1: Differentiation in several variables

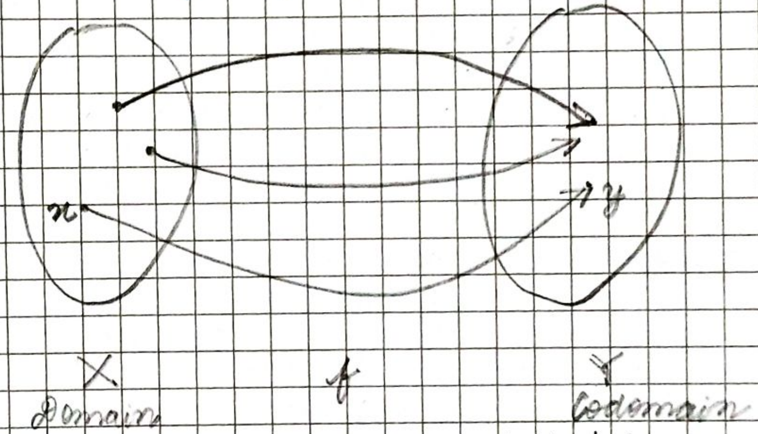
1.1 Functions of several variables

The volume and the surface area of a sphere depend on its radius, the formulas describing their relationships being $V = \frac{4}{3}\pi r^3$ and $S = 4\pi r^2$. (Here, V and S are, respectively, the volume and the surface area of the sphere and r its radius.) The equations define the volume and surface area as functions of the radius. The essential characteristic of a function is that the so called independent variable determines a unique value of the dependent variable. (V or S). Every element in the domain has one and only one image.

You can also think of many quantities that are determined uniquely ~~no~~ not only by one variable (as the volume of a sphere being determined by its radius) but by several: the area of a rectange $A = lb$, the volume of cylinder ~~r~~ or a cone, the average annual rainfall in Cleveland, or the national debt. Realistic modeling of the world requires that we understand the concept of a function of more than one variable and find meaningful ways to visualise such functions.

Definitions, Notations and examples.

A function, any function, has three features: (1) a domain set X, (2) a codomain set Y (3) a rule of assignment that associates to each element x in domain X a unique element, usually denoted $f(x)$, in the codomain Y. We will frequently use the notation $f: X \to Y$ for a function. Such notation indicates all the ingredients of a particular function, although it does not make the nature of the rule of assignment explicit. This notation also suggests the mapping nature of a function.



Example 1. Abstract definitions are necessary, but it is just as important to understand functions as they actually occur. Consider the act of assigning to each US citizen his or her social security number. This pairing defines a function: each citizen is assigned one social security number. The domain is the set of US citizens and the codomain is set of all nine-digit strings of numbers.

On the other hand, when a university assigns students to dormitory rooms, it is unlikely that it is creating a function from the set of available rooms to the set of students. This is because some rooms may have more than one student assigned to them, so that a particular room does not determine a unique student occupant.

Definition 1 (Range) The range of a function $f: X \to Y$ is the set of all elements of Y that are the actual values of Y. That is, the range of f is the set image of the set X under f. It consists of those $y \in Y$ such $y = f(x)$ for some $x \in X$.

$$\text{Range } f = \{ y \in Y \mid y = f(x) \text{ for some } x \in X \}.$$

In the social security ~~example~~ function of example 1, the range consists of those nine-digit numbers actually used as active social security numbers. For example, the number 000-00-0000 is not in the range, since no one is actually assigned this number.

Definition 1.2 (surjective function).
A function $f: X \to Y$ is said to be onto (or surjective) if every element of Y has a pre-image in X; if every $y \in Y$ is the element of some element of X, that is range $f = Y$.

The social security function is not surjective, since 000-00-000 is in the codomain but not in the range. Pictorially, ~~an onto function is sugg~~

Definition 1.3 (One-to-one function). A function $f: X \to Y$ is called one-to-one (injective) if no two distinct elements of the domain has the same image under f. For one-to-one functions, distinct elements have distinct images. That is, f is one-to-one if whenever $x_1, x_2 \in X$ and $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.

When you studied single-variable Calculus, the functions of interest were those whose domains and codomains were subsets of $\mathbb{R}$ (the real numbers). It was probably the case that only the rule of assignment was made explicit; it is generally assumed that domain is the largest possible subset in $\mathbb{R}$ for which the function makes sense. The codomain is generally taken to be all of $\mathbb{R}$.

Example 2. Suppose $f: \mathbb{R} \to \mathbb{R}$ is given by $f(x) = x^2$. Then the domain and codomain are, explicitly, all of $\mathbb{R}$, but the range of f is the subinterval $[0, \infty)$. Thus, f is not onto, since the codomain is strictly larger than the range. Note that f is not one-to-one since $f(2) = f(-2) = 4$, but $2 \neq -2$.

Example 3. Suppose g is a function such that $g(x) = \sqrt{x-1}$. Then, if we take the codomain to be all of $\mathbb{R}$, the domain cannot be larger than $[1, \infty)$. If the domain included any values less than one, the radicand would be negative and hence g would not be real-valued.

Now we're ready to think about functions of more than one variable. In the most general terms, these are functions whose domains are subsets X of $\mathbb{R}^n$ and whose codomains are subsets of $\mathbb{R}^m$ for some positive integers n and m. (For simplicity of notation, we'll take the codomains to be all of $\mathbb{R}^m$, except when specified otherwise.) That is, such a function is a mapping $f: X \subseteq \mathbb{R}^n \to \mathbb{R}^m$ that associates a vector (or point) x in X, a unique vector (point) $f(x)$ in $\mathbb{R}^m$.

Example 4. Let $T: \mathbb{R}^3 \to \mathbb{R}$ be defined by $T(x, y, z) = xy + yz + zx$. We can think of T as a sort of a temperature function. Given a point $x = (x, y, z)$ in $\mathbb{R}^3$, $T(x)$ calculates the temperature at that point.

Example 5. Let $L: \mathbb{R}^n \to \mathbb{R}$ be given by $L(x) = \|x\|$. This is a length function in that it computes the length of any vector x in $\mathbb{R}^n$. Note that L is not one-to-one, since $L(e_i) = L(e_j) = 1$, where e_i and e_j are any of the two standard basis vectors of $\mathbb{R}^n$. L fails to be onto, since the length of a vector is non-negative.

Example 6. Consider the function given by $N(x) = x / \|x\|$. where x is a vector in $\mathbb{R}^3$. Note that, N is not defined if $x = 0$, so the largest possible domain for N is $\mathbb{R}^3 - \{0\}$. The range of N consists of all unit vectors in $\mathbb{R}^3$. The function N is the normalization function, that is, the function that takes a non-zero vector in $\mathbb{R}^3$ and returns the unit vector that points in the same direction.

Example 7. 21/03/2020

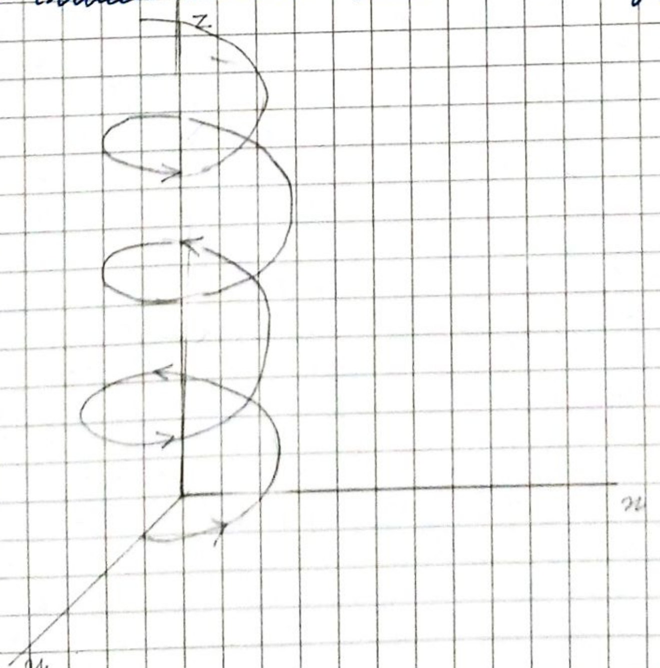
Sometimes a function may be given numerically by a table. One such example is the notion of a windchill - the apparent temperature one feels when taking into account both the actual air temperature and the speed of the wind. A standard table of windchill values is shown below. From it we see that if the air temperature is 20°F and the wind speed is 25 mph, the windchill temperature (how "cold" it feels) is 3°F. Similarly, if the air temperature is 35°F and the windspeed is 10 mph, then the windchill is 27°F. In other words, if s denotes the windspeed and t, the air temperature, then the windchill is a function $W(s, t)$.

The functions described in examples 4, 5 and 7 are scalar-valued functions, that is functions whose codomains are $\mathbb{R}$ or subsets of $\mathbb{R}$. Scalar-valued functions are our main concern for this chapter. Nonetheless, let's look at a few examples of functions whose codomains are $\mathbb{R}^m$, where $m > 1$.

Air Temp (deg F)	Windspeed (mph) 5	10	15	20	25	30	35	40	45	50	55	60
40	36	34	32	30	29	28	28	27	26	26	25	25
35	31	27	25	24	23	22	21	20	19	19	18	17
30	25	21	19	17	16	15	14	13	12	12	11	10
25	19	15	13	11	9	8	7	6	5	4	4	3
20	13	9	6	4	3	1	0	-1	-2	-3	-3	-4
15	7	3	0	-2	-4	-5	-7	-8	-9	-10	-11	-11
10	1	-4	-7	-9	-11	-12	-14	-15	-16	-17	-18	-19
5	-5	-10	-13	-15	-17	-19	-21	-22	-23	-24	-25	-26
-0	-11	-16	-19	-22	-24	-26	-27	-29	-30	-31	-32	-33
-5	-16	-22	-26	-29	-31	-33	-34	-36	-37	-38	-99	-40
-10	-22	-28	-32	-35	-37	-39	-41	-43	-44	-45	-46	-48
-15	-28	-35	-39	-42	-44	-46	-48	-50	-51	-52	-54	-55
-20	-34	-41	-45	-48	-51	-53	-55	-57	-58	-60	-61	-62
-25	-40	-47	-51	-55	-38	-60	-62	-64	-65	-67	-68	-69
-30	-46	-53	-58	-61	-64	-67	-69	-71	-72	-74	-75	-76
-35	-52	-59	-64	-68	-71	-73	-76	-78	-79	-81	-82	-84
-40	-57	-66	-71	-74	-78	-80	-82	-84	-86	-88	-89	-91
-45	-63	-72	-77	-81	-84	-87	-89	-91	-93	-95	-97	-98

Table of Windchill values

Example 8. Define $f : \mathbb{R} \to \mathbb{R}^3$ by $f(t) = (\cos t, \sin t, t)$. The range of f is the curve in $\mathbb{R}^3$ with parametric equations $x = \cos t$, $y = \sin t$, $z = t$. If we think of t as a time parameter, then this function traces out the corkscrew curve (called a helix) shown in the figure.



The helix. The arrow shows the direction of increasing t.

Example 9. We can think of the velocity of a fluid as a vector in $\mathbb{R}^3$. This vector depends depends on (atleast) the point at which one measures the velocity and also the time at which one makes the measurement. In other words, the velocity may be considered to be a function $v: X \subseteq \mathbb{R}^4 \to \mathbb{R}^3$. The domain X is a subset of $\mathbb{R}^4$ because three variables x, y, z are required to describe a point in a fluid and a fourth variable t is needed to keep track of time. For instance, such a function v might be given by the expression

$$v(x,y,z,t) = xyzt\,\hat{i} + (x^2-y^2)\hat{j} + (3z+t)\hat{k}.$$

You may have noted that the expression for v in example 9 is considerably more complicated than those for the functions given in examples 4-8. This is because all the variables and vector components have been written out explicitly. In general, if we have a function $f: X \subseteq \mathbb{R}^n \to \mathbb{R}^m$, then $x \in X$ can be written as $x = (x_1, x_2, \ldots, x_n)$ and f can be written in terms of its component functions $(f_1, f_2, \ldots, f_m)$. Each of the component functions are scalar-valued functions of $x \in X$ that define the components of the vector $f(x) \in \mathbb{R}^m$. What results is a morass of symbols:

$$\begin{aligned} f(x) &= f(x_1, x_2, \ldots, x_n) && \text{Emphasizing the variables} \\ &= (f_1(x), f_2(x), \ldots, f_m(x)) && \text{Emphasizing the component functions} \\ &= (f_1(x_1, \ldots, x_n), f_2(x_1, \ldots, x_n), && \text{Writing out all the components.} \\ &\quad \ldots, f_m(x_1, \ldots, x_n)) \end{aligned}$$

For example, the function L of example 5, when expanded becomes

$$L(x) = L(x_1, \ldots, x_n) = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

The function N of example 6 becomes

$$N(x) = \frac{x}{\|x\|} = \frac{(x_1, x_2, x_3)}{\sqrt{x_1^2 + x_2^2 + x_3^2}}$$

$$= \left(\frac{x_1}{\sqrt{x_1^2+x_2^2+x_3^2}}, \frac{x_2}{\sqrt{x_1^2+x_2^2+x_3^2}}, \frac{x_3}{\sqrt{x_1^2+x_2^2+x_3^2}} \right)$$

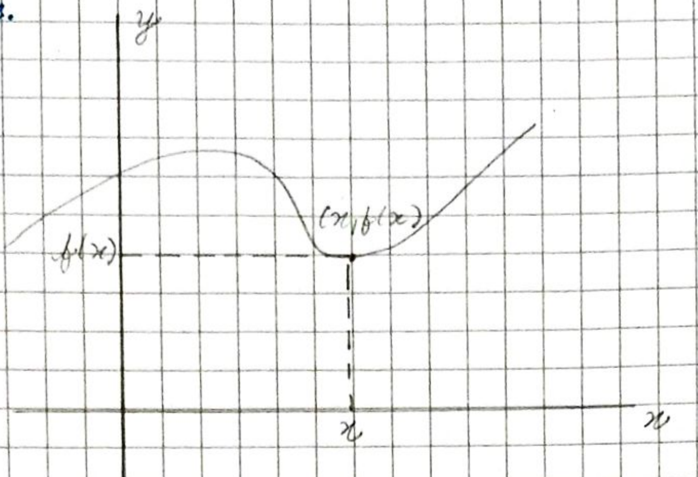
Although writing a function in terms of all its variables and components has the advantage of being explicit, quite a lot of ink and paper are used in the process. The use of vector notation not only saves space and time, but also helps to make the meaning of a function clear by emphasizing that the function ~~points~~ maps points in $\mathbb{R}^n$ to points in $\mathbb{R}^m$. Vector notation makes a function of 300 variables look "just like" a function of one variable. Try to avoid writing out components as much as you can. (Except when you want to impress your friends).

Visualising functions.

No doubt you have been graphing scalar-valued functions of one variable for so long that you give the matter little thought. Let's scrutinize what you've been doing, however. A function $f: X \subseteq \mathbb{R} \to \mathbb{R}$ takes a real number and returns another real number as suggested by the figure. The graph of f is something that lives in $\mathbb{R}^2$. It consists of points (x, y) such that $y = f(x)$. That is,

$$\text{Graph } f = \{x, f(x) \mid x \in X\} = \{(x,y) \mid x \in X, y = f(x)\}.$$

The important fact is that, in general, the graph of a scalar-valued function of a single-variable is a curve - a one-dimensional object sitting inside two dimensional space.



Now, suppose we have a function $f: X \subseteq \mathbb{R}^2 \to \mathbb{R}$, that is a function of two variables. We make essentially the same definition for the graph:

$$\text{Graph } f = \{(X, f(X)) \mid X \in \mathbb{R}\} \tag{1}$$

Of course $X = (x, y)$ is a point of $\mathbb{R}^2$. Thus, $\{(X, f(X))\}$ may also be written as

$$\{(x, y, f(x, y))\} \quad \text{or as} \quad \{(x, y, z) \mid (x, y) \in X,\ z = f(x, y)\}.$$

Hence, the graph of a scalar-valued function of two variables is something that sits in $\mathbb{R}^3$. Generally speaking, the graph will be a surface.

Example 10.
The graph of the function

$$f: \mathbb{R}^2 \to \mathbb{R}, \quad f(x, y) = \frac{1}{12}y^3 - y - \frac{1}{4}x^2 + \frac{7}{2}$$

is shown in the figure attached. For each $X = (x, y)$ in $\mathbb{R}^2$, the point in $\mathbb{R}^3$ with coordinates $(x, y, \frac{1}{12}y^3 - y - \frac{1}{4}x^2 + \frac{7}{2})$ is graphed.

Graphing functions of two variables is a much more difficult task than graphing functions of one variable. Of course, one method is to let a computer do the work. Nonetheless, if you want to get a feeling for functions of more than one variable, being able to sketch a rough graph by hand is still a valuable skill. The trick is to putting together a reasonable graph is to find a way to cut down on the dimensions involved. One way this can be achieved is by drawing certain special curves that lie on the surface $z = f(x, y)$. These special curves, called contour curves, are the ones obtain by intersecting the surface with horizontal planes $z = c$ for various values of the constant $z = c$. Some contour curves drawn on the surface of example 10 are shown in the figure. If we compress all contour curves onto the xy-plane (in essence if we look down along the positive z-axis), then we create a topographic map of the surface that is shown in the figure. These curves in the xy-plane are called the level curves of the original function f.

The point of the preceding discussion is that we can ~~...~~ reverse the process in order to sketch systematically the graph of a function f. ~~of~~ of two variables.

We first construct a topographic map in $\mathbb{R}^2$ by finding the level curves of f, then situating this curves in $\mathbb{R}^3$ at contour curves at the appropriate heights and finally complete the graph of the function. Before we give an example, let's state our terminology with greater precision.

Definition 1.4 Let $f: X \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a scalar valued function of two variables. The level curve at height c of f is the curve in $\mathbb{R}^2$ defined by the equation $f(x,y) = c$, where c is a constant. In mathematical notation,

$$\text{Level curve at height } c = \{(x,y) \in \mathbb{R}^2 \mid f(x,y) = c\}.$$

The contour curve at height c of f is the curve in $\mathbb{R}^3$ defined by the two equations $z = f(x,y)$ and $z = c$. Symbolized,

$$\text{Contour curve at height } c = \{(x,y,z) \in \mathbb{R}^3 \mid z = f(x,y) = c\}.$$

In addition to level and contour curves, consideration of the sections of a surface by the planes where x or y is held constant is also helpful. A section of a surface by a plane is just the intersection of the surface with that plane. Formally, we have the following definition.

Definition 1.5. Let $f: X \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a scalar-valued function of two variables. The section of the graph of f by the plane $x = c$ (where c is a constant, is a set of points (x,y,z), where $z = f(x,y,z)$, where $z = f(x,y)$ and $x = c$. Symbolized,

$$\text{Section by } x = c \text{ is } \{(x,y,z) \in \mathbb{R}^3 \mid z = f(x,y),\ x = c\}.$$

Similarly, the section of the graph of f by the plane $y = c$ is the set of points described as follows:

$$\text{Section by } y = c \text{ is } \{(x,y,z) \in \mathbb{R}^3 \mid z = f(x,y),\ y = c\}.$$

Example 11. We'll use level and contour curves to construct the graph of the function

$$f: \mathbb{R}^2 \to \mathbb{R},\quad f(x,y) = 4 - x^2 - y^2.$$

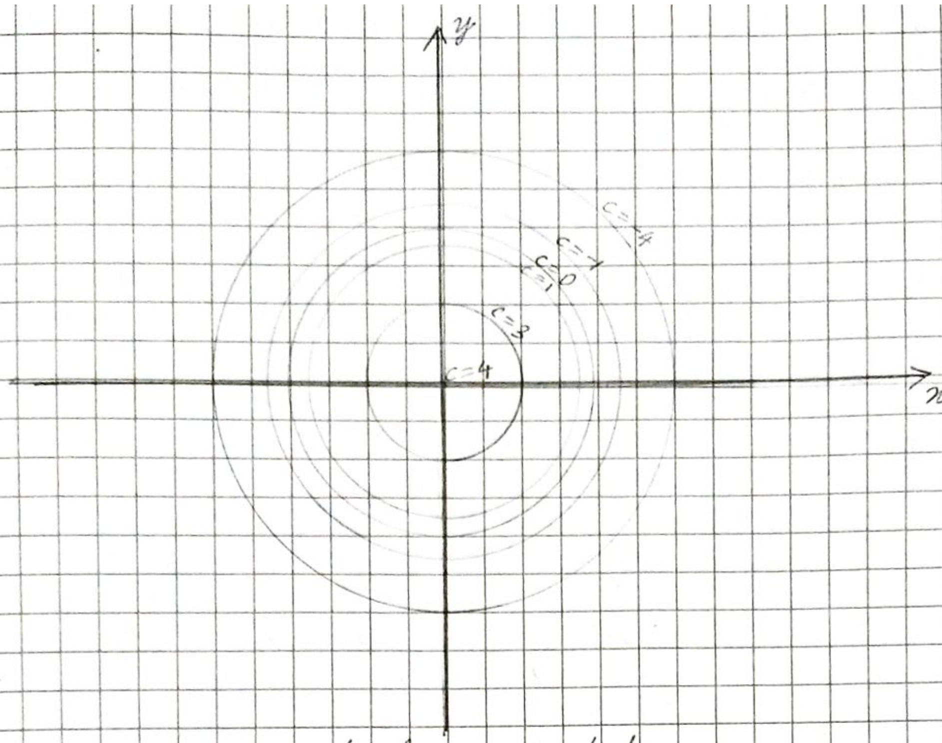
By definition 1.4, the level curve at height c is

$$\{(x,y) \in \mathbb{R}^2 \mid 4 - x^2 - y^2 = c\} = \{(x,y) \mid x^2 + y^2 = 4 - c\}.$$

Thus, we see that the level curves for $c < 4$ are circles centered at the origin of radius $\sqrt{4-c}$. The level curve at height $c = 4$ is not a curve at all but just a single point (the origin). Finally, there are no level curves at heights larger than 4 since the equation $x^2 + y^2 = 4 - 6$ has no real solutions in x and y. These remarks are summarized in the following table.

c	Level curve $x^2 + y^2 = 4 - c$
-5	$x^2 + y^2 = 9$
-1	$x^2 + y^2 = 5$
0	$x^2 + y^2 = 4$
1	$x^2 + y^2 = (\sqrt{3})^2$
3	$x^2 + y^2 = 1$
4	$x^2 + y^2 = 0$
$c > 4$	empty.

Thus, the family of level curves, the "topographic map" of the surface $z = 4 - x^2 - y^2$, is shown in the figure.



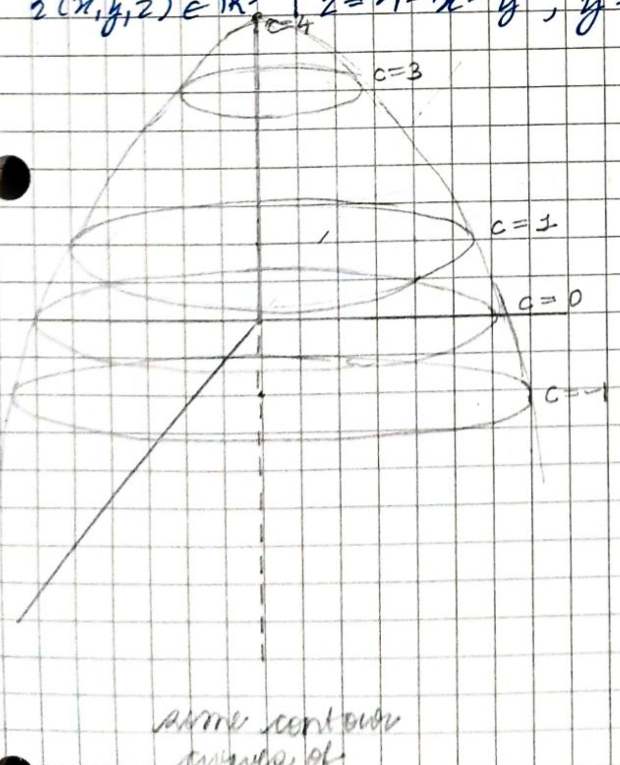
Level curves of f.

Some contour curves which sit in $\mathbb{R}^3$ are shown in the figure below where we can get a feeling for the complete graph of $z = 4 - x^2 - y^2$. It is a surface which looks like an inverted dish and is called a paraboloid. To make the picture clearer, we have also sketched in the sections of the surface by the planes $x = 0$ and $y = 0$. The section by $x = 0$ is given analytically by the set

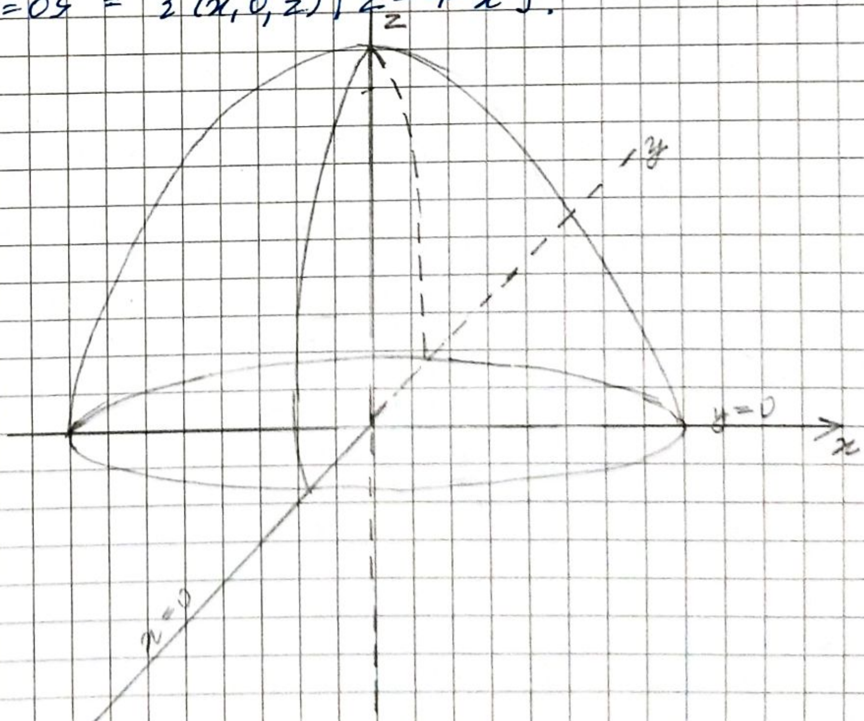
$$\{(x, y, z) \in \mathbb{R}^3 \mid z = 4 - x^2 - y^2,\ x = 0\} = \{(0, y, z) \mid z = 4 - y^2\}$$

Similarly, the section by $y = 0$ is

$$\{(x, y, z) \in \mathbb{R}^3 \mid z = 4 - x^2 - y^2,\ y = 0\} = \{(x, 0, z) \mid z = 4 - x^2\}.$$



some contour curves of $z = 4 - x^2 - y^2$



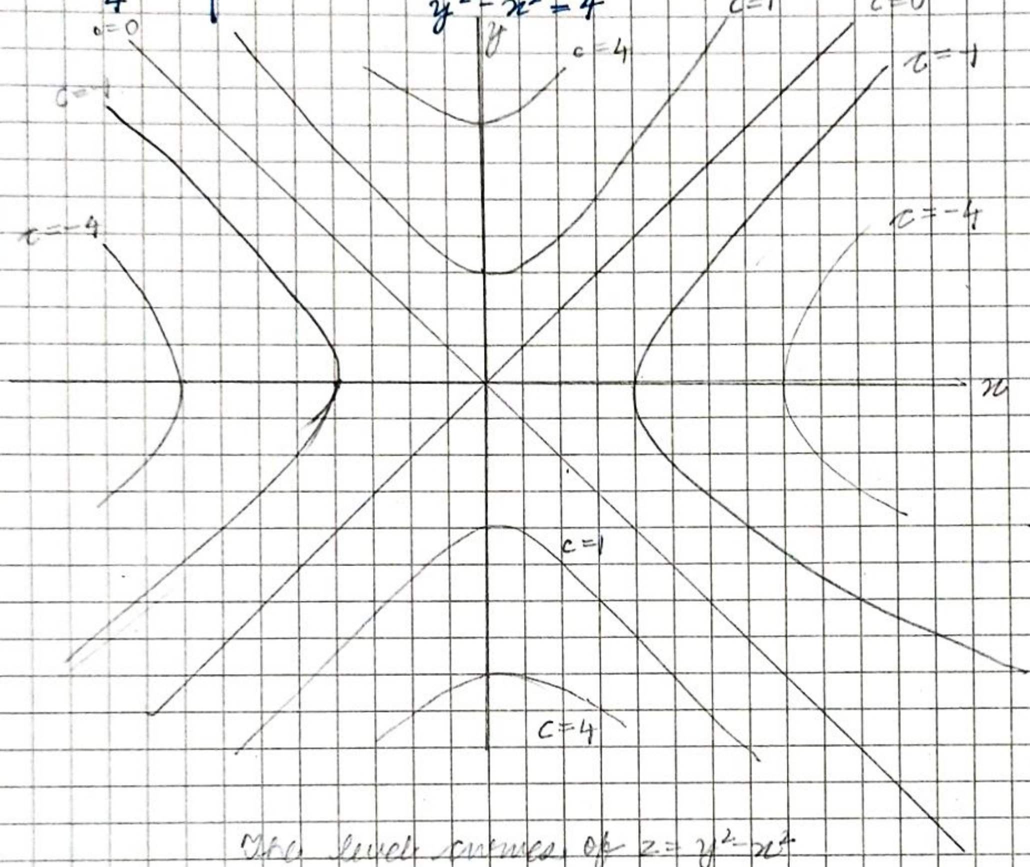
The graph of $z = 4 - x^2 - y^2$.

since the sections are parabolas, it is easy to see, how the surface obtained its name.

Example 12.

We'll graph the function $g: \mathbb{R}^2 \to \mathbb{R}$, $g(x,y) = y^2 - x^2$. The level curves are all hyperbolas, with the exception of the level curve at height 0, which is a pair of intersecting lines.

c	Level curve $y^2 - x^2 = c$
-4	$x^2 - y^2 = 4$
-1	$x^2 - y^2 = 1$
0	$(y-x)\cdot(y+x) = 0$
1	$y^2 - x^2 = 1$
4	$y^2 - x^2 = 4$



The level curves of $z = y^2 - x^2$

The collection of the level curves is graphed above. The sections by $x = c$ are

$$\{(x,y,z) \in \mathbb{R}^3 \mid z = y^2 - x^2, x = c\} = \{(c,y,z) \mid z = y^2 - c^2\}$$

These are clearly parabolas in the plane $x = c$. The sections by $y = c$ are

$$\{(x,c,z) \mid z = c^2 - x^2\}$$

which are again parabolas. The level curves and sections generate a surface called the hyperbolic paraboloid.

Example 13.

We compare the graphs of the functions $f(x,y) = 4 - x^2 - y^2$ of example 11 with that of

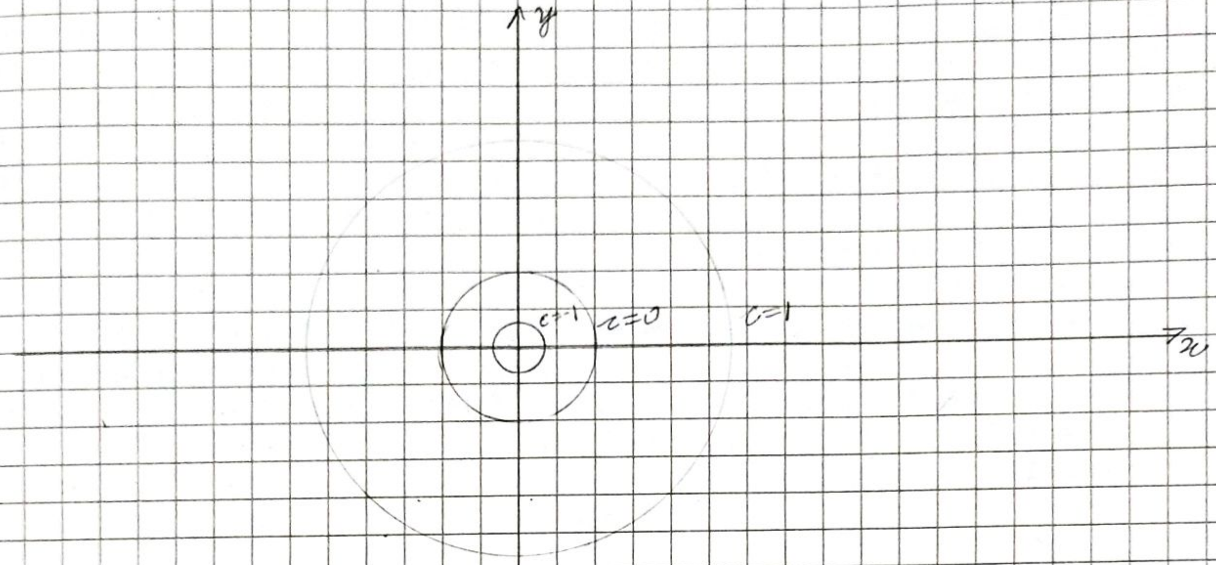
$$h: \mathbb{R}^2 - \{(0,0)\} \to \mathbb{R}, \quad h(x,y) = \ln(x^2 + y^2).$$

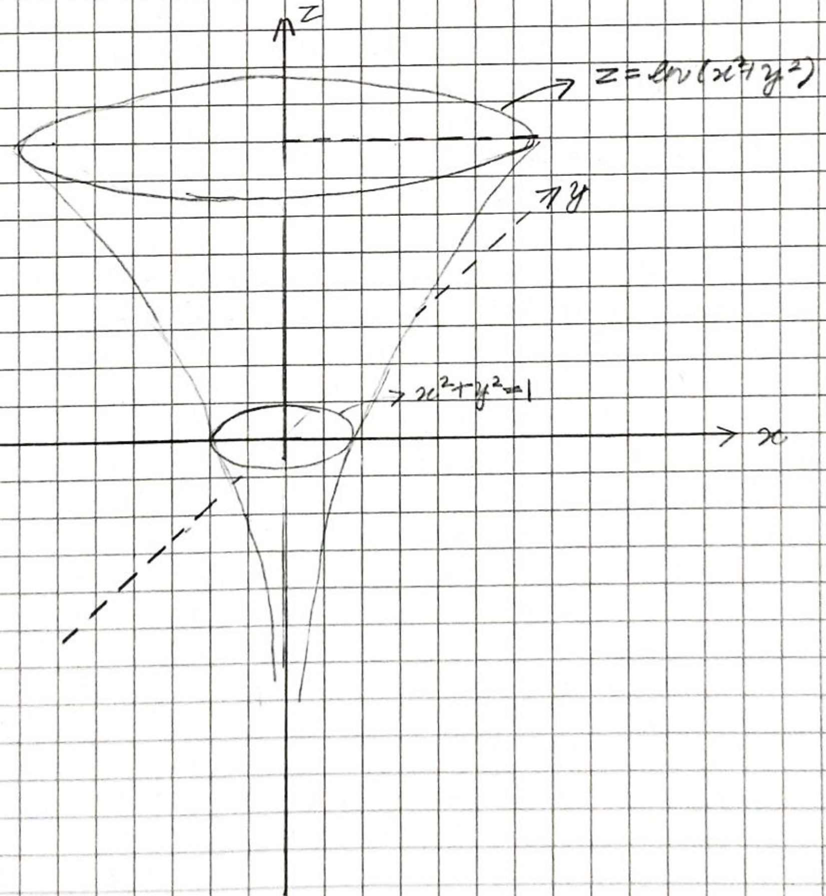
The level curve of h at height c is

$$\{(x,y) \in \mathbb{R}^2 \mid z = \ln(x^2+y^2) = c\} = \{(x,y) \mid x^2 + y^2 = e^c\}.$$

Since $e^c > 0$ for all $c \in \mathbb{R}$, we see that the level curve exists for all c and is a circle of radius $\sqrt{e^c} = e^{c/2}$.

c	Level curve $x^2+y^2=e^c$
-5	$x^2+y^2=e^{-5}$
-1	$x^2+y^2=e^{-1}$
0	$x^2+y^2=1$
1	$x^2+y^2=e$
3	$x^2+y^2=e^3$
4	$x^2+y^2=e^4$





The section of the graph by $x=0$ is

$$\{(x,y,z)\in\mathbb{R}^3 \mid z=\ln(x^2+y^2),\ x=0\} = \{(0,y,z)\in\mathbb{R}^3 \mid z=2\ln|y|\}$$

The section by $y=0$ is entirely similar:

$$\{(x,y,z)\in\mathbb{R}^3 \mid z=\ln(x^2+y^2),\ y=0\} = \{(x,0,z)\in\mathbb{R}^3 \mid z=2\ln|x|\}.$$

In fact, if we switch from Cartesian coordinates to cylindrical coordinates, it is quite easy to understand the surfaces in both examples 11 and 13. In view of the Cartesian / cylindrical relation $x^2+y^2=r^2$, we see that the function for f of example 11,

$$z=4-x^2-y^2=4-(x^2+y^2)=4-r^2.$$

For the function h of example 13, we have

$$z=\ln(x^2+y^2)=\ln(r^2)=2\ln(r).$$

where we assume the usual convention that the cylindrical coordinate r is non-negative. Thus, both these graphs are of surfaces of revolution obtained by revolving different curves around the z-axis (in case of example ~~ex~~ 11, a parabola $x^2=-(z-4)$ facing downward with ~~origin~~ center $(0,4)$, and in example 12, the curve $\ln|x|$). As a result, the level curves are, in general circular.

The preceding discussion has been devoted entirely to graphing scalar-valued functions of just two variables. However, all the ideas can be extended to more variables and higher dimensions. If $f: X\subseteq\mathbb{R}^n\to\mathbb{R}$ is a (scalar-valued) function of n variables, then the graph of f is the subset of $\mathbb{R}^{n+1}$ given by

$$\begin{aligned}\text{Graph } f &= \{(\mathbf{x}, f(\mathbf{x})) \mid \mathbf{x}\in X\} \\ &= \{(x_1,x_2,\dots,x_n, x_{n+1}) \mid (x_1,\dots,x_n)\in X \text{ and } x_{n+1}=f(x_1,\dots,x_n)\}.\end{aligned} \tag{2}$$

The compactness of the vector notation makes the definition of the graph of the function of n variables exactly the same as in (1). The level set at height c of such a function is defined by

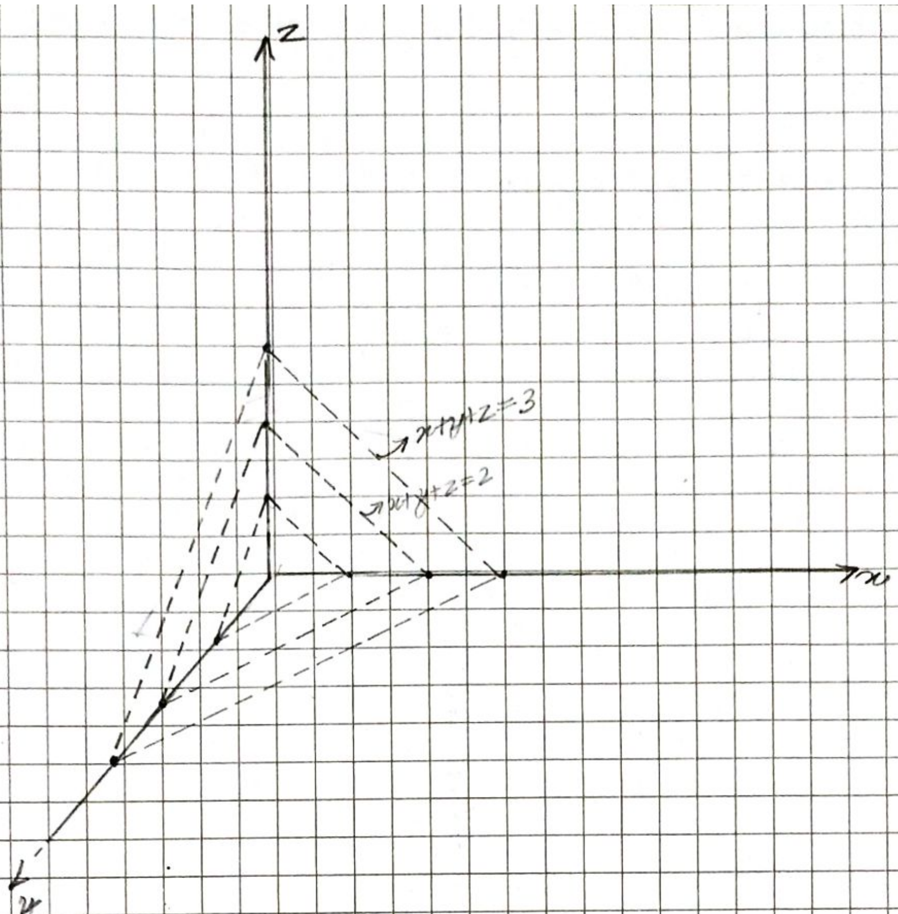
$$\begin{aligned}\text{Level set at height } c &= \{\mathbf{x}\in\mathbb{R}^n \mid f(\mathbf{x})=c\} \\ &= \{(x_1,x_2,\dots,x_n) \mid f(x_1,\dots,x_n)=c\}.\end{aligned}$$

While the graph of f is a subset of $\mathbb{R}^{n+1}$, a level set of f is a subset of $\mathbb{R}^n$. This makes it possible to get some geometric insight into graphs of functions of three variables, even though we cannot actually visualize them.

<u>Example 14.</u>
Let $F:\mathbb{R}^3\to\mathbb{R}$ be given by $F(x,y,z)=x+y+z$. Then, the graph of F is the set $\{(x,y,z,w) \mid w=x+y+z\}$ and is a subset of $\mathbb{R}^4$, called a hypersurface, which we cannot depict adequately. Nonetheless, look at the level sets of F, which are surfaces in $\mathbb{R}^3$.

$$\text{Level set at height } c = \{(x,y,z) \mid x+y+z=c\}.$$

Thus, the level sets form a family of parallel planes with the normal vector $\mathbf{i}+\mathbf{j}+\mathbf{k}$.



Surfaces in general

Not all curves in $\mathbb{R}^2$ can be described as the graph of a single function of one variables. Perhaps the most familiar example is the unit circle. Its graph cannot be determined by a single equation of the form $y=f(x)$ (or for that matter by one of the form $x=g(y)$). As we know, the graph of the circle may be described analytically by the equation $x^2+y^2=1$. In general, a curve in $\mathbb{R}^2$ is determined by an arbitrary equation in x and y, not necessarily the one that isolates y alone on one side in terms of x. In other words this means that, the general curve given by the equation of the form $F(x,y)=c$ (that is a level set of a function of two variables).

The analogous situation occurs with surfaces in $\mathbb{R}^3$. Frequently, a surface is determined by an equation of the form $F(x,y,z)=c$ (that is a level set of a function of three independent variables), not necessarily one of the form $z=f(x,y)$.

Example 15. A sphere is a surface in $\mathbb{R}^3$ whose points are all equidistant from a fixed point. If this fixed point is the origin, then the equation for the sphere

$$\|x-0\|=\|x\|=a \tag{3}$$

where a is positive constant and $x=(x,y,z)$ is a point on the sphere. If we square both sides of the equation (3) and expand the (implicit) dot product then we obtain perhaps the familiar equation of a sphere of radius a centered at the origin:

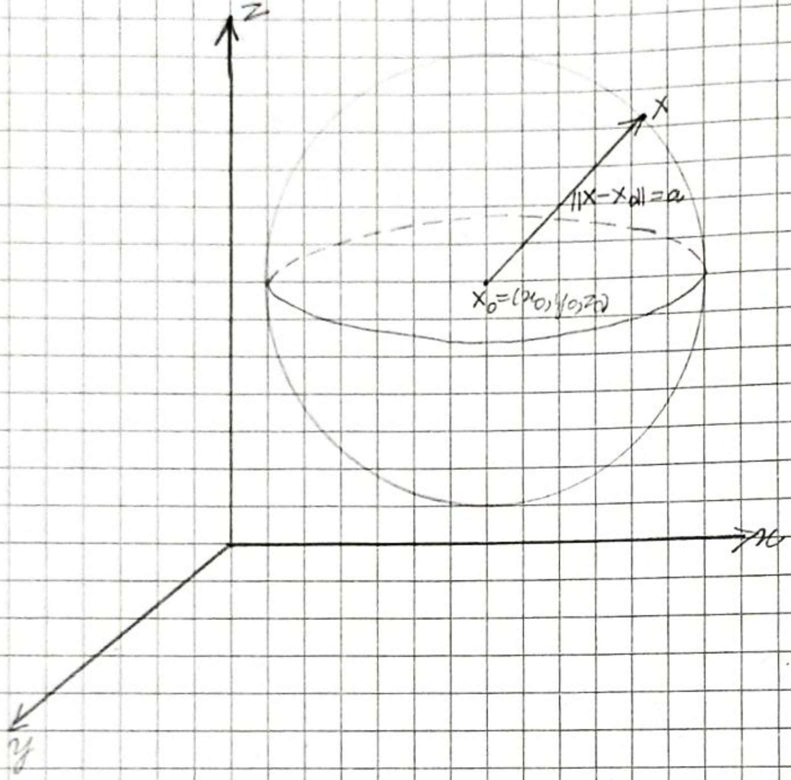
$$x^2+y^2+z^2=a^2 \tag{4}$$

If the center of the sphere is at point $x_0=(x_0,y_0,z_0)$, rather than the origin, then equation (3) should be modified to

$$\|X - X_0\| = a \tag{5}$$

When equation (5) is expanded, the following general equation for a sphere is obtained

$$(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = a^2 \tag{6}$$



In the equation for a sphere, there is no way to solve for z uniquely in terms of x and y. Indeed, if we try to isolate z in equation (4), then

$$z^2 = a^2 - x^2 - y^2,$$

so we are forced to make a choice of positive or negative square roots in order to solve for z:

$$z = \sqrt{a^2 - x^2 - y^2}$$
$$\text{or } z = -\sqrt{a^2 - x^2 - y^2}$$

The positive square root corresponds to the upper hemisphere and the negative square root to the lower one. In any case, the entire sphere cannot be the graph of a single function of two variables.

Of course the graph of a function of two variables describes a surface in the "level set" sense. If a surface happens to be given by an equation of the form

$$z = f(x, y).$$

for some appropriate function $f: X \subseteq \mathbb{R}^2 \to \mathbb{R}$, then we can move z to the opposite side obtaining

$$f(x,y) - z = 0.$$

If we define a new function F of three variables by

$$F(x,y,z) = f(x,y) - z$$

then the graph of f is precisely the level set at height 0 of F.

We reiterate this point, since it is all too often forgotten. The graph of a function of two variables is a surface in $\mathbb{R}^3$ and is a level set of a function of three variables. However, not all level sets of functions of three variables are graphs of functions of two variables.

Quadric surfaces.

Conic sections, those curves obtained from the intersection of a cone with various planes are among the simplest, yet also the most interesting of plane curves: they are the circle, the parabola, the ellipse and the hyperbola. Besides being produced in a similar geometric manner, conic sections have an elegant algebraic connection: every conic section is described analytically by a polynomial equation of degree two in two variables. That is every conic can be described by the equation that looks like:

$$Ax^2+Bxy+Cy^2+Dx+Ey+F=0$$

for suitable constants $A,\dots,F$.

In $\mathbb{R}^3$, the analytic analogue of the conic section is called a quadric surface. Quadric surfaces are those defined by equations that are polynomials of degree 2 in three variables.

$$Ax^2+Bxy+Cxz+Dy^2+Eyz+Fz^2+Gx+Hy+Iz+J=0.$$

To pass from this equation to the appropriate graph is, in general, a cumbersome process without the aid of either a computer or more linear algebra than we currently have at our disposal. So, instead we offer examples of those quadric surfaces whose axes of symmetry lie along the coordinate axes in $\mathbb{R}^3$ and whose corresponding analytic equations are relatively simple. In the discussion that follows, a, b and c are constants, which for convenience, we take to be positive.

Ellipsoid.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

This is three dimensional analogue of an ellipse in the plane. The sections of the ellipsoid by the planes perpendicular to the coordinate axes are all ellipses. For example, if the ellipsoid is intersected by with the plane $z=0$, one obtains the standard ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ z=0.$$

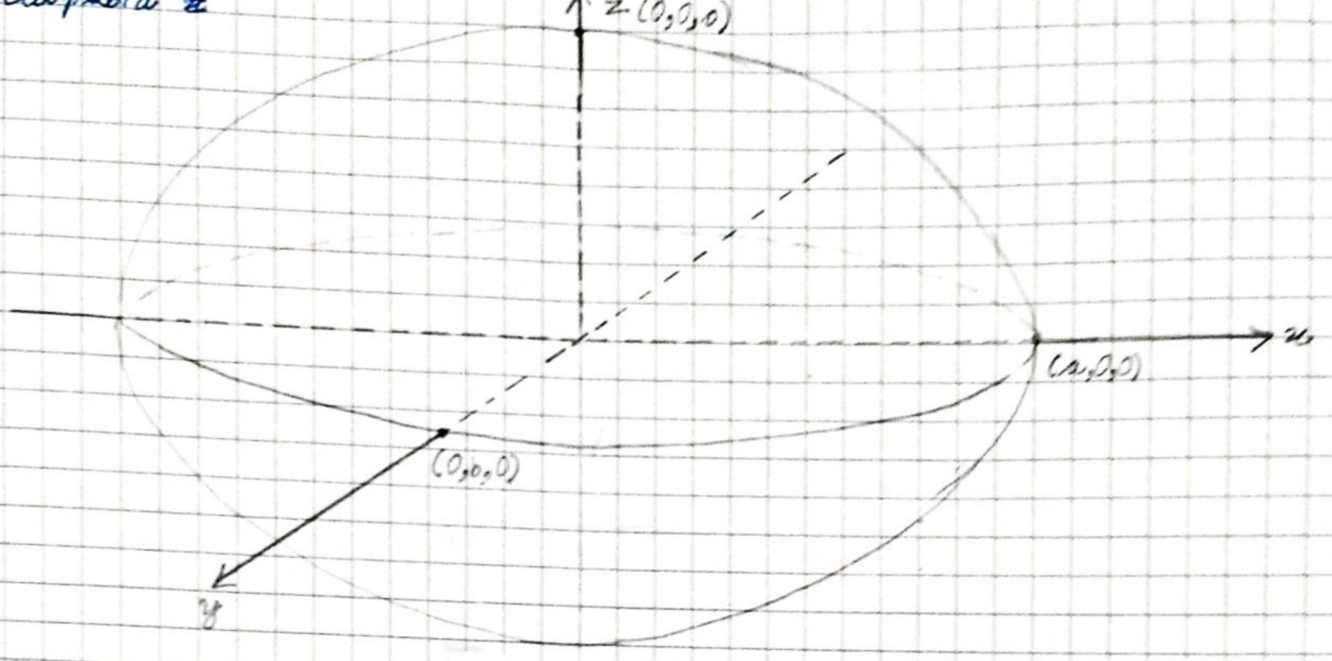
If $a=b=c$, then the ellipsoid is a sphere of radius a.

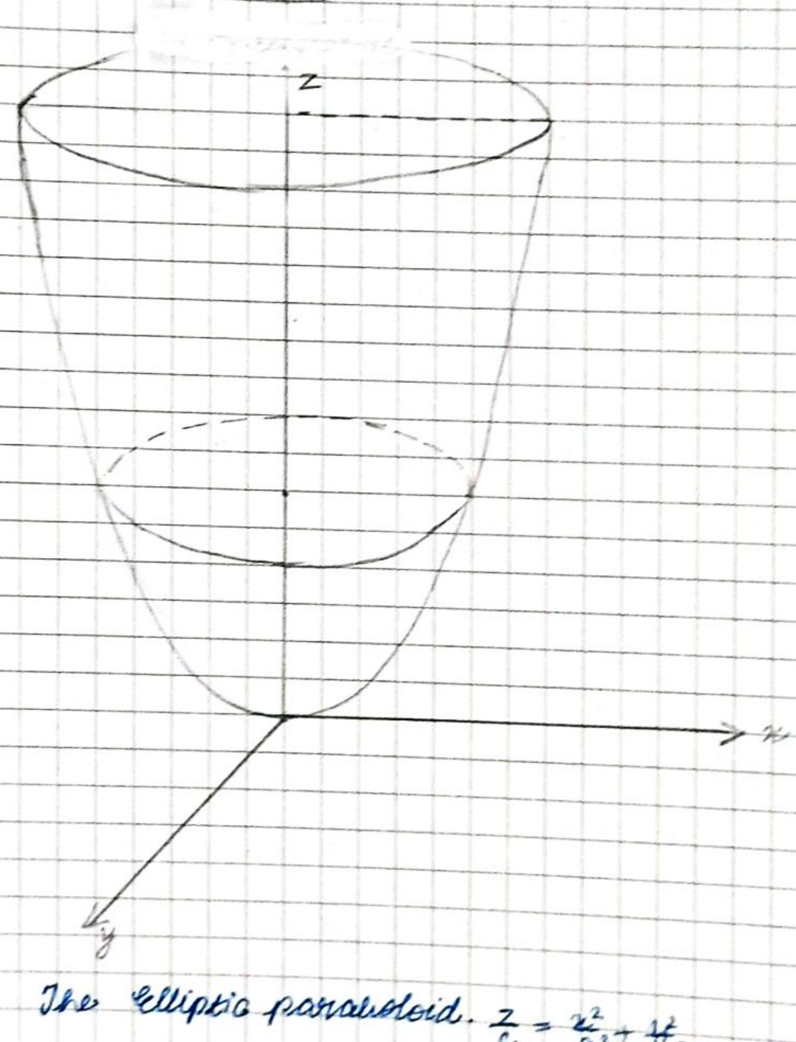
Elliptic Paraboloid.

$$\frac{z}{c}=\frac{x^2}{a^2}+\frac{y^2}{b^2}.$$

(The roles of x, y and z may be interchanged.) This surface is the graph of a function of x and y. The paraboloid ~~is elliptical~~ has elliptical sections by the planes "$z=$ constant" and parabolic sections by "$x=$ constant" and "$y=$ constant" planes. The constants a and b affect the aspect ratio of the elliptical cross-sections and the constant c affects the steepness of the dish. (Larger values of c produce steeper paraboloids).

The Ellipsoid





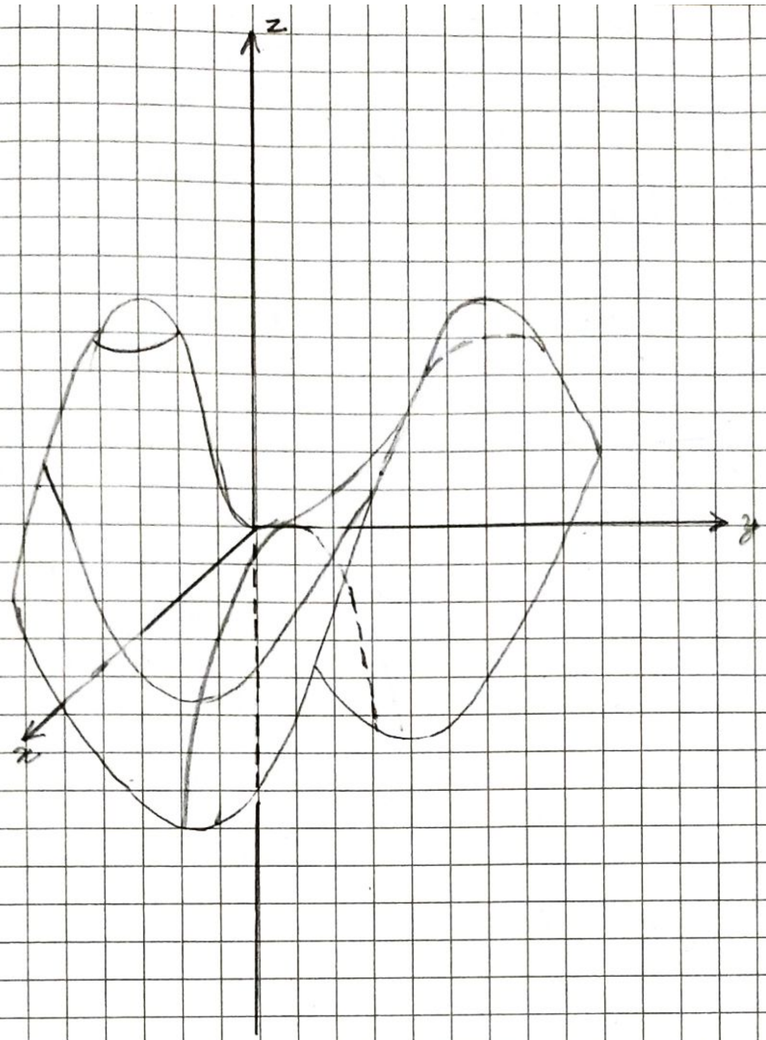
The Elliptic paraboloid. $\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

Hyperbolic paraboloid.

$$\frac{z}{c} = \frac{y^2}{b^2} - \frac{x^2}{a^2}.$$

(again the roles of x, y, z may be interchanged)

We saw the graph of this surface earlier in example 12 of this section. It is shaped like a saddle whose "x = constant" or "y = constant" sections are parabolas and z = "constant" sections are hyperbolas.

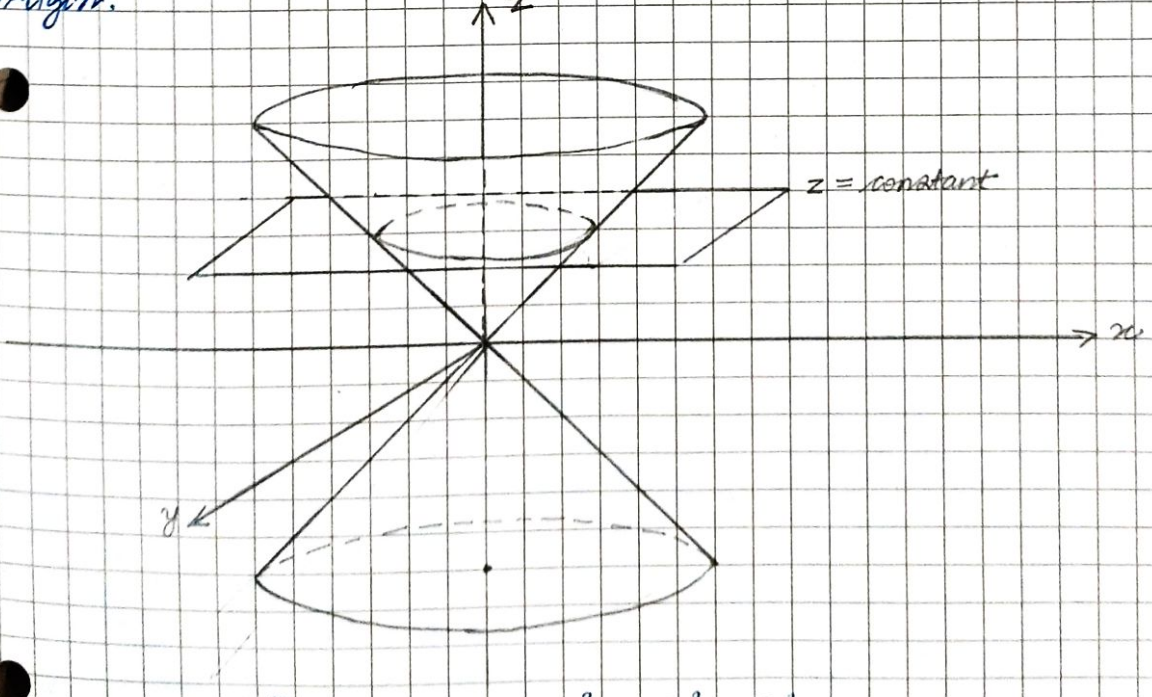


The hyperbolic paraboloid. $\frac{z}{c} = \frac{y^2}{b^2} - \frac{x^2}{a^2}$.

Elliptic Cone.

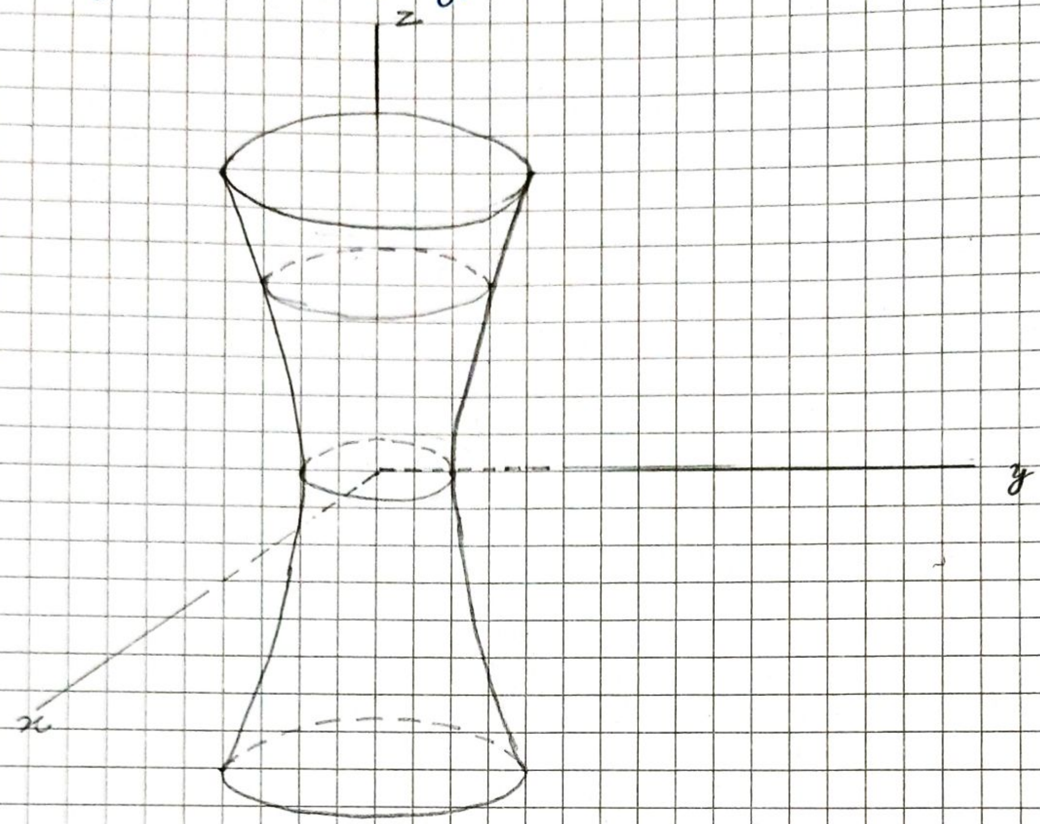
$$z^2/c^2 = x^2/a^2 + y^2/b^2.$$

The sections by z = constant planes are ellipses. The sections by $x = 0$ and $y = 0$ are each a pair of intersecting ~~s~~ lines, passing through the origin.



The elliptic cone $\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$.

Hyperboloid of one sheet. $x^2/a^2 + y^2/b^2 - z^2/c^2 = 1$.
The term "one sheet" signifies that the surface is connected (that is you can travel between any two points on the surface without having to leave the surface). The sections by "z = constant" planes are ellipses and those by x = constant and y = constant are hyperbolas, hence this surface's name.



The graph of the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ is a hyperboloid of one sheet.

Hyperboloid of two sheets. $z^2/c^2 - x^2/a^2 - y^2/b^2 = 1$.
The fact that the left hand side of the defining equation is the opposite of the left side of the previous hyperboloid is what causes this surface to consist of two pieces instead of one. More precisely, consider the sections of the surface by planes of the form $z = k$ for different constants k. These sections are thus given by

$$\frac{k^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

or equivalently

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{k^2}{c^2} - 1, \quad z = k$$

If $-c < k < c$, then $0 \leq k^2/c^2 < 1$. Thus, $k^2/c^2 - 1 < 0$ and so the preceding equation has no solutions in x and y. Hence the section by $z = k$ where $|k| < c$, is empty. If $|k| > c$, then the section is an ellipse.
The sections by "x = constant" or "y = constant" planes are hyperbolas.

In the same way that the hyperbolas

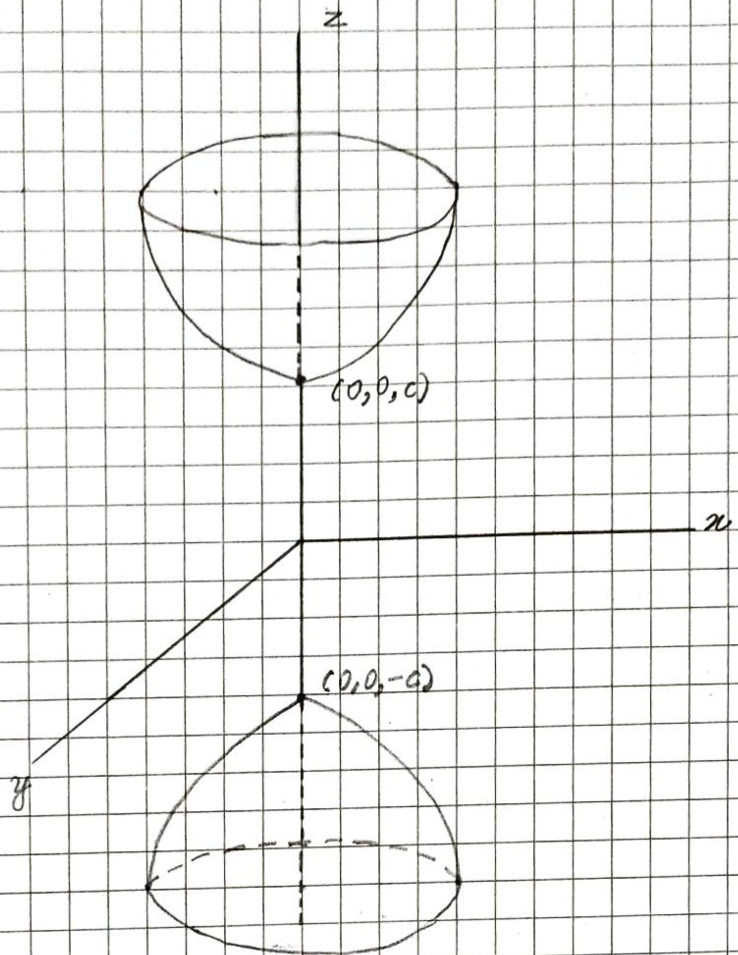
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \pm 1.$$

are asymptotic to the lines $y = \pm (b/a)x$, the hyperboloids

$$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2} \pm 1$$

are asymptotic to the cone

$$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}.$$



This is perhaps intuitively clear, but let's see how to prove this rigorously. In our present context, to say that the hyperboloids are asymptotic to the cone means they look more and more like cones as $|z|$ becomes arbitrarily large. Analytically, this should mean that the equations for the hyperboloids should approximate the equation for the cone for sufficiently large $|z|$. The equations for the hyperboloids can be written as follows –

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2} \pm 1 = \frac{z^2}{c^2}\left(1 \pm \frac{c^2}{z^2}\right).$$

As $|z| \to \infty$, $c^2/z^2 \to 0$, so the right hand side of the equation for the hyperboloids approaches z^2/c^2. Hence the equations for the hyperboloids approximate that of the cone as desired.

Problems.

1) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 2x^2+1$.
   (a) Find the domain and range of f.
   (b) Is f one-to-one?
   (c) Is f onto?

Solution.

The domain of f is $\mathbb{R}$. The range of f is $[1, \infty)$.

Let $2x_1^2+1 = 2x_2^2+1$

$x_1^2 = x_2^2$

$\therefore x_1 = x_2$ or $x_1 = -x_2$.

Hence, f is a two-to-one function. It is not one-to-one.
f is also not onto. For example, there is no x in X, such that $f(x) = 0$

2) Let $g : \mathbb{R}^2 \to \mathbb{R}$ be given by $g(x,y) = 2x^2 + 3y^4 - 7$.
(a) Find the domain and range of g.
(b) Find a way to restrict the domain to make a new function with the same rule of assignment as g that is one-to-one.
(c) Find a way to restrict the co-domain to make a new function with the same rule of assignment as g that is onto.

Solution.

a) The domain of g is $\mathbb{R}$. The range of g is $[-7, \infty)$.

b) If we restrict the domain to $[0, \infty)$, g g maps distinct elements to distinct images. g becomes one-to-one.

c) If we restrict the codomain to $[-7, \infty)$, g becomes surjective.

- Find the domain and range of each of the functions given in the problems 3-7.

3) $f(x,y) = \frac{x}{y}$.

Domain of $f = \{(x,y) \mid y \neq 0\}$
Range of $f = \{ z \mid z = f(x,y)\} = \mathbb{R}$.

so, f is defined on the entire plane $\mathbb{R}^2$ except the x-axis ($y = 0$).
The range of f is $\mathbb{R}$.

4) $f(x,y) = \ln(x+y)$.

The domain of f is all $(x,y) \in \mathbb{R}^2$ such that $x + y > 0$.
The range of f is the entire real line $\mathbb{R}$.

5) $g(x,y,z) = \sqrt{x^2 + (y-2)^2 + (z+1)^2}$

As the sum of squares of real numbers is always non-negative,
$x^2 + (y-2)^2 + (z+1)^2 \geq 0$.
so, g is defined for all $(x,y,z) \in \mathbb{R}^3$.
The range of g is $[0, \infty)$.

6) $g(x,y,z) = \dfrac{1}{\sqrt{4 - x^2 - y^2 - z^2}}$

The domain of g is all $(x,y,z) \in \mathbb{R}^3$, such that
$$4 - x^2 - y^2 - z^2 > 0$$
$$x^2 + y^2 + z^2 < 4.$$

so, the domain is a sphere of radius 2, excluding its surface.
The range of g is $\left[\frac{1}{2}, \infty\right)$.

7) $f(x,y) = \left(x + y, \dfrac{1}{y-1}, x^2 + y^2\right)$.

so.
The domain of $f = \{(x,y) \in \mathbb{R}^2 \mid y \neq 1\}$.
The range of $f = \{(u,v,w) \in \mathbb{R}^3 \mid u = x+y,\ v = 1/y-1,\ w = x^2+y^2\}$
$= \{(u,v,w) \in \mathbb{R}^3 \mid v^2 w = (uv - 1 - v)^2 + (v+1)^2,\ v \neq 0\}$

8) Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $f(x,y) = (x+y, ye^x, x^2y + 7)$.
Determine the component functions of f.

The component functions are $f = (f_1(x,y), f_2(x,y), f_3(x,y))$
where $f_1(x,y) = x+y$
$f_2(x,y) = ye^x$
$f_3(x,y) = x^2y+7$.

9) Determine the component functions of the function v in example 9.

$v(x,y,z,t) = (xyzt, x^2-y^2, 3z+t)$

Let $v = (v_1, v_2, v_3)$.
Then,
$v_1(x,y,z,t) = xyzt$
$v_2(x,y,z,t) = x^2-y^2$
$v_3(x,y,z,t) = 3z+t$.

10) Let $f: \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $f(x) = x+3j$. Write out the component functions of f in terms of the components of the vector x.

Let $x = (x_1, x_2, x_3) = x_1\hat{i} + x_2\hat{j} + x_3\hat{k}$.
$f(x) = x+3j = x_1\hat{i} + (x_2+3)\hat{j} + x_3\hat{k}$.

$f_1(x) = x_1$
$f_2(x) = x_2+3$
$f_3(x) = x_3$.

11) Consider the mapping that assigns to a non-zero vector x in $\mathbb{R}^3$ the vector of length 2 that points in the direction opposite to x.

(a) Give an analytic (symbolic) description of this mapping.
(b) If $x = (x, y, z)$, determine the component functions of this mapping.

Solution.
The unit vector with magnitude 1 and having the same direction as x is given by

$$u = \frac{x}{\|x\|}.$$

The vector of magnitude 2 having its direction opposite to x is,

$$-2u = -\frac{2x}{\|x\|}$$

f maps $x \mapsto -\frac{2x}{\|x\|}$. $f(x) = -\frac{2x}{\|x\|}$

Let $f = (f_1, f_2, f_3)$.
We can write f out in terms of its component functions

$$f = (f_1, f_2, f_3) = \left(\frac{-2x_1}{\sqrt{x_1^2+x_2^2+x_3^2}}, \frac{-2x_2}{\sqrt{x_1^2+x_2^2+x_3^2}}, \frac{-2x_3}{\sqrt{x_1^2+x_2^2+x_3^2}} \right)$$

12) Consider the function $f: \mathbb{R}^2 \to \mathbb{R}^3$ given by $f(x) = Ax$, where

$$A = \begin{bmatrix} 2 & -1 \\ 5 & 0 \\ -6 & 3 \end{bmatrix}$$

and the vector x in $\mathbb{R}^2$ is written as the 2×1 column matrix $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

(a) Explicitly determine the component functions of f in terms of the components x_1, x_2 of the vector (that is the column matrix) x.

(b) Describe the range of f.

Solution.

a)
$$Ax = \begin{bmatrix} 2 & -1 \\ 5 & 0 \\ -6 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 - x_2 \\ 5x_1 \\ 6x_1 + 3x_2 \end{bmatrix}$$

Let $f = (f_1, f_2, f_3)$
$f_1(x_1, x_2, x_3) = 2x_1 - x_2$
$f_2(x_1, x_2, x_3) = 5x_1$
$f_3(x_1, x_2, x_3) = -6x_1 + 3x_2$.

b) The range of f is the column space of A.
we find Range of $f = \left\{ \alpha \begin{pmatrix} 2 \\ 5 \\ -6 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix} : \alpha, \beta \in \mathbb{R} \right\}$.

Alternately suppose $y = (y_1, y_2, y_3)$ belongs to range (f).

$$2x_1 - x_2 = y_1$$
$$5x_1 = y_2$$
$$-6x_1 + 3x_2 = y_3$$

$5x_1 = y_2 \Rightarrow 2\left(\frac{y_2}{5}\right) - x_2 = y_1 \Rightarrow x_2 = \frac{2y_2}{5} - y_1$

substituting the values $x_1 = \frac{y_2}{5}$ and $x_2 = \frac{2y_2}{5} - y_1$ in $6x_1 + 3x_2 = y_3$, we have –

$$-\frac{6y_2}{5} + 3\left(\frac{2y_2}{5} - y_1\right) = y_3$$
$$-\frac{6y_2}{5} + \frac{6y_2}{5} - 3y_1 = y_3$$
$$y_3 = -3y_1$$

range $f = \{ (y_1, y_2, y_3 \in \mathbb{R}^3 \mid y_3 = -3y_1 \}$

3) consider the function $f : \mathbb{R}^4 \to \mathbb{R}^3$ given by $f(x) = Ax$ where

$$A = \begin{bmatrix} 2 & 0 & -1 & 1 \\ 0 & 3 & 0 & 0 \\ 2 & 0 & -1 & 1 \end{bmatrix}$$

and the vector x in $\mathbb{R}^4$ is written as the 4×1 column matrix

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

(a) Determine the component functions of f in terms of the components x_1, x_2, x_3, x_4 of the vector (that is the column matrix) x.

(b) Describe the range of f.

Solution.

$$f(x) = Ax = \begin{bmatrix} 2 & 0 & -1 & 1 \\ 0 & 3 & 0 & 0 \\ 2 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_1 - x_3 + x_4 \\ 3x_2 \\ 2x_1 - x_3 + x_4 \end{bmatrix}$$

$f_1(x_1, x_2, x_3, x_4) = 2x_1 - x_3 + x_4$
$f_2(x_1, x_2, x_3, x_4) = 3x_2$
$f_3(x_1, x_2, x_3, x_4) = 2x_1 - x_3 + x_4$

The range of $f = \{(y_1, y_2, y_3) \in \mathbb{R}^3 \mid y_1 = y_3\}$.

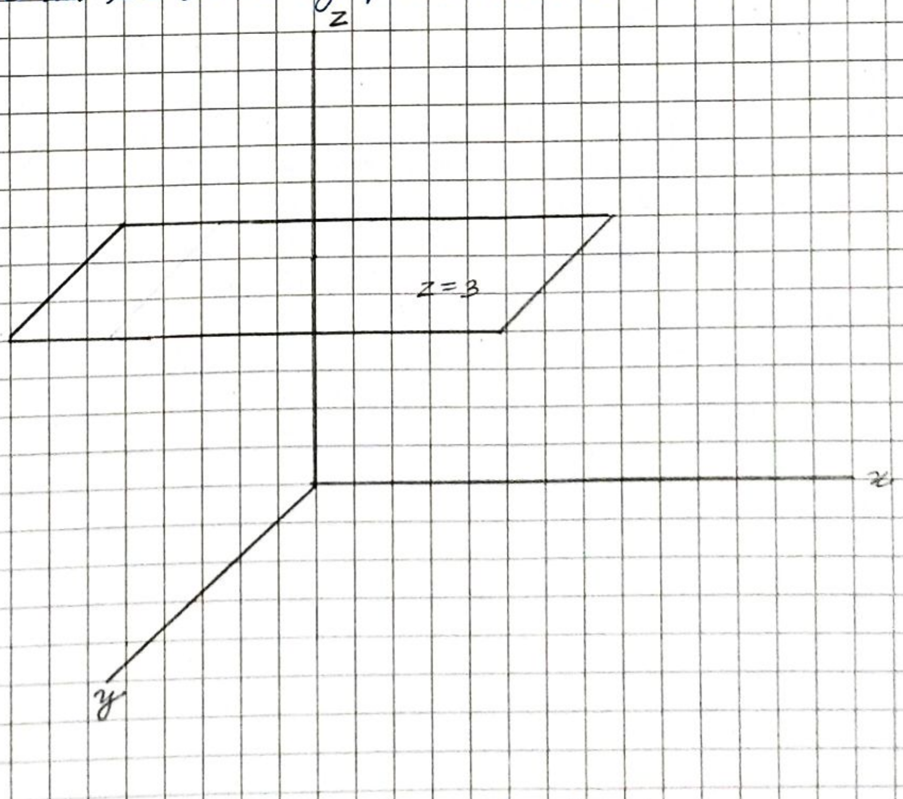
In each of the problems 14-23 (a) determine several level curves of the given function (make sure to indicate the height c of each curve); (b) use the information obtained in part (a) to sketch the graph of f.

14) $f(x, y) = 3$
Solution

The level curve at height c are given by

$\{$ Level curve at height $c = \{(x, y) \mid z = 3, z = c\}$

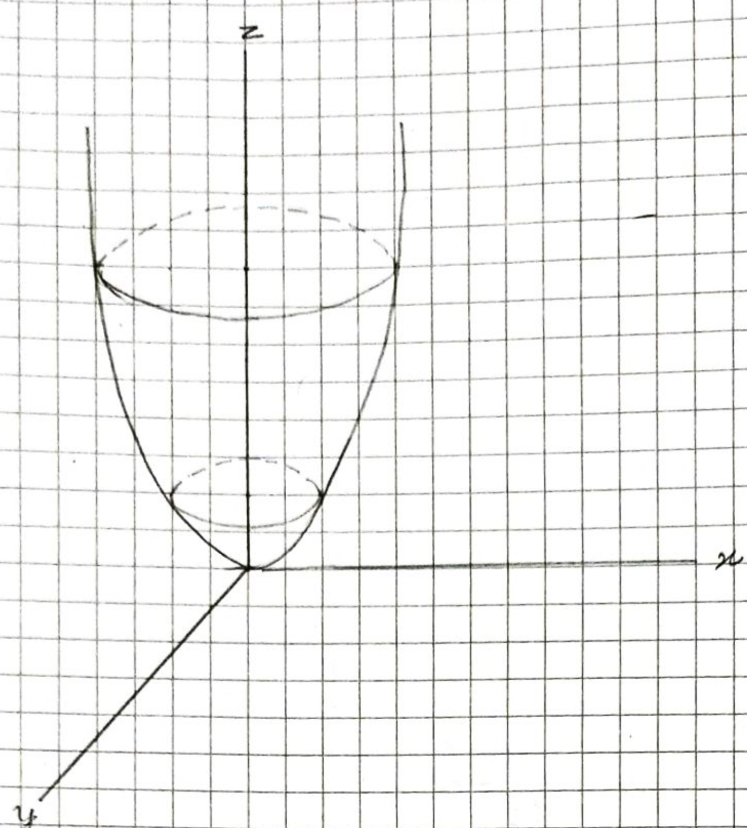
so, the level curve is the entire xy-plane when $c = 3$ or it is the empty set when $c \neq 3$.



15) $f(x, y) = x^2 + y^2$.

c	Level curve at height c
0	$(0, 0)$
1	$x^2 + y^2 = 1^2$
4	$x^2 + y^2 = 2^2$
9	$x^2 + y^2 = 3^2$.

Level curves at height $c = \{(x, y) \in \mathbb{R}^2 \mid z = x^2 + y^2, z = c\}$.



Paraboloid $z = x^2+y^2$

The section by $x=0$ is

$$\{(0,y,z) \mid z = x^2+y^2,\ x=0\} = \{(0,y,z) \mid z = y^2\}.$$

The section by $y=0$ is

$$\{(x,0,z) \mid z = x^2+y^2,\ y=0\} = \{(x,0,z) \mid z = x^2\}.$$

16) $f(x,y) = x^2+y^2-9.$

The level curve at height c is given by,

$$\text{Level curve at height } c = \{(x,y)\in\mathbb{R}^2 \mid z = x^2+y^2-9,\ z=c\}$$
$$= \{(x,y)\in\mathbb{R}^2 \mid x^2+y^2 = c+9\}.$$

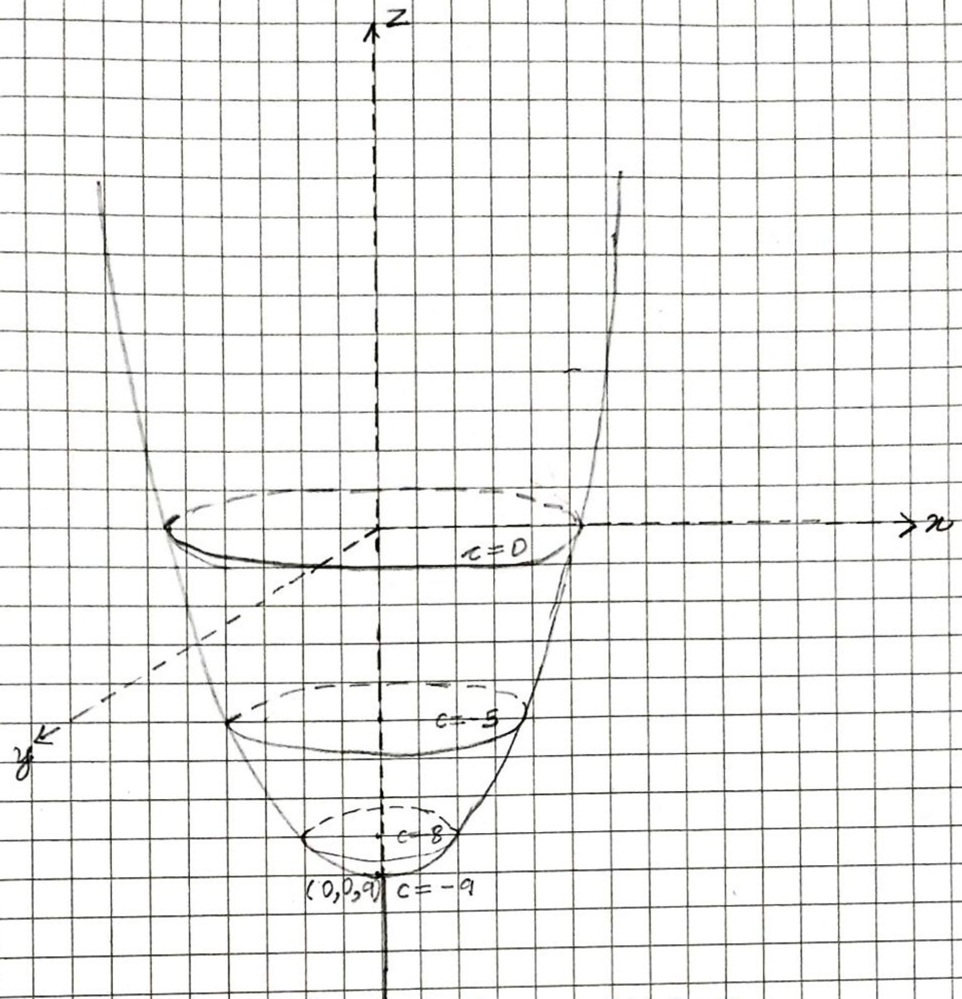
The family of level curves are:

c	Level curve at height c
-9	$(0,0)$
-8	$x^2+y^2=1^2$
-5	$x^2+y^2=2^2$
0	$x^2+y^2=3^2$
7	$x^2+y^2=4^2$
16	$x^2+y^2=5^2$.

17) $f(x,y) = \sqrt{x^2+y^2}.$

The level curve at height c is given by

$$\text{Level curve at height } c = \{(x,y)\in\mathbb{R}^2 \mid z = \sqrt{x^2+y^2},\ z=c\}.$$



Paraboloid $z=x^2+y^2-9$.

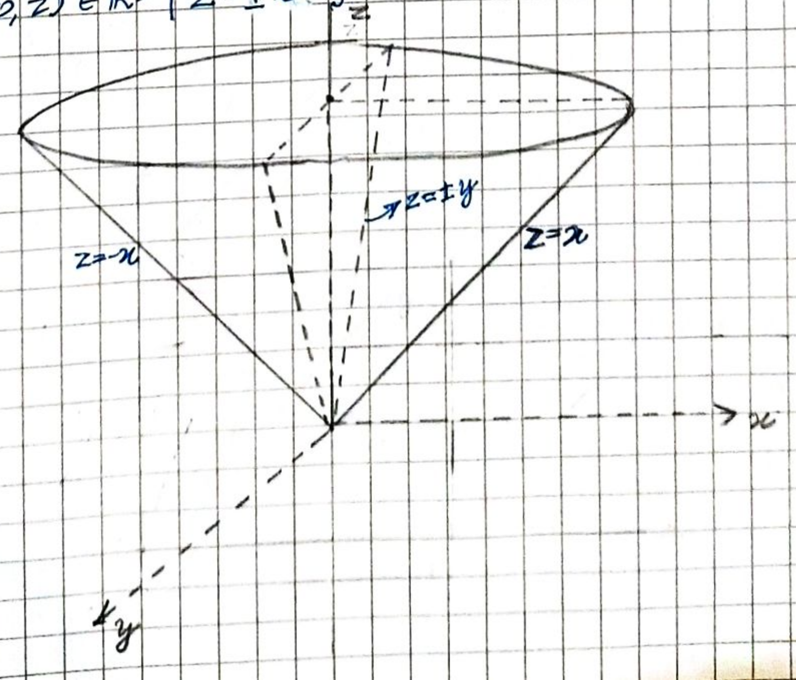
$f(x,y)=c=\sqrt{x^2+y^2}$	Level curve at height c.
0	$(0,0)$
1	$x^2+y^2=1^2$
2	$x^2+y^2=2^2$
3	$x^2+y^2=3^2$
4	$x^2+y^2=4^2$

The section by $x=0$ is

$$\{(x,y,z)\in\mathbb{R}^3 \mid z=\sqrt{x^2+y^2},\ x=0\}$$
$$=\{(0,y,z)\in\mathbb{R}^3 \mid z=\pm y\}$$

The section by $y=0$ is

$$=\{(x,y,z)\in\mathbb{R}^3 \mid z=\sqrt{x^2+y^2},\ y=0\}$$
$$=\{(x,0,z)\in\mathbb{R}^3 \mid z=\pm x\}$$



18) $z = f(x,y) = 4x^2 + 9y^2$.

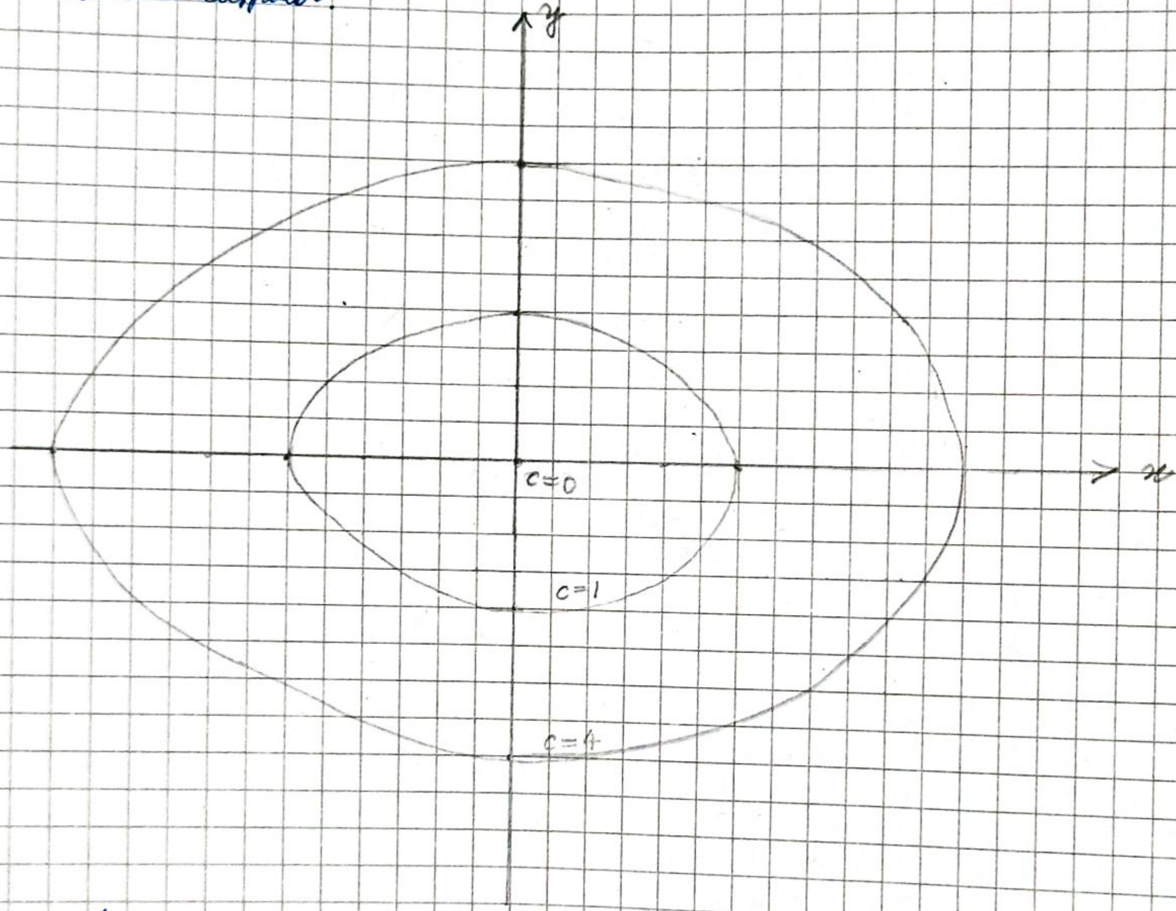
Level curve at height $c = \{(x,y) \in \mathbb{R}^2 \mid z = 4x^2 + 9y^2, z = c\}$
Simplifying this equation, $4x^2 + 9y^2 = c$

The function $f(x,y)$ can be written in the form $\frac{x}{\left(\frac{1}{2}\right)^2} + \frac{y}{\left(\frac{1}{3}\right)^2}$

as, $x^2/(\sqrt{c}/2)^2 + y^2/(\sqrt{c}/3)^2 = 1$.

c	Level curve at height c
0	$(0,0)$
1	$x^2/(1/2)^2 + y^2/(1/3)^2 = 1$
4	$x^2/1^2 + y^2/(2/3)^2 = 1$.
9	$x^2/(3/2)^2 + y^2/1^2 = 1$.
16	$x^2/2^2 + y^2/(4/3)^2 = 1$.

The Level curves are ellipses.



The section by $x=0$
$= \{(x,y,z) \mid z = 4x^2 + 9y^2, x = 0\} = \{(0,y,z) \mid z = 9y^2\}$

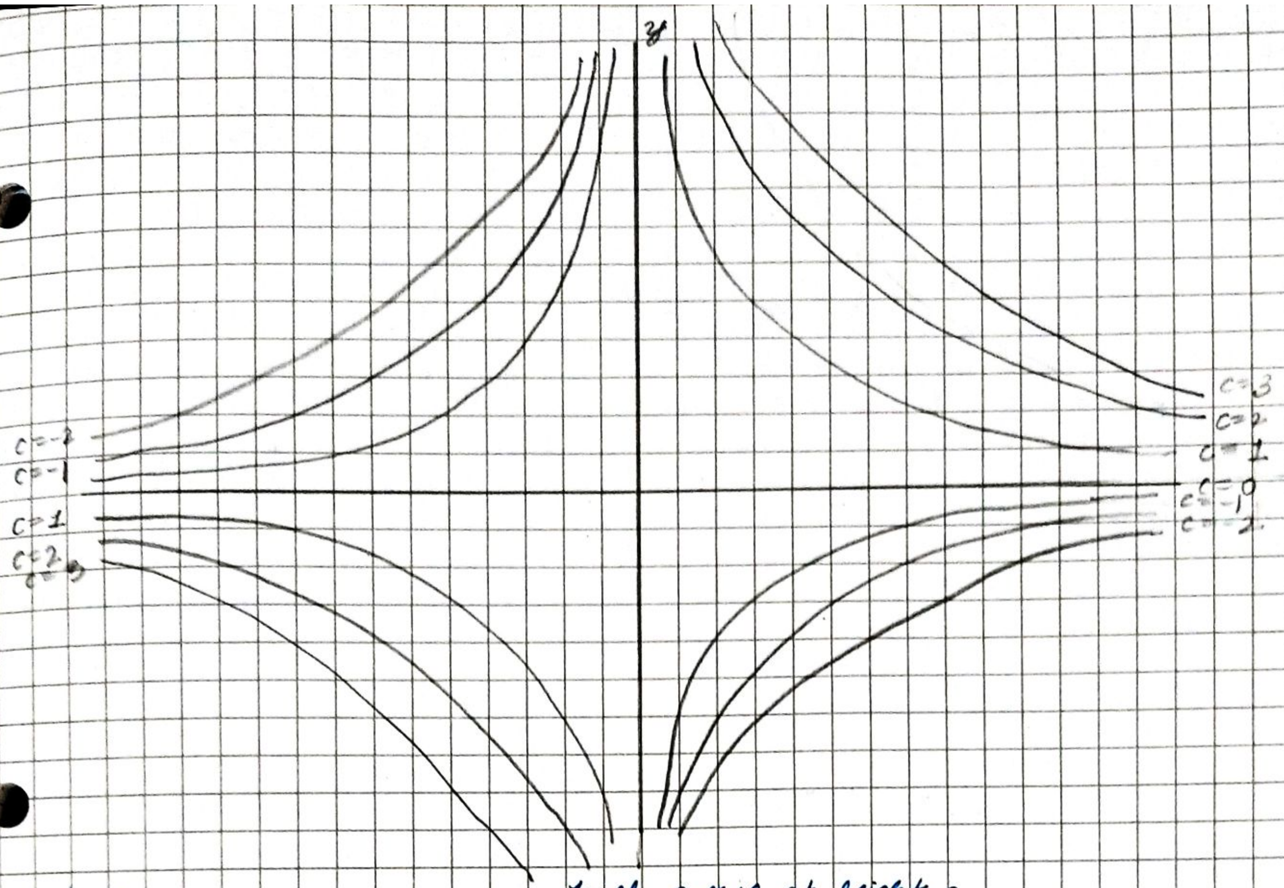
The section by $y=0$,
$= \{(x,y,z) \mid z = 4x^2 + 9y^2, y = 0\} = \{(x,0,z) \mid z = 4x^2\}$.

The sections are parabolas with the vertex at the origin.

Hence, the surface is an elliptic paraboloid with the ellipses having the ratio of the semi-major axis/semi-minor axis $= 3:2$.

19) $f(x,y) = xy$.

Level curve at height $c = \{(x,y) \mid z = xy, z = c\}$
$= \{(x,y) \mid y = \frac{c}{x}\}$



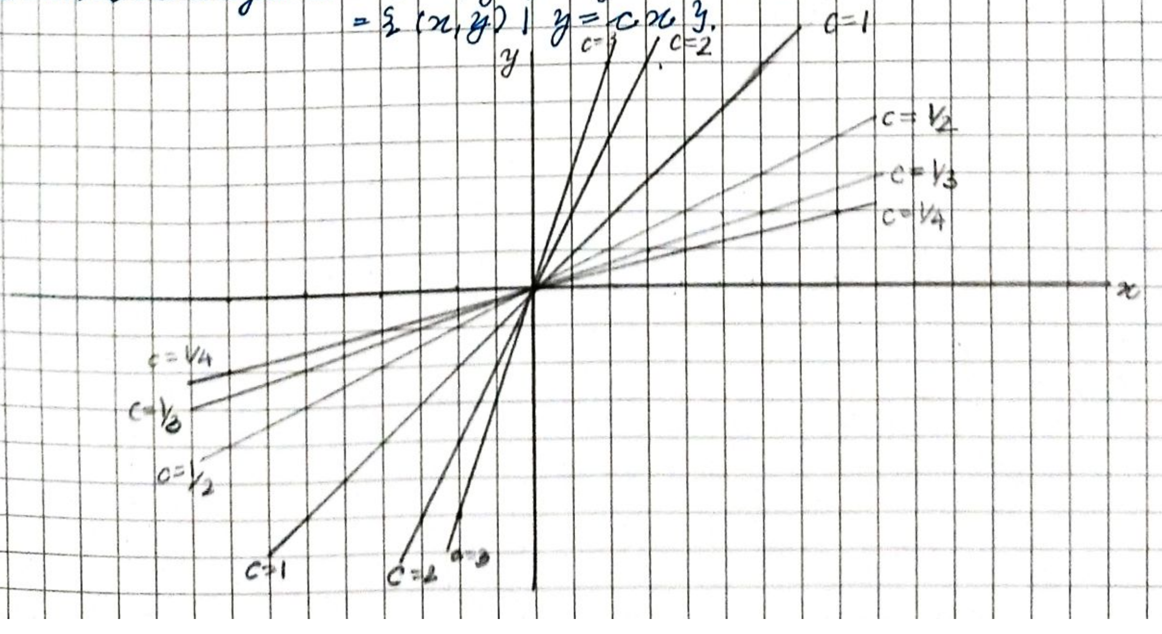
c	Level curve at height c
-3	$y = -3/x$
-2	$y = -2/x$
-1	$y = -1/x$
0	$y = 0$
1	$y = 1/x$
2	$y = 2/x$
3	$y = 3/x$

The section by $x=k$ is $= \{(x,y,z) \in \mathbb{R}^3 \mid z = xy,\ x = k\}$
$= \{(k,y,z) \in \mathbb{R}^3 \mid z = ky\}$

The section by $y=k$ is $= \{(x,y,z) \in \mathbb{R}^3 \mid z = xy,\ y = k\}$
$= \{(x,k,z) \in \mathbb{R}^3 \mid z = \cancel{xy}\ kx\}$.

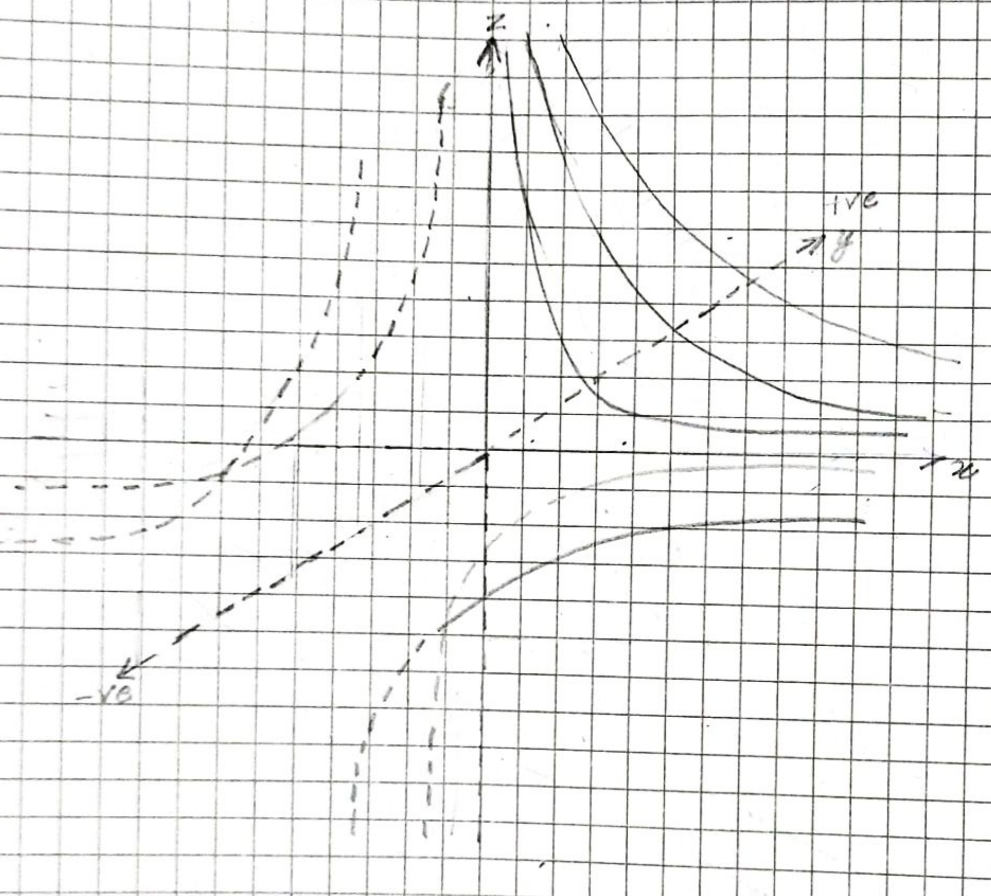
20) $f(x,y) = \frac{y}{x}$.

Level curve at height c $= \{(x,y) \mid z = y/x,\ z = c\}$
$= \{(x,y) \mid y = cx\}$.



c	Level curve at height c.
1/4	$y = x/4$
1/3	$y = x/3$
1/2	$y = x/2$
1	$y = x$
2	$y = 2x$
3	$y = 3x$
4	$y = 4x$

The section by $x = k$ is
$$\{(x,y,z) \in \mathbb{R}^3 \mid z = y/x,\ x = k\}$$
$$= \{(k,y,z) \in \mathbb{R}^3 \mid z = y/k\}$$
These are straight lines.
The section by $y = k$ is
$$\{(x,y,z) \in \mathbb{R}^3 \mid z = y/x,\ y = k\}$$
$$= \{(x,k,z) \in \mathbb{R}^3 \mid z = k/x\}$$
These are hyperbolas.



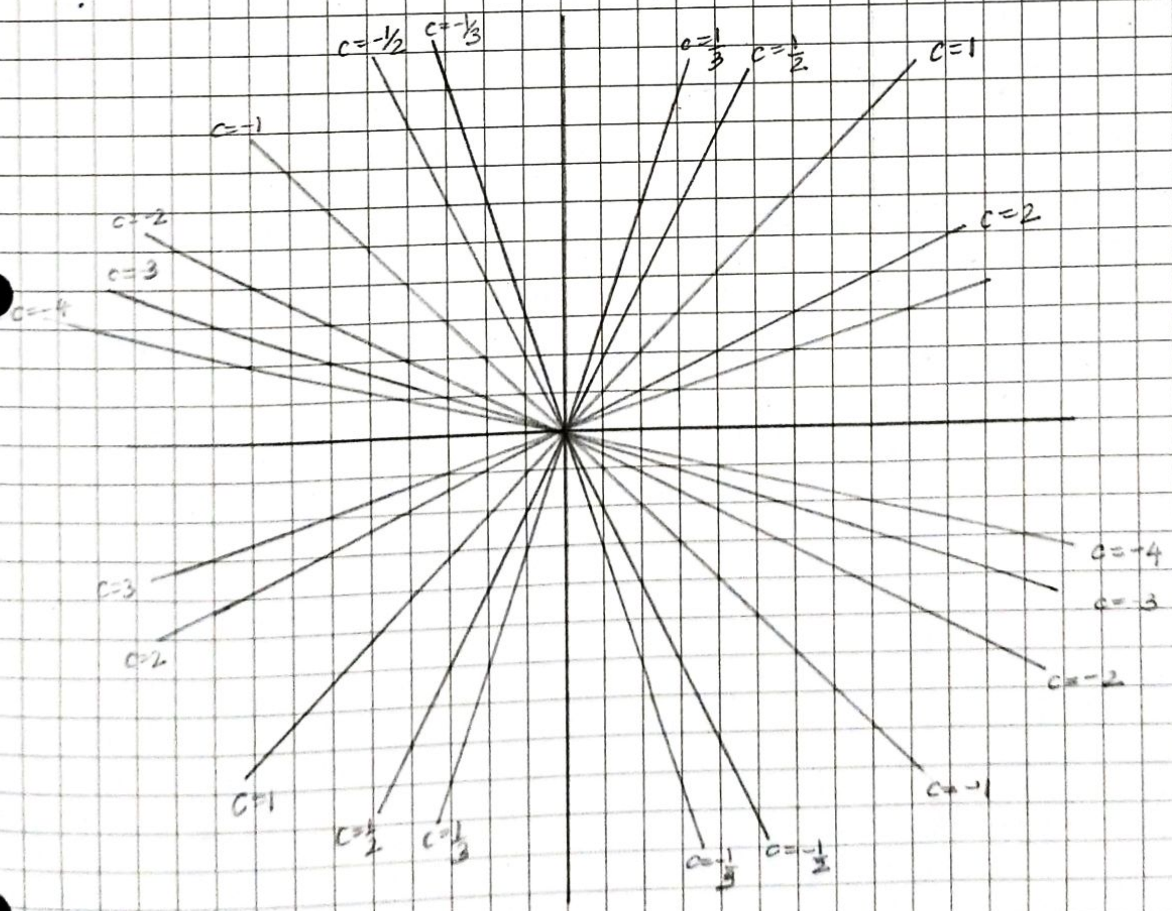
The sign in the different octants is as follows.

x	y = constant plane k	z
+ve	+ve	+ve
+ve	−ve	−ve
−ve	+ve	−ve
−ve	−ve	+ve.

21) $f(x,y) = \frac{x}{y}$.

Level curve at height c $= \{(x,y) \mid z = x/y,\ z = c\}$
$$= \{(x,y) \in \mathbb{R}^2 \mid x/y = c\}$$

c	Level curve at height c
-4	$y = -\frac{x}{4}$
-3	$y = -\frac{x}{3}$
-2	$y = -\frac{x}{2}$
-1	$y = -x$
$-1/2$	$y = -2x$
$-1/3$	$y = -3x$
$-1/4$	$y = -4x$
$c \to 0^-$	~~undefin~~ $-\infty$
$c \to 0^+$	$+\infty$
$c = 1/4$	$y = 4x$
$c = 1/3$	$y = 3x$
$c = 1/2$	$y = 2x$
$c = 1$	$y = x$
$c = 2$	$y = x/2$
$\vdots$	



Level curves

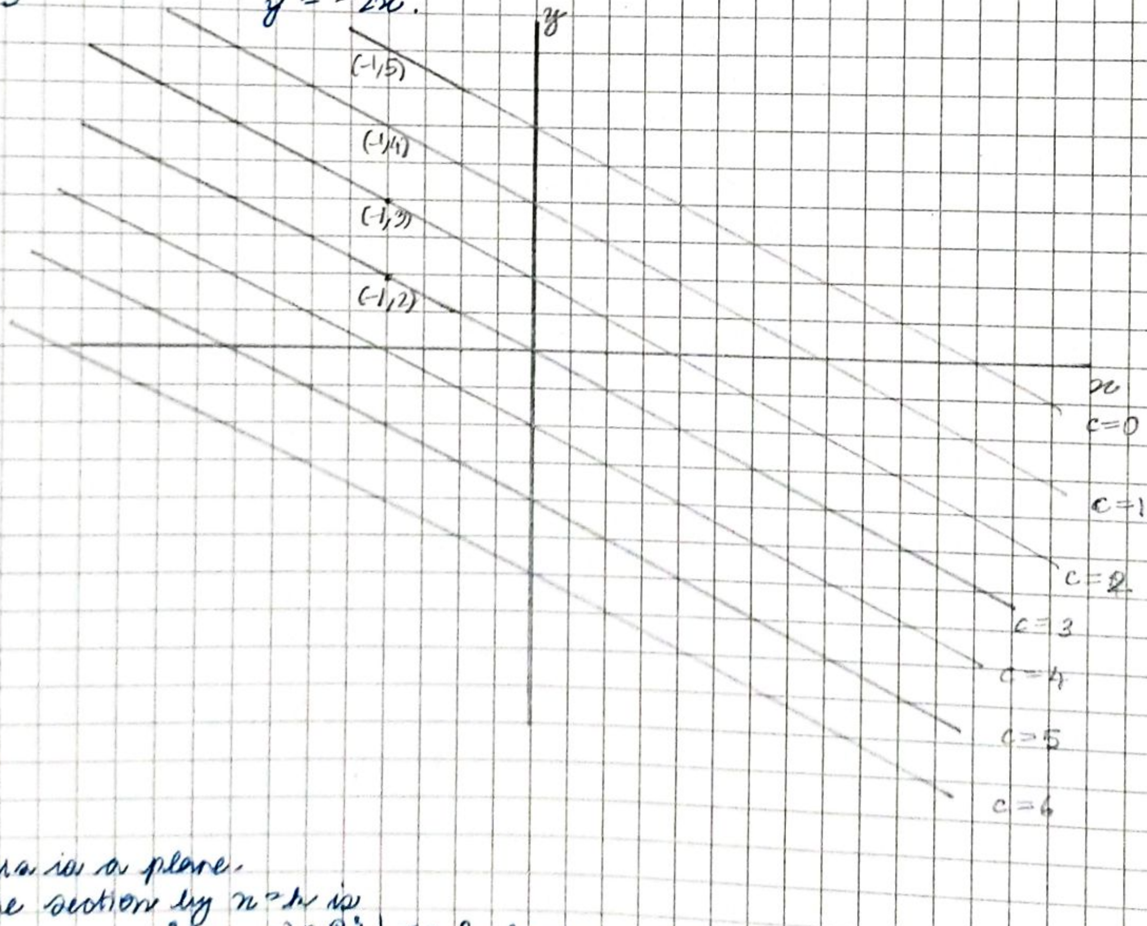
The section by $x=k$ is
$$\{(x,y,z)\in\mathbb{R}^3 \mid z=x/y, x=k\}$$
$$=\{(k,y,z)\in\mathbb{R}^3 \mid z=\frac{k}{y}\}$$
These are hyperbolas.
The section by $y=k$ is
$$\{(x,y,z)\in\mathbb{R}^3 \mid z=x/y, y=k\}$$
$$=\{(x,k,z)\in\mathbb{R}^3 \mid z=x/k\}.$$
These are straight lines.

This is the same as the plot in example 20, except that the hyperbolas are now in the yz-plane.

22) $f(x,y)=3-2x-y$.

The level curve at height c is,
$$\{(x,y)\in\mathbb{R}^2 \mid z=3-2x-y, z=c\}$$
$$\{(x,y)\in\mathbb{R}^2 \mid 2x+y=3-c\}=\{(x,y) \mid y=-2x+(3-c)\}$$

c	Level curve at height c
-3	$y=-2x+6$
-2	$y=-2x+5$
-1	$y=-2x+4$
0	$y=-2x+3$
1	$y=-2x+2$
2	$y=-2x+1$
3	$y=-2x$.



This is a plane.
The section by $x=k$ is
$$\{(x,y,z)\in\mathbb{R}^3 \mid z=3-2x-y, x=k\}$$
$$=\{(k,y,z)\in\mathbb{R}^3 \mid z=3-2k-y\}$$

This.

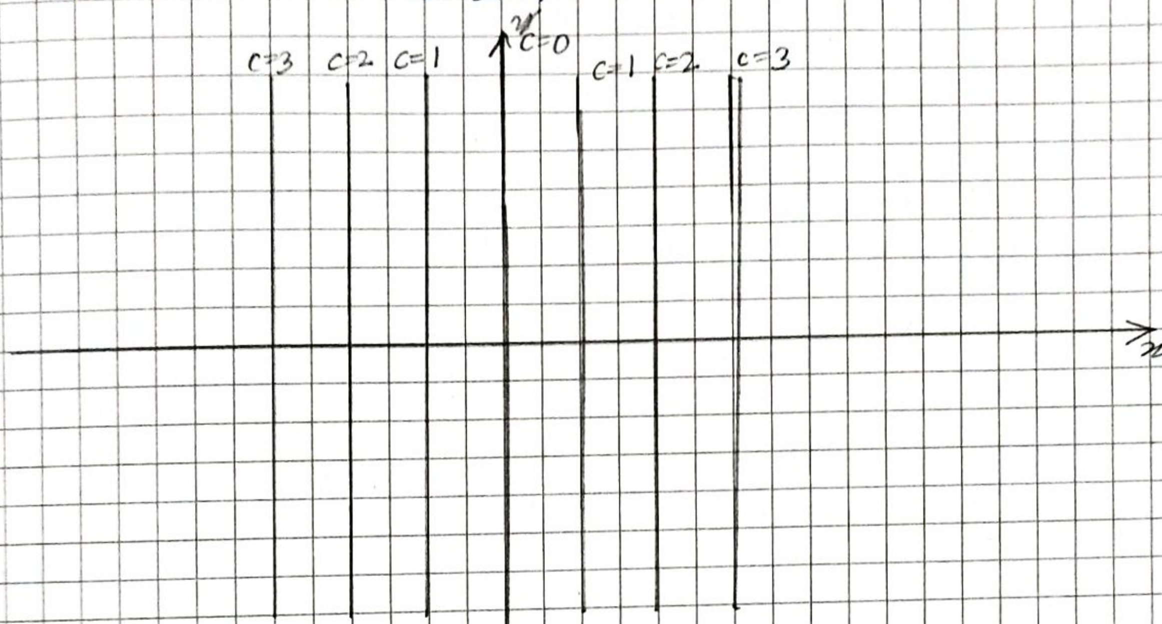
23) $f(x,y) = |x|$

The level curve at height c is

$$\{(x,y) \mid z = |x|, z = c\}$$
$$= \{(x,y) \mid |x| = c\}$$

Thus, the level curve at height c is the straight line $x = \pm c$ parallel to the y-axis.

c	Level curve at height c
0	$x = 0$
1	$x = \pm 1$
2	$x = \pm 2$
3	$x = \pm 3$
4	$x = \pm 4$



Level curves of $z = |x|$.