2. Sets and Functions.

Modern analysis, like most of modern mathematics is concerned with numbers, sets and geometry. We have already introduced one type of number system, the natural numbers. We will introduce two other number systems shortly, but for now we pause to introduce the concepts and notation of set theory, as they will be used increasingly heavily in later chapters.

While set theory is not the main focus of this course, almost every other branch of mathematics relies on set theory as its foundation, so it is important to get atleast some grounding in set theory before doing other advanced areas of mathematics. In this chapter, we present the more elementary aspects of axiomatic set theory, leaving more advanced topics such as discussion of infinite sets and the axiom of choice to a later chapter. A full treatment of the finer subtleties of set theory is beyond the scope of this course.

2.1 Fundamentals.

Definition. (Informal): We define a set to be any unordered collection of objects e.g. $\{3,8,5,2\}$ is a set. If x is an object, we say that x is an element of A or $x \in A$ if x lies in the collection; otherwise we say $x \notin A$. For example, $3 \in \{1,2,3,4,5\}$, but $7 \notin \{1,2,3,4,5\}$.

Axiom. (Sets are objects). If A is a set, then A is also an object. In particular, given two sets A and B, it is meaningful to ask if A is also an element of B.

Examples.
1. The set $\{3,\{3,4\},4\}$ is a set of three distinct elements, one of which happens to itself be a set of two elements. However, not all objects are sets. We do not consider a natural number 3 to be a set. (The more accurate statement is that natural numbers can be cardinalities of sets, rather than necessarily being sets themselves).

Definition (Equality of sets.). Two sets A and B are equal if and only if every element of A is an element of B and vice versa. To put it another way, $A=B$ if and only if every element x of A belongs also to B and every element y of B also belongs to A.

Thus, for instance, $\{1,2,3,4,5\}$ and $\{3,4,1,2,5\}$ are the same set, since they contain the exact same elements. (The set $\{3,3,1,5,2,4,2\}$ is also equal to $\{1,2,3,4,5\}$.)

One can easily verify that this notion of equality is reflexive, symmetric and transitive. Observe that if $x \in A$ and $A=B$, then ~~$x \in B$ by definition~~

(I) Reflexive.
$A=A$, since a set always equals a copy of itself.

(II) Symmetric.
If $A=B$, all elements x in A, belongs to B and vice versa. So, $B=A$.

(III) Transitive.
If $A = B$ and $B = C$, then $A = C$.

$A = B$ means that, for all elements x in A, x belongs to B and vice versa.
$B = C$ means that, for all elements y in B, y belongs to C and vice versa.
Thus, every element $x \in A$, it belongs to C and vice versa. So, $A = C$.

Axiom (Empty set). There exists a set ϕ known as the empty set, which contains no elements ie. for every object x we have $x \notin \phi$.

The empty set is also denoted by $\{\}$. Note that there can be only one empty set; if there were two sets ϕ and ϕ' which were both empty, then by the definition of sets they would be equal to each other.

If a set is not equal to an empty set, we call it non-empty. The following statement is very simple, but worth stating nevertheless.

Lemma (Single choice). Let A be a non-empty set. Then there exists an object x, such that $x \in A$.

Proof.
We prove by contradiction. Let A be a non-empty set.

Suppose that there does not exist x, such that $x \in A$. Then, for all objects x, $x \notin A$. From the axiomatic definition of empty set, we also know that $\forall x, x \notin \phi$. Therefore, $x \in A \Leftrightarrow x \in \phi$. So, $A = \phi$. But, this is a contradiction.

If A is a non-empty set, there exists x, such that $x \in A$.

The above lemma asserts that given any non-empty set A, we are allowed to choose an element x of A which demonstrates this non-emptyness. Later on, we will show, given any finite number of non-empty sets, say $A_1, A_2, \ldots, A_n$ it is possible to choose one element $x_1, x_2, \ldots, x_n$ from each set $A_1, A_2, \ldots, A_n$: this is known as finite choice. However in order to choose elements from an infinite number of sets, we need an additional axiom - the axiom of choice.

Axiom: singleton sets and pair sets.
If a is an object, then there exists a set $\{a\}$ whose only element is a ie. for every object y, we have $y \in \{a\}$ if and only if $y = a$; we refer to the set $\{a\}$ as the singleton set whose element is a. Furthermore, if a and b are objects, then there exists a set $\{a, b\}$, whose only elements are a and b ie for every object y, we have $y \in \{a, b\}$ if and only if $y = a$ or $y = b$; we refer to this set as the pair-set formed by $\{a, b\}$. a and b.

Just as there is only one empty set, there is only one singleton set for each object a, thanks to the ~~f~~ definition of equality of sets.

Verification:

Suppose there are two singleton sets $A = \{a\}$ and $A' = \{a\}$ for a object a. Then, since, all elements belonging to A are also members of A' and vice versa, $A = A'$.

The definition also ensures that $\{a, b\} = \{b, a\}$ and $\{a, a\} = \{a\}$.

Verification.

Sets are unordered collections of objects. Order does not matter. As all elements belonging to $\{a, b\}$ are also members of $\{b, a\}$ and vice versa, the two sets are equal. $\{a, b\} = \{b, a\}$.

Thus, the singleton set axiom is in fact redundant, being a consequence of the pair set axiom. Conversely, the pair set axiom will follow from the singleton set axiom and the pairwise union axiom below. One may wonder why don't go further and create triplet axioms, quadruplet axioms etc. However, there will be no need for this once we introduce the pairwise union axiom below.

Since ϕ is a set (and hence an object), so is the singleton set $\{\phi\}$ i.e. the set whose only element is ϕ (and it is not the same ~~as~~ set as ϕ, $\{\phi\} \neq \phi$). ~~Similarly, the singleton~~ This is because by definition, for all elements x, $x \notin \phi$. But, $\{\phi\}$ is a non-empty set. Similarly, the singleton set $\{\{\phi\}\}$ and the pair set $\{\phi, \{\phi\}\}$ are also sets. These three sets are not equal to each other.

As the above examples show, we can now create quite a few sets; however, the sets we make are still fairly small. (each set that we can ~~x~~ build consists of no more than two elements so far). The next axiom allows us to build somewhat larger sets than before.

Pairwise Union Axiom. Given any two sets A, B, there exists a set $A \cup B$, called the union ~~of~~ $A \cup B$ of A and B, whose elements consist of all the elements that belong to A or B or both. In other words,

$$A \cup B := \{x : (x \in A) \vee (x \in B)\}$$

Examples.

1) The set $\{1, 2\} \cup \{2, 3\}$ consists of those elements which either lie in $\{1, 2\}$ or in $\{2, 3\}$ or in both, or in other words the elements of this set are ~~are~~ simply $\{1, 2, 3\}$.

Remark. If A, B, A' are sets and $A = A'$, then $A \cup B = A' \cup B$. Why?

Verification

By the pairwise union axiom

$$A \cup B = \{x : (x \in A) \vee (x \in B)\}$$

$$A' \cup B = \{x : (x \in A') \vee (x \in B)\}$$

But, $A = A'$. That is, by definition of set equality;

$$x \in A \Leftrightarrow x \in A' \text{ for all elements } x.$$

So, $$(x \in A) \vee (x \in B) \Leftrightarrow (x \in A') \vee (x \in B) \text{ for all elements } x.$$

Thus, $$(A \cup B) = (A' \cup B)$$

Similarly, if B' is a set which is equal to B, then $(A \cup B)$ is equal to $(A \cup B')$. Thus, the operation of union obeys the axiom of substitution and is thus well-defined on sets.

Lemma. If a and b are objects, then $\{a, b\} = \{a\} \cup \{b\}$. If A, B, C are sets, then the union operation is commutative, that is $A \cup B = B \cup A$ and associative $(A \cup B) \cup C = A \cup (B \cup C)$. Also, we have $A \cup A = A \cup \emptyset = \emptyset \cup A = A$.

Proof.

Union of sets is commutative.

Suppose x is an element of $A \cup B$. By pair-wise union axiom, this means that atleast one of $x \in A$ or $x \in B$ is true. But this implies $x \in B \cup A$. The converse can also be proven similarly.

So, every element x of $(A \cup B)$ is an element of $(B \cup A)$ and vice versa. Therefore, by definition, $(A \cup B) = (B \cup A)$.

Union of sets is associative.

Suppose x is an element of $(A \cup B) \cup C$. By pair-wise union axiom, this means that atleast one of $x \in (A \cup B)$ or $x \in C$ is true. We now divide into two cases. If $x \in C$, then $x \in (B \cup C)$ and so $x \in A \cup (B \cup C)$. Now, suppose instead $x \in (A \cup B)$. Then, $x \in A$ or $x \in B$. If $x \in A$, then $x \in A \cup (B \cup C)$, while if $x \in B$, $x \in (B \cup C)$ and consequently $x \in A \cup (B \cup C)$. Thus in all cases, we see that every element x of $(A \cup B) \cup C$ lies in $A \cup (B \cup C)$. A similar argument shows that every element of $A \cup (B \cup C)$ lies in $(A \cup B) \cup C$, and so $(A \cup B) \cup C = A \cup (B \cup C)$.

Because of the above lemma, we do not need to use parenthesis to denote multiple unions, thus for instance, we can write $A \cup B \cup C$ instead of $(A \cup B) \cup C$ or $A \cup (B \cup C)$. Similarly for unions of four sets.

This axiom allows us to define triple sets, quadruplet sets and so forth: if a, b, c are objects, we define $\{a, b, c\} := \{a\} \cup \{b\} \cup \{c\}$; if a, b, c, d are four objects, then we define $\{a, b, c, d\} := \{a\} \cup \{b\} \cup \{c\} \cup \{d\}$, and so forth. On the other hand, we are not yet in a position to define sets consisting of n objects for any given natural number n; this would require iterating the above iteration construction "n times", but the concept of n fold iteration has not yet been rigorously defined.
For similar reasons, we cannot yet define sets consisting of

infinitely many objects because that would require iterating the axiom of pairwise union infinitely often and it is not clear at this stage that one can do this rigorously. Later on, we will introduce other axioms of set theory which allow us to construct arbitrarily large, infinite sets.

Clearly, some sets seem to be larger than others. One way to formalize this concept is through the notion of a subset.

Definition. Let A, B be sets. We say that A is a subset of B, denoted $A \subseteq B$, if and only if every element of A is also an element of B. That is,

For any object x, if $x \in A$, then $x \in B$.

We say that, A is a proper subset of B, denoted $A \subsetneq B$, if $A \subseteq B$ and $A \neq B$.

Examples.

1. We have $\{1,2,4\} \subseteq \{1,2,3,4,5\}$, because every element of $\{1,2,4\}$ is also an element of $\{1,2,3,4,5\}$. In fact, we also have $\{1,2,4\} \subsetneq \{1,2,3,4,5\}$, since the two sets $\{1,2,4\}$ and $\{1,2,3,4,5\}$ are not equal.

Proposition. Given any set A, we always have $A \subseteq A$ and $\emptyset \subseteq A$.

Proof.

Suppose an element x belongs to the set on the left hand side. Then, x is an element of the set on the right hand side.
$\Rightarrow A \subseteq A$ by definition.

Moreover, since $\emptyset \cup A = \{\} \cup A = A$, and $\emptyset \subseteq \emptyset \cup A$, we must have $\emptyset \subseteq A$.

The notion of subsets in set theory is similarly to the notion of less than equal to for numbers, as the following proposition demonstrates.

Proposition. Sets are partially ordered by set inclusion.

Let A, B, C be sets. If $A \subseteq B$ and $B \subseteq A$, then $A = B$. If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$. Finally, if $A \subsetneq B$ and $B \subsetneq C$ then $A \subsetneq C$.

Proof.

Claim: If $A \subseteq B$ and $B \subseteq A$, $A = B$.

If $A \subseteq B$, every element x of A is also an element of B.
If $B \subseteq A$, every element y of B is also an element of A.

$(A \subseteq B) \wedge (B \subseteq A) \Rightarrow (x \in A) \Leftrightarrow (x \in B)$ for all elements x.
Thus, $A = B$.

Claim: If $A \subseteq B$ and $B \subseteq C$, $A \subseteq C$.

Suppose $A \subseteq B$ and $B \subseteq C$.

Let us pick an arbitrary element x in A.

Then, since $A \subseteq B$, x must then be an element of B. But again since $B \subseteq C$, x must be an element of C. Since, x was an arbitrary element, this holds true for all elements in A.

so, $A \subseteq C$.

There is one important difference between the subset relation $\subseteq$ and the less than relation $<$. Given any two distinct natural numbers n, m, we know one of them is smaller than the other. However, given two distinct sets, it is not in general true, that one of them is a subset of the other. For ~~no~~ instance, take $A := \{2n : n \in \mathbb{N}\}$ to be the set of even natural numbers and $B := \{2n+1 : n \in \mathbb{N}\}$ to be the set of odd natural numbers. Then, neither set is a subset of the other. That is why, we say that sets are only partially ordered, whereas the natural numbers are totally ordered.

<u>Remark</u>. We should also caution that the subset relation $\subseteq$ is not the same as the element relation $\in$. The number 2 is an element of $\{1,2,3\}$ but not a subset; thus $2 \in \{1,2,3\}$ but $2 \notin \{1,2,3\}$. Indeed, 2 is not even a set. Conversely, while $\{2\}$ is a subset of $\{1,2,3\}$; it is not an element of the set; thus $\{2\} \subseteq \{1,2,3\}$ but $\{2\} \notin \{1,2,3\}$. The point is that the number 2 and the set $\{2\}$ are distinct objects. It is important to distinguish sets from their elements, as they can have different properties. For instance, it is possible to have an of infinite set consisting of finite numbers (the set $\mathbb{N}$ of natural numbers is one such example), and it is also possible to have a finite set of infinite objects (consider for instance the set $\{\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$ which has four elements, all of which are infinite.)

We now give an axiom which easily allows us to create subsets out of larger sets.

<u>Axiom of specification</u>. Let A be a set, and for each $x \in A$, let $P(x)$ be a property pertaining to x (that is $P(x)$ is either a true statement or false statement). Then there exists a set called $\{x \in A : P(x) \text{ is true}\}$ (or simply $\{x : P(x)\}$ for short) whose elements are precisely the elements x in A for which $P(x)$ is true. In other words, for any object y,

$$y \in \{x \in A : P(x) \text{ is true}\} \iff \{y \in A \text{ and } P(y) \text{ is true}\}.$$

This axiom is also known as the axiom of separation. Note that $\{x \in A : P(x) \text{ is true}\}$ is always a subset of A., though it could be as large as A, or as small as the empty set.

<u>Verification</u>.
Let us choose an arbitrary element in the set $\{x \in A : P(x) \text{ is true}\}$, and denote it by y. For all such y, $y \in A$. Thus, $\{x \in A : P(x)\} \subseteq A$.

<u>Examples</u>.

1. Let $S := \{1,2,3,4,5\}$. Then, the set $\{n \in S : n < 4\}$ is the set of those elements in S for which $n < 4$ is true. That is, $\{n \in S : n < 4\} = \{1,2,3\}$.

Similarly, the set $\{x \in S : x < 7\}$ is the same as S itself, while $\{x \in S ; x < 1\}$ is the empty set.

We sometimes write $\{x \in A \mid P(x)\}$ instead of $\{x \in A : P(x)\}$; this is useful when we are using colon : to denote something else, for instance to denote the range and domain of the function $f : X \to Y$. We can use this axiom of specification to define some further operations on sets, namely intersections and difference sets.

Definition (Intersections). The intersection $S_1 \cap S_2$ of two sets is defined to be the set

$$S_1 \cap S_2 := \{(x \in S_1) : (x \in S_2)\}$$

In other words, $S_1 \cap S_2$ consists of all elements which belong to both S_1 and S_2. Thus, for all objects x,

$$x \in S_1 \cap S_2 \Leftrightarrow (x \in S_1) \wedge (x \in S_2).$$

Examples. By the way, one should be careful with the English word "and": rather confusingly it can mean either union or intersection, depending on the context. For example, if one talks about a set of "boys and girls" one means the union of a set of boys ~~and~~ with a set of girls, but if one means about the set who are single and male, then one means the intersection of the set of single people with the set of male people. Another problem is that "and" is also used in English to denote addition, thus for instance one could say "2 and 3 is 5", while also saying that "the elements of $\{2\}$ and the elements of $\{3\}$ form the set $\{2,3\}$. This can certainly get confusing. One reason, we resort to mathematical symbols instead of English words such as "and" is that mathematical symbols always have a precise and unambiguous meaning, whereas one must look very carefully at the context in order to work out what an English word means.

Two sets A, B are said to be disjoint if $A \cap B = \phi$. Note that this is not the same concept as being distinct. For example, the sets $\{1,2,3\}$ and $\{2,3,4\}$ are distinct (there are distinct elements of one set which are not the elements of the other) but not disjoint (because their intersection is non-empty). Meanwhile, the sets ϕ and ϕ are disjoint, but not distinct.

Difference sets (Definition). Given two sets A and B, we define the set $A \setminus B$ or $A - B$ to be the set A with any elements of B removed.

$$A \setminus B := \{x : (x \in A) \wedge (x \notin B)\}$$

for instance $\{1,2,3,4\} \setminus \{2,4,6\} = \{1,3\}$.

We now give some basic properties of unions, intersections and difference sets.

Proposition. Sets form a boolean algebra.

Let A, B, C be sets and let X be a set containing A, B, C as subsets.

(a) Minimal element. We have $A \cup \phi = A$ and $A \cap \phi = \phi$.
(b) Maximal element. We have $A \cup X = X$ and $A \cap X = A$.
(c) Identity. We have $A \cup A = A$ and $A \cap A = A$.
(d) Commutativity. We have $A \cup B = B \cup A$ and $A \cap B = B \cap A$.
(e) Associativity. We have $(A \cup B) \cup C = A \cup (B \cup C)$. and $(A \cap B) \cap C = A \cap (B \cap C)$.
(f) Distributivity. We have $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
(g) Partition. We have $A \cup (X \setminus A) = X$ and $A \cap (X / A) = \phi$.

(h) De Morgan's laws. We have $X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B)$. and $X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$.

Proof.
(a) Claim. $A \cup \phi = A$.
$\Rightarrow$ direction.
Let x be an arbitrary element in $A \cup \phi$. Then by definition, this proposition will hold if and only if atleast one of $(x \in A)$ or $(x \in \phi)$ is true. But, $x \notin \phi$ for all elements x, since ϕ is empty. Thus, we are left only with $x \in A$.

$\Leftarrow$ direction.
A is always a subset of $A \cup \phi$.

So, $A \cup \phi = A$.

Claim. $A \cap \phi = \phi$
$\Rightarrow$ direction.
Consider an arbitrary element in $A \cap \phi$.
$x \in (A \cap \phi) \Rightarrow (x \in A) \wedge (x \in \phi)$
Both propositions must hold simultaneously.
But, by definition, $x \notin \phi$ for all elements x.
So, there are no elements x that satisfy both the propositions
$(x \in A) \wedge (x \in \phi)$.

Hence, $A \cap \phi$ is the empty set ϕ.

(b) Claim. $A \cup X = X$
$\Rightarrow$ direction.
Consider an arbitrary element $x \in (A \cup X)$

$x \in A \cup X \Rightarrow (x \in A) \vee (x \in X)$.

So, there are two possibilities. If $x \in A$, then $x \in X$, since $A \subseteq X$. If $x \in X$, then we are done. In both possibilities, $x \in X$.

$\Leftarrow$ direction.
Suppose $x \in X$. Since, $X \subseteq X \cup A$, it implies $x \in X \cup A$.

Hence, $A \cup X = X$.

(e) Identity.

Claim. $A \cup A = A$.

($\Rightarrow$ direction)

Consider an arbitrary element $x \in (A \cup A)$.

$x \in (A \cup A) \Rightarrow (x \in A) \vee (x \in A)$.

$\Rightarrow (x \in A)$.

For the left hand-side to be true, x must belong to A

($\Leftarrow$ direction

Consider an arbitrary element $x \in A$.

$x \in A \Rightarrow (x \in A) \vee (x \in A)$

$\Rightarrow (x \in A \cup A)$.

Claim. $A \cap A = A$.

($\Rightarrow$ direction.)

Consider an arbitrary element $x \in (A \cap A)$.

Then the proposition

$x \in (A \cap A)$

$\Rightarrow (x \in A) \wedge (x \in A)$.

$\Rightarrow (x \in A)$.

Both the propositions are identical and true. Hence, proved.

$\Leftarrow$ direction.

Consider any arbitrary element $x \in (A \cap A)$.

Then, the two cases hold simultaneously.

$x \in A \cap A)$

$\Rightarrow (x \in A) \wedge (x \in A)$

$\Rightarrow (x \in A)$.

This closes the proof.

(g) Partition.

Claim. $A \cup (X \setminus A) = X$.

Consider an arbitrary element $x \in A \cup (X \setminus A)$.

Then, atleast one of the below propositions hold:

(1) $x \in A$

(2) $x \in (X \setminus A)$.

If x is in A, and since $A \subseteq X$, x belonging to A implies $x \in X$.

If x is in $(X \setminus A)$, then $x \in X$, but $x \notin A$. Thus, x is still an element in X.

In both cases, $x \in X$.

In the opposite direction, let $x \in X$. A is a subset of X. so we partition it into two possibilities. Atleast one of these should be true.

$((x \in A) \wedge (x \in X)) \vee ((x \notin A) \wedge (x \in X))$

$\Rightarrow (x \in (A \cap X)) \vee (x \in X \setminus A)$

$\Rightarrow (x \in A) \vee (x \in X \setminus A)$.

$\Rightarrow x \in A \cup (X \setminus A)$

Claim. $A \cap (X \setminus A) = \phi$

($\Rightarrow$ direction).

Let x be an arbitrary element belonging to $A \cap (X \setminus A)$.
Then the below propositions hold simultaneously.

(1) $x \in A$

(2) $x \in (X \setminus A)$ that $x \in X$, $x \notin A$.

The two conditions are complementary and cannot hold simultaneously. Thus, no such x exists.

($\Leftarrow$ direction).

ϕ is a subset of any other set.

(h) De-Morgan's laws.

Claim $X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B)$.

($\Rightarrow$ direction).

Let x be an arbitrary element in $X \setminus (A \cup B)$.
Then,

$$x \in X \setminus (A \cup B)$$
$$\Rightarrow (x \in X) \wedge (x \notin A \cup B).$$
$$\Rightarrow (x \in X) \wedge \neg((x \in A) \vee (x \in B))$$
$$\Rightarrow ((x \in X) \wedge (x \notin A)) \wedge ((x \in X) \wedge (x \notin B))$$
$$\Rightarrow (x \in (X \setminus A)) \wedge (x \in (X \setminus B))$$
$$\Rightarrow x \in ((X \setminus A) \cap (X \setminus B))$$

($\Leftarrow$ direction.)

Let y be an arbitrary element in $(X \setminus A) \cap (X \setminus B)$.
Then,

$$y \in (X \setminus A) \wedge (X \setminus B)$$
$$\Rightarrow (y \in (X \setminus A)) \wedge (y \in X \setminus B)$$
$$\Rightarrow ((y \in X \wedge y \notin A)) \wedge ((y \in X) \wedge (y \notin B))$$
$$\Rightarrow (y \in X) \wedge \underbrace{(y \notin A) \wedge (y \notin B)}.$$

y belongs to none of the sets, is equivalent to saying, y is not in either of the sets.

$$\Rightarrow (y \in X) \wedge (y \notin (A \cup B))$$
$$\Rightarrow y \in X \setminus (A \cup B).$$

Claim. $X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$.

($\Rightarrow$ direction).

We will show that every element in $X \setminus (A \cup B)$ is contained in at least one of two sets $(X \setminus A)$ or $(X \setminus B)$.

Let x be an arbitrary element ~~to~~ in $X \setminus (A \cap B)$.
Then

$$x \in X \setminus (A \cap B)$$
$$\Rightarrow (x \in X) \wedge (x \notin (A \cap B))$$
$$\Rightarrow (x \in X) \wedge \neg (x \in A \cap B).$$
$$\Rightarrow (x \in X) \wedge ((x \notin A) \vee (x \notin B))$$
$$\Rightarrow ((x \in X) \wedge (x \notin A)) \vee ((x \in X) \wedge (x \notin B))$$
$$\Rightarrow (x \in (X \setminus A)) \vee (x \in (X \setminus B))$$
$$\Rightarrow x \in (X \setminus A) \cup (X \setminus B).$$

($\Leftarrow$ direction).

Let x be an arbitrary element in $(X \setminus A) \cup (X \setminus B)$. Then, atleast one of the below propositions hold.

(1) $x \in (X \setminus A)$.
(2) $x \in (X \setminus B)$.

$$(x \in X \setminus A) \vee (x \in X \setminus B)$$
$$\Rightarrow (x \in X \wedge x \notin A) \vee (x \in X \wedge x \notin B).$$
$$\Rightarrow (x \in X) \wedge (x \notin A \vee x \notin B).$$
$$\Rightarrow (x \in X) \wedge (x \notin A \cap B)$$
$$\Rightarrow x \in (X \setminus A \cap B).$$

(d) Commutativity.

Claim: $A \cup B = B \cup A$.

$\Rightarrow$ Let x be any arbitrary element in $A \cup B$. Then atleast one of the two possibilities hold: (1) $x \in A$ or (2) $x \in B$. If $x \in A$, then $x \in B \cup A$. If $x \in B$, then $x \in B \cup A$. In both cases, $x \in B \cup A$

It is an easy exercise to prove the converse.

Claim. $A \cap B = B \cap A$.

$\Rightarrow$ Let x be any arbitrary element in $A \cap B$. So, both $x \in A$ and $x \in B$ hold. Thus, $x \in B \cap A$.

$\Leftarrow$ direction
Let x be an arbitrary element of $B \cap A$.
$$\Rightarrow (x \in B) \wedge (x \in A)$$
$$\Rightarrow (x \in A) \wedge (x \in B)$$
$$\Rightarrow x \in (A \cap B)$$

(e) Associativity.

Claim. $A \cup (B \cup C) = (A \cup B) \cup C$.

Let x be any arbitrary element in $(A \cup B) \cup C$. We can divide it into two possibilities – either $x \in A$ or $x \in B \cup C$. If $x \in A$, then $x \in (A \cup B)$, and so x belongs to $(A \cup B) \cup C$. If $x \in B \cup C$, then atleast one of $(x \in B)$ and $(x \in C)$ holds. If $x \in B$, then $x \in A \cup B)$ and consequently $x \in (A \cup B) \cup C$. If $x \in C$, then $x \in (A \cup B) \cup C$. Thus, in all cases, we see that every element x of $A \cup (B \cup C)$ lies in $(A \cup B) \cup C$. A similar argument shows that every element x in $(A \cup B) \cup C$ lies in $A \cup (B \cup C)$.

Hence, $(A \cup B) \cup C = A \cup (B \cup C$.

Claim. $(A \cap B) \cap C = A \cap (B \cap C)$

Let x be any arbitrary element in $(A \cap B) \cap C$. Then, the element x lies in $(A \cap B)$ and it also lies in C. So, $(x \in A) \wedge (x \in B) \wedge (x \in C)$ Thus, $(x \in A) \wedge (x \in B \cap C)$. Consequently, $x \in A \cap (B \cap C)$.

It is an easy exercise to show that the converse is true. 23/11/2019.

Therefore, $(A\cap B)\cap C = A\cap(B\cap C)$.

The reader may observe a certain symmetry in the above laws between $\cup$ and $\cap$ and between X and ϕ. This is an example of duality – two distinct properties or objects being dual to each other. In this case, the duality is manifested by the complementation relation $A \to X\setminus A$; the de Morgan laws assert this relation converts unions into intersections and vice versa. It also interchanges X and the empty set. The above laws are collectively known as the laws of Boolean Algebra, after the mathematician George Boole (1815-1864) and are also applicable to a number of other objects other than sets; it plays a particularly important role in logic.

We have accumulated a number of axioms and results about sets, but there are still many things we are not able to do yet. One of the basic things we wish to do with a set is take each of the objects of that set and somehow transform each such object into a new object; for instance we may wish to start with a set of numbers, say $\{3,5,9\}$ and increment each one creating a new set $\{4,6,10\}$. This is not something we can do directly using only the axioms we have already, so we need a new axiom.

<u>Axiom of Replacement</u>. Let A be a set. For any object $x\in A$, and any object y, suppose we have a statement $P(x,y)$ pertaining to x and y, such that for each $x\in A$, there is at most one y for which $P(x,y)$ is true. Then, there exists a set $\{y : P(x,y)$ is true for some $x\in A\}$, such that for any object z,

$z \in \{y : P(x,y)$ is true for some $x\in A\}$
$\iff P(x,z)$ is true for some $x\in A$.

<u>Examples.</u>
1) Let $A := \{3,5,9\}$ and let $P(x,y)$ be the statement $y = x{+}{+}$ i.e. y is the successor of x. Observe that for every $x\in A$, there is exactly one y for which $P(x,y)$ is true – specifically, the successor of x. Thus, the above axiom asserts that the set $\{y : y = x{+}{+}$ for some $x\in A\}$ exists; in this case it is clearly the same set as $\{4,6,10\}$. (why?)

<u>Verification.</u>
$3{+}{+} = 4$
$5{+}{+} = 6$
$9{+}{+} = 10$

For each x in $\{3,5,9\}$, there is at most one y for which the statement $y = x{+}{+}$ is true.

2) Let $A = \{3,5,9\}$ and let $P(x,y)$ be the statement $y = 1$. Then again, for each $x\in A$, there is exactly one y for which $P(x,y)$ is true – specifically the number 1. In this case $\{y : y = 1$ for some $x\in\{3,5,9\}\}$ is just the singleton set $\{1\}$; we have replaced each element 3, 5, 9 of the original set A by the same object, namely 1.

Thus, this rather silly example shows that the set obtained by the above axiom can be "smaller" than the original set.

We often abbreviate a set of the form

$$\{y : y = f(x) \text{ for some } x \in A\}$$

as $\{f(x) : x \in A\}$ or $\{f(x) \mid x \in A\}$. Thus, for instance $A = \{3,5,9\}$ then $\{x{+}{+} : x \in A\}$ is the set $\{4,6,10\}$. We can of course combine the axiom of replacement with the axiom of specification, thus for instance we can create sets such as $\{f(x) : x \in A ; P(x)$ is true$\}$ by starting with the set A $\{x \in A : P(x)$ is true$\}$ and then applying the axiom of replacement to create $\{f(x) : x \in A$ ~~at~~; $P(x)$ is true$\}$. Thus, for instance $\{n{+}{+} : n \in \{3,5,9\}; n < 6\} = \{4,6\}$.

~~Axiom 6~~

For many of our examples, we have implicitly assumed that natural numbers are in fact objects. Let us formalize this as follows -

Axiom (Infinity). There exists a set $\mathbb{N}$, whose elements are called natural numbers, as well as an object 0 in $\mathbb{N}$, and an object $n{+}{+}$ assigned to every natural number $n \in \mathbb{N}$, such that the Peano axioms hold.

This the more formal version of assumption 2.6. It is called the axiom of infinity, because it introduces the most basic example of an infinite set, namely the set of natural numbers $\mathbb{N}$. We will formalize what finite and infinite mean shortly. From the axiom of infinity, we see that the numbers such as 3, 5, 7 etc are objects in set theory, and so (from the pair set axiom and the pairwise union axiom) we can legitimately construct sets such as $\{3,5,9\}$ as we have been doing in our examples.

One has to keep the concept of a set distinct from the elements of that set; for instance, the set $\{n+3 : n \in \mathbb{N}, 0 \le n \le 5\}$ is not same thing as the same as the expression or the function $n+3$. We emphasize this with an example:

Examples.

1) This example requires the notion of subtraction, which has not yet been formally introduced. The following two sets are equal,

$$\{n+3 : n \in \mathbb{N}, 0 \le n \le 5\} = \{8-n : n \in \mathbb{N}, 0 \le n \le 5\}.$$

even though the expressions $n+3$ and $8-n$ are never equal to each other for any natural number n. Thus, it is a good idea to remember to use curly braces $\{\}$ when you talk about sets, lest you accidentally confuse a set with its elements. One reason for this counter-intuitive situation is that the letter n is being used in two different ways for th on the two sides of the above equality. To clarify the situation, let us rewrite the set $\{8-n : n \in N, 0 \le n \le 5\}$ by replacing the letter n by the letter m, thus giving $\{8-m : m \in N; 0 \le m \le 5\}$. This is exactly the same set as before. (Why?) so we can ~~so~~ re-write it as:

$$\{n+3 : n \in N; 0 \le n \le 5\} = \{8-m : m \in N; 0 \le m \le 5\}.$$

Now, it is easy to see why this identity is true: every number of the form $n+3$, where n is a natural number between 0 and 5, is also of the form $8-m$ where $m := 5-n$ (note that m is therefore also a natural number between 0 and 5); conversely every number of the form $8-m$, where m is a natural number between 0 and 5, is also of the form $n+3$, where ~~n:5~~ $n := 5-m$ (note that therefore n is a natural number between 0 and 5). Observe how much more confusing the explanation would have been, if we had not changed one of the n's to an m first.

Problems.

1) Show that the definition of set equality is reflexive, symmetric and transitive.

Proof.

(1) Reflexive

$A = A$ holds.

This is because, for each n in the left hand side set A $\Leftrightarrow$ n belong to the right hand side set.

(2) Symmetric

If $A = B$, then $B = A$.

Assume that $A = B$.

Therefore, $(x \in A) \Leftrightarrow (x \in B)$,

so, $B = A$.

(3) Transitive.

If $A = B$ and $B = C$, then $A = C$.

Since $A = B$,

$$(x \in A) \Leftrightarrow (x \in B) \qquad (a)$$

Also since $B = C$,

$$(x \in B) \Leftrightarrow (x \in C). \qquad (b)$$

From (a) and (b),

$$(x \in A) \Leftrightarrow (x \in C).$$

$$\therefore A = C.$$

2) Let A, B be sets. Show that the three statements $A \subseteq B$, $A \cup B = B$ and $A \cap B = A$ are logically equivalent. (any one of them implies the other two).

Proof.

(a) $1 \Rightarrow 2$.

Assume that $A \subseteq B$.

Let x be an arbitrary element in A.
$A \subseteq B$ implies that, if $x \in A$, then $x \in B$.

By definition, $A \cup B$ is the set of all x for which at least one of $(x \in A)$ or $(x \in B)$ holds. If $x \in A$, then $x \in B$. If $x \in B$, certainly $(x \in B)$ holds. In both cases, $x \in B$. Further, the converse is also true since, $B \subseteq B \cup A$. Thus, $A \cup B = B$.

(b) $(2) \Rightarrow (3)$.

We are given that $A \cup B = B$ and we are interested to show that $A \cap B = A$.

$\Rightarrow$ direction.
$A \cap B \subseteq A$. This is trivial.

$\Leftarrow$ direction.
We are given that $A \cup B = B$. That means, if $x \in A \Rightarrow x \in B$.

Now, suppose x is an arbitrary element in A.

$$x \in A \Leftrightarrow (x \in A) \wedge (x \in B) \quad \text{as } x \in A \Rightarrow x \in B.$$
$$\Leftrightarrow x \in A \cap B.$$

Therefore, $A \subseteq A \cap B$.

We conclude, $A \cap B = A$.

A.

(c) $3 \Rightarrow 1$.

We are given that $A \cap B = A$
so, $(x \in A) \wedge (x \in B) \Rightarrow x \in A$.
$(x \in A) \Rightarrow (x \in A) \wedge (x \in B)$.

Now, suppose x is an arbitrary element in A.

$$(x \in A) \Leftrightarrow (x \in A) \wedge (x \in B)$$
$$\Leftrightarrow (x \in B) \text{ must be true.}$$

so, $A \subseteq B$.

3. Let A, B, C be sets. Show that $A \cap B \subseteq A$ and $A \cap B \subseteq B$. Furthermore, show that $C \subseteq A$ and $C \subseteq B$ if and only if $C \subseteq A \cap B$. In a similar spirit show that $A \subseteq (A \cup B)$ and $B \subseteq A \cup B$

Proof.

1) $A \cap B \subseteq A$ and $A \cap B \subseteq B$ is trivial.

By definition, $x \in (A \cap B)$

$\Leftrightarrow (x \in A) \wedge (x \in B)$.

This is equivalent to saying,

$(x \in A \cap B \Rightarrow x \in A) \wedge (x \in A \cap B \Rightarrow x \in B)$.

$\Leftrightarrow (A \cap B \subseteq A) \wedge (A \cap B \subseteq B)$.

This closes the proof.

2) Claim: $C \subseteq A$ and $C \subseteq B$ if and only if $C \subseteq A \cap B$.

$\Rightarrow$ direction.

$(C \subseteq A) \wedge (C \subseteq B)$

$\Leftrightarrow (x \in C \Rightarrow x \in A) \wedge (x \in C \Rightarrow x \in B)$

$\Leftrightarrow x \in C \Rightarrow ((x \in A) \wedge (x \in B))$

$\Leftrightarrow x \in C \Rightarrow x \in A \cap B$.

$\Leftrightarrow C \subseteq A \cap B$.

$\Leftarrow$ direction.

$C \subseteq A \cap B$

$\Leftrightarrow x \in C \Rightarrow x \in (A \cap B)$

$\Leftrightarrow x \in C \Rightarrow (x \in A) \wedge (x \in B)$

$\Leftrightarrow (x \in C \Rightarrow x \in A) \wedge (x \in C \Rightarrow x \in B)$

$\Leftrightarrow (C \subseteq A) \wedge (C \subseteq B)$.

This closes the proof.

3) By definition,

$A \cup B := \{x : x \in A \vee x \in B\}$.

so, if $x \in A \Rightarrow x \in A \cup B$.

$A \subseteq A \cup B$.

Also, if $x \in B \Rightarrow x \in A \cup B$.

$B \subseteq A \cup B$.

This closes the proof.

4. Let A, B be sets. Prove the absorption laws:

$A \cap (A \cup B) = A$.

$A \cup (A \cap B) = A$.

Proof.

Claim.

$A \cap (A \cup B) = A$.

($\Rightarrow$ direction).

$A \cap (A \cup B) \subseteq A$

This is trivial, as $A \cap X \subseteq A$ for any set X.

($\Leftarrow$ direction)

$A \subseteq A \cap (A \cup B)$.

Let x be an arbitrary element in A. Then, $x \in A$ and $x \in A \cup B$. So, $x \in A \cap (A \cup B)$.

so, $A \subseteq A \cap (A \cup B)$.

Claim. $A \cup (A \cap B) = A$.
($\Rightarrow$ direction).
Let x be an arbitrary element of the set $A \cup (A \cap B)$.
Then, atleast one of the statements $(x \in A)$ or
$x \in (A \cap B)$ must hold. If $x \in A$, then we are done.
If $x \in (A \cap B)$, then $(x \in A) \wedge (x \in B)$. so, $(x \in A)$ in
particular. Thus, in all cases, we see that every element
of $A \cup (A \cap B)$ lies in A.

($\Leftarrow$ direction)
Every element x of A lies in $A \cup (A \cap B)$. This is trivial.

$\therefore A \cup (A \cap B) = A$.

5. Let A, B, X be sets such that $A \cup B = X$ and $A \cap B = \phi$. Show that,
$A = X \setminus B$ and $B = X / A$.

Proof.

Claim $A = X \setminus B$.

($\Rightarrow$ direction)
Let x be an arbitrary element in A
$x \in A$
$\therefore x \in A \cup B$
$\therefore x \in X$
$\therefore (x \in B) \vee (x \in X \setminus B)$.
If $x \in B$, then $x \in A \cap B$. But, we are given that $A \cap B = \phi$.
so, $x \in X \setminus B$.
As a result, $A \subseteq X \setminus B$.

($\Leftarrow$ direction).
Let y be an arbitrary element in $X \setminus B$.
$(x \in X) \wedge (x \notin B)$
$\Leftrightarrow (x \in A \vee x \in B) \wedge (x \notin B)$
$\Leftrightarrow x \in A$.

so, $X \setminus B \subseteq A$.

Therefore, $A = X \setminus B$.

This closes the proof.

Thus, the sets A and B are complements with respect to X.
It is an easy exercise to prove that $B = X \setminus A$.

6. Let A and B be sets. Show that the three set $A \setminus B$, $A \cap B$ and
$B \setminus A$ are disjoint, and that their union is $A \cup B$.

Proof.

If $x \in A \setminus B$, then $(x = A) \wedge (x \notin B)$. Thus, the statement
$(x \in A) \wedge (x \in B)$ will not hold. so, $x \notin A \cap B$. These are
disjoint sets.

If $x \in A \cap B$
$\iff (x \in A) \wedge (x \in B)$

so, $(x \in B) \wedge (x \notin A)$ is false. Therefore, $x \notin B \setminus A$.
The two sets $A \cap B$ and $B \setminus A$ are disjoint.

Further, if $x \in A \setminus B$
$\iff (x \in A) \wedge (x \notin B)$

so, $(x \in B) \wedge (x \notin A)$ is false.

$A \setminus B$ and $B \setminus A$ are disjoint sets.

The second claim is easy to prove as well. Consider an arbitrary element x in the union of the three sets. Atleast, one of the below propositions must be true.

$(x \in A) \wedge (x \notin B) \Rightarrow x \in A \Rightarrow x \in A \cup B$.

$(x \in A) \wedge (x \in B) \Rightarrow x \in A \cup B$.

$(x \notin A) \wedge (x \in B) \Rightarrow x \in B \Rightarrow x \in B \cup A$.

In all three cases, $x \in A \cup B$.

In the opposite direction, if x is an arbitrary element in $A \cup B$, then we have a trichotomy. Atleast one of the three propositions must be true.

$$(x \in A \wedge x \notin B) \Rightarrow x \in A \setminus B$$
$$\vee \ (x \in A \wedge x \in B) \Rightarrow x \in A \cap B.$$
$$\vee \ (x \notin A \wedge x \in B) \Rightarrow x \in B \setminus A.$$

This completes the proof.

~~Russell's paradox~~
~~Many of the axioms introduced in the previous section have a similar flavour.~~

Cartesian Product (Definition). If A and B are non-empty sets, then the cartesian product $A \times B$ of A and B is the set of all ordered pairs (a, b) with $a \in A$ and $b \in B$. That is,

$$A \times B := \{(a, b) : a \in A, b \in B\}.$$

Thus, if $A = \{1, 2, 3\}$ and $B = \{1, 5\}$, then the set $A \times B$ is the set whose elements are the ordered pairs —

$$(1,1), (1,5), (2,1), (3,1), (3,5).$$