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A matrix in the row-echelon form is upper triangular.

For example,

$$U = \begin{bmatrix} 1 & 0 & 3 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is in the row-echelon form.

<u>The reduced row Echelon form.</u>

If a matrix in row-echelon form satisfies the following conditions, then it is said to be in reduced-row echelon form.

1) The matrix is in row-echelon form.

2) Each leading 1 is the only non-zero entry in its column.

The reduced row echelon form of the matrix discussed in the previous section is

$$R = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

<u>.1.2 Equivalent systems of Linear Equations.</u>

The sole objective of linear algebra is to solve the system of equations:

$$Ax = b$$

where $A$ is a matrix of order $m \times n$ over the field of reals, $x \in \mathbb{R}^n$, the right side vector $b \in \mathbb{R}^m$. We are interested to find $x = (x_1, x_2, \dots, x_n)$ that satisfies the above system of equations.

A solution of a linear system is therefore, an assignment of values to the variables $x_1, x_2, \dots, x_n$ such that each of the equations is satisfied. The set of all possible solutions is called the solution set.

Consider the system of equations

$$2x_1 + 3x_2 = 6$$
$$3x_1 + 2x_2 = 4$$

$$A = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad b = \begin{bmatrix} 6 \\ 4 \end{bmatrix}.$$

~~The ans~~

$$\begin{array}{r} 6x_1 + 9x_2 = 18 \\ 6x_1 + 4x_2 = 8 \\ \hline 5x_2 = 10 \\ x_2 = 2 \\ x_1 = 0. \end{array}$$

This system of equations has a solution $x = (0, 2)$. A system of equations that has a solution is called consistent. If it has no solution it is called inconsistent.

Consider

$$2x_1 + 3x_2 = 6$$
$$2x_1 + 3x_2 = 7$$

These are parallel lines. The system of equations has no solution.

Consider the system

$$2x_1 + 3x_2 = 6$$
$$4x_1 + 6x_2 = 12.$$

The lines $2x_1 + 3x_2 = 6$ and $4x_1 + 6x_2 = 12$ are coincident. This system has an infinite number of solutions.

Thus, there is a trichotomy:

- A system of equations may have no solution at all (for example parallel lines).
- A system of equations may have a unique solution.
- A system of equations may have an infinite number of solutions.

In general, a system of $m$ equations in $n$ unknowns ~~is said~~ is said to be underdetermined, if the number of equations are lesser than the number of unknowns $m < n$. If $m < n$, the system general has no solution or an infinite number of solutions. If $m = n$, in general, the system has a unique solution. If the number of equations exceeds the number of unknowns, $m > n$, it is overdetermined, and generally has no solution.

This is just to give a geometric viewpoint to the solutions of a system of linear equations.

Consider the system of equations

$$Ax = b.$$

Let's write this equation in the expanded form:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & & & & \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{bmatrix}$$

That is:

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n &= b_3 \\ \vdots \qquad\qquad & \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n &= b_m \end{aligned}\right\} \text{system I.}$$

These are $m$ equations in $n$ unknowns.

Suppose we take a linear combination of the first $m$ equations.

$$\alpha_1 (a_{11} x_1 + a_{12} x_2 + a_{13} x_3 + \dots + a_{1n} x_n)$$
$$+ \alpha_2 (a_{21} x_1 + a_{22} x_2 + a_{23} x_3 + \dots + a_{2n} x_n)$$
$$+ \alpha_3 (a_{31} x_1 + a_{32} x_2 + a_{33} x_3 + \dots + a_{3n} x_n)$$
$$+ \dots$$
$$+ \alpha_k (a_{k1} x_1 + a_{k2} x_2 + a_{k3} x_3 + \dots + a_{kn} x_n)$$

$$= \alpha_1 b_1 + \alpha_2 b_2 + \dots + \alpha_k b_k.$$

We can collect the coefficients of $x_1, x_2, \dots, x_n$.
$$x_1 (\alpha_1 a_{11} + \alpha_2 a_{21} + \dots + \alpha_k a_{k1}) + x_2 (\alpha_1 a_{12} + \alpha_2 a_{22} + \dots + \alpha_k a_{k2})$$
$$+ \dots + x_n (\alpha_1 a_{1n} + \alpha_2 a_{2n} + \dots + \alpha_k a_{kn}) = \alpha_1 b_1 + \alpha_2 b_2 + \dots + \alpha_k b_k.$$

Thus, if we take any solution $x = (x_1, x_2, \dots, x_n)$ that satisfies the original system of $m$ equations, the same ~~x vector~~ solution vector $x$ also satisfies the new equation. The new equation is a linear combination of the first '$k$' equations of the original system. Any solution of system I, will also be a solution of an equation obtained by taking linear combinations of equations of system I.

Consider a new system
$$R x = d$$

$$\begin{bmatrix} c_{11} & c_{12} & c_{13} & \dots & c_{1n} \\ c_{21} & c_{22} & c_{23} & \dots & c_{2n} \\ c_{31} & c_{32} & c_{33} & \dots & c_{3n} \\ \vdots & & & & \\ c_{m1} & c_{m2} & c_{m3} & \dots & c_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ d_m \end{bmatrix} \qquad \text{system II.}$$

~~Eq~~ Each equation of this new system is obtained by a certain linear combination of the $m$ equations of system I.

Any solution of system I will be a solution of system II. However, the converse in general is false.

Definition. Two systems of linear equations are said to be row equivalent, if each equation of one system is a linear combination of the equations of the other system.

Considering linear combinations of equations of system I, amounts to doing row operations.

Definition (Row-equivalent matrices). Let $A, B \in \mathbb{R}^{m \times n}$ be matrices over the field of real numbers. B is said to be row-equivalent to A, if B can be obtained by ~~a series of element~~ a finite sequence of elementary row operations on A. B is said to be row-equivalent to A, if there are elementary matrices P, Q such that

$$B = PAQ.$$

Row-equivalence is an RST relation.

a) A is row-equivalent to itself.

b) If is B is row-equivalent to A, A is row-equivalent to B.

$$B = PAQ$$
$$BQ^{-1} = PAQQ^{-1} = PA$$
$$P^{-1} B Q^{-1} = P^{-1} PA = A.$$

$\therefore$ A is row equivalent to B.

This holds because, P and Q are elementary matrices, and thus they are invertible.

c) If A is row-equivalent to B and B is row-equivalent to C, A is row-equivalent to C.

Finding the inverse of a matrix.
If A is an invertible square matrix of order $n$, then ~~we must be able~~ it must be the product of elementary matrices. It can be obtained by performing elementary row operations on $I_n$.

Let $A = E_k E_{k-1} \dots E_3 E_2 E_1$
Taking inverse on both sides:
$$A^{-1} = E_1^{-1} E_2^{-1} E_3^{-1} \dots E_{k-1}^{-1} E_k^{-1}.$$

Thus, we can interpret this equality in the following way. In order to get the inverse $A^{-1}$ of an invertible matrix, one can find a sequence of row operations that reduces A to I, and then perform the same sequence of operations on I.

We can join these two steps by first augmenting A and I, denote the resulting matrix by

$$[A \mid I]$$

and then apply the row operations to resulting matrix reducing the left part of it to I. The right part of it will be transformed to $A^{-1}$.

Theorem 11. Let $A, B \in \mathbb{R}^{m \times n}$ be matrices over the field of real numbers. ~~B~~ Suppose $B = PA$, where $P \in \mathbb{R}^{m \times m}$. Then, the row space (B) is a subspace of the row space (A).

Proof.
A, B are matrices such that $B = PA$ where P is a square matrix of order $m$, $P \in \mathbb{R}^{m \times m}$.

Look at $Ax$ for any $x \in \mathbb{R}^n$. Let $A = (A_1, A_2, \dots, A_n)$, ~~be~~ the $n$-columns of A.

$$Ax = [A_1\ A_2 \dots A_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = A_1x_1 + A_2x_2 + \dots + A_nx_n.$$

Thus, $Ax$ is a linear combination of the columns of A. This implies that the product $Ax$ is in the column space of A.

Let's go back to $B = PA$. After taking transposes, using the fact that transpose satisfies the reverse order law;

$$B^T = (PA)^T = A^T P^T = A^T Q, \quad Q = P^T.$$

Let $Q = (Q_1, Q_2, \dots, Q_m)$, the $m$ columns of Q. And let $B^T = (B_1, B_2, \dots, B_m)$ be the $m$ columns of B. (B has $m$ rows, $n$ columns, so $B^T$ has $n$ rows, $m$ columns).

$B^T = A^T Q$

$$(B_1, B_2, \ldots, B_m) = A^T (Q_1, Q_2, \ldots, Q_m)$$
$$= (A^T Q_1, A^T Q_2, \ldots, A^T Q_m)$$

if we push the matrix $A^T$ inside.

Now, any column vector $B_i$ is in the column space of $B^T$. Then, $B_i$ is in the row space of $B$. But, $B_i = A^T Q_i$.

$A^T Q_i$ is a matrix-vector product. Invoking the fact that, $Ax$ is in the column space of $A$, $A^T Q_i$ is in the column space of $A^T$. Therefore, $A^T Q_i$ is a linear combination of the columns of $A^T$. It implies that it is a linear combination of the rows of $A$.

So, $A^T Q_i$ is in the row space of $A$.

But, we already know that, $B_i = A^T Q_i$ is in the row space of $B$.

What we have therefore shown is, that if $B = PA$, the rows of $B$ are a linear combination of the rows of $A$.

If $B = PA$, then row space $(B) \subset$ row space $(A)$.

Lemma. If $B = PA$, with $P$ invertible, then row space $(B)$ = row space $(A)$. Row-equivalent matrices have the same row-space.

Proof.

If $P$ is an elementary matrix, $P$ is invertible. Then, pre-multiplying by $P^{-1}$, we obtain

$$P^{-1}B = P^{-1}PA$$
$$\therefore A = P^{-1}B = SB.$$

Applying the above theorem,

row space $(A) \subset$ row space $(B)$.

Thus, if $P$ is an elementary matrix, ~~then~~ and $B = PA$, row space $(B)$ = row space $(A)$. Row-equivalent matrices have the same row space. Elementary row operations on a matrix $A$, do not alter the row space of $A$.

Theorem.12 Let $A \in \mathbb{R}^{m \times n}$ be a matrix of order $m \times n$ over the field of reals. The row rank $(A)$ = column rank $(A)$.

Proof.

We would like to determine a basis for the row space of $A$. ~~If~~ We look at the row-reduced echelon form $R$ of the matrix $A$. Suppose $R$ has $r$ non-zero rows. The row space of $A$ is a subspace consisting of linear combinations of rows of $A$. $R$ has the form -

$$\begin{bmatrix}
0 & \cdots & 0 & 1 & & & & & \\
0 & \cdots & & \cdots & 0 & 1 & & & \\
0 & \cdots & & & & \cdots & 0 & 1 & \\
0 & \cdots & & & & & & 0 & 1 \\
0 & \cdots & & & & & & & 0 \\
0 & \cdots & & & & & & & 0 \\
\vdots & & & & & & & & \vdots \\
0 & \cdots & & & & & & & 0
\end{bmatrix}$$

$r$ non-zero rows.

Therefore, row space (R) is the space spanned by these r vectors. Since, A is row-equivalent to R, the row space of $A = r$.

Observe how R looks. The first r non-zero rows are linearly independent. For example, if

$$R = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \left.\right\} r \text{ non-zero rows}$$

then, $\{(0,0,0,1)\}$ is linearly independent. $(0,1,3,0)$ is not a scalar multiple of $(0,0,0,1)$, so $\{(0,0,0,1), (0,1,3,0)\}$ is a linearly independent set. Further, $(1,0,3,0)$ cannot be expressed as a linear combination of the row vectors below it. So, $(1,0,3,0) \notin \text{span}\{(0,0,0,1),(0,1,3,0)\}$. Therefore, $\{(0,0,0,1), (0,1,3,0), (1,0,3,0)\}$ is a linearly independent set. This closes the first part of the proof.

We need to show that the column rank of A is also r. We start all over again.

Let $A \in \mathbb{R}^{m\times n}$. Look at the system of equations $Ax = 0$. In expanded form –

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= 0 \\ a_{31}x_1 + a_{32}x_2 + \dots + a_{3n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= 0 \end{aligned}$$

A system of equations is called homogenous, if the right hand side vector b is the zero vector. This system has $(x_1, x_2, \dots, x_n) = (0, 0, \dots, 0)$ in which every $x_i = 0$ as its trivial solution. Thus, the question of importance for homogenous systems concerns the existence of a non-trivial solution.

Let S be the solution set of the homogenous system of equations $Ax = 0$.

$$S = \{X : AX = 0\}.$$

Remember, we can call this the kernel of the matrix A. S is a subspace of $\mathbb{R}^n$. I would like to calculate the dimension of this subspace. I would like to conclude that the dimension of this subspace is $n - r$. Then, it will follow that the column rank $= r$.

If R is the row-reduced echelon form of A, then the solution set S is:

$$S = \{X : RX = 0\}$$

Let's write down the equations $RX = 0$ in the expanded form. Let's use the following notation.

Define $J = \{1, 2, 3, \dots, n\} \setminus \{c_1, c_2, \dots, c_r\}$. Now, $x_{c_1}, x_{c_2}, \dots, x_{c_r}$ are the 'r' unknowns that correspond to the columns $c_1, c_2, \dots, c_r$. Remember, $c_1, c_2, c_3, \dots, c_r$ represent the columns in which the first ~~element~~ non-zero element in row $1, 2, \dots, r$ occurs.

$$\left.\begin{aligned}
x_{c_1} + \sum_{j\in J} \alpha_{1j} x_j &= 0.\\
x_{c_2} + \sum_{j\in J} \alpha_{2j} x_j &= 0.\\
&\vdots\\
x_{c_r} + \sum_{j\in J} \alpha_{rj} x_j &= 0.
\end{aligned}\right\} \text{r equations in 'n' unknowns}$$

The cardinality of $J$ is $(n-r)$. This is because we deleted $x_{c_1}, x_{c_2}, \dots, x_{c_r}$ from $x_1, x_2, \dots, x_n$.

We look at the variables $x_j$, $j \in J$. We assign arbitrary values to them. These are the free variables. If we substitute the ~~slack~~ (~~free~~) values of the slack (free) variables, we get $x_{c_1}, x_{c_2}, \dots, x_{c_r}$. This is one set of values. Take another set of values for the free variables $x_j$, $j \in J$. We may compute another distinct set of values for $x_{c_1}, x_{c_2}, \dots, x_{c_r}$.

Let e this set of values be denoted by a vector $s^j$, $j \in J$. How many $s^j$'s are there? Although the number of solutions to the above system $Rx=0$ are infinite, we can a linearly independent set of vectors. (infinite because, it has only $r$ equations, and $n$ unknowns, $r<n$, so it is underdetermined.)

The definition of $s^j$ is as follows. ~~$\forall$~~ suppose $s^j = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$. We look at one specific assignment of the free variables.

$$\left.\begin{aligned}
x_j &= 1, \text{ for some } j \in J.\\
x_i &= 0, \text{ for all } i \in J,\ i \neq J.
\end{aligned}\right\}$$

I take the first $j \in J$, and set $x_j = 1$. I set all other entries $x_i = 0$, $i \in J$, $i \neq j$. That is we are looking at a very specific assignment now. Pick a $j$ in $J$. Let us say, $1 \in J$. Then, I'll write down a vector, I'll call $s^1$. $s^1$

We substitute these values in the $r$ equations to get the values of the constrained variables $x_{c_1}, x_{c_2}, \dots, x_{c_r}$. I fill these values up in $s^j$.

I do this for each $j \in J$. Thus, we obtain $(n-r)$ $s^j$'s, $j \in J$. Each $s^j$ is a solution of $Rx = 0$. So, each $s^j \in S$.

$s^j \in S, \forall j \in J$.

We observe that these vectors are linearly independent. The reason is similar to the standard basis vectors, being linearly independent. For example, only one vector $s^j$ will have its $j$th entry $= 1$, all other free variables are zero. $s^{j+1}$ will have a 1 in $(j+1)$st coordinate, $j \in J$ and all other free variables 0. So, $s^j$ cannot be expressed as a linear combination of any other vectors.

$s^j_j$'s are linearly independent.

The next step is to show that these vectors span any solution.

Let the terms containing those unknowns $x_j$, with $j \neq c_1, c_2, \ldots, c_r$ be transferred to the right hand side, so that the left side of the equation $i$ is merely the term $x_{c_i}$. $(i = 1, 2, 3, \ldots, r)$.
We have then solved for $x_{c_1}, x_{c_2}, \ldots, x_{c_r}$ in terms of the remaining unknowns, which may be regarded as parameters. The system is consistent and all solutions can be found by assigning arbitrary values to the parameters and determining the values of $x_{c_1}, \ldots, x_{c_r}$.

For simplicity, let's assume. $1, 2 \in J$. Suppose we substitute $x_1 = a$, $x_2 = b$. Then, All other free variables are set to 0.

$$x_{c_1} = -\alpha_{11} x_1 - \beta\alpha_{12} x_2 = a(-\alpha_{11}) + b(-\alpha_{12})$$
$$x_{c_2} = -\alpha_{21} x_1 - \alpha_{22} x_2 = a(-\alpha_{21}) + b(-\alpha_{22})$$
$$\vdots \qquad\qquad \vdots$$
$$x_{c_r} = -\alpha_{r1} x_1 - \alpha_{r2} x_2 = a(-\alpha_{r1}) + b(-\alpha_{r2}).$$

The solution, therefore obtained is: a linear combination of the vectors $s^1, s^2$, that is $s^* = a s_1 + b s_2$. Any solution $x^*$ can be expressed as a linear combination of the $(n-r)$ solution vectors $s^j$, $j \in J$.

Thus, $\{s^j : j \in J\}$ forms a basis of the solution space $S$. The dimension of the solution space $S$ equals $(n-r)$.

$\dim S = (n-r)$

Let's define $T: \mathbb{R}^n \to \mathbb{R}^m$ by
$T(X) = Ax$, $x \in \mathbb{R}^n$.

Then, the rank nullity dimension (RND) theorem can be applied to $T$.
By rank-nullity dimension theorem:
$\operatorname{rank}(T) + \operatorname{nullity}(T) = n$.
Now, $\operatorname{rank}(T) = \dim \operatorname{range}(T)$
$= \dim \{y : y = T(X), x \in \mathbb{R}^n\}$
$= \dim \{y \in \mathbb{R}^m : y = AX, x \in \mathbb{R}^n\}$.

But, we have encountered this subspace ~~xxxx~~. Recall that, if $y = AX$, then $y$ belongs to the column space of $A$.

column rank of $A = \dim \{y \in \mathbb{R}^m : y = AX, X \in \mathbb{R}^n\}$.
$\therefore$ Column rank of $A = \operatorname{rank}(T)$.

$\operatorname{nullity}(T) = \dim \operatorname{null}(T)$
$= \dim \{X : T(X) = 0\}$
$= \dim \{X : AX = 0\}$
$= (n-r)$

So, $\operatorname{rank}(T) = r$
Therefore, column rank $(A) = r =$ row rank $(A)$.

$$B = \begin{bmatrix} I_r & O_{r\times(n-r)} \\ O_{(m-r)\times r} & O_{(m-r)\times(n-r)} \end{bmatrix}$$

Normal form of a matrix A (definition). $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ is called the normal form of a matrix A.

Theorem 13. Let e denote a single elementary row operation. Then, for all $A \in \mathbb{R}^{m\times n}$, $e(A) = EA$, where E is an elementary matrix given by $E = e(I)$.

Claim. The $(i,j)$ th entry of EA = The $(i,j)$ th entry of $e(I)$
$(EA)_{ij}$ $\qquad$ $(e(A))_{ij}$.

Proof.
Let e be the operation - replace row s by row s + $\alpha$ times row t.
so, I want to write down the matrix E. The element in the $(i,k)$ place of E is -

$$\text{Element } (i,k) = \begin{cases} \delta_{sk} + \alpha \cdot \delta_{tk} & i = s \\ \delta_{ik} & i \neq s \end{cases}$$

I'll use this definition and matrix multiplication. It will turn out to be what is there on the right hand side. So, let's do matrix multiplication to verify the equality.

The element in the $(i,j)$ th place of EA is given by -

$$\text{Element } (i,j) = \sum_{k=1}^{m} E_{ik} \cdot a_{kj}$$

$$= \begin{cases} \sum_k (\delta_{sk} + \alpha\, \delta_{tk}) \cdot a_{kj} & , i = s \\ \sum_k \delta_{ik} \cdot a_{kj} & , i \neq s \end{cases}$$

$$= \begin{cases} a_{sj} + \alpha\, a_{tj} & , i = s \\ a_{ij} & , i \neq s. \end{cases}$$

where $\delta_{ij}$ is the Kronecker's delta.

$$= (e(A))_{ij}$$

Theorem 14. Let $A, B \in \mathbb{R}^{m\times n}$. Then, B is row-equivalent to A, if and only if $B = PA$, where P is a product of elementary matrices.

Proof.
If I do a single elementary row operation and get B from A, then the matrix P will be an elementary matrix as we have already seen. We would like to extend this logic. You will see that this is useful later on as well. There are certain things preserved by elementary row operations. In the case of square matrices, for instance, we will look at certain numbers, the determinant, the rank, the inverse etc.

One could show that by [illegible] elementary row operations, certain numbers ~~such~~ like the determinant, the rank, the inverse remain the same. In proving these results, we will make use of this, that B can be written as P times A, $B = PA$, where P is a product of elementary matrices.

($\Leftarrow$ direction).
Suppose $B = PA$. To must show that B is row-equivalent to A.

Let $P = E_k E_{k-1} \dots E_2 E_1$. Then,

$$\begin{aligned} B &= PA \\ &= (E_k E_{k-1} \dots E_2 E_1) A \\ &= (E_k E_{k-1} \dots E_2) E_1 A \qquad \text{since matrix multiplication is associative} \end{aligned}$$

By virtue of the previous theorem, $E_1 A$ is row-equivalent to A, as $E_1$ is an elementary matrix,.

$E_1 A \sim A$.

Again consider

$$\begin{aligned} B &= PA \\ &= (E_k E_{k-1} \dots E_2) E_1 A \\ &= (E_k E_{k-1} \dots E_3) E_2 (E_1 A) \end{aligned}$$

$E_2 (E_1 A) \sim E_1 A \sim A$.

As row-equivalence is a transitive relation, $E_2 E_1 A \sim A$. Continuing in this fashion, $E_k E_{k-1} \dots E_2 E_1 A \sim A$. But, the left-hand side is precisely B. So, B is row-equivalent to A.

($\Rightarrow$ direction)
Suppose B is row equivalent to A. Then, B is obtained from A by a sequence of elementary row operations. $E_1, E_2, \dots, E_{l-1}, E_l$. Then, the first elementary row operation is $E_1 A$ pre-multiplying $A$ by $E_1$, $E_1 A$, the second elementary row operation results in $E_2 E_1 A$, the third elementary row operation is obtained by $E_3 E_2 E_1 A$ and so forth.

$$B = \underbrace{E_l E_{l-1} \dots E_3 E_2 E_1}_{Q} A$$

$$B = QA$$

where $Q = E_l E_{l-1} \dots E_2 E_1$, a product of elementary matrices.

1.3 <u>Invertibility of a Matrix.</u>
Let $A \in \mathbb{R}^{n \times n}$ be a matrix of order $n$. A is said to have a right inverse, if there exists another matrix $B \in \mathbb{R}^{n \times n}$ such that, $AB = I_n$. A left-inverse is defined similarly. ~~A~~ left-inverse A is said to have a left-inverse $\mathcal{E}$, there exists a matrix $C \in \mathbb{R}^{n \times n}$, such that $CA = I_n$.

Together, A is said to be invertible, if A has a right inverse and a left inverse.

<u>Theorem 15</u>. A square matrix A is said to be invertible, if A has both a left-inverse and a right-inverse. Both the left-inverse is equal to right-inverse and is the inverse of the matrix A.