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(g) Ages of a couple. An insurance company is interested in the age distribution of couples. Let x stand for the age of the husband, y for the age of the wife. Each observation results in a number-pair (x, y). For the sample space corresponding to a single observation, we take the first quadrant of the xy-plane, so that each point $x>0, y>0$ is a sample point. The event A, "husband is older than 40", is represented by all points ~~the~~ to the right of the line $x=40$; the event B, "husband is older than wife, is represented by the angular region between the x-axis and the bisector $y=x$, that is to say, by the aggregate of points with $x>y$; the event C, "wife is older than 40", is represented by the portion of the first quadrant above the line $y=40$. For a geometric representation of the joint age distributions of two couples we would require a four-dimensional space.

(h) Phase space. In statistical mechanics, each possible state of a system is called a point in "phase space." This is only a difference in terminology. The phase space is simply our sample space; its points are our sample points.

3. The sample space - events.
It should be clear from the preceding that we shall never speak of probabilities except in relation to a given sample space (or physically in relation to a given conceptual experiment). We start with the notion of a sample space and its points; from now on they will be considered given. They are primitive

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and undefined notions of the theory precisely as the notions of points and straight lines remain undefined in an axiomatic treatment of Euclidean geometry. The nature of the sample points does not enter our theory. The sample-space provides a model of an ideal experiment in the sense that, by definition every thinkable outcome of the experiment is completely described by one and only one sample point. It is meaningful to talk about an event A only when it is clear for every outcome of the experiment whether the event A has occurred or not. The collection of all those sample points representing outcomes where A has occurred completely describes the event. Conversely, any given aggregate A containing one or more sample points can be called an event; this event does or does not occur, according as the outcome of the experiment is, or is not represented by a point of the aggregate A. We therefore defined the word event to mean the same as an aggregate of sample points. We shall say that an event A consists of (or contains) ~~as~~ certain points, namely those representing outcomes of the ideal experiment in which A occurs.

Example. In the sample space of the example (2.a) consider the event U consisting of the points number 1, 5, 17. This is a formal, straight-forward definition, but U can be described in many equivalent ways. For example, U can be defined as the event that the following three conditions are satisfied (1) the second cell is empty (2) the ball a is in the first cell (3) the ball b does not appear after c. Each of these conditions itself describes an event. The event U_1 defined by the condition (1) alone consists of the points 1, 3, 5, 8, 11, 14, 17, 20. The event U_2 defined by (2) consists of 1, 4, 5, 10, 11, 16, 17, 22, 24.

and the event U_3 defined by (3) contains the points 1-9, 16 ~~For~~ The event U can also be described as the simultaneous realization of all three events U_1, U_2, U_3.

The terms sample point and event have an intuitive appeal, but they refer to the notions of point and point set common to all parts of mathematics.

We have seen in the preceding example and in (2.a) that new events can be defined in terms of two or more given events. With these examples in mind, we now proceed to introduce the notation of the formal algebra of events (that is algebra of point sets).

4. Relations among events.
We shall now suppose that an arbitrary, but fixed, sample space S is given. We use capitals to denote events, that is, sets of sample points. The fact that a point x is contained in the event A is denoted by $x \in A$. Thus, $x \in S$ for every point x. We write $A = B$, if and only if the two events consist of exactly the same points.

In general, events will be defined by certain conditions on their points, and it is convenient to have a symbol to express the fact that no point satisfies a specified set of conditions. The next definition serves this purpose.

Definition 1. We shall use the notation $A = \phi$ to express that the event A contains no sample points (is impossible).

To every event A, there corresponds another event defined by the condition that A does not occur. It contains all points not contained in A.

Definition 2. The event consisting of all points not contained in the event A will be called the complementary event of A and will be denoted by A^c. In particular, $S^c = \phi$.

With any two events A and B we can associate two new events defined by the conditions "both A and B occur" and "either A or B or both occur". These events will be denoted by $A \cap B$ and $A \cup B$ respectively. The event $A \cap B$ contains all sample points which are common to A and B. If A and B exclude each other, then there are no points common to A and B and the event $A \cap B$ is impossible. Analytically, this situation is described by the equation

$$A \cap B = \phi.$$

which should be read as A and B are mutually exclusive. The event $A \cap B^c$ means that both A and B^c occur or, in other words, that A but not B occurs. Similarly, $A^c \cap B^c$ means that ~~neig~~ neither A nor B occurs. The event $A \cup B$ means that atleast one of the events A and B occurs; it contains all sample points except those that belong neither to A nor to B.

In the theory of probability, we can describe the event $A \cap B$ as the simultaneous occurrence of A and B. In standard mathematical terminology $A \cap B$ is called the logical intersection of A and B. Similarly, $A \cup B$ is the union of A and B. Our notion carries over to the case of the events A, B, C, D, ...

Definition 3. To every collection $A, B, C, \ldots$ of events we define two new events as follows. The aggregate of the sample points which belong to all the given sets will be denoted by $A \cap B \cap C \cap \ldots$ and called the intersection (or simultaneous realization) of $A, B, C, \ldots$ The aggregate of sample points which belong to atleast one of the given sets will be denoted by $A \cup B \cup C \ldots$ and called the union (or realization of atleast one) of the given events. The events A, B, C are mutually exclusive if no two have a point in common, that is, if $A \cap B = \phi$, $A \cap C = \phi$, $B \cap C = \phi, \ldots$

We still require a symbol to express the statement that A cannot occur without B occurring, that is, the occurrence of A implies the occurrence of B. That means that every point of A is contained in B. Think of intuitive analogies like the aggregate of all mothers, which forms a part of the aggregate of all women. All mothers are women, but not all ~~mother~~ women are mothers.

Definition 4. The symbols $A \subset B$ and $B \supset A$ are equivalent and signify that every point of A is contained in B; they are read respectively "A implies B" and "B is implied by A". If this is the case we shall also write $B - A$ instead of $B \cap A^c$ to denote the event that B but not A occurs.

The event $B - A$ contains all those points which are in B but not in A. With these notations, we can write $A^c = S - A$ and $A \cap A^c = \phi$.

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Examples.

(a) If A and B are mutually exclusive, then the occurrence of A implies the non-occurrence of B and vice versa. Thus, $AB = \phi$ means the same as $A \subset B^c$ and as $B \subset A^c$.

(b) The event $A - AB$ means the occurrence of A but not of both A and B. Thus, $A - AB = AB^c$.

(c) In the example, (2·g), the event AB means the husband is older than 40 and older than his wife, AB^c means that he is older than 40, but not older than his wife. AB is represented by the infinite trapezoidal region region between the X-axis and the lines $x = 40$ and $y = x$, ~~the~~ and the event AB^c is represented by the angular domain between the lines $x = 40$ and $y = x$, the latter boundary included. The event AC means both the husband and wife are older than 40. The event $A \cup C$ means that atleast one of them is older than 40, and $A \cup B$ means that the husband is older than 40 or, if not atleast older than his wife.

(d) In example let E_i be the event that the cell number i is empty (here $i = 1, 2, 3$). Similarly, let S_i, D_i, T_i, respectively denote that the cell number i is occupied simply, doubly or triply. Then, $E_1 E_2 = T_3$, and $S_1 S_2 \subset S_3$ and $D_1 D_2 = \phi$. Note also that $T_1 \subset E_2$, etc. The event $D_1 \cup D_2 \cup D_3$ is defined by the condition that there exists atleast one doubly occupied cell.

(e) Bridge. Let A, B, C, D be events respectively, that North, South, East, West have atleast one ace. It is clear that atleast one player has an ace, so that one or more of the four the events must occur. Hence, $A \cup B \cup C \cup D = S$ is the whole sample space. The event ABCD occurs if, and only if, each player has an

"West has all four aces" means that none ace. The event "West has all four aces" means that none of the three events A, B, C has occurred. This is the same as the simultaneous occurrence of A^c and B^c and C^c, or the event $A^cB^cC^c$.

(f) In the example (2·g), we have $BC \subset A$; in words, if the husband is older than the wife (B) and the wife is older than 40 (C), then the husband is older than 40 (A). How can the event $A - BC$ be described in words?

5. Discrete sample spaces.

The simplest sample spaces are those containing only a finite number n, of points. If n is fairly small (as in the case of tossing a few coins), it is easy to visualize the space. The space of distribution of cards in bridge is more complicated, but we may imagine each sample point represented on a chip and may then consider the collection of all these chips as representing the sample space. An event A (like "North has two aces") is represented by a certain set of chips, the complement A^c by the ~~rep.~~ remaining ones. It takes only one step from here to imagine a bowl with infinitely many chips or a sample space with an infinite sequence of points.

Examples.

(a) Let us toss a coin as often as necessary to turn up one head. The points of the sample space are then $E_1 = H$, $E_2 = TH$, $E_3 = TTH$, $E_4 = TTTH$ etc. We may or may not consider as thinkable that H never appears. If we do this, the possibility should be represented by a point E_0.

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