4. The general theory of Linear Differential Equations. 06/11/2019

1. Linear Differential Operators.

We have already had occasion to remark that the operator D which maps a differentiable function onto its derivative is a linear transformation. The same is true of polynomials in D and even of more complicated expressions such as xD^2+D+x. Linear transformations of this sort which involve D and its powers are called linear differential operators. The study of such operators naturally leads to the theory of linear differential equations, the subject matter of this chapter and the chapters which follow.

To give a precise meaning to the term "linear differential operator", let I be an arbitrary interval of the real line, and for each non-negative integer n, let $\mathcal{C}^n(I)$ denote the vector space of all real-valued functions which have ~~a~~ a continuous nth derivative everywhere in I. Recall that the vectors in $\mathcal{C}^n(I)$ are real-valued functions whose first n derivatives exist and are continuous throughout I, and that vector addition and scalar multiplication in this space are defined by the equations

$$(f+g)(x) = f(x)+g(x)$$
$$(\alpha f)(x) = \alpha f(x).$$

for all x in I. By agreement, $\mathcal{C}^0(I) = \mathcal{C}(I)$.

In these terms, we now state -

Definition 1.1. A linear transformation $L: \mathcal{C}^n(I) \to \mathcal{C}(I)$ is said to be a linear differential operator of order n on the interval I if it can be expressed in the form

$$L = a_n(x)D^n + a_{n-1}(x)D^{n-1} + \cdots + a_1(x)D + a_0(x). \tag{4.1}$$

where the coefficients $a_0(x), a_1(x), \ldots, a_n(x)$ are continuous everywhere in I, and $a_n(x)$ is not identically zero on I. In addition, the transformation which maps every function in $\mathcal{C}^n(I)$ onto the zero function is also considered to be a linear differential operator. It however is not assigned an order.

Thus, the image of a function f in $\mathcal{C}^n(I)$ under the linear differential operator described above is the function $\mathcal{C}(I)$ defined by the identity

$$Lf(x) = a_n(x)\frac{d^n}{dx^n}f(x) + a_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}}f(x) + \cdots + a_1(x)\frac{d}{dx}f(x) + a_0(x)f(x), \tag{4.2}$$

or more simply by,

$$Ly = a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y \tag{4.3}$$

where $y', y'', \ldots, y^{(n)}$ are the first n derivatives of the function $y=f(x)$. Strictly speaking the left-hand side of (1.2) is the value of Lf at the point x, and a scrupulous regard for accuracy would require that it be written $(L(f))(x)$. For obvious reasons the extra parenthesis is always omitted. Moreover, we shall occasionally refer to $Lf(x)$ as the linear transformation L applied to the function $f(x)$, thereby following the familiar custom of confusing a function with its value at a point. This of course is just a linguistic convenience, and once understood causes no difficulty.

Examples.

1. The nth derivative operator, D^n, is the simplest example of a linear differential operator of order n on an arbitrary interval I. When $n=0$, D^0 is just the identity transformation and in general, D^n can be viewed as the nth power of a linear transformation D.

2. Any polynomial in D of degree n, with real coefficients is a linear differential operator of order n on every interval of the real line.

3. A linear differential operator of order 0 on I has the form

$$L = a_0(x) \tag{1.4}$$

where $a_0(x)$ is continuous and not identically zero on I. Thus, if f is any function on $\mathcal{C}(I)$,

$$Lf(x) = a_0(x) f(x).$$

which, of course, is just the product of the functions a_0 and f. Occasionally, one finds (1.4) written in the form $a_0(x) D^0$ to emphasize the fact is being viewed as an operator and not as a function in $\mathcal{C}(I)$.

4. The linear differential operator

$$xD^2 + 3\sqrt{x}\, D + 1$$

is of order 2 in $[0, \infty]$ or any of its subintervals. By way of contrast

$$(x+|x|)\, D^2 - \sqrt{x+1}\, D + \ln(x+1)$$

is of order 2 on $(-1, 1)$, but of order 1 on the subinterval $(-1, 0]$. Thus, the order of a linear differential operator may depend upon the interval in which it is being considered, as well as on the the algebraic form of the operator itself.

A linear differential operator is by definition, a linear transformation, and hence, under suitable hypotheses, it makes sense to talk about the product of two such operators. Such product are again linear differential operators although it is impossible to say very much about their order or the domain of definition (discussed later). We remind the reader that the usual precautions arising from the ~~no~~ noncommutativity of operator multiplication must also be observed in this setting. For instance, a product such as $(xD+2)(2xD+1)$ cannot be computed by multiplying the expressions $xD+2$ and $2xD+1$ according to the usual rules of algebra. Indeed, if it could, we would have $(xD+2)(2xD+1) = 2x^2D^2 + xD + 4xD + 2 = 2x^2D^2 + 5xD + 2$, when, in fact the correct answer is $2x^2D^2 + 7xD + 2$, as can be seen from the following computation:

$$\begin{aligned}
(xD+2)(2xD+1)y &= (xD+2)(2xy'+y) \\
&= xD(2xy'+y) + 2(2xy'+y) \\
&= x(2y' + 2xy'' + y') + 4xy' + 2y \\
&= 3xy' + 2x^2y'' + 4xy' + 2y \\
&= 2x^2y'' + 7xy' + 2y.
\end{aligned}$$

However, in the special case of operators with constant coefficients, products can be computed as though the operators were ordinary polynomials in D. As we shall see, this fact will ultimately enable us to solve all linear differential equations with constant coefficients.

Problems.

1. Evaluate each of the following expressions.

(a) $(D^2+D)e^{2x}$

(b) $(3D^2+2D+2)\sin x$

(c) $(xD-x)(2\ln x)$

(d) $(D+1)(D-x)(2e^x+\cos x)$.

Solutions.

a) $(D^2+D)e^{2x} = D^2e^{2x}+De^{2x}$
$= 4e^{2x}+2e^{2x} = 6e^{2x}$

b) $(3D^2+2D+2)\sin x = 3D^2\sin x + 2D\sin x + 2\sin x$
$= -3\sin x + 2\cos x + 2\sin x$
$= 2\cos x - \sin x$.

c) $(xD-x)(2\ln x)$
$= xD(2\ln x) - x(2\ln x)$
$= 2x\cdot\left(\frac{1}{x}\right) - 2x\ln x$
$= 2(1-x\ln x)$.

d) $(D+1)(D-x)(2e^x+\cos x)$
$= (D+1)D(2e^x+\cos x) - (D+1)x(2e^x+\cos x)$
$= (D+1)(2e^x-\sin x) - (D+1)x(2e^x+\cos x)$
$= D(2e^x-\sin x) + (2e^x-\sin x) - \cancel{(D+1)}\,D\,x(2e^x+\cos x)$
$- x(2e^x+\cos x)$.
$= (2e^x \cancel{-\cos x}) + (2e^x-\sin x) - (2e^x \cancel{+\cos x}) - x(2e^x-\sin x)$
$- x(2e^x+\cos x)$
$= 2e^x - 2\cos x - \sin x - x(4e^x - \cos x - \sin x)$.

2. Repeat exercise 1 for each of the following expressions.

(a) $(aD^2+bD+c)e^{kx}$, a, b, c, k constants.

$(aD^2+bD+c)e^{kx}$
$= aD^2e^{kx} + bDe^{kx} + ce^{kx}$
$= ak^2e^{kx} + bke^{kx} + ce^{kx}$
$= e^{kx}(ak^2+bk+c)$

(b) $(x^2D^2-2xD+4)x^k$
$= x^2D^2\cdot x^k - 2xD\cdot x^k + 4x^k$
$= x^2k(k-1)x^{k-2} - 2xkx^{k-1} + 4x^k$
$= x^k[k(k-1) - 2k + 4]$
$= x^k(k^2-k-2k+4)$
$= x^k(k^2-3k+4)$

(c) $(4x^2D^2+4xD+4x^2+1)\left(\frac{1}{\sqrt{x}}\sin x\right)$

$$= 4x^2 D^2\left(\frac{\sin x}{\sqrt{x}}\right) + 4x\, D\left(\frac{\sin x}{\sqrt{x}}\right) + 4x^2 \cdot \frac{\sin x}{\sqrt{x}} + \frac{\sin(x)}{\sqrt{x}}.$$

$$= 4x^2 \cdot D\left\{\left(\sqrt{x}\cos x - \frac{\sin x}{2\sqrt{x}}\right)\Big/ x\right\} + \cancel{4}\cancel{x}^{2} \cdot \frac{(2x\cos x - \sin x)}{\cancel{2}\cancel{x}\sqrt{x}} + \frac{4x^2 \sin x}{\sqrt{x}} + \frac{\sin x}{\sqrt{x}}.$$

$$= 4x^2\, D\left(\frac{2x\cos x - \sin x}{2x\sqrt{x}}\right) + 4\sqrt{x}\cos x - \frac{2\sin x}{\sqrt{x}} + \frac{4x^2 \sin x}{\sqrt{x}} + \frac{\sin x}{\sqrt{x}}$$

$$= 4x^2\, D\left(\frac{2x\cos x - \sin x}{2x\sqrt{x}}\right) + 4\sqrt{x}\cos x + 4x\sqrt{x}\sin x - \frac{\sin x}{\sqrt{x}}$$

$$= \cancel{4}\cancel{x^2}\left(\frac{2x\sqrt{x}(2\cos x - 2x\sin x - \cos x) - (2x\cos x - \sin x)\,3\sqrt{x}}{4x^{\cancel{3}}}\right)$$

$$+ 4\sqrt{x}\cos x + 4x\sqrt{x}\sin x - \frac{\sin x}{\sqrt{x}}$$

$$= 2\sqrt{x}(2\cos x - 2x\sin x - \cos x) - 3\frac{(2x\cos x - \sin x)}{\sqrt{x}}$$

$$+ 4\sqrt{x}\cos x + 4x\sqrt{x}\sin x - \frac{\sin x}{\sqrt{x}}.$$

$$= \cancel{4\sqrt{x}\cos x} - \cancel{4x\sqrt{x}\sin x} - \cancel{2\sqrt{x}\cos x} - \cancel{6\sqrt{x}}\cos x + 3\frac{\sin x}{\sqrt{x}}$$

$$+ \cancel{4\sqrt{x}\cos x} + \cancel{4x\sqrt{x}\sin x} - \frac{\sin x}{\sqrt{x}}$$

$$\frac{2\sin x}{\sqrt{x}}.$$

3. Find constants a, b, c such that $a+b+c=1$ and

$$[(1-x^2)D^2 - 2xD + 6](ax^2+bx+c) = 0.$$

<u>solution</u>

$$(1-x^2)D^2(ax^2+bx+c) - 2xD(ax^2+bx+c) + 6(ax^2+bx+c) = 0.$$

$$(1-x^2)(2a) - 2x(2ax+b) + 6(ax^2+bx+c) = 0$$

$$2a - \cancel{2ax^2} - \cancel{4ax^2} - 2bx + \cancel{6ax^2} + 6bx + 6c = 0$$

$$2a + 4bx + 6c = 0$$

$$a + 2bx + 3c = 0.$$

Assume, $b = 0$, Then,

$$a + c = 1$$

$$a + 3c = 0$$

$$2c = -1$$

$$c = -\frac{1}{2}, \quad a = \frac{3}{2}.$$

Thus, $a = \frac{3}{2}$, $b = 0$, $c = -\frac{1}{2}$.

4. Find each of the following linear differential operators in the standard form.

(a) $(D^2+1)(D-1)y$

$\cancel{= D^2(D-1) + (D-1)}$

$= (D^2+1)(Dy - y) = (D^2+1)(y' - y)$

$= D^2(y'-y) + (y'-y)$

$= y''' - y'' + y' - y.$

$\Leftrightarrow$ The standard form is $D^3 - D^2 + D - 1.$

(b) $xD(D-x)y$

$= xD(y' - xy)$

$= x(y'' - y - xy')$

$= xy'' - xy - x^2y'$

$= (xD^2 - x^2D - x)(y).$

(c) $(xD^2+D)^2$

$(xD^2+D)^2 y$

$= (xD^2+D)(xD^2+D)y.$

$= (xD^2+D)(xy^{(2)} + y')$

$= xD^2(xy''+y') + D(xy''+y')$

$= xD(y''+xy'''+y'') + (y''+xy'''+y'')$

$= xD(xy'''+2y'') + (xy'''+2y'')$

$= x(y'''+xy^{(4)}+2y''') + xy'''+2y''$

$= xy''' + x^2y^{(4)} + 2xy''' + xy''' + 2y''$

$= x^2y^{(4)} + 4xy''' + 2y''$

$= (x^2D^4 + 4xD^3 + 2D^2)y.$

(d) $D^2(xD-1)D.$

Solution.

$D^2(xD-1)Dy$

$= D^2(xD-1)y'$

$= D^2(xDy' - y')$

$= D^2(xy'' - y'')$

$= D(y''+xy'''-y'')$

$= D(xy''') = y''' + xy^{(4)}$

$= (xD^4 + D^3)y.$

(e) $D(De^x + D + e^x)y$

$= D(De^x+1)y + e^xy.$

$= D(De^xy + 1) + e^xy.$

$= D(e^xy + e^xy' + 1) + e^xy$

$= e^xy + e^xy' + e^xy' + e^xy'' + e^xy$

$= e^x[y'' + 2y' + 2y]$

$= e^x(D^2 + 2D + 2)y.$

5. Show that $D(xD) \neq (xD)D.$

Solution.

$D(xD)y = D(xy') = y' + xy'' = (D + xD^2)y$

$(xD)Dy = xD(y') = xy'' = xD^2(y).$

$D + xD^2 \neq xD^2$

$\Rightarrow D(xD) \neq (xD)D.$

6. (a) Prove that a linear differential operator of order n is a linear transformation from $\mathscr{C}^n(I)$ to $\mathscr{C}(I)$.
(b) Is this linear transformation one-to-one when $n>0$? Why?

Solution.

Let $L = a_n(x)D^n + a_{n-1}(x)D^{n-1} + \cdots + a_1(x)D + a_0(x)I$.
Let's two arbitrary functions whose nth derivative is continuous — $f, g \in \mathscr{C}^n(I)$.

$$L(f+g) = (a_n(x)D^n + a_{n-1}(x)D^{n-1} + \cdots + a_1(x)D + a_0 x I)(f+g)$$
$$= a_n(x)D^n(f+g) + a_{n-1}(x)D^{n-1}(f+g) + \cdots + a_1(x)D(f+g) + a_0(x)I(f+g)$$

since sum of linear transformations
$$(S+T)(v) = S(v) + T(v).$$

$$= a_n(x)D^n f + a_n(x)D^n g + a_{n-1}(x)D^{n-1}f + a_{n-1}(x)D^{n-1}g + \cdots + a_1(x)Df + a_1(x)Dg + a_0(x)f + a_0(x)g$$
$$= \{(a_n(x)D^n + a_{n-1}(x)D^{n-1} + \cdots + a_1(x)D + a_0(x)I)f\} + \{(a_n(x)D^n + a_{n-1}(x)D^{n-1} + \cdots + a_1(x)D + a_0(x)I)g\}$$
$$= Lf + Lg.$$

$$L(\alpha f) = (a_n(x)D^n + \cdots + a_0(x)I)(\alpha f)$$
$$= a_n(x)D^n(\alpha f) + \cdots + a_0(x)I(\alpha f)$$
$$= \alpha a_n(x)D^n f + \cdots + \alpha a_0(x)If.$$
$$= \alpha L(f).$$

Hence, L ~~is closed under vector~~ preserves additivity ~~addition~~ and multiplication by a scalar ~~scalar multiplication~~.
L is a linear transformation.

(b) ~~Consider~~ null L = $\{f : Lf = 0\}$.
L is not one-to-one.
since, for every constant $c \in \mathbb{R}$, if f is n times differentiable
$$L(f+c) = L(f).$$

7. Prove that $D^m(a(x)D^n)$ is a linear differential operator of order $m+n$ by expressing this product in the standard form as a polynomial in D. Assume the existence and continuity of all derivatives of $a(x)$.

Solution.

$$D^m(a(x)D^n)y = D^m a(x) D^n y$$
$$= a^m(x) \cdot D^n y + a(x) D^{m+n} y$$
$$= (a(x)D^{m+n} + a^m(x)D^n)y.$$

$$\therefore D^m(a(x)D^n) = a(x)D^{m+n} + a^m(x)D^n.$$

Hence, it is a linear differential operator of order $m+n$.

8. Prove that the sum of two linear differential operators defined on an interval I is the linear differential operator on I obtained by adding the corresponding coefficients in the standard polynomial representation of the given operators.

Solution.

Let $L_1, L_2 \in \mathcal{L}(\mathscr{C}^n(I), \mathscr{C}(I))$.
Suppose $L_1 = a_n(x)D^n + \cdots + a_1(x)D + a_0(x)$
$L_2 = b_n(x)D^n + \cdots + b_1(x)D + b_0(x)$

$(L_1 + L_2)y = L_1 y + L_2 y$ as linear transformation are just functions.
$= a_n(x) D^n y + a_{n-1}(x) D^{n-1} y + \dots + a_1(x) Dy + a_0(x) y$
$+ b_n(x) D^n y + b_{n-1}(x) D^{n-1} y + \dots + b_1(x) Dy + b_0(x) y$
$= (a_n(x) + b_n(x)) D^n y + (a_{n-1}(x) + b_{n-1}(x)) D^{n-1} y + \dots +$
$(a_1(x) + b_1(x)) Dy + (a_0(x) + b_0(x)) y.$

Note, that, the space $\mathcal{L}(C^n(I), C(I))$ of all linear differential operators is closed under vector addition and scalar multiplication. so, $L_1 + L_2$ is well defined and belongs to this vector space.

9. (a) Prove that
$$(aD^m)(bD^m) = (bD^n)(aD^m) = ab\, D^{m+n}$$
where a, b are constants.

Proof.
(a) $(aD^m)(bD^n) y = aD^m \cdot b \frac{d^n y}{dx^n} = ab \frac{d}{dx^m}\left(\frac{d^n y}{dx^n}\right) = ab \frac{d^{m+n} y}{dx^{m+n}} = ab\, D^{m+n} y.$

$(bD^n)(aD^m) y = b D^m a D^m y = ba\, D^{m+n} y = ab\, D^{m+n} y.$

Hence, $(aD^m)(bD^n) = (bD^n)(aD^m) = ab\, D^{m+n}.$

(b) Use (a) and the general distributivity formula for linear transformations that was established earlier to prove that the multiplication of constant coefficient linear differential operators is commutative. Deduce from this that the product of two such operators can be obtained by treating them as ordinary polynomials in D and using the usual rules of elementary algebra.

Proof.
Let $L_1 = a_n D^n + a_{n-1} D^{n-1} + \dots + a_1 D + a_0 I$
$L_2 = b_m D^m + b_{m-1} D^{m-1} + \dots + b_1 D + b_0 I$

$L_1 L_2 = (a_n D^n + a_{n-1} D^{n-1} + \dots + a_1 D + a_0 I)(b_m D^m + b_{m-1} D^{m-1} + \dots + b_1 D + b_0 I)$
$= a_n b_m D^{m+n} + (a_n b_{m-1} + b_m a_{n-1}) D^{m+n-1} + (a_n b_{m-2} + a_{n-1} b_{m-1} + a_{n-2} b_m) D^{m+n-2}$
$+ \dots$ using distributivity
$= (b_m D^m + b_{m-1} D^{m-1} + \dots + b_1 D + b_0)(a_n D^n + a_{n-1} D^{n-1} + \dots + a_1 D + a_0),$
$= L_2 L_1.$ since $a_n b_m D^{m+n} = (a_n D^n)(b_m D^m) = (b_m D^m)(a_n D^n)$
$a_n b_{m-1} D^{m+n-1} = (a_n D^n)(b_{m-1} D^{m-1}) = (b_{m-1} D^{m-1})(a_n D^n)$

$= L_2 L_1$

Thus, the product of two such operators can be obtained by treating them as ordinary polynomials in D and using the rules of elementary algebra.

10. Factor each of the linear differential operators into a product of irreducible factors of lower order.

(a) $D^2 - 3D + 2$
$= D^2 - 2D - D + 2$
$= D(D-2) - 1(D-2)$
$= (D-1)(D-2)$

(b) $2D^2 + 5D + 2$
$= 2D^2 + 4D + D + 2$
$= 2D(D+2) + (D+2)$
$= (2D+1)(D+2).$

(c) $4D^2+4D+1 \quad = (2D+1)^2.$

$= \cancel{4D^2+2D+2D+1}$

$\cancel{2D(2D+1)+1}$

(d) D^3-3D^2+4

$= D^3+D^2-4D^2+4$

$= D^2(D+1)-4(D^2-1)$

$= D^2(D+1)-4(D+1)(D-1)$

$= (D+1)(D^2-4D+4)$

$= (D+1)(D-2)^2.$

(e) $4D^4+4D^3-7D^2+D-2$

$$\begin{array}{r|l} & \cancel{4D^3+4D^2-7D+1} \\ \hline \cancel{D-1} & \cancel{4D^4+4D^3-7D^2+D-2} \\ & \cancel{-4D^3-4D^2+7D-1} \\ \hline & \cancel{8D^3-3D^2-6D+1} \end{array}$$

$$\begin{array}{r|l} & 4D^3+8D^2+D+2 \\ \hline D-1 & 4D^4+4D^3-7D^2+D-2 \\ & 4D^4-4D^3 \\ \hline & 8D^3-7D^2+D-2 \\ & 8D^3-8D^2 \\ \hline & D^2+D-2 \\ & D^2-D \\ \hline & 2D-2 \\ & 2D-2 \\ \hline & 0 \end{array}$$

$= (D-1)(4D^3+8D^2+D+2)$

$$\begin{array}{r|l} & 4D^2+1 \\ \hline D+2 & 4D^3+8D^2+D+2 \\ & 4D^3+8D^2 \\ \hline & D+2 \\ & D+2 \\ \hline & 0 \end{array}$$

$= (D-1)(D+2)(4D^2+1).$

(f) (D^4-1)

$= \cancel{(D-1)(D^3-D^2+D-1)}$

$= (D-1)(D^3+D^2+D+1)$

$= (D-1)(D^2+1)(D+1)$

$$\begin{array}{r|l} & D^3+D^2+D+1 \\ \hline D-1 & D^4-1 \\ & D^4-D^3 \\ \hline & D^3-1 \\ & D^3-D^2 \\ \hline & D^2-1 \\ & D^2-D \\ \hline & D-1 \end{array}$$

(g) (D^4+1)

$=$

11. Prove that

$D^2[f(x)g(x)] = f''(x)g(x)+2f'(x)g'(x)+f(x)g''(x)$

$D^3[f(x)g(x)] = f'''(x)+3f''(x)g'(x)+3f'(x)g''(x)+g'''(x)$

Can you make a conjecture as to the form of $D^n[f(x)g(x)]$. ?

Proof.

$$D^2[f(x)g(x)] = D[f'g + fg']$$
$$= f''g + f'g' + f'g' + fg''$$
$$= f''g + 2f'g' + fg''$$

$$D^3[f(x)g(x)] = D^3[fg] = D^2[f'g + fg']$$
$$= D[f''g + 2f'g' + fg'']$$
$$= [f'''g + f''g' + 2f''g' + 2f'g'' + f'g'' + fg''']$$
$$= f'''g + 3f''g' + 3f'g'' + fg'''$$

We conjecture that $D^n[f(x)g(x)]$ must have the form -

$$D^n(fg) = f^{(n)}g + \binom{n}{1}f^{(n-1)}g' + \binom{n}{2}f^{(n-2)}g^{(2)} + \dots + \binom{n}{n-1}f'g^{(n-1)} + fg^{(n)}$$

12. Use mathematical induction to prove Leibnitz's rule.

$$D^n[f(x)g(x)] = \sum_{k=0}^{n}\binom{n}{k}(D^{n-k}f(x))(D^k g(x)).$$

Proof.

a) $P(1)$ is true.
The left hand side is $D(fg) = f'g + fg'$
The right hand side is $\binom{1}{0}Df\cdot g + \binom{1}{1}f\,Dg$
$= f'g + fg'$

b) Assume $P(n)$ is true.

$$D^n(fg) = f^{(n)}g + \binom{n}{1}f^{(n-1)}g' + \binom{n}{2}f^{(n-2)}g^{(2)} + \dots + \binom{n}{k}f^{(n-k)}g^{(k)} + \dots + fg^{(n)}$$

c) Prove that $P(n+1)$ is true.

$$D^{n+1}(fg) = D\cdot D^n(fg) = D\left[f^{(n)}g + \binom{n}{1}f^{(n-1)}g' + \binom{n}{2}f^{(n-2)}g^{(2)} + \dots + \binom{n}{k}f^{(n-k)}g^{(k)} + \dots + \binom{n}{n}f g^{(n)}\right]$$

$$= f^{(n+1)}g + \binom{n}{0}f^{(n)}g' + \binom{n}{1}f^{(n)}g' + \binom{n}{1}f^{(n-1)}g^{(2)} + \binom{n}{2}f^{(n-1)}g^{(2)} + \dots$$
$$+ \underbrace{\binom{n}{k-1}f^{(n-k+1)}g^{(k)} + \binom{n}{k}f^{(n-k+1)}g^{(k)}} + \dots]$$

The general term is -

$$\left[\binom{n}{k-1} + \binom{n}{k}\right]f^{(n-k+1)}g^{(k)} = \binom{n+1}{k}f^{(n-k+1)}g^{(k)}$$

$\Rightarrow P(n+1)$ is true.

13. Use the result of the preceding exercise to express each of the following differential operators in the form $a_n(x) D^m + \dots + a_1(x) D + a_0(x)$.

(a) $D^3(xD)$
(b) $D^m(xD)$
(c) $D^5(xD^2 + e^x)$

Solution.

a) $D^3(xD) = x^{(3)} D + 3x^{(2)} D^2 + 3x^{(1)} D^3 + x D^4$
$= 3D^3 + x D^4$

b) $D^m(xD) = x^{(m)} D + \binom{m}{1} x^{(m-1)} D^2 + \binom{m}{2} x^{(m-2)} D^3$

$+ \dots + \binom{m}{m-1} x' D^{(m)} + x D^{m+1}$

$= m D^m + x D^{m+1}$

c) $D^5(xD^2 + e^x)$
$= D^5(xD^2) + D^5(e^x)$
$= D^5(xD^2) + e^x$
$= \cancel{5D^7 + D^7 + e^x}$. $= \cancel{5(D^2)^4 + x(D^2)^5} +$
$= \cancel{5D^5(D^2) + x}$

$= 5Dx \cdot D^4(D^2) + x D^5(D^2) + e^x$
$= 5D^6 + x D^7 + e^x$.

14. Prove that for any pair of non-negative integers k and m,

$$D^m x^k = \begin{cases} m! \binom{k}{m} x^{k-m}, & m \le k \\ 0, & m > k. \end{cases}$$

Solution

$D x^k = k x^{k-1}$
$D^2 x^k = k D x^{k-1} = k(k-1) x^{k-2}$
$\vdots$
$D^m x^k = k(k-1) \dots (k-m+1) x^{k-m}$
$= \frac{k!}{(k-m)!} x^{k-m}$

$= \frac{k!}{(k-m)!\, m!} \times m! \times x^{k-m}$

$= m! \binom{k}{m} x^{k-m}$

If $m > k$, $D^m x^k = 0$.

15 (a) Prove that, for any pair of non-negative integers k and m,

$(x^m D^m) x^k = k(k-1) \dots (k-m+1) x^k$

for any real number k.

(b) Prove that
$(a_2 x^2 D^2 + a_1 x D + a_0) x^k = [a_2 k(k-1) + a_1 k + a_0] x^k$
for any real number k, (a_0, a_1, a_2) are constants.

(c) Prove that $(xD)(x^3D^3) = (x^3D^3)(xD)$.

Solution.

(a) $(x^m D^m)\, x^k = x^m \cdot (D^m x^k)$

$= x^m \cdot \frac{k!}{m!}\, x^{(k-m)}$

$= x^k \cdot k(k-1)(k-2)\ldots(k-m+1).$

(b) $(a_2 x^2 D^2 + a_1 x D + a_0)\, x^k$

$= a_2 x^2 D^2 x^k + a_1 x D x^k + a_0 x^k$

$= a_2 k(k-1) x^k + a_1 k x^k + a_0 x^k$

$= (a_2 k(k-1) + a_1 k + a_0)\, x^k$

(c) $(xD)(x^3D^3)\, y$

$= (xD)\, x^3 y'''$

$= x(D x^3 y''')$

$= x(3x^2 \cdot y''' + x^3 y^{(4)})$

$= (3x^3 D^3 + x^4 D^4)\, y$

Also, $(x^3D^3)(xD)\, y$

$= x^3 D^3 (xDy)$

$= x^3 D^3\, x y'$

$= x^3 (D^3 x \cdot y' + 3D^2 x \cdot Dy' + 3Dx \cdot D^2 y' + x D^3 y')$

$= x^3 (3y''' + x y^{(4)})$

$= (3x^3 D^3 + x^4 D^4)\, y.$

$\Rightarrow (xD)(x^3D^3) = (x^3D^3)(xD).$

16. A linear differential operator is sometimes said to be equidimensional or an Euler operator if it can be written in the form $a_n x^n D^n + \cdots + a_1 x D + a_0$, where $a_0, \ldots, a_n$ are constants.

(a) Compute the value of Lx^k, k an arbitrary real number, when L is equidimensional.

(b) Prove that $(x^m D^m)(x^n D^n) = (x^n D^n)(x^m D^m)$ for any pair of non-negative integers m, n and hence deduce that the multiplication of equidimensional operators is commutative.

Solution.