Thus, $\max |\lambda_2| \le \|A\|$
$$\rho(A) \le \|A\|.$$

An interesting and useful result, which is similar to theorem is the following.

Theorem 2. For any $n\times n$ matrix A, and each arbitrary $\epsilon > 0$, there exists a natural norm $\|\cdot\|$, with the property that
$$\rho(A) \le \|A\| \le \rho(A) + \epsilon.$$

The proof of this result is beyond the scope of this course.

Examples.

1) Determine the l_2-norm of
$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}.$$

Solution.

To apply theorem 1, we need to compute the spectral radius of $A^t A$, so we first need the eigenvalues of $A^T A$:

$$A^t A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 6 & 4 \\ -1 & 4 & 5 \end{bmatrix}$$

The characteristic polynomial of the above matrix is
$$\det(A^t A - \lambda I) = 0$$
$$\begin{vmatrix} 3-\lambda & 2 & -1 \\ 2 & 6-\lambda & 4 \\ -1 & 4 & 5-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)\begin{vmatrix} 6-\lambda & 4 \\ 4 & 5-\lambda \end{vmatrix} - 2\begin{vmatrix} 2 & 4 \\ -1 & 5-\lambda \end{vmatrix} - 1\begin{vmatrix} 2 & 6-\lambda \\ -1 & 4 \end{vmatrix} = 0$$

$$(3-\lambda)(30 - 11\lambda + \lambda^2 - 16) - 2(2(5-\lambda)+4) - 1(8+6-\lambda) = 0.$$
$$(3-\lambda)(\lambda^2 - 11\lambda + 14) - 2(10 - 2\lambda + 4) - (14-\lambda) = 0$$
$$\cancel{(3-\lambda)}\ (3\lambda^2 - 33\lambda + 42 - \lambda^3 + 11\lambda^2 - 14\lambda) + 2(2\lambda - 14) + (\lambda - 14) = 0.$$
$$-\lambda^3 + 14\lambda^2 - 47\lambda + 42 + 4\lambda - 28 + \lambda - 14 = 0.$$
$$-\lambda^3 + 14\lambda^2 - 42\lambda = 0$$
$$\lambda^3 - 14\lambda^2 + 42\lambda = 0$$
$$\lambda(\lambda^2 - 14\lambda + 42) = 0$$
$$\lambda(\lambda^2 - 2\cdot\lambda\cdot 7 + 7^2 - 7) = 0$$
$$\lambda((\lambda-7)^2 - (\sqrt{7})^2) = 0$$
$$\lambda(\lambda - 7 + \sqrt{7})(\lambda - 7 - \sqrt{7}) = 0$$
$$\lambda_1 = 0,\ \lambda_2 = 7 - \sqrt{7},\ \lambda_3 = 7 + \sqrt{7}.$$

$$\rho(A^T A) = \max\{0, 7-\sqrt{7}, 7+\sqrt{7}\} = 7 + \sqrt{7}$$
$$\|A\|_2 = \sqrt{\rho(A^T A)} = \sqrt{7+\sqrt{7}} \approx 3.106.$$

3.2 Convergent Matrices

In studying iterative matrix techniques, it is of particular importance to know when powers of a matrix become small (that is when all the entries approach zero). Matrices of this type are called convergent.

Definition. An $n \times n$ matrix A is called convergent if,

$$\lim_{k \to \infty} A^k = 0$$

Examples.
1) Show that

$$A = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

is a convergent matrix.

Solution.

$$A^1 = \begin{bmatrix} 1/2 & 0 \\ 1/4 & 1/2 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1/2 & 0 \\ 1/4 & 1/2 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 1/4 & 1/2 \end{bmatrix} = \begin{bmatrix} 1/4 & 0 \\ 2/8 & 1/4 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1/4 & 0 \\ 2/8 & 1/4 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 1/4 & 1/2 \end{bmatrix} = \begin{bmatrix} 1/8 & 0 \\ 3/16 & 1/8 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 1/8 & 0 \\ 3/16 & 1/8 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 1/4 & 1/2 \end{bmatrix} = \begin{bmatrix} 1/16 & 0 \\ 4/32 & 1/16 \end{bmatrix}$$

In general,

$$A^k = \begin{bmatrix} 1/2^k & 0 \\ k/2^{k+1} & 1/2^k \end{bmatrix}$$

$$\lim_{k \to \infty} \frac{1}{2^k} = 0.$$

$$\lim_{k \to \infty} \frac{k}{2^{k+1}} = 0$$

so, $\lim_{k \to \infty} A^k = 0.$

Therefore, A is convergent matrix.

Theorem 3. The following statements are equivalent.
(i) A is a convergent matrix.
(ii) $\lim_{n \to \infty} \| A^n \| = 0$, for some natural norm.
(iii) $\lim_{n \to \infty} \| A^n \| = 0$, for all natural norms.
(iv) $\rho(A) < 1$
(v) $\lim_{n \to \infty} A^n x = 0$, for every x.

The proof of these statements is left to ~~an more~~ advanced undergraduate level course.

Problems.
1. Compute the eigenvalues and the associated eigenvectors of the following matrices.
a)

a) $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

The characteristic polynomial is given by

$$\det(A-\lambda I)=0$$

$$\begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)^2 - 1 = 0$$
$$\lambda^2-4\lambda+4-1=0$$
$$\lambda^2-4\lambda+3=0$$
$$\lambda^2-3\lambda-\lambda+3=0$$
$$\lambda(\lambda-3)-1(\lambda-3)=0$$
$$(\lambda-1)(\lambda-3)=0$$
$$\lambda_1=1,\ \lambda_2=3.$$

1) $\lambda_1=1$.
Solve $Ax=\lambda_1 x$
$(A-\lambda_1 I)x=0$
$(A-I)x=0$

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=0$$

$$A=\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

$x_1-x_2=0$
Here, x_2 is the free variable. Assigning an arbitrary value $x_2=1$, we find $x_1=1$.

$x^1=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is the eigenvector corresponding to the eigenvalue $\lambda_1(A)=1$.

2) $\lambda_2=3$.
Solve $Ax=\lambda x$
$(A-\lambda_2 I)x=0$
$(A-3I)x=0$

$$\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=0$$

$$A=\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \sim \begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$x_1+x_2=0$
Here, x_2 is the free-variable. Assigning arbitrary values $x_2=-1$, we find $x_1=1$.

$x^2=\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is the eigenvector corresponding to the eigenvalue $\lambda_2(A)=3$.

b) $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$

Solution.
The characteristic polynomial of A is,

$$\det(A-\lambda I)=0$$

$$\begin{vmatrix} -\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix}=0.$$

$-\lambda(1-\lambda)-1=0$

$-\lambda+\lambda^2-1=0$

$\lambda^2-\lambda-1=0$

$\lambda^2-2\cdot\lambda\cdot\left(\frac{1}{2}\right)+\left(\frac{1}{2}\right)^2-\frac{5}{4}=0$

$\left(\lambda-\frac{1}{2}\right)^2-\left(\frac{\sqrt{5}}{2}\right)^2=0.$

$\left(\lambda-\frac{1}{2}-\frac{\sqrt{5}}{2}\right)\left(\lambda-\frac{1}{2}+\frac{\sqrt{5}}{2}\right)=0.$

$\lambda=\frac{1}{2}+\frac{\sqrt{5}}{2}, \quad \lambda=\frac{1}{2}-\frac{\sqrt{5}}{2}.$

1) $\lambda_1=\frac{1}{2}+\frac{\sqrt{5}}{2}.$

solve $Ax=\lambda_1 x$

$(A-\lambda_1 I)x=0$

$$\begin{bmatrix} -(1+\sqrt{5})/2 & 1 \\ 1 & (1-\sqrt{5})/2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=0.$$

$$A=\begin{bmatrix} -(1+\sqrt{5})/2 & 1 \\ 1 & (1-\sqrt{5})/2 \end{bmatrix}\sim\begin{bmatrix} -(1+\sqrt{5})/2 & 1 \\ 0 & 0 \end{bmatrix}$$

$x_1\left(\frac{1+\sqrt{5}}{2}\right)-x_2=0$

Here, x_2 is the free variable. Assigning arbitrary values $x_2=1$, we find $x_1=\frac{1}{\left(\frac{1+\sqrt{5}}{2}\right)}$

$=\frac{1-\sqrt{5}}{1-\sqrt{5}}\cdot\frac{1}{(1+\sqrt{5})}\cdot 2$

$=\frac{(1-\sqrt{5})\cdot 2}{1^2-(\sqrt{5})^2}=-\left(\frac{1-\sqrt{5}}{2}\right)$

~~Also~~,

2) $\lambda_2=\frac{1-\sqrt{5}}{2}$

solve $Ax=\lambda x$

$(A-\lambda_2 I)x=0$

$$\begin{bmatrix} -(1-\sqrt{5})/2 & 1 \\ 1 & (1+\sqrt{5})/2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$$\begin{bmatrix} -(1-\sqrt{5})/2 & 1 \\ 1 & (1+\sqrt{5})/2 \end{bmatrix}\sim\begin{bmatrix} -(1-\sqrt{5})/2 & 1 \\ 0 & 0 \end{bmatrix}$$

$x_1\left(\frac{1-\sqrt{5}}{2}\right)-x_2=0.$

Here, x_2 is the free variable. Assigning arbitrary values, $x_2=1$, we find $x_1=\frac{1}{\left(\frac{1-\sqrt{5}}{2}\right)}$

$=\frac{1+\sqrt{5}}{1+\sqrt{5}}\cdot\frac{1}{1-\sqrt{5}}\cdot 2$

$=\frac{(1+\sqrt{5})\,2}{(-4)}=-\frac{1+\sqrt{5}}{2}.$

c) $\begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}$

The characteristic polynomial is

$p(\lambda)=0$

$\det(A-\lambda I)=0.$

$$\begin{vmatrix} -\lambda & 1/2 \\ 1/2 & -\lambda \end{vmatrix} = 0$$

$$\lambda^2 - \frac{1}{4} = 0$$

$$\lambda^2 - \left(\frac{1}{2}\right)^2 = 0$$

$$(\lambda - 1/2)(\lambda + 1/2) = 0$$

$$\lambda_1 = 1/2, \lambda_2 = -1/2.$$

1) $\lambda_1 = 1/2$.

solve $Ax = \lambda_1 x$.

$$(A - 1/2\, I)x = 0$$

$$\begin{bmatrix} -1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$$A = \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} \sim \begin{bmatrix} -1/2 & 1/2 \\ 0 & 0 \end{bmatrix}$$

$$(-1/2)x_1 + (1/2)x_2 = 0$$

$$-x_1 + x_2 = 0$$

Here, x_2 is the free variable. Assigning arbitrary values $x_2 = 1$, we find $x_1 = 1$.

$x^1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is the eigenvector corresponding to the eigenvalue $\lambda_1(A) = 1/2$.

2) $\lambda_2 = (-1/2)$.

solve $Ax = \lambda_2 x$.

$$(A - \lambda_2 I)x = 0$$

$$(A + 1/2\, I)x = 0$$

$$\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$$A = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \sim \begin{bmatrix} 1/2 & 1/2 \\ 0 & 0 \end{bmatrix}$$

$$x_1 + x_2 = 0.$$

Here, x_2 is the free variable. Assigning the arbitrary values $x_2 = -1$, we find $x_1 = 1$.

$x^2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is the eigenvector corresponding to the eigenvalue $\lambda_2(A) = -1/2$.

a) $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

The characteristic polynomial of A is:

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 2-\lambda & 1 & 0 \\ 1 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0.$$

$$(3-\lambda) \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)\left((2-\lambda)^2 - 1\right) = 0$$

$$(\lambda-3)(\lambda^2 - 4\lambda + 4 - 1) = 0$$

$$(\lambda-3)(\lambda^2 - 4\lambda + 3) = 0$$

$$(\lambda-3)(\lambda-1)(\lambda-3) = 0$$

$(\lambda-1)(\lambda-3)^2=0$

$\lambda_1=1$ with multiplicity 1.

$\lambda_2=3$ with multiplicity 2.

1) $\lambda_1=1$.

$(A-\lambda_1 I)x=0$.

$(A-I)x=0$.

$$\begin{bmatrix}1&1&0\\1&1&0\\0&0&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}.$$

The matrix A can be row-reduced to rref.

$$A=\begin{bmatrix}1&1&0\\1&1&0\\0&0&2\end{bmatrix}\sim\begin{bmatrix}1&1&0\\0&0&0\\0&0&2\end{bmatrix}$$

$x_1+x_2=0$

$x_3=0$.

Here, x_2 ~~and~~ is the free variable. Assigning arbitrary values $x_2=-1$, we find $x_1=1$.

$x^1=\begin{bmatrix}1\\-1\\0\end{bmatrix}$ is the eigenvector corresponding to the eigenvalue $\lambda_1(A)=1$.

~~II~~

2) $\lambda_2=3$

$(A-\lambda_2 I)x=0$

$(A-3I)x=0$

$$\cancel{A=}\begin{bmatrix}-1&1&0\\1&-1&0\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$A=\begin{bmatrix}1&-1&0\\1&-1&0\\0&0&0\end{bmatrix}\sim\begin{bmatrix}1&-1&0\\0&0&0\\0&0&0\end{bmatrix}$$

$x_1-x_2=0$, $x_3=0$

Here, x_2 is the free variable. Assigning arbitrary values $x_2=1$, we find $x_1=1$.

$x^2=\begin{bmatrix}1\\1\\0\end{bmatrix}$ is the eigenvector corresponding to the eigenvalue $\lambda_2=3$.

e) $A=\begin{bmatrix}-1&2&0\\0&3&4\\0&0&7\end{bmatrix}$

The characteristic polynomial of A is —

$\det(A-\lambda I)=0$.

$$\begin{vmatrix}-1-\lambda&2&0\\0&3-\lambda&4\\0&0&7-\lambda\end{vmatrix}=0$$

$$-1-\lambda\begin{vmatrix}3-\lambda&4\\0&7-\lambda\end{vmatrix}=0$$

$(-1-\lambda)(3-\lambda)(7-\lambda)=0$.

$(\lambda+1)(\lambda-3)(\lambda-7)=0$
$\lambda_1=-1, \lambda_2=3, \lambda_3=7.$

1) $\lambda_1=-1$
solve $Ax=\lambda_1 x$
$(A-\lambda_1 I)x=0$
$(A+I)x=0$

$$\begin{bmatrix}0&2&0\\0&4&4\\0&0&8\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}.$$

$$A=\begin{bmatrix}0&2&0\\0&4&4\\0&0&8\end{bmatrix}\sim\begin{bmatrix}0&2&0\\0&0&0\\0&0&8\end{bmatrix}$$

$x_2=0, x_3=0$
Here, x_1 is a free variable. Assigning the arbitrary value $x_1=1$.

$x^1=\begin{bmatrix}1\\0\\0\end{bmatrix}$ is the eigenvector corresponding to the eigenvalue $\lambda_1(A)=-1$.

2) $\lambda_2=3$
$Ax=\lambda_2 x$
$(A-\lambda_2 I)x=0$
$(A-3I)x=0$

$$\begin{bmatrix}-4&2&0\\0&0&4\\0&0&4\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}.$$

$$A=\begin{bmatrix}-4&2&0\\0&0&4\\0&0&4\end{bmatrix}\sim\begin{bmatrix}-4&2&0\\0&0&4\\0&0&0\end{bmatrix}$$

$4x_1-2x_2=0$
$x_3=0$
Here, x_2 is the free variable. Assigning the arbitrary value $x_2=2$, we find $x_1=1$.

$x^2=\begin{bmatrix}1\\2\\0\end{bmatrix}$ is the eigenvector corresponding to the eigenvalue $\lambda_2=3$.

3) $\lambda_3=7$.
$Ax=\lambda_3 x$
$(A-7I)x=0$

$$\begin{bmatrix}-8&2&0\\0&-4&4\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$A=\begin{bmatrix}-8&2&0\\0&-4&4\\0&0&0\end{bmatrix}\sim\begin{bmatrix}-8&2&0\\0&0&4\\0&0&0\end{bmatrix}$$

$x_3=0$
$8x_1-x_2=0$
Here, x_2 is the free variable. Assigning the arbitrary value $x_2=4$, we find $x_1=1$.

$x^3=\begin{bmatrix}1\\4\\0\end{bmatrix}$ is the eigenvector corresponding to the eigenvalue $\lambda_3=7$.

f)

$$A=\begin{bmatrix}2&1&1\\2&3&2\\1&1&2\end{bmatrix}$$

The characteristic polynomial of A is:

$$p(\lambda)=\det(A-\lambda I)=0$$

$$\begin{vmatrix}2-\lambda&1&1\\2&3-\lambda&2\\1&1&2-\lambda\end{vmatrix}=0.$$

$$(2-\lambda)\begin{vmatrix}3-\lambda&2\\1&2-\lambda\end{vmatrix}-1\begin{vmatrix}2&2\\1&2-\lambda\end{vmatrix}+1\begin{vmatrix}2&3-\lambda\\1&1\end{vmatrix}=0$$

$$(2-\lambda)(\lambda^2-5\lambda+6-2)-(2(2-\lambda)-2)+(2-(3-\lambda))=0.$$

$$(2-\lambda)(\lambda^2-5\lambda+4)-(4-2\lambda-2)+(2-3+\lambda)=0$$

$$(2-\lambda)(\lambda^2-5\lambda+4)+(2\lambda-2)+(\lambda-1)=0.$$

$$(2-\lambda)(\lambda-1)(\lambda-4)+2(\lambda-1)+(\lambda-1)=0$$

$$\{(2-\lambda)(\lambda-4)+3\}(\lambda-1)=0$$

$$(\lambda-1)\{2\lambda-8-\lambda^2+4\lambda+3\}=0$$

$$(\lambda-1)(\lambda^2-6\lambda+5)=0$$

$$(\lambda-1)(\lambda-5)(\lambda-1)=0$$

$$(\lambda-1)^2(\lambda-5)=0$$

$$\lambda_1=1,\ \lambda_2=5.$$

1) $\lambda_1=1$

solve $Ax=\lambda_1 x$

$(A-I)x=0$

$$\begin{bmatrix}1&1&1\\2&2&2\\1&1&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}.$$

$$A=\begin{bmatrix}1&1&1\\2&2&2\\1&1&1\end{bmatrix}\sim\begin{bmatrix}1&1&1\\0&0&0\\0&0&0\end{bmatrix}$$

$x_1+x_2+x_3=0.$

Here, x_2 and x_3 are the free variables. Assigning arbitrary values $x_2=-1$, $x_3=0$, we find $x_1=1$. Assigning the values $x_2=0$, $x_3=-1$, we find $x_1=1$.

$x^1=\begin{bmatrix}1\\-1\\0\end{bmatrix}$ and $x^2=\begin{bmatrix}1\\0\\-1\end{bmatrix}$ are the eigenvectors corresponding to the eigenvalue $\lambda_1(A)=1$.

2) $\lambda_2=5.$

$(A-5I)x=0$

$$\begin{bmatrix}-3&1&1\\2&-2&2\\1&1&-3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$B=\begin{bmatrix}-3&1&1\\2&-2&2\\1&1&-3\end{bmatrix}\sim\begin{bmatrix}1&1&-3\\1&-1&1\\-3&1&1\end{bmatrix}\sim\begin{bmatrix}1&1&-3\\0&-2&4\\0&4&-8\end{bmatrix}\sim\begin{bmatrix}1&1&-3\\0&-1&2\\0&1&-2\end{bmatrix}\sim\begin{bmatrix}1&0&-1\\0&-1&2\\0&0&0\end{bmatrix}$$

$x_1-x_3=0$

$x_2=2x_3=0.$

Here, x_3 is the free variable. Assigning the arbitrary value $x_3=1$, we find

$x_1=1,\ x_2=2.$

2. Compute the eigenvalues and eigenvectors of the following matrices

a) $\begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix}$

solution:

The characteristic polynomial is

$$\det(A-\lambda I)=0$$

$$\begin{vmatrix} 1-\lambda & 1 \\ -2 & -2-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(-2-\lambda)+2=0$$
$$(\lambda-1)(\lambda+2)+2=0$$
$$\lambda^2+2\lambda-\lambda-2+2=0$$
$$\lambda^2+\lambda=0$$
$$\lambda(\lambda+1)=0$$
$$\lambda_1=0,\ \lambda_2=-1.$$

1) $\lambda_1=0$.

solving $Ax=0.x$

$Ax=0$.

$$\begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$A=\begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$x_1+x_2=0$

Here, x_2 is the free variable. Assigning arbitrary values $x_2=-1$, we find $x_1=1$.

$x'=\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is the eigenvector corresponding to the eigenvalue $\lambda_1=0$.

2) $\lambda_2=-1$.

$(A+I)x=0$

$$\begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$$B=\begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$$

$2x_1+x_2=0$.

Here, x_2 is the free variable. Assigning the arbitrary values $x_2=-2$, we find $x_1=1$

$x'=\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ is the eigenvector corresponding to the eigenvalue $\lambda_2=-1$.

b) $\begin{bmatrix} -1 & -1 \\ 1/3 & 1/6 \end{bmatrix}$.

solution.

The characteristic polynomial is

$$p(\lambda)=0$$
$$\det(A-\lambda I)=0$$

$$\begin{vmatrix} -1-\lambda & -1 \\ 1/3 & 1/6-\lambda \end{vmatrix}=0.$$

$$(-1-\lambda)\left(\frac{1}{6}-\lambda\right)+\frac{1}{3}=0.$$

$$(-1-\lambda)(1-6\lambda)+2=0.$$

$(\lambda+1)(6\lambda-1)+2=0$

$6\lambda^2-\lambda+6\lambda-1+2=0$

$6\lambda^2+5\lambda+1=0$

$6\lambda^2+3\lambda+2\lambda+1=0$

$3\lambda(2\lambda+1)+1(2\lambda+1)=0$

$(2\lambda+1)(3\lambda+1)=0$

$\lambda_1=-\frac{1}{2},\ \lambda_2=-\frac{1}{3}$.

1) $\lambda_1=-\frac{1}{2}$.

$(A+1/2\,I)x=0$

$$\begin{bmatrix}-1/2 & -1\\ 1/3 & 2/3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}.$$

$$A'=\begin{bmatrix}-1/2 & -1\\ 1/3 & 2/3\end{bmatrix}\sim\begin{bmatrix}-1 & -2\\ 1 & 2\end{bmatrix}\sim\begin{bmatrix}-1 & -2\\ 0 & 0\end{bmatrix}$$

$x_1+2x_2=0$

Here, x_2 is the free variable. Assigning arbitrary values ~~to~~ $x_2=-1$, we find $x_1=2$.

$X^1=\begin{bmatrix}2\\ -1\end{bmatrix}$ is the eigenvector corresponding to the eigenvalue $\lambda_1=-1/2$.

2) $\lambda_2=-\frac{1}{3}$

$\left(A+\frac{1}{3}I\right)x=0.$

$$\begin{bmatrix}-2/3 & -1\\ 1/3 & 1/2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$$A'=\begin{bmatrix}-2/3 & -1\\ 1/3 & 1/2\end{bmatrix}\sim\begin{bmatrix}2 & 3\\ 2 & 3\end{bmatrix}\sim\begin{bmatrix}2 & 3\\ 0 & 0\end{bmatrix}$$

$2x_1+3x_2=0$

Here, x_2 is the free variable. Assigning the arbitrary value $x_2=-2$, $2x_1=6$, $\Rightarrow x_1=3$.

$X^2=\begin{bmatrix}3\\ -2\end{bmatrix}$ is the eigenvector corresponding to the eigenvalue $\lambda_2=-1/3$.

c) $\begin{bmatrix}3 & 4\\ 1 & 0\end{bmatrix}$

The characteristic polynomial is

$\det(A-\lambda I)=0.$

$$\begin{vmatrix}3-\lambda & 4\\ 1 & -\lambda\end{vmatrix}=0.$$

$(3-\lambda)(-\lambda)-4=0$

$\lambda(\lambda-3)-4=0$

$\lambda^2-3\lambda-4=0$

$\lambda^2-4\lambda+\lambda-4=0$

$\lambda(\lambda-4)+1(\lambda-4)=0$

$(\lambda+1)(\lambda-4)=0.$

$\lambda_1=-1,\ \lambda_2=4.$

1) $\lambda_1=-1$.

$(A+I)x=0$

$$\begin{bmatrix}4 & 4\\ 1 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=0.$$

$A' = \begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 4 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

$x_1 + x_2 = 0$

Here, x_2 is the free variable. Assigning the arbitrary value $x_2 = -1$, we find $x_1 = 1$.

$x' = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is the eigenvector corresponding to the eigenvalue $\lambda_1(A) = -1$.

2) $\lambda_2 = 4$.

$(A - 4I)x = 0$.

$$\begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$A' = \begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix} \sim \begin{bmatrix} -1 & 4 \\ 0 & 0 \end{bmatrix} \sim$

$x_1 - 4x_2 = 0$

Here, x_2 is the free variable. Assigning the arbitrary value $x_2 = 1$, we find $x_1 = 4$.

So, $x^2 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ is the eigenvector corresponding to the eigenvalue $\lambda_2 = 4$.

d) $\begin{bmatrix} 3 & 2 & -1 \\ 1 & -2 & 3 \\ 2 & 0 & 4 \end{bmatrix}$.

The characteristic polynomial is

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 3-\lambda & 2 & -1 \\ 1 & -2-\lambda & 3 \\ 2 & 0 & 4-\lambda \end{vmatrix} = 0$$

$$2 \begin{vmatrix} 2 & -1 \\ -2-\lambda & 3 \end{vmatrix} + (4-\lambda) \begin{vmatrix} 3-\lambda & 2 \\ 1 & -2-\lambda \end{vmatrix} = 0.$$

$2(6 - (\lambda+2)) + (4-\lambda)((3-\lambda)(-2-\lambda) - 2) = 0$

$2(4-\lambda) + (4-\lambda)((\lambda-3)(\lambda+2) - 2) = 0$

$(4-\lambda)\{\cancel{2} + (\lambda-3)(\lambda+2) - \cancel{2}\} = 0$

$(\lambda+2)(\lambda-3)(\lambda-4) = 0$

$\lambda_1 = -2, \lambda_2 = 3, \lambda_3 = 4$.

1) $\lambda_1 = -2$

$(A + 2I)x = 0$

$$\begin{bmatrix} 5 & 2 & -1 \\ 1 & 0 & 3 \\ 2 & 0 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$A' = \begin{bmatrix} 5 & 2 & -1 \\ 1 & 0 & 3 \\ 2 & 0 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 5 & 2 & -1 \\ 2 & 0 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & -16 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -8 \\ 0 & 0 & 0 \end{bmatrix}$.

$x_1 + 3x_3 = 0$

$x_2 - 8x_3 = 0$

Here, x_3 is the free variable. Assigning arbitrary values $x_3 = 1$, we find that: $x_2 = 8$, $x_1 = -3$.

$x' = \begin{bmatrix} -3 \\ 8 \\ 1 \end{bmatrix}$ is eigenvector corresponding to the eigenvalue $\lambda_1(A) = -2$.

2) $\lambda_2 = 3$.

$(A - 3I)x = 0$.

$$\begin{bmatrix} 0 & 2 & -1 \\ 1 & -5 & 3 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

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$$A' = \begin{bmatrix} 0 & 2 & -1 \\ 1 & -5 & 3 \\ 2 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & 3 \\ 0 & 2 & -1 \\ 2 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & 3 \\ 0 & 2 & -1 \\ 0 & 10 & -5 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$2x_1 + x_3 = 0$
$2x_2 - x_3 = 0$

Here, x_3 is the free variable. Assigning arbitrary values $x_3 = 2$, we find $x_1 = -1, x_2 = 1$.

$X^2 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$ is the eigenvector corresponding to the eigenvalue $\lambda_2 = 3$.

3) $\lambda_3 = 4$.
~~to~~ solve $(A - 4I)x = 0$

$$\begin{bmatrix} -1 & 2 & -1 \\ 1 & -6 & 3 \\ 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$A' = \begin{bmatrix} -1 & 2 & -1 \\ 1 & -6 & 3 \\ 2 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} -1 & 2 & -1 \\ 0 & -4 & 2 \\ 0 & 4 & -2 \end{bmatrix} \sim \begin{bmatrix} -2 & 4 & -2 \\ 0 & -4 & 2 \\ 0 & 4 & -2 \end{bmatrix} \sim \begin{bmatrix} -2 & 0 & 0 \\ 0 & -4 & 2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = 0,\ 2x_2 - x_3 = 0$

Here, x_3 is the free variable. Assigning the arbitrary value $x_3 = 2$, we find $x_2 = 1$.

$X^3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ is the eigenvector corresponding to the eigenvalue $\lambda_3 = 4$.

e) $A = \begin{bmatrix} 1/2 & 0 & 0 \\ -1 & 1/2 & 0 \\ 2 & 2 & -1/3 \end{bmatrix}$.

Solution

The characteristic polynomial is –

$$\det(A - \lambda I) = 0.$$

$$\begin{vmatrix} 1/2-\lambda & 0 & 0 \\ -1 & 1/2-\lambda & 0 \\ 2 & 2 & -1/3-\lambda \end{vmatrix} = 0$$

$$(1/2-\lambda) \begin{vmatrix} 1/2-\lambda & 0 \\ 2 & -1/3-\lambda \end{vmatrix} = 0.$$

$$\left(\frac{1}{2} - \lambda\right) \left\{ \left(\lambda - \frac{1}{2}\right)\left(\lambda + \frac{1}{3}\right) \right\} = 0$$

$$\left(\lambda - \frac{1}{2}\right)^2 \left(\lambda + \frac{1}{3}\right) = 0.$$

$$\lambda_1 = \frac{1}{2},\ \lambda_2 = -\frac{1}{3}.$$

1) $\lambda_1 = 1/2$.

$(A - 1/2\, I)\, x = 0$

$$\begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 2 & 2 & -5/6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$A' = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 2 & 2 & -5/6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 2 & -5/6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & -5/6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 12 & -5 \end{bmatrix}.$$

$x_1 = 0,\ 12x_2 - 5x_3 = 0$

Here, x_3 is the free variable. Assigning the arbitrary value $x_3 = 12$, ~~x_2~~ $12x_2 = 60$, we find $x_2 = 5$.

$X^1 = \begin{bmatrix} 0 \\ 5 \\ 12 \end{bmatrix}$ is the eigenvector corresponding to the eigenvalue $\lambda_1(A) = 1/2$.

2) $\lambda_2 = -1/3$.

$(A + 1/3\, I)x = 0$.

$$\begin{bmatrix} 5/6 & 0 & 0 \\ -1 & 5/6 & 0 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A' = \begin{bmatrix} 5/6 & 0 & 0 \\ -1 & 5/6 & 0 \\ 2 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 5 & 0 & 0 \\ -6 & 5 & 0 \\ 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = 0,\ x_2 = 0$

Here, x_3 is the free variable. Assigning the arbitrary value $x_3 = 1$, we find that -

$X^2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is the eigenvector corresponding to the eigenvalue $\lambda_2 = -1/3$.

f) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & -1 & 2 \end{bmatrix}$

The characteristic polynomial is

$$\det(A - \lambda I) = 0.$$

$$\begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 2-\lambda & 4 \\ 0 & -1 & 2-\lambda \end{vmatrix} = 0.$$

$$2-\lambda \begin{vmatrix} 2-\lambda & 4 \\ -1 & 2-\lambda \end{vmatrix} = 0$$

$(\lambda-2)\{(\lambda-2)^2 + 4\} = 0$

$(\lambda-2)(\lambda^2 - 4\lambda + 4 + 4) = 0$

$(\lambda-2)(\lambda^2 - 4\lambda + 8) = 0$.

$\lambda_1 = 2$ or $\lambda = \dfrac{4 \pm \sqrt{16 - 4(8)}}{2} = \dfrac{4 \pm 4i}{2} = 2 \pm 2i$

1) $\lambda = 2$

$(A - 2I)x = 0$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & -1 & 0 \end{bmatrix}$$

$x_2 = 0,\ x_3 = 0$

Here x_1 is the free variable. ~~Assi~~ We can assign an arbitrary value to x_1.

Assume $x_1 = 1$.

$X' = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is the eigenvector corresponding to the eigenvalue $\lambda_1(A) = 2$.

2) $\lambda_2 = 2 + 2i$

$(A - 2 - 2i)x = 0$

$$\begin{bmatrix} -2i & 0 & 0 \\ 0 & -2i & 4 \\ 0 & -1 & -2i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$A' = \begin{bmatrix} -2i & 0 & 0 \\ 0 & -2i & 4 \\ 0 & -1 & -2i \end{bmatrix} \sim \begin{bmatrix} 2i & 0 & 0 \\ 0 & +2i & -4 \\ 0 & 1 & 2i \end{bmatrix} \sim \begin{bmatrix} 2i & 0 & 0 \\ 0 & 2i & -4 \\ 0 & 2i & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2i & -4 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & i & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = 0,\ ix_2 - 2x_3 = 0$.

Here, x_3 is the free variable. Assigning the arbitrary value $x_3 = i$, $x_2 = 2$.

$X^2 = (0, 2, i)$ is the eigenvector corresponding to the eigenvalue $\lambda_2 = 2 + 2i$

3) $\lambda_3 = 2-2i$

$(A-2+2i)x=0$

$$\begin{bmatrix} 2i & 0 & 0 \\ 0 & 2i & 4 \\ 0 & -1 & 2i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$A' = \begin{bmatrix} 2i & 0 & 0 \\ 0 & 2i & 4 \\ 0 & -1 & 2i \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & i & 2 \\ 0 & -1 & 2i \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & i & 2 \\ 0 & -i & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & i & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

$x_1 = 0$

$ix_2 + 2x_3 = 0$. Here x_3 is the free variable

Assigning the arbitrary value $x_3 = i$, $x_2 = -2$.

$x^3 = (0, -2, i)$ is the eigenvector corresponding to the eigenvalue $\lambda_3 = 2-2i$.

8. Find the complex eigenvalues and the associated eigenvectors for the following matrices.

a) $\begin{bmatrix} 2 & 2 \\ -1 & 2 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix}$

Solution

a) $\det(A-\lambda I)=0$

$$\begin{vmatrix} 2-\lambda & 2 \\ -1 & 2-\lambda \end{vmatrix} = 0$$

$(\lambda-2)^2 + 2 = 0$

$(\lambda-2)^2 + (\sqrt{2})^2 = 0$

$(\lambda-2)^2 - (\sqrt{2}i)^2 = 0$

$(\lambda-2+\sqrt{2}i)(\lambda-2-\sqrt{2}i)=0$

$\lambda_1 = 2-\sqrt{2}i$, $\lambda_2 = 2+\sqrt{2}i$.

1) $\lambda_1 = 2-\sqrt{2}i$

$(A-2+\sqrt{2}i)x=0$

$$\begin{bmatrix} \sqrt{2}i & 2 \\ -1 & \sqrt{2}i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$$A' = \begin{bmatrix} \sqrt{2}i & 2 \\ -\sqrt{2}i & -2 \end{bmatrix} \sim \begin{bmatrix} \sqrt{2}i & 2 \\ 0 & 0 \end{bmatrix}$$

$\sqrt{2}i\,x_1 + 2x_2 = 0$

Here, x_1 is a free variable. Assigning the arbitrary value $x_1 = -1$, $2x_2 = \sqrt{2}i$, $x_2 = \frac{1}{\sqrt{2}}i$.

$x' = \left(-1, \frac{1}{\sqrt{2}}i\right)$ is the eigenvector corresponding to the eigenvalue $\lambda_1(A) = 2-\sqrt{2}i$.

2) $\lambda_2 = 2+\sqrt{2}i$.

$(A-2-\sqrt{2}i)x = 0$

$$\begin{bmatrix} -\sqrt{2}i & 2 \\ -1 & -\sqrt{2}i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

~~Assigning~~ A $A' = \begin{bmatrix} -\sqrt{2}i & 2 \\ -1 & -\sqrt{2}i \end{bmatrix} \sim \begin{bmatrix} -\sqrt{2}i & 2 \\ \sqrt{2}i & -2 \end{bmatrix} \sim \begin{bmatrix} -\sqrt{2}i & 2 \\ 0 & 0 \end{bmatrix}$

$\sqrt{2}i x_1 - 2x_2 = 0$

Here, x_1 is the free variable. Assigning arbitrary values $x_1 = 1$, $2x_2 = \sqrt{2}i$, $x_2 = \frac{1}{\sqrt{2}}i$

$x^2 = \left(1, \frac{i}{\sqrt{2}}\right)$ is the eigenvector corresponding to the eigenvalue $\lambda_2 = 2+\sqrt{2}i$

b) $\det(A-\lambda I)=0$

$$\begin{vmatrix} 1-\lambda & 2 \\ -1 & 2-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(2-\lambda)+2=0$$
$$\lambda^2-3\lambda+2+2=0$$
$$\lambda^2-3\lambda+4=0$$
$$\lambda^2-2\cdot\lambda\left(\frac{3}{2}\right)+\left(\frac{3}{2}\right)^2+4-\left(\frac{3}{2}\right)^2=0$$
$$\left(\lambda-\frac{3}{2}\right)^2+\frac{16}{4}-\frac{9}{4}=0$$
$$\left(\lambda-\frac{3}{2}\right)^2+\left(\frac{\sqrt{7}}{2}\right)^2=0$$
$$\left(\lambda-\frac{3}{2}+\frac{\sqrt{7}i}{2}\right)\left(\lambda-\frac{3}{2}-\frac{\sqrt{7}i}{2}\right)=0.$$
$$\lambda_1=\frac{3}{2}-\frac{\sqrt{7}i}{2},\ \lambda_2=\frac{3}{2}+\frac{\sqrt{7}i}{2}.$$

1) $\lambda_1=(3-\sqrt{7}i)/2$

$$\left(A-\frac{(3-\sqrt{7}i)}{2}I\right)x=0$$

$$\begin{bmatrix} (-1/2+\sqrt{7}/2)i & 2 \\ -1 & (1/2+\sqrt{7}/2)i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$$A'=\begin{bmatrix} (-1/2+\sqrt{7}/2)i & 2 \\ -(-1/2+\sqrt{7}/2)i & (\frac{\sqrt{7}i}{2})^2-(\frac{1}{2})^2 \end{bmatrix} \sim \begin{bmatrix} (-1/2+\sqrt{7}/2)i & 2 \\ -(-1/2+\sqrt{7}/2)i & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} (-1/2+\sqrt{7}/2i) & 2 \\ 0 & 0 \end{bmatrix}$$

$$1+\sqrt{7}i-\frac{(-4)}{-1+\sqrt{7}i}$$
$$(1+\sqrt{7}i)+\frac{4(-1-\sqrt{7}i)}{(-1)^2-(\sqrt{7}i)^2}$$
$$=1+\sqrt{7}i-\frac{2}{3}(1+\sqrt{7}i)$$
$$=\frac{1}{3}(1+\sqrt{7}i)$$

$$x_1\left(\frac{-1}{2}+\frac{\sqrt{7}}{2}i\right)+2x_2=0.$$

Here, x_1 is the free variable. Assigning the arbitrary value $x_1=2$, $x_2=\frac{1}{2}-\frac{\sqrt{7}i}{2}$.

$x'=\left(2,\frac{1}{2}-\frac{\sqrt{7}i}{2}\right)$ is the eigenvector corresponding to eigenvalue $\lambda_1=\frac{3-\sqrt{7}i}{2}$.

2) $\lambda_2=\frac{3}{2}+\frac{\sqrt{7}i}{2}$

$$\left(A-\frac{3}{2}-\frac{\sqrt{7}i}{2}I\right)x=0.$$

$$\begin{bmatrix} -1/2-\sqrt{7}i/2 & 2 \\ -1 & 1/2-\sqrt{7}i/2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$$A'=\begin{bmatrix} 1/2+\sqrt{7}i/2 & -2 \\ -(1/2+\sqrt{7}i/2) & (1/2)^2-(\sqrt{7}i/2)^2 \end{bmatrix} \sim \begin{bmatrix} 1/2+\sqrt{7}i/2 & -2 \\ 0 & 0 \end{bmatrix}$$

$x^2=\left(2,\frac{1}{2}+\frac{\sqrt{7}i}{2}\right)$ is the eigenvector corresponding to the eigenvalue $\lambda_2=\frac{3+\sqrt{7}i}{2}$.

4. Find the complex eigenvalues and associated eigenvectors for the following matrices.

a) $A=\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix}$

b) $\begin{bmatrix} 0 & 1 & -2 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$

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Solution

a) $\det(A-\lambda I)=0$

$$\begin{vmatrix} 1-\lambda & 0 & 2 \\ 0 & 1-\lambda & -1 \\ -1 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\begin{vmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} + 2\begin{vmatrix} 0 & 1-\lambda \\ -1 & 1 \end{vmatrix} = 0$$

$$(1-\lambda)\{(1-\lambda)^2+1\} + 2(1-\lambda) = 0$$

$$(1-\lambda)[(1-\lambda)^2+1+2]$$

$$(\lambda-1)(\lambda^2-2\lambda+1+3)$$

$$(\lambda-1)(\lambda^2-2\lambda+4)=0$$

$\lambda_1 = 0$ or $\lambda^2-2\lambda+4=0$

$$(\lambda-1)^2+(\sqrt{3})^2=0$$

$$\lambda_2 = 1+\sqrt{3}i,\quad \lambda_3 = 1-\sqrt{3}i.$$

1) $\lambda_1 = 1$

$$Ax = \lambda_1 x$$

$$(A-\lambda_1 I)x = 0$$

$$(A-I)x=0$$

$$\begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A' \Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 - x_2 = 0,\ x_3 = 0$

Here, x_2 is the free variable. Assigning an arbitrary value $x_2 = 1$, we find $x_1 = 1$.
Thus, $X^1 = (1,1,0)$ is the eigenvector corresponding to the eigenvalue $\lambda_1(A) = 1$.

2) $\lambda_2 = 1+\sqrt{3}i$

$$Ax = (1+\sqrt{3}i)x$$

$$(A-(1+\sqrt{3}i)I)x = 0$$

$$\begin{bmatrix} -\sqrt{3}i & 0 & 2 \\ 0 & -\sqrt{3}i & -1 \\ -1 & 1 & -\sqrt{3}i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$A = \begin{bmatrix} -\sqrt{3}i & 0 & 2 \\ 0 & -\sqrt{3}i & -1 \\ -1 & 1 & -\sqrt{3}i \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & \sqrt{3}i \\ 0 & -\sqrt{3}i & -1 \\ -\sqrt{3}i & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & \sqrt{3}i \\ 0 & -\sqrt{3}i & -1 \\ 0 & -\sqrt{3}i & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & \sqrt{3}i \\ 0 & -\sqrt{3}i & -1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} -\sqrt{3}i & \sqrt{3}i & -3 \\ 0 & -\sqrt{3}i & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} -\sqrt{3}i & 0 & -\cancel{3}4 \\ 0 & -\sqrt{3}i & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sqrt{3}i\,x_1 + 4x_3 = 0$$

$$\sqrt{3}i\,x_2 + x_3 = 0$$

Here, x_3 is a free variable. Assigining the arbitrary value $x_3 = \sqrt{3}i$, we obtain

$\sqrt{3}i x_1 + 4\sqrt{3}i = 0$ $\qquad$ $\sqrt{3}i\,x_2 + \sqrt{3}i = 0$

$x_1 + 4 = 0$ $\qquad$ $x_2 = -1.$

$x_1 = -4$

$X^2 = (4, 1, -\sqrt{3}i)$ is the eigenvector corresponding to the eigenvalue $\lambda_2(A) = 1+\sqrt{3}i$.

3) $\lambda_3 = 1-\sqrt{3}i$

$Ax = (1-\sqrt{3}i)x$

$(A-(1+\sqrt{3}i)I)x = 0$

$$\begin{bmatrix} \sqrt{3}i & 0 & 2 \\ 0 & \sqrt{3}i & -1 \\ -1 & 1 & \sqrt{3}i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$A' = \begin{bmatrix} \sqrt{3}i & 0 & 2 \\ 0 & \sqrt{3}i & -1 \\ -1 & 1 & \sqrt{3}i \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -\sqrt{3}i \\ 0 & \sqrt{3}i & -1 \\ \sqrt{3}i & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -\sqrt{3}i \\ 0 & \sqrt{3}i & -1 \\ 0 & \sqrt{3}i & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -\sqrt{3}i \\ 0 & \sqrt{3}i & -1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} \sqrt{3}i & -\sqrt{3}i & 3 \\ 0 & \sqrt{3}i & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} \sqrt{3}i & 0 & 2 \\ 0 & \sqrt{3}i & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\sqrt{3}ix_1 + 2x_2 = 0$

$\sqrt{3}ix_2 - x_3 = 0$

Here, x_3 is the free variable. Assigning the arbitrary value $x_3 = \sqrt{3}i$,

$x_1 = -2$, $x_2 = 1$.

$X3 = (-2, 1, \sqrt{3}i)$ is the eigenvector corresponding to the eigenvalue $\lambda_3(A) = 1-\sqrt{3}i$.

b) $\det(A-\lambda I) = 0$

$$\begin{vmatrix} -\lambda & 1 & -2 \\ 1 & -\lambda & 0 \\ 1 & 1 & 1-\lambda \end{vmatrix} = 0.$$

$$R_3 \to R_3 - R_2 \quad \begin{vmatrix} -\lambda & 1 & -2 \\ 1 & -\lambda & 0 \\ 0 & 1+\lambda & 1-\lambda \end{vmatrix} = 0$$

$-\lambda[-\lambda(1-\lambda)] - 1[(1-\lambda)+2(1+\lambda)] = 0$

$\lambda^2(1-\lambda) - \{1-\lambda+2+2\lambda\} = 0$

$\lambda^2(1-\lambda) - (\lambda+3) = 0$

$\lambda^2 - \lambda^3 - \lambda - 3 = 0$

$\lambda^3 - \lambda^2 + \lambda + 3 = 0$

$\lambda^3 + \lambda^2 - 2\lambda^2 + \lambda + 3 = 0$

$\lambda^2(\lambda+1) - (2\lambda^2 - \lambda - 3) = 0$

$\lambda^2(\lambda+1) - (2\lambda^2 - 3\lambda + 2\lambda - 3) = 0$

$\lambda^2(\lambda+1) - [\lambda(2\lambda-3) + 1(2\lambda-3)] = 0$

$\lambda^2(\lambda+1) - (\lambda+1)(2\lambda-3) = 0$

$(\lambda+1)(\lambda^2 - 2\lambda + 3) = 0$

$(\lambda+1)(\lambda^2 - 2\lambda + 1 + 2) = 0$

$(\lambda+1)[(\lambda-1)^2 + (\sqrt{2})^2] = 0$

$(\lambda+1)(\lambda - 1 + \sqrt{2}i)(\lambda - 1 - \sqrt{2}i) = 0$

$\lambda_1 = -1$, $\lambda_2 = 1+\sqrt{2}i$, $\lambda_3 = 1-\sqrt{2}i$.

1) $\lambda_1 = -1$.

$Ax = \lambda_1 x$

$Ax = -x$

$(A+I)x = 0$

$$\begin{bmatrix} 1 & 1 & -2 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$A' = \begin{bmatrix} 1 & 1 & -2 \\ 0 & 0 & 2 \\ 0 & 0 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 + x_2 = 0$
$x_3 = 0$
Here, x_2 is the free variable. Assigning arbitrary values to x_2, we find ~~t~~
let $x_2 = -1$, then $x_1 = 1$.

$X^1 = (1, -1, 0)$ is the eigenvector corresponding to the eigenvalue $\lambda_1(A) = -1$.

2) $\lambda_2 = 1 + \sqrt{2}i$
$(A - \lambda_2 I)x = 0$
$(A - 1 - \sqrt{2}iI)x = 0$.

$$\begin{bmatrix} -1-\sqrt{2}i & 1 & -2 \\ 1 & -1-\sqrt{2}i & 0 \\ 1 & 1 & -\sqrt{2}i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$A' \sim \begin{bmatrix} 1 & 1 & -\sqrt{2}i \\ 1 & -(1+\sqrt{2}i) & 0 \\ -(1+\sqrt{2}i) & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -\sqrt{2}i \\ 0 & -(2+\sqrt{2}i) & \sqrt{2}i \\ 0 & 2+\sqrt{2}i & -\sqrt{2}i \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -\sqrt{2}i \\ 0 & -(2+\sqrt{2}i) & \sqrt{2}i \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2+\sqrt{2}i & 2+\sqrt{2}i & 2-2\sqrt{2}i \\ 0 & -(2+\sqrt{2}i) & \sqrt{2}i \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 2+\sqrt{2}i & 0 & 2-\sqrt{2}i \\ 0 & -(2+\sqrt{2}i) & \sqrt{2}i \\ 0 & 0 & 0 \end{bmatrix}$$

$(2+\sqrt{2}i)x_1 + (2-\sqrt{2}i)x_3 = 0$
$-(2+\sqrt{2}i)x_2 - (\sqrt{2}i)x_3 = 0.$

Here, x_3 is the free variable. Assigning the arbitrary value $x_3 = (2+\sqrt{2}i)$, we find that —

$x_1\cancel{(2+\sqrt{2}i)} + (2-\sqrt{2}i)\cancel{(2+\sqrt{2}i)} = 0$
$x_1 = -(2-\sqrt{2}i)$

$x_2 - \sqrt{2}i = 0$
$x_2 = \sqrt{2}i$

$X^2 = (-(2-\sqrt{2}i), \sqrt{2}i, 2+\sqrt{2}i)$ is the eigenvector corresponding to the eigenvalue $\lambda_2(A) = 1+\sqrt{2}i$

3) $\lambda_3 = 1-\sqrt{2}i$
$(A - (1-\sqrt{2}i)I)x = 0$

$$\begin{bmatrix} -1+\sqrt{2}i & 1 & -2 \\ 1 & -1+\sqrt{2}i & 0 \\ 1 & 1 & \sqrt{2}i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$A \sim \begin{bmatrix} 1 & 1 & \sqrt{2}i \\ 1 & -1+\sqrt{2}i & 0 \\ -1+\sqrt{2}i & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & \sqrt{2}i \\ 0 & -2+\sqrt{2}i & -\sqrt{2}i \\ 0 & 2-\sqrt{2}i & \sqrt{2}i \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & \sqrt{2}i \\ 0 & -2+\sqrt{2}i & -\sqrt{2}i \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2-\sqrt{2}i & 2-\sqrt{2}i & 2+\sqrt{2}i \\ 0 & -2+\sqrt{2}i & -\sqrt{2}i \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 2-\sqrt{2}i & 0 & 2+\sqrt{2}i \\ 0 & -2+\sqrt{2}i & -\sqrt{2}i \\ 0 & 0 & 0 \end{bmatrix}.$$

$x_1(2-\sqrt{2}i) + x_3(2+\sqrt{2}i) = 0$
$x_2(-2+\sqrt{2}i) + x_3(-\sqrt{2}i) = 0.$

~~Let $x_3 =$~~ Here, x_3 is a free variable. ~~Assigning an arb~~ arbitrary value $x_3 = (2-\sqrt{2}i)$. We find $x_1 = -(2+\sqrt{2}i)$. $-x_2\{-(2-\sqrt{2}i) + x_3(-\sqrt{2}i)(2-\sqrt{2}i) = 0$

So, ~~x_2~~ $-x_2 + x_3(-\sqrt{2}i) = 0$
$x_2 = -\sqrt{2}i.$

$x^3 = (-2-\sqrt{2}i, -\sqrt{2}i, 2-\sqrt{2}i)$ is the eigenvector corresponding to the eigenvalue $\lambda_3(A) = 1-\sqrt{2}i$.

5) Find the spectral radius for each matrix in problem 1.
Solution:
a) $\rho(A) = \max|\lambda_i| = \max\{1, 3\} = 3.$
b) $\rho(A) = \max|\lambda_i| = \max\left\{\left|\frac{1+\sqrt{5}}{2}\right|, \left|\frac{1-\sqrt{5}}{2}\right|\right\} = \frac{1+\sqrt{5}}{2}.$
c) $\rho(A) = \max|\lambda_i| =$ ~~max~~
$= \max\left\{\left|\frac{1}{2}\right|, \left|-\frac{1}{2}\right|\right\} = \frac{1}{2}.$

d) $\rho(A) = \max|\lambda_i| = \max\{1, 3\} = 3$
e) $\rho(A) = \max|\lambda_i| = \max\{|-1|, |3|, |7|\} = 7.$
f) $\rho(A) = \max|\lambda| = \max\{|1|, |5|\} = 5.$

6) Find the spectral radius for each matrix in problem 2.
Solution:
a) $\rho(A) = \max|\lambda_i| = \max\{0, 1\} = 1$
b) $\rho(A) = \max|\lambda_i| = \max\left\{\frac{1}{2}, \frac{1}{3}\right\} = \frac{1}{2}$

c) $\rho(A) = \max|\lambda_i| = \max\{1, 4\} = 4.$
d) $\rho(A) = \max|\lambda_i| = \max\{2, 3, 4\} = 4.$
e) $\rho(A) = \max|\lambda_i| = \max\left\{\frac{1}{2}, \frac{1}{3}\right\} = \frac{1}{2}$

f) $\rho(A) = \max|\lambda_i| = \max\{2, |2+2i|, |2-2i|\}$
$= \max\{2, 2\sqrt{2}, 2\sqrt{2}\} = 2\sqrt{2}.$

7) Which of the matrices in problem 1 are convergent?
The matrix $A = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}$ is convergent.

8) Which of the matrices in problem 2 are convergent?
None of the matrices in problem 2 are convergent.

9) Find the l_2 norm of the matrices in problem 1.

$\|A\|_2 = [\rho(A^tA)]^{1/2}$

a) $A^TA = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}^T \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

$= \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}$

The eigenvalues of A^tA are given by,
$\det(A^tA - \lambda I) = 0$

$\begin{vmatrix} 5-\lambda & -4 \\ -4 & 5-\lambda \end{vmatrix} = 0$

$(5-\lambda)^2 - 16 = 0$
$(\lambda-5)^2 - 4^2 = 0$ $\quad (\lambda-1)(\lambda-9) = 0.$ $\quad \lambda_1 = 1, \lambda_2 = 9.$

$\rho(A^TA) = 9$

$\|A\|_2 = \sqrt{\rho(A^TA)} = 3.$

b) $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$

$A^tA = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$

$\det(A^TA - \lambda I) = 0$

$\begin{vmatrix} 1-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$

$(\lambda-1)(\lambda-2) - 1 = 0$

$\lambda^2 - 3\lambda + 2 - 1 = 0$

$\lambda^2 - 3\lambda + 1 = 0$

$\lambda^2 - 2\cdot\lambda\cdot\left(\frac{3}{2}\right) + \left(\frac{3}{2}\right)^2 + 1 - \left(\frac{3}{2}\right)^2 = 0$

$\left(\lambda - \frac{3}{2}\right)^2 - \frac{5}{4} = 0$

$\left(\lambda - \frac{3}{2}\right) - \left(\frac{\sqrt{5}}{2}\right)^2 = 0$

$\left(\lambda - \frac{3}{2} + \frac{\sqrt{5}}{2}\right)\left(\lambda - \frac{3}{2} - \frac{\sqrt{5}}{2}\right) = 0$

$\lambda_1 = \frac{3-\sqrt{5}}{2}, \ \lambda_2 = \frac{3+\sqrt{5}}{2}$

$\rho(A^tA) = \max|\lambda_i| = \frac{3+\sqrt{5}}{2}.$

$\|A\|_2 = \sqrt{\rho(A^tA)} = \sqrt{\frac{3+\sqrt{5}}{2}}.$

c) $A = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}.$

$A^tA = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}.$

$= \begin{bmatrix} 1/4 & 0 \\ 0 & 1/4 \end{bmatrix}$

$\det(A^tA - \lambda I) = 0$

$\begin{vmatrix} 1/4-\lambda & 0 \\ 0 & 1/4-\lambda \end{vmatrix} = 0$

$\left(\lambda - \frac{1}{4}\right)^2 = 0$

$\lambda_1 = \lambda_2 = \frac{1}{4}.$

$\rho(A^tA) = \max|\lambda_i| = \frac{1}{4}.$

$\|A\|_2 = \sqrt{\rho(A^tA)} = \frac{1}{2}.$

d)
$$A=\begin{bmatrix}2&1&0\\1&2&0\\0&0&3\end{bmatrix}$$

$$A^tA=\begin{bmatrix}2&1&0\\1&2&0\\0&0&3\end{bmatrix}\begin{bmatrix}2&1&0\\1&2&0\\0&0&3\end{bmatrix}=\begin{bmatrix}5&4&0\\4&5&0\\0&0&9\end{bmatrix}$$

$\det(A^tA-\lambda I)=0$

$$\begin{vmatrix}5-\lambda&4&0\\4&5-\lambda&0\\0&0&9-\lambda\end{vmatrix}=0$$

$$(9-\lambda)\begin{vmatrix}5-\lambda&4\\4&5-\lambda\end{vmatrix}=0$$

$(\lambda-9)\{(\lambda-5)(\lambda-5)-16\}=0$
$(\lambda-9)\{(\lambda-5)^2-4^2\}=0$
$(\lambda-9)(\lambda-1)(\lambda-9)=0$
$(\lambda-1)(\lambda-9)^2=0$
$\lambda_1=1,\ \lambda_2=9$
$\rho(A^tA)=\max|\lambda_i|=9$
$\|A\|_2=\sqrt{\rho(A^tA)}=3.$

e)
$$A=\begin{bmatrix}-1&2&0\\0&3&4\\0&0&7\end{bmatrix}$$

$$A^tA=\begin{bmatrix}-1&0&0\\2&3&0\\0&4&7\end{bmatrix}\begin{bmatrix}-1&2&0\\0&3&4\\0&0&7\end{bmatrix}=\begin{bmatrix}1&-2&0\\-2&13&12\\0&12&65\end{bmatrix}$$

$\det(A^tA-\lambda I)=0$

$$\begin{vmatrix}+1-\lambda&-2&0\\-2&13-\lambda&12\\0&12&65-\lambda\end{vmatrix}=0$$

$$(1-\lambda)\begin{vmatrix}13-\lambda&12\\12&65-\lambda\end{vmatrix}+2\begin{vmatrix}-2&12\\0&65-\lambda\end{vmatrix}=0$$

-144

$(1-\lambda)(\lambda^2-78\lambda+845)-4(65-\lambda)=0.$
$\lambda^2-78\lambda+701-\lambda^3+78\lambda^2-701\lambda-260+4\lambda=0.$
$-\lambda^3+79\lambda^2-775\lambda+441=0.$
$\lambda^3-79\lambda^2+775\lambda-441=0$

$\lambda_1=67.63840,\ \lambda_2=10.7553,\ \lambda_3=0.6062.$

$\rho(A^tA)=\max|\lambda_i|=67.63840$
$\|A\|_2=\sqrt{\rho(A^tA)}=8.2242.$

10) Find the l_2-norm for the matrices in problem 2.

Solution:

a)
$$\begin{bmatrix}1&1\\-2&-2\end{bmatrix}$$

The $A^tA=\begin{bmatrix}1&-2\\1&-2\end{bmatrix}\begin{bmatrix}1&1\\-2&-2\end{bmatrix}=\begin{bmatrix}5&5\\5&5\end{bmatrix}$

$\det(A^tA-\lambda I)=0$

$$\begin{vmatrix} 5-\lambda & 5 \\ 5 & 5-\lambda \end{vmatrix} = 0$$

$(5-\lambda)^2 - 25 = 0$

$(\lambda-5)^2 - 5^2 = 0$

$\lambda(\lambda-10) = 0$

$\lambda = 0$ or $\lambda = 10$

But, $\lambda \neq 0$. so, $\lambda_1 = 10$.

$\rho(A^tA) = \max |\lambda_i| = 10$

$\|A\|_2 = \sqrt{\rho(A^tA)} = \sqrt{10}$.

b) $A = \begin{bmatrix} -1 & -1 \\ 1/3 & 1/6 \end{bmatrix}$

$A^tA = \begin{bmatrix} -1 & -1 \\ 1/3 & 1/6 \end{bmatrix} \begin{bmatrix} -1 & 1/3 \\ -1 & 1/6 \end{bmatrix} = \begin{bmatrix} 2 & -1/2 \\ -1/2 & 5/36 \end{bmatrix}$

$\det(A^tA - \lambda I) = 0$

$$\begin{vmatrix} 2-\lambda & -1/2 \\ -1/2 & 5/36-\lambda \end{vmatrix} = 0.$$

$(2-\lambda)(5/36-\lambda) - \frac{1}{4} = 0$

$(\lambda-2)\left(\lambda - \frac{5}{36}\right) - \frac{1}{4} = 0$

$\lambda^2 - \frac{5}{36}\lambda - \frac{72}{36}\lambda + \frac{10}{36} - \frac{1}{4} = 0$

$\lambda^2 - \frac{77\lambda}{36} + \frac{1}{36} = 0.$

~~$\lambda^2 - 2.13$~~

$36\lambda^2 - 77\lambda + 1 = 0$

$\lambda = \frac{77 \pm \sqrt{77^2 - 4(36)}}{2(36)}$

$\lambda_1 = \frac{77 + 76.0591}{72} = 2.125822$

$\lambda_2 = 0.013066$

$\rho(A^tA) = \max |\lambda_i| = 2.125822$

$\|A\|_2 = \sqrt{\rho(A^tA)} = \sqrt{2.125822} = 1.458019.$

c) $A = \begin{bmatrix} 3 & 4 \\ 1 & 0 \end{bmatrix}$

$A^tA = \begin{bmatrix} 3 & 1 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 1 & 0 \end{bmatrix}$

$= \begin{bmatrix} 10 & 12 \\ 12 & 16 \end{bmatrix}$

$\det(A^tA - \lambda I) = 0$

$$\begin{vmatrix} 10-\lambda & 12 \\ 12 & 16-\lambda \end{vmatrix} = 0$$

$(10-\lambda)(16-\lambda) - 144 = 0$

$\lambda^2 - 26\lambda + 160 - 144 = 0$

$\lambda^2 - 26\lambda + 16 = 0$
$\lambda^2 - 2\lambda(13) + 13^2 + 16 - 13^2 = 0$
$(\lambda - 13)^2 - 153 = 0$
$(\lambda - 13)^2 - \cancel{17}(3\sqrt{17})^2 = 0$
$(\lambda - 13 + 3\sqrt{17})(\lambda - 13 - 3\sqrt{17}) = 0$
$\lambda_1 = 13 - 3\sqrt{17}, \ \lambda_2 = 13 + 3\sqrt{17}.$
$\rho(A^tA) = \max|\lambda_i| = 13 + 3\sqrt{17}$
$\|A\|_2 = \sqrt{\rho(A^tA)} = \sqrt{13 + 3\sqrt{17}} = 5.02679.$

d)
$$A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & -2 & 3 \\ 2 & 0 & 4 \end{bmatrix}$$
$$A^tA = \begin{bmatrix} 3 & 1 & 2 \\ 2 & -2 & 0 \\ -1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 3 & 2 & -1 \\ 1 & 2 & -3 \\ 2 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 14 & 8 & 2 \\ 4 & 0 & 4 \\ 8 & 4 & 8 \end{bmatrix}$$

$\det(A^tA - \lambda I) = 0$

$$\begin{vmatrix} 14-\lambda & 8 & 2 \\ 4 & -\lambda & 4 \\ 8 & 4 & 8-\lambda \end{vmatrix} = 0$$

$$(14-\lambda)\begin{vmatrix} -\lambda & 4 \\ 4 & 8-\lambda \end{vmatrix} - 8\begin{vmatrix} 4 & 4 \\ 8 & 8-\lambda \end{vmatrix} + 2\begin{vmatrix} 4 & -\lambda \\ 8 & 4 \end{vmatrix} = 0.$$

$(14-\lambda)[\lambda(\lambda-8) - 16] - 8[32 - 4\lambda - 32] + 2[16 + 8\lambda] = 0.$
$(14-\lambda)(\lambda^2 - 8\lambda - 16) - 8(-4\lambda) + 32 + 16\lambda = 0$
$(14-\lambda)(\lambda^2 - 8\lambda - 16) + 32\lambda + 16 + 16\lambda = 0$
$\cancel{(14-\lambda}\ 14\lambda^2 - 112\lambda - 224 - \lambda^3 + 8\lambda^2 + 16\lambda + 48\lambda + 32 = 0$
$-\lambda^3 + 22\lambda^2 - 48\lambda - 192 = 0$
$\lambda^3 - 22\lambda^2 + 48\lambda + 192 = 0$

$\lambda_1 = -2, \ \lambda_2 = 5.07179, \ \lambda_3 = 18.92820.$
$\rho(A^tA) = \max|\lambda_i| = 18.92820.$
$\|A\|_2 = \sqrt{\rho(A^tA)} = \sqrt{18.92820} = 4.2757.$

e)
$$A = \begin{bmatrix} 1/2 & 0 & 0 \\ -1 & 1/2 & 0 \\ 2 & 2 & -1/3 \end{bmatrix}$$
$$A^tA = \begin{bmatrix} 1/2 & 0 & 0 \\ -1 & 1/2 & 0 \\ 2 & 2 & -1/3 \end{bmatrix}^T \begin{bmatrix} 1/2 & 0 & 0 \\ -1 & 1/2 & 0 \\ 2 & 2 & -1/3 \end{bmatrix} = \begin{bmatrix} 1/2 & -1 & 2 \\ 0 & 1/2 & 2 \\ 0 & 0 & 1/3 \end{bmatrix} \begin{bmatrix} 1/2 & 0 & 0 \\ -1 & 1/2 & 0 \\ 2 & 2 & -1/3 \end{bmatrix}$$
$$= \begin{bmatrix} 21/4 & 3/2 & -2/3 \\ 3/2 & 17/4 & -2/3 \\ -2/3 & 2/3 & -1/9 \end{bmatrix}$$

$\det(A^tA - \lambda I) = 0.$

$$\begin{vmatrix} 21/4-\lambda & 3/2 & -2/3 \\ 3/2 & 17/4-\lambda & -2/3 \\ 2/3 & 2/3 & -1/9-\lambda \end{vmatrix} = 0.$$

$\lambda_1 = 6.19394, \ \lambda_2 = 3.16126, \ \lambda_3 = 0.03369.$
$\rho(A^tA) = \max(\lambda) = 6.19394$
$\|A\|_2 = \sqrt{\rho(A^tA)} = \sqrt{6.19394} = 2.488.$

11) Let $A_1 = \begin{bmatrix} 1 & 0 \\ 1/4 & 1/2 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 1/2 & 0 \\ 16 & 1/2 \end{bmatrix}$.

Show that A_1 is not convergent, but that A_2 is convergent.

Proof

$$A_1 = \begin{bmatrix} 1 & 0 \\ 1/4 & 1/2 \end{bmatrix}$$

$$A_1^2 = \begin{bmatrix} 1 & 0 \\ 1/4 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1/4 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3/8 & 1/4 \end{bmatrix}$$

$$A_1^3 = \begin{bmatrix} 1 & 0 \\ 3/8 & 1/4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1/4 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 7/16 & 1/8 \end{bmatrix}$$

$$A_1^4 = \begin{bmatrix} 1 & 0 \\ 7/16 & 1/8 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1/4 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 15/32 & 1/16 \end{bmatrix}$$

$$A_1^k = \begin{bmatrix} 1 & 0 \\ 2^k - 1/2^{k+1} & 1/2^k \end{bmatrix}$$

$$\lim_{k \to \infty} (1) = 1$$

$$\lim_{k \to \infty} (0) = 0$$

$$\lim_{k \to \infty} \frac{2^k - 1}{2^{k+1}} = \lim_{k \to \infty} \left(\frac{1}{2} - \frac{1}{2^{k+1}} \right) = \frac{1}{2}$$

$$\lim_{k \to \infty} \left(\frac{1}{2^k} \right) = 0.$$

$$\lim_{k \to \infty} A^k = \begin{bmatrix} 1 & 0 \\ 1/2 & 0 \end{bmatrix}$$

Hence, A_1 is not convergent.

$$A_2 = \begin{bmatrix} 1/2 & 0 \\ 16 & 1/2 \end{bmatrix}$$

$$A_2^2 = \begin{bmatrix} 1/2 & 0 \\ 16 & 1/2 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 16 & 1/2 \end{bmatrix} = \begin{bmatrix} 1/4 & 0 \\ 16 & 1/4 \end{bmatrix}$$

$$A_2^3 = \begin{bmatrix} 1/4 & 0 \\ 16 & 1/4 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 16 & 1/2 \end{bmatrix} = \begin{bmatrix} 1/8 & 0 \\ 3/4(16) & 1/8 \end{bmatrix}$$

$$A_2^4 = \begin{bmatrix} 1/8 & 0 \\ 3/4(16) & 1/8 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 16 & 1/2 \end{bmatrix} = \begin{bmatrix} 1/16 & 0 \\ 1/2(16) & 1/16 \end{bmatrix}$$

$$A_2^5 = \begin{bmatrix} 1/16 & 0 \\ 8 & 1/16 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 16 & 1/2 \end{bmatrix} = \begin{bmatrix} 1/32 & 0 \\ 5/16(16) & 1/32 \end{bmatrix}$$

$$A_2^6 = \begin{bmatrix} 1/32 & 0 \\ 5 & 1/32 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 16 & 1/2 \end{bmatrix} = \begin{bmatrix} 1/64 & 0 \\ 6/32(16) & 1/64 \end{bmatrix}$$

$$A_2^7 = \begin{bmatrix} 1/64 & 0 \\ 3 & 1/64 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 16 & 1/2 \end{bmatrix} = \begin{bmatrix} 1/128 & 0 \\ 7/64(16) & 1/128 \end{bmatrix}$$

$$A_2^8 = [$$

$$A^k = \begin{bmatrix} 1/2^k & 0 \\ k/2^{k-1} \cdot 16 & 1/2^k \end{bmatrix}.$$

$$\lim_{k \to \infty} \left(\frac{1}{2^k} \right) = 0$$

$$\lim_{k \to \infty} 16 \left(\frac{k}{2^{k-1}} \right) = 0$$

$\lim_{k \to \infty}$ Thus, $\lim_{k \to \infty} A^k = 0$.

7) Which of the matrices in problem 1 are convergent?

Solution.

$\lim_{k \to \infty} A^k =$ Let's see if $A = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}$ is convergent.

$$A^2 = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix} = \begin{bmatrix} 1/4 & 0 \\ 0 & 1/4 \end{bmatrix} \quad A^k = \begin{bmatrix} 1/2^k & 0 \\ 0 & 1/2^k \end{bmatrix}$$

$\lim_{k\to\infty} A^k = 0$

so, A is convergent.

8) Which of the matrices in problem 2 are convergent?

Let's see if $A = \begin{bmatrix} -1 & -1 \\ 1/3 & 1/6 \end{bmatrix}$ is convergent.

$A^2 = \begin{bmatrix} -1 & -1 \\ 1/3 & 1/6 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 1/3 & 1/6 \end{bmatrix} = \begin{bmatrix} +2/3 & 5/6 \\ -5/18 & -11/36 \end{bmatrix}$

$A^3 = \begin{bmatrix} +2/3 & 5/6 \\ -5/18 & -11/36 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 1/3 & 1/6 \end{bmatrix} = \begin{bmatrix} -7/18 & -19/36 \\ 19/108 & 49/216 \end{bmatrix}$

$A^4 = \begin{bmatrix} -7/18 & -19/36 \\ 19/108 & 49/216 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 1/3 & 1/6 \end{bmatrix} = \begin{bmatrix} 23/108 & 65/216 \\ -65/648 & -179/1296 \end{bmatrix}$

All of the denominators are increasing exponentially.

$\lim_{k\to\infty} A^k = 0.$

A is convergent.

Let's also test example (e) where $A = \begin{bmatrix} 1/2 & 0 & 0 \\ -1 & 1/2 & 0 \\ 2 & 2 & -1/3 \end{bmatrix}$ for convergence.

$A^2 = \begin{bmatrix} 1/2 & 0 & 0 \\ -1 & 1/2 & 0 \\ 2 & 2 & -1/3 \end{bmatrix} \begin{bmatrix} 1/2 & 0 & 0 \\ -1 & 1/2 & 0 \\ 2 & 2 & -1/3 \end{bmatrix} = \begin{bmatrix} 1/4 & 0 & 0 \\ -1 & 1/4 & 0 \\ -5/3 & 1/3 & 1/9 \end{bmatrix}$

$A^3 = \begin{bmatrix} 1/4 & 0 & 0 \\ -1 & 1/4 & 0 \\ -5/3 & 1/3 & 1/9 \end{bmatrix} \begin{bmatrix} 1/2 & 0 & 0 \\ -1 & 1/2 & 0 \\ 2 & 2 & -1/3 \end{bmatrix} = \begin{bmatrix} 1/8 & 0 & 0 \\ -3/4 & 1/8 & 0 \\ -17/18 & 7/18 & -1/27 \end{bmatrix}$

$A^4 = \begin{bmatrix} 1/8 & 0 & 0 \\ -3/4 & 1/8 & 0 \\ -17/18 & 7/18 & -1/27 \end{bmatrix} \begin{bmatrix} 1/2 & 0 & 0 \\ -1 & 1/2 & 0 \\ 2 & 2 & -1/3 \end{bmatrix} = \begin{bmatrix} 1/16 & 0 & 0 \\ -4/8 & 1/16 & 0 \\ -101/108 & 13/108 & 1/81 \end{bmatrix}$

~~A~~ $\lim_{k\to\infty} A^k = 0$

All terms (a_{ij}) diminish and approach zero.
so, A is convergent.

12) An $n \times n$ matrix is called nilpotent, if an integer m exists with $A^m = 0$. Show that if λ is eigenvalue of a nilpotent matrix, then $\lambda = 0$.

Solution

Suppose the integer m is such that

$A^m = 0$ (the zero matrix).

And λ is an eigenvalue of A^m.

This means that

$A^m x = \lambda x, \quad x \neq 0.$

But, $A^m x = 0$ vector.

Therefore, $\lambda x = 0$ vector, $x \neq 0$.

This is possible, if and only if $\lambda = 0$. (scalar)

Applied problems.

13) The contribution of a female beetle of a certain type made to the future years beetle population could be expressed in terms of the matrix

$A = \begin{bmatrix} 0 & 0 & 6 \\ 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \end{bmatrix}$

where (a_{ij}) = probabilistic contribution of a beetle of age j onto the next year's female population of age i.

a) Does the matrix have any real eigenvalues? If so determine them and any associated eigenvectors.

b) If a sample of this species was needed for laboratory test purposes that would have a constant proportion in each age group from year-to-year, what criteria would be imposed on the initial population to ensure that this requirement would be satisfied?

Solution

a) The characteristic polynomial of A is

$$\det(A-\lambda I)=0$$

$$\begin{vmatrix} -\lambda & 0 & 6 \\ 1/2 & -\lambda & 0 \\ 0 & 1/3 & -\lambda \end{vmatrix} = 0$$

$$-\lambda \begin{vmatrix} -\lambda & 0 \\ 1/3 & -\lambda \end{vmatrix} + 6 \begin{vmatrix} 1/2 & -\lambda \\ 0 & 1/3 \end{vmatrix} = 0$$

$$-\lambda(\lambda^2) + 6(1/6) = 0$$

$$\lambda^3 - 1 = 0$$

$$(\lambda-1)(\lambda^2+\lambda+1)=0$$

$$(\lambda-1)\left[\left(\lambda+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2\right]=0$$

$$(\lambda-1)\left(\lambda+\frac{1+\sqrt{3}i}{2}\right)\left(\lambda+\frac{1-\sqrt{3}i}{2}\right)=0$$

$\lambda_1 = 1$, $\lambda_2 = \frac{-1-\sqrt{3}i}{2}$, $\lambda_3 = \frac{-1+\sqrt{3}i}{2}$.

$\lambda_1 = 1$ is a real eigenvalue.

solving $Ax=\lambda_1 x$, $Ax = x \Rightarrow (A-I)x=0$

$$\begin{bmatrix} -1 & 0 & 6 \\ 1/2 & -1 & 0 \\ 0 & 1/3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A' = \begin{bmatrix} -1 & 0 & 6 \\ 1/2 & -1 & 0 \\ 0 & 1/3 & -1 \end{bmatrix} \sim \begin{bmatrix} -1 & 0 & 6 \\ 0 & -1 & 3 \\ 0 & 1/3 & -1 \end{bmatrix} \sim \begin{bmatrix} -1 & 0 & 6 \\ 0 & -1 & 3 \\ 0 & 1 & -3 \end{bmatrix} \sim \begin{bmatrix} -1 & 0 & 6 \\ 0 & -1 & 3 \\ 0 & 0 & 0 \end{bmatrix}.$$

$x_1 - 6x_3 = 0$
$x_2 - 3x_3 = 0$

Here, x_3 is the free variable. Assigning arbitrary values to x_3, $x_3=1$, we find $x_1=6$, $x_2=3$.

$x' = (6,3,1)$ is the eigenvector corresponding to the eigenvalue $\lambda=1$.

b) Let π be the ~~initial~~ proportion of female beetles at time zero. The jth component of $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$ denotes the proportion of female beetles of age j.

The matrix A represents the probabilistic contribution of a female beetle of a certain type made to the future years ~~beetle of~~ beetle population.

$(a_{ij}) = P(i|j)$

so, $A\pi$ is the age distribution of female beetles in 1 year.
$A^2\pi$ is the proportion of female beetles in 2 years.
$\vdots$

In order that π remains constant, we require that

$$A\pi = \pi.$$

But, this is an eigen value problem.

π is the eigenvector corresponding to the eigenvalue $\lambda = 1$.

$$\pi^{(0)} = (6, 3, 1).$$

14) consider a female beetle population suppose the . 2 matrix

$$P = \begin{bmatrix} 0 & 1/8 & 1/4 & 1/2 \\ 1/2 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 0 \\ 0 & 0 & 1/8 & 0 \end{bmatrix}$$

where the entries p_{ij} denote the contribution that beetles of age j will make to the next year's female beetles of age i.

a) Find the characteristic polynomial of A.
b) Find the spectral radius $\rho(A)$
c) Given the initial population $x = (x_1, x_2, x_3, x_4)$ what will eventually happen?

Solution.

a) $\det(A - \lambda I) = 0$

$$\begin{vmatrix} -\lambda & 1/8 & 1/4 & 1/2 \\ 1/2 & -\lambda & 0 & 0 \\ 0 & 1/4 & -\lambda & 0 \\ 0 & 0 & 1/8 & -\lambda \end{vmatrix} = 0$$

$$-\lambda \begin{vmatrix} -\lambda & 0 & 0 \\ 1/4 & -\lambda & 0 \\ 0 & 1/8 & -\lambda \end{vmatrix} + \left(-\frac{1}{2}\right) \begin{vmatrix} 1/8 & 1/4 & 1/2 \\ 1/4 & -\lambda & 0 \\ 0 & 1/8 & -\lambda \end{vmatrix} = 0$$

$$2\lambda \left\{ -\lambda \begin{vmatrix} -\lambda & 0 \\ 1/8 & -\lambda \end{vmatrix} \right\} + \left\{ \frac{1}{8} \begin{vmatrix} -\lambda & 0 \\ 1/8 & -\lambda \end{vmatrix} - \frac{1}{4} \begin{vmatrix} 1/4 & 1/2 \\ 1/8 & -\lambda \end{vmatrix} \right\} = 0$$

$$2\lambda \{ -\lambda (\lambda^2) \} + \frac{1}{8}\lambda^2 - \frac{1}{4}\left(-\frac{\lambda}{4} - \frac{1}{16} \right) = 0$$

$$-2\lambda^4 + \frac{1}{8}\lambda^2 + \frac{\lambda}{16} + \frac{1}{64} = 0$$

$$\lambda^4 - \frac{1}{16}\lambda^2 - \frac{\lambda}{32} - \frac{1}{128} = 0$$

b) The eigenvalues are

$\lambda_1 = -0.25$ $|\lambda_1| = 0.25$

$\lambda_2 = 0.423909$ $|\lambda_2| = 0.43905$

$\lambda_3 = -0.0869525 - 0.25i$ $|\lambda_3| = 0.264689$

$\lambda_4 = -0.0869525 + 0.25i$ $|\lambda_4| = 0.264689.$

The spectral radius $\rho(A^TA)$ $\rho = \max |\lambda_i| = 0.48905.$

c)

Consider

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$$A^2 = \begin{bmatrix} 0 & 1/8 & 1/4 & 1/2 \\ 1/2 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 0 \\ 0 & 0 & 1/8 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1/8 & 1/4 & 1/2 \\ 1/2 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 0 \\ 0 & 0 & 1/8 & 0 \end{bmatrix} = \begin{bmatrix} 1/16 & 1/16 & 1/16 & 0 \\ 0 & 1/16 & 1/8 & 1/4 \\ 1/8 & 0 & 0 & 0 \\ 0 & 1/32 & 0 & 0 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1/16 & 1/16 & 1/16 & 0 \\ 0 & 1/16 & 1/8 & 1/4 \\ 1/8 & 0 & 0 & 0 \\ 0 & 1/32 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1/8 & 1/4 & 1/2 \\ 1/2 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 0 \\ 0 & 0 & 1/8 & 0 \end{bmatrix} = \begin{bmatrix} 1/32 & 3/128 & 1/64 & 1/32 \\ 1/32 & 1/32 & 1/32 & 0 \\ 0 & 1/64 & 1/32 & 1/16 \\ 1/64 & 0 & 0 & 0 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 1/32 & 3/128 & 1/64 & 1/32 \\ 1/32 & 1/32 & 1/32 & 0 \\ 0 & 1/64 & 1/32 & 1/16 \\ 1/64 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1/8 & 1/4 & 1/2 \\ 1/2 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 0 \\ 0 & 0 & 1/8 & 0 \end{bmatrix} = \begin{bmatrix} 3/256 & 1/128 & 3/256 & 1/64 \\ 1/64 & 3/256 & 1/128 & 1/64 \\ 1/128 & 1/128 & 1/128 & 0 \\ 0 & 1/512 & 1/256 & 1/128 \end{bmatrix}$$

The matrix A will likely converge to zero.

$$\lim_{k\to\infty} A^k x = (0,0,0,0)$$

Theoretical problems.

15. Show that the characteristic polynomial $p(\lambda) = \det(A-\lambda I)$ for the $n\times n$ matrix is an nth degree polynomial.

[Hint: Expand $\det(A-\lambda I)$ along the first row and use mathematical induction on n].

solution.

The characteristic polynomial of A is –

$$\det(A-\lambda I) = \begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33}-\lambda & \cdots & a_{3n} \\ \vdots & & & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn}-\lambda \end{vmatrix}$$

- A determinant is the sum of

$$\det(P) = \sum p_{1a_1} p_{2a_2} \cdots p_{na_n}$$

where $(a_1, a_2, \ldots a_n)$ is some permutation of $(1, 2, 3, \ldots, n)$.

Thus, $\det(A-\lambda I)$ will contain atleast one term which the product of the terms on the principal diagonal. The express

$$(a_{11}-\lambda)(a_{22}-\lambda)\ldots(a_{nn}-\lambda)$$

is a polynomial of degree n. As a result, the characteristic polynomial has degree n.

16 a) show that if A is a $n\times n$ matrix, then

$$\det(A) = \prod_{i=1}^{n} \lambda_i$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A. [Hint: Consider $p(0)$]

b) show that A is singular if and only if $\lambda = 0$ is an eigenvalue of A.

solution.

a) If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A, they must be the roots of the characteristic polynomial. So, $p(\lambda)$ has factors $(\lambda-\lambda_1), (\lambda-\lambda_2), \ldots, (\lambda-\lambda_n)$.

$$p(\lambda) = \det(A-\lambda I) = (\lambda_1-\lambda)(\lambda_2-\lambda)\ldots(\lambda_n-\lambda)$$
$$p(0) = \det(A) = \lambda_1 \cdot \lambda_2 \cdot \ldots \cdot \lambda_n$$

Note that, as A has n distinct eigenvalues, it $\lambda_1, \ldots, \lambda_n$, it is diagonalisable, its matrix with respect to the basis $B = \{u^1, \ldots, u^n\}$ where u^i is the eigenvector corresponding to the eigenvalue λ_i is $\mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. So, $p(\lambda) = \det(A-\lambda I)$ takes the above form.

b) If $\lambda = 0$ is an eigenvalue of A, using the above result

$$\det(A) = 0.$$

so, A is singular.

In the opposite direction, if $\det(A) = 0$, then atleast one of terms in the product $\prod_i \lambda_i$ must be zero. So, $\lambda = 0$ must be one of the eigenvalues of A.

17) Let λ be an eigenvalue of the $n\times n$ matrix A and $x \neq 0$ be an associated eigenvector.

a) show that λ is also an eigenvalue of A^t.

Proof

a) Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}$$

The characteristic polynomial of A is

$$\det(A-\lambda I) = \begin{vmatrix} a_{11}-\lambda & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22}-\lambda & \ldots & a_{2n} \\ \\ a_{n1} & a_{n2} & \ldots & a_{nn}-\lambda \end{vmatrix}$$

But, $\det(A-\lambda I) = \det((A-\lambda I)^t) = \det(A^t - \lambda I_n)$.

This may not be immediately obvious. To see that $\det(X) = \det(X^T)$, remember that the determinant of a matrix is the sum of products of the form $a_{\pi(1),1}\, a_{\pi(2),2} \ldots a_{\pi(n),n}$, where π is any permutation of the numbers $\{1, 2, 3, \ldots, n\}$. But, this stays the same, if we replace columns by rows. So, $\det(X^t) = \det(X)$.

Therefore, $\det(A-\lambda I) = 0$
$\Rightarrow \det(A^t - \lambda I) = 0$.

As a result, A^t has the same eigenvalues as A

b) Show that for any integer $k \geq 1$, λ^k is an eigen value of A^k with the eigen vector x.

Proof.

$$\begin{aligned} A^k x &= A^{k-1}\cdot(Ax) \\ &= A^{k-1}\cdot(\lambda x) \\ &= \lambda(A^{k-1}x) \\ &= \lambda\cdot A^{k-2}(Ax) \\ &= \lambda\cdot A^{k-2}\cdot\lambda x \\ &= \lambda^2 A^{k-2}\cdot x \\ &\vdots \\ &= \lambda^k x. \end{aligned}$$

Thus, $A^k x = \lambda^k x$.
λ^k is an eigenvalue of A^k corresponding to the eigenvector x.

c) Show that if A^{-1} exists, then $1/\lambda$ is an eigenvalue of A^{-1} with eigenvector x.

Proof.

$$\begin{aligned} Ax &= \lambda x \\ A^{-1}Ax &= \lambda A^{-1}x. \\ Ix &= \lambda A^{-1}x \\ \therefore A^{-1}x &= \left(\frac{1}{\lambda}\right)x. \end{aligned}$$

$\frac{1}{\lambda}$ is the eigenvalue of A^{-1} corresponding to the eigenvector x.

d) Generalize parts (b) and (c) to $(A^{-1})^k$ for integers $k \geq 2$.

$$\begin{aligned} A^{-1}x &= \frac{1}{\lambda}x \\ A^{-1}A^{-1}x &= \frac{1}{\lambda}A^{-1}x \\ (A^{-1})^2 x &= \left(\frac{1}{\lambda^2}\right)x \\ \therefore (A^{-1})^k x &= \left(\frac{1}{\lambda^k}\right)x. \end{aligned}$$

so, $(A^{-1})^k$ has the eigenvalue $\frac{1}{\lambda^k}$ corresponding to the eigenvector x.

e) Given the polynomial $q(x) = q_0 + q_1 x + \dots + q_k x^k$, define $q(A)$ to be the matrix $q(A) = q_0 I + q_1 A + \dots + q_k A^k$. Show that $q(\lambda)$ is an eigenvalue of $q(A)$.

Proof.

$$\begin{aligned} q(A)x &= (q_0 I + q_1 A + \dots + q_k A^k)x \\ &= q_0 Ix + q_1 Ax + \dots + q_k A^k x \\ &= q_0 x + q_1 \lambda x + q_2\lambda^2 x + \dots + q_k \lambda^k x \\ &= (q_0 + q_1\lambda + q_2\lambda^2 + \dots + q_k\lambda^k)x \\ &= q(\lambda)x. \end{aligned}$$

As $I, A, A^2, \dots, A^k$ are all linear operators, their sum is defined in the usual way functions are added.
$(S+T)(x) = Sx + Tx$.

so, $q(A)$ has the eigenvalue $q(\lambda)$ with respect to the eigenvector x.