5. Equations with constant coefficients.

Linear differential equations with constant coefficients, that is equations of the form

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = h(x) \qquad (5.1)$$

in which $a_0, a_1, \dots, a_n \neq 0$ are real constants are in many respects the simplest of all differential equations. For one thing they can be discussed entirely within the context of linear algebra, and form the only substantial class of equations of order greater than one which can be explicitly solved. This, plus the fact that such equations arise in a surprisingly wide variety of physical problems, accounts for the special place they occupy in the theory of linear differential equations.

We shall begin the discussion of this chapter by considering the homogenous version of equation (5.1), which can be written as

$$(D^n + a_{n-1}D^{n-1} + a_{n-2}D^{n-2} + \dots + a_1 D + a_0 I)\, y = 0, \qquad (5.2)$$

or

$$Ly = 0. \qquad (5.3)$$

where L is constant coefficient linear differential operator

$$D^n + a_{n-1}D^{n-1} + \dots + a_1 D + a_0 I.$$

Algebraically, such operators behave exactly as if they were ordinary polynomials in D, and therefore can be factored according to the rules of elementary algebra. In particular, it follows that every linear differential operator with constant coefficients can be expressed as a product of constant coefficient operators of degree one and two. And that's because in algebra, any polynomial of degree ~~has~~ n has n roots. We accept this result without proof at this time. As we shall see therefore, it reduces the task of solving the linear differential equation of order n in (5.2) to the second order case, where complete results can be obtained with relative ease.

This done, we will take up the problem of finding a particular solution of $Ly = h$ given the general solution of the associated homogenous equation $Ly = 0$. Here, the restriction on the coefficients of L will be dropped and much more far reaching results obtained. The language of operator theory and the ideas of linear algebra will dominate this portion of our discussion and furnish just that measure of insight needed to make it intelligible. Finally, we shall conclude the chapter with some special results involving constant coefficient equations and a number of applications to problems in elementary physics.

Problems.

1 (a) Prove that if the product of two complex numbers $a+bi$ and $c+di$ is real if and only if either
(i) $b = d = 0$ or
(ii) $a = c$ and $b = -d$.

Solution

$(a+bi)(c+di) = (ac-bd) + (ad+bc)i$

Thus, the product is zero if $ad + bc = 0$.
If $b = d = 0$, $ad + bc = 0$.
If $a = c$ and $b = -d$, $ad + bc = ad + (-d)(a) = 0$.

(b) Let $P(x)$ be a polynomial with real coefficients and suppose that $P(x)$ has $a+bi$, $b > 0$ as a root, that $P(a+bi) = 0$. Prove that $(a-bi)$ is also a root of $P(x)$.

Solution.

Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + \dots + a_1 z + a_0$

Let ξ be a root of the polynomial.

Then, $P(\xi) = 0$

$$a_n \xi^n + a_{n-1} \xi^{n-1} + \dots + a_1 \xi + a_0 = 0.$$

It needs to be shown that $P(\bar{\xi}) = 0$.

We can write

$$\sum_{i=0}^{n} a_i \xi^i = 0$$

$$\therefore \overline{\sum_{i=0}^{n} a_i \xi^i} = 0 \Rightarrow \sum_{i=0}^{n} \overline{a_i \xi^i} = 0.$$

Since a_i are real coefficients, we must have

$$\sum_{i=0}^{n} a_i \overline{\xi^i} = 0$$

But, $\overline{\xi^i} = (\bar{\xi})^i$ by the property of complex conjugation.

Therefore,

$$P(\bar{\xi}) = 0$$

If ξ is a root of $P(x) = 0$, $\bar{\xi}$ is also a root of $P(x) = 0$.

2. Let $P(x)$ be a polynomial of degree n, $n > 0$, with real coefficients. Use the fact that $P(x)$ has exactly n roots in the complex number system to prove that $P(x)$ can be factored into a product of linear and quadratic factors with real ~~co~~ coefficients. [Hint- See exercise 1(b) above].

Solution.

3. Find the second degree polynomial which has $a+bi$ and $a-bi$, $b>0$ as roots.

<u>Solution.</u> Let α, β be the roots of the equation.

$\therefore (x-\alpha)(x-\beta)=0$

$x^2-(\alpha+\beta)x+\alpha\beta=0$

$x^2-(a+bi+a-bi)x+(a+bi)(a-bi)=0$

$x^2-2ax+a^2+b^2=0.$

4. Prove that every polynomial of odd degree with real coefficients has atleast one real root.

<u>Proof.</u>

Consider a polynomial of degree n. A polynomial of degree n has n roots. If it has a complex root, it conjugate $\bar{z}$ is also a root of the polynomial. ~~Therefore~~, So, complex roots occur in pairs.

If the degree of the polynomial n is odd, then $n=2m+1$, where m is any natural number. Thus, the polynomial can have atmost $2m$ complex roots. It will have atleast one real root.

5. Write each of the following linear differential operators as the product of operators of degree one and two.

(a) D^3+4D^2+5D+2

<u>Solution</u>

D^3+4D^2+5D+2

$= D^3+D^2+3D^2+5D+2$

$= D^3+D^2+3D^2+3D+2D+2$

$= D^2(D+1)+3D(D+1)+2(D+1)$

$= (D+1)(D^2+3D+2)$

$= (D+1)(D+2)(D+1)$

$= (D+1)^2(D+2)$

(b) D^3-D^2+D-1

<u>Solution.</u>

D^3-D^2+D-1

$= D^2(D-1)+(D-1)$

$= (D^2+1)(D-1).$

$=$

(c) $D^4+2D^3-10D-25$

<u>Solution.</u>

$D^4+2D^3-D^2+D^2-10D-25$

$= \cancel{D^4}+\cancel{D^3}+\cancel{D^3}$

$= D^2(D^2+2D-1)+(D-5)^2$

(d) D^4-5D^2+4

<u>Solution.</u>

$D^4-4D^2-D^2+4$

$= D^2(D-4)-1(D^2-4)$

$= (D^2-1)(D^2-4)$

$= (D-1)(D+1)(D-2)(D+2$

(e) $D^4+2D^2+10.$

<u>Solution.</u>

$= D^4+2D^2+1+9$

$= D^4+2D^2+1+(3)^2$

$= (D^2+1)^2+3^2$

$= (D^2+1+3i)(D^2+1-3i)^2$