11. $\sin\left(\frac{x+1}{x-1}\right)$.

The function $f(x) = \sin\left(\frac{x+1}{x-1}\right)$ is not defined at $x=1$. Hence, it does not belong to $C[-1,1]$.

12. $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$

Solution.

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} |x| = \lim_{x\to 0} (x) = 0$

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} |x| = \lim_{x\to 0} (-x) = 0$.

$f(0) = 0$.

f is continuous at $x=0$. Hence f belongs to $C[a,b]$.

2. Vector spaces.

Before defining what a vector space is, let's establish the definition of a few important sets in all of ... mathematics.

The set of all real numbers in mathematics is denoted by $\mathbb{R}$.

$$\mathbb{R} := \{x : -\infty < x < \infty\}$$

The set of all ordered pairs of real numbers, which you can think of as a cartesian plane is denoted by $\mathbb{R}^2$.

$$\mathbb{R}^2 := \{(x_1, x_2) : x_1, x_2 \in \mathbb{R}\}$$

The set $\mathbb{R}^3$ which you can think of as ordinary space, consists of all ordered triples of real numbers:

$$\mathbb{R}^3 := \{(x_1, x_2, x_3) : x_1, x_2, x_3 \in \mathbb{R}\}.$$

To generalize $\mathbb{R}^2$ and $\mathbb{R}^3$ to higher dimensions, we first need to discuss the concept of tuples. Suppose n is a non-negative integer, $n \geq 0$. A list of length n, is an ordered collection of n objects (which might be numbers, other lists or more abstract entities). A list of length n looks like this

$$(x_1, x_2, \ldots, x_n).$$

Thus, a list of length 2 is an ordered pair, a list of length 3 is an ordered triple. For $j = \{1, 2, \ldots, n\}$, we say x_j is jth coordinate of the list above.

To define higher dimensional analogues of $\mathbb{R}^2$ and $\mathbb{R}^3$ we simply replace $\mathbb{R}$ with $\mathbb{F}$ (which equals $\mathbb{R}$ or $\mathbb{C}$) and replace 2 or 3 with n. We define $\mathbb{F}^n$ to be the set of all n-tuples, consisting of elements of $\mathbb{F}$:

$$\mathbb{F}^n = \{(x_1, x_2, \ldots, x_n) : x_j \in \mathbb{F} \ \forall j \in \{1, 2, 3, \ldots, n\}\}.$$

If $n \geq 4$, we cannot easily visualize $\mathbb{R}^n$ as a physical object. The same problem arises with complex numbers: $\mathbb{C}$ can be thought of as a plane, but for $n \geq 2$, the human brain cannot provide geometric models of $\mathbb{C}^n$. However, even when n is large, we can perform algebra on vectors in $\mathbb{F}^n$ as easily as in $\mathbb{R}^2$ or $\mathbb{R}^3$.

For example, addition is defined on $\mathbb{F}^n$ by adding the corresponding coordinates:

$$(x_1, \dots, x_n) + (y_1, \dots, y_n) = (x_1 + y_1, \dots, x_n + y_n).$$

Definition of a Vector space.

A vector space V over a numeric field $\mathbb{F}$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$) is a non-empty set V, whose elements are called vectors and in which two operations are defined; called addition and scalar multiplication that enjoy the following properties –

Axioms for addition.

(A1) commutativity

$x + y = y + x$ for all $x, y \in V$.

(A2) associativity

$(x+y) + z = x + (y+z)$ for all $x, y, z \in V$.

(A3) Additive identity exists.

There exists a vector 0 in V, called the zero vector such that:

$x + 0 = x$ for all x in V.

(A4) Additive inverse exists.

For each x in V, there exists a vector $-x$ in V such that,

$x + (-x) = 0$.

Axioms for scalar multiplication.

(M1) scalar multiplication be associative.

$(\alpha\beta)x = \alpha(\beta x)$

(M2) multiplicative identity holds

$1x = x$ for all $x \in V$.

It is also required that, scalar multiplication mixes well with vector addition.

(D1) $\alpha(x+y) = \alpha x + \alpha y$

(D2) $(\alpha + \beta)x = \alpha x + \beta x$.

The student should not be discouraged by the formality of the definition; it looks much worse than it really is. The issue here is simply that in order to deserve the name, a vector space must have elementary and eminently reasonable properties in common with $\mathbb{R}^2$. We have already seen this happen in the case of the space of continuous functions $C[a,b]$ and before embarking on the general study of this subject we give several examples of such variety as to convince the reader that vector spaces are very common objects indeed in mathematics. still others will be found at the end of this section. In each case, we leave the verification of the axioms (i) through (viii) as an exercise to aid the beginner in assimilating the various requirements of this definition.

Examples:

1) Let $V = \mathbb{R} = \mathbb{F}$. We define addition in the usual way real numbers are added. scalar multiplication is also the usual multiplication. Then, $\mathbb{R}$ is a vector space over the field of real numbers.

2) Let $V = \mathbb{C} = \mathbb{F}$. We define addition to be complex number addition and scalar multiplication as complex number multiplication.

Then, $\mathbb{C}$ is a vector space over itself.

Proof.

(A1) Addition is commutative.

Let $z_1 = (x_1 + iy_1)$, $z_2 = (x_2 + iy_2)$ be arbitrary elements of V.

$$\begin{aligned} z_1 + z_2 &= (x_1 + iy_1) + (x_2 + iy_2) \\ &= (x_1 + x_2) + i(y_1 + y_2) \\ &= (x_2 + x_1) + i(y_2 + y_1) \\ &= (x_2 + iy_2) + (x_1 + iy_1) \\ &= z_2 + z_1 \end{aligned}$$

(A2) Addition is associative.

$$\begin{aligned} (z_1 + z_2) + z_3 &= \{(x_1 + iy_1) + (x_2 + iy_2)\} + (x_3 + iy_3) \\ &= \{(x_1 + x_2) + i(y_1 + y_2)\} + (x_3 + iy_3) \\ &= x_1 + x_2 + x_3 + i(y_1 + y_2 + y_3) \\ &= (x_1 + iy_1) + \{(x_2 + x_3) + i(y_2 + y_3)\} \\ &= (x_1 + iy_1) + \{(x_2 + iy_2) + (x_3 + iy_3)\} \\ &= z_1 + (z_2 + z_3) \end{aligned}$$

for all $z_1, z_2, z_3 \in V$.

(A3) Additive identity.

Let the zero vector be defined as $0 := 0 + 0i$.

Consider any element $z \in \mathbb{C}$.

$$\begin{aligned} z + 0 &= (x + iy) + (0 + 0i) \\ &= (x + 0) + i(y + 0) \\ &= x + iy \\ &= z. \end{aligned}$$

There exists an zero element.

(A4) Additive inverse.

If ~~x~~ z is any vector in V, there must be corresponding negative elements $(-z)$ such that $z + (-z) = 0$.

Let $z = x + iy$ be an arbitrary vector. and $-z := (-x) + (-iy)$.

$$\begin{aligned} z + (-z) &= (x + iy) + ((-x) + i(-y)) \\ &= (x + (-x)) + i(y + (-y)) \\ &= 0 + 0i \\ &= 0. \end{aligned}$$

for all z.

Further, if $\alpha \in \mathbb{F} = \mathbb{C}$ and $z \in \mathbb{C}$, we know that $\alpha z \in \mathbb{C}. \forall z$.

(M1) scalar multiplication is associative.

$$\begin{aligned} (\alpha\beta) z &= \{(\alpha_1 + i\alpha_2)(\beta_1 + i\beta_2)\} (x + iy). \\ &= \{(\alpha_1\beta_1 - \alpha_2\beta_2) + i(\alpha_1\beta_2 + \alpha_2\beta_1)\} (x + iy). \\ &= \{(\alpha_1\beta_1 - \alpha_2\beta_2) x - (\alpha_1\beta_2 + \alpha_2\beta_1) y\} \\ &\quad + i\{(\alpha_1\beta_1 - \alpha_2\beta_2) y + (\alpha_1\beta_2 + \alpha_2\beta_1) x\}. \\ &= \alpha_1\{(\beta_1 x - \beta_2 y) + i(\beta_1 y + \beta_2 x)\} \\ &\quad + i\alpha_2\{(\beta_1 x - \beta_2 y) + i(\beta_1 y + \beta_2 x)\} \\ &= (\alpha_1 + i\alpha_2)\{(\beta_1 + i\beta_2)(x + iy)\} \\ &= \alpha(\beta z). \end{aligned}$$

(M2) Multiplicative identity holds.

$$1z = 1(x+iy) \\ = x+iy \\ = z.$$

(D1)-(D2). It is an easy exercise to show that vector addition and scalar multiplication mix well.

3) In general, any field $\mathbb{F}$ is a vector space over itself. $\mathbb{F}(\mathbb{F})$ is a vector space.

4) Trivial Vector Space.

Let $V = \{0\}$ be a vector space consisting of a single element, the zero vector. $\mathbb{F}$ is the field of real numbers. Vector addition is defined as

$$0+0=0$$

scalar multiplication is defined as

$$\alpha \cdot 0 = 0.$$

Thus, $V(\mathbb{F})$ is a vector space.

5) Let $V = \mathbb{F}^2$ be the set of all all column vectors, which have just two components (coordinates).

$$\mathbb{F}^2 := \left\{ x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : x_1, x_2 \in \mathbb{F} \right\}$$

Let $x, y \in \mathbb{F}^2$. Assume, $x = (x_1, x_2)$, $y = (y_1, y_2)$. And let $\alpha \in \mathbb{F}$.
We define vector addition as

$$x+y = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1+y_1 \\ x_2+y_2 \end{pmatrix}.$$

We define scalar multiplication as:

$$\alpha x = \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \end{pmatrix}.$$

It is an easy exercise to verify that $\mathbb{F}^2$ is a vector space over $\mathbb{F}$, with respect to coordinate wise vector addition and scalar multiplication defined above.

6) Generalisation.

Let $V = \mathbb{F}^n$ be the set of all n-tuples.

$$\mathbb{F}^n = \left\{ x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} : x_1, x_2, \ldots, x_n \in \mathbb{F} \right\}$$

Let $x, y \in \mathbb{F}^n$. We define vector addition as before, coordinate-wise.

$$x+y = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1+y_1 \\ x_2+y_2 \\ \vdots \\ x_n+y_n \end{pmatrix}.$$

We define scalar multiplication similarly.

$$\alpha \cdot x = \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_n \end{pmatrix}.$$

Again, it is an easy exercise to prove that, $\mathbb{F}^n$ is a vector space over $\mathbb{F}$.

7) Let $\mathbb{F}^{m \times n}$ be the set of all $m \times n$ matrices.

$$\mathbb{F}^{m\times n} = \left\{ A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ a_{31} & a_{32} & \dots & a_{3n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \;\middle|\; a_{ij} \in \mathbb{F} \right\}$$

Let x, y be vectors in $\mathbb{F}^{m\times n}$. We define vector addition in the usual way matrices are added.

$$x+y = \begin{pmatrix} x_{11}+y_{11} & x_{12}+y_{12} & \dots & x_{1n}+y_{1n} \\ x_{21}+y_{21} & x_{22}+y_{22} & \dots & x_{2n}+y_{2n} \\ x_{31}+y_{31} & x_{32}+y_{32} & \dots & x_{3n}+y_{3n} \\ \vdots & & & \vdots \\ x_{m1}+y_{m1} & x_{m2}+y_{m2} & \dots & x_{mn}+y_{mn} \end{pmatrix}.$$

We define scalar multiplication as—

$$\alpha \cdot x = \begin{pmatrix} \alpha \cdot x_{11} & \alpha x_{12} & \dots & \alpha x_{1n} \\ \alpha x_{21} & \alpha x_{22} & \dots & \alpha x_{2n} \\ \alpha x_{31} & \alpha x_{32} & \dots & \alpha x_{3n} \\ \vdots & & & \\ \alpha x_{m1} & \alpha x_{m2} & \dots & \alpha x_{mn} \end{pmatrix}.$$

It is an easy exercise to show that $\mathbb{F}^{m\times n}$ is a vector space over $\mathbb{F}$, with respect to matrix addition and scalar multiplication defined above.

8) Let $V = P_n(\mathbb{R})$ be the set of all polynomials in the real variable t, with degree not exceeding n.

Let $p, q \in P_n(\mathbb{R})$.
$p = p_0 + p_1 t + p_2 t^2 + \dots + p_n t^n$
$q = q_0 + q_1 t + q_2 t^2 + \dots + q_n t^n$.

We define vector addition as:

$$(p+q) = (p_0+q_0) + (p_1+q_1)t + (p_2+q_2)t^2 + \dots + (p_n+q_n)t^n.$$

We define scalar multiplication as:

$$\alpha p = \alpha p_0 + \alpha p_1 t + \alpha p_2 t^2 + \dots + \alpha p_n t^n.$$

Let us verify that $\mathcal{P}_n(\mathbb{R})$ is a real vector space.

Proof.

(A1) Addition is commutative.

$$\begin{aligned} p+q &= (p_0 + p_1 t + p_2 t^2 + \dots + p_n t^n) + (q_0 + q_1 t + q_2 t^2 + \dots + q_n t^n) \\ &= (p_0+q_0) + (p_1+q_1)t + (p_2+q_2)t^2 + \dots + (p_n+q_n)t^n \\ &= (q_0+p_0) + (q_1+p_1)t + (q_2+p_2)t^2 + \dots + (q_n+p_n)t^n \\ &= (q_0 + q_1 t + q_2 t^2 + \dots + q_n t^n) + (p_0 + p_1 t + p_2 t^2 + \dots + p_n t^n) \\ &= q + p. \end{aligned}$$

(A2) Addition is associative.

$$\begin{aligned} p + (q+r) &= (p_0 + p_1 t + \dots + p_n t^n) + \{(q_0 + q_1 t + \dots q_n t^n) + (r_0 + r_1 t + \dots + r_n t^n)\} \\ &= (p_0 + p_1 t + \dots + p_n t^n) + \{(q_0 + r_0) + (q_1 + r_1)t + \dots + (q_n + r_n)t^n\} \\ &= (p_0 + q_0 + r_0) + (p_1 + q_1 + r_1)t + \dots + (p_n + q_n + r_n)t^n \\ &= (p_0 + q_0) + (p_1 + q_1)t + \dots + (p_n + q_n)t^n \\ &\quad + r_0 + r_1 t + \dots + r_n t^n \\ &= \{p_0 + p_1 t + \dots + p_n t^n + q_0 + q_1 t + \dots + q_n t^n\} + r_0 + r_1 t + \dots + r_n t^n \\ &= (p+q) + r. \end{aligned}$$

(A3) Additive identity.

Let the vector 0 be the zero polynomial with all coefficients 0.

$$\underset{\substack{\uparrow \\ \text{vector}}}{0} := 0 + 0t + \dots + 0t^n$$

Then,

$$\begin{aligned} p + 0 &= p_0 + p_1 t + \dots + p_n t^n \\ &\quad + 0 + 0t + \dots + 0t^n \\ &= (p_0 + 0) + (p_1 + 0)t + \dots + (p_n + 0)t^n \\ &= p_0 + p_1 t + \dots + p_n t^n \\ &= p. \end{aligned}$$

(A4) Additive inverse.

Let $p = p_0 + p_1 t + p_2 t^2 + \dots + p_n t^n$. We define the additive inverse as the polynomial $(-p)$ with coefficients

$$(-p) = (-p_0) + (-p_1)t + (-p_2)t^2 + \dots + (-p_n)t^n$$

$$\begin{aligned} p + (-p) &= p_0 + p_1 t + p_2 t^2 + \dots + p_n t^n \\ &\quad + (-p_0) + (-p_1)t + (-p_2)t^2 + \dots + (-p_n)t^n \\ &= (p_0 + (-p_0)) + (p_1 + (-p_1))t + (p_2 + (-p_2))t^2 \\ &= 0 + 0t + 0t^2 + \dots + 0t^n \\ &= 0. \end{aligned}$$

(M1) Scalar multiplication is associative.

$$\begin{aligned} (\alpha\beta)p &= (\alpha\beta)p_0 + (\alpha\beta_1)p_1 t + \dots + (\alpha\beta)p_n t^n \\ &= \alpha(\beta p_0) + \alpha(\beta p_1)t + \dots + \alpha(\beta p_n)t^n \\ &= \alpha(\beta p_0 + \beta p_1 t + \beta p_2 t^2 + \dots + \beta p_n t^n) \\ &= \alpha(\beta p). \end{aligned}$$

(M2) Multiplicative identity.

$1\cdot p = (1\cdot p_0) + (1\cdot p_1)t + (1\cdot p_2)t^2 + \dots + (1\cdot p_n)t^n$

$= p_0 + p_1 t + p_2 t^2 + \dots + p_n t^n$

$1\cdot p = p.$

(D1) $\alpha(p+q) = \alpha(p_0 + p_1 t + p_2 t^2 + \dots + p_n t^n + q_0 + q_1 t + q_2 t^2 + \dots + q_n t^n)$

$= \alpha((p_0+q_0) + (p_1+q_1)t + (p_2+q_2)t^2 + \dots + (p_n+q_n)t^n)$

$= \alpha(p_0+q_0) + \alpha(p_1+q_1)t + \alpha(p_2+q_2)t^2 + \dots + \alpha(p_n+q_n)t^n$

$= (\alpha p_0 + \alpha q_0) + (\alpha p_1 + \alpha q_1)t + (\alpha p_2 + \alpha q_2)t^2 + \dots + (\alpha p_n + \alpha q_n)t^n$

$= \alpha p_0 + \alpha p_1 t + \dots + \alpha p_n t^n + \alpha q_0 + \alpha q_1 t + \alpha q_2 t^2 + \dots + \alpha q_n t^n$

$= \alpha p + \alpha q.$

(D2) $(\alpha+\beta)p = (\alpha+\beta)p_0 + (\alpha+\beta)p_1 t + (\alpha+\beta)p_2 t^2 + \dots + (\alpha+\beta)p_n t^n$

$= (\alpha p_0 + \beta p_0) + (\alpha p_1 + \beta p_1)t + (\alpha p_2 + \beta p_2)t^2 + \dots + (\alpha p_n + \beta p_n)t^n$

$= \alpha p_0 + \alpha p_1 t + \alpha p_2 t^2 + \dots + \alpha p_n t^n + \beta p_0 + \beta p_1 t + \beta p_2 t^2 + \dots + \beta p_n t^n$

$= \alpha p + \beta p.$

Thus, $P_n(\mathbb{R})$ is a real vector space.

9) Let $V = C([a,b])$ be the set of all real-valued continuous functions on the closed interval $[a,b]$.
For $p, q \in C([a,b])$, we define vector addition to be
$(p+q)(x) = p(x) + q(x)$

We define scalar multiplication to be -
$(\alpha p)(x) = \alpha p(x).$

It can again be shown that $C([a,b])$ is a real vector space.

10) Let $V = C^k((a,b))$ be the set of all real valued functions f with the property that, the kth derivative is continuous.

We define vector addition and scalar-multiplication point-wise as before. since, the sum of two continuous functions is continuous, $f+g$ is indeed a vector in $C^k((a,b))$. Moreover, αf is a continuous function. Thus, $C^k((a,b))$ is a real vector space.

11) Let $V = C^\infty((a,b))$ be the set of all real valued continuous functions that are infinitely many times differentiable in any open interval (a,b). It is important to observe that the sum of two differentiable functions is differentiable. Also, αf is differentiable.

$\therefore$ $C^\infty((a,b))$ is a real vector space, with respect to the operations of addition and scalar multiplication defined in the usual manner.

12) Let $F((a,b), \mathbb{R}) := \{f : (a,b) \to \mathbb{R}\}$ be the set of all real valued functions defined over the open interval (a,b).

In this, we look at all the functions that are Riemann-integrable over (a,b).

$V = \{f \in F : \int_a^b f(t)\,dt \text{ exists}\}.$

Then, with respect to the operations of addition and scalar multiplication, we can show that V is a real vector space. This follows from the fact that if f is Riemann integrable and g is Riemmann integrable $(f+g)$ is Riemann integrable.

13) Let $A \in \mathbb{R}^{m \times n}$.
We define
$$V = \{x \in \mathbb{R}^n : Ax = 0\} \subseteq \mathbb{R}^n.$$

V is the set of all solutions of the system of linear homogenous equations $Ax = 0$.

$\mathbb{R}^n$ already has vector addition and scalar multiplication defined.

Let x, y be any two solutions of system of linear homogenous equations. Thus, $x, y \in V$. That is,
$$Ax = 0$$
$$Ay = 0$$
$$\therefore Ax + Ay = 0$$
$$\therefore A(x+y) = 0$$ (from the properties of matrix multiplication)
$$\Leftrightarrow x + y \in V.$$

Thus, $x+y$ is also a solution of the system of equations.

If x is a solution of the system of equations, and α is any scalar, $\alpha \in \mathbb{R}$, then
$$Ax = 0$$
$$\alpha(Ax) = 0$$
$$A(\alpha x) = 0$$

αx is also a solution of the system of equations.

Thus, V is a vector space.

14) Let us look at an operator D defined on a function $y = f(x)$, which is atleast n times differentiable. y is a function of the independent variable x.

$$D(y) \equiv a_n \frac{d^n}{dx^n}(y) + a_{n-1} \frac{d^{n-1}}{dx^{n-1}}(y) + \dots + a_1 \frac{d}{dx}(y)$$

The coefficients $a_1, \dots, a_n$ are functions of x. D is called the differential operator.

Consider $V = \{y : D(y) = 0\}$ be the set of all solutions that satisfy the given linear differential equation.

Let y_1 and y_2 be any two arbitrary solutions of the differential equation. Then,
$$D(y_1) = 0 \text{ and } D(y_2)$$

$$a_n \frac{d^n}{dx^n}(y_1) + a_{n-1} \frac{d^{n-1}}{dx^{n-1}}(y_1) + \dots + a_1 \frac{d}{dx}(y_1) = 0$$
$$a_n \frac{d^n}{dx^n}(y_2) + a_{n-1} \frac{d^{n-1}}{dx^{n-1}}(y_2) + \dots + a_1 \frac{d}{dx}(y_2) = 0.$$

$\Leftrightarrow a_n \frac{d^n}{dx^n}(y_1+y_2) + a_{n-1}\frac{d^{n-1}}{dx^{n-1}}(y_1+y_2) + \cdots + a_1 \frac{d}{dx}(y_1+y_2) = 0.$

(since the derivative of a sum is a sum of the two derivatives).

$\Leftrightarrow y_1+y_2$ is a solution of the linear differential equation.

Similarly, if y_1 is a solution, then αy_1 is also a solution of a linear differential equation. Thus, V - the space of solutions of a linear differential equation form a vector space.

15) Let $V = \{x \in \mathbb{R}^2 : x_2 = \alpha x_1\}$. where α is a fixed constant.
We collect all vectors, which have the property that the second coordinate is α times the first coordinate. Geometrically, this set represents the set of all points lying on a straight line passing through the origin. It can be shown that V is a vector space.

16) $V = \{x \in \mathbb{R}^3 : ax_1 + bx_2 + cx_3 = 0\}$
Geometrically, this set represents all points lying on the plane $ax_1 + bx_2 + cx_3 = 0$ passing through the origin in 3D-space. It can be shown that V is a vector space.

17) $V = \left\{ x \in \mathbb{R}^3 : \frac{x_1}{l} = \frac{x_2}{m} = \frac{x_3}{n},\ x_1, x_2, x_3 \in \mathbb{R},\ l, m, n \text{ are fixed constants} \right\}$

Geometrically, this set represents the set of all points lying on a straight line with direction numbers l, m, n passing through the origin. It can be shown that V is a vector space.

These examples and many others illustrate how the linear space concept permeates throughout algebra, geometry and analysis. When a theorem is deduced from the axioms of a linear space, we obtain in one stroke a result valid for each concrete example.

Elementary consequences of the axioms.
The following properties are easily deduced for a linear space.

Proposition. Uniqueness of zero element.
In any linear space, there is one and only one zero element.

Proof.
Axiom (A3) tells us that there exists atleast one zero vector in a vector space. Suppose that there were two zero vectors 0 and 0'.
Then,

$$x + 0 = x$$

setting $x = 0'$,

$$0' + 0 = 0'$$

But, $0' + 0 = 0$ since, $0'$ is a zero element.

$$\Leftrightarrow 0 = 0'.$$

Proposition. Uniqueness of negative elements.
~~If~~ In any linear space, every element has exactly one negative element. That is, for every x, there is one and only one y, such that $x+y = 0$.

Proof.

Axiom (A4) tells us that each x has atleast one negative, namely $-x$. Suppose x has two negatives x' and x''.

Then,

$$x + x' = 0$$
$$x + x'' = 0$$

$$\Leftrightarrow \quad x + x' + x'' = x''$$
$$x + x'' + x' = x'' \quad \text{(addition is commutative)}$$
$$0 + x' = x'' \quad \text{(x'' is the negative of x)}$$
$$x' = x''$$

Proposition.
In a given linear space, let x and y denote arbitrary vectors and let α and β denote scalars from the field F. Then, we have the following properties.

a) $0x = 0$ vector.
b) $\alpha 0 = 0$ vector.
c) $(-\alpha)x = -(\alpha x) = \alpha(-x)$.
d) If $\alpha x = 0$, then either $\alpha = 0$ scalar or $x = 0$ vector.
e) If $\alpha x = \alpha y$, then either $\alpha = 0$ scalar or $x = y$.
f) If $\alpha x = \beta x$, and $x \neq 0$ vector, $\alpha = \beta$.
g) $-(x+y) = (-x) + (-y) = -x - y$.
h) $x + x = 2x$
$x + x + x = 3x$
and in general $\sum_{i=1}^{n} x = nx$.

i) $(-1)(-x) = x = -(-x)$.

Proof.
a) $0x = (0+0)x$
$= 0x + 0x$

Let $z = 0x$
$\Leftrightarrow z + z = z$
Adding $(-z)$ on both sides
$\Leftrightarrow z + z + (-z) = z + (-z)$
$\Leftrightarrow z + 0 = 0$
$\Leftrightarrow z = 0$
$\Leftrightarrow 0x = 0$

b) $\alpha 0 = \alpha(0+0)$
$= \alpha 0 + \alpha 0$
Let $z = \alpha 0$.
$\Leftrightarrow z + z = z$
Adding $-z$ on both sides:
$z + z + (-z) = z + (-z)$
$z + 0 = 0$
$z = 0$.

c) $(-\alpha)x + (\alpha x) = (-\alpha)x + (\alpha)x$
$= (-\alpha + \alpha)x$
$= 0x$
$= 0$.

$\Leftrightarrow (-\alpha)x$ is the negative of αx.
$\Leftrightarrow (-\alpha)x = -(\alpha x)$.

Also,

$$\alpha(-x) + \alpha x = \alpha(x + (-x))$$
$$= \alpha 0$$
$$= 0$$

$\Leftrightarrow$ $\alpha(-x)$ is the negative of αx.

$\Leftrightarrow$ $\alpha(-x) = -(\alpha x)$.

$\Leftrightarrow$ $(-\alpha)x = -(\alpha x) = \alpha(-x)$.

f) $\alpha x = \beta x,\ x \neq 0$

$$\alpha x + (-\beta)x = \beta x + (-\beta x)$$
$$(\alpha - \beta)x = 0$$

$\Rightarrow$ $\alpha = \beta$.

e) $\alpha x = \alpha y$

$$\alpha x + (-\alpha y) = \alpha y + (-\alpha)y$$
$$= 0$$
$$\alpha(x + (-y)) = 0 = \alpha 0$$

Thus,

$$x + (-y) = 0$$
$$x + (-y) + y = y$$
$$x + 0 = y$$
$$x = y.$$

d) If $\alpha x = 0$ then either $\alpha = 0$ or $x = 0$.

Either $\alpha x = 0 = 0x$

$\Leftrightarrow$ $\alpha = 0$

Or $\alpha x = 0 = \alpha 0$

$\Leftrightarrow$ $x = 0$

g) $-(x+y) = (-x) + (-y)$.

Let $z = -(x+y)$.

$\Leftrightarrow$ $z + (x+y) = 0$

$\Leftrightarrow$ $z + x + y + (-x) = (-x)$

$\Leftrightarrow$ $z + x + (-x) + y = (-x)$

$\Leftrightarrow$ $z + 0 + y = (-x)$

$\Leftrightarrow$ $z + y = (-x)$

$\Leftrightarrow$ $z + y + (-y) = (-x) + (-y)$

$\Leftrightarrow$ $z + 0 = (-x) + (-y)$

$\Leftrightarrow$ $z = (-x) + (-y)$

$\Leftrightarrow$ $-(x+y) = (-x) + (-y)$.

Also, $-x - y = (-x) + (-y)$.

h) $x + x = 1x + 1x$

$$= (1+1)x$$
$$= 2x.$$

$$x + x + x = x + (x + x)$$
$$= x + 2x$$
$$= 1x + 2x$$
$$= (1+2)x$$
$$= 3x.$$

In general, $\sum_{i=1}^{n} x = x + \sum_{i=1}^{n-1} x$

$$= x + x + \sum_{i=1}^{n-2} x$$
$$= 2x + \sum_{i=1}^{n-2} x = 2x + x + \sum_{i=1}^{n-3} x = 3x + \sum_{i=1}^{n-3} x = \cdots = nx.$$