One could show that by ~~taking~~ elementary row operations, certain numbers like the determinant, the rank, the inverse remain the same. In proving these results, we will make use of this, that B can be written as P times A, $B = PA$, where P is a product of elementary matrices.

($\Leftarrow$ direction).
Suppose $B = PA$. To must show that B is row-equivalent to A.

Let $P = E_k E_{k-1} \dots E_2 E_1$. Then,

$$
\begin{aligned}
B &= PA \\
&= (E_k E_{k-1} \dots E_2 E_1) A \\
&= (E_k E_{k-1} \dots E_2) E_1 A \qquad \text{since matrix multiplication is associative}
\end{aligned}
$$

By virtue of the previous theorem, $E_1 A$ is row-equivalent to A, as E_1 is an elementary matrix,

$E_1 A \sim A$.

Again consider

$$
\begin{aligned}
B &= PA \\
&= (E_k E_{k-1} \dots E_2) E_1 A \\
&= (E_k E_{k-1} \dots E_3) E_2 (E_1 A)
\end{aligned}
$$

$E_2 (E_1 A) \sim E_1 A \sim A$.

As row-equivalence is a transitive relation, $E_2 E_1 A \sim A$. Continuing in this fashion, $E_k E_{k-1} \dots E_2 E_1 A \sim A$. But, the left-hand side is precisely B. So, B is row-equivalent to A.

($\Rightarrow$ direction)
Suppose B is row equivalent to A. Then, B is obtained from A by a sequence of elementary row operations $E_1, E_2, \dots, E_{\ell-1}, E_\ell$. Then, the first elementary row operation is $E_1 A$ pre-multiplying ~~E~~ A by E_1, $E_1 A$, the second elementary row operation results in $E_2 E_1 A$, the third elementary row operation is obtained by $E_3 E_2 E_1 A$ and so forth.

$$B = \underbrace{E_\ell E_{\ell-1} \dots E_3 E_2 E_1}_{} A$$

$$B = QA$$

where $Q = E_\ell E_{\ell-1} \dots E_2 E_1$, a product of elementary matrices.

1.3 Invertibility of a Matrix.
Let $A \in \mathbb{R}^{n\times n}$ be a matrix of order n. A is said to have a right inverse, if there exists another matrix $B \in \mathbb{R}^{n\times n}$ such that, $AB = I_n$. A left-inverse is defined similarly. ~~A left-inverse~~ A is said to have a left-inverse C, if there exists a matrix $C \in \mathbb{R}^{n\times n}$, such that $CA = I_n$.

Together, A is said to be invertible, if A has a right inverse and a left inverse.

Theorem 15. A square matrix A is said to be invertible, if A has both a left-inverse and a right-inverse. ~~Both~~ The left-inverse is equal to right-inverse and is the inverse of the matrix A.

Proof.
There exists B such that $AB = I_n$
There exists C such that $CA = I_n$.
Then,

$$\begin{aligned} B &= I_n B \\ &= (CA)B \\ &= C(AB) \\ &= C(I_n) \\ B &= C. \end{aligned}$$

Theorem 16. Every elementary matrix is invertible.

Proof.
Let e be an elementary row operation and

$$E = e(I).$$

Define E' (the inverse of the elementary matrix A) to $e'(A)$.

$$E' := e'(I).$$

Consider

$$\begin{aligned} EE' &= e(E') \\ &= e(e'(I)) \\ &= (e \circ e')(I) \\ &= i(I) \qquad e \circ e' \text{ is the identity operation.} \\ &= I. \end{aligned}$$

$$\begin{aligned} E'E &= e'(E) \\ &= e'(e(I)) \\ &= (e' \circ e) I \\ &= i(I) \\ &= I. \end{aligned}$$

Thus, $EE' = I = E'E$.

Examples
1) Find the inverses of all five elementary matrices of of order 2.

Solution
The elementary matrices of order 2 are:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix}, \begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ \alpha & 1 \end{bmatrix}.$$

The inverse of these matrices are:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1/\alpha \end{bmatrix}, \begin{bmatrix} 1/\alpha & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1/\alpha \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -1/\alpha & 1 \end{bmatrix}.$$

Theorem 17
For $A \in \mathbb{R}^{n \times n}$, the following conditions are equivalent:
(a) A is invertible.
(b) A is row-equivalent to the identity matrix.
(c) A is a product of elementary matrices.

Proof. ($a \Rightarrow b$)
Suppose that A is invertible. To show that A is row-equivalent to I.
Let R be the row-reduced echelon form matrix, row equivalent to A.

$A \sim R$

$\therefore R = PA$

where R is a finite product of elementary matrices. $\because$ the product of elementary matrices is invertible, is invertible. Given that A is invertible, ~~B is invertible~~. PA is invertible. so, R is invertible. But, we know that, if this happens, (R is a row-reduced echelon matrix) ~~and~~, it must equal the identity matrix. so,

$$R = I.$$

$\therefore A \sim I.$

This proves (a) $\Rightarrow$ (b).

(b $\Rightarrow$ c)

Given that $A \sim I$.

Then, $I = PA$, where P is a product of elementary matrices. Now, let

$$I = \underbrace{E_l E_{l-1} \dots E_2 E_1}_{P} A$$

Pre-multiplying by E_l^{-1},

$$E_l^{-1} I = E_l^{-1} E_l E_{l-1} \dots E_2 E_1 A$$

$$E_l^{-1} = E_{l-1} E_{l-2} \dots E_2 E_1 A.$$

Pre-multiplying by E_{l-1}^{-1}

$$E_{l-1}^{-1} E_l^{-1} = E_{l-1}^{-1} E_{l-1} E_{l-2} \dots E_2 E_1 A.$$

$$E_{l-1}^{-1} E_l^{-1} = E_{l-2} \dots E_2 E_1 A.$$

Continuing in this fashion -

$$E_1^{-1} E_2^{-1} \dots E_{l-1}^{-1} E_l^{-1} = A.$$

Thus, A is a product of elementary matrices, since the inverse of an elementary matrix is again a elementary matrix.

(c $\Rightarrow$ a)

Let A be a product of elementary matrices $E_1, E_2, \dots, E_s$.

$$A = E_1 E_2 \dots E_s.$$

where each E_i is an elementary matrix. Since, each E_i is invertible, and the product of invertible matrices is invertible, the product on the right is invertible. Therefore, A is invertible.

Corollary. Let $A \in \mathbb{R}^{n \times n}$ be a square invertible matrix. Then, the same sequence of elementary row operations on A yielding I, when applied to the identity matrix I, yield A^{-1}.

Proof: As A is invertible, there exist a sequence of operations $E_1, E_2, \dots, E_{l-1}, E_l$ s.t.

$$I = E_l E_{l-1} \dots E_2 E_1 A.$$

Since A^{-1} exists, by ~~a~~ post-multiplying with A^{-1}, I get -

$$A^{-1} = E_l E_{l-1} \dots E_2 E_1 (AA^{-1})$$
$$= E_l E_{l-1} \dots E_2 E_1 I$$

This tells us, that A^{-1} is obtained by performing the same row operations on I.

Theorem 18.

For $A \in \mathbb{R}^{n \times n}$, the following statements are equivalent –

(a) The square matrix A is invertible.

(b) The homogenous system of equations $Ax = 0$ has only the trivial solution $x = 0$.

(c) The non-homogenous system of equations $Ax = b$ has a solution for all right-hand sides $b \in \mathbb{R}^n$.

Proof.

(a) $\Leftrightarrow$ (b).

(1) Let us show that (a) $\Rightarrow$ (b).

Assume that the matrix A is invertible.

Consider the homogenous system of equations

$$Ax = 0.$$

I know that, A^{-1} exists. I shall pre-multiply both sides by A^{-1}.

$$A^{-1}(Ax) = A^{-1}(0)$$

$$(A^{-1}A)x = 0. \rightarrow \text{vector}$$

$$Ix = 0 \text{ vector}$$

$$x = 0 \text{ vector}.$$

So, if A is invertible, then the homogenous system of linear equations $Ax = 0$ has $x = 0$ as the only solution.

(2) I must prove the converse (b) $\Rightarrow$ (a).

Assume that the homogenous system $Ax = 0$ has $x = 0$ as the only solution.

If rank$(A) = r$, by rank-nullity-dimension (RND) theorem, nullity$(A) = n - r$.

As $Ax = 0$ has only the trivial solution, nullity$(A) = n - r = 0$.

Thus, $r = n$. Thus, A is row-equivalent to the identity matrix I_n.

Hence, A is invertible.

(a) $\Leftrightarrow$ (c).

(1) We shall first prove (a) $\Rightarrow$ c.

Consider $x = A^{-1}b$. given any right side b. I know that A is invertible, that's statement (a). So, I look at the vector x defined as $A^{-1}b$. Then,

$$\begin{aligned} Ax &= A(A^{-1}b) \\ &= (AA^{-1})(b) \\ &= Ib \\ &= b. \end{aligned}$$

Thus, $x = A^{-1}b$ solves $Ax = b$ for all right sides $b \in \mathbb{R}^n$.

(2) I need to show (c) $\Leftarrow$ (a). I want to show that A is invertible.

Let R be the row-reduced echelon matrix row-equivalent to A. We will show that R has its last row non-zero. This would mean that R is the identity matrix, A is row-equivalent to I and by appealing to the previous theorem, A is invertible.

To show that the last row of R is non-zero, we need to show that, the below system has a solution:

$$Rx = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

This follows from the fact, that if the last row of R is zero, then the last element of ~~by matrix vector~~ the right side vector b, must also be zero by vector-matrix multiplication. So, if I show that the system $Rx = \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix}$ has a solution, x, then it follows that the last row of R is not zero.

To show that $Rx = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$ has a solution:

R is the row-reduced echelon form of A., so $R = PA$, where P is a finite product of elementary matrices.

~~Let~~ Define $b^* = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$.

$$Rx = b^* \qquad (I)$$

$$\Leftrightarrow PAx = b^* \qquad (II)$$

~~But~~ Since P is a product of elementary matrices; elementary matrices are invertible, product of invertible matrices is invertible, so P is invertible. Pre-multiplying by P^{-1},

$$Ax = P^{-1} b^* = b^{**} \qquad (III)$$

Does this system have a solution? We know that, whatever be the right hand-side vector, b^{**}, $Ax = b^{**}$ always has a solution x. So, system (II) has a solution. Therefore, system (I) has a solution. So, the last row of R is non-zero. So, ~~A~~ $R = I$ and $A \sim I$, so that by the previous theorem A is invertible.

Corollary. Let $A \in \mathbb{R}^{n \times n}$. If A has a left-inverse or a right inverse, then A is invertible.

Proof.
Suppose that A has a left-inverse B. Then,
$$BA = I.$$
I want to show that A is invertible. I will appeal to the previous theorem, which connects invertibility with homogeneous systems. I will show that the system $Ax = 0$ has $x = 0$ as the only solution.

Consider
$$Ax = 0.$$
Pre-multiplying by B
$$BAx = B(0)$$
$$Ix = 0 \text{ vector}$$
$$x = 0.$$

Thus, A is invertible.

Suppose that A has a right-inverse. There exists ~~C~~ C such that,
$$AC = I.$$
Thus, C has a left-inverse A. By appealing to the first part, I know that C is invertible. Post-multiplying by C^{-1},
$$AC = I$$
$$ACC^{-1} = IC^{-1}$$
$$A = C^{-1}$$
$$A^{-1} = C.$$

So,