3. Subspaces.

(1) Consider $P_n(\mathbb{R})$: the set of all polynomials with coefficients $a_0, a_1, \ldots, a_n$ of degree not exceeding n. We have seen that $P_n(\mathbb{R})$ is a real vector space.

It is easy to see that for $1 \leq k \leq n$, P_k is subset of P_n. Moreover, P_k is a vector space in its own right, with respect to same operations of polynomial addition and scalar multiplication defined coordinate wise.

(2) Consider $V \subseteq \mathbb{R}^2$ defined as
$V := \{(x,y) : y = 5x,\ x,y \in \mathbb{R}\}$.
V is the ~~set of~~ subset of points in $\mathbb{R}^2$ that lie on the straight line $y = 5x$ passing through the origin. We have seen that V is a vector space in its own right.

This motivates a discussion on subspaces.

Definition of a subspace.
A subset W of V is called a subspace of V, if W is also a vector space (using the same operations of vector addition and and scalar multiplication as on V). For example,

$$\{(x_1, x_2, 0) : x_1, x_2 \in \mathbb{F}\}$$

is a subspace of $\mathbb{F}^3$.

Theorem. A subset W of a vector space V is a subspace of V, if and only if
(i) W is closed with respect to vector addition.
If $x, y \in W \Rightarrow x + y \in W$.

(ii) W is closed with respect to scalar multiplication.
If $x \in W$ and $\alpha \in \mathbb{F}$, then $\alpha x \in W$.

Proof.
If W is a subspace of V, then W is a vector space. So, (i) and (ii) hold. W is a vector space in its own right.

Converse:
Suppose W is a subset of V such that the conditions (i) and (ii) hold. We must show that W is a subspace.

We are given that, W is closed with respect to vector addition and scalar multiplication.

If $x, y \in W$, then $x + y \in W$
If $\alpha \in \mathbb{F}$ and $x \in W$, then $\alpha x \in W$.

(A1) Addition must be commutative.
Suppose $x, y \in W$. Then, $x, y \in V$, since $W \subseteq V$.
$x + y = y + x$, as addition is commutative in V.

(A2) Addition must be associative.
Suppose $x, y, z \in W$. As W is a subset of V, $x, y, z \in V$.
Since, vector addition is associative in V,
$(x + y) + z = x + (y + z)$

(A3) There exists a zero vector in W, such that $x + 0 = x \ \forall x \in W$.
This we don't have presently. What we presently have is - if x is in V, there is a zero vector in V, such that $x + 0 = x \ \forall x \in V$. Why should the zero vector belong to W, if conditions (i) and (ii) hold?

(A4) For each $x \in W$, there exists a negative element $(-x)$, such that $x + (-x) = 0$.
Again, we only know that, if x belongs to V, its negative element $(-x)$ belongs to V. Why should the negative element belong to W?

(M1) Multiplication should be associative.
Let $\alpha, \beta \in \mathbb{F}$ and $x \in W$.
Then, as $W \subseteq V$, $x \in V$.
Scalar multiplication is associative in V, as V is a vector space.
Therefore, $(\alpha\beta)x = \alpha(\beta x) \quad \forall x \in W$.

(M2) Multiplication with identity.
Let $x \in W$ and $\alpha \in \mathbb{F}$. Since, $W \subseteq V$, $x \in V$.
In V, we know that, when $\alpha = 1$,
$1x = x$.
$\Rightarrow$ This property holds for all $x \in W$.

(D1) Let $\alpha \in \mathbb{F}$, $x, y \in W$.
As W is a subset of V, if $x, y \in W \Rightarrow x, y \in V$.
In V, $\alpha(x+y) = \alpha x + \alpha y$.
Thus, it holds for all elements x, y in W.

(D2) Let $\alpha, \beta \in \mathbb{F}$, $x \in W$.
Then, $(\alpha+\beta)x = \alpha x + \beta x$, since $x \in W$ and $W \subseteq V \Rightarrow x \in V$ and this property holds in the vector space V.

We only need to verify (A3) and (A4).

We know that, W is closed under scalar multiplication. If $\alpha \in \mathbb{F}$, $x \in W \Rightarrow \alpha x \in W$.

(i) Set $\alpha = 0$.
Then, $\alpha x = 0x = 0$.
$\Rightarrow 0 \in W$.

(ii) Set $\alpha = -1$.
Then, $\alpha x = (-1)x = -x$.
$\Rightarrow -x \in W$ for each $x \in W$.

Examples.
1) Let $A \in \mathbb{R}^{m \times n}$.
Define $= \{x \in \mathbb{R}^n : Ax = 0\} \subseteq \mathbb{R}^n$. $\to$ 0 vector $\in \mathbb{R}^m$.
$\mathbb{R}^n$ already has vector addition ~~can~~ and scalar multiplication defined.
Show that V is a vector space (subspace).
Proof
We need to prove that the solutions of the system of linear homogeneous equations

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= 0 \\ \vdots \qquad\qquad\qquad\qquad & \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= 0 \end{aligned}\right\} \text{m equations in n unknowns.}$$

form a vector space.

$\mathbb{R}^n$ is a vector space. $V \subseteq \mathbb{R}^n$.

(i) V is closed with respect to vector addition.
Let $x, y \in V$. Then, $Ax = 0$ and $Ay = 0$.
By the properties of matrices—

$$Ax + Ay = 0$$
$$\Leftrightarrow A(x+y) = 0.$$
$$\Leftrightarrow x+y \in V.$$

(ii) V is closed with respect to scalar multiplication.
Let $x \in V$ and $\alpha \in \mathbb{F}$.

$$Ax = 0$$

Multiplying by the scalar α on both sides:

$$\alpha A x = 0$$
$$A(\alpha x) = 0$$
$$\Leftrightarrow \alpha x \in V.$$

Thus, V is a vector space in its own right. V is a subspace of $\mathbb{R}^n$.

2) $V = \mathbb{R}^{m \times n}$
Let us define $W = \{ A \in \mathbb{R}^{m \times n} : a_{ij} = a_{ji}$ for all $1 \leq i,j \leq n$, $A = (a_{ij}) \}$.

These matrices are called symmetric matrices.
Prove that the set of symmetric matrices forms a subspace.
<u>Proof</u>
A matrix is symmetric, when $A = A^T$.

(i) Let A, B be two symmetric matrices. Then,

$$\begin{aligned} A+B &= (a_{ij}) + (b_{ij}) \\ &= (a_{ij} + b_{ij}) \\ &= (a_{ji} + b_{ji}) \quad (\text{since } a_{ij} = a_{ji} \text{ and } b_{ij} = b_{ji}) \\ &= (A+B)^T \end{aligned}$$

$\Leftrightarrow$ A+B is a symmetric matrix.

(ii) Let A be a symmetric matrix and α be a scalar.

$$\begin{aligned} \alpha A &= \alpha(a_{ij}) \\ &= (\alpha a_{ij}) \\ &= (\alpha a_{ji}) \\ &= (\alpha A)^T. \end{aligned}$$

$\Leftrightarrow$ αA is a symmetric matrix.

So, W is a subspace of V.

3) Let $V = \mathbb{C}^{m \times n}$, be the set of all $m \times n$ matrices whose entries are complex numbers, the underlying field is understood to be $\mathbb{C}$.

Let us define
$$W = \{ A \in V : a_{ij} = \overline{a_{ji}} \}$$
(called the complex conjugate)

that is $A^* = A$, where $A^* = (\overline{A})^T$. A^* is called the conjugate transpose. You take the complex conjugate of the matrix first and then you take the transpose. Such matrices are called Hermitian matrices.

Is W a subspace of V?

Solution.
Intuitively, we would say yes. But, the answer is no. This is not a subspace of V. The reason is the following -

You take a Hermitian matrix. The diagonal entries of a Hermitian matrix must be real numbers. For if,
$$z = \overline{z}$$
$$x + iy = x - iy$$
$$2iy = 0$$
$$y = 0.$$
$\Leftrightarrow$ z is a real number. The imaginary part of z is zero.

Let's write down a 2×2 Hermitian matrix.

$$A = \begin{pmatrix} \alpha & \gamma + i\delta \\ \gamma - i\delta & \beta \end{pmatrix}.$$

V is a complex vector space. The scalars come from the underlying field $\mathbb{C}$. Let's look at αA, where $\alpha = i$

$$\alpha A = iA = \begin{pmatrix} i\alpha & i(\gamma + i\delta) \\ i(\gamma - i\delta) & i\beta \end{pmatrix}.$$

$$= \begin{pmatrix} i\alpha & -\delta + i\gamma \\ \delta + i\gamma & i\beta \end{pmatrix}.$$

The diagonal entries are not real. So, iA is not a Hermitian matrix. W is not a complex subspace. If, however, the underlying field were $\mathbb{R}$, then W is a real subspace.

Consider $A, B \in W$.
$$A + B = (a_{ij} + b_{ij})$$
$$= (\overline{a_{ji}} + \overline{b_{ji}})$$
$$= (\overline{a_{ji} + b_{ji}})$$
$$= (\overline{A + B})^T = (A + B)^*$$

$\Leftrightarrow$ $A + B \in W$.

Further, if $\alpha \in \mathbb{R}$ and $A \in W$,
$$\alpha A = (\alpha a_{ij})$$
$$= (\alpha \overline{a_{ji}})$$
$$= (\overline{\alpha a_{ji}})$$
$$= (\overline{\alpha A})^T = (\alpha A)^*.$$

αA is a Hermitian matrix and belongs to W.

Therefore, W is a real subspace.

4) Let $W = \{x \in \mathbb{R}^2 : x_2 = cx_1\}$, c is a fixed constant. It is an easy exercise to show that W is a subspace of $\mathbb{R}^2$.

Proof.

(i) W has an additive identity.
The zero vector $0 = (0,0)$ belongs to W,
since $0\ (0,0)$ satisfies $x_2 = cx_1$.

(ii) W is closed under vector addition.
Assume $x, y \in W$. Then, $x_2 = cx_1$ and $y_2 = cy_1$.

$\Leftrightarrow \quad x_2 + y_2 = cx_1 + cy_1$

$\Leftrightarrow \quad x_2 + y_2 = c(x_1 + y_1)$

$\Leftrightarrow \quad x + y \in W.$

(iii) W is closed under scalar multiplication.
Assume $x, y \in W$. and $\alpha \in F$.
Then, $x_2 = cx_1$

$\alpha x_2 = c(\alpha x_1).$

$\Rightarrow \alpha x \in W.$

W is a subspace of $\mathbb{R}^2$.

4. Sums and Direct Sums.

In later chapters, we will find that the notion of ~~sums and~~ vector space sums and direct sums are useful. We define these

Suppose $U_1, U_2, \ldots, U_m$ are subspaces of V. The sum of $U_1, U_2, \ldots, U_m$ denoted by $U_1 + \ldots + U_m$, is defined to be the set of all possible sums of elements of $U_1, U_2, \ldots, U_m$. More precisely,

$$U_1 + U_2 + \ldots + U_m := \{u_1 + u_2 + \ldots + u_m : u_1 \in U_1, u_2 \in U_2, \ldots, u_m \in U_m\}.$$

Let us verify that, if $U_1, U_2, \ldots, U_m$ are the subspaces of V, $U_1 + U_2 + \ldots + U_m$ is a subspace of V.

Proof.

Consider $u, w \in U_1 + U_2 + \ldots + U_m$.
Say, $u = u_1 + u_2 + \ldots + u_m$, $u_i \in U_i$
$w = w_1 + w_2 + \ldots + w_m$, $w_i \in W_i$

Then,

$$u + w = (u_1 + w_1) + (u_2 + w_2) + \ldots + (u_m + w_m)$$

$u_1 + w_1 \in U_1$

$u_2 + w_2 \in U_2$

$\vdots$

$u_m + w_m \in U_m$

as $U_1, U_2, \ldots, U_m$ are subspaces of V and are closed under addition.
By definition, therefore $u + w \in V$.

Also, suppose $\alpha \in F$ and $u = u_1 + u_2 + \ldots + u_m$, $u_i \in U_i$ be an arbitrary element in the sum of subspaces $U_1 + U_2 + \ldots + U_m$

$\alpha u = \alpha(u_1 + \ldots + u_m)$

$= \alpha u_1 + \ldots + \alpha u_m$ (distributivity).

$\alpha u_i \in U_i \quad \forall i = 1, 2, \ldots, m$

$\Rightarrow \alpha u \in U_1 + U_2 + \ldots + U_m$