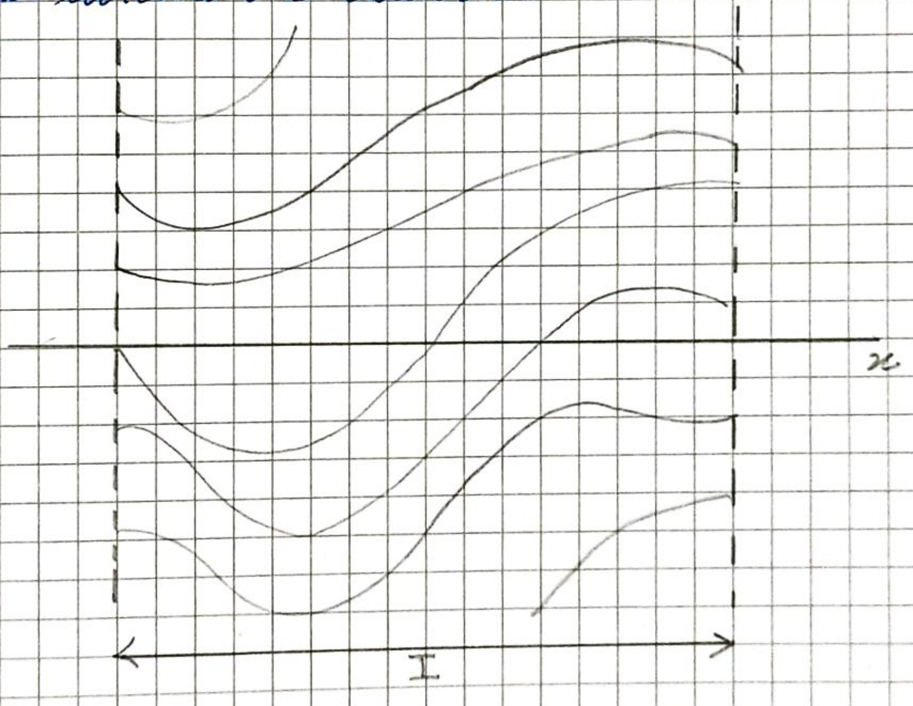
4. Existence and uniqueness of solutions

On the strength of the results of the preceding section we can assert that every first order linear differential equation which is normal on an interval I has solutions. In fact, it has infinitely many, one for each value of c in the expression

$$y = \left[c + \int \frac{h(x)}{a_1(x)} e^{\int [a_0(x)/a_1(x)]\,dx} \, dx \right] e^{-\int [a_0(x)/a_1(x)]\,dx}, \tag{4.30}$$

and the general solution of such an equation therefore is a one-parameter family of plane curves which traverse the strip of the xy-plane determined by I as shown in the below figure. Even more important, it is easy to see that there is a solution curve passing through any pre-assigned point (x_0, y_0) in this strip, since (4.30) can be solved for c, when $x = x_0$, $y = y_0$.

The problem of finding a function $y = y(x)$ which is the solution of a normal first-order linear differential equation and which also satisfies the condition $y(x_0) = y_0$ is called an initial-value problem for the given equation. This terminology is designed to serve as a reminder of the physical interpretation which views such a solution as a path, or trajectory, of a moving particle which started at the point (x_0, y_0) and whose subsequent motion was governed by the equation in question. In these terms, our earlier results can be summarized by saying that every initial value problem involving a linear first-order differential equation has at least one solution.



At this point, it is only natural to ask whether or not such a problem can admit more than one solution. This is the so-called uniqueness problem for first-order linear differential equations and is anything but an idle question. Indeed in applications of differential equations to the natural sciences, it is often essential to be able to guarantee that the problem being investigated has a unique solution since any attempt to predict the future behavior of a physical system governed by an initial value problem relies upon this knowledge. In the case at hand, that the desired uniqueness obtains (problem 14) and hence, so the above assertion can be amended to read as follows.

Theorem 4.1 Every initial-value problem involving a linear first-order differential equation has precisely one unique solution.

The general theory of linear differential equations can properly be said to begin with the theorem which generalizes this result to linear differential equations of order n. In the special case above, the theorem was proved by the simple expedient of exhibiting all of the solutions at issue. Unfortunately, it is impossible to give an argument of this type for equations of higher order, and though the asserted theorem is true, its proof is not conspicuously easy. Thus, rather than become involved in a long and somewhat arid discussion at this time, we content ourselves with a formal statement of the result.

Theorem 4.2. Existence and uniqueness theorem for Linear Differential Equations.

Let

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x) = h(x) \qquad (4.31)$$

be a linear differential equation of order n, defined on an interval I. $a_0(x), a_1(x), \ldots, a_n(x)$ and $h(x)$ are each continuous functions of I, and $a_n(x) \neq 0$, when $x \in I$. Then, there exists one and only one unique solution $y(x)$ of the linear differential equation (4.31), satisfying the set of initial conditions

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \ldots, y^{(n-1)}(x_0) = y_{n-1} \qquad (4.32)$$

As in the case of first-order equations, the problem of finding a solution of (4.31) which satisfies the n additional conditions given (4.32) is called an initial-value problem with initial conditions

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \ldots, y^{(n-1)}(x_0) = y_{n-1}.$$

This theorem is an existence and uniqueness theorem, because it gives the conditions under which a solution of (4.31) satisfying (4.32) must exist. It is also a uniqueness theorem because it gives the conditions under which the solution of (4.31) satisfying (4.32) is unique. As remarked previously, the proof of this theorem is postponed to a ~~latte~~ later lesson.

It is also worth noting that the theorem (4.2) can be phrased in the language of linear operators, in which case it assumes the following suggestive form.

If $L: \mathcal{C}^n(I) \to \mathcal{C}(I)$ is a linear differential operator of order n, there exists a unique inverse operator $G: \mathcal{C}(I) \to \mathcal{C}^n(I)$ such that

(i) $L[G(h)] = h$, for all $h \in \mathcal{C}(I)$.

(ii) $G(h)(x_0) = y_0, \; G(h)'(x_0) = y_1, \ldots, G(h)^{(n-1)}(x_0) = y_{n-1}$.

When stated in these terms, it is clear that the task of solving an initial-value problem for linear differential equations comes ~~to~~ down to finding an explicit form for the inverse operator G, since once G is known, the problem

$$Ly = h;$$

$y(x_0) = y_0, y'(x_0) = y_1, \ldots, y^{(n-1)}(x_0) = y_{n-1}$

can be solved by computing the value of $G(h)$.

$$Ly = h$$
$$GLy = Gh.$$
$$Iy = Gh$$
$$y = Gh.$$

This point of view will be exploited in later chapters where much of our work will be directed towards finding G for specific classes of linear differential operators. As we shall see, G will turn out to be an integral operator.

Right now we content ourselves with the proofs of three important properties of linear differential equations that we shall need for a clearer understanding of the remaining lessons of this chapter.

Problems.

1. Find the solution of each of the following initial value problems and specify the domain of the solution.

1. $xy' + 2y = 0, \quad y(1) = -1.$

Solution.

$xy' + 2y = 0$

Dividing throughout by x, we have—

$\frac{dy}{dx} + \frac{2}{x}(y) = 0 \quad (x \neq 0).$

The integrating factor is $e^{\int (2/x).dx} = e^{2\ln(x)} = x^2$. Therefore, multiplying by x^2, we get.

$x^2\frac{dy}{dx} + 2xy = 0$

$\frac{d}{dx}(x^2y) = 0$

$x^2y = \int 0 \cdot dx + c$

$x^2y = c$

$y = \left(\frac{c}{x^2}\right).$

The initial condition is $y(1) = -1$.

Thus, $-1 = \frac{c}{1}$

$c = -1$

Hence, the desired solution is

$y = \left(-\frac{1}{x^2}\right).$

2. $(\sin x)\, y' + (\cos x)\, y = 0, \quad y\left(\frac{3\pi}{4}\right) = 2.$

solution.

The above expression of the left is an exact differential.

$$d(y \sin x) = 0 \cdot dx$$

$$\int d(y \cdot \sin x) = \int 0 \cdot dx + c$$

$$y \sin x = c$$

$$y = \left(\frac{c}{\sin x}\right)$$

$$y\left(\frac{3\pi}{4}\right) = \frac{c}{\sin(3\pi/4)} = \frac{c}{1/\sqrt{2}} = \sqrt{2}\, c = 2$$

$$\therefore \quad c = \frac{1}{\sqrt{2}}$$

$$\Rightarrow y = \left(\frac{1}{\sqrt{2} \sin x}\right)$$

$y = 0$

3. $2y' + 3y = e^{-x}, \quad y(-3) = -3.$

solution.

$$2\frac{dy}{dx} + 3y = e^{-x}$$

$$\frac{dy}{dx} + \frac{3}{2} y = \frac{e^{-x}}{2}$$

The integrating factor is $e^{\int 3/2\, dx} = e^{3x/2}$

Multiplying throughout by the integrating factor

$$e^{3x/2}\frac{dy}{dx} + \frac{3}{2} e^{3x/2} \cdot (y) = \frac{e^{x/2}}{2}$$

$$\frac{d}{dx}\left(y e^{3x/2}\right) = \frac{e^{x/2}}{2}$$

$$y e^{3x/2} = \frac{1}{2}\int e^{x/2}\, dx + c$$

$$= e^{x/2} + c$$

$$y = e^{-x/2} + c e^{-3x/2}$$

The initial condition is $y(-3) = -3$.

$$\therefore \quad -3 = e^{-(-3/2)} + c e^{-3(-3/2)}$$

$$-3 = e^{3/2} + c e^{9/2}$$

$$c e^{9/2} = -3 - e^{3/2}$$

$$c = -3e^{-9/2} - e^{-3}$$

$$= -\left(\frac{3}{(e^{3/2})^3} + \frac{1}{e^3}\right)$$

$$\Rightarrow \quad y = e^{-x/2} - (e^{-3} + 3e^{-9/2}) e^{-3x/2}$$

4. $(x^2+1) y' - (1-x^2) y = e^{-x}, \quad y(-2) = 0.$

Solution

$$\frac{dy}{dx} - \left(\frac{1-x^2}{1+x^2}\right) y = \left(\frac{e^{-x}}{1+x^2}\right)$$

The integrating factor is $e^{\int -\left(\frac{1-x^2}{1+x^2}\right) dx}$.

$$\text{Let } I_1 = \int \frac{x^2-1}{x^2+1} dx = \int \left(1 - \frac{2}{1+x^2}\right) dx$$

$$= x - 2\arctan x.$$

Multiplying throughout by $e^{x - 2\arctan x}$:

$$e^{x-2\arctan x} \frac{dy}{dx} + y \cdot \left(\frac{x^2-1}{x^2+1}\right) e^{x-2\arctan x} = \left(\frac{e^{-2\arctan x}}{1+x^2}\right)$$

$$\frac{d}{dx}\left(y \cdot e^{x-2\arctan x}\right) = \frac{e^{-2\arctan x}}{1+x^2}$$

Now, let $I_2 = \int \frac{e^{-2\arctan x}}{1+x^2} dx$

Let $\arctan x = t$

$$\frac{dx}{1+x^2} = dt$$

$$I_2 = \int e^{-2t} dt = \frac{e^{-2t}}{-2} + c$$

$$= -\frac{1}{2} e^{-2\arctan x} + c.$$

$$\Rightarrow y \cdot e^{x} = -\frac{1}{2} + c e^{2\arctan x}$$

Use the given general solution to solve each of the following initial-value problems.

6. $y'' - k^2 y = 0$, $y(0) = y'(0) = 1$, $y = c_1 \sinh kx + c_2 \cosh kx$, $k \neq 0$.

Solution

The initial conditions are:

$$y(0) = 1$$
$$c_1(0) + c_2(1) = 1$$
$$\therefore c_2 = 1.$$

$$y'(0) = 1$$
$$y(x) = c_1 \sinh kx + c_2 \cosh kx$$
$$y'(x) = c_1 k \cosh kx + c_2 k \sinh kx$$
$$y'(0) = kc_1(1) + kc_2(0) = 1$$
$$kc_1 = 1$$
$$c_1 = \frac{1}{k}, \quad k \neq 0.$$

The desired solution satisfying the initial conditions is:

$$y(x) = \frac{1}{k} \sinh kx + \cosh kx$$

7. $(1-x^2)y'' - 2xy' = 0$, $y(-2) = 0$, $y'(-2) = 1$, $y = c_1 + c_2 \log \left|\frac{x-1}{x+1}\right|$.

Solution.

The initial conditions are -

$$y(-2) = 0$$
$$c_1 + c_2 \log \left|\frac{-2-1}{-2+1}\right| = 0.$$

$$c_1 + c_2 \log 3 = 0 \qquad (1)$$

$$y(x) = c_1 + c_2 \log \left|\frac{x-1}{x+1}\right|$$

$$y'(x) = c_2 \cdot \left(\frac{x+1}{x-1}\right) \cdot \frac{(x+1)-(x-1)}{(x+1)^2}$$
$$= \frac{2c_2}{x^2-1}$$

$$y'(-2) = 1$$

$$\frac{2c_2}{(-2)^2-1} = 1$$
$$2c_2 = 3$$
$$c_2 = \frac{3}{2}$$

$$\Rightarrow c_1 + \frac{3}{2} \log 3 = 0$$

$$c_1 = -\frac{3}{2} \log 3.$$

$$\Rightarrow y(x) = -\frac{3}{2} \log 3 + \frac{3}{2} \log \left|\frac{x-1}{x+1}\right| = \frac{3}{2}\left(\log \frac{x-1}{3(x+1)}\right)$$

8: $xy'' + y' + xy = 0$, $y(1) = y'(1) = 1$; $y = c_1 J_0(x) + c_2 Y_0(x)$, where J_0 and Y_0 are linearly independent solutions of the equation on $(0,\infty)$.

Solution.

$$y = c_1 J_0(x) + c_2 Y_0(x)$$

$$y(1) = c_1 J_0(1) + c_2 Y_0(1) = 1$$

$$y' = c_1 J_0'(x) + c_2 Y_0'(x)$$

$$y'(1) = c_1 J_0'(1) + c_2 Y_0'(1) = 1$$

By Cramer's rule, $\Delta_{c_1} = \begin{vmatrix} Y_0(1) & 1 \\ Y_0'(1) & 1 \end{vmatrix}$

$$\Delta_{c_2} = \begin{vmatrix} J_0(1) & 1 \\ J_0'(1) & 1 \end{vmatrix}$$

$$\Delta = \begin{vmatrix} J_0(1) & Y_0(1) \\ J_0'(1) & Y_0'(1) \end{vmatrix}$$

$$\frac{c_1}{\Delta_{c_1}} = \frac{-c_2}{\Delta_{c_2}} = \frac{1}{\Delta}$$

$$c_1 = \frac{\Delta_{c_1}}{\Delta} = \frac{\begin{vmatrix} Y_0(1) & 1 \\ Y_0'(1) & 1 \end{vmatrix}}{\begin{vmatrix} J_0(1) & Y_0(1) \\ J_0'(1) & Y_0'(1) \end{vmatrix}}$$

$$c_2 = \frac{\Delta_{c_2}}{\Delta} = \frac{\begin{vmatrix} J_0(1) & 1 \\ J_0'(1) & 1 \end{vmatrix}}{\begin{vmatrix} J_0(1) & Y_0(1) \\ J_0'(1) & Y_0'(1) \end{vmatrix}}.$$

9. $4x^2 y'' + 4xy' + (4x^2 - 1)y = 0$, $y\left(\frac{\pi}{6}\right) = -1$, $y'\left(\frac{\pi}{6}\right) = 0$

Solution

$$y = \sqrt{\frac{2}{\pi x}}\,(c_1 \sin x + c_2 \cos x).$$

$$y(x) = \sqrt{\frac{2}{\pi x}}\,(c_1 \sin x + c_2 \cos x)$$

$$y\left(\frac{\pi}{6}\right) = \sqrt{\frac{2}{\pi(\pi/6)}}\left(c_1 \sin\frac{\pi}{6} + c_2 \cos\frac{\pi}{6}\right) = -1$$

$$\sqrt{\frac{12}{\pi^2}}\left(c_1 \cdot \frac{1}{2} + c_2 \cdot \frac{\sqrt{3}}{2}\right) = -1$$

$$\frac{c_1 + \sqrt{3}\,c_2}{2} = -\frac{\pi}{2\sqrt{3}}$$

$$c_1 + \sqrt{3}\,c_2 = -\frac{\pi}{\sqrt{3}}. \qquad (1)$$

$$y'(x) = \sqrt{\frac{2}{\pi}} \cdot \left(\frac{c_1 \sin x + c_2 \cos x}{\sqrt{x}}\right)'$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{\sqrt{x}\,(c_1 \cos x - c_2 \sin x) - (c_1 \sin x + c_2 \cos x)/2\sqrt{x}}{(\sqrt{x})^2}$$

$$= \frac{1}{x}\sqrt{\frac{2}{\pi}} \frac{2x(c_1\cos x - c_2\sin x) \mp (c_1\sin x + c_2\cos x)}{2\sqrt{x}}$$

$$y'\left(\frac{\pi}{6}\right) = \frac{1}{(\pi/6)}\sqrt{\frac{2}{\pi}} \cdot \frac{1}{2\sqrt{\pi/6}}\left\{2\left(\frac{\pi}{6}\right)\left(c_1\cos\frac{\pi}{6} - c_2\sin\frac{\pi}{6}\right) \mp \left(c_1\sin\frac{\pi}{6} + c_2\cos\frac{\pi}{6}\right)\right\} = 0$$

$$\therefore \frac{\pi}{3}\left(\frac{\sqrt{3}c_1 - c_2}{2}\right) - \left(\frac{c_1 + \sqrt{3}c_2}{2}\right) = 0$$

$$\pi(\sqrt{3}c_1 - c_2) - 3(c_1 + \sqrt{3}c_2) = 0$$

$$\pi(\sqrt{3}c_1 - c_2) - 3\left(-\frac{\pi}{\sqrt{3}}\right) = 0$$

$$(\sqrt{3}c_1 - c_2) + \sqrt{3} = 0$$

$$\sqrt{3}c_1 - c_2 = -\sqrt{3}.$$

Thus, $c_1 + \sqrt{3}c_2 = \frac{-\pi}{\sqrt{3}}$

$$\sqrt{3}c_1 - \sqrt{3}c_2 = -3$$

$$4c_1 = \frac{-\pi + 3\sqrt{3}}{\sqrt{3}} = -\frac{9 + \sqrt{3}\pi}{\sqrt{3}}$$

$$c_1 = -\frac{9 + \sqrt{3}\pi}{12}$$

$$c_2 = \sqrt{3}c_1 + \sqrt{3}$$
$$= -\frac{9\sqrt{3} + 3\pi}{12} + \sqrt{3}$$
$$= \frac{3\sqrt{3} - 3\pi}{12}$$
$$= \frac{\sqrt{3} - \pi}{4}$$

So, $y(x) = \sqrt{\frac{2}{\pi x}}\left[-\frac{9 + \sqrt{3}\pi}{12}\sin x + \frac{\sqrt{3} - \pi}{4}\cos x\right]$

is the desired solution.

10. (a) Let y_1 and y_2 be solutions of a linear first order differential equation on an interval I. Prove that $y_1 - y_2$ is either identical zero or different from zero everywhere on I.

(b) Use the result in (a) to deduce that every initial-value problem involving a first order linear differential equation has at most one solution.

Solution.

(a) Let $a_1(x)\frac{dy}{dx} + a_0(x)y = h(x)$

is a first-order linear differential equation. The general solution is

$$y e^{\int [a_1(x)/a_0(x)]dx} = \int \frac{h(x)}{a_1(x)} e^{\int [a_1(x)/a_0(x)]dx} + c.$$

As y_1 and y_2 are solutions of the differential equation, they must have the above form.

$$y_1 e^{\int [a_1(x)/a_0(x)] dx} = \int \frac{a_1(x)}{h(x)} \cdot e^{\int [a_1(x)/a_0(x)]} dx + c_1$$

$$y_2 e^{\int [a_1(x)/a_0(x)] dx} = \int \frac{a_1(x)}{h(x)} e^{\int [a_1(x)/a_0(x)]} dx + c_2$$

$$\Leftrightarrow \quad (y_1 - y_2) e^{\int [a_1(x)/a_0(x)] dx} = c_1 - c_2.$$

$$y_1 - y_2 = (c_1 - c_2) e^{-\int [a_1(x)/a_0(x)] dx}$$

Since, $a_1(x) \neq 0$ does not vanish for all $x \in I$, $e^{-\int [a_1(x)/a_0(x)] dx}$ is non-zero in I.

Hence, there are two possibilities

(1) $c_1 = c_2$
$\Leftrightarrow y_1 - y_2 = 0. \quad \forall x \in I.$

(2) $c_1 \neq c_2$
$y_1 \neq y_2$ in I.

Thus, y_1 differs from y_2 over the entire interval I.

(b) Suppose we are subject to the initial condition

$$y(x_0) = y_0.$$

Assume that there are two solutions that satisfy this initial value problem — $y_1(x)$ and $y_2(x)$.

$$y_1(x_0) - y_2(x_0) = y_0 - y_0 = 0$$
$$(c_1 - c_2) e^{-\int a_1(x_0)/a_0(x_0) dx} = 0$$

But, $e^{-\int [a_1(x_0)/a_0(x_0)] dx} \neq 0$ in the interval I.

$\therefore c_1 - c_2 = 0$
$c_1 = c_2.$

Hence, $y_1(x) = y_2(x)$.

Every initial value problem involving a linear first order differential equation has at most one solution.

11. Let $y_1(x)$ and $y_2(x)$ be distinct solutions of a normal first-order linear differential equation on an interval I. Prove that the general solution of the equation on I is

$$\frac{y - y_2}{y_1 - y_2} = c.$$

where c is an arbitrary constant.

Hint. See problem 10 a.

Solution.

12. Prove that a non-trivial solution of homogeneous first-order linear differential equation cannot intersect the x-axis. [Hint: Use theorem 4.1].

Proof.

Let the differential equation be

$$\frac{dy}{dx} + P(x)y = 0.$$

The integrating factor is $e^{\int P(x)\,dx}$.

Thus,

$$e^{\int P(x)dx}\frac{dy}{dx} + P(x)e^{\int P(x)dx}\cdot y = 0$$

$$d\left(y\cdot e^{\int P(x)dx}\right) = 0$$

$$y\cdot e^{\int P(x)dx} = c$$

$$y = ce^{-\int P(x)\,dx}.$$

The function $e^{-\int P(x)dx}$ is different from 0 on the entire real line.

Thus, $y(x) = ce^{-\int P(x)dx}$ does not intersect x-axis.

13. Use the results of this section to prove that $y = c_1 \sin x + c_2 \cos x$ is the general solution of $y'' + y = 0$ on $(-\infty, \infty)$. [Hint: If $u(x)$ is any solution on $(-\infty, \infty)$, show that c_1 and c_2 can be chosen so that $y(0) = u(0)$, $y'(0) = u'(0)$ and then apply theorem 4.2].

Proof

Clearly,
$$y_1 = \sin x \quad , \quad y_2 = \cos x$$
$$y_1' = \cos x \quad ; \quad y_2' = -\sin x$$
$$y_1'' = -\sin x \quad ; \quad y_2'' = -\cos x .$$

y_1 and y_2 are solutions of the second order linear differential equation
$$y'' - y = 0.$$
Moreover, if $c_1 \sin x + c_2 \cos x = 0$ for all x, then setting $x = 0$, $c_2 = 0$. Setting $x = \pi/2$, $c_1 = 0$. Hence, $\{\sin x, \cos x\}$ is a linearly independent set.

Therefore, the general solution of the second-order linear differential equation is
$$y = c_1 \sin x + c_2 \cos x .$$

We need two conditions to determine the parameters c_1 and c_2.

$$y(0) = u(0) = c_2$$
$$y' = c_1 \cos x - c_2 \sin x$$
$$y'(0) = u'(0) = c_1 .$$

So, the desired solution satisfying the initial conditions is
$$y(x) = u'(0) \sin x + u(0) \cos x .$$

14. Show that the two distinct solution of a normal first order linear differential equation cannot have a point of intersection.

Proof.

Let $y_1(x)$ and $y_2(x)$ be two distinct solutions of first-order linear differential equation. The solutions must be of the form:
$$y_1 e^{\int [a_0(x)/a_1(x)]\,dx} = \int \frac{h(x)}{a_1(x)} e^{\int [a_0(x)/a_1(x)]\,dx}\,dx + c_1$$
$$y_2 e^{\int [a_0(x)/a_1(x)]\,dx} = \int \frac{h(x)}{a_1(x)} e^{\int [a_0(x)/a_1(x)]\,dx}\,dx + c_2 .$$

$$[y_1(x) - y_2(x)]\, e^{\int [a_0(x)/a_1(x)]\,dx} = c_1 - c_2 .$$

If $c_1 \neq c_2$, so $y_1(x) - y_2(x) \neq 0 \ \forall x \in I$. $\therefore$ Thus, $y_1(x)$ is different from $y_2(x)$ for all x. $y_1(x) \neq y_2(x)$ for any x.

The solutions of a normal first order linear differential equation cannot have a point of intersection.

15. Prove that every non-trivial solution $u(x)$ of a normal second-order linear differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

has only simple zeros. [A point x_0 is said to be a zero of a function $u(x)$ if and only if $u(x_0) = 0$. A zero of $u(x)$ is simple if and only if $u'(x_0) \neq 0$.].

Proof.

4.5 Dimension of the Solution Space.

In this section, we shall use the existence and uniqueness theorem stated above to give a simple yet elegant proof of the fact that the dimension of the solution space of every normal homogenous linear differential equation is equal to the order of the equation. The reader should note however, that this result fails in the case of an equation whose leading coefficient vanishes somewhere in the interval under consideration.

This said, we now prove -

Theorem 4.3.

If $a_0(x), a_1(x), \ldots, a_n(x)$ and $h(x)$ are each continuous functions of x on a common interval I and $a_n(x) \neq 0$ when x is in I, then:

1. The homogenous linear differential equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots a_1(x)\frac{dy}{dx} + a_0(x)y = 0 \qquad (4.33)$$

defined on the interval I has n linearly indipendent solutions $y_1(x), y_2(x), \ldots, y_n(x)$. It is an n-dimensional subspace of $\mathcal{C}(I)$.

2. The linear combination of these solutions

$$y_c(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_0(x)$$