5. (a) Verify that $\sin^3 x$ and $\sin x - \frac{1}{3}\sin 3x$ are solutions of

$$y'' + (\tan x - 2\cot x)\, y' = 0$$

on any interval where $\tan x$ and $\cot x$ both are defined. Are these solutions linearly independent?

(b) Find the general solution of this equation.

Solution.

(a) Let $f(x) = \sin^3 x$

$$f'(x) = 3\sin^2 x \cos x$$
$$f''(x) = 3(2\sin x \cos x)\cos x + 3\sin^2 x(-\sin x)$$
$$= 6\sin x\cos^2 x - 3\sin^3 x$$

$$f''(x) + (\tan x - 2\cot x) f'(x)$$
$$= 6\sin x\cos^2 x - 3\sin^3 x + \left(\frac{\sin x}{\cos x} - 2\frac{\cos x}{\sin x}\right)(3\sin^2 x\cos x)$$
$$= 6\sin x\cos^2 x - 3\sin^3 x + \left(\frac{\sin^2 x - 2\cos^2 x}{\sin x \cancel{\cos x}}\right)(3\sin^{\cancel{2}} x\, \cancel{\cos x})$$
$$= 6\sin x\cos^2 x - 3\sin^3 x + 3\sin^3 x - 6\sin x\cos^2 x$$
$$= 0.$$

Let $g(x) = \sin x - \frac{1}{3}\sin 3x$

$$g'(x) = \cos x - \frac{1}{\cancel{3}}\cdot \cancel{3}\cos 3x = \cos x - \cos 3x.$$
$$g''(x) = -\sin x + 3\sin 3x = 3\sin 3x - \sin x.$$

$$g''(x) + (\tan x - 2\cot x) g'(x)$$
$$= (-\sin x + 3\sin 3x) + \left(\frac{\sin x}{\cos x} - \frac{2\cos x}{\sin x}\right)(\cos x - \cos 3x)$$
$$= -\sin x + 3(3\sin x - 4\sin^3 x) + \left(\frac{\sin^2 x - 2\cos^2 x}{\sin x\cos x}\right)(\cos x - 4\cos^3 x + 3\cos x)$$
$$= -\sin x + 9\sin x - 12\sin^3 x + \left(\frac{\sin^2 x - 2\cos^2 x}{\sin x\, \cancel{\cos x}}\right)(4\cos x - 4\cos^{3} x)$$
$$= 8\sin x(1-\sin^2 x) - 4\sin^3 x + 4\sin x(\sin^2 x - 2\cos^2 x)$$
$$= 8\sin x\cos^2 x - 4\sin^3 x + 4\sin^3 x - 8\sin x\cos^2 x$$
$$= 0.$$

(b) The general solution of the ODE is –

$$y(x) = c_1\left(\sin x - \frac{1}{3}\sin 3x\right) + c_2 \sin^3 x.$$

3. Linear first order differential equations.

Let

$$a_1(x)\frac{dy}{dx} + a_0(x)\, y = h(x) \qquad (4.15)$$

be a normal first-order linear differential equation defined on an interval I of the x axis. Then, as we know, the general solution of this equation can be expressed in the form

$$y = y_p(x) + y_h(x) \qquad (4.16)$$

where $y_p(x)$ is any particular solution and $y_h(x)$ is the general solution of the homogenous equation. –

$$a_1(x)\frac{dy}{dx} + a_0(x)y = 0. \tag{4.17}$$

Since, $a_1(x) \neq 0$ everywhere in I, (4.17) may be re-written as

$$\frac{1}{y}\frac{dy}{dx} = -\frac{a_0(x)}{a_1(x)}, \quad y \neq 0.$$

and integrated to yield

$$\int \frac{dy}{y} = -\int \frac{a_0(x)}{a_1(x)}\,dx$$

$$\ln|y| = -\int \frac{a_0(x)}{a_1(x)}\,dx$$

$$|y| = e^{-\int [a_0(x)/a_1(x)]\,dx}.$$

Hence, by the theorem cited in the preceding section, the general solution of (4.17) is

$$y = ce^{-\int [a_0(x)/a_1(x)]\,dx}$$

where c is an arbitrary constant.

To obtain a particular solution of (4.15), we write the equation as

$$\frac{dy}{dx} + \frac{a_0(x)}{a_1(x)}y = \frac{h(x)}{a_1(x)} \tag{4.18}$$

and multiply it $e^{\int [a_0(x)/a_1(x)]\,dx}$ to obtain –

$$\left(\frac{dy}{dx} + \frac{a_0(x)}{a_1(x)}y\right)e^{\int [a_0(x)/a_1(x)]dx} = \frac{h(x)}{a_1(x)}e^{\int [a_0(x)/a_1(x)]dx}. \tag{4.19}$$

But,

$$\frac{d}{dx}\left(y\,e^{\int [a_0(x)/a_1(x)]dx}\right) = \left(\frac{dy}{dx} + \frac{a_0(x)}{a_1(x)}y\right)e^{\int [a_0(x)/a_1(x)]dx}$$

and so (4.19) may be replaced by, the equivalent equation

$$\frac{d}{dx}\left(y\,e^{\int [a_0(x)/a_1(x)]dx}\right) = \frac{h(x)}{a_1(x)}e^{\int [a_0(x)/a_1(x)]dx}.$$

$$\therefore \quad y\,e^{\int [a_0(x)/a_1(x)]dx} = \int \frac{h(x)}{a_1(x)}e^{\int [a_0(x)/a_1(x)]dx}\,dx$$

It follows from (4.16), that the general solution to (4.15) is

$$y = y_p(x) + y_h(x)$$

$$= \left[\int \frac{h(x)}{a_1(x)}e^{\int [a_0(x)/a_1(x)]dx}\right]e^{-\int [a_0(x)/a_1(x)]dx} + ce^{-\int [a_0(x)/a_1(x)]dx}$$

$$y(x) = \left[c + \int \frac{h(x)}{a_1(x)}e^{\int [a_0(x)/a_1(x)]dx}\,dx\right]e^{-\int [a_0(x)/a_1(x)]dx}.$$

Thus, to find the general solution of a normal first-order linear differential equation, rewrite the equation in the form

$$\frac{dy}{dx} + \frac{a_0(x)}{a_1(x)} y = \frac{h(x)}{a_1(x)}$$

multiply by $e^{\int [a_0(x)/a_1(x)]dx}$ and integrate.

Examples.

1. Find the general solution of

$$\frac{dy}{dx} + 2x\,[y] = x.$$

Solution

The integrating factor is $e^{\int 2x dx} = e^{x^2}$.
Multiplying by e^{x^2}, we obtain

$$\frac{dy}{dx} e^{x^2} + (2x e^{x^2})\, y = x e^{x^2}.$$

$$\frac{d}{dx} (e^{x^2} y) = x e^{x^2}$$

$$e^{x^2} y = \int x e^{x^2} dx + c$$

$$= \frac{1}{2} \int (2x)\, e^{x^2} dx + c$$

$$= \frac{1}{2} e^{x^2} + c$$

$$y(x) = \left(\frac{e^{x^2}}{2} + c\right) e^{-x^2}$$

$$y = \frac{1}{2} + c e^{-x^2}$$

2. Solve the equation

$$x \frac{dy}{dx} + y = x$$

Solution

Since the leading coefficient of y this equation vanishes when $x = 0$, the above method applies on the intervals $(0, \infty)$ and $(-\infty, 0)$. There, however, we can write -

$$\frac{dy}{dx} + \frac{1}{x} \cdot (y) = 1$$

The integrating factor is $e^{\int \frac{dx}{x}} = e^{\ln x} = x$.

So, $x \dfrac{dy}{dx} + y = x$

$$\frac{d}{dx}(xy) = x$$

$$xy = \int x\, dx + c$$

$$= \frac{x^2}{2} + c$$

$$y = \frac{x}{2} + \frac{c}{x}$$