1. What are partial differential equations. 03/05/2020

Let us begin by delineating our field of study. A differential equation is an equation that relates the derivatives of a (scalar) function depending on one or more variables. For example,

$$\frac{d^4 u}{dx^4} + \frac{d^2 u}{dx^2} + u^2 = \cos x \tag{1.1}$$

is a differential equation for the function $u(x)$ depending on a single variable x, while

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - u, \quad u = u(t,x,y). \tag{1.2}$$

is a differential equation involving a ~~variable~~ function $u(t,x,y)$ of three variables.

A differential equation is called ordinary if the function u depends on only a single variable and partial if it depends on more than one variable. Usually (but not quite always) the dependence of u can be inferred from the derivatives that appear in the differential equation. The order of a differential equation is that of the highest order derivative that appears in the equation. Thus, (1.1) is a fourth-order ordinary differential equation, while (1.2) is a second-order partial differential equation.

Remark: A differential has order 0 if it contains no derivatives of the function u. These are more properly treated as algebraic equations, which, while of great interest in their own right are not the subject of this text. To be a bona fide differential equation, it must contain atleast one derivative of u and hence have order ≥ 1.

1. Brief Historical Comments.

Historically, partial differential equations originated from the study of surfaces in geometry and a wide variety of problems in mechanics. During the second half of the nineteenth century, a large number of famous mathematicians became actively involved in the investigation of numerous problems presented by partial differential equations. The primary reason for this research was that partial differential equations express many fundamental laws of nature and frequently arise in the mathematical analysis of diverse problems in science and engineering.

The next phase of the development of linear partial differential equations was characterised by efforts to develop the general theory and various methods of solution of linear equations. In force, partial differential equations have been found to be essential to the theory of surfaces on the one hand and to the solution of physical problems on the other. These two areas of mathematics can be seen as linked by the bridge of the calculus of variations. With the discovery of basic concepts and properties of distributions, the modern theory of linear partial differential equations is now well established. The subject plays a central role in modern mathematics especially in physics, geometry and analysis.

Almost all physical phenomena obey mathematical laws that can be formulated by differential equations. This striking fact was first discovered by Issac Newton (1642-1727) when he formulated the laws of mechanics and applied them to describe the motion of planets. During the three centuries since Newton's fundamental discoveries, many partial differential equations that govern physical, chemical and biological phenomena have been found and successfully solved by numerous methods. These equations include Euler's equations for the dynamics of rigid bodies and for the motion of an ideal fluid, Lagrange's equations of motion, Hamilton's equation of motion in analytical mechanics, Fourier's equation for the diffusion of heat, Cauchy's equation of motion and Navier's equation of motion in elasticity, the Navier-Stokes equation for the motion of viscous fluids, the Cauchy-Riemann equations in complex function theory, the Cauchy-Green equations for the static and dynamic behaviour of elastic solids, Kirchoff's equation for electric circuits, Maxwell's equation for electromagnetic fields, and the Schrodinger equation and Dirac equation in quantum mechanics. This is only a sampling, and the recent scientific and mathematical literature reveals an almost unlimited number of differential equations that have been discovered to model physical, chemical and biological systems and processes.

From the very beginning of the study considerable attention has been given to the geometric ~~fact~~ approach to the solution of differential equations. The fact that families of curves and surfaces can be defined by a differential equation means that the equation can be studied geometrically in terms of these curves and surfaces. The curves involved, known as characteristic curves, are very useful in determining, whether it is or is not possible to find a surface containing a given curve and satisfying a given differential equation. This geometric approach to differential equations was begun by by Joseph Louis Lagrange (1736-1813) and Gaspard Monge (1746-1818). Indeed, Monge first introduced the ideas of characteristic surfaces and characteristic cones (Monge cones). He also did some work on second-order linear, homogeneous partial differential equations.

The study of first-order partial differential equations began to receive some serious attention as early as 1739, when Alexis-Claude Clairaut (1713-1765) encountered these equations in his work on the shape of the earth. On the other hand in the 1770s, Lagrange first initiated a systematic study of the first order non-linear partial in the form

$$f(x, y, u, u_x, u_y) = 0.$$

where $u = u(x,y)$ is a function of two independent variables. Motivated by research on gravitational effects of bodies of different shapes and mass distributions, another major impetus for work in partial differential equations originated from potential theory. Perhaps the most important partial differential equation in applied mathematics is the potential equation, also known as the Laplace equation $u_{xx} + u_{yy} = 0$, where subscripts denote partial derivatives. This equation arises in steady state heat conduction problems involving homogenous solids. James Clerk Maxwell (1831-1879) also gave a new initiative to potential theory through his famous equations known as Maxwell's equations for electromagnetic fields.

~~La~~ Lagrange developed analytical mechanics as the application of partial differential equations to the motion of rigid bodies. He also described the geometrical content of a first-order partial differential ~~diff~~ equation and developed the method of characteristics for finding the general solution of quasi-linear equations. At the same time, the specific solution of physical interest was obtained by formulating an initial-value problem (or a Cauchy problem) that satisfies certain supplementary conditions. The solution of an initial-value problem still plays an important role in applied mathematics, science and engineering. The fundamental role of characteristics was soon recognized in the study of quasi-linear and non-linear partial differential equations. Physically, the first-order quasi-linear equations often represent conservation laws which describe the conservation of some physical quantities of a system.

In its early stages of development, the theory of second-order linear partial differential equations was concentrated on applications to mechanics and physics. All such equations can be classified into three basic categories — the wave equation, the heat equation and the Laplace equation (or potential equation). Thus, a study of these different kinds of equations yields much information about the more general second-order linear partial differential equations. Jean D'Alembert (1717-1783) first derived the one dimensional wave equation for the vibration of an elastic string and solved this equation in 1746. ~~How~~ His solution is now known as the d'Alembert solution. The wave equation is one of the oldest equations in mathematical physics. Some form of this equation, or its various generalizations, almost inevitably arises in any mathematical analysis of phenomena involving the propagation of waves in a continuous medium. In fact, the studies of water waves, acoustic waves, elastic waves in solids and electromagnetic waves are all based on this equation. A technique known as the method of ~~sepa~~ separation of variables is perhaps one of the oldest systematic methods for solving partial differential equations including the wave equation. The wave equation and its methods of solution attracted the attention of many famous mathematicians including Leonard Euler (1707-1783), James Bernoulli (1667-1748), Daniel Bernoulli (1700-1782), J.L. Lagrange (1736-1813) and Jacques Hadamard (1865-1963). They discovered solutions in several different forms, and the merit of their solutions and relations among these solutions were argued in a series of papers extending over more than twenty-five years; most concerned the nature of the kinds of functions that can be represented by the trigonometric (or Fourier) series. These controversial problems were finally resolved during the nineteenth century.

It was Joseph Fourier (1768-1830) who made the first major step towards developing a general method of solutions of the equation describing the conduction of heat in a solid body in the early 1800s. Although Fourier is most celebrated for his work on the conduction of heat, the mathematical methods involved, particularly trigonometric series, are important and useful in many other situations. He created a coherent mathematical method by which the different components of an equation and its solution in series ~~wer~~ were neatly identified with the different aspects of the physical solution being analysed. In spite of the striking success of Fourier analysis as one of the most useful mathematical methods, J.D. Lagrange and S.D. Poisson (1781-1840) hardly recognized Fourier's work because of its lack of rigour. Nonetheless, Fourier was eventually recognized for his pioneering work after publication of his monumental treatise titled La Theorie Analytique de la Chaleur in 1822.

It is generally believed that the concept of an integral transform originated from the integral theorem as stated by Fourier in his 1822 treatise. It was the work of Augustin Cauchy (1789-1857) that contained the exponential form of the Fourier Integral transform.

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} \left[\int_{-\infty}^{\infty} e^{-ik\xi} \cdot f(\xi)\, d\xi \right] dk.$$

This theorem has been expressed in several slightly different forms to better adapt it to particular applications. It has been recognized, almost from the start, however that the form ~~that combines~~ which best combines mathematically simplicity and complete generality makes use of the exponential oscillating function $\exp(ikx)$. Indeed, the Fourier integral formula is regarded as one of the most fundamental results of modern mathematical analysis and it has widespread physical and engineering applications. The generality and importance of the theorem is well expressed by Kelvin and Tait who said: "Fourier's theorem, which is not only one of the most beautiful results of modern analysis, but may be said to furnish an indispensable instrument in the treatment of nearly every recondite question in modern physics. To mention only sonorous vibrations, the propagation of electric signals along a telegraph wire, and the conduction of heat by the earth's crust, as subjects in their generality intractable without it, is to give but a feeble idea of its importance. This integral formula is usually used to define the classical Fourier transform of a function and the inverse fourier transform. No doubt, the ~~sci~~ scientific achievements of Joseph Fourier have not only provided the fundamental basis for the study of the ~~k~~ heat equation, Fourier series, and Fourier integrals, but for the modern developments of the theory and applications of the partial differential equations.

One of the most important of all partial differential equations involved in applied mathematics and mathematical physics is that associated with the name Pierre-Simon-Laplace (1749-1827). This equation was first discovered by Laplace while he was involved in an extensive study of gravitational attraction of arbitrary bodies in space. Although the main field of Laplace's research was celestial research, he ~~so~~ also made important contributions to the theory of probability and its applications. This work introduced the method known later as the Laplace Transform, a simple and elegant method of solving differential and integral equations. Laplace first introduced the concept of potential, which is invaluable in a wide range of subjects, such as gravitation, electromagnetism, hydrodynamics and acoustics. Consequently the Laplace equation is often referred to as the potential equation. This equation is also an important special case of both the wave equation and the heat equation in two or three dimensions. It arises in the study of many physical phenomena including electrostatic or gravitational potential, the velocity potential for an incompressible fluid flows, the steady state heat equation, and the equilibrium (time-independent) displacement ~~to~~ field of a two- or three-dimensional elastic membrane. The Laplace equation also occurs in other branches of applied mathematics and mathematical physics.

Since there is no time dependence in any of the mathematical problems stated above, there are no initial data to be satisfied by the solutions ~~of the~~ Laplace equation. They must however satisfy certain boundary conditions on the boundary curve or ~~boundary~~ surface of a region in ~~eq~~ which the Laplace equation is to be solved. The problem of finding a solution of the Laplace's equation that takes on the given boundary values is known as the Dirichlet boundary value problem, after Peter Gustav Lejeune Dirichlet (1805-1859). On the other hand if the values of the normal derivative are prescribed on the boundary, the problem is known as Neumann Boundary-value problem, in honor of Karl Gottfried Neumann (1832-1925). Despite great efforts by many mathematicians including Gaspard Monge (1746-1818), Adrien Marie Legendre (1752-1833), Carl Friedrich Gauss (1777-1855), Simeon-Denis Poisson (1781-1840), and Jean Victor Poncelet (1788-1867), very little was known about the general properties of the solutions of Laplace's equation until 1828, when George Green (1793-1841) and Mikhail Ostrogradsky independently investigated the properties of a class of solutions known as harmonic functions. On the other hand Augustin Cauchy (1789-1857) and Bernhard Riemann (1826-1866) derived a set of first-order partial differential equations, known as Cauchy Riemann equations, in their independent work on functions of complex variables. These equations led to the Laplace equation, and functions satisfying this equation in a domain are called harmonic functions in that domain. Both Cauchy and Riemann occupy a special place in the history of mathematics. Riemann made enormous contributions to almost all areas of pure and applied mathematics. His extra-ordinary achievements simulated further developments not only in mathematics, but also in mechanics, physics and the natural sciences as a whole.

Augustin Cauchy is universally recognized for his fundamental contributions to complex analysis. He also provided the first systematic and rigorous investigation of differential equations and gave a rigorous proof for the existence of power series solutions of a differential equation in the 1820s. In 1841 Cauchy developed what is known as the method of majorants for proving that a solution of a partial differential equation exists in the form of a power series in the independent variables. The method of majorants was also introduced independently by Karl Weierstrass (1815-1896) in that same year in application to a system of differential equations. Subsequently, Weierstrass's student Sophie Kowalewskaya (1850 — 1891) used the method of majorants and a normalization theorem of Carl Gustav Jacobi (1804-1851) to prove an exceedingly elegant theorem, known as the Cauchy-Kowalewskaya theorem. This theorem quite generally asserts the local existence of solutions of a system of partial differential equations with initial conditions on a non-characteristic surface. This theorem seems to have little practical importance because it does not distinguish between well-posed and ill-posed problems; it covers situations where a small change in the initial data leads to a large change in the solution. Historically, however it is the first existence theorem for a general class of partial differential equations.

The general theory of partial differential equations was initiated by A.R. Forsyth (1858-1942) in the fifth and sixth volumes of his Theory of Differential Equations and E.J.B. Goursat (1858-1936) in his books entitled Cours d'analyse mathematiques (1918) and his Lecons sur l'integration des equations aux dérivées, volume 1 (1890) and volume 2 (1896). Another notable contribution to the subject was made by E. Cartan's book, Lecons sur les invariants intégraux, published in 1922. Joseph Liouville (1809-1822) formulated a more tractable partial differential equation of the form

$$u_{xx} + u_{yy} = ke^{au}$$

and obtained a general solution of it. This equation has a large number of applications. It is a special case of the equation derived by J.L. Lagrange for the stream function ψ in the case of two-dimensional steady vortex motion in an incompressible fluid, that is,

$$\psi_{xx} + \psi_{yy} = F(\psi).$$

where $F(\psi)$ is an arbitrary function of ψ. When $\psi = u$ and $F(u) = ke^{au}$, the above equation reduces to the Liouville equation. In view of the special mathematical interest in the non-homogenous non linear equations of the above form type, a number of famous mathematicians including Henri Poincaré, E Picard (1856-1941), Cauchy (1789 - 1857), Sophus Lie (1844 -1899), L.M.H. Navier (1785 - 1836) and G.G. Stokes (1819-1903) made major contributions to partial differential equations.

Historically, Euler first solved the eigenvalue problem when he developed a simple mathematical models describing the 'buckling modes' of a vertical elastic beam. The general theory of eigenvalue problems for second-order differential equations, now known as Sturm - Liouville theorems, originated from the study of a class of boundary-value problems due to Charles Sturm (1803-1855) and Joseph Liouville (1809-1882). They showed that in general, there is an infinite set of eigenvalues satisfying the given equation and associated boundary conditions, and that these eigenvalues increase to infinity. Corresponding to these eigenvalues, there is an infinite set of corresponding orthogonal eigenfunctions so that the linear superposition principle to fo can be applied to find the convergent infinite series solution of the given problem. Indeed, the Sturm Liouville theory is a natural generalization of the theory of Fourier series that greatly extends the scope of the method of separation of variables. In 1926, the WKB approximation method was developed by Gregor Wentzel, Hendrik Kramers and Marcel Louis Brillouin for finding the approximate eigenvalues and eigenfunctions of the one dimensional Schrödinger equation in quantum mechanics. This method is now known as the short wave approximation or the geometrical optics approximation in wave propagation theory.

At the end of the seventeenth century, many important questions and problems in geometry and mechanics involved minimizing ~~and~~ or maximizing of certain integrals for two reasons. The first of these were several existence problems such as Newton's problem of missile of least resistance, Bernoulli's isoperimetric problem, Bernoulli's problem of brachistochrone (brachistos means shortest, chronos means time), the problem of minimal surfaces due to Joseph Plateau (1801-1883), and Fermat's principle of least time. Indeed, the variational principle as applied to the propagation and reflection of light in a medium was first enunciated in 1662 by one of the greatest mathematicians of the seventeenth century, Pierre Fermat (1601-1665). According to his principle a ray of light travels in a homogeneous medium from one point to another along a path in a minimum time. The second reason is somewhat philosophical, that is how to discover a minimizing principle in nature. The following 1744 statement of Euler is characteristic of the philosophical origin of what is known as the principle of least action: "As the construction of the universe is the most perfect possible, being the handiwork of all-wise Maker, nothing can be met with in the world in which some maximal or minimal property is not displayed. There is, consequently, no doubt but all the effects of the world can be derived by the method of maxima and minima from their final causes as well as from their efficient ones." In the middle of the eighteenth century, Pierre de Maupertuis (1698-1759) stated a fundamental principle, known as the principle of least action, as a guide to the nature of the universe. A still more precise and general formulation of Maupertuis' principle of least action was given by Lagrange in his analytical mechanics published in 1788. He formulated it as

$$\delta S = \delta \int_{t_1}^{t_2} (2T)\, dt = 0.$$

where T is the kinematic energy of a dynamical system with the constraint that the total energy, (T+V) is constant along the trajectories, where V is the potential energy of the system. He also derived the celebrated equation for a holonomic dynamical system

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i$$

where q_i are the generalized coordinates, $\dot{q}_i$ is the velocity, and Q_i is the force. For a conservative dynamical system, $Q_i = -\frac{\partial V}{\partial q_i}$, $V = V(q_i)$, $\frac{\partial V}{\partial \dot{q}_i} = 0$,

then the above equation can be expressed ~~as~~ in terms of the Lagrangian, $L = T - V$, as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

This principle was reformulated by Euler in a way that made it useful in mathematical and physics.

With the rapid development of the theory and applications of differential equations, the closed form analytical solutions of many different types of equations were hardly possible. However, it is extremely important and absolutely necessary to provide some insight into qualitative and quantitative nature of solutions subject to initial and boundary conditions. This insight usually takes the form of numerical and graphical representative of solutions. It was E. Picard (1856-1941) who first developed the method of successive approximations for the solutions of differential equations in the most general form and later made it an essential part of his treatment of differential equations in the second volume of his Traité d'Analyse published in 1896. During the last two centuries, the calculus of finite differences in various forms played a significant role in finding the numerical solutions of differential equations. Historically, many well-known integration formulas and numerical methods including the Euler Maclaurin formula, Gregory integration formula, Simpson's rule, Adam-Bashforth's method, the Jacobi iteration, the Gauss-Siedel method and the Runge-Kutta method have been developed and then generalized in various forms.

With the development of modern calculators and high-speed electronic computers, there has been an increasing trend in research ~~toward~~ towards the numerical solution of ordinary and partial differential equations during the twentieth century. Many well known numerical methods including the Crank Nicolson methods, the Lax-Wendroff method, Richtmyer's method and Stone's implicit iterative technique have been developed in the second-half of the twentieth century. All finite difference methods reduce differential equations to discrete forms. In recent years more modern and powerful computational methods such as the finite element method and the boundary element method have been developed to handle curved or irregularly shaped domains.

2. Introduction

There are two common notations for partial derivatives and we shall employ them interchangeably. The first used in (1.1) and (1.2) is the familiar Leibnitz notation that employs a d ~~dee~~ to denote ordinary derivatives and the ∂ symbol (usually also pronounced 'del') for partial derivatives of functions of more than one variable. An alternative more compact notation employs subscripts to indicate partial derivatives. For example, u_t represents $\partial u/\partial t$, while u_{xx} is used for $\partial^2 u/\partial x^2$ and $\partial^3 u/\partial^2 x \partial y$ for u_{xxy}. Thus, in subscript notation, the partial differential equation (1.2) is written

$$u_t = u_{xx} + u_{yy} - u. \tag{1.3}$$

We will similarly abbreviate partial differential operators sometimes writing $\partial/\partial x$ as ∂_x while $\partial^2/\partial x^2$ can be written as either ∂_x^2 or ∂_{xx}