

# Second-Order Logic

Final seminar for "Logic in Computer Science"

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# Outline

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# Introduction

- First-order logic allows "iteration" over the *elements* of a structure
- Happens thanks to **quantifiers**:  $\forall, \exists$ 
  - ▶  $\forall x. \phi(x) \rightarrow$  "For each  $x$ ,  $x$  satisfies the formula  $\phi$ "
  - ▶  $\exists x. \phi(x) \rightarrow$  "There exists  $x$  s.t. the formula  $\phi$  is satisfied"
- Limiting since we may only need to range over *subsets* or "*combinations*" (e.g. *Cartesian product*)

# A brief recall

- Second-order logic "extends" first-order logic
- Since that, let's recall the basics of first-order logic
- Two key parts:
  - ▶ *Syntax*: Which sequences constitute **well-formed** expressions
  - ▶ *Semantic*: The **meaning** behind this expressions

# Syntax

# Introduction

- Two base types:
  - ▶ **Terms:** Represents *objects*
  - ▶ **Formulas:** Represents *predicates*
- Both formed by *symbol* concatenation
- All symbols together form the **alphabet** of the language
- Can divide symbols in two categories
  - ▶ *Logical* symbols
  - ▶ *Non-logical* symbols

# Logical symbols

- Infinite set of **variables**:  $x, y, z, \dots, x_0, x_1, \dots$  (Lowercase letters)
- **Connectives**:  $\wedge, \vee, \Rightarrow, \neg$
- **Quantifiers**:  $\forall, \exists$
- **Equality** (or *Identity*):  $=$
- **Auxiliary symbols**:  $(; ); .$  (dot);  $,$  (comma)

# Non-logical symbols

- Represents *predicates* (or *relations*), *functions* and *constants*
- $\forall n \in \mathbb{Z}^*$  we have a set of  $n$ -ary **predicate symbols**

$$P_0^n, P_1^n, \dots \quad (\text{Uppercase letters})$$

- $\forall n \in \mathbb{Z}^*$  there exist infinite  $n$ -ary **function symbols**

$$f_0^n, f_1^n, \dots \quad (\text{Lowercase letters})$$



# Formation rules (1)

## Definition (Terms formation)

The set **TERM** of *terms* can be inductively defined by the following rules:

- ① If  $x$  is a variable, then  $x \in \text{TERM}$
- ② Any expression  $f(t_1, \dots, t_n)$ , with  $t_1, \dots, t_n \in \text{TERM}$ , is a term.  
Since that, the following statement holds

$$f(t_1, \dots, t_n) \in \text{TERM}$$

## Formation rules (2)

### Definition (Formulas formation)

The set FORM of *formulas* can be inductively defined by the following rules:

- 1 If  $P \in \text{PRED}^a$  and  $t_1, \dots, t_n \in \text{TERM}$ , then  $P(t_1, \dots, t_n) \in \text{FORM}$
- 2 If  $t_1, t_2 \in \text{TERM}$ , then  $t_1 = t_2 \in \text{FORM}$
- 3 If  $\phi \in \text{FORM}$ , then  $\neg\phi \in \text{FORM}$
- 4 If  $\phi, \psi \in \text{FORM}$ , then  $\phi \square \psi \in \text{FORM}$  (with  $\square \in \{\wedge, \vee, \Rightarrow\}$ )
- 5 If  $\phi \in \text{FORM}$  and  $x$  is a variable, then  $Qx.\phi \in \text{FORM}$  (with  $Q \in \{\forall, \exists\}$ )

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<sup>a</sup>The set of *predicate symbols*

# Variables (1)

## Definition (Free and Bound variables)

The *free* and *bound* variable occurrences in a formula are defined inductively by the following rules:

- 1 If  $\phi$  is *atomic*, then any variable  $x \in \text{Var}(\phi)$  is *free*
- 2  $x$  is *free/bound* in  $\neg\phi$  iff  $x$  is *free/bound* in  $\phi$
- 3  $x$  is *free/bound* in  $\phi \square \psi$  iff  $x$  is *free/bound* in either  $\phi$  or  $\psi$  (with  $\square \in \{\wedge, \vee, \Rightarrow\}$ )
- 4  $x$  is *free* in  $Qy.\phi$  iff  $x$  is *free* in  $\phi$  and  $y \neq x$
- 5  $x$  is *bound* in  $Qy.\phi$  iff  $x$  is *bound* in  $\phi$

## Variables (2)

- More easily, a variable  $x$  is *bounded* if it occurs in a quantification,  $x$  is *free* otherwise
- A variable can be both *free* and *bounded* in the same formula, e.g.

$$P(x, y) \Rightarrow \exists x. Q(x)$$

- 1 In the **LHS**  $x$  is *free*
  - 2 In the **RHS**  $x$  is *bounded*
  - 3 Even so, the formula is still *well-formed*
- A formula with no *free* variables is called a **sentence**

# Semantic

# Structure and Interpretation

## Definition (Structure)

A *structure* is formed by a *domain*  $D$ ,  $\mathbb{P} = \{P_1, \dots, P_n\}$  predicates on  $D$ ,  $\mathbb{F} = \{f_1, \dots, f_n\}$  **total** functions on  $D$  and a set  $\mathbb{C} \subseteq D$  of constants

## Definition (Interpretation)

Given a *structure*  $\mathfrak{D}$  and a *map*  $(\cdot)^{\mathfrak{D}}$  s.t.

- for all  $c$  in my language,  $(c)^{\mathfrak{D}} = c^{\mathfrak{D}} \in \mathbb{C}$
- for all  $k$ -ary function  $f$  in my language,  $(f)^{\mathfrak{D}} = f^{\mathfrak{D}} : D^k \rightarrow D \in \mathbb{F}$
- for all  $n$ -ary predicate  $P$  in my language,  $(P)^{\mathfrak{D}} = P^{\mathfrak{D}} \subseteq D^k \in \mathbb{P}$

we call  $\langle \mathfrak{D}, (\cdot)^{\mathfrak{D}} \rangle$  an *interpretation*.

# Evaluation (1)

- Given an *interpretation* and an **assignment**  $\bar{a}$ , it is possible to evaluate a formula
- The **evaluation** process *maps* the whole formula to a **truth value**
- The *assignment*  $\bar{a}$  associates each *free variable* with a *truth value*
- If the formula is a *sentence*,  $\bar{a}$  does not affect the *truth value* of the formula
- Next slides shows the evaluation steps

# Evaluation (2)

- ① Extend  $\bar{a}$  to all terms of the language with the following rules:
  - ▶ Each variable  $x$  evaluates to  $\bar{a}(x)$
  - ▶ Given  $\{t_1, \dots, t_n\} \in \text{TERM}$  evaluated to  $\{d_1, \dots, d_n\}$ , a function  $f(t_1, \dots, t_n)$  evaluates to  $(f)^{\mathfrak{D}^1}(d_1, \dots, d_n)$
- ② Assign each formula to a *truth value* with the following (inductive) rules:
  - ▶ (*Continues in next slides*)

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<sup>1</sup>Supposing we're evaluating  $f$  in a structure  $\mathfrak{D}$



## Evaluation (3)

- An *atomic formula*  $P(t_1, \dots, t_n)$  is associated with a *truth value*, depending on the truth of the following:

$$\langle v_1, \dots, v_n \rangle \in (P)^{\mathfrak{D}}$$

where  $v_1, \dots, v_n$  represents the evaluation of the predicate terms

- An *atomic formula*  $t_1 = t_2$  evaluates to a *truth value* depending if  $v_1 = v_2$  in  $D$ , where  $v_1, v_2$  represents the evaluation of the terms

## Evaluation (4)

- A formula containing *logical connectives* (e.g.  $\phi \Box \psi^2, \neg\phi$ ) is evaluated according to the *truth table* of the connective
- A formula  $\exists x.\phi$  is evaluated true iff exists an assignment  $\bar{a}'$  s.t. it differs from  $\bar{a}$  only for the assignment of  $x$  and  $\phi$  is evaluated true via the  $\bar{a}'$  assignment, false otherwise
- A formula  $\forall x.\phi$  is evaluated true iff exists an assignment  $\bar{a}'$  s.t. it differs from  $\bar{a}$  only for the assignment of  $x$  and  $\phi$  is evaluated true for all values in  $\bar{a}'$ , false otherwise

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<sup>2</sup>with  $\Box \in \{\wedge, \vee, \Rightarrow\}$

# Evaluation (5)

- Given a *structure*  $\mathfrak{D}$ , evaluation can be seen as a *map*

$$\rho_{\mathfrak{D}} : \text{Var} \rightarrow D$$

- We can use the  $\llbracket \cdot \rrbracket$  notation to express the evaluation of a *term*

## Definition

Given a structure  $\mathfrak{D}$  and  $\llbracket \cdot \rrbracket_{\rho_{\mathfrak{D}}} : \text{TERM} \rightarrow D$ , we can define

- $\llbracket c \rrbracket_{\rho_{\mathfrak{D}}} = c^{\mathfrak{D}}$
- $\llbracket x \rrbracket_{\rho_{\mathfrak{D}}} = \rho_{\mathfrak{D}}(x)$
- $\llbracket f(t_1, \dots, t_n) \rrbracket_{\rho_{\mathfrak{D}}} = f^{\mathfrak{D}}(\llbracket t_1 \rrbracket_{\rho_{\mathfrak{D}}}, \dots, \llbracket t_n \rrbracket_{\rho_{\mathfrak{D}}})$

# Satisfiability

## Definition (Satisfiability relation)

Given a *structure*  $\mathfrak{D}$  we can recursively define the *satisfiability relation*  $\models$  as:

- 1  $\rho_{\mathfrak{D}} \not\models \perp$
- 2  $\rho_{\mathfrak{D}} \models P(t_1, \dots, t_n) \Leftrightarrow \langle \llbracket t_1 \rrbracket_{\rho_{\mathfrak{D}}}, \dots, \llbracket t_n \rrbracket_{\rho_{\mathfrak{D}}} \rangle \in \mathbb{R}$
- 3  $\rho_{\mathfrak{D}} \models t_1 = t_2 \Leftrightarrow \llbracket t_1 \rrbracket_{\rho_{\mathfrak{D}}} = \llbracket t_2 \rrbracket_{\rho_{\mathfrak{D}}}$
- 4  $\rho_{\mathfrak{D}} \models (\phi \Rightarrow \psi) \Leftrightarrow \rho_{\mathfrak{D}} \not\models \phi \text{ o } \rho_{\mathfrak{D}} \models \psi$
- 5  $\rho_{\mathfrak{D}} \models (\phi \wedge \psi) \Leftrightarrow \rho_{\mathfrak{D}} \models \phi \text{ e } \rho_{\mathfrak{D}} \models \psi$
- 6  $\rho_{\mathfrak{D}} \models (\phi \vee \psi) \Leftrightarrow \rho_{\mathfrak{D}} \models \phi \text{ o } \rho_{\mathfrak{D}} \models \psi$
- 7  $\rho_{\mathfrak{D}} \models \forall x. \phi(x) \Leftrightarrow \forall a \in D : \rho_{\mathfrak{D}}[x/a] \models \phi$
- 8  $\rho_{\mathfrak{D}} \models \exists x. \phi(x) \Leftrightarrow \exists a \in D : \rho_{\mathfrak{D}}[x/a] \models \phi$

# Natural Deduction

- Just as reference, we introduce the *quantification rules* for the First-Order natural deduction system

$$\frac{\phi(x)}{\forall y.\phi(y)} \forall I$$

$$\frac{\phi(y)}{\exists x.\phi(x)} \exists I$$

$$\frac{\forall x.\phi(x)}{\phi(y)} \forall E$$

$$\frac{\begin{array}{c} [\phi(x)] \\ \Pi \\ \exists y.\phi(y) \end{array} \quad \psi}{\psi} \exists E$$

# Second-Order Logic

# Introduction

- As said before, Second-Order Logic *extends* First-Order Logic (respectively SOL and FOL from now on)
- The *terms* and *formulas* are pretty much the same of FOL
- To represent this "*extension*" we need a brief redefinitin of what we've seen in the previous section

# Syntax



# Alphabet

- Consists of the following symbols:
  - ▶ **Individual variables:**  $x_0, x_1, \dots$
  - ▶ **Individual constants:**  $c_0, c_1, \dots$
  - ▶ **Predicate variables:**  $X_0^n, X_1^n, \dots$
  - ▶ **Predicate constants:**  $\perp, P_0^n, P_1^n, \dots$
  - ▶ **Connectives:**  $\wedge, \vee, \Rightarrow, \neg$
  - ▶ **Quantifiers:**  $\forall, \exists$
  - ▶ **Auxiliary symbols:**  $(;); .$  (dot);  $,$  (comma)

# Formulas

- The set FORM of Second-Order *formulas* is inductively defined as follows
  - ▶  $X_i^0, P_i^0, \perp \in \text{FORM}$
  - ▶  $\forall n \in \mathbb{Z}^*. X^n(t_1, \dots, t_n) \in \text{FORM}$
  - ▶  $\forall n \in \mathbb{Z}^*. P^n(t_1, \dots, t_n) \in \text{FORM}$
- FORM is *closed* under **connectives** and **quantifiers**

# Semantic

## Definition (Second-Order Structure)

A *structure* is formed by a *domain*  $D$ , a set  $D^* = \langle D_n \mid n \in \mathbb{N} \rangle$  with  $D_n \subseteq \mathcal{P}(A^n)$ , a set  $\mathbb{R} = \{R_1^n, \dots, R_k^n\}$  of *predicates* s.t.  $R_i^n \in D_n$  and a set of *constants*  $\mathbb{C} \subseteq D$

- If  $D_n$  contains **all**  $n$ -ary predicates ( $D_n = \mathcal{P}(D^n)$ ) we call the structure *full*
- Even if the *elements* of a Second-Order structure are slightly different from the elements of a First-Order structure, we can use the same rules for **interpretation** and **evaluation**

# Satisfiability

## Definition (Second-Order Satisfiability relation)

Given a *Second-Order structure*  $\mathfrak{D}$  and a *language*  $\mathcal{L}$  that defines a name  $\bar{S}$  for all  $S \in D$ , we can define the *satisfiability relation*  $\models_2$  as:

- ①  $\rho_{\mathfrak{D}} \not\models_2 \perp$
- ②  $\rho_{\mathfrak{D}} \models_2 \bar{S}^n(\bar{s}_1, \dots, \bar{s}_n) \Leftrightarrow \langle s_1, \dots, s_n \rangle \in S^{na}$
- ③ All connectives follow the same rules of First-Order Logic
- ④ Quantification over a *variable* follow the same rules of First-Order Logic
- ⑤  $\rho_{\mathfrak{D}} \models_2 \forall X_i^n. \phi(P_i^n) \Leftrightarrow \forall S^n \in D_n : \rho_{\mathfrak{D}} \models \phi(S^n)$
- ⑥  $\rho_{\mathfrak{D}} \models_2 \exists X_i^n. \phi(P_i^n) \Leftrightarrow \exists S^n \in D_n : \rho_{\mathfrak{D}} \models \phi(\bar{S}^n)$

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<sup>a</sup>By using this notation, we can handle all types of *predicates* with one rule

# Natural Deduction

- We need to add a set of *rules* that allows to validly **derive** the new "extended" *quantifications*

$$\frac{\phi}{\forall X^n.\phi} \forall^2 I$$

$$\frac{\phi^*}{\exists X^n.\phi} \exists^2 I$$

$$\frac{\forall X^n.\phi}{\phi^*} \forall^2 E$$

$$\frac{\begin{array}{c} [\phi] \\ \Pi \\ \exists X^n.\phi \end{array} \quad \psi}{\psi} \exists^2 E$$

- $\phi^*$  is  $\phi[X^n(t_1, \dots, t_n)/\psi(t_1, \dots, t_n)]$ , where  $\psi$  is a generic formula
- No  $t_i$  becomes *bounded* during the substitution above, so  $\psi$  cannot *quantify* any term  $t_1, \dots, t_n$

# Comprehension schema (1)

- States that "Any definable subclass of a set is a set", formally:

## Definition (Axiom schema of comprehension)

Given a *formula*  $\psi$  with  $FV[\psi] \in \{x, t_1, \dots, t_n\}$ , and a set  $A$ , the following holds

$$\forall t_1, \dots, t_n. \forall A. \exists B. \forall x. (x \in B \Leftrightarrow (x \in A \wedge \psi(x, t_1, \dots, t_n, A)))$$

- Since the schema holds for all  $x$  and for all  $A$ , for the generality of  $\psi$ , it is always possible to define a set from another set
- $\exists^2 I$  gives us this schema, since we can substitute  $P^n$  with  $\psi$ , obtaining the "filtered" subset

## Comprehension schema (2)

### Proof idea.

Since we can derive

$$\forall t_1, \dots, t_n. (\phi(t_1, \dots, t_n) \Leftrightarrow \phi(t_1, \dots, t_n))$$

the following is correct

$$\frac{\forall t_1, \dots, t_n. (\phi(t_1, \dots, t_n) \Leftrightarrow \phi(t_1, \dots, t_n))}{\exists X^n. \forall t_1, \dots, t_n. (\phi(t_1, \dots, t_n) \Leftrightarrow P^n(t_1, \dots, t_n))} \exists^2 I$$



- **Strong result!** We can derive *Second-Order quantification* the same way our deduction system derived *First-Order quantification*
- Same proof concept can be applied for  $\forall^2 E$ , since we can define  $\forall^2$  from  $\exists^2$



# Flattening SOL (1)

- Given the *First-Order predicates*  $Ap_0, Ap_1, \dots$  s.t.  $Ap_n$  is  $(n + 1)$ -ary
- $Ap_n(X, t_1, \dots, t_n)$  can be seen as  $X^n(t_1, \dots, t_n)$ , so  $Ap_0$  is the First-Order version of  $X^0$
- We also use some *unary predicates*
  - ▶  $V \rightarrow$  "is an element"
  - ▶  $U_0 \rightarrow$  "is an 0-ary predicate"
  - ▶  $U_1 \rightarrow$  "is an 1-ary predicate"
  - ▶ *and so on...*

## Flattening SOL (2)

- ①  $\forall x, y, z. (U_i(x) \wedge U_j(y) \wedge V(z) \Rightarrow x \neq u \wedge y \neq z \wedge z \neq x)$  for all  $i \neq j$
- ②  $\forall X, y_1, \dots, y_n. (Ap_n(X, y_1, \dots, y_n) \Rightarrow U_n(P) \wedge \bigwedge_i V(y_i))$  for  $n \geq 1$
- ③  $U_0(C_0, V(C_{2^{i+1}}))$  for  $i \geq 0$  and  $U_n(C_{3^{i5^n}})$  for  $i, n \geq 0$   
 $\forall z_1, \dots, z_m. \exists P. ($ 
  - ④  $U_n(X) \wedge \forall y_1, \dots, y_n. \left( \bigwedge V(y_i) \Rightarrow (\phi^* \Leftrightarrow Ap_n(X, y_1, \dots, y_n)) \right)$   
 $)$  where  $X \notin FV[\phi^*]$ ,  $FV[\phi] \in \{z_1, \dots, z_m, y_1, \dots, y_n\}$
- ⑤  $\neg Ap_0(C_0)$

## Flattening SOL (3)

- 1 The  $i$ -ary and  $j$ -ary predicates are pairwise disjoint and disjoint from the element
- 2 If  $\langle X, y_1, \dots, y_n \rangle \in Ap_n$  then  $X$  is a *predicate* and  $y_1, \dots, y_n$  are it's *elements*
- 3 We can have both *element* and *predicate* constants
- 4 First-Order equivalent of the *comprehension schema*
- 5 The 0-ary predicate for "*false*", equivalent to  $\perp$

# Flattening SOL (4)

- SOL can be translated preserving **derivability**
- We can assign symbols to the ones in SOL alphabet, in order to convert strings inductively

- ▶  $(x_i)^* \leftarrow x_{2^{i+1}}$
- ▶  $(c_i)^* \leftarrow c_{2^{i+1}}$
- ▶  $(X_i^n)^* \leftarrow x_{3^i 5^n}$
- ▶  $(P_i^n)^* \leftarrow c_{3^i 5^n}$
- ▶  $(X_i^0)^* \leftarrow Ap_0(x_{3^i})$
- ▶  $(P_i^0)^* \leftarrow Ap_0(c_{3^i})$
- ▶  $(\perp)^* \leftarrow Ap_0(c_0)$

with  $i, n \geq 0$

# Flattening SOL (5)

- Doing this, we can "*translate*" Second-Order formulas as follows:

- ▶  $(\phi \Box \psi)^* \leftarrow \phi^* \Box \psi^*$
- ▶  $(\neg \phi)^* \leftarrow \neg \phi^*$
- ▶  $(\forall x_i. \phi(x_i))^* \leftarrow \forall x_i^*. (V(x_i^*) \Rightarrow \phi^*(x_i^*))$
- ▶  $(\exists x_i. \phi(x_i))^* \leftarrow \exists x_i^*. (V(x_i^*) \wedge \phi^*(x_i^*))$
- ▶  $(\forall X_i^n. \phi(X_i^n))^* \leftarrow \forall (X_i^n)^*. (U_n((X_i^n)^*) \Rightarrow \phi^*((X_i^n)^*))$
- ▶  $(\exists X_i^n. \phi(X_i^n))^* \leftarrow \exists (X_i^n)^*. (U_n((X_i^n)^*) \wedge \phi^*((X_i^n)^*))$

# Model

## Definition (Model)

Given a *structure*  $\mathfrak{D}$ , it is called a **model** of SOL if it admits (a.k.a is *valid*) the *comprehension schema*.

If  $\mathfrak{D}$  is *full*, then it is called a **principal** (or *standard*) **model**

- From the definition above we get two distinct notions of "validity" in SOL
  - ▶ true in all models
  - ▶ true in all *principal* models
- We'll use the first one as default