3. Iterative Techniques in ~~xxx~~ Matrix Algebra

3.1 Spectral radius ρ.

The spectral radius $\rho(A)$ of a matrix A is defined by –

$$\rho(A) = \max |\lambda|$$

where λ is the eigenvalue of A.

Examples.

1) Determine the eigenvalues, eigenvectors and the spectral radius of the matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 2 \\ 1 & -1 & 4 \end{bmatrix}$$

The characteristic polynomial $p(\lambda)$ for this matrix is given by

$$\det(A-\lambda I) = 0$$

$$\begin{vmatrix} 2-\lambda & 0 & 0 \\ 1 & 1-\lambda & 2 \\ 1 & -1 & 4-\lambda \end{vmatrix} = 0.$$

$$2-\lambda \begin{vmatrix} 1-\lambda & 2 \\ -1 & 4-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(\lambda^2 - 5\lambda + 4 + 2) = 0$$

$$(\lambda-2)(\lambda^2 - 5\lambda + 6) = 0$$

$$(\lambda-2)(\lambda-3)(\lambda-2) = 0$$

$$(\lambda-2)^2(\lambda-3) = 0.$$

$\lambda_1 = 2$ with multiplicity 2.

$\lambda_2 = 3$ with multiplicity ~~2~~ 1.

The eigenvalues of A are $\lambda_1 = 2$, $\lambda_2 = 3$.

1) $\lambda_1 = 2$

solve $Ax = \lambda x$.

$$(Ax - \lambda I)x = 0$$

$$(A - 2I)x = 0$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 2 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The matrix A can be reduced to rref.

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & -1 & 2 \end{bmatrix}$$

$$x_1 - x_2 + 2x_3 = 0.$$

Here, x_2 and x_3 are the free variables. Assigning ~~the~~ arbitrary values $x_2 = 1$, $x_3 = 0$, we get $x_1 = 1$. Assigning the ~~ab~~ arbitrary values $x_2 = 0$, $x_3 = 1$, $x_1 = -2$.

$$x^1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad x^2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}.$$ are the eigenvectors corresponding to $\lambda_1(A) = 2$.

2) $\lambda_2 = 3$

Solve $Ax = \lambda x$

$$(A - \lambda I)x = 0$$
$$(A - 3I)x = 0$$

$$\begin{bmatrix} -1 & 0 & 0 \\ 1 & -2 & 2 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -2 & 2 \\ 1 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$x_1 = 0$

$-x_2 + x_3 = 0$

Here x_3 is the free variable. Assigning the arbitrary value $x_3 = 1$, we find $x_2 = 1$.

$x^3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ is the eigenvector corresponding to the eigenvalue $\lambda_2(A) = 3$.

The spectral radius $\rho(A) = \max\{2, 3\} = 3$.

Theorem. 1.

If A is an $n \times n$ matrix then

(i) $\|A\|_2 = [\rho(A^t A)]^{1/2}$

(ii) $\rho(A) \leq \|A\|$, for any natural norm A.

Proof.

(i) We are interested to prove that $\rho(A) = \|A\|_2 = \sqrt{\rho(A^t A)}$ ... eigenvector x.

suppose we pick a vector x such that $\|x\|_2 = 1$. (1)

The l_2-norm of the matrix A is:

$$\|A\|_2^2 = \max_{x \neq 0} \|Ax\|_2^2 = \max (Ax)^t (Ax)$$
$$= \max x^t (A^t A) x$$
$$= \max_{x \neq 0} \frac{(A^t A) x}{x^t x} \quad (2)$$

The matrix $A^t A$ is symmetric. Because, $(A^t A)^t = A^t \cdot (A^t)^t = A^t A$. Another elementary result in linear algebra is that the eigenvectors of a symmetric matrix are orthogonal. To see this, let λ_1, λ_2 be two distinct eigenvalues, corresponding to the eigenvectors x^1, x^2.

$$Ax^1 = \lambda_1 x^1$$
$$Ax^2 = \lambda_2 x^2.$$

$$\langle x^1, Ax^2 \rangle = (Ax^2)^t x^1 = (x^2)^t A^t x^1$$
$$= (x^2)^t A x^1$$
$$= \lambda_1 (x^2)^t x^1.$$

But,

$$\langle x^1, Ax^2 \rangle = (Ax^2)^t x^1 = (\lambda_2 x^2)^t x^1$$
$$= \lambda_2 (x^2)^t x_1$$

$$\therefore (\lambda_1 - \lambda_2)(x^2)^t x^1 = 0$$

As $\lambda_1 \neq \lambda_2$, $(x^2)^t x^1 = 0$.

$$\therefore \langle x_1, x_2 \rangle = 0.$$

Since, A^tA is symmetric it has a complete set of n orthogonal eigenvectors, say $u_1, u_2, \ldots, u_n$ such that
$$\langle u_j, u_k\rangle = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$$

That is,
$$\langle u_j, u_k\rangle = \delta_{jk}.$$

Let
$$(A^tA)\,u_s = \lambda_s u_s.$$
The multiplication of the above by u_s on the left yields further –
$$u_s^t (A^tA)\,u_s = \lambda_s u_s^t \cdot u_s = \lambda_s \geq 0$$
(Non-negative).

Every vector has a unique expansion in the basis $\{u_1, \ldots, u_n\}$. Say in particular, that
$$x = \sum_{s=1}^{n} \alpha_s u_s.$$
Then, the equation (2) becomes
$$\begin{aligned}
\|A\|_2^2 &= \left(\sum_{i=1}^{n} \alpha_i u_i\right)^t A^tA \left(\sum_{j=1}^{n} \alpha_j u_j\right) \\
&= \sum_i^n \alpha_i^t \cdot u_i^t \cdot \sum_{j=1}^{n} \alpha_j (A^tA)\,u_j \\
&= \sum_{i=1}^{n} \alpha_i u_i^t \cdot \sum_{j=1}^{n} \alpha_j \lambda_j u_j \\
&= \sum_i \lambda_i |\alpha_i|^2. \\
&\leq \max_i \lambda_i \sum_i |\alpha_i|^2
\end{aligned}$$
But, x is a ~~unit~~ vector, such that $\|x\|_2 = 1$ with respect to any orthonormal basis. So, $\sum_i |\alpha_i|^2 = 1$.
$$= \max_i \lambda_i = \rho(A^tA).$$
Thus, $\sqrt{\rho(A^tA)}$ is an upper bound $\|A\|_2$. However, using $x = u_s$ where $\lambda_s = \rho(A^tA)$, we get
$$\begin{aligned}
\|Au_s\|_2 &= (u_s^t A^tA\,u_s)^{1/2} \\
&= (\lambda_s)^{1/2} \\
&= (\rho(A^tA))^{1/2}
\end{aligned}$$

(ii) We saw above, that the spectral radius of symmetric matrices is equal to their euclidean norm. However, in general, this is not true.

For any natural norm, $\|\cdot\|$ and square matrix A,
$$\rho(A) \leq \|A\|.$$

The spectral radius a sort of infimum of all norms.

For each eigenvalue $\lambda_s(A)$ there is a corresponding eigenvector, say u_s
$$\|A\| = \max_{x\neq 0} \frac{\|Au_s\|}{\|u_s\|} \geq \frac{\|Au_s\|}{\|u_s\|} = \frac{\|\lambda_s u_s\|}{\|u_s\|} = |\lambda_s| \frac{\|u_s\|}{\|u_s\|}$$
$$\therefore \|A\| \geq |\lambda_s| \text{ for all } \|\cdot\|.$$

Thus, $\max |\lambda_i| \leq \|A\|$
$$\rho(A) \leq \|A\|.$$

An interesting and useful result, which is similar to theorem is the following

Theorem 2. For any $n \times n$ matrix A, and each arbitrary $\epsilon > 0$, there exists a natural norm $\|\cdot\|$, with the property that
$$\rho(A) \leq \|A\| \leq \rho(A) + \epsilon.$$

The proof of this result is beyond the scope of this course.

Examples.

1) Determine the l_2-norm of
$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}.$$

Solution.

To apply theorem 1, we need to compute the spectral radius of $A^t A$, so we first need the eigenvalues of $A^T A$.

$$A^t A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 6 & 4 \\ -1 & 4 & 5 \end{bmatrix}$$

The characteristic polynomial of the above matrix is
$$\det(A^t A - \lambda I) = 0$$

$$\begin{vmatrix} 3-\lambda & 2 & -1 \\ 2 & 6-\lambda & 4 \\ -1 & 4 & 5-\lambda \end{vmatrix} = 0$$

$$3-\lambda \begin{vmatrix} 6-\lambda & 4 \\ 4 & 5-\lambda \end{vmatrix} - 2 \begin{vmatrix} 2 & 4 \\ -1 & 5-\lambda \end{vmatrix} - 1 \begin{vmatrix} 2 & 6-\lambda \\ -1 & 4 \end{vmatrix} = 0$$

$$(3-\lambda)(30 - 11\lambda + \lambda^2 - 16) - 2(2(5-\lambda) + 4) - 1(8 + 6 - \lambda) = 0.$$
$$(3-\lambda)(\lambda^2 - 11\lambda + 14) - 2(10 - 2\lambda + 4) - (14 - \lambda) = 0$$
$$\cancel{(3-\lambda)}\ (3\lambda^2 - 33\lambda + 42 - \lambda^3 + 11\lambda^2 - 14\lambda) + 2(2\lambda - 14) + (\lambda - 14) = 0.$$
$$-\lambda^3 + 14\lambda^2 - 47\lambda + 42 + 4\lambda - 28 + \lambda - 14 = 0.$$
$$-\lambda^3 + 14\lambda^2 - 42\lambda = 0$$
$$\lambda^3 - 14\lambda^2 + 42\lambda = 0$$
$$\lambda(\lambda^2 - 14\lambda + 42) = 0$$
$$\lambda(\lambda^2 - 2\cdot\lambda\cdot 7 + 7^2 - 7) = 0$$
$$\lambda((\lambda-7)^2 - (\sqrt{7})^2) = 0$$
$$\lambda(\lambda - 7 + \sqrt{7})(\lambda - 7 - \sqrt{7}) = 0$$
$$\lambda_1 = 0,\ \lambda_2 = 7 - \sqrt{7},\ \lambda_3 = 7 + \sqrt{7}.$$

$$\rho(A^T A) = \max\{0, 7-\sqrt{7}, 7+\sqrt{7}\} = 7 + \sqrt{7}$$
$$\|A\|_2 = \sqrt{\rho(A^T A)} = \sqrt{7+\sqrt{7}}. \approx 3.106.$$

3.2 Convergent Matrices

In studying iterative matrix techniques, it is of particular importance to know when powers of a matrix become small (that is when all the entries approach zero). Matrices of this type are called convergent.