Therefore, W is a real subspace.

4) Let $W = \{x \in \mathbb{R}^2 : x_2 = cx_1\}$, c is a fixed constant. It is an easy exercise to show that W is a subspace of $\mathbb{R}^2$.

Proof.

(i) W has an additive identity.
The zero vector $0 = (0,0)$ belongs to W,
since $0\ (0,0)$ satisfies $x_2 = cx_1$.

(ii) W is closed under vector addition.
Assume $x, y \in W$. Then, $x_2 = cx_1$ and $y_2 = cy_1$.
$\Leftrightarrow \quad x_2 + y_2 = cx_1 + cy_1$
$\Leftrightarrow \quad x_2 + y_2 = c(x_1 + y_1)$
$\Leftrightarrow \quad x + y \in W$.

(iii) W is closed under scalar multiplication.
Assume $x, y \in W$. and $\alpha \in F$.
Then, $x_2 = cx_1$
$\alpha x_2 = c(\alpha x_1)$.
$\Rightarrow \quad \alpha x \in W$.

W is a subspace of $\mathbb{R}^2$.

4. Sums and Direct Sums

In later chapters, we will find that the notion of sums and vector space sums and direct sums are useful. We define these

Suppose $U_1, U_2, \dots, U_m$ are subspaces of V. The sum of $U_1, U_2, \dots, U_m$ denoted by $U_1 + \dots + U_m$, is defined to be the set of all possible sums of elements of $U_1, U_2, \dots, U_m$. More precisely,

$$U_1 + U_2 + \dots + U_m := \{u_1 + u_2 + \dots + u_m : u_1 \in U_1, u_2 \in U_2, \dots, u_m \in U_m\}.$$

Let us verify that, if $U_1, U_2, \dots, U_m$ are the subspaces of V, $U_1 + U_2 + \dots + U_m$ is a subspace of V.

Proof.

Consider $u, w \in U_1 + U_2 + \dots + U_m$.
Say, $u = u_1 + u_2 + \dots + u_m$, $u_i \in U_i$
$w = w_1 + w_2 + \dots + w_m$, $w_i \in W_i$

Then,
$u + w = (u_1 + w_1) + (u_2 + w_2) + \dots + (u_m + w_m)$
$u_1 + w_1 \in U_1$
$u_2 + w_2 \in U_2$
$\vdots$
$u_m + w_m \in U_m$

as $U_1, U_2, \dots, U_m$ are subspaces of V and are closed under addition.
By definition, therefore $u + w \in V$.

Also, suppose $\alpha \in F$ and $u = u_1 + u_2 + \dots + u_m$, $u_i \in U_i$ is an arbitrary element in the sum of subspaces $U_1 + U_2 + \dots + U_m$
$\alpha u = \alpha(u_1 + \dots + u_m)$
$= \alpha u_1 + \dots + \alpha u_m$ (distributivity).
$\alpha u_i \in U_i \quad \forall i = 1, 2, \dots, m$
$\Rightarrow \alpha u \in U_1 + U_2 + \dots + U_m$

Thus, $U_1 + U_2 + \dots + U_m$ is a subspace of V.

Let's look at some examples of sums of subspaces. Suppose, U is the set of all elements of $\mathbb{F}^3$ whose second and third coordinates equal 0, and W is the set of all elements in $\mathbb{F}^3$ whose first and third coordinates equal 0:

$U = \{(x, 0, 0) \in \mathbb{F}^3 : x \in \mathbb{F}\}$ and $W = \{(0, y, 0) \in \mathbb{F}^3 : y \in \mathbb{F}\}$.

Then, $U + W = \{(x, y, 0) \in \mathbb{F}^3 : x, y \in \mathbb{F}\}$.

As another example, suppose U is as above and W is set of all elements of $\mathbb{F}^3$ whose first and second coordinates equal each other and whose third coordinate equals 0:

$$W = \{(y, y, 0) \in \mathbb{F}^3 : y \in \mathbb{F}\}.$$

Let (a_1, a_2, a_3) be an arbitrary element in $U + W$.

$$\begin{aligned}(a_1, a_2, a_3) &= (x, 0, 0) + (y, y, 0)\\ &= (x+y, y, 0)\end{aligned}$$

The third coordinate $a_3 = 0$.

If we choose $y = a_2$ and $x + y = a_1$ that is $x = a_1 - a_2$, then we can always write all vectors of the form $(a_1, a_2, 0)$ as the sum of two vectors $u \in U$ and $w \in W$. Hence,

$$U + W = \{(x, y, 0) \in \mathbb{F}^3 : x, y \in \mathbb{F}\}.$$

The sum of subspaces in the theory of vector spaces is analogous to the union of subsets in set theory. Given two subspaces of a vector space, the smallest subspace containing them is their sum. Analogously, given two subsets of a set, the smallest subset containing them is their union.

Suppose $U_1, \dots, U_m$ are subspaces of V. Clearly, $U_1, \dots, U_m$ are all contained in $U_1 + \dots + U_m$. To see this consider the sums $u_1 + u_2 + \dots + u_m$ where all except one of u's are zero. Conversely, any subspace of V containing $U_1, U_2, \dots, U_m$ must contain $U_1 + U_2 + \dots + U_m$ (because subspaces must contain all finite sums of their elements). Hence, $U_1 + U_2 + \dots + U_m$ is smallest subspace of V containing $U_1, \dots, U_m$.

Suppose $U_1, U_2, \dots, U_m$ are subspaces of V such that $V = U_1 + U_2 + \dots + U_m$. Thus, every element of V can be written in the form

$$u_1 + u_2 + \dots + u_m,$$

where each $u_j \in U_j$.

~~Let's look at some ex~~

We will be especially interested in cases where each vector in V can be uniquely represented in the form above. This situation is so important that we give it a special name: direct sum. Specifically, we say that V is the direct sum of subspaces $U_1, \dots, U_m$, written $V = U_1 \oplus U_2 \dots \oplus U_m$, if each element of V can be uniquely written as a sum $u_1 + u_2 + \dots + u_m$, where each $u_j \in U_j$.

Let's look at some examples of direct sums. Suppose U is a subspace of $\mathbb{F}^3$ consisting of those vectors whose last coordinate equals 0, and W is the subspace of $\mathbb{F}^3$ consisting of those vectors whose first two coordinates equal 0:

$U = \{(x, y, 0) \in \mathbb{F}^3 : x, y \in \mathbb{F}\}$ and $W = \{(0, 0, z) \in \mathbb{F}^3 : z \in \mathbb{F}\}$.

Then, $\mathbb{F}^3 = U \oplus W$.

<u>Verification</u>

Consider any arbitrary element $(x, y, z) \in \mathbb{F}^3$. (x, y, z) can be uniquely expressed as the sum of two vectors -

$$(x, y, z) = \underbrace{(x, y, 0)}_{\text{Belongs to } U} + \underbrace{(0, 0, z)}_{\text{Belongs to } W.}$$

This representation is unique. Hence, $\mathbb{F}^3 = U \oplus W$.

As another example, suppose U_j is the subspace of $\mathbb{F}^n$ consisting of those vectors whose coordinates are all 0, except possibly in the j^{th} slot. (for example $U_2 = \{(0, x_2, 0, \ldots, 0) \in \mathbb{F}^n : x_2 \in \mathbb{F}\}$). Then,

$$\mathbb{F}^n = U_1 \oplus U_2 \oplus \ldots \oplus U_n.$$

As a final example, consider the vector space of all polynomials with coefficients in $\mathbb{F}$. Let U_e denote the subspace of $\mathcal{P}(\mathbb{F})$ consisting of all polynomials p of the form

$$p(z) = a_0 + a_2 z^2 + a_4 z^4 + \ldots + a_{2m} z^{2m}$$

and let U_o denote the subspace of $\mathcal{P}(\mathbb{F})$ consisting of all polynomials p of the form

$$p(z) = a_1 z + a_3 z^3 + a_5 z^5 + \ldots + a_{2m+1} z^{2m}.$$

The notations U_e and U_o should remind you of the odd and even powers of z.

Let's verify that $\mathcal{P}(\mathbb{F}) = U_e \oplus U_o$.

<u>Proof.</u>

Let $p \in \mathcal{P}(\mathbb{F})$ be an arbitrary polynomial of degree $2m+1$, $m \in \mathbb{N}$.

$$\begin{aligned} p(z) &= a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + a_5 z^5 + \ldots + a_i z^i + \ldots + a_{2m} z^{2m} + a_{2m+1} z^{2m+1} \\ &= (a_0 + a_2 z^2 + a_4 z^4 + \ldots + a_{2m} z^{2m}) + (a_1 z + a_3 z^3 + a_5 z^5 + \ldots + a_{2m+1} z^{2m+1}) \\ &= u_e + u_o \end{aligned}$$

where $u_e \in U_e$ and $u_o \in U_o$.

This representation is unique.

Sometimes, nonexamples add to our understanding as much as examples. Consider the following three subspaces of $\mathbb{F}^3$.

$$\begin{aligned} U_1 &= \{(x, y, 0) \in \mathbb{F}^3 : x, y \in \mathbb{F}\} \\ U_2 &= \{(0, 0, z) \in \mathbb{F}^3 : z \in \mathbb{F}\} \\ U_3 &= \{(0, y, y) \in \mathbb{F}^3 : y \in \mathbb{F}\}. \end{aligned}$$

Clearly, $\mathbb{F}^3 = U_1 + U_2 + U_3$ because an arbitrary vector $(x, y, z) \in \mathbb{F}^3$ can be written as

$$(x, y, z) = (x, y, 0) + (0, 0, z) + (0, 0, 0)$$

where $(x, y, 0) \in U_1$
$(0, 0, z) \in U_2$
$(0, 0, 0) \in U_3$.

However, $\mathbb{F}^3$ does not equal the direct sum of U_1, U_2, U_3, because the vector $(0, 0, 0)$ can be written in two different ways as a sum $u_1 + u_2 + u_3$, with each $u_j \in U_j$. Specifically, we have

$$(0, 0, 0) = (0, 0, 0) + (0, 0, 0) + (0, 0, 0)$$

and

$$(0, 0, 0) = (0, 1, 0) + (0, 0, 1) + (0, -1, -1)$$

In the example above, we showed that something is not a direct sum by showing that 0 does not have a unique representation as a sum of appropriate vectors. The definition of direct sum requires that every vector in the space have a unique representation as an appropriate sum.

Proposition. Suppose that $U_1, U_2, \ldots, U_m$ are subspaces of V. Then, $V = U_1 \oplus U_2 \oplus \ldots \oplus U_m$, if and only if both the following conditions hold:

(a) $V = U_1 + U_2 + \cdots + U_m$.
(b) The only way to write 0 as a sum $u_1 + u_2 + \cdots + u_m$, where each $v_j \in U_j$ is by taking all u_j's equal to 0.

Proof.
First suppose that $V = U_1 \oplus \ldots \oplus U_n$. Clearly (a) holds because of how sum and direct sum are defined. To prove (b), suppose that $u_1 \in U_1, \ldots, u_n \in U_n$ and

$$0 = u_1 + u_2 + \cdots + u_n.$$

Then each u_j must be 0.

This follows from the uniqueness part of the definition of direct sum because $0 + 0 + \ldots + 0 = 0$ and $0 \in U_1, 0 \in U_2, \ldots, 0 \in U_n$ and this representation must be unique.

Converse:
Suppose that (a) and (b) hold. Let $v \in V$. By (a), we can write—

$$v = u_1 + u_2 + \cdots + u_n.$$

for some $u_1 \in U_1, u_2 \in U_2, \ldots, u_n \in U_n$.

To show that this representation is unique, suppose that we also have—

$$v = ~~u_1~~\ v_1 + v_2 + \cdots + v_m.$$

where $v_1 \in U_1, v_2 \in U_2, \ldots, v_n \in U_n$. Subtracting these two equations, we have

$$0 = (u_1 - v_1) + (u_2 - v_2) + \cdots + (u_n - v_n)$$

Clearly, $u_1 - v_1 \in U_1$
$u_2 - v_2 \in U_2$
$\vdots$
$u_n - v_n \in U_n$.

So, the equation above and (b) imply that each $u_j - v_j = 0$. Thus, $u_1 = v_1, u_2 = v_2, \ldots, u_n = v_n$ as desired.

Q.E.D.

Sums of subspaces are analogous to unions of subsets. Similarly, direct sums of subspaces are analogous to disjoint unions of subsets. No two subspaces of a vector space V can be disjoint, because they both must ~~so~~ contain 0. So, disjointedness is replaced, atleast in the case of two subspaces with the requirement that the intersection equals $\{0\}$.

The next proposition gives a simple condition for testing which pairs of subspaces give a direct sum. Note that, this proposition deals only with the case of two subspaces. When asking about a possible direct sum with more than two subspaces, it is not enough to test that any two of the subspaces intersect only at 0. To see this consider the non-example presented before. In that non-example, we had $\mathbb{F}^3 = U_1 + U_2 + U_3$, but $\mathbb{F}^3$ did not equal the direct sum of U_1, U_2, U_3. However, in that ~~example~~ non example we have:

$$U_1 \cap U_2 = \{0\}$$
$$U_2 \cap U_3 = \{0\}$$
$$U_3 \cap U_1 = \{0\}.$$

The next example shows that with just two subspaces, we get a nice necessary and sufficient condition for a direct sum.

<u>Proposition.</u> Suppose that U and W are subspaces of V. Then, $V = U \oplus W$ if and only if $V = U + W$ and $U \cap W = \{0\}$.

<u>Proof.</u>
First suppose that $V = U \oplus W$. Then, by the definition of a direct sum $V = U + W$. Also, if $v \in U \cap W$ then $0 = v + (-v)$, where $v \in U$ and $-v \in W$. Thus, $0 \in V$. By the unique representation of 0 as a sum of a vector in U and a vector in W, we must have $v = 0$. Thus, $U \cap W = \{0\}$, completing the proof in one direction.

To prove the other direction, now suppose that $V = U + W$ and $U \cap W = \{0\}$. To prove that $V = U \oplus W$, suppose that

$$0 = u + w$$

where $u \in U$ and $w \in W$. To complete the proof, we need only show that $u = w = 0$. The equation above implies that $u = -w \in W$. Thus, $u \in U \cap W$, and hence $u = 0$. This along with equation $u = -w$, implies that $w = 0$, completing the proof.

<u>Problems.</u>
1. For each of the following subsets of $\mathbb{F}^3$, determine whether it is a subspace of $\mathbb{F}^3$:
(a) $\{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1 + 2x_2 + 3x_3 = 0\}$;

Solution.
(i) Additive identity.
$0 \in W$
since $0 + 2(0) + 3(0) = 0$.

(ii) Closure under vector addition.
Assume $x, y \in W$.
$x_1 + 2x_2 + 3x_3 = 0$
$y_1 + 2y_2 + 3y_3 = 0$.

$(x_1+y_1) + 2(x_2+y_2) + 3(x_3+y_3) = 0$.
$\Leftrightarrow$ $x + y \in W$.

(iii) Closure under scalar multiplication.
Assume $\alpha \in \mathbb{F}$ and $x \in W$.
$\alpha x = (\alpha x_1, \alpha x_2, \alpha x_3)$.

As $x \in W$, $x_1 + 2x_2 + 3x_3 = 0$
$\alpha x_1 + 2\alpha x_2 + 3\alpha x_3 = 0$.
$\Rightarrow$ $\alpha x \in W$.

(b) $\{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1 + 2x_2 + 3x_3 = 4\}$;
~~Assume $\alpha \in \mathbb{F}$ and~~

(i) Additive identity.
$0 \notin W$.
since $0 + 2\cdot 0 + 3\cdot 0 = 0$.

(c) $\{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1 x_2 x_3 = 0\}$;
(i) Additive identity.
$0 \in W$.
since $(0)(0)(0) = 0$.

(ii) Closure under vector addition.
Assume $x, y \in W$.
$x_1 x_2 x_3 = 0$
$y_1 y_2 y_3 = 0$

$x_1 x_2 x_3 + y_1 y_2 y_3 = 0$

However, this does not imply $x + y \in W$.
since, $(x_1+y_1)(x_2+y_2)(x_3+y_3)$
$= (x_1 x_2 + x_1 y_2 + x_2 y_1 + y_1 y_2)(x_3 + y_3)$
$= (x_1 x_2 x_3 + y_1 y_2 y_3 + \dots)$.

(d) $\{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1 = 5x_3\}$.
(i) Additive identity.
$0 \in W$.
since $0 = 5\cdot 0$.

(ii) If $x, y \in W$, then $x_1 = 5x_3$ and $y_1 = 5y_3$
$x_1 + y_1 = 5x_3 + 5y_3$
$= 5(x_3 + y_3)$.

$\Leftrightarrow$ $x + y \in W$.

(iii) If $\alpha \in \mathbb{F}$ and $x \in W$,
~~at~~ $\alpha x \in W$.
Hence, W is a subspace of $\mathbb{R}^3$.

2. Give an example of a non-empty subset U of $\mathbb{R}^2$ such that U is closed under addition and under taking additive inverses (meaning $-u \in U$ whenever $u \in U$), but U is not a subspace of $\mathbb{R}^2$.

Solution

Let $U = \{ (x_1, x_2) \in \mathbb{R}^2 :$

?

3. Prove that the intersection of any collection of subspaces of V is a subspace of V.

Solution.

Let $U_1, U_2, \ldots, U_m$ be subspaces of V.
Consider the subset $U_1 \cap U_2 \cap \ldots \cap U_m$.

(i) Additive identity.
$U_1, \ldots, U_m$ are subspaces of V and hence they contain the zero vector. so, $0 \in U_1$, $0 \in U_2$, $\ldots$, $0 \in U_m \Rightarrow 0 \in (U_1 \cap U_2 \cap \ldots \cap U_m)$.

(ii) Closure under addition.
Let $x, y \in (U_1 \cap \ldots \cap U_m)$.
$\Leftrightarrow$ x, y belong to each of U_is.
$\Leftrightarrow$ since U_i is closed under addition, $x+y \in U_i$ for all $i = 1, \ldots, m$.
$\Leftrightarrow$ $x+y \in U_1 \cap \ldots \cap U_m$.

(iii) Scalar multiplication.
Let $x \in \bigcap_{i=1}^{m} U_i$. $\alpha \in F$.

Then, since U_i is closed under scalar multiplication, if $x \in U_i$ and $\alpha \in \mathbb{F}$, it implies $\alpha x \in U_i$.
$\Leftrightarrow$ $\alpha x \in \bigcap_{i=1}^{m} U_i$

$U_1 \cap \ldots \cap U_m$ is also a subspace of V.

4. Prove that the union of two subspaces of V is a subspace of V, if and only if one of the subspaces is contained in the other.

Proof.

$\Leftarrow$ direction.

Assume that U and W are two subspaces of V and $U \subseteq W$.
Consider the subset $U \cup W$.

$$U \cup W = \{x : x \in U \text{ or } x \in W\}$$
$$= \{x : x \in W\} \text{ since if } x \in U \Rightarrow x \in W, \text{ as } U \subseteq W.$$
$$= W.$$

W is a subspace of V.

$\Leftrightarrow$ $U \cup W$ is a subspace of V.

$\Rightarrow$ direction.

Assume that U and W are two subspaces of V, such that $U \cup W$ is also a subspace.

If $U \cup W$ is a subspace of V, it satisfies the properties

(i) Additive identity
$0 \in U \cup W$.

(ii) Closed over vector addition.
If $x, y \in U \cup W \Rightarrow x + y \in U \cup W$.

(iii) Closed over scalar multiplication.
If $x \in U \cup W$ and $\alpha \in F$, $\Rightarrow \alpha x \in U \cup W$.

Suppose $x \in U$ and $y \in W$. $x, y \in U \cup W$. $\Rightarrow x + y \in U \cup W$.

Case I. $x + y \in U$ only.
If $y \in W$ and $x + y \in U \Rightarrow y$ must also belong to U.
$\Leftrightarrow$ Whenever y is in W, y is also in U.
$\Leftrightarrow$ $W \subset U$.

Case II. $x + y \in W$ only.
$x \in U$ and $x + y \in W \Rightarrow x$ must also belong to W.
$\Leftrightarrow$ $U \subset W$.

Case III. $x + y \in U \cap W$.
$\Rightarrow$ x belongs to W and y belongs to U as well.
$U = W$.

Hence, either $U \subseteq W$ or $W \subseteq U$.

5. Suppose that U is a subspace of V. What is $U + U$?

Solution.

$$U + U = \{u_1 + u_2 : u_1 \in U, u_2 \in U\}$$
$$= U$$

since U is closed under vector addition, the set of all possible sums of pairs of vectors in U must belong to U.