the force. For a conservative dynamical system, $Q_i = -\frac{\partial V}{\partial q_i}$, $V = V(q_i)$, $\frac{\partial V}{\partial \dot{q}_i} = 0$,

then the above equation can be expressed ~~as~~ in terms of the Lagrangian, $L = T - V$, as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

This principle was reformulated by Euler in a way that made it useful in mathematics and physics.

With the rapid development of the theory and applications of differential equations, the closed form analytical solutions of many different types of equations were hardly possible. However, it is extremely important and absolutely necessary to provide some insight into qualitative and quantitative nature of solutions subject to initial and boundary conditions. This insight usually takes the form of numerical and graphical representatives of solutions. It was E. Picard (1856-1941) who first developed the method of successive approximations for the solutions of differential equations in the most general form and later made it an essential part of his treatment of differential equations in the second volume of his Traité d'Analyse published in 1896. During the last two centuries, the calculus of finite differences in various forms played a significant role in finding the numerical solutions of differential equations. Historically, many well-known integration formulas and numerical methods including the Euler-Maclaurin formula, Gregory integration formula, Simpson's rule, Adam-Bashforth's method, the Jacobi iteration, the Gauss-Siedel method and the Runge-Kutta method have been developed and then generalized in various forms.

With the development of modern calculators and high-speed electronic computers, there has been an increasing trend in research ~~toward~~ towards the numerical solution of ordinary and partial differential equations during the twentieth century. Many well known numerical methods including the Crank-Nicolson methods, the Lax-Wendroff method, Richtmyer's method and Stone's implicit iterative technique have been developed in the second-half of the twentieth century. All finite difference methods reduce differential equations to discrete forms. In recent years more modern and powerful computational methods such as the finite element method and the boundary element method have been developed to handle curved or irregularly shaped domains.

2. Introduction

There are two common notations for partial derivatives and we shall employ them interchangeably. The first used in (1.1) and (1.2) is the familiar Leibnitz notation that employs a d ~~der~~ to denote ordinary derivatives and the ∂ symbol (usually also pronounced 'del') for partial derivatives of functions of more than one variable. An alternative more compact notation employs subscripts to indicate partial derivatives. For example, u_t represents $\partial u/\partial t$, while u_{xx} is used for $\partial^2 u/\partial x^2$ and $\partial^3 u/\partial^2 x \partial y$ for u_{xxy}. Thus, in subscript notation, the partial differential equation (1.2) is written

$$u_t = u_{xx} + u_{yy} - u. \qquad (1.3)$$

We will similarly abbreviate partial differential operators sometimes writing $\partial/\partial x$ as ∂_x while $\partial^2/\partial x^2$ can be written as either ∂_x^2 or ∂_{xx}

and $\partial^3/\partial x^2 \partial y$ becomes $\partial_{xxy} = \partial_x^2 \partial y$.

It is worth pointing out that the preponderence of differential equations arising in applications, in science, in engineering, and within mathematics itself are of either first or second order, with the latter being by far the most prevalent. Third-order equations arise when modelling waves in dispersive media e.g. water waves or plasma waves. Fourth-order equations show up in elasticity, particularly plate and beam mechanics and in image processing.

A basic pre-requisite for studying this text is the ability to solve simple ordinary differential equations: linear constant coefficient equations and linear systems. In addition, we shall assume some familiarity with the basic theorems concerning the existence and uniqueness of solutions to initial value problems. Partial differential equations are considerably more demanding and can challenge the analytical skills of even the most accomplished mathematician. Many of the most effective strategies rely on reducing the partial differential equation to one or more ordinary differential equations. Thus, in the course of our study of partial differential equations, we will need to develop, ab initio, some of the more advanced aspects of the theory of ordinary differential equations, including boundary-value problems, eigenvalue problems, series solutions, singular functions and special functions.

Following the introductory remarks in the present chapter, the exposition begins in earnest with simple first-order equations concentrating on those arising that arise in as models of wave phenomena. Most of the remainder of the text will be devoted to understanding and solving the three essential linear second-order partial differential equations in one, two and three space dimensions: the heat equation modelling thermodynamics in a continuous medium, as well as the diffusion of animal population and chemical pollutants; the wave equation modelling vibrations of bars, strings, plates and solid bodies as well as acoustic, fluid, and electromagnetic vibrations; and the Laplace equation (Potential equation) and its inhomogenous counterpart, the Poisson equation, governing mechanical and thermal equilibria of bodies, as well as fluid-mechanic and electromagnetic potentials.

Each increase in dimension requires an increase in mathematical sophistication, as well as the development of additional analytic tools, as well as the development of additional analytic tools — although the key ideas will have all appeared once we reach our physical, three-dimensional universe. The three starring examples, heat/wave and Laplace/Poisson — are not only essential to a wide range of applications, but also serve as instructive paradigms for the three principal classes of ~~non~~-linear partial differential equations — parabolic, hyperbolic and elliptic. Some interesting non-linear partial differential equations, including first-order transport equations modelling shock waves, the second-order Burgers's equation governing non-linear diffusion processes, and the third-order Korteweg-De-Vries equation governing dispersive waves will also be discussed. But, in such an introductory course, the further reaches of the vast realm of non-linear partial differential equations must remain unexplored awaiting the readers more advanced mathematical excursions.

More generally, a system of differential equations is a collection of one or more equations relating the derivatives of one or more functions

It is essential that all functions occurring in the system depend on the same set of variables. The symbols representing these functions are known as the dependent variables, while the variables they depend on are called independent variables. Systems of differential equations are called ordinary or partial according to whether there are one or more independent variables.

The order of the system is the highest order derivative occurring in any one of its equations.

For example, the three-dimensional Navier-Stokes equations

$$
\begin{aligned}
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} &= -\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \\
\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} &= -\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) \qquad (1.4) \\
\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} &= -\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right).
\end{aligned}
$$

is a second-order system of differential equations that involves four functions $u(t,x,y,z)$, $v(t,x,y,z)$, $w(t,x,y,z)$, $p(t,x,y,z)$, each depending on four variables while $\nu \geq 0$ is a fixed constant. (The function p necessarily depends on t, even though no t derivative of it appears in the system.) The independent variables are t representing time and x, y, z representing space coordinates. The dependent variables are u, v, w, p with $\vec{v} = (u, v, w)$ representing the velocity vector field of an incompressible fluid flow e.g. water and p the accompanying pressure. The parameter ν measures the viscosity of the fluid. The Navier-Stokes equations are fundamental in fluid mechanics and notoriously difficult to solve either analytically or numerically. ~~Either establishing~~ Indeed establishing the existence or non-existence of solutions for all future times remains a major unsolved problem in mathematics, whose resolution will earn you a \$1,000,000 prize (see http://claymath.org) for details. The Navier-Stokes equations first appeared in the early 1800s in works of the French applied mathematician/engineer Claude-Louis Navier and later the British applied mathematician George Stokes, whom you already know from his eponymous multivariable calculus theorem. The inviscid case, $\nu = 0$, is known as Euler's equations in honor of their discoverer, the incomparably influential eighteenth century Swiss Mathematician Leonard Euler.

We shall be employing a few basic notational conventions regarding the variables that appear in our differential equations. We always use t to denote time, while (x, y, z) represent (Cartesian) space coordinates. Polar coordinates r, θ, cylindrical coordinates r, θ, z and spherical coordinates r, θ, ϕ will also be used when needed. An equilibrium equation models an unchanging physical system and so involves only the space variables. The time variable appears when modeling dynamical, meaning time-varying, processes. Both time and space coordinates are usually independent variables. The dependent variables will mostly be denoted by u, v, w although occasionally - particularly in representing particular physical quantities - other letters may be employed, e.g. the pressure p in (1.4). On the other hand, the letters f, g, h, typically represent specified functions of the independent variables e.g. forcing or boundary or initial conditions.

In this introductory text, we must confine our attention ~~to~~ to the most basic analytic and numerical solution techniques for a select few of the most important partial differential equations. More advanced topics, including all systems of partial differential equations, must be deferred to graduate and research-level texts. In fact many important issues remain incompletely resolved and / or poorly understood making partial differential equations one of the most active and exciting fields of contemporary mathematical research. One of my goals is that, by reading this book you will be both inspired and equipped to venture much further into this fascinating and essential area of mathematics and/or its remarkable range of applications throughout science, engineering, economics, biology and beyond.

Problems.

1) Classify each of the following differential equations as ordinary, partial and equilibrium or dynamic; then write down its order.

(a) $\frac{du}{dx} + xu = 1$

Ordinary differential equation, first order.
Equilibrium equation.

b) $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = x,$

First-order partial differential equation
Dynamic equation.

c) $u_{tt} = 9u_{xx}.$
Second-order partial differential equation
Dynamic equation.

d) $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}.$

Second-order partial differential equation
Dynamic equation.

e) $-\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = x^2 + y^2$

Second-order partial differential equation
Dynamic equation.

f) $\frac{d^2 u}{dt^2} + 3u = \sin t$

Second-order ordinary differential equation.
Dynamic equation.

g) $u_{xx} + u_{yy} + u_{zz} + (x^2+y^2+z^2)u = 0,$

Second-order partial differential equation
~~Dynamic~~ Equilibrium equation.

h) $u_{xx} = x + u^2.$

Second-order partial differential equation
Equilibrium equation.

i) $\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} + u\frac{\partial u}{\partial x} = 0,$

Third-order partial differential equation
Dynamic equation

j) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y \partial z} = u,$

Second-order partial differential equation
Equilibrium equation.

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A) $u_{tt} = u_{xxxx} + 2u_{xxyy} + u_{yyyy}$.
Fourth order partial differential equations.
& Dynamic equation.

2) In two space dimensions, the Laplacian is defined as the second-order partial differential operator $\Delta = \partial_x^2 + \partial_y^2$. Write out the following partial differential equations in (i) Leibniz notation (ii) subscript notation:
(a) the Laplace equation $\Delta u = 0$; (b) the Poisson equation $-\Delta u = f$; the two dimensional heat equation $\partial_t u = \Delta u$; (d) the Von Karman plate equation $\Delta^2 u = 0$.

Solution.

(a) $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$
$$u_{xx} + u_{yy} = 0$$

(b) $$-\frac{\partial^2 u}{\partial x^2} + -\frac{\partial^2 u}{\partial y^2} = f(x,y)$$
$$-u_{xx} - u_{yy} = f(x,y).$$

(c) $$\frac{\partial u}{\partial t} = -\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$
$$u_t = -(u_{xx} + u_{yy}).$$

(d) $$\frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} = 0.$$
$$u_{xxxx} + 2u_{xxyy} + u_{yyyy} = 0.$$

3) Answer exercise (1.2) for the three-dimensional Laplacian.
Solution.

(a) $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$
$$u_{xx} + u_{yy} + u_{zz} = 0.$$

(b) ~~$-u_{xx} - u$~~
$$-\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial z^2} = f(x,y,z)$$
$$-u_{xx} - u_{yy} - u_{zz} = f.$$

(c) $$\frac{\partial u}{\partial t} = -\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right).$$
$$u_t = -(u_{xx} + u_{yy} + u_{zz}).$$

(d) $$\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4} + \frac{\partial^4 u}{\partial z^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + 2\frac{\partial^4 u}{\partial y^2 \partial z^2} + 2\frac{\partial^4 u}{\partial x^2 \partial z^2} = 0.$$
$$u_{xxxx} + u_{yyyy} + u_{zzzz} + 2u_{xxyy} + 2u_{yyzz} + 2u_{zzxx} = 0.$$

4) Identify the independent variables, the dependent variables and the order of the following systems of partial differential equations.

(a) $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x};$$

This is a first-order system of differential equations. The independent variables are x, y. The dependent variables are u, v. The independent variables are x, y.

b) $u_{xx} + u_{yy} = \cos(x+y)$,
$u_x v_y - u_y v_x = 1$

Here This is a second-order system of differential equations.
x, y are the independent variables.
u, v are the dependent variables.

c) $\frac{\partial u}{\partial t} = \frac{\partial v}{\partial x}, \frac{\partial^2 v}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$.

This is a second-order system of differential equations.
u, v are the dependent variables.
x, t are the independent variables.

d) $u_t + u u_x + v u_y = p_x$
$v_t + u v_x + v v_y = p_y$,
$u_x + v_y = 0$;

This is a first-order system of differential equations.
u, v, p are the dependent variables.
t, x, y are the independent variables.

e) $u_t = v_{xxx} + v(1-v)$,
$v_t = u_{xxy} + vw$
$w_t = u_x + u_y$.

This a third order system of differential equations.
Here, u, v, w are the dependent variables.
t, x, y are the independent variables.

3. Classical solutions.

Let us now focus our attention on a single differential equation involving a single scalar-valued function u that depends on one or more independent variables. The function u is usually real-valued, although complex functions can and do, play a role in the analysis. Everything that we say in this section will, when suitably adapted, apply to systems of differential equations.

By a solution, we mean a sufficiently smooth function u of the independent variables that satisfies the differential equation at every point of its domain of definition. We do not necessarily require that the solution be defined for all possible values of the independent variable. Indeed, usually the differential equation is imposed on some domain D contained in the space of independent variables, and we seek a solution defined only on D. In general, the domain D will be an open subset usually connected, and in particular in equilibrium equations, often bounded, with a reasonably nice boundary, denoted by ∂D.

We will call a function smooth if it can be differentiated sufficiently often, atleast so that all of the derivatives appearing in the equation are well-defined on the domain of interest D. More specifically, if the differential equation has order n, then we require that the solution u must belong to function space $C^n(D)$, which means that it and all its derivatives of order $\leq n$ are continuous functions in D and such that the differential equation that relates the derivatives of u holds throughout D. However, on occasion when dealing with shock waves, we will consider the most general types of solutions. The most important such class consists of the so-called weak-solutions to be introduced in section 10.4. To emphasize the distinction, the smooth solutions described