5.3 Homogeneous Equations of Arbitrary Order.

The technique for solving homogeneous constant coefficient linear differential equations is now all but complete. For instance, to solve

$$(D^4 - 2D^3 + 2D^2 - 2D + 1)\,y = 0 \tag{5.9}$$

we first decompose the operator into linear and quadratic factors, as suggested in section (5.1) to obtain the equivalent function

$$(D^4 - D^3 - D^3 + D^2 + D^2 - 2D + 1)\,y = 0$$
$$[D^3(D-1) - D^2(D-1) + (D-1)^2]\,y = 0$$
$$(D-1)[D^3 - D^2 + D - 1]y = 0$$
$$(D-1)[D^2(D-1) + 1(D-1)]\,y = 0$$
$$(D-1)^2\,(D^2+1)\,y = 0.$$

and then invoke lemma (5.1) to assert that the solution space of each of the second-order equations $(D-1)^2 y = 0$ and $(D^2+1)y = 0$ is contain in the solution space of (5.9). Thus, e^x, xe^x, $\cos x$ and $\sin x$ are solutions of (5.9) and since the functions are linearly independent in $C(-\infty,\infty)$, the general solution of the given equation is

$$y = c_1 e^x + c_2 x e^x + c_3 \cos x + c_4 \sin x$$

This, in brief is how all homogeneous constant cofficient linear differential equations are solved and save for the difficulty occasioned by the equations

$$(D-1)^4 y = 0$$

and

$$(D^2+1)^2 y = 0.$$

where the above argument fails to yield the required number of linearly independent solutions, we are done. But, recalling our experience with the equation $(D-\alpha)^2 y = 0$, it is not difficult to guess, that the missing solutions for the above equations are respectively $x^2 e^x$, $x^3 e^x$ and $x \sin x$, $x \cos x$. Both of these conjectures are correct and we shall now prove the relevant generalization of this fact for arbitrary equations of the form

$$(D^n + a_{n-1} D^{n-1} + \dots + a_0)\,y = 0. \tag{5.10}$$

where $a_0, \dots, a_{n-1}$ are constants.

We begin by decomposing the operator into linear and quadratic factors, the linear factors being determined by the real roots of the auxiliary or characteristic equation

$$m^n + a_{n-1} m^{n-1} + \dots + a_0 = 0.$$

the quadratic factors by its complex roots. Then by lemma 5.1 we can find the solutions of (5.10) by finding the null space of each factor of form $(D-\alpha)^m$ corresponding to the real root α, and of each factor of the form $[D^2 - 2aD + (a^2+b^2)]^m$ corresponding to the pair of complex roots $a \pm bi$, $b > 0$. This we accomplish in the following theorem, which, it should be noted also includes the case $(D-\alpha)^2$ discussed earlier.

Theorem 5.1. If $y(x)$ belongs to the null space of a constant coefficient linear differential operator L, then $x^{m-1} y(x)$ belongs to the null space of L^m.

Proof.

To establish this result, we must compute the value of the linear differential operator L^m applied to the product $x^{m-1}y$, and we therefore begin by giving a formula for evaluating all such expressions.

Let L be the constant coefficient linear differential operator given by

$$L = D^n + a_{n-1} D^{n-1} + a_{n-2} D^{n-2} + \dots$$

If we were to formally differentiate L with respect to D, we get a series of constant coefficient linear operators L, L', L'', L''' which are:

$$L = D^n + a_{n-1} D^{n-1} + a_{n-2} D^{n-2} + \cdots$$
$$L' = n \cdot D^{n-1} + a_{n-1} \cdot (n-1) D^{n-2} + a_{n-2} (n-2) D^{n-3} + \cdots$$
$$L'' = n(n-1) D^{n-2} + a_{n-1} (n-1)(n-2) D^{n-3} + a_{n-2} (n-2)(n-3) D^{n-4} + \cdots$$
$$L''' = n(n-1)(n-2) D^{n-3} + a_{n-1} (n-1)(n-2)(n-3) D^{n-4} + a_{n-2} (n-2)(n-3)(n-4) D^{n-4} + \cdots$$

Consider the constant coefficient linear differential operator L applied to a product of two functions u and v.

$$L(uv) = \underbrace{D^n (uv)}_{\text{first term}} + a_{n-1} \underbrace{D^{n-1} (uv)}_{\text{second term}} + a_{n-2} \underbrace{D^{n-2} (uv)}_{\text{third term}} + \cdots$$

Applying Leibniz's rule to the first term, we obtain–

$$D^n (uv) = \left[(D^n u)(v) + \binom{n}{1} (D^{n-1} u)(Dv) + \binom{n}{2} (D^{n-2} u)(D^2 v) + \cdots \right] \quad (1)$$

Applying Leibniz's rule to the second term, we have

$$a_{n-1} D^{n-1} (uv) = a_{n-1} \left[(D^{n-1} u)(v) + \binom{n-1}{1} (D^{n-2} u)(Dv) + \binom{n-1}{2} (D^{n-3} u)(D^2 v) + \cdots \right] \quad (2)$$

Applying Leibniz's rule to the third term, we have –

$$a_{n-2} D^{n-2} (uv) = a_{n-2} \left[(D^{n-2} u)(v) + \binom{n-2}{1} (D^{n-3} u)(Dv) + \binom{n-2}{2} (D^{n-4} u)(D^2 v) + \cdots \right] \quad (3)$$

Adding these 'n' equations, the result is as follows. On the left-hand side the sum

$$D^n (uv) + a_{n-1} D^{n-1} (uv) + a_{n-2} D^{n-2} (uv) + \cdots + a_1 D(uv) + a_0 (uv)$$
$$= (D^n + a_{n-1} D^{n-1} + a_{n-2} D^{n-2} + \cdots + a_1 D + a_0)(uv)$$
$$= L(uv).$$

Collecting the first terms on the right hand side,

$$(D^n u) v + a_{n-1} (D^{n-1} u) v + a_{n-2} (D^{n-2} u) v + \cdots + a_1 (Du) v + a_0 (u) v$$
$$= (D^n u + a_{n-1} D^{n-1} u + a_{n-2} D^{n-2} u + \cdots + a_1 D u + a_0 u) v$$
$$= (Lu) v$$

Collecting the second terms on the right side,

$$n (D^{n-1} u)(Dv) + a_{n-1} (n-1)(D^{n-2} u)(Dv) + a_{n-2} (n-2)(D^{n-3} u)(Dv) + \cdots$$
$$= ((n D^{n-1} + a_{n-1} (n-1) D^{n-2} + a_{n-2} (n-2) D^{n-3} + \cdots)(u))(Dv)$$
$$= (L' u)(Dv).$$

Collecting the third terms on the right side:

$$\frac{n(n-1)}{2!} (D^{n-2} u)(D^2 v) + a_{n-1} \frac{(n-1)(n-2)}{2!} (D^{n-3} u)(D^2 v) + a_{n-2} \frac{(n-2)(n-3)}{2!} (D^{n-4} u)(D^2 v) + \cdots$$

$$= \frac{1}{2!} \left[n(n-1)(D^{n-2} u) + a_{n-1} (n-1)(n-2)(D^{n-3} u) + a_{n-2} (n-2)(n-3)(D^{n-4} u) \right] (D^2 v)$$

$$= \frac{1}{2!} (L'' u)(D^2 v).$$

Thus, we have the general formula

$$L(uv) = (Lu)v + (L'u)(Dv) + \frac{1}{2!}(L''u)(D^2v) + \frac{1}{3!}(L'''u)(D^3v) + \cdots \quad (5.12)$$

Accepting the validity of (5.12), let L and $y(x)$ be as in the statement of the theorem; that is $Ly = 0$. We must prove that

$$L^m(x^{m-1}y) = 0.$$

To this end, we set $L^m = M$. and apply the above formula to obtain:

$$M(yx^{m-1}) = (My)x^{m-1} + (M'y)Dx^{m-1} + \frac{1}{2!}(M''y)(D^2x^{m-1}) + \cdots$$

To show that this expression is zero, we first note that $D^r x^{m-1} = 0$ whenever $r \geq m$. Moreover whenever $r < m$, the r^{th} formal derivative of M with respect to D, $M^{(r)}$, consists of a sum of terms, each of which contains the factor L^{m-r}.

$$M = L^m$$
$$M' = mL^{m-1}\left[nD^{n-1} + a_{n-1}(n-1)D^{n-2} + a_{n-2}(n-2)D^{n-3} + \cdots + a_1\right]$$
$$M'' = m(m-1)L^{m-2}\left[n(n-1)D^{n-2} + a_{n-1}(n-1)(n-2)D^{n-3} + a_{n-2}(n-2)(n-3)D^{n-4} + \cdots\right]$$

Hence, since $Ly = 0$ and since L is a constant coefficient operator, lemma (5.1) applies. That is, $M^{(r)} = L^{(m-r)}\{\text{something}\}$. By lemma (5.1), since $L^{(m-r)}(y) = 0$, $M^{(r)}y = 0$. Thus, all of the terms in the above expression are zero and the theorem is proved.

Q.E.D.

As a consequence of this theorem we can now assert that the null space of $(D-\alpha)^m$ contains the functions

$$e^{\alpha x},\ xe^{\alpha x},\ x^2e^{\alpha x},\ \ldots,\ x^{m-1}e^{\alpha x}$$

and that the null space of $[D^2 - 2aD + (a^2+b^2)]^m$ contains

$$e^{ax}\sin bx,\ xe^{ax}\sin bx,\ \ldots,\ x^{m-1}e^{ax}\sin bx$$
$$e^{ax}\cos bx,\ xe^{ax}\cos bx,\ \ldots,\ x^{m-1}e^{ax}\cos bx.$$

And it is out of just such functions that the general solution of every homogeneous constant coefficient linear differential equation is constructed. The construction depends, of course, upon the fact that the various functions obtained in this way are linearly independent in $\mathcal{C}(-\infty,\infty)$. They are, but unfortunately there is no really brief proof of this assertion. One particular elegant proof will be given in section (B.7) as an illustration of the ideas introduced there, and in the meantime we will content ourselves with indicating an alternate approach in example 5 below.

Examples:

1) Find the general solution of

$$(D^3+1)y = 0. \quad (5.13)$$

Solution:

Here the factorization of the operator is $(D+1)(D^2-D+1)$, and it follows that the roots of the characteristic equation

$$m^3+1 = 0$$

are -1, $\frac{1}{2} + \frac{\sqrt{3}}{2}i$, $\frac{1}{2} - \frac{\sqrt{3}}{2}i$. Thus, the general solution of (5.13) is

$$y = c_1e^{-x} + e^{x/2}\left(c_2\cos\frac{\sqrt{3}}{2}x + c_3\sin\frac{\sqrt{3}}{2}x\right).$$

2) Solve

$$y^{(7)} - 2y^{(5)} + y^{(3)} = 0.$$

Solution.
In operator notation this equation becomes

$$(D^7 - 2D^5 + D^3)y = 0$$

and since

$$D^7 - 2D^5 + D^3$$
$$= D^3(D^4 - 2D^2 + 1)$$
$$= D^3(D^2-1)^2$$
$$= D^3(D+1)^2(D-1)^2$$

The roots of the characteristic equation are 0 with multiplicity 3, 1 with multiplicity 2 and -1 with multiplicity 2. So, the general solution of the equation is:

$$y = (c_1 + c_2 x + c_3 x^2) + (c_4 + c_5 x)e^{x} + (c_6 + c_7 x)e^{-x}.$$

3) Find a constant coefficient linear differential equation which has e^{2x} and xe^{-3x} among its solutions.

Solution.
In this case, we must find a constant coefficient linear differential operator L with the property that $Le^{2x} = 0$ and $L(xe^{-3x}) = 0$. In more picturesque terminology it is sometimes said that L annihilates these functions. Clearly any operator which contains the factors $(D-2)$ and $(D+3)^2$ will answer the problem and hence

$$(D-2)(D+3)^2 y = 0$$
$$\Rightarrow (D-2)(D^2+6D+9)y = 0$$
$$\Rightarrow (D^3 + 6D^2 + 9D - 2D^2 - 12D - 18)y = 0$$
$$(D^3 + 4D^2 - 3D - 18)y = 0$$

will be an equation of the required type.

4) Find a constant coefficient linear differential operator L which, when applied to the equation

$$(D^2+1)(D-1)y = e^{x} + 2 - 7x\sin x$$

produces the homogenous equation

$$L(D^2+1)(D-1)y = 0.$$

Solution.
Since L must annihilate the functions e^{x}, 2, and $x\sin x$, it must contain the factors $D-1$, D and $(D^2+1)^2$. By lemma (5.1) we can therefore set

$$L = D(D-1)(D^2+1)^2.$$

5) As our final example, we shall prove that the various solutions of the equation

$$(D-2)(D+5)^3(D^2-4D+13)y = 0$$

obtained using theorem (5.1) are linearly independent in $C(-\infty,\infty)$.

Solution.
The solutions in this case are:

e^{2x} corresponding to the factor $D-2$.
$e^{-5x}, xe^{-5x}, x^2e^{-5x}$ corresponding to the factor $(D+5)^3$.
$e^{2x}\cos 3x$, $e^{2x}\sin 3x$ corresponding to the factor $D^2-4D+13$,

and it is obvious that they are somewhat too numerous to permit their Wronskian to be computed easily. Instead, we reason as follows:

Let $c_1, c_2, \ldots, c_6$ be constants such that

$$c_1 e^{2x} + (c_2 + c_3 x + c_4 x^2) e^{-5x} + e^{2x}(c_5 \cos 3x + c_6 \sin 3x) = 0. \quad (5.14)$$

for all x. Apply the operator $(D+5)^3 (D^2 - 4D + 13)$ to this expression and annihilate every term but the first, thereby obtaining.

$$c_1 (D+5)^3 (D^2 - 4D + 13) e^{2x} \equiv 0.$$

But since, $(D+5)^3 (D^2 - 4D + 13)$ does not annihilate e^{2x}, it follows that $c_1 = 0$ and that (5.14) reduces to

$$(c_2 + c_3 x + c_4 x^2) e^{-5x} + e^{2x}(c_5 \cos 3x + c_6 \sin 3x) \equiv 0. \quad (5.15)$$

Next, apply the operator $(D+5)^2 (D^2 - 4D + 13)$ to annihilate every term in (5.15) except $c_4 x^2 e^{-5x}$. This gives

$$c_4 (D+5)^2 (D^2 - 4D + 13) x^2 e^{-5x} \equiv 0.$$

Since $(D+5)^2 (D^2 - 4D + 13)$ is not identically equal to zero, we conclude that $c_4 = 0$. Hence, (5.15) becomes

$$(c_2 + c_3 x) e^{-5x} + e^{2x}(c_5 \cos x + c_6 \sin x).$$

Continue the argument using the operators $(D+5)(D^2 - 4D + 13)$ and $(D^2 - 4D + 13)$ in turn to deduce that $c_3 = 0$ and $c_2 = 0$. Finally from the identity

$$e^{2x}(c_5 \cos 3x + c_6 \sin 3x) \equiv 0$$

we conclude directly that $c_5 = c_6 = 0$, and the linear independence of this particular set of solutions has been proved.

In point of fact, this argument can be refined to give a general proof of the linear independence of the functions appearing in the solutions of constant coefficient homogenous linear differential equations. We refrain from doing so, however, since the problem will be considered in a later chapter where an entirely different proof will be given.

<u>Problems.</u>

Find the general solution of each of the following differential equations.

1) $y''' + 3y'' - y' - 3y = 0.$

<u>Solution</u>

The characteristic equation is

$$D^3 + 3D^2 - D - 3 = 0$$
$$D^2(D+3) - 1(D+3) = 0$$
$$(D^2 - 1)(D+3) = 0$$
$$(D-1)(D+1)(D+3) = 0.$$

The roots of the characteristic equation are—

$\alpha_1 = 1$, $\alpha_2 = -1$ and $\alpha_3 = -3$.

The general solution of the equation is

$$y = c_1 e^{-x} + c_2 e^{-3x} + c_3 e^{x}.$$

2) $y''' + 5y'' - 8y' - 12y = 0.$

<u>Solution.</u>

$$D^3 + 5D^2 - 8D - 12D = 0$$
$$D^3 + D^2 + 4D^2 - 8D - 12 = 0$$
$$D^3 + D^2 + 4D^2 - 12D + 4D - 12 = 0$$
$$D^2(D+1) + 4D(D-3) + 4(D-3) = 0$$
$$D^2(D+1) + 4(D+1)(D-3) = 0$$
$$(D+1)(D^2 + 4D - 12) = 0.$$
$$(D+1)(D^2 + 6D - 2D - 12) = 0$$
$$(D+1)[D(D+6) - 2(D+6)] = 0$$
$$(D+1)(D-2)(D+6) = 0.$$

The roots of the characteristic equation are:

$\alpha_1 = -1, \alpha_2 = 2, \alpha_3 = -6.$

The general solution of the equation is

$y = c_1 e^{-x} + c_2 e^{2x} + c_3 e^{-6x}.$

3) $4y''' + 12y'' + 9y' = 0.$

solution.

The characteristic equation is -

$4D^3 + 12D^2 + 9D = 0$

$D(4D^2 + 12D + 9) = 0$

$D(4D^2 + 6D + 6D + 9) = 0.$

$D[2D(2D+3) + 3(2D+3)] = 0$

$D(2D+3)^2 = 0.$

The roots of the equation are:

$\alpha_1 = 0, \alpha_2 = -\frac{3}{2}$. with multiplicity 2.

The general solution of the equation is:

$y = c_1 + (c_2 + c_3 x) e^{-3/2 x}.$

4) $y''' + 6y'' + 13y' = 0.$

solution.

The characteristic equation is :

$D^3 + 6D^2 + 13D = 0$

$D(D^2 + 6D + 13) = 0.$

$D(D^2 + 2(D)(3) + 3^2 + 4) = 0$

$D[(D+3)^2 + 2^2] = 0.$

The roots of the characteristic equation are:

$\alpha_1 = 0, \alpha_2 = -3 + 2i, \alpha_3 = -3 - 2i.$

The general solution of the equation is:

$y = c_1 + e^{-3x}(c_2 \cos 2x + c_3 \sin 2x).$

5) $2y''' + y'' - 8y' - 4y = 0.$

solution

The characteristic equation is -

$2D^3 + D^2 - 8D - 4 = 0.$

$2D^3 - 4D^2 + 5D^2 - 8D - 4 = 0.$

$2D^3 - 4D^2 + 5D^2 - 10D + 2D - 4 = 0$

$2D^2(D-2) + 5D(D-2) + 2(D-2) = 0$

$(2D^2 + 5D + 2)(D-2) = 0.$

$(2D^2 + 4D + D + 2)(D-2) = 0$

$(2D(D+2) + 1(D+2))(D-2) = 0$

$(2D+1)(D+2)(D-2) = 0$

The roots of the characteristic equation are:

$\alpha_1 = -\frac{1}{2}, \alpha_2 = -2, \alpha_3 = 2.$

The general solution of the equation is -

$y_1 = c_1 e^{-x/2} + c_2 e^{-2x} + c_3 e^{2x}.$

6) $y''' + 3y'' + y' + 3y = 0.$

solution.

The characteristic equation is -

$D^3 + 3D^2 + D + 3 = 0.$

$D^2(D+3) + 1(D+3) = 0$

$(D^2+1)(D+3) = 0$

The roots of the characteristic equation are
$\alpha_1 = -3,\ \alpha_2 = i,\ \alpha_3 = -i$
The general solution of the equation is -
$y = c_1 e^{-3x} + \quad (c_2 \cos x + c_3 \sin x).$

7) $y^{(IV)} - y'' = 0.$
Solution:
The ~~general solution of the eqn~~
characteristic equation is:
$D^4 - D^2 = 0$
$D^2(D^2 - 1) = 0$
The roots of the characteristic equation are
$\alpha_1 = 0$ with multiplicity 2
$\alpha_2 = 1$ with multiplicity 2.
So, the general solution of the equation is:
$y = c_1 + c_2 x + e^{x}(c_3 + c_4 x).$

8) $y^{(IV)} - 8y'' + 16y = 0.$
Solution:
The characteristic equation is:
$D^4 - 8D^2 + 16 = 0$
$(D^2 - 4)^2 = 0$
$(D-2)^2 (D+2)^2 = 0$
The roots of the characteristic equation are:
$\alpha_1 = -2$ with multiplicity 2.
$\alpha_2 = 2$ with multiplicity 2.
The general solution of the equation is:
$y = e^{-2x}(c_1 + c_2 x) + e^{2x}(c_3 + c_4 x).$

9) $y^{(4)} - 18y'' + 81y = 0$
Solution:
The characteristic equation is -
$(D^2 - 9)^2 = 0$
$(D-3)^2 (D+3)^2 = 0.$
The roots of the characteric equation are -
$\alpha_1 = 3$ with multiplicity 2.
$\alpha_2 = -3$ with multiplicity -2.
The general solution of the equation is
$y = e^{-3x}(c_1 + c_2 x) + e^{3x}(c_3 + c_4 x).$

10) $4y^{(4)} - 8y''' - y'' + 2y' = 0.$
Solution:
The characteristic equation is -
$4D^4 - 8D^3 - D^2 + 2D = 0$
$4D^3(D-2) - D(D-2) = 0$
$(D-2)(4D^3 - D) = 0$
$D(D-2)(4D^2 - 1) = 0$
$D(D-2)(2D-1)(2D+1) = 0$
The roots of the characteristic equation are -
$\alpha_1 = 0,\ \alpha_2 = -\frac{1}{2},\ \alpha_3 = \frac{1}{2},\ \alpha_4 = 2.$
The general solution of the equation is:

$y = c_1 e^{-x/2} + c_2 + c_3 e^{x/2} + c_4 e^{2x}.$

11) $y^{(4)} + y^{(3)} + y'' = 0.$
Solution.

The characteristic equation is:

$$D^4 + D^3 + D^2 = 0$$
$$D^2(D^2 + D + 1) = 0$$
$$D^2\left[\left(D+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2\right] = 0$$

The roots of the characteristic equation are—

$\alpha_1 = 0$ with multiplicity 2

$$\alpha_2 = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$$
$$\alpha_3 = -\frac{1}{2} - \frac{\sqrt{3}}{2} i$$

The general solution of the equation is—

$$y = c_1 + c_2 x + e^{-x/2}\left[\cos\left(\frac{\sqrt{3}}{2}x\right) + \sin\left(\frac{\sqrt{3}}{2}x\right)\right].$$

12) $y^{(4)} = 0$

solution.

The characteristic equation is:

$$D^4 = 0$$

The roots of the characteristic-equation are

$\alpha_1 = 0$ with multiplicity 4.

The general solution of the equation is:

$$y = c_1 + c_2 x + c_3 x^2 + c_4 x^3.$$

13) $y^{(4)} - 4y^{(3)} + 6y'' - 4y' + y = 0.$

solution.

The characteristic equation is

$$D^4 - 4D^3 + 6D^2 - 4D + 1 = 0$$
$$(D-1)^4 = 0$$

The roots of the characteristic equation are:

$\alpha_1 = 1$ with multiplicity 4.

The general solution of the equation is—

$$y = e^x(c_1 + c_2 x + c_3 x^2 + c_4 x^3).$$

14) $y^{(5)} + 2y^{(3)} + y' = 0$

solution.

The characteristic equation is

$$D^5 + 2D^3 + D = 0$$
$$D(D^4 + 2D^2 + 1) = 0.$$
$$D(D^2+1)^2 = 0.$$

The roots of the characteristic equation are

$\alpha_1 = 0$

$\alpha_2 = -i$ with multiplicity 2

$\alpha_3 = i$ with multiplicity 2.

The general solution of the equation is—

$$y = c_1 + (c_2 + c_3 x)(c_4 \cos x + c_5 \sin x).$$

15) $y^{(5)} + 6y^{(4)} + 15y^{(3)} + 26y'' + 36y' + 24y = 0$

solution.

The characteristic equation is:

$$D^5 + 6D^4 + 15D^3 + 26D^2 + 36D + 24 = 0.$$

Let $f(D) = D^5 + 6D^4 + 15D^3 + 26D^2 + 36D + 24$

We find that —

$$f(-2) = (-2)^5 + 6(-2)^4 + 15(-2)^3 + 26(-2)^2 + 36(-2) + 24$$
$$= -32 + 96 - 120 + 104 - 72 + 24$$
$$= 64 - 16 - 48$$
$$= 0.$$

Thus, $D+2$ is a factor of $f(D)$. By long division -

$$\begin{array}{r|l} & D^4 + 4D^3 + 7D^2 + 12D + 12 \\ D+2 & D^5 + 6D^4 + 15D^3 + 26D^2 + 36D + 24 \\ & D^5 + 2D^4 \\ \hline & 4D^4 + 15D^3 \\ & 4D^4 + 8D^3 \\ \hline & 7D^3 + 26D^2 \\ & 7D^3 + 14D^2 \\ \hline & 12D^2 + 36D \\ & 12D^2 + 24D \\ \hline & 12D + 24 \\ & 12D + 24 \\ \hline & 0 \end{array}$$

$$\therefore f(D) = (D+2)(D^4 + 4D^3 + 7D^2 + 12D + 12).$$
$$= (D+2)\, g(D).$$

where

$$g(D) = D^4 + 4D^3 + 7D^2 + 12D + 12.$$

We find that:

$$g(-2) = (-2)^4 + 4(-2)^3 + 7(-2)^2 + 12(-2) + 12$$
$$= 16 - 32 + 28 - 24 + 12$$
$$= -16 + 4 + 12$$
$$= 0.$$

$(D+2)$ is a factor of $D^4 + 4D^3 + 7D^2 + 12D + 12$. By long division -

$$\begin{array}{r|l} & D^3 + 2D^2 + 3D + 6 \\ D+2 & D^4 + 4D^3 + 7D^2 + 12D + 12 \\ & D^4 + 2D^3 \\ \hline & 2D^3 + 7D^2 \\ & 2D^3 + 4D^2 \\ \hline & 3D^2 + 12D \\ & 3D^2 + 6D \\ \hline & 6D + 12 \\ & 6D + 12 \\ \hline & 0. \end{array}$$

$$\therefore f(D) = (D+2)^2 (D^3 + 2D^2 + 3D + 6)$$
$$= (D+2)^2 [D^2(D+2) + 3(D+2)]$$
$$= (D+2)^2 [(D^2+3)(D+2)]$$
$$= (D+2)^3 (D^2+3).$$

The roots of the characteristic equation

$$(D+2)^3 (D^2+3) = 0$$

are:

$\alpha_1 = -2$ with multiplicity 3.

$\alpha_2 = \sqrt{3}\, i$

$\alpha_3 = -\sqrt{3}\, i$

The general solution of the equation is:

$$y = (c_1 + c_2 x + c_3 x^2) e^{-2x} + (c_4 \cos \sqrt{3} x + c_5 \sin \sqrt{3} x).$$

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16) Find the general solution of
$(D^4+1)y = 0$.

Solution.
The characteristic equation is
$$D^4 + 1 = 0.$$
We would like to find the fourth root of ~~and~~ -1.
$$D^4 = -1$$
$$\therefore D^4 = \cos(2n+1)\pi + i\sin(2n+1)\pi.$$
$$D = [\cos(2n+1)\pi + i\sin(2n+1)\pi]^{1/4}$$
By De-moivre's theorem,
$$D = \cos(2n+1)\frac{\pi}{4} + i\sin(2n+1)\frac{\pi}{4}.$$

Substituting $n=0$, $\alpha_1 = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1}{\sqrt2} + \frac{1}{\sqrt2}i$

$n=1$, $\alpha_2 = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4} = \frac{-1}{\sqrt2} + \frac{1}{\sqrt2}i$

$n=2$, $\alpha_3 = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4} = \frac{-1}{\sqrt2} - \frac{1}{\sqrt2}i$

$n=3$, $\alpha_4 = \cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4} = \frac{1}{\sqrt2} - \frac{1}{\sqrt2}i$

The general solution of the equation is
$$y = e^{-x/\sqrt2}\left[c_1\cos\frac{x}{\sqrt2} + c_2\sin\frac{x}{\sqrt2}\right] + e^{x/\sqrt2}\left[c_3\cos\frac{x}{\sqrt2} + c_4\sin\frac{x}{\sqrt2}\right].$$

17) Find the linear differential operators which annihilate each of the following functions.
~~solution~~.
(a) $x^2 e^{(x+1)}$
(b) $3e^{2x}\cos 2x$
(c) $x(2x+1)\sin x$.
(d) $3 + 4x - 2e^{-2x}$
(e) $x^2 \sin x \cos x$
(f) $x^2 e^x \sin^2 x$.

solution.
(a) Since L must annihilate the function $x^2 e^{(x+1)}$, it must contain the factor
$(D-1)^2$.
Thus, $L = (D-1)^2$.

(b) As L must annihilate the function $e^{2x}\cos 2x$, it must contain the factor
$(D-2)^2 + 2^2$
$= D^2 - 4D + 4 + 4$
$= (D^2 - 4D + 8)$.
$\therefore L = D^2 - 4D + 8.$

(c) $x(2x+1)\sin x$.
As L must annihilate the function $x\sin x$ and $x^2\sin x$, it must contain the factor $(D-1)^2$ $(a=0, b=1)$. $(D^2+1)^3$.
Hence, $L = (D^2+1)^3$.

(d) $3 + 4x - 2e^{-2x}$
As L must annihilate the functions 1, x, e^{-2x}, it must contain the ~~function~~ factors D^2 and $(D+2)$.
$L = D^2(D+2)$.

(e) As L must annihilate the function

$$x^2 \sin x \cos x$$
$$= \frac{1}{2} x^2 . 2\sin x \cos x$$
$$= \frac{1}{2} x^2 \sin 2x.$$

it must contain the factor $(D^2+4)^3$.
$L = (D^2+4)^3$.

(f) $x^2 e^x \sin^2 x$.
$$= x^2 e^x \left(\frac{1-\cos 2x}{2}\right)$$
$$= x^2 e^x - \frac{1}{2} x^2 e^x \cos 2x.$$

As L must annihilate the functions
$x^2 e^x$ - it must contain the factor $(D-1)^3$.
$x^2 e^x \cos 2x$ - it must contain the factor $(D^2-2D+5)^3$.

$L = (D-1)^3 (D^2-2D+5)^3$.

(g) $x \sin(x+1)$.
As L must annihilate the function $x \sin(x+1)$, it must contain the factor $(D^2+1)^2$.
$L = (D^2+1)^2$.

(h) $(x^2-1)(\cos x+1)e^{3x}$
$$= e^{3x}(x^2\cos x + x^2 - \cos x - 1)$$
$$= x^2 e^{3x}\cos x + x^2 e^{3x} - e^{3x}\cos 2x - e^{3x}.$$

As L must annihilate the functions
$e^{3x}\cos 2x$, $x^2 e^{3x}\cos 2x$ - it must contain the factor $(D^2-6D+10)^3$.
e^{3x}, $x^2 e^{3x}$ - it must contain the factor $(D-3)^3$.

$L = (D-3)^3(D^2-6D+10)^3$.

(i) $(xe^x+1)^3$.
$= (x^3 e^{3x} + 3x^2 e^{2x} + 3xe^x + 1)$

As L must annihilate the functions
1 - it must contain the factor D.
xe^x - it must contain the factor $(D-1)^2$
$x^2 e^{2x}$ - it must contain the factor $(D-2)^3$
$x^3 e^{3x}$ - it must contain the factor $(D-3)^4$

$\therefore L = D(D-1)^2(D-2)^3(D-3)^4$.

(j) $(x+e^x)^n$.
$$= x^n + \binom{n}{1} x^{n-1} e^x + \binom{n}{2} x^{n-2} e^{2x} + \dots + \binom{n}{n-1} x e^{(n-1)x} + e^{nx}.$$

As L must annihilate the functions
x^n - It must contain the factor D^{n+1}.
$x^{n-1} e^x$ - It must contain the factor $(D-1)^n$
$x^{n-2} e^{2x}$ - It must contain the factor $(D-2)^{n-1}$

$x e^{(n-1)x}$ - It must contain the factor $(D-(n-1))^2$
e^{nx} - It must contain the factor $(D-n)$

$$L = D^{n+1}(D-1)^{n}(D-2)^{n-1} \cdots (D-(n-1))^2 (D-n).$$
$$= \prod_{i=0}^{n} (D-i)^{(n+1-i)}$$

18) Verify formula (5.12) when
(a) $L = D$
(b) $L = D^2 + 2D + 1$
(c) $L = D^n$.

Solution.
For any linear differential operator L, we have
$$L(uv) = (Lu)v + (L'u)(Dv) + \frac{1}{2!}(L''u)(D^2v) + \frac{1}{3!}(L'''u)(D^3v) + \cdots$$

(a) $L = D$.
The left hand side is $L(uv) = D(uv) = (Du)v + u(Dv)$.
The right hand side is
$$(Lu)(v) + (L'u)(Dv) + \frac{1}{2!}(L''u)(D^2v) + \cdots$$
$$= (Du)(v) + (D'u)(Dv) + \frac{1}{2!}(D''u)(D^2v) + \cdots$$

$D' = 1$
$D'' = 0.$

So, we have
$$(Du)(v) + u(Dv) + 0 + \cdots$$
$$= (Du)v + u(Dv).$$

(b) $D^2 + 2D + 1$.
The left hand side is:
$$(D^2+2D+1)(uv) = D^2(uv) + 2D(uv) + I(uv)$$
$$= u''v + 2u'v' + uv'' + 2(u'v + uv') + uv$$
$$= (D^2u)v + 2(Du)(Dv)$$

The right hand side becomes:

$$(Lu)v + (L'u)(Dv) + \frac{1}{2!}(L''u)(D^2v) + \cdots$$
$$= ((D^2+2D+1)(u))v + ((2D+2)(u))(Dv) + \frac{1}{2!}(2Du)(D^2v) + \cdots$$
$$= (D^2u + 2Du + u)v + (2Du + 2u)(Dv) + (Du)(D^2v)$$
$$= (D^2u)v + (2Du)v + 2u(Dv) + 2(Du)(Dv) + (Du)(D^2v) + uv.$$

(c) $L = D^n$
The left-hand side is:
$$D^n(uv) = (D^nu)(v) + \binom{n}{1}(D^{n-1}u)(Dv) + \binom{n}{2}(D^{n-2}u)(D^2v) + \cdots$$

The right-hand side is –
$$(Lu)(v) + (L'u)(Dv) + \frac{1}{2!}(L''u)(D^2v) + \cdots$$
$$= (D^nu)v + n(D^{n-1}u)(Dv) + \frac{n(n-1)}{2}(D^{n-2}u)(D^2v) + \cdots$$
$$= (D^nu)v + \binom{n}{1}(D^{n-1}u)(Dv) + \binom{n}{2}(D^{n-2}u)(D^2v) + \cdots$$

19. (a) Show that the Wronskian of the functions $e^{k_1 x}, e^{k_2 x}, \dots, e^{k_n x}$ is

$$e^{(k_1+k_2+\dots+k_n)x}\begin{vmatrix} 1 & 1 & \cdots & 1 \\ k_1 & k_2 & & k_n \\ k_1^2 & k_2^2 & & k_n^2 \\ k_1^3 & k_2^3 & & k_n^3 \\ \vdots & \vdots & & \vdots \\ k_1^{n-1} & k_2^{n-1} & & k_n^{n-1} \end{vmatrix}$$

(b) The determinant appearing in (a) is known as the Vandermonde determinant. Prove that every such determinant is non-zero whenever $k_1, k_2, \dots, k_n$ are distinct.
[Hint: Let V denote the determinant in question, and consider k_1 as a variable. Use an inductive argument to show that for each n, V can be viewed as polynomial of degree $(n-1)$ in k_1, which has $k_2, \dots, k_n$ as roots.]
(c) By direct computation prove that every 3×3 Vandermonde determinant is different from zero, if k_1, k_2, k_3 are distinct.
(d) What conclusion can be drawn about the solutions of constant coefficient linear differential equations from the results of (a) and (b)

Solution.

a)

$$W[e^{k_1 x}, e^{k_2 x}, \dots, e^{k_n x}] = \begin{vmatrix} e^{k_1 x} & e^{k_2 x} & \cdots & e^{k_n x} \\ k_1 e^{k_1 x} & k_2 e^{k_2 x} & & k_n e^{k_n x} \\ k_1^2 e^{k_1 x} & k_2^2 e^{k_2 x} & & k_n^2 e^{k_n x} \\ k_1^3 e^{k_1 x} & k_2^3 e^{k_2 x} & & k_n^3 e^{k_n x} \\ \vdots & \vdots & & \vdots \\ k_1^{n-1} e^{k_1 x} & k_2^{n-1} e^{k_2 x} & & k_n^{n-1} e^{k_n x} \end{vmatrix}$$

$$= e^{(k_1+k_2+k_3+\dots+k_n)x}\begin{vmatrix} 1 & 1 & \cdots & 1 \\ k_1 & k_2 & & k_n \\ k_1^2 & k_2^2 & & k_n^2 \\ k_1^3 & k_2^3 & & k_n^3 \\ k_1^4 & k_2^4 & & k_n^4 \\ \vdots & \vdots & & \vdots \\ k_1^{n-1} & k_2^{n-1} & & k_n^{n-1} \end{vmatrix}$$

(b) The Vandermonde's determinant can be simplified as:

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ k_1 & k_2 & & k_n \\ k_1^2 & k_2^2 & & k_n^2 \\ k_1^3 & k_2^3 & & k_n^3 \\ \vdots & \vdots & & \vdots \\ k_1^{n-1} & k_2^{n-1} & & k_n^{n-1} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & \cdots & 0 \\ k_1 & k_2 - k_1 & & k_n - k_1 \\ k_1^2 & k_2^2 - k_1^2 & & k_n^2 - k_1^2 \\ k_1^3 & k_2^3 - k_1^3 & & k_n^3 - k_1^3 \\ \vdots & \vdots & & \vdots \\ k_1^{n-1} & k_2^{n-1} - k_1^{n-1} & & k_n^{n-1} - k_1^{n-1} \end{vmatrix}$$

$$= (k_2-k_1)(k_3-k_1)\cdots(k_n-k_1)\begin{vmatrix} 1 & 1 & \cdots & 1 \\ k_2+k_1 & k_3+k_1 & & k_n+k_1 \\ k_2^2+k_1k_2+k_1^2 & k_3^2+k_3k_1+k_1^2 & & k_n^2+k_nk_1+k_1^2 \\ \vdots & \vdots & & \vdots \end{vmatrix}$$

$$= (k_2-k_1)(k_3-k_1)\cdots(k_n-k_1)\begin{vmatrix} 1 & 0 & \cdots & 1 \\ k_2+k_1 & k_3-k_2 & & k_n-k_2 \\ & k_3^2-k_2^2 & & k_n^2-k_2^2 \\ & \vdots & & \vdots \end{vmatrix}$$

$$= (k_2-k_1)(k_3-k_1)\cdots(k_n-k_1)(k_3-k_2)(k_4-k_2)\cdots(k_n-k_2)\begin{vmatrix} 1 & 1 & \cdots & 1 \\ k_3+k_2 & k_4+k_2 & & k_n+k_2 \end{vmatrix}$$

Continuing in this fashion, the determinant can be reduced to:

$$= \prod_{i<j}(k_j-k_i).$$

The determinant is non-zero whenever $k_i \neq k_j$ for all i, j. Thus, the determinant is non-zero, if $k_1, k_2, \ldots, k_n$ are distinct.

(a) By direct computation, we have:-

$$W[e^{k_1x}, e^{k_2x}, e^{k_3x}] = e^{(k_1+k_2+k_3)x}\begin{vmatrix} 1 & 1 & 1 \\ k_1 & k_2 & k_3 \\ k_1^2 & k_2^2 & k_3^2 \end{vmatrix}$$

$$= e^{(k_1+k_2+k_3)x}\begin{vmatrix} 1 & 0 & 0 \\ k_1 & k_2-k_1 & k_3-k_1 \\ k_1^2 & k_2^2-k_1^2 & k_3^2-k_1^2 \end{vmatrix}$$

$$= e^{(k_1+k_2+k_3)x}\begin{vmatrix} k_2-k_1 & k_3-k_1 \\ k_2^2-k_1^2 & k_3^2-k_1^2 \end{vmatrix}$$

$$= e^{(k_1+k_2+k_3)x}(k_2-k_1)(k_3-k_1)\begin{vmatrix} 1 & 1 \\ k_2+k_1 & k_3+k_1 \end{vmatrix}$$

$$= e^{(k_1+k_2+k_3)x}(k_2-k_1)(k_3-k_1)\begin{vmatrix} 1 & 0 \\ k_2+k_1 & k_3-k_2 \end{vmatrix}$$

$$= e^{(k_1+k_2+k_3)x}(k_2-k_1)(k_3-k_1)(k_3-k_2).$$

If k_1, k_2, k_3 are distinct, the Wronskian is not identically equal to zero.

(a) The solutions $e^{k_1x}, e^{k_2x}, \ldots, e^{k_nx}$ of a homogeneous constant coefficient linear differential equation are linearly independent in $(-\infty, \infty)$.