(b) If $a_i = a_j$, ~~the~~

We can simplify the determinant as follows –

$$= e^{(a_1+a_2+\cdots+a_n)x} \begin{vmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ a_1 & a_2-a_1 & a_3-a_1 & a_4-a_1 & \cdots & a_n-a_1 \\ a_1^2 & a_2^2-a_1^2 & a_3^2-a_1^2 & a_4^2-a_1^2 & \cdots & a_n^2-a_1^2 \\ \vdots & & & & & \vdots \\ a_1^{n-1} & a_2^{n-1}-a_1^{n-1} & a_3^{n-1}-a_1^{n-1} & a_4^{n-1}-a_1^{n-1} & \cdots & a_n^{n-1}-a_1^{n-1} \end{vmatrix}$$

$$= e^{(a_1+a_2+\cdots+a_n)x} \begin{vmatrix} a_2-a_1 & a_3-a_1 & a_4-a_1 & \cdots & a_n-a_1 \\ a_2^2-a_1^2 & a_3^2-a_1^2 & a_4^2-a_1^2 & \cdots & a_n^2-a_1^2 \\ \vdots & & & & \\ a_2^{n-1}-a_1^{n-1} & a_3^{n-1}-a_1^{n-1} & a_4^{n-1}-a_1^{n-1} & \cdots & a_n^{n-1}-a_1^{n-1} \end{vmatrix}$$

$$= e^{(a_1+\cdots+a_n)x} (a_2-a_1)(a_3-a_1)\cdots(a_n-a_1) \begin{vmatrix} 1 & 1 \\ a_2+a_1 & a_3+a_1 \\ a_2^2+a_1a_2+a_1^2 & a_3^2+a_1a_3+a_1^2 \\ \vdots & \vdots \end{vmatrix}$$

$$= e^{(a_1+a_2+\cdots+a_n)x} (a_2-a_1)(a_3-a_1)\cdots(a_n-a_1) \begin{vmatrix} 1 & 0 & 0 \\ a_2+a_1 & a_3-a_2 & a_n-a_2 \\ & a_3^2-a_2^2+a_1(a_3-a_2) & \\ & & \end{vmatrix}$$

$$= e^{(a_1+a_2+\cdots+a_n)x} (a_2-a_1)\cdots(a_n-a_1)(a_3-a_2)(a_4-a_2)(a_5-a_2)\cdots$$

$$= e^{(a_1+a_2+\cdots+a_n)x} \prod_{i>j} (a_i-a_j)$$

If $a_i = a_j$, the Wronskian $W[e^{a_1x}, e^{a_2x}, \ldots, e^{a_nx}]$ is identically equal to zero.

4.7 Abel's Formula.

According to theorem 4.6, the wronskian of a set of solutions of a linear homogenous differential equation either vanishes identically or not at all. This fact can also be deduced from the following theorem, which gives an explicit formula for the Wronskian in this case.

Theorem 4.7. Let $y_1, y_2, \ldots, y_n$ be solutions on an interval I of a linear, homogenous n-th order differential equation –

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_0(x)y = 0. \quad (4.48)$$

and suppose $a_n(x) \neq 0$ everywhere in I. Then,

$$W[y_1(x), y_2(x), \ldots, y_n(x)] = c e^{-\int [a_{n-1}(x)/a_n(x)]\, dx}. \qquad (4.49)$$

for an appropriate constant c. This result is known as the Abel's formula for the Wronskian.

Proof:.

In order to avoid using general properties of determinants, we shall prove (4.49) only in the case $n=2$. The general proof is identical, except that it uses the formula for the derivative of an nth-order derivative determinant.

Suppose that y_1 and y_2 are the solutions of

$$a_2(x)\frac{d^2y}{dx^2} + a(x)\frac{dy}{dx} + a_0(x)y = 0.$$

on an interval I, in which $a_2(x)$ does not vanish. Then,

$$\frac{d}{dx} W[y_1(x), y_2(x)] = \frac{d}{dx}\begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$$

$$= \frac{d}{dx}\left(y_1(x)\, y_2'(x) - y_2(x)\, y_1'(x)\right)$$

$$= \cancel{y_1'(x)\cdot y_2'(x)} + y_1(x)\cdot y_2''(x) - \cancel{y_2'(x)\, y_1'(x)} - y_2(x)\, y_1''(x)$$

$$= y_1(x)\cdot y_2''(x) - y_2(x)\, y_1''(x).$$

But, $y_1(x)$ and $y_2(x)$ satisfy the differential equation

$$a_2(x)\, y'' + a_1(x)\, y' + a_0(x)\, y = 0.$$

so, $$y_2'' = -\frac{a_1(x)}{a_2(x)} y_2' - \frac{a_0(x)}{a_2(x)} y_2$$

$$\Leftrightarrow \frac{d}{dx}\left[W(y_1(x), y_2(x))\right] = y_1(x)\left[\frac{-a_1(x)}{a_2(x)} y_2'(x) - \frac{a_0(x)}{a_2(x)} y_2(x)\right]$$

$$- y_2(x)\left[\frac{-a_1(x)}{a_2(x)} y_1'(x) - \frac{a_0(x)}{a_2(x)} y_1(x)\right]$$

$$= \frac{a_1(x)}{a_2(x)} y_1(x)\cdot y_2'(x) - \cancel{\frac{a_0(x)}{a_2(x)} y_1(x) y_2(x)} - y_1'(x) y_2(x)\frac{a_1(x)}{a_2(x)} + \cancel{y_1(x) y_2(x)\cdot\frac{a_0(x)}{a_2(x)}}$$

$$= -\frac{a_1(x)}{a_2(x)}\left[y_1(x)\cdot y_2'(x) - y_1'(x)\cdot y_2(x)\right].$$

$$= -\frac{a_1(x)}{a_2(x)} W[y_1(x), y_2(x)].$$

Thus, $W[y_1(x), y_2(x)]$ is differentiable on I and satisfies the first order linear differential equation.

$$\frac{dW}{dx} + \frac{a_1(x)}{a_2(x)} W(x) = 0$$

$$\frac{dW}{W} = -\frac{a_1(x)}{a_2(x)} dx$$

$$\log W[y_1(x), y_2(x)] = -\int [a_1(x)/a_2(x)]\, dx.$$

$$W[y_1(x), y_2(x)] = c e^{-\int [a_1(x)/a_2(x)] dx}$$

This closes the proof.

If x_0 is any fixed point in I and if $W(x_0)$ denotes the value of the Wronskian of $y_1, \ldots, y_n$ at x_0, then Abel's formula may be written in the form

$$W[y_1(x), y_2(x), \ldots, y_n(x)] = W(x_0) \cdot e^{-\int_{x_0}^{x} [a_{n-1}(x)/a_n(x)] dx}. \tag{4.50}$$

This formula shows that the Wronskian of any basis for the solution space of a homogeneous linear differential equation is determined upto a multiplicative constant by the equation itself and does not depend upon the particular basis used to compute it. This simple observation will be important later on.

Examples

1. Since

$$x \frac{d^2y}{dx^2} + \frac{dy}{dx} + xy = 0$$

is normal on $(0, \infty)$, the Wronskian of any two solutions y_1, y_2 of this equation must be of the form:

$$\begin{aligned} W[y_1(x), y_2(x)] &= c e^{-\int (\frac{1}{x}) dx} \\ &= c e^{-\ln(x)} \\ &= \left(\frac{c}{x}\right). \end{aligned}$$

If in addition, y_1 and y_2 satisfy the initial conditions—

$$y_1(x_0) = a_0, \quad y_1'(x_0) = a_1$$
$$y_2(x_0) = b_0, \quad y_2'(x_0) = b_1.$$

at some point $x_0 > 0$, then

$$W(x_0) = \frac{c}{x_0} = \begin{vmatrix} a_0 & b_0 \\ a_1 & b_1 \end{vmatrix} = a_0 b_1 - a_1 b_0$$

$$\therefore c = x_0 (a_0 b_1 - a_1 b_0).$$

Thus, $W[y_1(x), y_2(x)] = \dfrac{x_0(a_0 b_1 - a_1 b_0)}{x}$

As our first substantial application of theorem (4.7), we shall use Abel's formula to find the general solution of a second-order homogeneous linear differential equation given one non-trivial solution of the equation. Thus, let $y_1 \neq 0$ be a solution of

$$a_2(x) y'' + a_1(x) y' + a_0(x) y = 0 \tag{4.51}$$

Then, every solution y_2 of (4.51) must satisfy the equation

$$W[y_1(x), y_2(x)] = c e^{-\int [a_1(x)/a_2(x)] dx}.$$

By ~~(4.20)~~ and $y_2(x)$ can thus be found out by solving the non-homogeneous first-order equation.

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$$\begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = ce^{-\int[a_1(x)/a_2(x)]\,dx}$$

$$y_1(x)\cdot y_2'(x) - y_2(x)\,y_1'(x) = ce^{-\int[a_1(x)/a_2(x)]\,dx}$$

$$\therefore\ y_1(x)\frac{d}{dx}[y_2(x)] - y_1'(x)[y_2(x)] = ce^{-\int[a_1(x)/a_2(x)]\,dx}.$$

By ~~E~~ (4·20), the general solution of this first order linear differential equation is:

$$\frac{y_1(x)\frac{d}{dx}y_2(x) - y_1'(x)\cdot y_2(x)}{(y_1(x))^2} = \frac{ce^{-\int[a_1(x)/a_2(x)]\,dx}}{y_1(x)^2}$$

$$d\left[\frac{y_2(x)}{y_1(x)}\right] = \frac{ce^{-\int[a_1(x)/a_2(x)]\,dx}}{y_1(x)^2}\cdot dx$$

$$\int d\left[\frac{y_2(x)}{y_1(x)}\right] = \int\frac{ce^{-\int[a_1(x)/a_2(x)]\,dx}}{y_1(x)^2}\,dx + k$$

$$\frac{y_2(x)}{y_1(x)} = \int\frac{ce^{-\int[a_1(x)/a_2(x)]\,dx}}{y_1(x)^2}\,dx + k.$$

$$y_2(x) = c\,y_1(x)\int\frac{e^{-\int[a_1(x)/a_2(x)]\,dx}}{y_1(x)^2}\,dx + k\,y_1(x).$$

where k is an arbitrary constant and since this formula is valid on any sub-interval of I in which $y_1 \neq 0$, it can be used to determine a second solution of (4·51) on such a sub-interval. In particular, the function

$$y_2(x) = y_1(x)\int\frac{e^{-\int[a_1(x)/a_2(x)]\,dx}}{y_1(x)^2}\,dx$$

will be such a solution and is clearly linearly independent of y_1 as desired. Thus, we have proved the following theorem.

Theorem 4·8. If y_1 is a non-trivial solution of equation (4·51) on an interval in which $a_2(x)$ does not vanish, then

$$y_2(x) = y_1(x)\int\frac{e^{-\int[a_1(x)/a_2(x)]\,dx}}{y_1(x)^2}\,dx \qquad (4·53)$$

is a solution of the equation on any subinterval of I in which $y_1 \neq 0$. Moreover, y_2 is linearly independent of y_1 and the general solution and the general solution of (4·51) is

$$y = c_1 y_1 + c_2 y_2.$$

where c_1, c_2 are arbitrary constants.

~~2) By direct ant~~