subsets of $\mathbb{R}$. Scalar-valued functions are our main concern for this chapter. Nonetheless, let's look at a few examples of functions whose codomains are $\mathbb{R}^m$, where $m > 1$.

| Air Temp (deg F) | Windspeed (mph) 5 | 10 | 15 | 20 | 25 | 30 | 35 | 40 | 45 | 50 | 55 | 60 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 40 | 36 | 34 | 32 | 30 | 29 | 28 | 28 | 27 | 26 | 26 | 25 | 25 |
| 35 | 31 | 27 | 25 | 24 | 23 | 22 | 21 | 20 | 19 | 19 | 18 | 17 |
| 30 | 25 | 21 | 19 | 17 | 16 | 15 | 14 | 13 | 12 | 12 | 11 | 10 |
| 25 | 19 | 15 | 13 | 11 | 9 | 8 | 7 | 6 | 5 | 4 | 4 | 3 |
| 20 | 13 | 9 | 6 | 4 | 3 | 1 | 0 | -1 | -2 | -3 | -3 | -4 |
| 15 | 7 | 3 | 0 | -2 | -4 | -5 | -7 | -8 | -9 | -10 | -11 | -11 |
| 10 | 1 | -4 | -7 | -9 | -11 | -12 | -14 | -15 | -16 | -17 | -18 | -19 |
| 5 | -5 | -10 | -13 | -15 | -17 | -19 | -21 | -22 | -23 | -24 | -25 | -26 |
| -0 | -11 | -16 | -19 | -22 | -24 | -26 | -27 | -29 | -30 | -31 | -32 | -33 |
| -5 | -16 | -22 | -26 | -29 | -31 | -33 | -34 | -36 | -37 | -38 | -39 | -40 |
| -10 | -22 | -28 | -32 | -35 | -37 | -39 | -41 | -43 | -44 | -45 | -46 | -48 |
| -15 | -28 | -35 | -39 | -42 | -44 | -46 | -48 | -50 | -51 | -52 | -54 | -55 |
| -20 | -34 | -41 | -45 | -48 | -51 | -53 | -55 | -57 | -58 | -60 | -61 | -62 |
| -25 | -40 | -47 | -51 | -55 | -38 | -60 | -62 | -64 | -65 | -67 | -68 | -69 |
| -30 | -46 | -53 | -58 | -61 | -64 | -67 | -69 | -71 | -72 | -74 | -75 | -76 |
| -35 | -52 | -59 | -64 | -68 | -71 | -73 | -76 | -78 | -79 | -81 | -82 | -84 |
| -40 | -57 | -66 | -71 | -74 | -78 | -80 | -82 | -84 | -86 | -88 | -89 | -91 |
| -45 | -63 | -72 | -77 | -81 | -84 | -87 | -89 | -91 | -93 | -95 | -97 | -98 |

Table of Windchill values

Example 8. Define $f : \mathbb{R} \to \mathbb{R}^3$ by $f(t) = (\cos t, \sin t, t)$. The range of $f$ is the curve in $\mathbb{R}^3$ with parametric equations $x = \cos t$, $y = \sin t$, $z = t$. If we think of $t$ as a time parameter, then this function traces out the corkscrew curve (called a helix) shown in the figure.




The helix. The arrow shows the direction of increasing $t$.

Example 9. We can think of the velocity of a fluid as a vector in $\mathbb{R}^3$. This vector depends depends on (atleast) the point at which one measures the velocity and also the time at which one makes the measurement. In other words, the velocity may be considered to be a function $v: X \subseteq \mathbb{R}^4 \to \mathbb{R}^3$. The domain $X$ is a subset of $\mathbb{R}^4$ because three variables $x, y, z$ are required to describe a point in a fluid and a fourth variable $t$ is needed to keep track of time. For instance, such a function $v$ might be given by the expression

$$v(x,y,z,t) = xyzt\,\hat{i} + (x^2-y^2)\hat{j} + (3z+t)\hat{k}.$$

You may have noted that the expression for $v$ in example 9 is considerably more complicated than those for the functions given in examples 4-8. This is because all the variables and vector components have been written out explicitly. In general, if we have a function $f: X \subseteq \mathbb{R}^n \to \mathbb{R}^m$, then $x \in X$ can be written as $x = (x_1, x_2, \ldots, x_n)$ and $f$ can be written in terms of its component functions $(f_1, f_2, \ldots, f_m)$. Each of the component functions are scalar-valued functions of $x \in X$ that define the components of the vector $f(x) \in \mathbb{R}^m$. What results is a morass of symbols:

$$\begin{aligned} f(x) &= f(x_1, x_2, \ldots, x_n) && \text{Emphasizing the variables} \\ &= (f_1(x), f_2(x), \ldots, f_m(x)) && \text{Emphasizing the component functions} \\ &= (f_1(x_1, \ldots, x_n), f_2(x_1, \ldots, x_n), && \text{Writing out all the components.} \\ &\quad \ldots, f_m(x_1, \ldots, x_n)) \end{aligned}$$

For example, the function $L$ of example 5, when expanded becomes

$$L(x) = L(x_1, \ldots, x_n) = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

The function $N$ of example 6 becomes

$$N(x) = \frac{x}{\|x\|} = \frac{(x_1, x_2, x_3)}{\sqrt{x_1^2 + x_2^2 + x_3^2}}$$

$$= \left( \frac{x_1}{\sqrt{x_1^2+x_2^2+x_3^2}}, \frac{x_2}{\sqrt{x_1^2+x_2^2+x_3^2}}, \frac{x_3}{\sqrt{x_1^2+x_2^2+x_3^2}} \right)$$

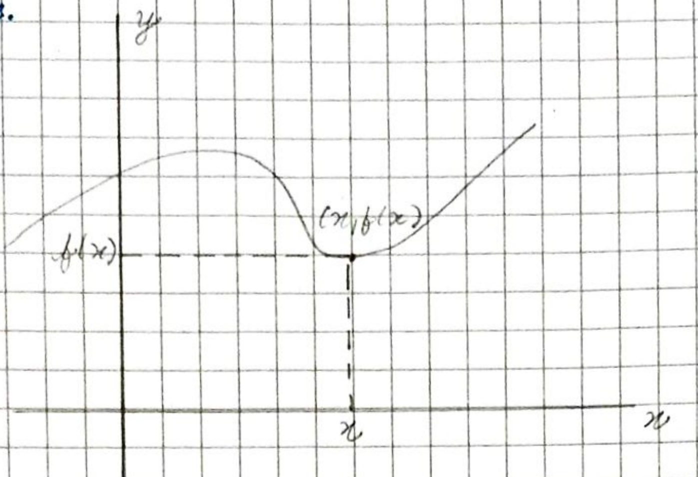
Although writing a function in terms of all its variables and components has the advantage of being explicit, quite a lot of ink and paper are used in the process. The use of vector notation not only saves space and time, but also helps to make the meaning of a function clear by emphasizing that the function ~~points~~ maps points in $\mathbb{R}^n$ to points in $\mathbb{R}^m$. Vector notation makes a function of 300 variables look "just like" a function of one variable. Try to avoid writing out components as much as you can. (Except when you want to impress your friends).

Visualising functions.

No doubt you have been graphing scalar-valued functions of one variable for so long that you give the matter little thought. Let's scrutinize what you've been doing, however. A function $f: X \subseteq \mathbb{R} \to \mathbb{R}$ takes a real number and returns another real number as suggested by the figure. The graph of $f$ is something that lives in $\mathbb{R}^2$. It consists of points $(x, y)$ such that $y = f(x)$. That is,

$$\text{Graph } f = \{x, f(x) \mid x \in X\} = \{(x,y) \mid x \in X, y = f(x)\}.$$

The important fact is that, in general, the graph of a scalar-valued function of a single-variable is a curve - a one-dimensional object sitting inside two dimensional space.




Now, suppose we have a function $f : X \subseteq \mathbb{R}^2 \to \mathbb{R}$, that is a function of two variables. We make essentially the same definition for the graph:

$$\text{Graph } f = \{(X, f(X)) \mid X \in \mathbb{R}\} \tag{1}$$

Of course $X = (x, y)$ is a point of $\mathbb{R}^2$. Thus, $\{(X, f(X))\}$ may also be written as
$$\{(x, y, f(x, y))\} \text{ or as } \{(x, y, z) \mid (x, y) \in X, z = f(x, y)\}.$$

Hence, the graph of a scalar-valued function of two variables is something that sits in $\mathbb{R}^3$. Generally speaking, the graph will be a surface.

Example 10.
The graph of the function
$$f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x, y) = \frac{1}{12} y^3 - y - \frac{1}{4} x^2 + \frac{7}{2}$$
is shown in the figure attached. For each $X = (x, y)$ in $\mathbb{R}^2$, the point in $\mathbb{R}^3$ with coordinates $(x, y, \frac{1}{12} y^3 - y - \frac{1}{4} x^2 + \frac{7}{2})$ is graphed.

Graphing functions of two variables is a much more difficult task than graphing functions of one variable. Of course, one method is to let a computer do the work. Nonetheless, if you want to get a feeling for functions of more than one variable, being able to sketch a rough graph by hand is still a valuable skill. The trick to putting together a reasonable graph is to find a way to cut down on the dimensions involved. One way this can be achieved is by drawing certain special curves that lie on the surface $z = f(x, y)$. These special curves, called contour curves, are the ones obtain by intersecting the surface with horizontal planes $z = c$ for various values of the constant $c$. Some contour curves drawn on the surface of example 10 are shown in the figure. If we compress all contour curves onto the xy-plane (in essence if we look down along the positive z-axis), then we create a topographic map of the surface that is shown in the figure. These curves in the xy-plane are called the level curves of the original function $f$.

The point of the preceding discussion is that we can ~~reverse~~ reverse the process in order to sketch systematically the graph of a function $f$. ~~or~~ of two variables.

We first construct a topographic map in $\mathbb{R}^2$ by finding the level curves of $f$, then situation this curves in $\mathbb{R}^3$ at contour curves at the appropriate heights and finally complete the graph of the function. Before we give an example, let's make our terminology with greater precision.

Definition 1.4 Let $f: X \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a scalar valued function of two variables. The level curve at height $c$ of $f$ is the curve in $\mathbb{R}^2$ defined by the equation $f(x,y) = c$, where $c$ is a constant. In mathematical notation,

$$\text{Level curve at height } c = \{(x,y) \in \mathbb{R}^2 \mid f(x,y) = c\}.$$

The contour curve at height $c$ of $f$ is the curve in $\mathbb{R}^3$ defined by the two equations $z = f(x,y)$ and $z = c$. Symbolized,

$$\text{Contour curve at height } c = \{(x,y,z) \in \mathbb{R}^3 \mid z = f(x,y) = c\}.$$

In addition to level and contour curves, consideration of the sections of a surface by the planes where $x$ or $y$ is held constant is also helpful. A section of a surface by a plane is just the intersection of the surface with that plane. Formally, we have the following definition.

Definition 1.5. Let $f: X \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a scalar-valued function of two variables. The section of the graph of $f$ by the plane $x = c$ (where $c$ is a constant, is a set of points $(x,y,z)$, where $z = f(x,y,z)$, where $z = f(x,y)$ and $x = c$. Symbolized,

$$\text{Section by } x = c \text{ is } \{(x,y,z) \in \mathbb{R}^3 \mid z = f(x,y),\ x = c\}.$$

Similarly, the section of the graph of $f$ by the plane $y = c$ is the set of points described as follows:

$$\text{Section by } y = c \text{ is } \{(x,y,z) \in \mathbb{R}^3 \mid z = f(x,y),\ y = c\}.$$

Example 11. We'll use level and contour curves to construct the graph of the function

$$f: \mathbb{R}^2 \to \mathbb{R},\quad f(x,y) = 4 - x^2 - y^2.$$

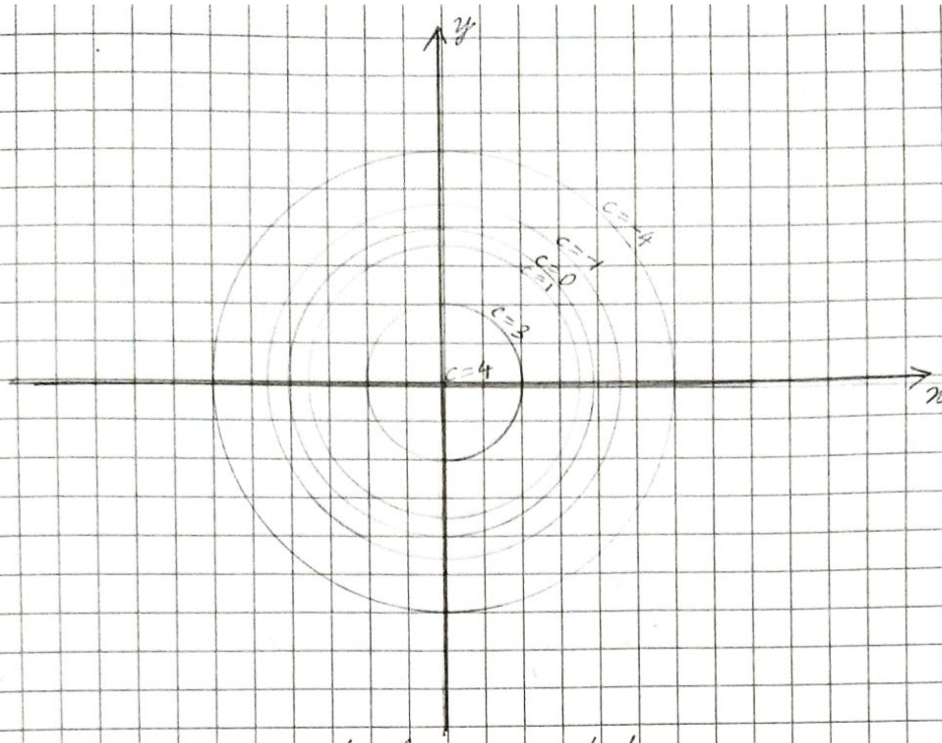
By definition 1.4, the level curve at height $c$ is

$$\{(x,y) \in \mathbb{R}^2 \mid 4 - x^2 - y^2 = c\} = \{(x,y) \mid x^2 + y^2 = 4 - c\}.$$

Thus, we see that the level curves for $c < 4$ are circles centered at the origin of radius $\sqrt{4-c}$. The level curve at height $c = 4$ is not a curve at all but just a single point (the origin). Finally, there are no level curves at heights larger than 4 since the equation $x^2 + y^2 = 4 - 6$ has no real solutions in $x$ and $y$. These remarks are summarized in the following table.

| $c$ | Level curve $x^2 + y^2 = 4 - c$ |
|---|---|
| $-5$ | $x^2 + y^2 = 9$ |
| $-1$ | $x^2 + y^2 = 5$ |
| $0$ | $x^2 + y^2 = 4$ |
| $1$ | $x^2 + y^2 = (\sqrt{3})^2$ |
| $3$ | $x^2 + y^2 = 1$ |
| $4$ | $x^2 + y^2 = 0$ |
| $c > 4$ | empty. |

Thus, the family of level curves, the "topographic map" of the surface $z = 4 - x^2 - y^2$, is shown in the figure.



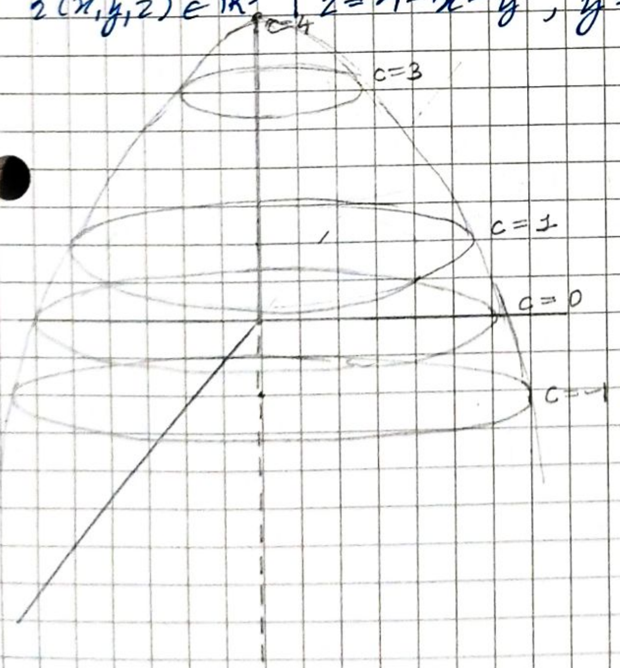

Level curves of f.

Some contour curves which sit in $\mathbb{R}^3$ are shown in the figure below where we can get a feeling for the complete graph of $z = 4 - x^2 - y^2$. It is a surface which looks like an inverted dish and is called a paraboloid. To make the picture clearer, we have also sketched in the sections of the surface by the planes $x = 0$ and $y = 0$. The section by $x = 0$ is given analytically by the set

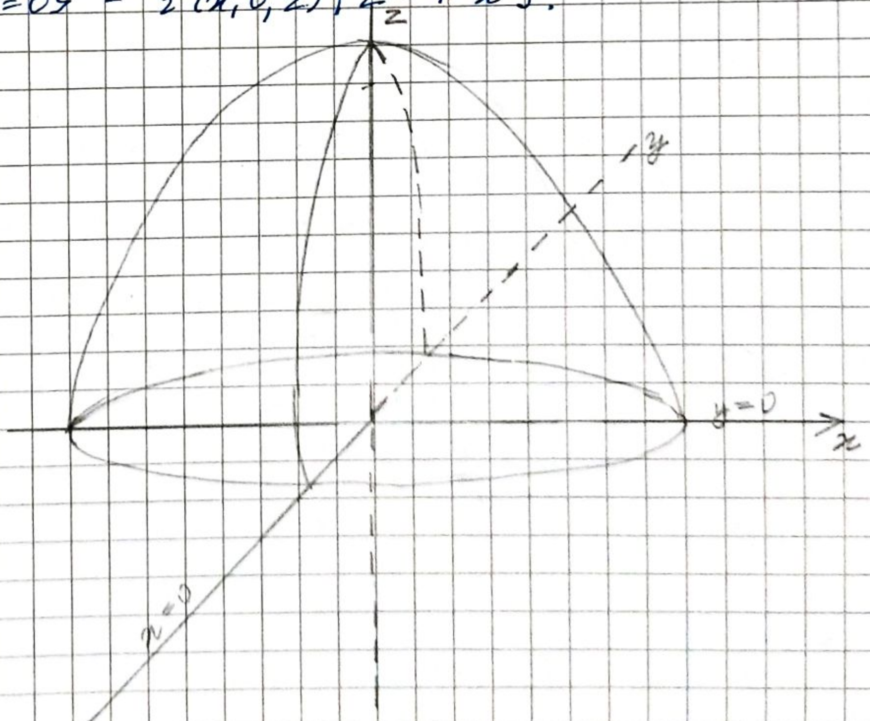
$$\{(x,y,z) \in \mathbb{R}^3 \mid z = 4 - x^2 - y^2,\ x = 0\} = \{(0,y,z) \mid z = 4 - y^2\}$$

Similarly, the section by $y = 0$ is

$$\{(x,y,z) \in \mathbb{R}^3 \mid z = 4 - x^2 - y^2,\ y = 0\} = \{(x,0,z) \mid z = 4 - x^2\}.$$




some contour curves of $z = 4 - x^2 - y^2$

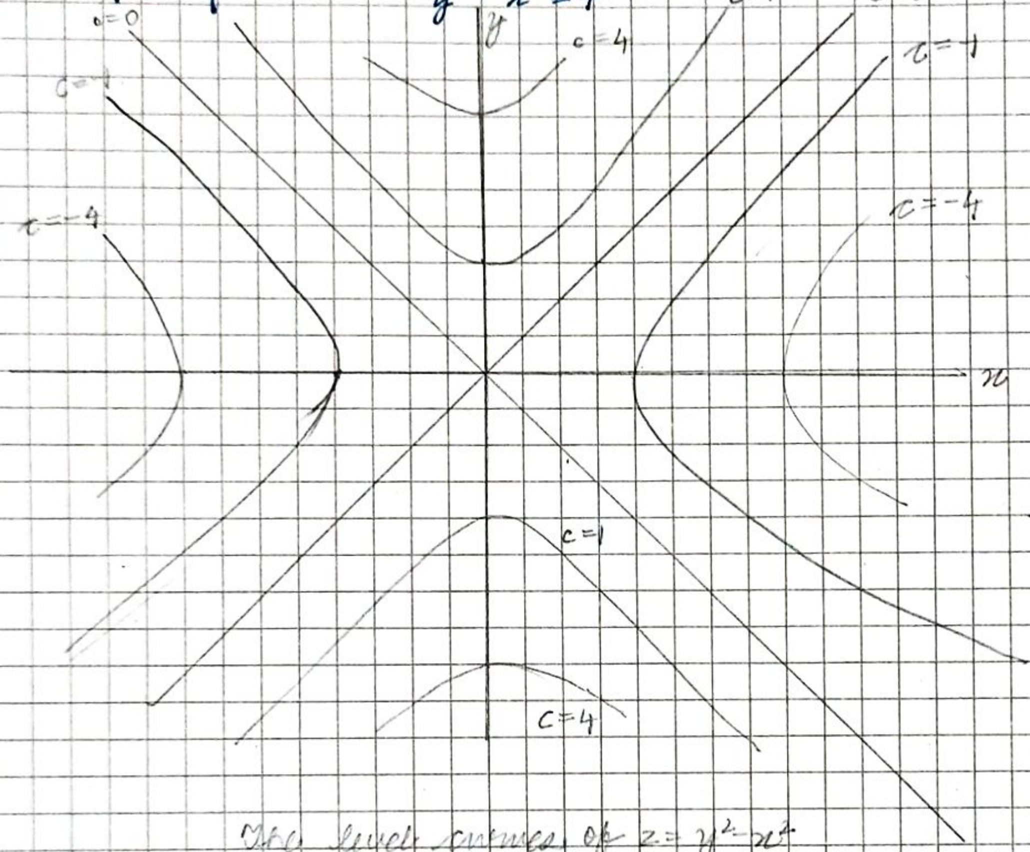



The graph of $z = 4 - x^2 - y^2$.

since the sections are parabolas, it is easy to see, how the surface obtained its name.

Example 12.
We'll graph the function $g: \mathbb{R}^2 \to \mathbb{R}$, $g(x,y) = y^2 - x^2$. The level curves are all hyperbolas, with the exception of the level curve at height 0, which is a pair of intersecting lines.

| $c$ | Level curve $y^2 - x^2 = c$ |
|---|---|
| $-4$ | $x^2 - y^2 = 4$ |
| $-1$ | $x^2 - y^2 = 1$ |
| $0$ | $(y-x)\cdot(y+x) = 0$ |
| $1$ | $y^2 - x^2 = 1$ |
| $4$ | $y^2 - x^2 = 4$ |




The level curves of $z = y^2 - x^2$

The collection of the level curves is graphed above. The sections by $x = c$ are

$$\{(x,y,z) \in \mathbb{R}^3 \mid z = y^2 - x^2,\ x = c\} = \{(c,y,z) \mid z = y^2 - c^2\}$$

These are clearly parabolas in the plane $x = c$. The sections by $y = c$ are

$$\{(x,c,z) \mid z = c^2 - x^2\}$$

which are again parabolas. The level curves and sections generate a surface called the hyperbolic paraboloid.

Example 13.
We compare the graphs of the functions $f(x,y) = 4 - x^2 - y^2$ of example 11 with that of

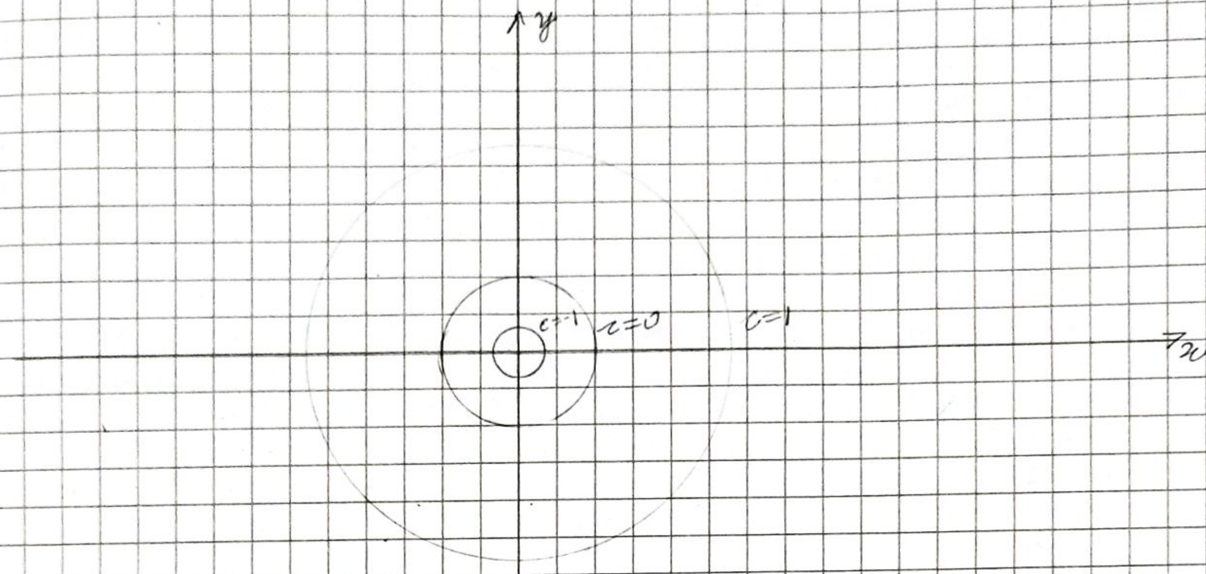
$$h: \mathbb{R}^2 - \{(0,0)\} \to \mathbb{R}, \quad h(x,y) = \ln(x^2 + y^2).$$

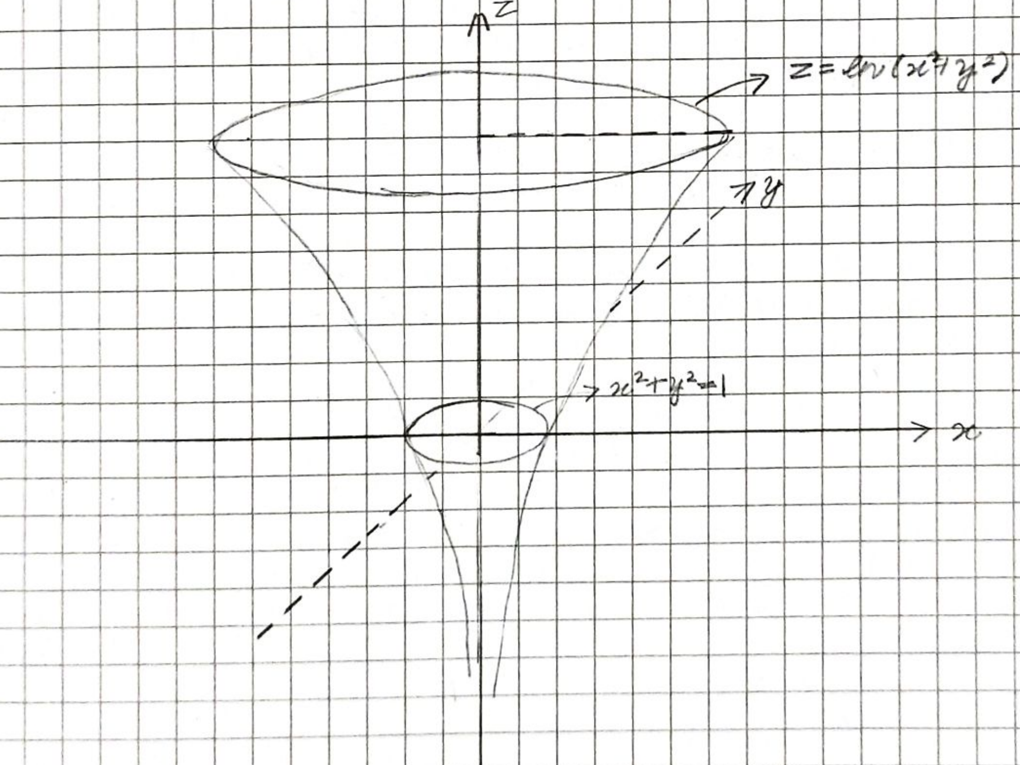
The level curve of $h$ at height $c$ is

$$\{(x,y) \in \mathbb{R}^2 \mid z = \ln(x^2+y^2) = c\} = \{(x,y) \mid x^2 + y^2 = e^c\}.$$

Since $e^c > 0$ for all $c \in \mathbb{R}$, we see that the level curve exists for all $c$ and is a circle of radius $\sqrt{e^c} = e^{c/2}$.

| $c$ | Level curve $x^2 + y^2 = e^c$ |
|---|---|
| $-5$ | $x^2 + y^2 = e^{-5}$ |
| $-1$ | $x^2 + y^2 = e^{-1}$ |
| $0$ | $x^2 + y^2 = 1$ |
| $1$ | $x^2 + y^2 = e$ |
| $3$ | $x^2 + y^2 = e^3$ |
| $4$ | $x^2 + y^2 = e^4$ |



The section of the graph by $x=0$ is

$$\{(x,y,z)\in\mathbb{R}^3 \mid z=\ln(x^2+y^2),\ x=0\} = \{(0,y,z)\in\mathbb{R}^3 \mid z=2\ln|y|\}$$

The section by $y=0$ is entirely similar:

$$\{(x,y,z)\in\mathbb{R}^3 \mid z=\ln(x^2+y^2),\ y=0\} = \{(x,0,z)\in\mathbb{R}^3 \mid z=2\ln|x|\}.$$

In fact, if we switch from Cartesian coordinates to cylindrical coordinates, it is quite easy to understand the surfaces in both examples 11 and 13. In view of the Cartesian / cylindrical relation $x^2+y^2=r^2$, we see that the function for $f$ of example 11,

$$z=4-x^2-y^2=4-(x^2+y^2)=4-r^2.$$

For the function $h$ of example 13, we have

$$z=\ln(x^2+y^2)=\ln(r^2)=2\ln(r).$$

where we assume the usual convention that the cylindrical coordinate $r$ is non-negative. Thus, both these graphs are of surfaces of revolution obtained by revolving different curves around the $z$-axis (in case of example ~~ex~~ 11, a parabola $x^2=-(z-4)$ facing downward with ~~origin~~ center $(0,4)$, and in example 12, the curve $\ln|x|$). As a result, the level curves are, in general circular.

The preceding discussion has been devoted entirely to graphing scalar-valued functions of just two variables. However, all the ideas can be extended to more variables and higher dimensions. If $f: X\subseteq\mathbb{R}^n\to\mathbb{R}$ is a (scalar-valued) function of $n$ variables, then the graph of $f$ is the subset of $\mathbb{R}^{n+1}$ given by

$$\text{Graph } f = \{(\mathbf{x}, f(\mathbf{x})) \mid \mathbf{x}\in X\} = \{(x_1,x_2,\dots,x_n,x_{n+1}) \mid (x_1,\dots,x_n)\in X \text{ and } x_{n+1}=f(x_1,\dots,x_n)\}. \quad (2)$$

The compactness of the vector notation makes the definition of the graph of the function of $n$ variables exactly the same as in (1). The level set at height $c$ of such a function is defined by

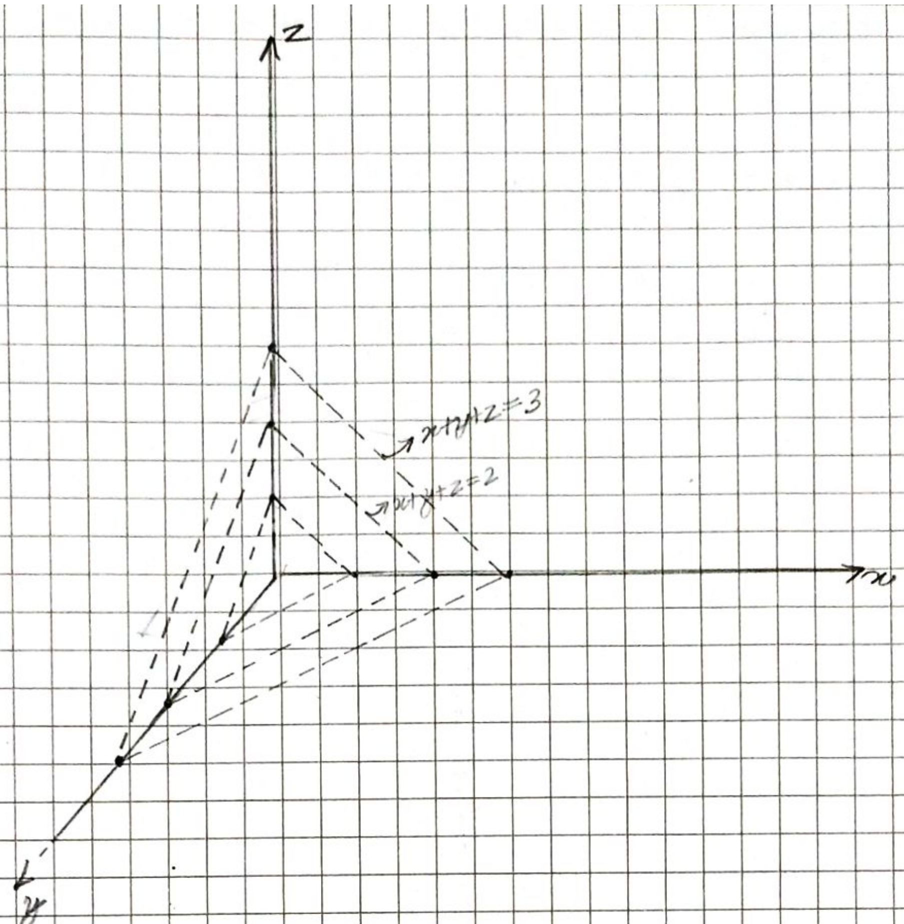
$$\text{Level set at height } c = \{\mathbf{x}\in\mathbb{R}^n \mid f(\mathbf{x})=c\} = \{(x_1,x_2,\dots,x_n) \mid f(x_1,\dots,x_n)=c\}.$$

While the graph of $f$ is a subset of $\mathbb{R}^{n+1}$, a level set of $f$ is a subset of $\mathbb{R}^n$. This makes it possible to get some geometric insight into graphs of functions of three variables, even though we cannot actually visualize them.

<u>Example 14.</u>
Let $F:\mathbb{R}^3\to\mathbb{R}$ be given by $F(x,y,z)=x+y+z$. Then, the graph of $F$ is the set $\{(x,y,z,w) \mid w=x+y+z\}$ and is a subset of $\mathbb{R}^4$, called a hypersurface, which we cannot depict adequately. Nonetheless, look at the level sets of $F$, which are surfaces in $\mathbb{R}^3$.

$$\text{Level set at height } c = \{(x,y,z) \mid x+y+z=c\}.$$

Thus, the level sets form a family of parallel planes with the normal vector $\mathbf{i}+\mathbf{j}+\mathbf{k}$.




Surfaces in general

Not all curves in $\mathbb{R}^2$ can be described as the graph of a single function of one variables. Perhaps the most familiar example is the unit circle. Its graph cannot be determined by a single equation of the form $y=f(x)$ (or for that matter by one of the form $x=g(y)$). As we know, the graph of the circle may be described analytically by the equation $x^2+y^2=1$. In general, a curve in $\mathbb{R}^2$ is determined by an arbitrary equation in $x$ and $y$, not necessarily the one that isolates $y$ alone on one side in terms of $x$. In other words this means that, the general curve given by the equation of the form $F(x,y)=c$ (that is a level set of a function of two variables).

The analogous situation occurs with surfaces in $\mathbb{R}^3$. Frequently, a surface is determined by an equation of the form $F(x,y,z)=c$ (that is a level set of a function of three independent variables), not necessarily one of the form $z=f(x,y)$.

Example 15. A sphere is a surface in $\mathbb{R}^3$ whose points are all equidistant from a fixed point. If this fixed point is the origin, then the equation for the sphere

$$\|x-0\|=\|x\|=a \tag{3}$$

where $a$ is positive constant and $x=(x,y,z)$ is a point on the sphere. If we square both sides of the equation (3) and expand the (implicit) dot product then we obtain perhaps the familiar equation of a sphere of radius $a$ centered at the origin:

$$x^2+y^2+z^2=a^2 \tag{4}$$

If the center of the sphere is at point $x_0=(x_0,y_0,z_0)$,