Republic of the Philippines

UNIVERSITY OF RIZAL SYSTEM

Province of Rizal

DIFFERENTIAL CALCULUS

Author

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Nurturing Tomorrow's Noblest

Course Guide

MATH 2E: DIFFERENTIAL CALCULUS

1ST Semester, AY 2020 - 2021

INTRODUCTION

I welcome you all to this online class in Differential Calculus. This is a three (3) unit course with fifty four (54) hour time allotment the whole semester. Many were asking when they use Calculus in the real world. Well, this subject finds applications in many disciplines like physics, engineering, economics, statistics, and medicine. But why is it most people find the subject very hard? Calculus by itself is an introductory course, but solution of many problems requires ones knowledge and competencies in Algebra, Trigonometry and both Solid and Analytic geometry, making it appears to be a tough course. But don't you worry guys; you will not be left alone for I will be with you all the way as your course facilitator this 1st Semester of Academic Year 2020-2021. With the COVID-19 pandemic ravaging the globe and with the necessity of observing physical distancing measures, the traditional schooling experience that requires students to attend classes in person is prohibited. But, it is truly important and is urgent indeed at this time of crisis to safeguard your generation of students of the opportunity to learn and be educated. As mandated by law amidst this pandemic situation, educational institutions have to go for online teaching and learning even though its implementation posed different problems and challenges to both the teachers and students. Educational institutions and this includes the University of Rizal System, were given the academic freedom by the Commission on Higher Education to implement available distance learning, e-learning and other alternative modes of delivery to students. This is said to be the new normal in education, using blended and flexible learning modalities. I dearly advise you all to practice hard work, perseverance, patience and all other positive attitudes needed to facilitate understanding of the modules and instructional videos especially developed for your needs and I do hope these learning materials will develop in you the sense of independent learning. So, hang on guys, and hope you will all enjoy these online learning experiences.

THE FACULTY

Hi everyone! Yours truly is DR. ELVIRA C. CATOLOS, Professor IV at the University of Rizal System (URS)-Morong Campus. I am an engineer by profession; but, God's plan for me is to become a teacher, spending the 36 years of my existence passionately mentoring and molding young minds. I got my bachelor degree in Chemical Engineering from the University of Santo Tomas in Manila, earned my Master of Engineering Education, major in Chemical Engineering from the University of the Philippines in Diliman, Quezon City as an EDPITAF scholar; and, my Doctorate Degree in Educational Management from the University of Rizal System where I was an Academic Excellence and Best Research Award Recipient. The GURONASYON Foundation, Inc. in the Province of Rizal gave me the recognition as the 2014 Most Outstanding College Faculty and by the Local Government Unit of Tanay, Rizal as the Natatanging Anak ng Tanay sa Larangan ng Edukasyon during the 2015 Parangal sa Natatanging Anak ng Tanay (PANATA). The University of Rizal System bestowed me with the Exemplary Behaviour Award and the Best Research Award (Developmental Category), respectively, during the September 2019 and 2012 Program on Awards and Incentives for Service Excellence (PRAISE) awarding ceremonies. My passion for teaching becomes my catalyst to author books in Plane and Spherical Trigonometry, Analytic Geometry, General Mathematics for Senior High School, Mathematics in the Modern World; and to develop more instructional materials, namely: Worktext in Precalculus, Worktext in Basic Calculus, Worktext in Differential Calculus; and, Worktext in Integral Calculus which are all copyrighted and with ISBN.

COURSE DESCRIPTION

Differential Calculus is an introductory course covering the core concepts of limit, continuity and differentiability of functions involving one or more variables. This also includes the application of differential calculations in solving problems on optimization, rates of change, related rates, tangents and normal, curve tracing, and approximations.

COURSE OBJECTIVES

To understand the core concepts of limit, continuity and differentiability of functions involving one or more variables, the differentiation process of algebraic and transcendental functions and apply derivatives of functions in solving problems on rates of change, rectilinear motion, angle of intersection of curves, optimization, time-rates, tangents and normal; and, limit evaluation using L' Hospital's Rule.

SPECIFIC OBJECTIVES:

After completing this course, the students must be able to:

1. Understand concept of function, its, domain, range, range of correspondence, graph and limit.
2. Evaluate limit of a function analytically and graphically.
3. Identify the point of discontinuity of a function.
4. Understand concept of increment and derivative
5. Solve problem on rate of change using derivative of a function.
6. Know the differentiation formulas for algebraic and transcendental functions and be able to apply them on problems on rectilinear motion, angle of intersection of curves, optimization, time-rates, tangents and normal; and, limit evaluation using L' Hospital's Rule.
7. Understand the concept of partial differentiation and apply to it in getting partial derivative of a function in two or more variables.

COURSE PREREQUISITE

This is an introductory course, it has no pre-requisite course. However, students are expected to be equipped with the essential knowledge and competencies in Algebra, Trigonometry, Solid and Analytic Geometry to facilitate learning of the course.

COURSE MATERIALS

At the start of the semester, your email address will be asked from you so I can invite you to join the Google classroom that will be created for our class and each of you needs to log-in to have your own student account. Our class group chat will also be created to facilitate sharing of messages and information. Furthermore, we can also enroll on the URS Learning Management System (LMS) and log-in at agreed day and time for chat is most welcome. This LMS is called MOODLE (Modular Object–Oriented Dynamic Learning Environment. The MOODLE can be accessed if you log-in to http://www.urs.....and click on the MOODLE link. Course enrolment key will be provided or can be requested from the

Technical Support of the URS Management Information System (MIS).Student should have access to the internet since online and asynchronous discussion and sharing, concerns and inquiries may be sent to an online platform to be identified later. Aside from the modules especially developed for your need, instructional videos on the different modules can be viewed at my you tube channel. And, I suggest also that you browse the internet so you may get hold of other supplementary reading materials to enhance your learning. All these could be of help to you while preparing your assignments and activities. Our virtual classroom, GC and LMS will serve as the avenue so you can access with copy of the modules especially developed for your need, copy of the course guide, can submit all course requirements and even can take quiz and examination online.

COURSE STRUCTURE

The entire course consists of six (6) units divided into twenty (20) modules.

UNIT 1 - Function and Limit of a Function

Module 1. Relation and Function
Module 2. Function, its Continuity and Limit

UNIT 2 – Increment and Derivative of a Function

Module 3. Increment and Derivative
Module 4. Rate of Change

Unit 3 - Derivative of Algebraic Function

Module 5. Differentiation Formulas for Algebraic Function
Module 6. Slope of Tangent and Normal Line
Module 7. Angle of Intersection of Curves
Module 8. Rectilinear Motion
Module 9. Higher Order Derivative
Module 10. Implicit Differentiation
Module 11. Chain Rule of Differentiation
Module 12. Maximum and Minimum Value of Function
Module 13. Optimization Problems
Module 14. Time-Rates

UNIT 4 – Derivative of Transcendental Function

Module 15. Derivatives of Trigonometric Function and their Applications
Module 16. Derivatives of Inverse Trigonometric Function and their Applications
Module 17. Derivatives of Exponential and Logarithmic Functions and their Applications
Module 18. Derivative of a Variable Raised to Another Variable

Unit 5 – Indeterminate Forms

Module 19. Concept of Indeterminate Forms

Unit 6 – Partial Differentiation

Module 20. Concept of Partial Differentiation

COURSE SCHEDULE

DATE	ACTIVITY	MEDIUM
August 24, 2020	Creation of class group chat, Course Orientation/URS Mission, Vision, Quality Policy, URS Hymn	LMS
August 26, 2020	Reading and Viewing of instructional video of Module 1 and Module 2	Study From Home
September 2, 2020	Open Forum on the submitted answers to SAQ and Activity	Zoom platform/ Class group chat
September 7, 2020	Quiz No. 1	Google Classroom
September 14, 2020	Reading and Viewing of instructional video of Module 3 and Module 4	Study From Home
September 16, 2020	Open Forum on the submitted answers to SAQ and Activity and Quiz No. 2	Zoom platform Class group chat
September 21, 2020	Reading and Viewing of instructional video of Module 5 and Module 6	Study From Home
September 23, 2020	Open Forum on the submitted answers to SAQ and Activity and Quiz No. 3	Zoom platform Class group chat
September 28, 2020	Reading and Viewing of instructional video of Module 7	Study From Home
September 30, 2020	**Preliminary Examination**	Google Classroom
October 5, 2020	Reading and Viewing of instructional video of Module 8 and Module 9	Study From Home
October 7, 2020	Open Forum on the submitted answers to SAQ and Activity and Quiz No. 4	Zoom platform Class group chat Google Classroom
October 12, 2020	Reading and Viewing of instructional video of Module 10 and Module 11	Study From Home
October 14, 2020	Open Forum on the submitted answers to SAQ and Activity and Quiz No. 5	Zoom platform Class group chat Google Classroom
October 19, 2020	Reading and Viewing of instructional video of Module 12	Study From Home
October 21, 2020	Open Forum on the submitted answers to SAQ and Activity	Zoom platform Class group chat
October 26, 2020	Quiz No. 6	Google Classroom
October 28, 2020 November 2, 2020	Reading and Viewing of instructional video of Module 13 and Module 14	Study From Home
November 4, 2020	Open Forum on the SAQ and Activity	Zoom platform Class group chat

November 9, 2020	Quiz No. 7	Google Classroom
November 11, 2020	**Mid-term Examination**	Google Classroom
November 16, 2020	Reading and Viewing of instructional video of Module 15 and Module 16	Study From Home
November 18, 2020	Open Forum on the SAQ and Activity	Zoom platform Class group chat
November 23, 2020	Quiz No. 8	Google Classroom
November 25, 2020	Reading and Viewing of instructional video of Module 17 and Module 18	Study From Home
November 30, 2020	Open Forum on the SAQ and Activity	Zoom platform Class group chat
December 2, 2020	Quiz No. 9	Google Classroom
December 7, 2020	Reading and Viewing of instructional video of Module 19 and Module 20	Study From Home
December 9, 2020	Open Forum on the SAQ and Activity and Quiz No. 10	Zoom platform Class group chat
December 11, 2020	Final Examination	Google Classroom

COURSE REQUIREMENTS AND GRADING SYSTEM:

To pass the course, the students should satisfactorily pass the following requirements according to the existing University rules and regulations. These course requirements are to be taken/submitted using any of the online platforms like messenger, e-mail, University LMS

CLASS STANDING	……………………..		**60%**
▪ Quizzes	……………	30%	
▪ Projects (Activity Sheets)	……………	20%	
▪ Recitation	……………	10%	
MAJOR EXAMINATION	………………………		**40%**
TOTAL			**100%**

ACADEMIC INTEGRITY

Academic dishonesty: Any form of cheating or plagiarism in this course will result in zero on the exam, assignment or project. Allowing others access to your work potentially involves you in cheating. Working with others to produce very similar reports is plagiarism regardless of intent.

FOR QUESTIONS AND INQUIRIES: email address: elviracatolos@gmail.com
FB Account: Elvira Catolos

MODULE 1

RELATION AND FUNCTION

Specific Objectives:

At the end of the module, students must be able to:

1. Understand concept of function, its, domain, range, range of correspondence and graph.
2. Differentiate a relation from a function.
3. To determine the domain and range using analytical and graphical method.
4. Evaluate a function at a given value of the independent variable.

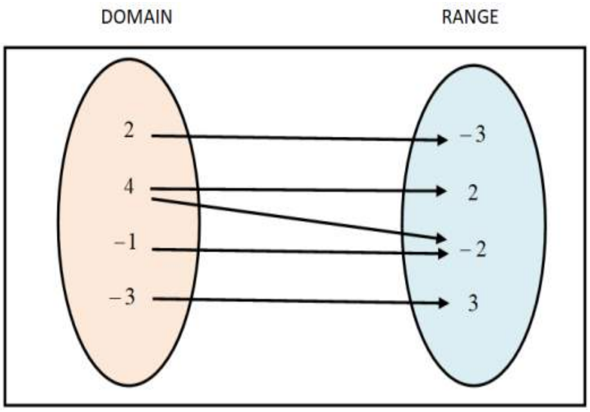
INTRODUCTION

In our daily living, we often encounter quantities that do come in pair. For example, the number of kilograms of rice and the amount of money needed to purchase. Furthermore, the number of miles a car travelled and the liters of gasoline consumed. Likewise, the plant growth in centimeters and the amount of rainfall it received. When one quantity changed, the other also changed. These pairings are best represented as ordered pairs.

RELATION

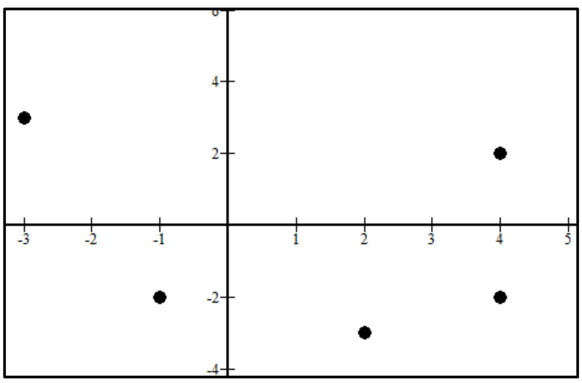
If we let the ordered pair be(x, y), we call a set of ordered pairs as a **relation**. The set of all the first elements (the values of x) in the ordered pairs is referred to as the **domain** of the relation while the set of all the second elements (the values of y) forms the **range**. Thus, in a relation, there is a correspondence between the domain and range, such that to each element of the domain there is assigned one or more elements of the range.

The given mapping diagram better explains the definition of relation, its domain and range. This relation consists of five ordered pairs, namely: $(2,-3), (4,2), (4,-2)$, $(-1,-2)$ and$(-3,3)$. Its domain is set $\{2,4,-1,-3\}$ and its range is set$\{-3,-2,2,3\}$.



GRAPH OF A RELATION

There is a one-to-one correspondence between the ordered-pairs $(x, y)$and the points on the rectangular or Cartesian plane. Each point on the plane corresponds to one and only one ordered pair(x, y). While the domain of a relation is usually apparent from the definition of the relation, the range is often determined from its graph. The graph of the above relation consisting of points is shown at the right.

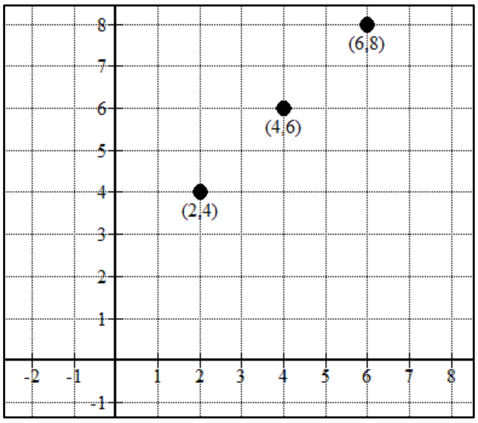


RULE OF CORRESPONDENCE

The rule of correspondence is any equation describing how the elements of the domain and range of any relation are paired. It virtually gives the range of the relation. Let us consider three relations described by the same rule of correspondence but having different domains.

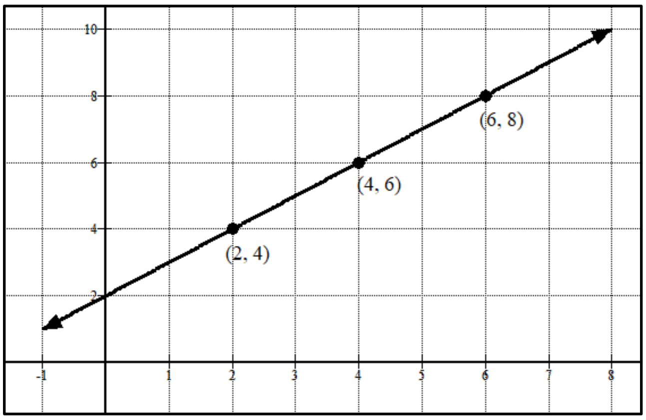
Example 1. Given: Relation$A = \{(x, y)|y = x + 2, x = 2, 4, 6\}$.

Relation A is a set of ordered pairs consisting of all the possible pairings of the elements of the domain and range that are formed according to the given rule of correspondence. Hence, the elements of relation A are ordered pairs$(2, 4), (4, 6), \text{and } (6, 8)$. Thus, the domain of relation A is set$\{2, 4, 6\}$, its range is set$\{4, 6, 8\}$ and its graph consists of only three points.



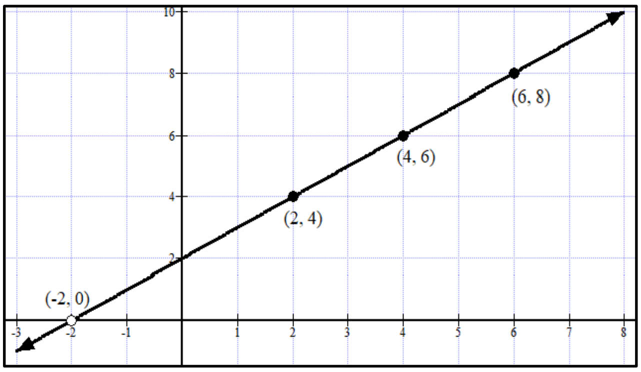
Example 2. Relation $B = \{(x, y)|y = x + 2. x, y \in R\}$.

Relation B which can simply be denoted as $B = \{(x, y)|y = x + 2\}$ consists of an infinite number of ordered pairs. It is a general rule that if the domain is not indicated, it means that it consists of all real numbers without any exception. Any real number that is excluded in the domain must be clearly indicated in the notation used. This matter is exhibited on the graph of relation B which is a line represented by linear equation $y = x + 2$ extending indefinitely up to the right and down to the left. The domain of relation B is $\{x|x \in R\}$, where R the real number set is and its range is$\{y|y \in R\}$. Its graph is shown below.



Example 3. Relation $C = \{(x,y)|y = x + 2. x \neq -2\}$.

It is understood that the domain of C consists of all real values except $x = -2$. Hence, point $(-2, 0)$is not in set C. This fact is revealed on the graph by drawing an open circle around the point. Therefore, domain $\{x|x \neq -2\}$and range $\{y|y \neq 0\}$belong to relation D.



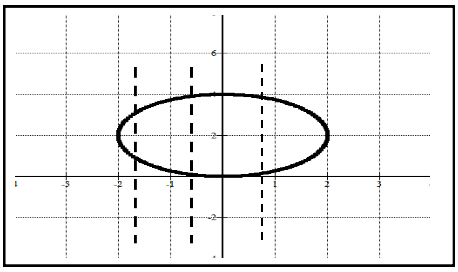
FUNCTION

Function is a special kind of relation. It is a set of ordered-pairs (x, y) of real numbers in which no two pairs have the same first element x. Furthermore, it is a relation in which each x-element has only one y-element associated with it. Relations A, B and C described on the above examples are all functions since for every value of the first element x, there is one and only one corresponding value of the second element y.

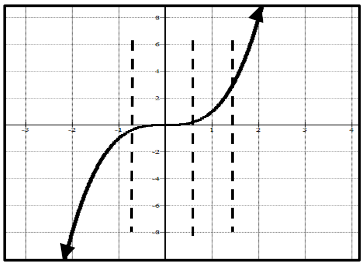
VERTICAL LINE TEST

The vertical line test tells whether a relation is a function. Given the graph of the relation, if every vertical line drawn crosses the graph in only one point, then, the relation is a function. On the contrary, if one can draw a vertical line that goes through two points, y is not a function of x.

The graph of relations W and Q shown below reveals that W is a function since any vertical line drawn through its graph intersects it in one and only one point. Moreover, Q is not a function since any vertical line drawn through its graph crosses it in more than one point.



Graph of Q



Graph of W

CONSTANT AND VARIABLE

In Mathematics, a **constant** is a quantity that maintains a fixed value throughout a particular problem. Absolute constants such as $\pi, e, \sqrt{2}, 3$ retain the same values in all problems. Arbitrary constants remain constant in a particular problem but may assume different values in other problems.

A **variable** is a quantity that may assume various values in the course of a problem. In equation $y = 1 - x$, letter x whose values would be freely assumed is called the independent variable and letter y whose value depends on the assumed value of x is called the dependent variable.

FUNCTION NOTATION

To be able to discuss functions and their properties, we use a symbol, usually a letter of the alphabet to stand for a function. The most often used are $f, g, H, M, \alpha, \beta$. Sometimes, subscripts are employed so that, for example f_1, f_2, f_3 and f_4 would stand for four different functions. To write a function, we enclose the independent variable in parentheses preceded by a chosen letter. In symbol form, $f(x)$, read "function of x", with the chosen letter f indicating that there exists a relationship between variable x and another variable.

In equation $y = -\sqrt{x+1}$, $y = \alpha(x)$ is read "y is a function of x", with the Greek letter α indicating a relationship between dependent variable y and independent variable x, hence, the ordered pair (x, y) can be denoted by $[x, \alpha(x)]$ or $\left(x, -\sqrt{x+1}\right)$. Function α is single-valued function.

Moreover, in function $f(x) = \sqrt{x+1}$, $f(x)$ is a double-valued function. For example, when $x = 3$, $f(x) = \pm 2$.

A function that depends on two or more independent variables is symbolically represented in a similar manner. Hence, a function of variables x and z is written as $\beta(x, z)$ and is read function "β of x and z". The function $\beta(x, z)$ when $x = 2$ and $z = 0$ is denoted by $\beta(2, 0)$.

FUNCTION EVALUATION

This is the process of finding value of function, say $h(x)$, given value of the independent variable x. The notation $h(-1)$ refers to the value of function h when $x = -1$. Likewise, in $f(x, y)$, $f(-2, 4)$ means the value of the function f when $x = 2$ and $y = -4$.

Example 4. Suppose that f is a function defined by the equation $f(x) = x^2 - 2x - 3$. Evaluate $f(0), f(-1), f(-2), f(1), f(2), f(3), f[f(x)]$. Draw the graph of f for the portion of the domain $-2 \le x \le 3$.

Solution: Substituting the given value of the independent variable , we have

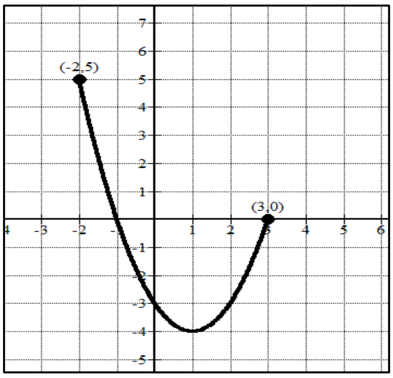
- $f(0) = (0^2 - 2(0) - 3 = -3$
- $f(-1) = (-1)^2 - 2(-1) - 3 = 1 + 2 - 3 = 0$

- $f(-2) = (-2)^2 - 2(-2) - 3 = 4 + 4 - 3 = 5$
- $f(1) = (1)^2 - 2(1) - 3 = -4$
- $f(2) = (2)^2 - 2(2) - 3 = 4 - 4 - 3 = -3$
- $f(3) = (3)^2 - 2(3) - 3 = 9 - 6 - 3 = 0$
- $f[f(x)] = f(x^2 - 2x - 2) = (x^2 - 2x - 3)^2 - 2(x^2 - 2x - 3) - 3$
  $= x^4 + 4x^2 + 9 - 4x^3 - 6x^2 + 12x - 2x^2 + 4x + 6 - 3$
  $= x^4 - 4x^3 - 4x^2 + 16x + 12$

Tabulating the x values and the corresponding y or $f(x)$ values,

x	-2	-1	0	1	2	3
$f(x) = y$	5	0	-3	-4	-3	0
(x, y)	$(-2,5)$	$(-1,0)$	$(0,-3)$	$(1,-4)$	$(2,-3)$	$(3,0)$

The graph of $f(x) = x^2 - 2x - 3$ is a parabola with vertex at $(1, -4)$.



Example 5. Find the value of $\dfrac{f(x+h) - f(x)}{h}, h \neq 0$, given function $f(x) = \dfrac{1}{x^2}$.

Solution: Evaluate $f(x+h) = \dfrac{1}{(x+h)^2}$

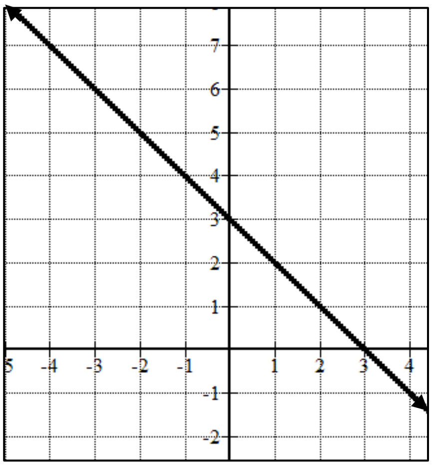
Therefore, $\dfrac{f(x+h) - f(x)}{h} = \dfrac{\dfrac{1}{(x+h)^2} - \dfrac{1}{x^2}}{h} = \dfrac{\dfrac{x^2 - (x+h)^2}{x^2(x+h)^2}}{h}$

$$= \frac{f(x+h) - f(x)}{h} = \frac{x^2 - x^2 - 2hx - h^2}{hx^2(x+h)^2} = \frac{-h(2x+h)}{hx^2(x+h)^2} = \frac{-(h+2x)}{x^2(x+h)^2}$$

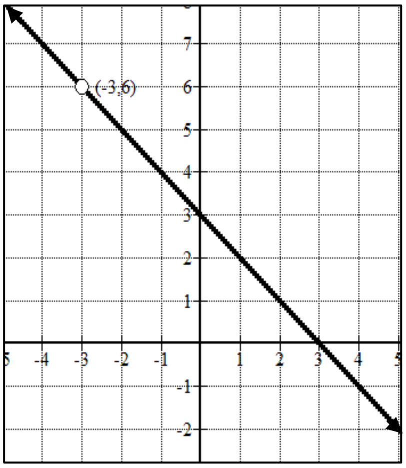
Example 6. Discuss the distinction between the given functions $H(x)$ and $G(x)$ defined $H(x) = \dfrac{9 - x^2}{3 + x}$, and, $G(x) = 3 - x$

Solution:

At the first glance, it appears that the functions are the same since $9-x^2$ is factorable. However, the domain of $G(x)$ is $x \in R$, meaning, x is any real number. However, for the function $H(x)$, the values of both numerator and denominator are zero when $x=-3$. Therefore, $H(x)$ and $G(x)$ are identical for all x-values except $x=-3$. The graph of $H(x)$ has an open circle drawn around the point $(-3, 6)$ since this point does not lie on its graph.



Graph of $G(x) = 3 - x$



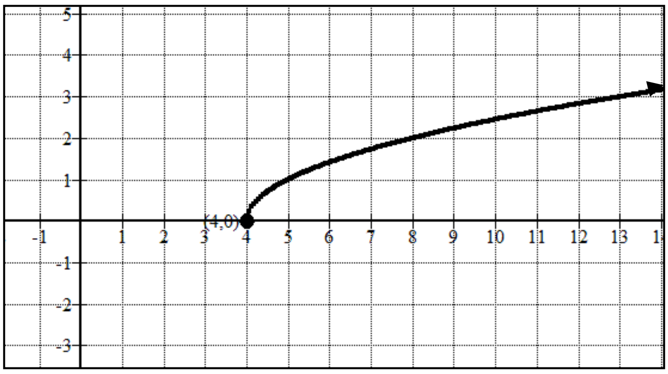
Graph of $H(x) = \frac{9-x^2}{3+x}$, $x \neq -3$

Example 7: Find the domain and range of function $\beta(x)=\sqrt{x-4}$.

Solution:

The function $\beta(x)=\sqrt{x-4}$ is defined only at x-values equal or greater than 4. That is, for the function to be a real number, the radicand $x-4 \geq 0$ or $x \geq 4$ Hence, the domain of the function is $\{x \mid x \geq 4\}$.

The definition of the given function shows that at values of x in the interval $x \geq 4$ corresponding value of the function is zero or more than zero. That is same as saying the range of the function is $\{y|y \geq 0\}$. The graph of function is the upper half of the parabola with vertex at $(4,0)$.

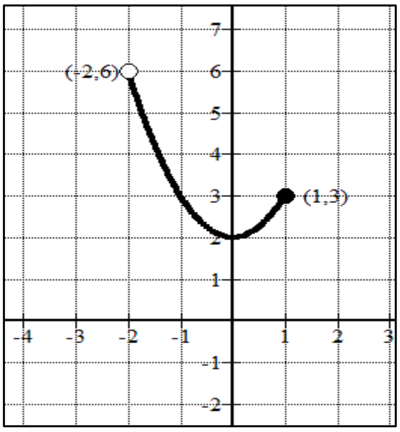


Example 8. Find the domain and range for the function defined as $f(x) = x^2 + 2$, for $-2 < x \leq 1$.

Solution:

The domain of the function $f(x) = x^2 + 2$ is $\{x \mid -2 < x \leq 1\}$. To find the range, when $x = -2$, $f(-2) = 6$. It could be observed from the graph that range is all real numbers more than 2 but less than 6. In symbol form, range is $\{y \mid 2 \leq y < 6\}$. The graph has an open circle at $(-2,6)$ indicating that the domain excludes $x = -2$ and the range does not include $y = 6$.

The graph of the function is a portion of parabola $f(x) = x^2 + 2$ having vertex at $(0,2)$ opening upward.



Note: If the rule of correspondence defining a given function does not explicitly point out the domain, one should be sharp enough to identify it. Say for example, $f(x) = \frac{x}{x^2 - 4}$ is a function defined for all values of x except $x = \pm 2$, since division by zero is undefined. Similarly, if $h(x) = \sqrt{1 - x^2}$, the domain consists of x values that satisfy the quadratic inequality $1 - x^2 \geq 0$. Solution of this inequality and the domain of the function is the interval $-1 \leq x \leq 1$. The graph of the function is the upper half of the circle having center at the origin and of radius equal to one.

PIECEWISE-DEFINED FUNCTION

This is a function whose domain is divided into parts and each part is defined by a different function rule. It is defined on a series of intervals. The word piecewise is used to describe any property of a piecewise-defined function that holds for each piece but may not hold for the whole domain of the function.

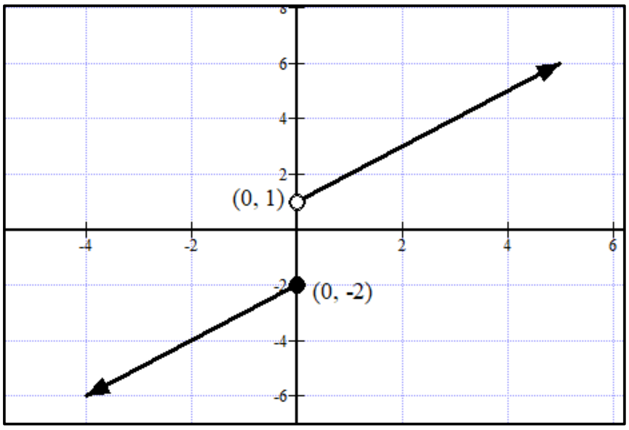
A common piecewise-defined function is the absolute value.

$$|x| = \begin{cases} -x & , x < 0 \\ 0 & , x = 0 \\ x & , x > 0 \end{cases}$$

Example 9. Find the domain and range of given piecewise-defined function $f(x)$. Draw its graph and find value of f when $x = -3$ and $x = 6$.

$$f(x) = \begin{cases} x+1 & ,x > 0 \\ x-2, & ,x \leq 0 \end{cases}$$

Solution: Based on the given parts of the domain, we say that the domain of the given piecewise-defined function is $\{x \mid x \in R\}$. Let us draw the graph of the given function $f(x)$.

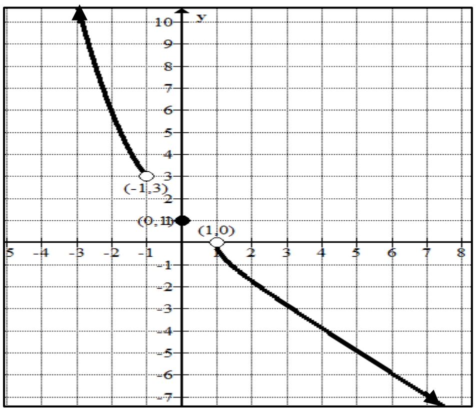


The graph above shows the range of $f(x)$ is $\{y|y \leq -2 \cup y > 1\}$ and when $x = -3$, $f(-3) = -3 - 2 = -5$. Furthermore, when $x = 6$, $f(6) = 6 + 1 = 7$.

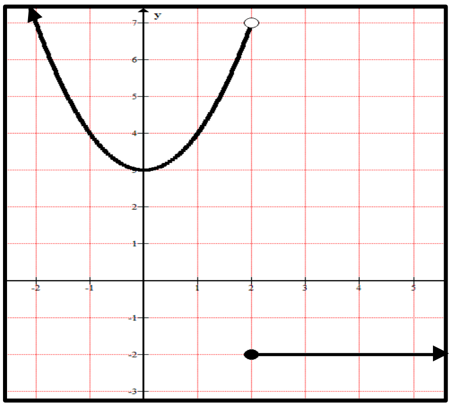
Example 10. Find the domain and range of function defined below and draw its graph.

$$H(x) = \begin{cases} -\sqrt{x^2 - 1} & ,x > 1 \\ 1 & ,x = 0 \\ x^2 + 2 & ,x < -1 \end{cases}$$

The domain of the given function is $\{x|x > 1 \cup x = 0 \cup x < -1\}$. Below is the graph of $H(x)$. Based on the graph of the function $H(x)$ shown below, it is evident that the range is $\{y|y > 3 \ \cup y = 1 \ \cup y < 0\}$.

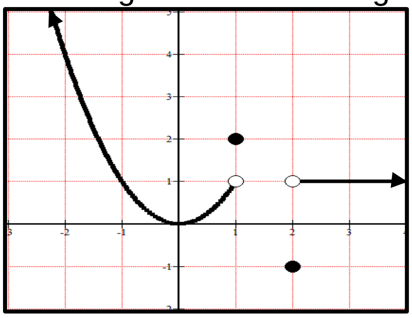


Example 11: Find the domain and range of piecewise-defined function $H(x)$ whose graph is shown below. Evaluate $H(2)$.



Solution: Domain $D = \{x|x \in R\}$and the Range is $\{y|y \geq 3 \cup y = -2\}$. From the given graph, the value of the function when $x = 2$ is $H(2) = -2$.

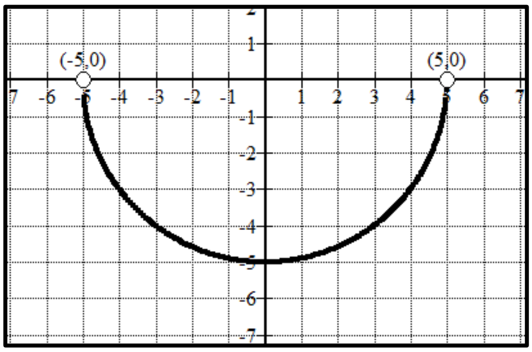
Example 12. Find the domain and range of the function graphed below.



Solution: The domain is $\{x|x \geq 2 \ \cup x \ \leq\leq 1\}$ while the range is $\{y|y = -2 \cup y \geq 0\}$.

Example 13. Find the domain and range of $y = -\sqrt{25 - x^2}$.

Solution: The graph of the function is the lower half of circle $x^2 + y^2 = 25$ having its center at the origin $(0, 0)$ and radius equal to 5. For the value of y to be real, $25 - x^2 > 0$. This inequality has solution $-5 < x < 5$. Hence, the domain of the function is $\{x| -5 < x < 5\}$. The value $x = 5$ and $x = -5$ are excluded on the domain as indicated by the open circles at those values of x. And from its graph below, it is evident that the range of the function is $\{y| -5 \leq y < 0\}$.



SAQ1

ACTIVITY 1.1 – A

NAME: ____ SCORE: _____________

SECTION: __________DATE: ______________ PROF: _______________________________

Identify which of the following representations is/are a function and not a function. Write F if a function and NF if not a function on the space provided before each number.

__________ 1. $A = \{(1,4),\ (4,7),(9,12),(13,16)\}$

__________ 2. $B = \{(0,1),\ (2,5),(3,10),(4,17),(5,26),(6,37)\}$

__________ 3. $C = \{(0,10),(6,8),(8,6),(6,-8)\}$

__________ 4. $D = \{\{(x,y)|y = x + 1\}\}$

__________ 5. $E = \{\{(x,y)|y = x^2 - 1\}\}$

__________ 6. $F = \left\{\{(x,y)|y = -\sqrt{x+2}\}\right\}$

__________ 7. $G = \left\{\left\{(x,y)\middle|y = \frac{1}{x}\right\}\right\}$

__________ 8. $H = \left\{\left\{(x,y)\middle|y = \frac{x+3}{x^2-4}\right\}\right\}$

__________ 9. $I = \{\{(x,y)|y^2 = 4 - x^2\}\}$

__________ 10. $J = \left\{\{(x,y)|y = \pm\sqrt{x^2 - 16}\}\right\}$

ASAQ1

ACTIVITY 1.1 – A

NAME: ____ SCORE: ______________

SECTION: ___________DATE: _______________ PROF: ____________________________________

Identify which of the following representations is/are a function and not a function. Write F if a function and NF if not a function on the space provided before each number.	**ANSWER**
__________ 1. $A = \{(1,4),\ (4,7),(9,12),(13,16)\}$	**F**
__________ 2. $B = \{(0,1),\ (2,5),(3,10),(4,17),(5,26),(6,37)\}$	**F**
__________ 3. $C = \{(0,10),(6,8),(8,6),(6,-8)\}$	**NF**
__________ 4. $D = \{\{(x,y)\|y = x+1\}\}$	**F**
__________ 5. $E = \{\{(x,y)\|y = x^2 - 1\}\}$	**F**
__________ 6. $F = \left\{\{(x,y)\|y = -\sqrt{x+2}\}\right\}$	**F**
__________ 7. $G = \left\{\left\{(x,y)\middle\|y = \frac{1}{x}\right\}\right\}$	**F**
	F
__________ 8. $H = \left\{\left\{(x,y)\middle\|y = \frac{x+3}{x^2-4}\right\}\right\}$	**F**
__________ 9. $I = \{\{(x,y)\|y^2 = 4 - x^2\}\}$	**NF**
__________ 10. $J = \left\{\{(x,y)\|y = \pm\sqrt{x^2-16}\}\right\}$	**NF**

SAQ2

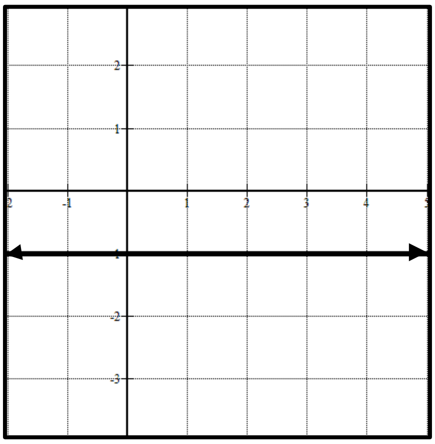
ACTIVITY 1.1 – B

NAME: ____ SCORE: ____________

SECTION: __________DATE: _____________ PROF: _______________________________

Given the graph of a relation, determine its domain and range. Write answer on the space provided under the given graph.

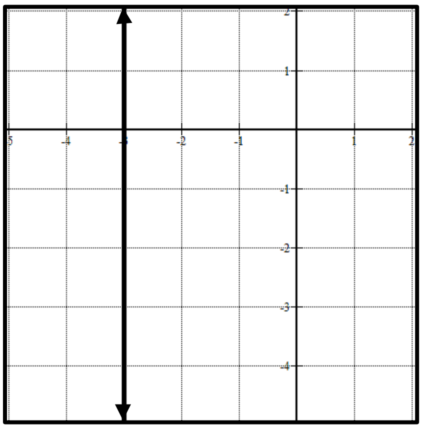
1.



Domain:___________________________

Range:_____________________________

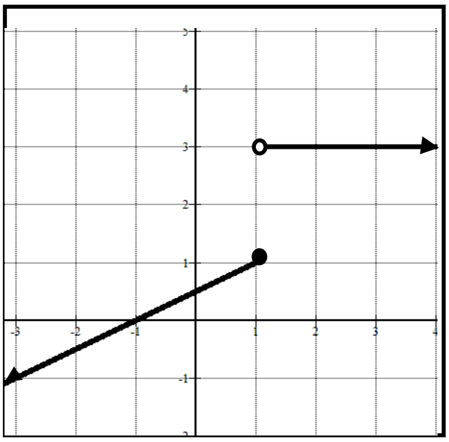
2.



Domain:_________________________

Range: ___________________________

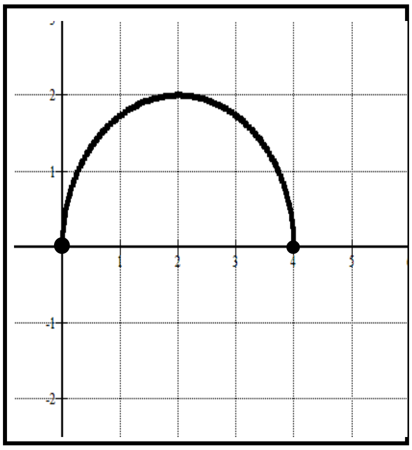
3



Domain:___________________________

Range: ____________________________

4.



Domain: _________________________

Range: ___________________________

ASAQ2

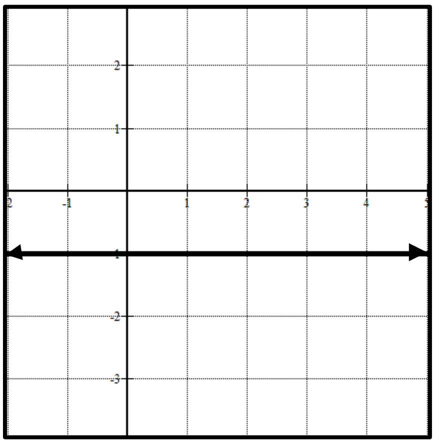
ACTIVITY 1.1 – B

NAME: ____ SCORE: _______________

SECTION: ____________DATE: ________________ PROF: ____________________________________

Given the graph of a relation, determine its domain and range.

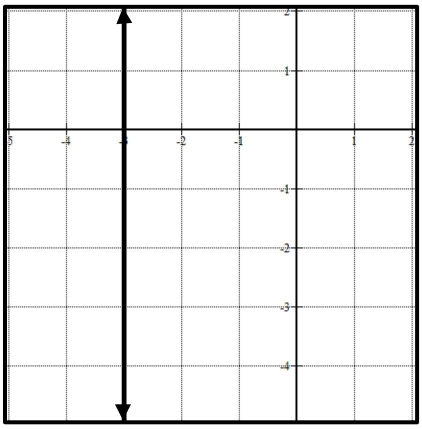
1.



Domain: $\{x|x \in R\}$

Range: $\{y|y = -1\}$

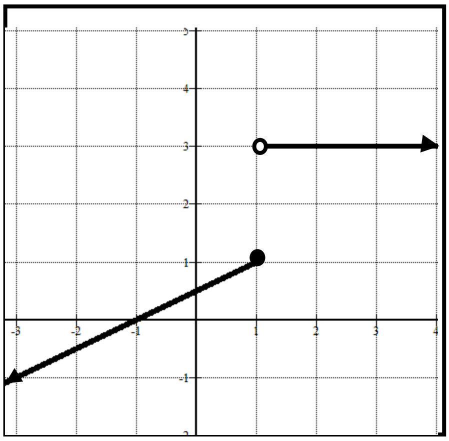
2.



Domain: : $\{x|x = -3\}$

Range: $\{y|y \in R\}$

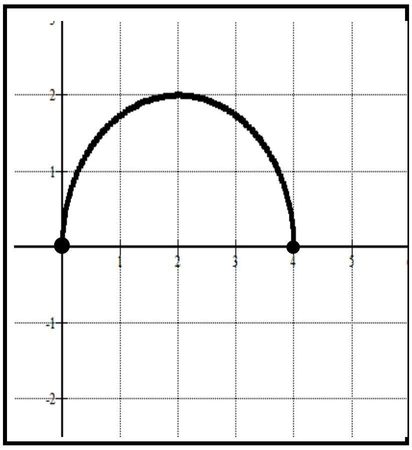
3



Domain: $\{x|x \in R\}$

Range: $\{y|y = 3 \ \cup y \leq 1\}$

4.



Domain: $\{x|0 \leq x \leq 4\}$

Range: $\{y|0 \leq y \leq 2\}$

ACTIVITY 1.1 – C

NAME: ____ SCORE: _______________

SECTION: ___________DATE: ________________ PROF: ____________________________________

Given the following relations, identify the domain and range and draw their graphs.

1. $A = \{(3,6),(0,3),(-2,1),(-4,-1)\}$

2. $B = \{(x,y) | y = 2x - 4\}$

3. $C = \{(x,y) | y = 2x - 4, x \neq -1\}$

4. $D = \left\{(x,y) \middle| y = \frac{1}{x}\right\}$

5. $E = \{(x,y) | y = \sqrt{x+3}\}$

6. $F = \{(x, y) | y = 3 - 4x^2\}$

7. $G = \{(x, y) | y = -\sqrt{3 - 2x}\}$

8. $H = \{(x, y) | y = \sqrt{4x + 1}\}$

9. $I = \{(x, y) | x^2 + y^2 = 4\}$

10. $J = \{(x, y) | y = \sqrt{24 + 2x - x^2}\}$

ACTIVITY 1.1 – D

NAME: ____ SCORE: _______________

SECTION: ____________DATE: ________________ PROF: ____________________________________

Given the piecewise-defined functions, draw the graph and identify the domain and range.

1. $G(x) = \begin{cases} 2x+1 & ,x \leq -2 \\ -2 & ,x > 2 \end{cases}$

2. $m(x) = \begin{cases} -4 & ,x > 0 \\ 2 & ,-2 \leq x \leq 0 \\ 4 & ,x < -2 \end{cases}$

3. $h(x) = \begin{cases} x^2+4, & x < 1 \\ x-1, & x \geq 1 \end{cases}$

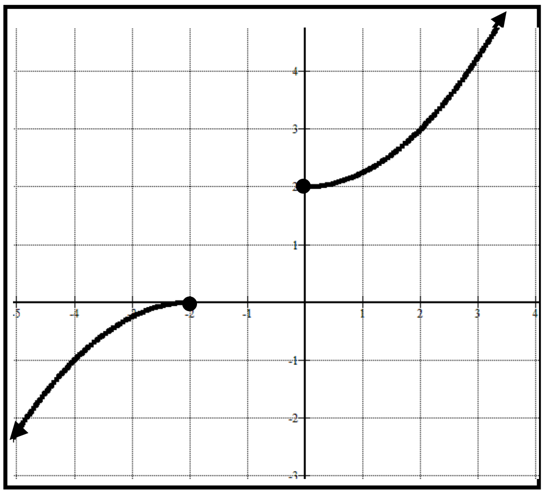
ACTIVITY 1.1 – E

NAME: ____ SCORE: ____________

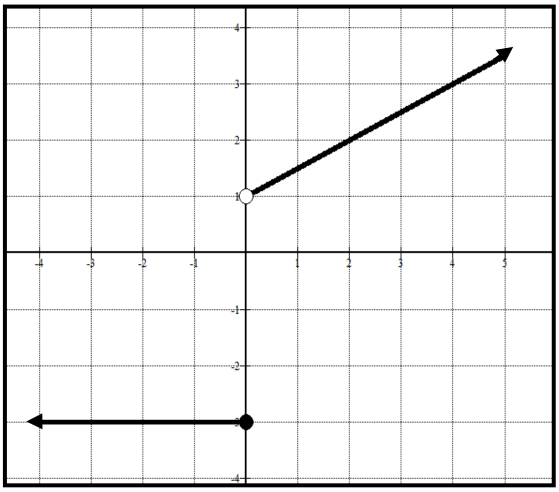
SECTION: ___________DATE: _______________ PROF: ______________________________

Given the graph of piecewise-defined functions, determine its domain and range.

1.



2.



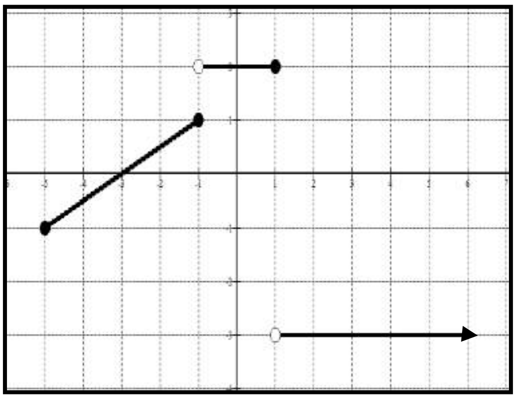
Domain: ________________________________

Range: _________________________________

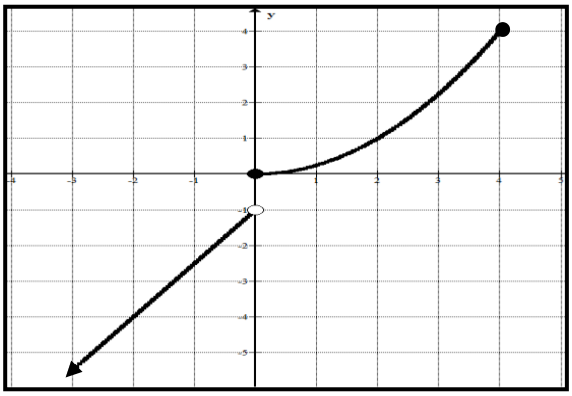
Domain: ________________________________

Range: _________________________________

3.



4.



Domain: ________________________________

Range: _________________________________

Domain: ________________________________

Range: _________________________________

ACTIVITY 1.1 – F

NAME: ____ SCORE: ______________

SECTION: ___________DATE: _______________ PROF: ___________________________________

Evaluate the given functions at the indicated values of x.

1. Given: $f(x) = 2 - x^2$, find:

a. $f(-3)$ = _____ c. $f(-1)$ = _____ e. $f(1)$ = _____

b. $f(-2)$ = _____ d. $f(0)$ = _______ f. $f(2)$ = _____

Draw the graph of $f(x)$ for $-3 \le x < 2$.

2. Given: $g(x) = x^2 + 2x - 1$, find:

a. $g(-4)$ = _____ c. $g(-2)$ = _____ e. $g(0)$ = _____ g. $g(2)$ = _____ i. $g(4)$ = _______

b. $g(-3)$ = _____ d. $g(-1)$ = _____ f. $g(1)$ = _____ h. $g(3)$ = _____ j. $g(a-1)$ = ____

Draw the graph of $g(x)$ for $-4 < x \le 4$.

3. Given: $\alpha(x) = \dfrac{3x-4}{2x+3}$, find:

a. $\alpha(-4)$ = _____ d. $\alpha(-1)$ = _____ g $\alpha(2)$ = _____

b. $\alpha(-3)$ = _____ e. $\alpha(0)$ = _____ h. $\alpha(3)$ = _____

c. $\alpha(-2)$ = _____ f. $\alpha(1)$ = _____ i. $\alpha(4)$ = _____

Which value of x is not an element of the domain? Draw the graph of $\alpha(x)$ for x on $[-4,4]$ using the values above and additional values, if needed.

ACTIVITY 1.1 – G

NAME: ____ SCORE: ______________

SECTION: ___________DATE: _______________ PROF: ____________________________________

1. Given the function $g(x) = x^2$ and $h(x) = \frac{x+1}{1-x}$, find $h[g(x)]$ and $g\left[h(\frac{3}{2})\right]$.

2. Given: $f(x) = x^2 - x + 4$, find $f[f(-2)]$.

3. Given the function $g(x) = x^2$ and $h(x) = \frac{x+1}{1-x}$, find $h[g(-3)]$ and $g\left[h\left(\frac{1}{2}\right)\right]$.

ACTIVITY 1.1 – H

NAME: ____ SCORE: _______________

SECTION: ____________DATE: ________________ PROF: ____________________________________

A. Prove: If $f(x)=\dfrac{2x^2}{x^4+16}$, show that $f(-x)=f(x)$.

B. If $r(x)=\sqrt{x}$, show that $\frac{r(a)-r(b)}{a-b}=\frac{1}{r(a)+r(b)}$.

C. Given $h(x)=x^2+4x-5$, what is the domain of function h? Plot the graph of h for x values in the interval $(-2,6)$.

MODULE 2

CONTINUITY AND LIMIT OF A FUNCTION

Specific Objectives:

At the end of the module, students must be able to:

1. Understand concept of continuity and limit of a function.
2. Evaluate limit of a function, given the rule of correspondence.
3. Identify the limit of a function, given its graph.

LIMIT OF A FUNCTION

In Calculus, the idea of limit is very important. The concept of limit is at the foundation of almost all mathematical analysis, and an understanding of it is absolutely essential. Deep understanding of limit is very rewarding since it facilitates a good grasp of all the basic processes of Calculus.

Let us consider a particular function, say $f(x)=\frac{x^2+x-2}{x-1}$. This function is defined for all values of x except $x=1$ since at $x=1$, both numerator and denominator take zero value or $f(x)=\frac{0}{0}$, which is a meaningless expression. We will study how the function f behaves when we assume values of x getting closer and closer to 1. There are two ways by which value of x may approach 1, one is by assuming values less than 1 and approaching 1; the other way is by taking values greater than 1, still approaching 1.

To get a better idea of what is happening as x takes values approaching 1, consider the constructed Table 1 at the right.

By means of factoring, we can write $f(x)$ in the form

$$f(x)=\frac{(x+2)(x-1)}{x-1}$$

If $x\neq 1$, we are allowed to divide both numerator and denominator by $(x-1)$. Therefore, $f(x)=x+2$, provided $x\neq 1$.

The value of this function as seen on the Table 1 approaches a value of 3 when the variable x approaches 1 by assuming values of x less than 1. That is, as

$x \to 1^-$, $f(x) \to 3$, provided, $x \neq 1$.

Table 1

x	$f(x)=\frac{x^2+x-2}{x-1}$
0	2
0.50	2.50
0.25	2.25
0.80	2.80
0.90	2.90
0.99	2.99
0.999	2.999
0.9999	2.9999
0.99999	2.99999
0.999999	2.999999

In symbol form, $\lim_{x\to 1^-} f(x) = 3$. This is read "limit of $f(x)$ as x approaches 1 through values less than 1 is equal to 3". The value of the limit of the function, in particular, is called the *Left-Hand Limit*.

We observe that as x gets closer and closer to 1, $f(x)$ gets closer and closer to 3; and the closer x is to 1, the closer $f(x)$ is to 3. We can see that we can make the value of $f(x)$ as close to 3 as we please by taking x close enough to 1. Another way of saying this is that we can make the absolute value of the difference between $f(x)$ and 3 as small as we please by making the absolute value of the difference between x and 1 small enough. That is, $|f(x)-3|$ can be made as small as we please by making $|x-1|$ small enough. But bear in mind that f(x) never takes on the value 3.

It is apparent that $f(x)$ can be made as close to 3 as we please by taking x sufficiently close to 1, and this property of the function f does not depend on f being defined when $x = 1$. This fact gives the distinction between limit of $f(x)$ as x approaches 1 and the function value at $x = 1$; that is limit of $f(x)$ as x approaches 1 is 3, but $f(1)$ does not exist. Take note that the limit of the function as x approaches a certain value, say for example $x = a$ may not be the function value when $x = a$.

Similarly, taking a look at Table 2, when x approaches 1 through values greater than 1, the value of the function f gets closer and closer to 3 but not equal to 3. That is, when $x \to 1^+$, $f(x) \to 3$, provided $x \neq 1$, in symbol form, $\lim_{x \to 1^+} f(x) = 3$.

This is read "the limit of $f(x)$ as x approaches 1 through values greater than 1 is equal to 3". This resulting value of the limit of $f(x)$ as $x \to 1^+$ is specifically called the *Right-Hand Limit* of the function.

The graph of the function f appears to be a straight line with a "hole" (an open circle) at the point $(1,3)$, that is, at $x = 1$, $f(1) = 3$.

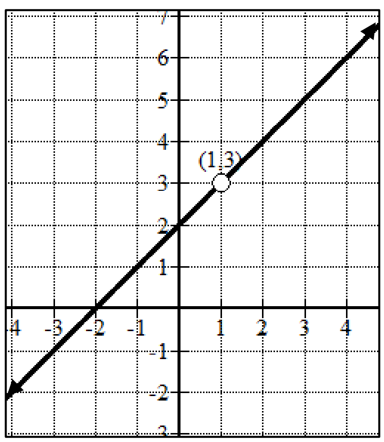


Table 2

x	$f(x) = \dfrac{x^2 + x - 2}{x - 1}$
2	4
1.5	3.5
1.25	3.25
1.10	3.10
1.01	3.01
1.001	3.001
1.0001	3.0001
1.00001	3.00001
1.000001	3.000001
1.0000001	3.0000001

DEFINITION OF LIMIT OF A FUNCTION

Given a function f and numbers a and L, we say that $f(x)$ approaches L as a limit as x approaches a if for each positive number ε (read "epsilon") there is appositive number δ (read "delta") such that $f(x)$ is defined and

$$|f(x) - L| < \varepsilon, \quad \text{whenever } 0 < |x - a| < \delta,\ \varepsilon > 0 \text{ and } \delta > 0.$$

In abbreviated notation, for the definition of limit, we write $f(x) \to L$ as $x \to a$. (This means "x nears but is never equal to a.) In symbol form, $\lim_{x \to a} f(x) = L$.

The values of ε are arbitrarily chosen and can be as small as desired, and that the value of δ is dependent on the ε chosen. It should be pointed out that the smaller the value of ε, the smaller will be the corresponding value of δ.

The above definition states that the function values $f(x)$ approach a limit L as x approaches a number a if the absolute value of the difference between $f(x)$ and L can be made as small as we please by taking x sufficiently near a but not equal to a.

Moreover, it is important to realize that the above definition does not mention about the value of the function when $x=a$. That is, it is not essential that the function is defined for $x=a$ in order for the limit to exist.

In addition, even if the function is defined for $x=a$, it is possible for the limit of $f(x)$ to exist even without having the same value for $f(a)$.

THEOREMS ON LIMIT OF A FUNCTION

If $\lim_{x\to a} f(x) = A$ and $\lim_{x\to a} g(x) = B$, then,

1. $\lim_{x\to a} c = c,$, where $c \in R$
2. $\lim_{x\to a} x = a$
3. $\lim_{x\to a} [f(x) \pm g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = A \pm B$
4. $\lim_{x\to a} [f(x) \cdot g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) = A \cdot B$
5. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \dfrac{A}{B}, provided\ B \neq 0$
6. $\lim_{x\to a} [f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n = [A]^n$
7. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)} = \sqrt[n]{A}$

Note: The symbols $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$ are referred to as the **One -Sided Limit of $f(x)$.**

However, $\lim_{x\to a} f(x)$ is called the **Two-Sided Limit of $f(x)$**.

Theorem: The two-sided limit of the function as x approaches say value a exists if both the one-sided limits of the function exist and are equal. That is,

$$\boxed{\lim_{x\to a} f(x) = \lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x)}$$

Example 13. Evaluate the following limits.

a. $\lim_{x\to -2} (3x-1)$

Use Theorem 1. $\lim_{x\to -2} (3x-1) = \lim_{x\to -2} 3x - \lim_{x\to -2} 1$

Use Theorems 1 and 4. $\lim_{x\to -2}(3x-1)=\lim_{x\to -2}3\cdot\lim_{x\to -2}x-1$

Use Theorems 1 and 2. $\lim_{x\to -2}(3x-1)=3(-2)-1=-6-1=-7$

b. $\lim_{x\to 3}\dfrac{x}{3-5x}$

Use Theorem 5. $\lim_{x\to 3}\dfrac{x}{3-5x}=\dfrac{\lim_{x\to 3}x}{\lim_{x\to 3}(3-5x)}$

Use Theorems 2 and $\lim_{x\to 3}\dfrac{x}{3-5x}=\dfrac{3}{\lim_{x\to 3}3-\lim_{x\to 3}5x}$

Use Theorems 1 and 4. $\lim_{x\to 3}\dfrac{x}{3-5x}=\dfrac{3}{3-\lim_{x\to 3}5\cdot\lim_{x\to 3}x}=\dfrac{3}{3-5(3)}=\dfrac{3}{3-15}=\dfrac{3}{-12}=-\dfrac{1}{4}$

c. $\lim_{x\to 1}\sqrt{\dfrac{4-x^2}{1+x^3}}$

Use Theorem 7. $\lim_{x\to 1}\sqrt{\dfrac{4-x^2}{1+x^3}}=\dfrac{1}{2}\left[\lim_{x\to 1}\dfrac{4-x^2}{1+x^3}\right]$

Use Theorem 5 and 3. $\lim_{x\to 1}\sqrt{\dfrac{4-x^2}{1+x^3}}=\sqrt{\dfrac{\lim_{x\to 1}(4-x^2)}{\lim_{x\to 1}(1+x^3)}}=\sqrt{\dfrac{\left[\lim_{x\to 1}4-\lim_{x\to 1}x^2\right]}{\left[\lim_{x\to 1}1+\lim_{x\to 1}x^3\right]}}$

Use Theorems 1 and 6. $\lim_{x\to 1}\sqrt{\dfrac{4-x^2}{1+x^3}}=\sqrt{\dfrac{4-\left(\lim_{x\to 1}x\right)^2}{1+\left(\lim_{x\to 1}x\right)^3}}=\sqrt{\dfrac{4-(1)^2}{1+(1)^3}}=\sqrt{\dfrac{3}{2}}=\dfrac{\sqrt{6}}{2}$

Take note that the answers to the examples presented above were obtained by directly substituting the value approached by the variable. Consider now the following illustrative examples.

d. $\lim_{x\to 2}\dfrac{3x^2-x-10}{x^2-4}$

Solution: By direct substitution method, $\lim_{x\to 2}\dfrac{3x^2-x-10}{x^2-4}=\dfrac{0}{0}$.

However, the expression $\dfrac{0}{0}$ is an indeterminate.

Eliminate the indeterminate form by factoring both the numerator and denominator. The purpose of which is to remove factor from the numerator and denominator that has zero value at $x=2$.

$$= \lim_{x\to 2} \frac{(x-2)(3x+5)}{(x-2)(x+2)}$$

$$= \lim_{x\to 2} \frac{3x+5}{x+2} = \frac{3(2)+5}{2+2} = \frac{11}{4}$$

e. $\lim_{x\to 3} \frac{x^4-81}{2x^2-5x-3}$

Solution: $\lim_{x\to 3} \frac{x^4-81}{2x^2-5x-3} = \frac{0}{0}$

The factor $(x-3)$ which is zero when $x=3$ needs to be eliminated from the numerator and denominator.

$$= \lim_{x\to 3} \frac{(x^2-9)(x^2+9)}{(x-3)(2x+1)} = \lim_{x\to 3} \frac{(x-3)(x+3)(x^2+9)}{(x-3)(2x+1)} = \lim_{x\to 3} \frac{(x+3)(x^2+9)}{2x+1}$$

$$= \frac{(3+3)[(3)^2+9]}{2(3)+1} = \frac{6(9+9)}{7} = \frac{6(18)}{7} = \frac{108}{7}$$

f. $\lim_{x\to 4} \frac{3-\sqrt{x+5}}{x-4}$

Solution: $\lim_{x\to 4} \frac{3-\sqrt{x+5}}{x-4} = \frac{0}{0}$

Factor $(x-4)$ must be removed from the numerator and denominator. To do it, multiply the members of the fraction by the conjugate of the numerator to eliminate the radical of index two. Then, recall the product of a sum and difference of two terms: $(a+b)(a-b)=a^2-b^2$.

$$= \lim_{x\to 4} \frac{3-\sqrt{x+5}}{x-4} \cdot \frac{3+\sqrt{x+5}}{3+\sqrt{x+5}} = \lim_{x\to 4} \frac{9-(x+5)}{(x-4)(3+\sqrt{x+5})} = \lim_{x\to 4} \frac{4-x}{(x-4)(3+\sqrt{x+5})}$$

$$= \lim_{x\to 4} \frac{-(x-4)}{(x-4)(3+\sqrt{x+5})} = \lim_{x\to 4} \frac{-1}{3+\sqrt{x+5}} = \frac{-1}{3+3} = -\frac{1}{6}$$

g. $\lim_{x\to 0} \frac{x^3-7x}{x^3}$

Solution: $\lim_{x\to 0} \frac{x^3-7x}{x^3} = \frac{0}{0}$

Eliminate x since it is the cause of zero value on the numerator and denominator.

$$= \lim_{x\to 0} \frac{x(x^2-7)}{x^3} = \lim_{x\to 0} \frac{x^2-7}{x^2} = \frac{-7}{0} = -\frac{7}{0} = -\infty$$

Note: The numerator approaches -7 and the denominator is a positive quantity approaching 0. The quantity $-\infty$ is NOT a real number and is NOT an indeterminate form. Hence, the limit of the given function does not exist.

h. $\lim_{x \to 0} \frac{\cos 2x - 1}{cosx - 1}$

Solution: $\lim_{x \to 0} \frac{cos2x - 1}{cosx - 1} = \frac{0}{0}$

Recall the trigonometric identity: $\cos 2x = 2\cos^2 x - 1$

Substitution into the given expression results to:

$$= \lim_{x \to 0} \frac{(2cos^2x - 1) - 1}{cosx - 1} = \lim_{x \to 0} \frac{2cos^2x - 2}{cosx - 1}$$

$$= \lim_{x \to 0} \frac{2(cos^2x - 1)}{cosx - 1}$$

$$= \lim_{x \to 0} \frac{2(cosx + 1)(cosx - 1)}{cosx - 1}$$

$$= \lim_{x \to 0} 2(cosx + 1) = 2(1 + 1) = 4$$

SAQ3

ACTIVITY 1.1 – I

NAME: ____ SCORE: _______________

SECTION: ____________DATE: ________________ PROF: _____________________________________

Evaluate the limit of the given functions.

1. $\lim_{x \to 3} \frac{4x - 5}{5x - 1}$

2. $\lim_{x \to -1} \frac{2x + 1}{x^2 - 3x + 4}$

3. $\lim_{x \to 4} \sqrt{\frac{x^2 - 3x + 4}{2x^2 - x - 1}}$

4. $\lim_{x \to -\frac{3}{2}} \frac{4x^2 - 9}{2x + 3}$

ASAQ3

ACTIVITY 1.2 – I

NAME: ____ SCORE: _____________

SECTION: ___________DATE: ______________ PROF: _______________________________

Evaluate the limit of the given functions using the direct substitution method.

1. $\lim_{x \to 3} \frac{4x-5}{5x-1} \quad \mathit{Answer}: \frac{1}{2}$

2. $\lim_{x \to -1} \frac{2x+1}{x^2-3x+4} \quad \mathit{Answer}: -\frac{1}{8}$

3. $\lim_{x \to 4} \sqrt{\frac{x^2-3x+4}{2x^2-x-1}} \quad \mathit{Answer}: \frac{8}{27}$

4. $\lim_{x \to -\frac{3}{2}} \frac{4x^2-9}{2x+3} \quad \mathit{Aswer}: -6$

ACTIVITY 1.2 – J

NAME: ____ SCORE: ________________

SECTION: ____________DATE: _________________ PROF: _______________________________________

Evaluate the limit of the given functions.

1. $$\lim_{x \to 4} \frac{3x^2 - 17x + 20}{4x^2 - 25x + 36}$$

2. $$\lim_{t \to 1} \frac{t^3 - 1}{t - 1}$$

3. $$\lim_{x \to 1} \frac{\sqrt{x} - 1}{x - 1}$$

4. $$\lim_{h \to -1} \frac{\sqrt{h + 5} - 2}{h + 1}$$

5. $\lim_{x \to 1} \frac{\sqrt[3]{x} - 1}{x - 1}$

6. $\lim_{x \to -2} \frac{x^3 - x^2 - x + 10}{x^2 + 3x + 2}$

7. $\lim_{x \to -1} \frac{2x^2 - x - 3}{x^3 + 2x^2 + 6x + 5}$

8. $\lim_{y \to 4} \frac{2y^3 - 11y^2 + 10y + 8}{3y^3 - 17y^2 + 16y + 16}$

LIMIT OF A FUNCTION INVOLVING INFINITY

If we consider the function $f(x)=\frac{1}{x}$, it is an observation that as $x \to 0$ through positive values, the corresponding values of the function get bigger and bigger. In case like this, it indicates the behaviour of the function. We say that $f(x)$ increases without limit or $f(x) \to \infty$ as $x \to 0^+$. In symbol form:

$$\boxed{\lim_{x\to 0^+} f(x) = +\infty}$$

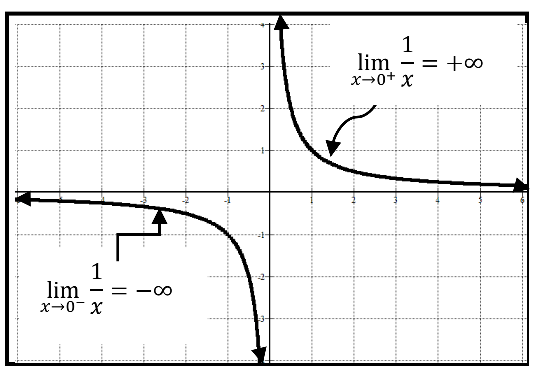
Likewise, as $x \to 0$ through negative values, the value of the function decreases without limit. Thus, in symbol from: $\lim_{x\to 0^-} f(x) = -\infty$

The introduction of the symbol ∞ does not in any way justify its use as a number. It is best to mention that the word "infinite" signifies only a state of being non-finite. Division by zero is a meaningless operation and it is not intended that the symbol ∞ represents $\frac{1}{0}$. Once again, it is to be stressed out that $+\infty$ is not a symbol for a real number.

When the limit of the function as x approaches a certain value, say a, is positive infinity, we say that the limit of the function does not exist. The symbol $+\infty$ indicates the behaviour of the function $f(x)$ as x gets closer and closer to value a.

In the same manner, getting $-\infty$ for the limit of the function simply indicates that the behaviour of the function whose function values decrease without bound. Getting $-\infty$ once again tells us that the limit of the function does not exist.

The behaviour of the function $f(x)=\frac{1}{x}$ is graphically shown below.



Example 14. $\lim_{x\to 0^-} 4^{\frac{1}{x}} = 4^{-\infty} = \frac{1}{4^{\infty}} = \frac{1}{\infty} = 0$

Example 15. $\lim_{x\to 0^+} 4^{\frac{1}{x}} = 4^{+\infty} = +\infty$

Theorems on Limit of a Function Involving Infinity

1. $\lim_{x\to+\infty} cx = \infty, (c > 0)$
2. $\lim_{x\to+\infty} cx = -\infty, (c < 0)$
3. $\lim_{x\to\pm\infty} \frac{c}{x} = 0$
4. $\lim_{x\to 0^+} \frac{c}{x} = +\infty, (c > 0)$
5. $\lim_{x\to 0^-} \frac{c}{x} = -\infty, (c > 0)$

Example 16. Evaluate the following:

a. $\lim_{x\to+\infty} \frac{4x-5}{6x+7}$

Solution: Substitution of $+\infty$ for x results to the indeterminate form $\frac{\infty}{\infty}$. In case like this, we use a standard technique in working with infinite limits by dividing each term on the numerator and denominator by the highest power of the variable x. Then, use Theorem 3 on limit of function involving infinity. Thus,

$$= \lim_{x\to+\infty} \frac{4x-5}{6x+7} = \lim_{x\to+\infty} \frac{4-\frac{5}{x}}{6+\frac{7}{x}}, \left(\text{provided } x \neq 0 \text{ and } x \neq -\frac{7}{6}\right)$$

$$= \lim_{x\to+\infty} \frac{4x-5}{6x+7} = \frac{4-0}{6+0} = \frac{4}{6} = \frac{2}{3}$$

b. $\lim_{x\to-\infty} \frac{4x+3}{3x^2+1}$

Solution: The limit takes the indeterminate form $-\frac{\infty}{\infty}$. Use the technique described on the previous illustrative example by dividing both numerator and denominator by x^2 and then using Theorem 3.

$$= \lim_{x\to-\infty} \frac{4x+3}{3x^2+1} = \lim_{x\to-\infty} \frac{\frac{4}{x}+\frac{3}{x^2}}{3+\frac{1}{x^2}}$$

$$= \frac{0+0}{3+0} = \frac{0}{3} = 0 = 0$$

c. $\lim_{x\to+\infty}\frac{6x^3+x^2+2x-1}{x^2+x+2}$

Solution: Divide each term on the numerator and denominator by x^3, the highest power of x and then use Theorem 3 since the evaluated limit of the given function equals $\frac{\infty}{\infty}$. Hence,

$$=\lim_{x\to+\infty}\frac{6x^3+x^2+2x-1}{x^2+x+2}=\lim_{x\to\infty}\frac{6+\frac{1}{x}+\frac{2}{x^2}-\frac{1}{x^3}}{\frac{1}{x}+\frac{1}{x^2}+\frac{2}{x^3}}=\frac{6+0+0+0}{0+0+0}=\frac{6}{0}=+\infty$$

Note: In illustrative Example 15, a, b and c, you will observe that in evaluating limit of function of the form $\frac{f(x)}{g(x)}$as x approaches $\pm\infty$, if:

i. The degree of the numerator equals the degree of the denominator; the limit of $\frac{f(x)}{g(x)}$ as $x$approaches $+\infty$ or $-\infty$ equals the ratio of the numerical coefficient of the highest power of x on the numerator to the numerical coefficient of the highest power of x on the denominator.

ii. The degree of the numerator is less than the degree of the denominator; the limit of $\frac{f(x)}{g(x)}$as $x$approaches $+\infty$ or $-\infty$ equals zero.

iii. The degree of the numerator is greater than the degree of the denominator, the limit of $\frac{f(x)}{g(x)}$ as $x$approaches $+\infty$ or $-\infty$ equals either ∞ or $-\infty$ as the case may be.

d. $\lim_{x\to 0^+}\sqrt{x}=0$

Take note that in evaluating the limit of the given function as $x\to 0^+$, the values taken by x are all greater than zero but approaching zero.

e. $\lim_{x\to 0^-}\sqrt{x}=\text{does not exist}$

The limit of the given function as $x\to 0^-$ does not exist since the values taken by x are all less than zero but approaching zero. Hence, the corresponding values of the given function are imaginary or not real numbers.

f. $\lim_{x\to 4^+}\frac{5}{x-4}=+\infty$

As x takes values greater than 4 but approaching 4, the denominator $(x-4)$is always greater than zero but approaching zero.

g. $\lim_{x \to 4^-} \frac{5}{x-4} = -\infty$

However, when x assumes values less than 4 but approaching 4, the denominator $(x-4)$ takes values less than zero but approaching zero.

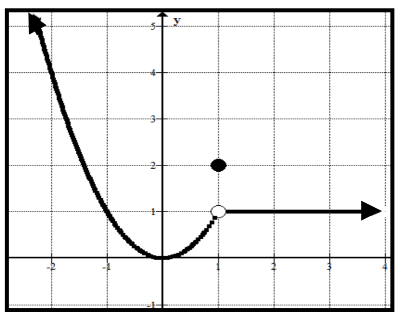
CONTINUITY OF A FUNCTION

A function $f(x)$ is said to be continuous at $x = a$ if and only if the following conditions are satisfied by the given function:

1. $f(a)$ exists;
2. $\lim_{x \to a} f(x)$ exists; and,
3. $f(a) = \lim_{x \to a} f(x)$

Consider the graph of function $f(x)$ shown below. Observe that at $x = 1$ the function is discontinuous since the conditions for the continuity of a function is not satisfied. The function when $x = 1$, that is $f(1) = 2$ while $\lim_{x \to 1} f(x) = 1$.

Therefore, $x = 2$ is the point of discontinuity of the given function since $f(1) \neq \lim_{x \to 1} f(x)$.



Remember: A rational function in x is a continuous function for all values of x except those values for which the denominator is zero.

Example 17. Find the point/s of discontinuity of the following functions. State the condition/s of continuity of a function, if any, which is/are not satisfied.

a. $f(x) = \frac{x+1}{x-3}$

The point of discontinuity of $f(x)$ is at $x = 3$ since the denominator on the rule of correspondence of $f(x)$ equals zero at $x = 3$. Hence, $f(3)$ does not exist or is undefined. To determine the $\lim_{x \to 3} f(x)$, evaluate $\lim_{x \to 3^+} f(x)$ and $\lim_{x \to 3^-} f(x)$.

And to do this, how about assuming values of x approaching 3 through values more than 3; and then, assume values of x approaching 3 through values less than 3. Remember the right-hand and the left-hand limit of a function?

As can be gleaned from the tables below, as $x \to 3^+$, $f(x) \to +\infty$ and while $x \to 3^-$, $f(x) \to -\infty$. Both limits do not exist, the function behaves differently both for $x \to +\infty$ and $x \to -\infty$. Hence, $\lim_{x\to3} f(x)$ does not exist. Evidently, the conditions for continuity of the function at $x = 3$ are not satisfied.

x	$f(x) = \frac{x+1}{x-3}$
3.1	41
3.01	401
3.001	4001
3.0001	40001
3.00001	400001
3.000001	4000001
3.0000001	40000001

x	$f(x) = \frac{x+1}{x-3}$
2.9	- 39
2.99	- 399
2.999	- 3999
2.9999	- 39999
2.99999	- 399999
2.999999	- 3999999
2.9999999	- 39999999

c. $\phi(x) = \dfrac{x^2 - 25}{x^2 - 5x - 6}$

The factors of the denominator are $(x-6)(x+1)$. Equating these factors to zero will give $x = 6$ and $x = -1$.These values are the points of discontinuity of the given

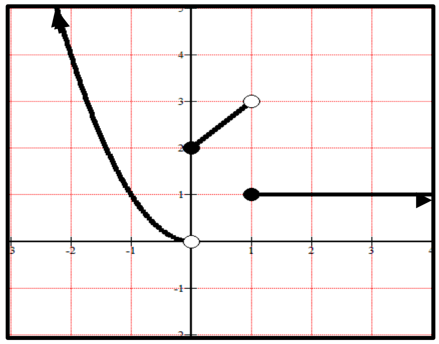
b. $g(x) = \dfrac{x}{\tan x}$

To determine the point of discontinuity, we look for value of the variable that will make the denominator equal to zero. We set $\tan x = 0$ and then, solve for x. This is a trigonometric equation whose solutions in the interval $0 \le x \le 2\pi$ are $x = 0, \pi$ and 2π. Therefore, the points of discontinuity are at $x = 0^o,\ \pi,\ 2\pi$.

c. $\phi(x) = \dfrac{x^2 - 25}{x^2 - 5x - 6}$

The factors of the denominator are $(x-6)(x+1)$. Equating these factors to zero will give $x = 6$ and $x = -1$.These values are the points of discontinuity of the given function $\phi(x)$ since at these values the function values are not defined.

Example 17. Graph of $h(x)$ is shown below.



From the graph, the following properties of the given function can be extracted:

1. $h(1) = 1$
2. $\lim_{x \to 1^+} h(x) = 1$
3. $\lim_{x \to 1^-} h(x) = 3$
4. $\lim_{x \to 1} h(x)$ does not exist
5. $\lim_{x \to 0^+} h(x) = 2$
6. $\lim_{x \to 0^-} h(x) = 0$
7. $\lim_{x \to 0} h(x) = \text{does not exist}$
8. $\lim_{x \to +\infty} h(x) = 1$
9. $\lim_{x \to -\infty} h(x) = +\infty$
10. Points of discontinuity:
    (a) $x = 1$ since $h(1) \neq \lim_{x \to 1} h(x)$; and,

    (b) $x = 0$ since $h(0) \neq \lim_{x \to 0} h(x)$.

SAQ4

ACTIVITY 1.2 – K

NAME: ____ SCORE: ______________

SECTION: ___________DATE: _______________ PROF: ___________________________________

Evaluate the limit of the following functions:

1. $\lim_{x\to\infty} \frac{x^2}{1-x^2}$

2. $\lim_{x\to+\infty} \frac{2x+3}{3x+5}$

3. $\lim_{x\to\infty} \frac{2x^2+7x+3}{x^3+3x+1}$

4. $\lim_{x\to+\infty} \frac{x^4+2x^2+5}{2x^3-6x+1}$

5. $\lim_{x\to+\infty} \frac{\sqrt{x^2+4}}{x+1}$

ASAQ4

ACTIVITY 1.2 – K

NAME: ____ SCORE: ______________

SECTION: ___________DATE: _______________ PROF: ___________________________________

Evaluate the limit of the following functions:

1. $\lim_{x \to \infty} \frac{x^2}{1 - x^2}$ $\quad Answer: -1$

2. $\lim_{x \to \infty} \frac{2x + 3}{3x + 5}$ $\quad Answer: \frac{2}{3}$

3. $\lim_{x \to \infty} \frac{2x^2 + 7x + 3}{x^3 + 3x + 1}$ $\quad Answer: 0$

4. $\lim_{x \to +\infty} \frac{x^4 + 2x^2 + 5}{2x^3 - 6x + 1}$ $\quad Answer: \infty$

5. $\lim_{x \to +\infty} \frac{\sqrt{x^2 + 4}}{x + 1}$ $\quad Answer: 1$

SAQ5

ACTIVITY 1.2 – L

NAME: ____ SCORE: _____________

SECTION: __________DATE: ______________ PROF: _______________________________

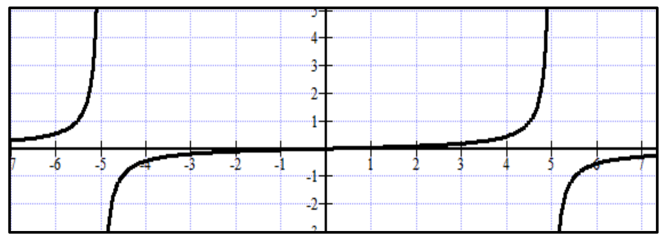
Evaluate the following limits of function. The graph of the given function may be useful on limit evaluation.

________ 1. a. $\lim_{x\to 5^-} \frac{x}{25-x^2}$

________ 1. b. $\lim_{x\to 5^+} \frac{x}{25-x^2}$

________ 1. c. $\lim_{x\to \infty} \frac{x}{25-x^2}$

________ 1. d. $\lim_{x\to -\infty} \frac{x}{25-x^2}$

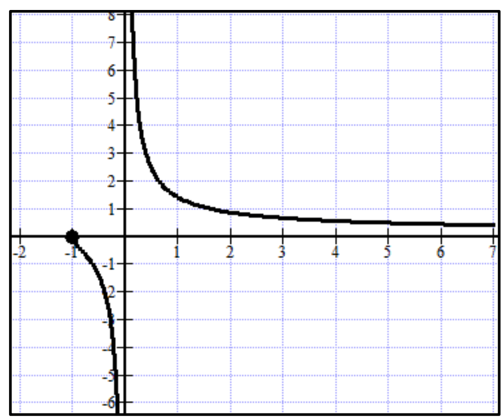


Graph of $y = \frac{x}{25-x^2}$

________ 2. a. $\lim_{x\to 0^+} \frac{\sqrt{x+1}}{x}$

________ 2. b. $\lim_{x\to 0^-} \frac{\sqrt{x+1}}{x}$

________ 2. c. $\lim_{x\to \infty} \frac{\sqrt{x+1}}{x}$



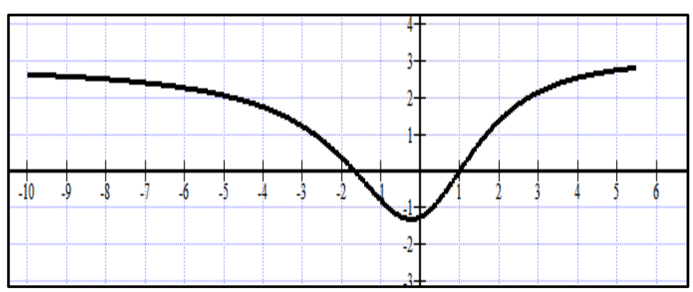
Graph of $y = \frac{\sqrt{x+1}}{x}$

________ 3. a. $\lim_{x\to -\infty} \frac{3x^2+2x-5}{x^2+4}$

________ 3. b. $\lim_{x\to \infty} \frac{3x^2+2x-5}{x^2+4}$

________ 3. c. $\lim_{x\to 0^-} \frac{3x^2+2x-5}{x^2+4}$

________ 3. d. $\lim_{x\to 0^+} \frac{3x^2+2x-5}{x^2+4}$



Graph of $y = \frac{3x^2+2x-5}{x^2+4}$

ASAQ5
ACTIVITY 1.2 – L

NAME: ____ SCORE: ____________

SECTION: __________DATE: ______________ PROF: _____________________________

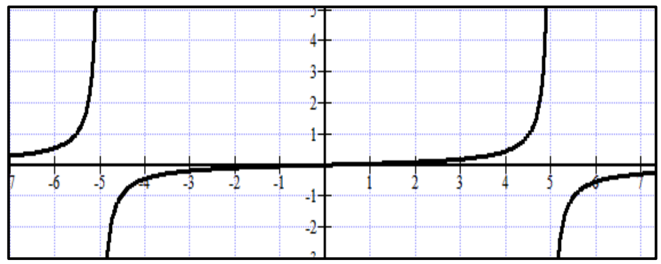
Evaluate the following limits of function. The graph of the given function may be useful on limit evaluation.

________$1.a.\ \lim_{x\to 5^-} \frac{x}{25-x^2}$ $Answer: -\infty$

________$1.b.\ \lim_{x\to 5^+} \frac{x}{25-x^2}$ $Answer: \infty$

________$1.c.\ \lim_{x\to \infty} \frac{x}{25-x^2}$ $Answer: 0$

________$1.d.\ \lim_{x\to -\infty} \frac{x}{25-x^2}$ $Answer: 0$

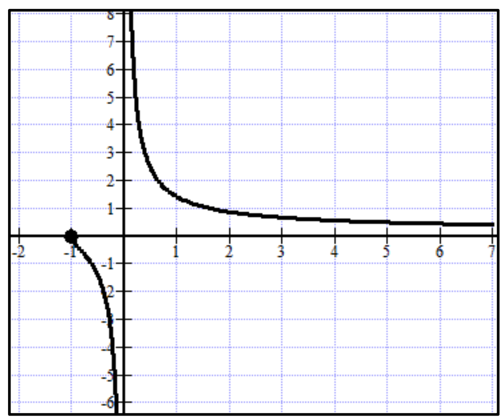


Graph of $y = \frac{x}{25-x^2}$

________$2.a.\ \lim_{x\to 0^+} \frac{\sqrt{x+1}}{x}$ $Answer: \infty$

________$2.b.\ \lim_{x\to 0^-} \frac{\sqrt{x+1}}{x}$ $Answer: -\infty$

________$2.c.\ \lim_{x\to \infty} \frac{\sqrt{x+1}}{x}$ $Answer: 0$



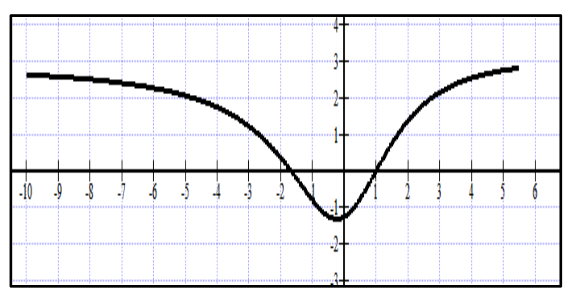
Graph of $y = \frac{\sqrt{x+1}}{x}$

________$3.a.\ \lim_{x\to \infty} \frac{3x^2+2x-5}{x^2+4}$ $Answer: 3$

________$3.b.\ \lim_{x\to -\infty} \frac{3x^2+2x-5}{x^2+4}$ $Answer: 3$

________$3.c.\ \lim_{x\to 0^-} \frac{3x^2+2x-5}{x^2+4}$ $Answer: -\frac{5}{4}$

________$3.d.\ \lim_{x\to 0^+} \frac{3x^2+2x-5}{x^2+4}$ $Answer: -\frac{5}{4}$



Graph of $y = \frac{3x^2+2x-5}{x^2+4}$

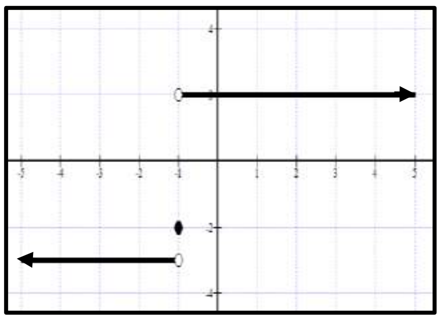
ACTIVITY 1.2 – M

NAME: ____ SCORE: _______________

SECTION: ___________DATE: ________________ PROF: ____________________________________

Based on the given graphs of $f(x)$, extract the indicated properties of the function and write answer on the space provided..

1.

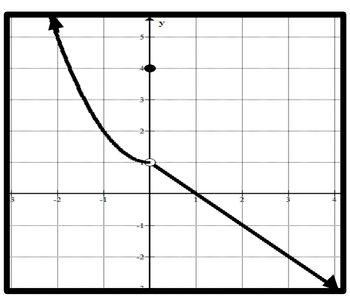


$b.\ f(-1) =$ ________________

$b.\ \lim_{x \to -1^-} f(x) =$ _____________

$c.\ \lim_{x \to -1^+} f(x) =$ _____________

2.



$a.\ \ f(0) =$ ______________

$b.\ \ \lim_{x \to 0^+} f(x) =$ _________

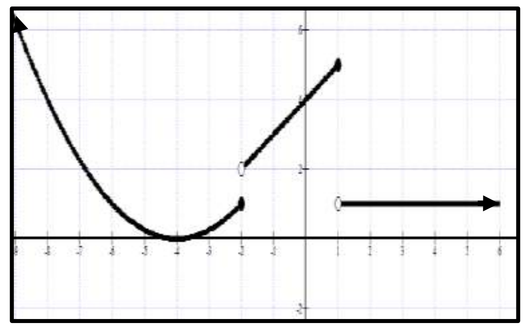
$c.\ \ \lim_{x \to 0^-} f(x) =$ _________

$d.\ \ \lim_{x \to 0} f(x) =$ ___________

$e.\ \ \lim_{x \to +\infty} f(x) =$ _________

$f\ \ \lim_{x \to -\infty} f(x) =$ _________

3.



$a.\ \ f(-2) =$ _______________

$b.\ \ f(1) =$ ________________

$c.\ \lim_{x \to -2^+} f(x) =$ _____________

$d.\ \lim_{x \to -2^-} f(x) =$ ____________

$e.\ \lim_{x \to 2} f(x) =$ _______________

$f.\ \lim_{x \to 1^+} f(x) =$ ______________

$g.\ \lim_{x \to 1^-} f(x) =$ ______________

$h.\ \lim_{x \to 1} f(x) =$ _______________

$i.\ \lim_{x \to +\infty} f(x) =$ _____________

$j.\ \lim_{x \to -\infty} f(x) =$ _____________

MODULE 3

INCREMENT AND DERIVATIVE

Specific Objectives:

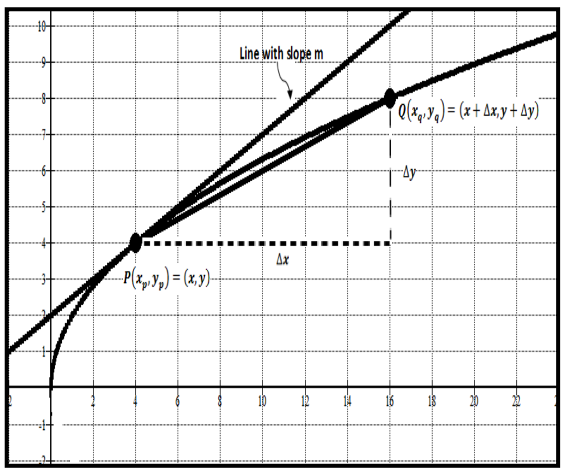
At the end of the module, students must be able to:

1. Understand concept of increment and derivative
2. Apply the increment method to find derivative of a given function.

INCREMENT

An increment is a small, unspecified, nonzero change in the value of a quantity. The symbol most commonly used is the uppercase Greek letter delta (Δ).

Consider the case of the graph of a function *y* = *f* (x) in Cartesian (rectangular) coordinates, as shown in the figure. The slope of this curve at a specific point *P* is defined as the limit of $m = \frac{\Delta y}{\Delta x}$, as Δx (read "delta x") approaches zero, provided the function is continuous (the curve is not "broken"). The value $\frac{\Delta y}{\Delta x}$ depends on defining two points in the vicinity of *P*. In the illustration, one of the points is *P* itself, defined as (x_P, y_P) and the other is $Q(x_q, y_q)$, which is near *P*. The increments here are $\Delta y = y_q - y_p$ and $\Delta x = x_q - x_p$. As point *Q* approaches point *P*, both of these increments approach zero, and the ratio of increments $\frac{\Delta y}{\Delta x}$ approaches the slope of the curve at point *P*. When the increment is positive, it means "increase in the value of the quantity" while a negative increment signifies a "decrease".



The term increment is occasionally used in physics and engineering to represent a small change in a parameter such as temperature T (ΔT), electric current I (ΔI) or time t (Δt).

DERIVATIVE

We will extend our discussion of limits and examine the idea of the derivative, the basis of differential calculus. We will assume a particular function of x, such that $y = f(x) = x^2$

If x is assigned the value 5, the corresponding value of y will be $(5)^2$ or 25. Now, if we increase the value of x by 3, making it 8, we have increment $\Delta x = 3$. This results in an increase in the value of y, and we call this increase an increment or Δy. From this we write

$$y + \Delta y = (x + \Delta x)^2 = (5+3)^2 = 64.$$

Thus,

$$\Delta y = (x + \Delta x)^2 - y$$

$$\Delta y = (5 + \Delta x)^2 - 25$$

We are interested in the ratio $\frac{\Delta y}{\Delta x}$ because the limit of this ratio as Δx approaches zero is the derivative of function f with respect to x.

As we recall from the discussion of limits, as Δx is made smaller, Δy gets smaller also. In our example, the ratio $\frac{\Delta y}{\Delta x}$ approaches 10 as shown on the table below. Let x = 5, correspondingly, y = 25, then assume values of Δx that tend to approach zero. Take note that as $\Delta x \to 0$, $\frac{\Delta y}{\Delta x} \to 10$.

Variable	Δx	1	0.1	0.01	0.001	0.0001	1x10^{-5}
	$\Delta y = (x + \Delta x)^2 - x^2$	11	1.01	0.1001	0.010001	0.001	1x10^{-4}
	$\frac{\Delta y}{\Delta x}$	11	10.1	10.01	10.001	10.0001	10.00001

The symbol $\frac{\Delta y}{\Delta x}$ gives the average rate of change of y with respect to x, that is, with x changing from x to $x + \Delta x$, and with y correspondingly changing from y to $y + \Delta y$. In effect, the value of the function $f(x)$ becomes $y = f(x + \Delta x)$. Furthermore, if for a fixed value of x, the quotient $\frac{\Delta y}{\Delta x}$ approaches a limit as the increment Δx approaches zero, this limit is called the derivative of y with respect to x for the given value of x. This is denoted by symbol $\frac{dy}{dx}$ or $\frac{d}{dx} f(x), y', f'(x), D_x y, D_x f(x)$.

Thus, by definition,

$$\boxed{\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}}$$

Differentiation is the process of finding $\frac{dy}{dx}$ when $y = f(x)$. If the derivative of $f(x)$ exists, then, $f(x)$ is said to be a differentiable function of x.

Note: If $y = f(x)$, the instantaneous rate of change of y per unit change in x at x_1 is $f'(x_1)$, or, equivalently, the derivative of y with respect to x at x_1, if it exists.

THE FOUR-STEP RULE OR THE INCREMENT METHOD OF DIFFERENTIATION

This is the long process of finding the derivative of a given function using the increment of a variable and it may be formulated as follows:

1. Replace $(x + \Delta x)$ for x and $(y + \Delta y)$ for y.
2. To get Δy, subtract the original function of x, that is $f(x)$, from the new function of $(x + \Delta x)$, which is $f(x + \Delta x)$. Thus, $\Delta y = f(x + \Delta x) - f(x)$.
3. Divide both sides of the resulting equation in Step 2 by Δx to define $\frac{\Delta y}{\Delta x}$.
4. Take the limit as Δx approaches zero of all the terms in the equation from Step 3. The resulting equation is the derivative of f(x) with respect to x or simply $\frac{dy}{dx}$.

Example 1. Using the 4-Step Rule or the Increment Method, find the derivative of y with respect to x or $\frac{dy}{dx}$.

a. $y = x^2 + 2x - 3$ ---(1)

Step 1: $y + \Delta y = (x + \Delta x)^2 + 2(x + \Delta x) - 3$ ----------------------(2)

Step 2: Subtracting (2) to (1) will give Δy.

$$\Delta y = (x + \Delta x)^2 + 2(x + \Delta x) - 3 - (x^2 + 2x - 3)$$
$$\Delta y = \left[x^2 + 2x\Delta x + (\Delta x)^2\right] + 2x + 2\Delta x - 3 - x^2 - 2x + 3$$
$$\Delta y = 2x\Delta x + (\Delta x)^2 + 2\Delta x$$
$$\Delta y = \Delta x(2x + \Delta x + 2)$$

Step 3: Divide the resulting equation in Step 2 by Δx to define $\frac{\Delta y}{\Delta x}$.

$$\frac{\Delta y}{\Delta x} = 2x + \Delta x + 2$$

Step 4. Take the limit of all the terms in the resulting equation in Step 3as x approaches zero.

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 2x + 2 = 2(x + 1)$$

b. $y = \frac{1}{(x-1)^2}$, when x = 2.

$$y + \Delta y = \frac{1}{(x+\Delta x-1)^2}$$

$$\Delta y = \frac{1}{(x+\Delta x-1)^2} - \frac{1}{(x-1)^2}$$

Recall: Square of a Trinomial
$(a+b+c)^2 = a^2+b^2+c^2+2ab+2ac+2bc$

$$\Delta y = \frac{(x-1)^2-(x+\Delta x-1)^2}{(x+\Delta x-1)^2(x-1)^2}$$

$$\Delta y = \frac{(x^2-2x+1)-(x^2+(\Delta x)^2+1+2x\Delta x-2x-2\Delta x)}{(x+\Delta x-1)^2(x-1)^2}$$

$$\Delta y = \frac{-(\Delta x)^2-2x\Delta x+2\Delta x}{(x+\Delta x-1)^2(x-1)^2}$$

$$\Delta y = \frac{\Delta x(-\Delta x-2x+2)}{(x+\Delta x-1)^2(x-1)^2}$$

$$\frac{\Delta y}{\Delta x} = \frac{-\Delta x-2x+2}{(x+\Delta x-1)^2(x-1)^2}$$

$$\frac{\Delta y}{\Delta x} = \frac{-\Delta x-2x+2}{(x+\Delta x-1)^2(x-1)^2}$$

Therefore,

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0}\frac{-\Delta x-2x+2}{(x+\Delta x-1)^2(x-1)^2}$$

$$\frac{dy}{dx} = \frac{0-2x+2}{(x+0-1)^2(x-1)^2} = \frac{-2x+2}{(x-1)^4} = \frac{-2(x-1)}{(x-1)^4} = \frac{-2}{(x-1)^3}$$

When$x = 2$: $\frac{dy}{dx} = \frac{-2}{(2-1)^3} = \frac{-2}{(1)^3} = -2$

Example 2. At what point on the curve $y = \sqrt{x}$ is the derivative of y with respect to x equal to $\frac{1}{4}$.

Solution: We use the Increment Method to define $\frac{dy}{dx}$.

$$y + \Delta y = \sqrt{x + \Delta x}$$
$$\Delta y = \sqrt{x + \Delta x} - \sqrt{x}$$

$$\Delta y = \left(\sqrt{x+\Delta x} - \sqrt{x}\right) \cdot \frac{\sqrt{x+\Delta x} + \sqrt{x}}{\sqrt{x+\Delta x} + \sqrt{x}}$$

$$\Delta y = \frac{\left(\sqrt{x+\Delta x} - x\right)\left(\sqrt{x+\Delta x} + x\right)}{\left(\sqrt{x+\Delta x} + x\right)}$$

Recall: Product of Sum and Difference of Two Squares
$$(a+b)(a-b) = a^2 - b^2$$

$$\Delta y = \frac{\left(\sqrt{x+\Delta x}\right)^2 - \left(\sqrt{x}\right)^2}{\sqrt{x+\Delta x} + \sqrt{x}}$$

$$\Delta y = \frac{x + \Delta x - x}{\sqrt{x+\Delta x} + \sqrt{x}} = \frac{\Delta x}{\sqrt{x+\Delta x} + \sqrt{x}}$$

$$\frac{\Delta y}{\Delta x} = \frac{1}{\sqrt{x+\Delta x} + \sqrt{x}}$$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{1}{\sqrt{x+\Delta x} + \sqrt{x}}$$

But by definition,

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{1}{\sqrt{x+0} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

To determine the unknown point on the curve, we use the given condition that

$\frac{dy}{dx} = \frac{1}{4}$.Thus, $$\frac{1}{2\sqrt{x}} = \frac{1}{4}$$

Cross-multiply and simplify. $4 = 2\sqrt{x}$ $\quad 2 = \sqrt{x}$

Square both sides of the above equation to solve for x. $x = 4$

Hence, the unknown point on the curve $y = \sqrt{x}$ where the derivative of y with respect to x equals $\frac{1}{4}$ is $(4,2)$.

SAQ6
ACTIVITY 2.3 – A

NAME: ____ SCORE: _______________

SECTION: ____________DATE: ________________ PROF: ____________________________________

Find $\frac{dy}{dx}$ using the Increment Method of Differentiation.

1. $y = 3x^2 - 2x + 5$

2. $y = x^3 - 4x$

3. $y = \dfrac{5}{2 - x}$

4. $y = \frac{1}{7x^2}$

5. $y = \frac{2x-3}{x+1}$

6. $y = \sqrt{4-x}$

ASAQ6
ACTIVITY 2.3 – A

NAME: ____ SCORE: _______________

SECTION: ____________DATE: ________________ PROF: _____________________________________

Find $\frac{dy}{dx}$ using the Increment Method of Differentiation.

1. $y = 3x^2 - 2x + 5$ $\quad Answer: 2(3x-1)$

2. $y = x^3 - 4x$ $\quad Answer: 3x^2 - 4$

3. $y = \dfrac{5}{2-x}$ $\quad Answer: \dfrac{5}{(2-x)^2}$

4. $y = \dfrac{1}{7x^2}$ $\quad Answer: -\dfrac{2}{7x^3}$

5. $y = \dfrac{2x-3}{x+1}$ $\quad Answer: \dfrac{5}{(x+1)^2}$

6. $y = \sqrt{4-x}$ $\quad Answer: -\dfrac{\sqrt{4-x}}{2(4-x)}$

ACTIVITY 2.3 – B

NAME: ____ SCORE: _______________

SECTION: ____________DATE: ________________ PROF: ____________________________________

In each of the following, find $\frac{dy}{dx}$ at the indicated value of x using the Increment Method.

1. $y = x(x + 1)$; $x = 3$

2. $y = \dfrac{4}{\sqrt{x-1}}$; $x = 5$

3. $y = \dfrac{1}{4x}$; $x = 2$

MODULE 4

RATE OF CHANGE

Specific Objectives:

At the end of the module, students must be able to:

1. Understand concept of rate of change.
2. Solve rate of change problems using derivative of a function.

RATE OF CHANGE

Recall that the symbol $\frac{\Delta y}{\Delta x}$ gives the average rate of change of y with respect to x. That is with x changing from x to $x+\Delta x$, y correspondingly changes from y to $y+\Delta y$. If a quantity say A changes with quantity D, the rate of change of A with respect to D represented by $\frac{dA}{dD}$, at a particular instant, say when $D=3$, is called the instantaneous rate of change of A with respect to D.

Example 1. For the function $y=\frac{1}{x}$, at what value of x is the rate of change of y with respect to x equal to -4.

Solution: The instantaneous rate of change of y with respect to x is represented by $\frac{dy}{dx}$ at the unknown value of x.

$$y+\Delta y=\frac{1}{x+\Delta x}$$

$$\Delta y=\frac{1}{x+\Delta x}-\frac{1}{x}$$

$$\Delta y=\frac{x-(x+\Delta x)}{x(x+\Delta x)}$$

$$\Delta y=\frac{-\Delta x}{x(x+\Delta x)}$$

$$\frac{\Delta y}{\Delta x}=\frac{-1}{x(x+\Delta x)}$$

$$\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{-1}{x(x+\Delta x)}=-\frac{1}{x^2}$$

Using the given condition that the rate of change of y with respect to x equals -4 will result to the unknown value of x.

$$\frac{dy}{dx}=-4$$

$$\frac{-1}{x^2}=-4$$

$$x^2=\frac{1}{4}$$

$$x=\pm\frac{1}{2}$$

Example 2. Find the rate at which the volume V of the cube changes with respect to its edge e when the edge measures 2 cm.

Here, we like to find the instantaneous rate of change of the volume with respect to the edge of cube when the edge measures 2 cm, that is, $\frac{dV}{de}$ when $e = 2$.

Recall the formula to find volume V of a cube having edge e, that is, $V = e^3$.Using the Increment Method of differentiation,

$$
\begin{aligned}
V + \Delta V &= (e + \Delta e)^3 \\
\Delta V &= (e + \Delta e)^3 - e^3 \\
\Delta V &= e^3 + 3e^2\Delta e + 3e(\Delta e)^2 + (\Delta e)^3 - e^3 \\
\Delta V &= 3e^2\Delta e + 3e(\Delta e)^2 + (\Delta e)^3 \\
\Delta V &= \Delta e\left[3e^2 + 3e\Delta e + (\Delta e)^2\right] \\
\frac{\Delta V}{\Delta e} &= 3e^2 + 3e\Delta e + (\Delta e)^2
\end{aligned}
$$

$$\lim_{\Delta e \to 0} \frac{\Delta V}{\Delta e} = \lim_{\Delta e \to 0} [3e^2 + 3e\Delta e + (\Delta e)^2]$$

When $e = 2$ cm, $\frac{dV}{de} = 3(2)^2 = 12\ \frac{cm^3}{cm}$

Example 3. Boyle's Law for the expansion of a gas is $PV = 4$, where P is the number of pounds per square unit of pressure, V is the number of cubic units in the volume of the gas, and k is a constant. Find the instantaneous rate of change of the volume with respect to the pressure when $V = 8$.

Solution: To find the instantaneous rate of change of the volume V with respect to the pressure P, first, we express the volume as a function of pressure. Thus, $V = \frac{4}{P}$. Then, using the Increment Method, we derive expression for $\frac{dV}{dP}$.

$$
\begin{aligned}
V + \Delta V &= \frac{4}{P + \Delta P} \\
\Delta V &= \frac{4}{P + \Delta P} - \frac{4}{P} \\
\Delta V &= \frac{4P - 4(P + \Delta P)}{P(P + \Delta P)}
\end{aligned}
$$

$$\Delta V = \frac{-4\Delta P}{P(P+\Delta P)}$$

$$\frac{\Delta V}{\Delta P} = \frac{-4}{P(P+\Delta P)}$$

$$\frac{dV}{dP} = \lim_{\Delta P \to 0} \frac{\Delta V}{\Delta P} = \lim_{\Delta P \to 0} \frac{-4}{P(P+\Delta P)} = \frac{-4}{P^2}$$

When $V = 8, P = \frac{4}{8} = \frac{1}{2}$. Substitution on the above equation will yield:

$$\frac{dV}{dP} = \frac{-4}{\left(\frac{1}{2}\right)^2} = \frac{-4}{\frac{1}{4}} = -4 \cdot (4) = -16$$

SAQ7
ACTIVITY 2.3 – C

NAME: ____ SCORE: ______________

SECTION: ___________DATE: _______________ PROF: __________________________________

Use the Increment Method of differentiation to solve for the unknown.

1. For the function $y = \dfrac{1}{2-x}$, at what values of x will the rate of change of y with respect to x be equal to $\frac{1}{16}$?

2. Find the rate of change of the area of a circle with respect to its diameter when the radius is 4 cm.

   Area of a circle $A = \pi R^2 = \frac{\pi}{4} D^2$.

3. Find the rate of change of the volume of a right circular cone of constant altitude of 15 cm with respect to its base radius when the radius measures 2 cm. Volume of a right circular cylinder $V = \pi R^2 H$.

4. Find the rate of change of the hypotenuse of a right triangle having a constant base of 2cm with respect to its altitude when the altitude measures $\sqrt{21}$ cm. Hint: Use Pythagorean Theorem.

5. Find the rate of change of the area of an equilateral triangle when the side measures 16 cm.

Area of an equilateral triangle $A = \frac{\sqrt{3}}{4}\, s^2$.

ASAQ7
ACTIVITY 2.3 – C

NAME: ____ SCORE: _______________

SECTION: ____________DATE: ________________ PROF: ____________________________________

Use the Increment Method of differentiation to solve for the unknown.

1. For the function $y = \dfrac{1}{2 - x}$, at what values of x will the rate of change of y with respect to x be equal to $\frac{1}{16}$? $Answer\!: x = -2;\ x = 6$

2. Find the rate of change of the area of a circle with respect to its diameter when the radius is 4 cm.

   Area of a circle $A = \pi R^2 = \frac{\pi}{4} D^2$. $Answer\!: 4\pi$

3. Find the rate of change of the volume of a right circular cone of constant altitude of 15 cm with respect to its base radius when the radius measures 2 cm. Volume of a right circular cylinder $V = \pi R^2 H$. $Answer\!: 60\pi$

4. Find the rate of change of the hypotenuse of a right triangle having a constant base of 2cm with respect to its altitude when the altitude measures $\sqrt{21}$ cm. Hint: Use Pythagorean Theorem. $Answer$: $\frac{\sqrt{21}}{5}$

5. Find the rate of change of the area of an equilateral triangle when the side measures 16 cm.

Area of an equilateral triangle $A = \frac{\sqrt{3}}{4}\, s^2$. $Answer$: $8\sqrt{\ }$

MODULE 5

DIFFERENTIATION FORMULAS FOR ALGEBRAIC FUNCTION

Specific Objectives:

At the end of the module, students must be able to:

1. Know the differentiation formulas for algebraic function.
2. How to use correctly the differentiation formulas for algebraic function.

ALGEBRAIC FUNCTION

An algebraic function is one formed by a finite number of algebraic operations on constants and/or variables. These algebraic operations include addition, subtraction, multiplication, division, raising to powers, and extracting roots. Polynomial and rational functions are particular kinds of algebraic functions.

Polynomial function is defined by $f(x)=a_nx^n+a_{n-1}x^{n-1}+a_{n-2}x^{n-2}+...+a_1x+a_0$ Where $a_0,a_1,...,a_n$ are real numbers $(a^n\neq 0)$ and n is a non-negative integer. Function f is a polynomial function of degree n.

Rational function is a function that can be expressed as a quotient of two polynomial functions.

Example 1. $f(x)=\frac{x-1}{x^2-4}$ Example 2. $h(x)=\frac{x^2-25}{x+3}$

DIFFERENTIATION FORMULAS OF ALGEBRAIC FUNCTIONS

The following differentiation formulas were derived using the Increment Method of differentiation. On the list below, let u,v and w be functions of x; while c and n are constants.

1. Derivative of a Constant $\frac{d}{dx}(c)=0$	6. Derivative of Square Root of a Function $\frac{d}{dx}\sqrt{u}=\frac{1}{2\sqrt{u}}\frac{d}{dx}(u)$
2. Derivative of x with respect to x $\frac{d}{dx}(x)=1$	7. Derivative of a Product of Two Factors $\frac{d}{dx}(uv)=u\frac{d}{dx}(v)+v\frac{d}{dx}(u)$
3. Derivative of a Power of x $\frac{d}{dx}(cx^n)=cnx^{n-1}$	8. Derivative of a Product of Three Factors $\frac{d}{dx}(uvw)=uv\frac{d}{dx}w+uw\frac{d}{dx}v+vw\frac{d}{dx}u$
4. Derivative of a Sum/Difference of Terms $\frac{d}{dx}(u\pm v)=\frac{d}{dx}(u)\pm\frac{d}{dx}(v)$	9. Derivative of a Quotient $\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{d}{dx}(u)-u\frac{d}{dx}(v)}{v^2}$
5. Derivative of a General Power $\frac{d}{dx}(cu^n)=cnu^{n-1}\frac{d}{dx}(u)$	10. Derivative of a Constant Over a Function $\frac{d}{dx}\left(\frac{c}{u}\right)=\frac{-c}{u^2}\frac{d}{dx}(u)$

Let me illustrate how to use the listed differentiation formulas. Most people find the differentiation process hard. Students failed to arrive at the correct derivative because of inadequate knowledge of trigonometry, geometry and algebra. But do not be threatened, illustrative examples have steps presented in detailed way.

Example 1. Find the derivative $\frac{dy}{dx}$ or $f'(x)$ of the given algebraic functions.

a. $y = x^3 - 4x^2 + 6x - 8$

The given function is a sum of terms. To differentiate, we use Formula 4: $\frac{d}{dx}(u \pm v) = \frac{d}{dx}(u) \pm \frac{d}{dx}(v)$, followed with the use of Formula 3: $\frac{d}{dx}(cx^n) = cnx^{n-1}$ as well as Formula 2: $\frac{d}{dx}(x) = 1$ and Formula 1: $\frac{d}{dx}(c) = 0$.

$$\frac{dy}{dx} = \frac{d}{dx}(x^3 - 4x^2 + 6x - 8)$$

$$= \frac{d}{dx}x^3 - \frac{d}{dx}4x^2 + \frac{d}{dx}6x - \frac{d}{dx}8$$

$$= 3x^{3-1} - 4(2)x^{2-1} + 6(x)^{1-1} - 0$$

$$\frac{dy}{dx} = 3x^2 - 8x + 6 \quad \text{(Recall: } x^0 = 1)$$

b. $y = (2x^2 - 3)^2$

There are two possible ways of finding the derivative of the given function which is a general power of x.

Method 1. First, we transform the given function to a sum of terms by expanding the right side of the given equation above using the special product called square of a binomial: $(a \pm b)^2 = a^2 \pm 2ab + b^2$. Hence,

$$y = (4x^4 - 12x^2 + 9)^2 = 4x^4 - 12x^2 + 9$$

$$\frac{dy}{dx} = \frac{d}{dx}4x^4 - \frac{d}{dx}12x^2 + \frac{d}{dx}9 = 4(4)x^{4-1} - 12(2)x^{2-1} + 0$$

$$\frac{dy}{dx} = 16x^3 - 24x = 8x(2x^2 - 3)$$

Method 2. We use directly the Formula 5: $\frac{d}{dx}(cu^n) = cnu^{n-1}\frac{d}{dx}(u)$, with $c = 1, u = 2x^2 - 3, n = 2$

$$\frac{dy}{dx} = 2(2x^2 - 3)^{2-1}\frac{d}{dx}(2x^2 - 3) = 2(2x^2 - 3)\left(\frac{d}{dx}2x^2 - \frac{d}{dx}3\right)$$

$$\frac{dy}{dx} = 2(2x^2 - 3)[2(2)x^{2-1} - 0] = \frac{dy}{dx} = 2(2x^2 - 3)(4x) = 8x(2x^2 - 3)$$

Observe that both methods of finding the derivative give to same result.

c. $y = \dfrac{(4x-1)^3}{x^2}$

This time, the given function is a quotient having the numerator a special product called a cube of binomial.

Method 1. First, we expand the special product $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$, the result of which, then divided by x^2. Doing so results to:

$$y = \frac{(4x)^3 - 3(4x)^2(1) + 3(4x)(1)^{2-} - (1)^3}{x^2}$$
$$y = \frac{64x^3 - 48x^2 + 12x - 1}{x^2} = 64x - 48 + 12x^{-1} - x^{-2}$$

The simplified form is now a sum of terms and we differentiate using Formula 4:
$\dfrac{d}{dx}(u \pm v) = \dfrac{d}{dx}(u) \pm \dfrac{d}{dx}(v)$.

$$\frac{dy}{dx} = \frac{d}{dx}(64x) - \frac{d}{dx}(48) + \frac{d}{dx}\left(12x^{-1}\right) - \frac{d}{dx}\left(x^{-2}\right)$$
$$\frac{dy}{dx} = 64(1)x^0 - 0 + 12(-1)x^{-2} - (-2)x^{-3}$$
$$\frac{dy}{dx} = 64 - \frac{12}{x^2} + \frac{2}{x^3}$$

Recall: $\frac{1}{a^n} = a^{-n}$. Hence: $x^{-3} = \frac{1}{x^3}$

Method 2. We use directly the quotient Formula 9: $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{v\dfrac{d}{dx}(u) - u\dfrac{d}{dx}(v)}{v^2}$, with:

$u = (4x-1)^3$ and $v = x^2$. We need $\frac{du}{dx}$ and $\frac{dv}{dx}$ for substitution on the formula.

$$u = (4x-1)^3 \qquad\qquad v = x^2$$
$$\frac{du}{dx} = 3(4x-1)^2 \frac{d}{dx}(4x-1) = 3(4x-1)^2\left(4x^0 - 0\right) \qquad\qquad \frac{dv}{dx} = 2x$$
$$\frac{du}{dx} = 3(4x-1)^2(4) = 12(4x-1)^2$$

Substitution on the quotient formula gives

$$y' = \frac{x^2\left[12(4x-1)^2\right] - (4x-1)^3(2x)}{\left(x^2\right)^2}$$

Simplify the numerator by bringing-out the common monomial factor $2x(4x-1)^2$.

$$y'=\frac{2x(4x-1)^2[6x-(4x-1)]}{x^4}=\frac{2x(4x-1)^2(6x-4x+1)}{x^4}$$

$$y'=\frac{2x(4x-1)^2(2x+1)}{x^4}=\frac{2x(16x^2-8x+1)(2x+1)}{x^4}$$

$$y'=\frac{2(32x^3-16x^2+2x+16x^2-8x+1)}{x^3}$$

$$y'=\frac{64x^3-12x+2}{x^3}=\frac{64x^3}{x^3}-\frac{12x}{x^3}+\frac{2}{x^3}$$

$$y'=64-\frac{12}{x^2}+\frac{2}{x^3}$$

Note: The derivative was expanded to show that the result of Method 2 is right. However, it is always best to express the derivative of a function in its factored-form.

d. $y=(3x^4-2x^2+4x-1)(x^5-4x+2)$

Method 1. Get the product of the factors on the right side of the given equation to bring the product to a sum of terms, then, use the Formula 4: $\frac{d}{dx}(u\pm v)=\frac{d}{dx}(u)\pm\frac{d}{dx}(v)$.

$$y=3x^9-2x^7+4x^6-13x^5+6x^4+8x^3-20x^2+12x-2$$

$$\frac{dy}{dx}=\frac{d}{dx}3x^9-\frac{d}{dx}2x^7+\frac{d}{dx}4x^6-\frac{d}{dx}13x^5+\frac{d}{dx}6x^4+\frac{d}{dx}8x^3-\frac{d}{dx}20x^2+\frac{d}{dx}12x-\frac{d}{dx}2$$

$$\frac{dy}{dx}=3(9)x^8-2(7)x^6+4(6)x^5-13(5)x^4+6(4)x^3+8(3)x^2-20(2)x+12$$

$$\frac{dy}{dx}=27x^8-14x^6+24x^5-65x^4+24x^3+24x^2-40x+12$$

Method 2. Use the product Formula 9: $\frac{d}{dx}(uv)=u\frac{d}{dx}(v)+v\frac{d}{dx}(u)$ with:

$$u=3x^4-2x^2+4x-1 \qquad v=x^5-4x+2$$

$$\frac{du}{dx}=12x^3-4x+4 \qquad \frac{dv}{dx}=5x^4-4$$

Substitute now the above on Formula 9.

$$\frac{dy}{dx}=\left(3x^4-2x^2+4x-1\right)\left(5x^4-4\right)+\left(x^5-4x+2\right)\left(12x^3-4x+4\right)$$

$$\frac{dy}{dx}=15x^8-10x^6+20x^5-5x^4-12x^4+8x^2-16x+4$$

$$+\left(12x^8-48x^4+24x^3-4x^6+16x^2-8x+4x^5-16x+8\right)$$

$$\frac{dy}{dx}=27x^8-14x^6+24x^5-65x^4+24x^3+24x^2-40x+12$$

$$\frac{dy}{dx}=27x^8-14x^6+24x^5-65x^4+24x^3+24x^2-40x+12$$

e. $y=\sqrt{x^3-4x^2-6x+7}$

Solution: To differentiate, we use the Formula 6: $\frac{d}{dx}\sqrt{u}=\frac{1}{2\sqrt{u}}\frac{d}{dx}(u)$ with:

$u=x^3-4x^2-6x+7$, $\frac{du}{dx}=3x^2-8x-6$. Substitution of these on Formula 6 results to

$$y'=\frac{1}{2\sqrt{x^3-4x^2-6x+7}}\frac{d}{dx}\left(x^3-4x^2-6x+7\right)$$

$$y'=\frac{1}{2\sqrt{x^3-4x^2-6x+7}}\left(3x^2-8x-6\right)$$

$$y'=\frac{3x^2-8x-6}{2\sqrt{x^3-4x^2-6x+7}}$$

Rationalize the fraction. $y'=\frac{\left(3x^2-8x-6\right)\sqrt{x^3-4x^2-6x+7}}{2\left(x^3-4x^2-6x+7\right)}$

f. $y=\frac{4}{x^2-25}$

The differentiation of the given function can be dome in three ways.

Method 1. We apply the Quotient Formula 9: $\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{d}{dx}(u)-u\frac{d}{dx}(v)}{v^2}$, with:

$u=4$ $\qquad$ $v=x^2-25$

$\frac{du}{dx}=0$, $\qquad$ $\frac{dv}{dx}=2x$,

Substitution yields $y'=\frac{\left(x^2-25\right)(0)-4(2x)}{\left(x^2-25\right)^2}=\frac{-8x}{\left(x^2-25\right)^2}$

Method 2. We use the special Quotient Formula 10: $\frac{d}{dx}\left(\frac{c}{u}\right)=\frac{-c}{u^2}\frac{d}{dx}(u)$,with:

$$c=4, u=x^2-25, \frac{du}{dx}=2x.$$

$$y'=-\frac{4}{\left(x^2-25\right)^2}\frac{d}{dx}\left(x^2-25\right)=\frac{-4}{\left(x^2-25\right)^2}(2x)$$

$$y'==\frac{-8x}{\left(x^2-25\right)^2}$$

Method 3. We may rewrite the given function to $y=4\left(x^2-25\right)^{-1}$, then, use the general power Formula 5: $\frac{d}{dx}\left(cu^n\right)=cnu^{n-1}\frac{d}{dx}(u)$, with $c=4, n=-1, n-1=-2, u=\left(x^2-25\right), \frac{du}{dx}=2x$

$$y'=4(-1)\left(x^2-25\right)^{-2}(2x)=\frac{-8x}{\left(x^2-25\right)^2}$$

Observe that all methods of differentiation presented yield same result.

g. $y=\frac{2x}{\sqrt{4x-5}}$

The given function can be differentiated in two different ways.

Method 1: Rewrite the given function to $y=2x(4x-5)^{-\frac{1}{2}}$, then, apply Product Formula 7: $\frac{d}{dx}(uv)=u\frac{d}{dx}(v)+v\frac{d}{dx}(u)$.

$$u=2x \qquad \frac{du}{dx}=2$$

$$v=(4x-5)^{-\frac{1}{2}}$$

$$\frac{dv}{dx}=-\frac{1}{2}(4x-5)^{-\frac{1}{2}-1}\frac{d}{dx}(4x-5)$$

$$\frac{dv}{dx}=-\frac{1}{2}(4x-5)^{-\frac{3}{2}}(4)=-\frac{2}{(4x-5)^{\frac{3}{2}}}$$

Therefore, after doing the necessary substitutions, we got

$$y'=2x\left[\frac{-2}{(4x-5)^{\frac{3}{2}}}\right]+(4x-5)^{-\frac{1}{2}}(2)$$

$$y'=\frac{-4x}{(4x-5)^{\frac{3}{2}}}+\frac{2}{\sqrt{4x-5}}=\frac{-4x}{(4x-5)\sqrt{4x-5}}+\frac{2}{\sqrt{4x-5}}$$

$$y' = \frac{-4x + 2(4x-5)}{(4x-5)\sqrt{4x-5}} = \frac{-4x+8x-10}{(4x-5)\sqrt{4x-5}} = \frac{4x-10}{(4x-5)\sqrt{4x-5}}$$
$$y' = \frac{2(2x-5)}{(4x-5)\sqrt{4x-5}}$$

Method 2. We use the Quotient Formula 9: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{d}{dx}(u) - u\frac{d}{dx}(v)}{v^2}$

$$v = \sqrt{4x-5}$$
$$\frac{dv}{dx} = \frac{1}{2\sqrt{4x-5}}(4) = \frac{2}{\sqrt{4x-5}}$$
$$y' = \frac{\sqrt{4x-5}(2) - 2x\left(\frac{2}{\sqrt{4x-5}}\right)}{4x-5}$$

$$u = 2x$$
$$\frac{du}{dx} = 2$$

$$y' = \frac{2(4x-5) - 4x}{(4x-5)\sqrt{4x-5}} = \frac{4x-10}{(4x-5)\sqrt{4x-5}}$$
$$y' = \frac{2(2x-5)}{(4x-5)\sqrt{4x-5}}$$

h. $y = \left(\frac{x^3+8}{2x^3-1}\right)^4$

Solution: To differentiate the given function, apply the general power Formula 5: $\frac{d}{dx}(cu^n) = cnu^{n-1}\frac{d}{dx}(u)$, with: $n = 4,\ \ n-1 = 3,\ \ u = \frac{x^3+8}{2x^3-1}$.

Using the quotient formula to find $\frac{du}{dx}$.

$$\frac{du}{dx} = \frac{(2x^3-1)(3x^2) - (x^3+8)(6x^2)}{(2x^3-1)^2} = \frac{-51x^2}{(2x^3-1)^2}.$$

Therefore, using Formula 5 yields: $\frac{dy}{dx} = 4\left(\frac{x^3+8}{2x^3-1}\right)^3\left[\frac{-51x^2}{(2x^3-1)^2}\right] = \frac{-204x^2(x^3+8)^3}{(2x^3-1)^3(2x^3-1)^2}$

$$\frac{dy}{dx} = \frac{-204x^2(x^3+8)^3}{(2x^3-1)^5}$$

SAQ8
ACTIVITY 3.5 – A

NAME: ____ SCORE: ______________

SECTION: ___________DATE: _______________ PROF: ___________________________________

Find the derivative of the following functions. If ever possible, express final answer in factored form.

1. $y = 3x^5 - 3x^4 + 7x^3 - 5x^2 + x - 8$	4. $y = \dfrac{4-9x}{4+9x}$
2. $y = 3x^5 - 4x^3 - \dfrac{5}{x^2}$	5. $y = \left(\dfrac{x^3 - 2}{x^3 + 1}\right)^3$
3. $y = 2(x^3 - 2x + 5)^4$	6. $y = \sqrt{3x} - \dfrac{1}{\sqrt[3]{5x}}$

ASAQ8
ACTIVITY 3.5 – A

NAME: ____ SCORE: ____________

SECTION: __________DATE: _____________ PROF: ______________________________

Find the derivative of the following functions. If ever possible, express final answer in factored form.

1. $y = 3x^5 - 3x^4 + 7x^3 - 5x^2 + x - 8$ $Answer: 15x^4 - 12x^3 + 21x^2 - 10x + 1$	4. $y = \dfrac{4-9x}{4+9x}$ $Answer: -\dfrac{72}{(4+9x)^2}$
2. $y = 2x^5 - 4x^3 - \dfrac{5}{x^2}$ $Answer: 10x^4 - 12x^2 + \dfrac{10}{x^3}$	5. $y = \left(\dfrac{x^3 - 2}{x^3 + 1}\right)^3$ $Answer: \dfrac{27x^2(x^3-2)^2}{(x^3+1)^3}$
3. $y = 2(x^3 - 6x + 5)^4$ $Answer: 24(x^3 - 6x + 5)^3(x^2 - 2)$	6. $y = \sqrt{3x} - \sqrt[3]{5x}$ $Answer: \dfrac{3}{2\sqrt{3x}} - \dfrac{5}{3\sqrt[3]{25x^2}}$

ACTIVITY 3.5 – B

NAME: ____ SCORE: ____________

SECTION: __________DATE: _____________ PROF: ______________________________

Find the derivative of the following functions. If ever possible, express final answer in factored form.

1. $y=\sqrt{3x^2-5x+1}$	4. $y=x^5(1+2x)^7$
2. $y=(3x^4-2x^2+4)^2$	5. $y=\sqrt[3]{2x^2-4x+3}$
3. $y=(x^3-6x)(2-4x^3)$	6. $y=\dfrac{2-3x-2x^2}{x+3}$

$7. y = \frac{3}{(x^2+4)^3}$	$10. y = \sqrt{\frac{x}{2x+1}}$
$8. y = \frac{\sqrt{x^2-1}}{x}$	$11. y = (x^3-2)\sqrt{3x^2+4}$
$9. y = \frac{x^4}{5} + \frac{5}{x^4}$	$12. y = (x^3-2)\sqrt{3x^2+4}$

MODULE 6

SLOPE OF TANGENT AND NORMAL LINE

Specific Objectives:

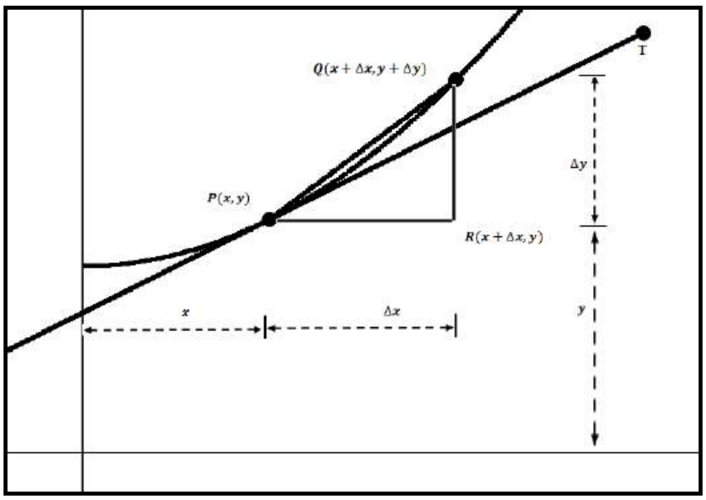
At the end of the module, students must be able to:

1. Know the relation of derivative of a function to slope of tangent and normal line to the graph of a function.
2. Find equation of the tangent and normal line to a curve at a given value of the independent variable.

THE SLOPE OF THE TANGENT AND THE DERIVATIVE

Consider two points $P(x, y)$, a fixed point and a variable point $Q(x + \Delta x, y + y)$ on the graph of function $y = f(x)$. Line PQ is a secant line and line PT the tangent line to the curve at point P. Let point Q approach point P along the curve. From the figure, we see the slope of the secant line $PQ = \frac{QR}{PR} = \frac{\Delta y}{\Delta x}$. As $Q \to P$, that is, as $\Delta x \to 0$, the slope of PQ takes the slope of the tangent line at P as its limit. Thus, by definition,

$$\text{Slope of tangent line at } P(x, y \quad \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \quad = \frac{dy}{dx} \text{ at point } P(x, y)$$



Note: The slope of the tangent line defines the slope of the curve at the point of tangency. Thus,

Slope of the curve $y = f(x)$ at point (x_1, y_1) equals $f'(x_1)$

Recall that a normal line to the curve of $y = f(x)$ at point of tangency (x_1, y_1) is perpendicular to the tangent at the point of tangency. Therefore, in Analytic Geometry, recall that the slope of the normal line is the negative reciprocal of the slope of the tangent. In symbol form, slope of normal line at $(x_1, y_1) = -\frac{1}{\frac{dy}{dx}}$ at (x_1, y_1). To get the equation of the tangent and normal to the curve at the point of tangency (x_1, y_1), we use the point-slope form of the equation of the line, that is $y - y_1 = m(x - x_1)$.

Equation of tangent line at (x_1, y_1): $y - y_1 = \frac{dy}{dx}(x - x_1)$

Equation of the normal line at (x_1, y_1): $y - y_1 = -\frac{1}{\frac{dy}{dx}}(x - x_1)$

Example 2. Find the slope of the given curve at the indicated point.

a. $y=\left(4x^2+3\right)^2$; $\left(\frac{1}{2},16\right)$

Solution: Use the power formula $\frac{d}{dx}\left(cu^n\right)=cnu^{n-1}\frac{d}{dx}(u)$ to find $f'(x)=\frac{dy}{dx}$.

$$f'(x)=\frac{dy}{dx}=2\left(4x^2+3\right)(8x)=16x\left(4x^2+3\right)$$

At point $\left(\frac{1}{2},16\right)$, slope of the tangent line is given below.

$$=f'\left(\frac{1}{2}\right)=16\left(\frac{1}{2}\right)\left[4\left(\frac{1}{2}\right)^2+3\right]=8\left[4\left(\frac{1}{4}\right)+3\right]=8(1+3)=32$$

Since the slope of the curve at the point of tangency is defined by the slope of the tangent line at that point, therefore, slope of the curve at $\left(\frac{1}{2},16\right)$ equals 32.

b. $y=2x^2\sqrt{4-x}$; $(0,0)$

Solution: Use the product formula of differentiation, $\frac{d}{dx}(uv)=u\frac{d}{dx}(v)+v\frac{d}{dx}(u)$

$$\frac{dy}{dx}=f'(x)=2x^2\left[\frac{1}{2\sqrt{4-x}}(-1)\right]+\sqrt{4-x}(4x)$$

$$\frac{dy}{dx}=\frac{-x^2}{\sqrt{4-x}}+4x\sqrt{4-x}$$

$$\frac{dy}{dx}=\frac{-x^2+4x(4-x)}{\sqrt{4-x}}$$

$$\frac{dy}{dx}=\frac{-x^2+16x-4x^2}{\sqrt{4-x}}=\frac{16x-5x^2}{\sqrt{4-x}}$$

At point $(0,0)$, $f'(0)=\frac{dy}{dx}=\frac{0}{2}=0$. Therefore, slope of the curve at $(0,0)$=slope of the tangent at $(0,0)$=0. This implies that the tangent to the curve at $(0,0)$is a horizontal line.

Example 3. Find the equation of the tangent line and normal line to the graph of $y=\dfrac{4}{x+1}$ at $(1,2)$.

Solution: Use the differentiation formula $\dfrac{d}{dx}\left(\dfrac{c}{u}\right)=\dfrac{-c}{u^2}\dfrac{d}{dx}(u)$, $y'=\dfrac{-4}{(x+1)^2}(1)$. At point $(1,2)$, $\dfrac{dy}{dx}=\dfrac{-4}{(1+1)^2}=\dfrac{-4}{4}=-1$. Hence, slope of the tangent line at $(1,2)$ equals -1 while slope of the normal line at the point is 1. Using the point-slope form of the equation of a line,

Equation of tangent line: $y-2=-1(x-1)$

$$y-2=-x+1$$

$$x+y-3=0$$

Equation of the normal line at $(1,2)$: $y-2=1(x-1)$

$$y-2=x-1$$

$$x-y+1=0$$

Example 4: Find the equation of the tangent line to the curve $y=x^2-x-6$ at the points of intersection of the curve with the x-axis.

Solution: First, find the point of intersection of the curve and the x-axis which is represented by $y=0$.

Therefore, $0=x^2-x-6=(x+2)(x-3)$

Solve for x. $x=-2$ and $x=3$

Thus, points of intersection are $(-2,0)$ and $(3,0)$.

Differentiate. $\dfrac{dy}{dx}=f'(x)=2x-1$

Slope of tangent at $(-2,0)$: $f'(-2)=2(-2)-1=-4-1=-5$

Equation of tangent line at $(-2,0)$: $y-0=-5(x+2)$

$$y=-5x-10$$

$$5x+y+10=0$$

Slope of tangent line at $(3,0)$: $f'(3)=2(3)-1=6-1=5$.

Similarly, equation of tangent at $(3,0)$: $y-0=5(x-3)$

$$5x-y-15=0$$

Example 5. At what points are the tangent to curve $y = x^3 + 4$

(a) parallel to line $y - 12x + 1 = 0$,
(b) perpendicular to line $18y + 6x + 7 = 0$.

Solution: (a). Reducing the given equation of line $y - 12x + 1 = 0$ to slope-intercept form $y = mx + b$ gives slope of line equal to 12 which is equal to the slope of the tangent line at the unknown points. Therefore,

$$f'(x) = 3x^2$$
$$12 = 3x^2$$
$$4 = x^2$$

Substitute x-value on equation $y = x^3 + 4$ of the curve to get corresponding y-value. When $x = 2, y = 12$ while $x = -2, y = -4$. So, the points on the curve where the tangent is parallel to $y - 12x + 1 = 0$ are $(2,12)$ and $(-2,-4)$.

Solution: (b). The slope of line $18y + 6x + 7 = 0$ after reduction to $y = mx + b$ is $-\frac{1}{3}$. Hence, slope of the tangent to the curve perpendicular to $18y + 6x + 7 = 0$ is 3 which is equal to y'. Therefore,

$$y'(x) = 3$$
$$3x^2 = 3$$
$$x^2 = 1$$
$$x = \mp 1$$

Substitution of $x = 1$ to the equation of curve $y = x^3 + 4$ gives $y = 5$ while when $x = -1, y = 3$. Thus, the points on the curve $y = x^3 + 4$ where the tangent line is perpendicular to line $18y + 6x + 7 = 0$ are $(-1,3)$ and $(1,5)$.

SAQ9
ACTIVITY 3.6 – C

NAME: ____ SCORE: _______________

SECTION: ____________DATE: ________________ PROF: _____________________________________

I. Find the equation of the tangent line to the given curves at the given point of tangency.

1. $y = x^3(x-1)^4$ at the point $(2,8)$

2. $y = \sqrt{x^2 - 25}$ at the point $(13,12)$

II. Find the equation of the normal line to the curves at the given point of tangency.

1. $y = (x^2 - 1)^2$ at the point $(-2,9)$

2. $y = \dfrac{x^2 - 4}{x+1}$ at the point $(0,-4)$

ASAQ9
ACTIVITY 3.6 – C

NAME: ____ SCORE: _______________

SECTION: ____________DATE: ________________ PROF: ____________________________________

I. Find the equation of the tangent line to the given curves at the given point of tangency.

1. $y = x^3(x-1)^4$ at the point $(2,8)$ $Answer: 44x - y - 80 = 0$

2. $y = \sqrt{x^2 - 25}$ at the point $(13,12)$ $Answer: 13x - 12y - 25 = 0$

II. Find the equation of the normal line to the curves at the given point of tangency.

1. $y = (x^2 - 1)^2$ at the point $(-2,9)$ $Answer: x - 24y - 218 = 0$

2. $y = \dfrac{x^2 - 4}{x + 1}$ at the point $(0,-4)$ $Answer: x - 4y + 16 = 0$

ACTIVITY 3.6 – D

NAME: ____ SCORE: _____________

SECTION: ___________DATE: ______________ PROF: _______________________________

Solve the following problems.

1. At what value/s of x will the normal to the curve $y = x^3 - \frac{1}{2}x^2 + 5x + 1$ perpendicular to the line $y = 9x - 2$?

2. Find equation of the normal to the curve $y = x^3 - 4x$ that is parallel to the line $x + 8y - 8 = 0$.

3. Find the points where the tangent to the curve $y = \sqrt[3]{7x - 6}$ is perpendicular to the line $12x + 7y + 2 = 0$.

4. At what point/s will the tangent/s to the curve $y = \dfrac{2x-1}{2x+1}$ be parallel to the line $y - 4x + 3 = 0$?

MODULE 7

ANGLE OF INTERSECTION OF CURVES

Specific Objectives:

At the end of the module, students must be able to:

1. Find point of intersection of curves of given functions.
2. Determine the angle of intersection between two intersecting curves.

ANGLE OF INTERSECTION OF TWO CURVES

Angle of intersection denoted by Greek letter phi (∅) is defined as the angle between the tangents to the curves at their point of intersection. Unless otherwise specified, the angle between the tangents will be taken as the acute angle between the lines. It is found by doing the following steps.

(1). Determine the point/s of intersection of the given curves using elimination or substitution method.

(2). Find the slopes m_1 and m_2 of the tangent lines to the curves at their point of intersection by evaluating $\frac{dy}{dx}$ of each function at their point of intersection.

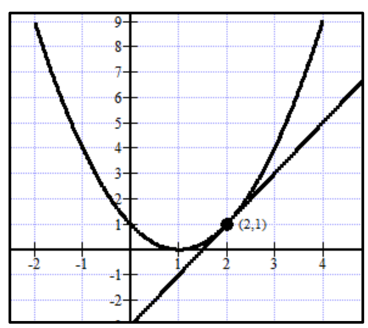
(3). Then, use the formula $\tan\phi = \frac{m_2 - m_1}{1 + m_2 m_1}$.

- If $\tan\phi > 0$, ϕ is the acute angle of intersection.
- If $\tan\emptyset < 0$, $\emptyset = 180^o - Tan^{-1}|\tan\emptyset|$ is the obtuse angle of intersection.
- If $m_2 m_1 = -1$, the tangents are perpendicular lines, hence $\phi = 90^o$. Moreover, If $m_2 = m_1 = 0$, then, $\emptyset = 0^o$ and this signifies that the tangents are coincident lines.

Example 6. Find the angle of intersection of curves (1) $y = x^2 - 2x + 1$ and (2) $y = 2x - 3$.

Solution: Their point of intersection is obtained using substitution method.

$$x^2 - 2x + 1 = 2x - 3$$
$$x^2 - 4x + 4 = 0$$
$$(x-2)^2 = 0$$
$$x = 2$$



When $x = 2$, $y = 1$. Hence, the point of intersection of the given parabola and line is $(2,1)$.

Differentiating each given function and evaluating the derivative at $x = 2$,

$$\frac{dy}{dx} = f_1'(x) = m_1 = 2x - 2$$

At $x = 2$, $f_1'(2) = 2(2) - 2 = 4 - 2 = 2$

This means the tangent to the parabola at the point where $x = 2$ has a slope of 2.

Likewise, differentiating $y = 2x - 3$ results to $\frac{dy}{dx} = f_2'(x) = m_2 = 2$. This signifies that the slope of the tangent line to line $y = 2x$ is always equal to 2 at any point on it.

Since $m_1 = m_2 = 2$, then, tangent lines at the point of intersection of the parabola and line are coincident lines, hence, $\phi = 0^o$.

Example 7. Find the acute angle of intersection of curves $y = x + \frac{1}{x}$ and $y = 1 + x^2$.

Solution: Find their point of intersection using substitution method.

$$x + \frac{1}{x} = 1 + x^2$$

$$x^2 + 1 = x + x^3$$

$$x^3 - x^2 + x - 1 = 0$$

The polynomial above is factorable by grouping.

$$(x^3 - x^2) + (x - 1) = 0$$
$$x^2(x - 1) + (x - 1) = 0$$
$$(x - 1)(x^2 + 1) = 0$$

Hence, $x = 1$, and at this value of x, $y = 2$.
The roots of equation $x^2 + 1 = 0$ are imaginary numbers, hence, rejected.
Again, slope of tangents is represented by $\frac{dy}{dx}$ or $f'(x)$.

Differentiating, $f^1(x) = m_1 = 1 - \frac{1}{x^2}$. Furthermore, $f_2'(x) = m_2 = 2x$

At $(1,2,)$, $f_1'(1) = m_1 = 1 - 1 = 0$ while $f_2'(1) = m_2 = 2(1) = 2$. Therefore,

$$\tan\phi = \frac{2 - 0}{1 + 0(2)} = 2$$

$$\emptyset = Tan^{-1}(2)$$

$$\phi = 63.43^o$$

SAQ10
ACTIVITY 3.7 – E

NAME: ____ SCORE: _______________

SECTION: ____________DATE: ________________ PROF: _____________________________________

Find the acute angle/s of intersection of the given curves.

(a). $y = \sqrt{4x - x^2}$ and $y = \sqrt{8 - x^2}$

(b). $y = 2x^2 - 11$ and $y = x^2 - 4x + 10$

ASAQ10
ACTIVITY 3.7 – E

NAME: ____ SCORE: _______________

SECTION: ____________DATE: ________________ PROF: _____________________________________

Find the acute angle/s of intersection of the given curves.

(a). $y = \sqrt{4x - x^2}$ and $y = \sqrt{8 - x^2}$
$Answer$: 45^o

(b). $y = 2x^2 - 11$ and $y = x^2 - 4x + 10$
$Answer$: 1.13^o, 4.57^o

MODULE 8

RECTILINEAR MOTION

Specific Objectives:

At the end of the module, students must be able to:

1. Understand concept of velocity and acceleration in relation to derivative of a function.
2. Solve using derivative velocity and acceleration problems.

RECTILINEAR MOTION

It is the motion of a particle along a straight line path. Equation of the form $s = f(t)$called equation of motion completely described the motion of the particle, where $t \geq 0$is the time and$s$is the displacement of the particle at any particular time, measured from a chosen fixed point in its path called reference point. Most of the time, the starting point of motion (the position of the particle when $t = 0$) is the selected reference point. The velocity $\boldsymbol{v}$and acceleration $\boldsymbol{a}$of the particle at time t can be computed using the equations below.

$$\boxed{v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}} \qquad \boxed{a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt}}$$

HORIZONTAL RECTILINEAR MOTION

This is motion of the particle is along a horizontal straight line. Since displacement, velocity and acceleration are all vector quantities, we will use the following sign convention:

- If $s > 0$, the particle is positioned at the right of the reference point
- If $s < 0$, the particle is positioned at the left of the reference point
- If $v > 0$, the particle is moving in the direction of increasing s(moving to the right away from the reference point, $\rightarrow$)
- If $v < 0$, it is moving in the direction of decreasing s(moving to the left, towards or away from the reference point, $\leftarrow$)
- If $v = 0$, the particle is at rest at that particular time
- If $a > 0$, velocity is increasing
- If $a < 0$, velocity is decreasing which means the particle is decelerating

VERTICAL RECTILINEAR MOTION

Free-fall motion is a good example of a vertical rectilinear motion. The moving particle, called a freely-falling body, is acted upon only by its weight with the air resistance considered negligible. Its acceleration is due to gravity.

Hence, $a = -g = -32\frac{ft}{s^2} = -10\frac{m}{s^2} = -980\frac{cm}{s^2}$.

The first two values are rounded-off to the nearest integer for computational convenience. For s and v, the following sign convention applies:

- If $s > 0$, the particle is positioned above the reference point
- If $s < 0$, the particle is positioned below the reference point
- If $v > 0$, the particle is moving in the direction of increasing s, $(\uparrow)$
- If $v < 0$, it is moving in the direction of decreasing s, $(\downarrow)$

In Physics, the position of the freely-falling body at any time t is described by equation

$$\boxed{s = v_0 t + \frac{1}{2} a t^2}$$

where: s = displacement at a particular time t

v_o = initial velocity of the moving particle (its velocity when t = 0)

$a = -g$

Example 8. A particle is moving along a horizontal straight line according to equation of motion $s = 2t^3 - 4$, where s, in meters, is the displacement of the particle at t seconds. Find the displacement, velocity and acceleration when $t = 2$ seconds.

Solution: Considering the given equation of motion $s(t) = 2t^3 - 4$, when $t = 2$, $s(2) = 2(2)^3 - 4 = 16 - 4 = 12$. This implies that the particle is positioned 12 meters at the right of the chosen reference point.

Differentiate the given equation of motion with respect to time t to get the velocity equation. Thus,

$$v(t) = \frac{ds}{dt} = 2(3)t^2 = 6t^2$$

When $t = 2$, $v(2) = 6(2)^2 = 6(4) = 24\ \frac{m}{\sec}(\rightarrow)$. This means the particle is moving to the right when $t = 2$.

Differentiate the velocity equation with respect to time t to get the acceleration equation.

Hence, $a(t) = \frac{dv}{dt} = 6(2)t = 12t$

When $t = 2$, $a(2) = 12(2) = 24\ \frac{m}{s^2}$. The positive sign of the computed acceleration means the particle is speeding-up when $t = 2$ seconds.

Example 9. At what time will the acceleration of the particle moving according to the law $s = t^3 - 9t^2 + 15t$ be equal zero? Where is the particle at that particular time?

Solution: Equation of motion: $s(t) = t^3 - 9t^2 + 15t$

Velocity equation: $v(t) = \frac{ds}{dt} = 3t^2 - 18t + 15$

Acceleration equation: $a(t) = \frac{dv}{dt} = 6t - 18$

At the required time, the given condition is $a = 0$.

$$0 = 6t - 18$$
$$6t = 18$$
$$t = \frac{18}{6} = 3$$

Position of the particle when $t = 3$:

$$s = (3)^3 - 9(3)^2 + 15(3) = 27 - 81 + 45$$
$$s = -9$$

Therefore, when $t = 3$, the particle is moving at constant speed (zero acceleration) and is 9 units to the left of the chosen reference point (the starting point).

Example 10.A particle moves such that $s = \frac{1}{4}t^4 - \frac{5}{3}t^3 + 2t^2$. Find its position when it is at rest.

Solution: The particle when it is at rest means its velocity equals zero.

Velocity equation: $v(t) = \frac{ds}{dt} = \frac{1}{4}(4)t^3 - \frac{5}{3}(3)t^2 + 4t$

$$v(t) = t^3 - 5t^2 + 4t$$

Condition at the required position: $v = 0$

$$t^3 - 5t^2 + 4t = 0$$
$$t(t^2 - 5t + 4) = 0$$
$$t(t-4)(t-1) = 0$$

Therefore, the time when the particle is at rest is when $t = 0$, $t = 1$ and $t = 4$.

Finding its position at each time when the particle is at rest:

$s(0) = 0$ (At the reference point)

$s(1) = \frac{1}{4} - \frac{5}{3} + 2 = \frac{3 - 20 + 24}{12} = \frac{7}{12}$ (The particle is at the right of the reference point)

$s(4) = \frac{1}{4}(4)^4 - \frac{5}{3}(4)^3 + 2(4)^2 = 64 - \frac{320}{3} + 32$

$= 96 - \frac{320}{3} = \frac{288 - 320}{3} = -\frac{32}{3}$ (The particle is at the left of the reference point)

Example 11. A stone moves in a vertical line under gravity alone, with negligible air resistance. If it is thrown initially upward at a velocity of $10\frac{m}{s}$, how far and for how long a time the stone will rise?

Solution: This kind of motion is a free-fall motion. Hence, the stone is moving as a freely-falling body. Use the equation of motion $s = v_o t + \frac{1}{2}at^2$.

Since initial velocity $v_o = +10m/s(\uparrow)$, then, $a = -g = -10m/s^2$. Thus, the equation of motion is:

$$s = +10t + \frac{1}{2}(-10)t^2$$
$$s = +10t - 5t^2$$

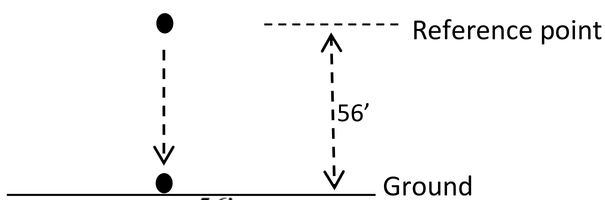
Differentiating s with respect to time t, the velocity equation is $v(t) = +10 - 10t$.
However, at the highest point of the stone's flight, the stone momentarily stops. Hence, at the maximum displacement, $v = 0$.

$$10 - 10t = 0$$
$$t = 1 \text{ sec}$$

Using $t = 1$sec, the maximum displacement $S_{\max} = 10(1) - 5(1)^2 = 10 - 5 = 5m.$

Example 12. From a point 56 feet above the ground, at what velocity will a stone strike the ground if it hits the ground after one second?

Solution: The starting point, the chosen reference point, is 56' above the ground.



$s = -56'$

$t = 1s$

$v = ?$

When the stone hits the ground, the displacement s is negative since its position is below the chosen reference point, the starting point of motion. The acceleration $a = -32\frac{ft}{s^2}$. Therefore,

$$s = v_o t + \frac{1}{2}(-32)t^2$$
$$s = v_o t - 16t^2$$

When on the ground, $t = 1s$, $s = -56'$: $-56 = v_o(1) - 16(1)^2$

$$v_o = -56 + 16 = -40\, ft/s(\downarrow)$$

Since calculated v_o is negative, it implies that the stone is thrown initially downward. Thus,

$$s = -40t - 16t^2$$

To find the velocity when it hits the ground:

$$v = \frac{ds}{dt} = -40 - 16(2)t$$
$$v = -40 - 32t$$

When it strikes the ground, $t = 1s$, $v = -40 - 32(1) = -72\,ft/s(\downarrow)$.

Computed velocity of the stone when it hits the ground is negative, indicating that the direction of motion at that particular time is downward.

Example 13. A stone is dropped without giving any push on it from the top of a building 256 feet high. Find (a) the velocity of the stone 3 seconds after it was dropped, (b) the time and the velocity of the stone when it reaches the ground.

Solution: Since the stone is dropped without giving any push on it, then, initial velocity v_o equals zero, with acceleration $a = -g = -32\frac{ft}{s^2}$.

$$s = v_o t + \frac{1}{2}at^2$$
$$s = -16t^2$$

Velocity equation: $v = -32t$

When $t = 3s$, $v = -32(3) = -96\,ft/s(\downarrow)$

When it strikes the ground, $s = -256\,ft$,

$$-256 = -16t^2$$
$$t^2 = \frac{256}{16} = 16$$
$$t^2 - 16 = 0$$
$$(t+4)(t-4) = 0$$

$t = 4s$ $\quad$ $t = -4s$ (Rejected)

When $t = 4s$, velocity $v = -32(4) = -128\,ft/s(\downarrow)$.

SAQ11
ACTIVITY 3.8 – F

NAME: ____ SCORE: _______________

SECTION: ____________DATE: ________________ PROF: ____________________________________

Solve for the unknown/s.

1. A particle moves horizontally according to the law $s=\frac{1}{3}t^3-\frac{1}{2}t^2-12t+1$. Where is the particle and at what velocity is it moving when $t=1$? When will the particle come to rest and what is its acceleration at that time?

2. The path taken by a particle is a horizontal line as it moves according to the law $s=\frac{1}{6}t^4-\frac{7}{6}t^3-7t^2+\frac{1}{2}t+1$. At what time will its acceleration be equal to one? What is its velocity at that time?

3. Two particles leave the same point at the same time and both are moving along a horizontal line according to the law $s=4t^3-2t^2+3$ and $s=-7+28t-10t^2$, respectively. When are they moving at the same speed? Find their positions, velocity and acceleration at that time?

4. A ball is thrown vertically downward from a height of 512 feet with a velocity of $64\frac{ft}{s}$. How long will it take the ball to reach the ground and at what velocity?

5. A stone is thrown vertically upward with an initial velocity of $80\frac{ft}{s}$ from the top of a building. How high will it rise? When and at what velocity will it strike the ground if the height of the building is 576 feet?

ASAQ11
ACTIVITY 3.8 – F

NAME: ____ SCORE: ____________

SECTION: __________DATE: ______________ PROF: ______________________________

Solve for the unknown/s.

1. A particle moves horizontally according to the law $s = \frac{1}{3}t^3 - \frac{1}{2}t^2 - 12t + 1$. At what velocity is the particle moving when $t = 1$? When will the particle come to rest and what is its acceleration at that time? $Answer: v(1) = -12,\ t = 4,\ \ a(4) = 7$

2. The path taken by a particle is a horizontal line as it moves according to the law $s = \frac{1}{6}t^4 - \frac{7}{6}t^3 - 7t^2 + \frac{1}{2}t + 1$. At what time will its acceleration be equal to one? What is its velocity at that time? $Answer: t = 5, a(5) = -\frac{221}{3}$

3. Two particles leave the same point at the same time and both are moving along a horizontal line according to the law $s = 4t^3 - 2t^2 + 3$ and $s = -7 + 28t - 10t^2$, respectively. When are they moving at the same speed? Find their positions and acceleration at that time?
$Answer: t = 5, s_1 = 5, s_2 = 11, a_1 = 20, a_2 = -20$

4. A ball is thrown vertically downward from a height of 512 feet with a velocity of $64\frac{ft}{s}$. How long will it take the ball to reach the ground and at what velocity? $Answer: t = 4, v = -192$

5. A stone is thrown vertically upward with an initial velocity of $80\frac{ft}{s}$ from the top of a building. How high will it rise? When and at what velocity will it strike the ground if the height of the building is 576 feet? $Answer: t = 9, v = -208$

ACTIVITY 3.8 – G

NAME: ____ SCORE: _______________

SECTION: ____________DATE: ________________ PROF: ____________________________________

1. A stone is dropped from rest from a point 125 meters from the ground. Find:
(a). its velocity when it falls on the ground.
(b). the time it takes the stone to reach the point 80 meters above the ground.

2. A diver jumps upward at a velocity of 32 $\frac{\text{ft}}{\text{sec}}$ from a diving board that is 48 feet above the water. At what time and at what velocity will the diver hit the water?

3. From a point 160 feet above the ground, a body is thrown vertically upward. When it is 32 feet above the starting point, it is moving downward at $16\frac{ft}{s}$. Find:
(a). the initial velocity of motion.
(b). the time when it returns back to the starting point.
(c). the time and velocity when the body hits the ground.
(d). the highest point of its flight measured from the ground.

4. A ball is thrown vertically from a point 30 feet above the ground.
(a). At what velocity must it be thrown for it to strike the ground in 2 seconds? Is the ball initially thrown vertically upward or downward?
(b). What is its velocity when it strikes the ground?
(c). Where is the ball one second after it was thrown initially? Is it moving upward or already on its way down? (Hint: Take the time and displacement at the maximum point of its flight by setting $v = 0$.)

5. A stone thrown vertically upward from the ground reaches a height of 5 m in one second. Find how high the ball will rise. Assume the stone moves as a freely-falling body.

6. A ball is thrown vertically upward from a point 13 m from the horizontal ground. When and at what velocity will it strike the ground if its velocity 3 m above the point of origin is 2 m/sec and is still moving upward?

MODULE 9

HIGHER ORDER DERIVATIVES

Specific Objectives:

At the end of the module, students must be able to:

1. Understand concept of higher order derivative.
2. Use an appropriate differentiation formula to find higher order derivative of a given function.

DERIVATIVES OF HIGHER ORDER

If $y = f(x)$ is a differentiable function of x, then, y' or f' is sometimes termed the first derivative of y with respect to x. If y' is a differentiable function, then, $\frac{d}{dx}y' = y'' = f''$ (read as "y double prime"). This is called the second derivative of y with respect to x. Likewise, $\frac{d}{dx}(y'') = y''' = f'''$ is the third derivative of y with respect to x, provided y'' exists. Moreover, $\frac{d}{dx}y^{n-1} = y^{(n)} = f^{(n)}$ is the n^{th} derivative of y with respect to x, where n is a positive integer greater than 1.

Based on Leibniz notation, $\frac{dy}{dx}$ is the first derivative, $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2(y)}{dx^2}$. The n^{th} derivative of y with respect to x has the notation $\frac{d^n y}{dx^n}$ or $D_x{}^n y$ or $\frac{d^n f(x)}{dx^n}$.

Example 14. Given: $y = 2x^5 - 3x^4 - 6x^3 - x^2 - 1$. Find $y^{(4)}$.

$$y' = 10x^4 - 12x^3 - 18x^2 - 2x$$
$$y'' = 40x^3 - 36x^2 - 36x - 2$$
$$y''' = 120x^2 - 72x - 36$$
$$y^{(4)} = 240x - 72 = 24(10x - 3)$$

Example 15. Given: $y = \frac{4}{2x-1}$, Find y'''.

Use Formula 10: $\frac{d}{dx}\left(\frac{c}{u}\right) = \frac{-c}{u^2}\frac{d}{dx}(u)$.

$$y' = \frac{-4}{(2x-1)^2}\frac{d}{dx}(2x-1) = \frac{-4}{(2x-1)^2}(2) = \frac{-8}{(2x-1)^2}$$

Use again Formula 10:

$$y'' = \frac{-(-8)}{(2X-1)^4}(2)(2x-1)(2) = \frac{32}{(2x-1)^3}$$

For the 3rd time, use Formula 10.

$$y''' = \frac{-32}{(2x-1)^6}(3)(2x-1)^2(2) = \frac{-192}{(2x-1)^4}$$

Example 16. Given: $y = (1-3x)(2x+5)^3$.

Use Formula 7: $\frac{d}{dx}(uv) = u\frac{d}{dx}(v) + v\frac{d}{dx}(u)$

$$y' = (1-3x)(3)(2x+5)^2(2) + (2x+5)^3(-3)$$

Bring-out the common monomial factor. $y' = 3(2x+5)^2[2(1-3x)-(2x+5)]$

Simplify further. $y' = 3(2x+5)^2(-8x-3) = -3(2x+5)^2(8x+3)$

Differentiate again: $y'' = -3[(2x+5)^2(8)+(8x+3)(2)(2x+5)(2)]$

Bring-out the common monomial factor. $y'' = -3(4)(2x+5)[2(2x+5)+(8x+3)]$

$y'' = -12(2x+5)(12x+13)$

Differentiate again. $y''' = -12[(2x+5)(12)+(12x+13)(2)]$

$y''' = -12(2)[6(2x+5)+(12x+13)]$

$y''' = -24(12x+30+12x+13)$

$y''' = -24(24x+43)$

Example 17. Find $\frac{d^2y}{dx^2}$, given $y=\sqrt{x^2+4}$.

Differentiate using the formula $\frac{d}{dx}\sqrt{u} = \frac{1}{2\sqrt{u}}\frac{du}{dx}$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x^2+4}}\frac{d}{dx}\left(x^2+4\right) = \frac{1}{2\sqrt{x^2+4}}(2x) = \frac{x}{\sqrt{x^2+4}}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{x}{\sqrt{x^2+4}}\right)$$

Using the quotient formula $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{d}{dx}(u)-u\frac{d}{dx}(v)}{v^2}$

$$\frac{d^2y}{dx^2} = \frac{\sqrt{x^2+4}(1)-x\left(\frac{1}{2\sqrt{x^2+4}}\right)(2x)}{\left(\sqrt{x^2+4}\right)^2} = \frac{\sqrt{x^2+4}-\frac{x^2}{\sqrt{x^2+4}}}{x^2+4}$$

$$\frac{d^2y}{dx^2} = \frac{\left(\sqrt{x^2+4}\right)^2-x^2}{\left(x^2+4\right)\sqrt{x^2+4}} = \frac{x^2+4-x^2}{\left(x^2+4\right)\sqrt{x^2+4}} = \frac{4}{\sqrt{\left(x^2+4\right)^3}}$$

But $y=\sqrt{x^2+4}$, therefore, $\frac{d^2y}{dx^2} = \frac{4}{y^3}$

SAQ12
ACTIVITY 3.9 – H

NAME: ____ SCORE: ______________

SECTION: ___________DATE: _______________ PROF: __________________________________

I. Find the second derivative of the given function.

1. $y = (2 + 3x)^2(1 - 4x)$

2. $y = \sqrt{1 + 8x}$

3. $y = \frac{1+5x}{1-5x}$

ASAQ12
ACTIVITY 3.9 – H

NAME: ____ SCORE: _______________

SECTION: ___________DATE: _______________ PROF: ___________________________________

II. Find the second derivative of the given function.

1. $y = (2 + 3x)^2(1 - 4x)$ $\quad Answer: -8(1 - 4x)$

2. $y = \sqrt{1 + 8x}$ $\quad Answer: \frac{-16}{(1+8x)\sqrt{1+8x}}$

3. $y = \frac{1+5x}{1-5x}$ $\quad Answer: \frac{100}{(1-5x)^3}$

MODULE 10

IMPLICIT DIFFERENTIATION

Specific Objectives:

At the end of the module, students must be able to:

1. Understand concept of implicit differentiation.
2. Perform implicit differentiation to find derivative of a given function.

IMPLICIT DIFFERENTIATION

We encounter some equations in x and y that do not explicitly define y as a function of x. It is not easy manipulating the equation to solve for y in terms of x, even though such function exists. The technique of finding $\frac{dy}{dx}$ without solving the given function for y is termed implicit differentiation. Generally, if the given function takes the form $F(x, y) = 0$, we find the $\frac{dy}{dx}$ by following the steps listed below.

a. Whenever possible, we solve the given equation of the curve for y and then, differentiate y with respect to x. This is true only for very simple equations; for complicated functions, this step is to be avoided.
b. Considering y as a function of x, differentiate each term of the given equation with respect to x, bearing in mind that y is a function of x, and solve the resulting equation for y'. This process of finding the derivative is implicit differentiation.

Example 18. Find y', given $y^2 - 3x^3y - 5x^2 + 4 = 0$

It is not possible to solve the given equation for y as a function of x. Thus, we differentiate implicitly.

$$\frac{d}{dx}y^2 - 3\frac{d}{dx}x^3y - \frac{d}{dx}5x^2 + \frac{d}{dx}4 = 0$$

$$2y\frac{dy}{dx} - 3\left[x^3\frac{dy}{dx} + y\left(3x^2\right)\right] - 10x = 0$$

Collect terms having $\frac{dy}{dx}$.
$$2y\frac{dy}{dx} - 3x^3\frac{dy}{dx} - 9x^2y - 10x = 0$$

Factor-out $\frac{dy}{dx}$.
$$\frac{dy}{dx}\left(2y - 3x^3\right) = 10x + 9x^2y$$

$$\frac{dy}{dx} = \frac{10x + 9x^2y}{2y - 3x^3}$$

$$\frac{dy}{dx} = \frac{x(10 + 9xy)}{2y - 3x^3}$$

Example 19.Find y'', given $2x^2 - y^2 = 5$

Using implicit differentiation:
$$\frac{d}{dx}\left(2x^2\right) - \frac{d}{dx}\left(y^2\right) = \frac{d}{dx}(5)$$

$$4x - 2y\frac{dy}{dx} = 0$$

$$-2y\frac{dy}{dx} = -4x$$

$$\frac{dy}{dx}=\frac{4x}{2y}=\frac{2x}{y}\text{, provided}\left(y\neq 0\right)$$

Using the quotient formula, differentiate $\frac{dy}{dx}$ implicitly with respect to x to get $\frac{d^2y}{dx^2}$ or y''.

Simplify using $\frac{dy}{dx}=\frac{2x}{y}$.

$$y''=2\left[\frac{y\frac{dx}{dx}-x\frac{dy}{dx}}{y^2}\right]=2\left[\frac{y-x\frac{dy}{dx}}{y^2}\right]$$

$$y''=\frac{2\left[y-x\left(\frac{2x}{y}\right)\right]}{y^2}=\frac{2\left(\frac{y^2-2x^2}{y}\right)}{y^2}$$

$$y''=\left[\frac{2\left(y^2-2x^2\right)}{y}\right]\cdot\frac{1}{y^2}$$

$$y''=\frac{2\left(y^2-2x^2\right)}{y^3}=\frac{-2\left(2x^2-y^2\right)}{y^3}$$

However, from the given, $2x^2-y^2$ can be replaced by 5. Substituting and simplifying, we got:

$$y''=\frac{-2(5)}{y^3}=\frac{-10}{y^3}$$

Example 20. Find slope and equation of the tangent line to curve $2x^2-3xy+2y^2=2$ at point $\left(-1,-\frac{3}{2}\right)$.

Implicitly differentiate:

$$\frac{d\left(2x^2\right)}{dx}-3\frac{d}{dx}(xy)+\frac{d\left(2y^2\right)}{dx}=\frac{d(2)}{dx}$$

$$4x-3\left(x\frac{dy}{dx}+y\right)+4y\frac{dy}{dx}=0$$

$$4x-3x\frac{dy}{dx}-3y+4y\frac{dy}{dx}=0$$

$$\frac{dy}{dx}(4y-3x)=3y-4x$$

$$\frac{dy}{dx}=\frac{3y-4x}{4y-3x}$$

Hence, at point $\left(-1,-\frac{3}{2}\right)$, the slope of the tangent line is

$$\frac{dy}{dx} = y' = \frac{3\left(-\frac{3}{2}\right)-4(-1)}{4\left(-\frac{3}{2}\right)-3(-1)} = \frac{\frac{-9}{2}+4}{-6+3}$$

$$\frac{dy}{dx} = \frac{\frac{-9+8}{2}}{-3} = \frac{\frac{-1}{2}}{-3} = \frac{-1}{2}\cdot\frac{1}{-3} = \frac{1}{6}$$

Equation of the tangent line using the point-slope form $y - y_1 = m(x - x_1)$:

$$y-\left(-\frac{3}{2}\right) = \frac{1}{6}[x-(-1)]$$

$$y+\frac{3}{2} = \frac{1}{6}(x+1)$$

$$\frac{2y+3}{2} = \frac{x+1}{6}$$

$$2y+3 = \frac{x+1}{3}$$

$$6y+9 = x+1$$

$$x-6y-8 = 0$$

Example 21. At what point of the curve $xy = 6$ is the slope of the tangent line equal to $-\frac{1}{6}$.

Use implicit differentiation: $x\frac{dy}{dx} + y = 0$

$\frac{dy}{dx} = \frac{-y}{x}$ = slope of the tangent line

But $y = \frac{6}{x}$ and $\frac{dy}{dx} = -\frac{1}{6}$:

$$-\frac{1}{6} = \frac{-\frac{6}{x}}{x} = \frac{-6}{x^2}$$

$$x^2 = 36$$

$$x^2 - 36 = 0$$

$$(x-6)(x+6) = 0$$

$$x = 6 \qquad x = -6$$

$$y = 1 \qquad y = -1$$

Thus, at points $(6,1)$ and $(-6,-1)$, the slope of the tangent line is equal to $-\frac{1}{6}$.

Example 22. Find the angle of intersection of circles $x^2 + y^2 = 4$ and $x^2 + y^2 - 2y = 0$.

First, find the point of intersection of the given circles using substitution method. But $x^2 + y^2 = 4$

$$x^2 + y^2 - 2y = 0$$
$$(4) - 2y = 0$$
$$y = 2$$

When $y = 2$, $x = 0$. Since they intersect at only a point, therefore, the circles are tangent circles at point $(0,2)$. Implicitly differentiating $x^2 + y^2 = 4$ results to:

$$2x + 2yy' = 0$$
$$y' = -\frac{x}{y} = m_1$$

Likewise, differentiating $x^2 + y^2 - 2y = 0$:

$$2x + 2yy' - 2y' = 0$$
$$2y'(y-1) = -2x$$
$$y' = \frac{-2x}{2(y-1)} = \frac{-x}{y-1} = m_2$$

At point $(0,2)$, $m_1 = m_2 = 0$. Therefore, $\tan\phi = \frac{0-0}{1+0} = 0.$

$$\phi = Arc\tan(0) = 0$$

Therefore, the tangent lines to the tangent circles at point $(0,2)$ are coincident lines, thus, $\phi = 0^o$.

SAQ13
ACTIVITY 3.10 – I

NAME: ____ SCORE: ______________

SECTION: ___________DATE: _______________ PROF: ___________________________________

I. Using implicit differentiation, find $\frac{dy}{dx}$.

1. $x^3y = 2$

2. $xy = x + y$

3. $xy^2 - x^2 + 2x^2y = 3$

4. $2x^3 = (4xy - 1)^2$

II. Using implicit differentiation, find y''(x).

1. $2y^2 + 5 = 3x^2$

2. $3x^2 + 5y^2 = 4$

III. Find equation of the tangent and normal to $x^2 + 1 + 3xy + y^2 = 0$ at point$(-1, 1)$.

IV. At what points are tangents to circle $x^2 + y^2 = 25$ of slope equal to $\frac{3}{4}$?

ASAQ13
ACTIVITY 3.10 – I

NAME: ____ SCORE: _______________

SECTION: ____________DATE: ________________ PROF: _____________________________________

V. Using implicit differentiation, find $\frac{dy}{dx}$.

2. $x^3y = 2$ $\quad Answer: -\frac{3y}{x}$

2. $xy = x + y$ $\quad Answer: \frac{1-y}{x-1}$

3. $xy^2 - x^2 + 2x^2y = 3$ $\quad Answer: \frac{2x - y^2 - 4xy}{2(x+y)}$

4. $2x^3 = (4xy-1)^2$ $\quad Answer: \frac{3x^2 - 16xy^2 + 4y}{16x^2y - 4x}$

VI. Using implicit differentiation, find y''(x).

3. $2y^2 + 5 = 3x^2 \quad Answer: -\dfrac{15}{4y^3}$

4. $3x^2 + 5y^2 = 4 \quad Answer: -\dfrac{12}{25y^3}$

VII. Find equation of the tangent and normal to $x^2 + 1 + 3xy + y^2 = 0$ at point$(-1, 1)$.

$Answer: x - y + 2 = 0; \;\; x + y = 0$

VIII. At what points are tangents to circle $x^2 + y^2 = 25$ of slope equal to $\frac{3}{4}$?

$Answer: (3, -4); \;\; (-3, 4)$

ACTIVITY 3.10 – J

NAME: ____ SCORE: _______________

SECTION: ____________DATE: ________________ PROF: ____________________________________

I. Find $\frac{d^2y}{dx^2}$ using implicit differentiation.

1. $4y^2 - 3x^2 = 8$

2. $6x^2 + 7y^2 = 3$

II. Find equation of the tangent and normal to curve $3x^2 - 2xy - y^2 = 3$ at the point $(1,0)$.

III. At what point does the line $y = 2x + 1$ cross the tangent line to the curve $2x^3 - 3xy + 2y^2 = 0$ at point $(-1,-2)$?

IV. Find the point of intersection of the tangents to circle $x^2 + y^2 = 10$ at $(3,-1)$ and to parabola $y^2 = 2x$ at $(2,2)$.

V. Find the angle of intersection

(a). between parabolas $x = y - y^2$ and $x = y^2 - 3$.

(b). circles $x^2 + y^2 - 8x - 2y + 7 = 0$ and $x^2 + y^2 - 3x - 7y + 12 = 0$. (Hint: Eliminate the quadratic terms to find their point of intersection.)

MODULE 11

CHAIN RULE OF DIFFERENTIATION

Specific Objectives:

At the end of the module, students must be able to:

1. Understand concept of chain rule of differentiation.
2. Perform chain rule of differentiation to find derivative of a given function.

CHAIN RULE OF DIFFERENTIATION

We are used to have $y = f(x)$ in finding the derivative of y with respect to x. If this is not the given case, the Chain Rule is one of the most important tools in differentiation.

Given	Differentiation Formula
$(a).y = f(u)$ and $x = g(u)$	$\frac{dy}{dx} = \frac{\frac{dy}{du}}{\frac{dx}{du}}$
$(b).y = f(u)$ and $u = g(x)$	$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$
$(c).x = f(y)$	$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$

Note: In Case (b), in many instances, it is conveniently possible to express $\frac{dy}{dx}$ in terms of x alone.

Example 23. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$, given $y = t^3 - 3$, $t = \sqrt{2x-1}$

Solution: The given functions fall on Case (b). Hence, $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$

$t = \sqrt{2x-1}$	$y = t^3 - 3$	$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$
$\frac{dt}{dx} = \frac{1}{2\sqrt{2x-1}}(2)$ $\frac{dt}{dx} = \frac{1}{\sqrt{2x-1}}$	$\frac{dy}{dt} = 3t^2$	$\frac{dy}{dx} = (3t^2) \cdot \frac{1}{\sqrt{2x-1}} = \frac{3t^2}{\sqrt{2x-1}}$ But, $t = \sqrt{2x-1}$. Therefore, $\frac{dy}{dx} = \frac{3(\sqrt{2x-1})^2}{\sqrt{2x-1}} = 3\sqrt{2x-1}.$

To find $\frac{d^2y}{dx^2}$, use Formula 6: $\frac{d}{dx}\sqrt{u} = \frac{1}{2\sqrt{u}} \frac{d}{dx}(u)$.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}y' = 3\left[\frac{1}{2\sqrt{2x-1}}\right](2) = \frac{3}{\sqrt{2x-1}} = \frac{3\sqrt{2x-1}}{2x-1}.$$

Example 24. If $y = (w^3+1)^5$ and $x = \frac{2}{w+1}$, find $\frac{dy}{dx}$.

The given functions fall under Case (a), so, $\frac{dy}{dx}=\frac{\frac{dy}{du}}{\frac{dx}{du}}$.

Use $\frac{d}{dx}\left(\frac{c}{u}\right)=\frac{-c}{u^2}\frac{d}{dx}(u)$. $\qquad \frac{dx}{dw}=\frac{-2}{(w+1)^2}(1)=\frac{-2}{(w+1)^2}$

Use the general power formula 5: $\frac{d}{dx}\left(cu^n\right)=cnu^{n-1}\frac{d}{dx}(u)$.

$$\frac{dy}{dw}=5\left(w^3+1\right)^4\left(3w^2\right)=15w^2\left(w^3+1\right)^4$$

Therefore,

$$\frac{dy}{dx}=\frac{\frac{dy}{dw}}{\frac{dx}{dw}}=\frac{15w^2\left(w^3+1\right)^4}{\frac{-2}{(w+1)^2}}$$

Divide the fractions. $\qquad \frac{dy}{dx}=15w^2\left(w^3+1\right)^4\cdot\frac{(w+1)^2}{-2}$

Finally, $\qquad \frac{dy}{dx}=-\frac{15w^2\left(w^3+1\right)^4(w+1)^2}{2}$

Example 25. Find equation of the tangent line to curve $x=\frac{y}{y^2-2}$ at point $(1,2)$.

Solution: Slope of the tangent line TL is given by $\frac{dy}{dx}$ which is equal to $\frac{1}{\frac{dx}{dy}}$.

The given function falls under Case (c). $\frac{dy}{dx}=\frac{1}{\frac{dx}{dy}}$.

$$\frac{dx}{dy}=\frac{\left(y^2-2\right)(1)-y(2y)}{\left(y^2-2\right)^2}=\frac{y^2-2-2y^2}{\left(y^2-2\right)^2}=\frac{-2-y^2}{\left(y^2-2\right)^2}=-\frac{2+y^2}{\left(y^2-2\right)^2}$$

Therefore, $\qquad \frac{dy}{dx}=\frac{1}{\frac{dx}{dy}}=\frac{1}{-\frac{\left(2+y^2\right)}{\left(y^2-2\right)^2}}=-\frac{\left(y^2-2\right)^2}{y^2+2}=m_{TL}$

At point $(1,2)$, $m_{TL} = -\dfrac{\left[(2)^2 - 2\right]^2}{(2)^2 + 2} = -\dfrac{(2)^2}{6} = -\dfrac{4}{6} = -\dfrac{2}{3}$

Thus, equation of tangent line is:

$$y - 2 = -\frac{2}{3}(x - 1)$$

$$3y - 6 = -2x + 2$$

$$2x + 3y - 8 = 0$$

SAQ14
ACTIVITY 3.11 – K

NAME: ____ SCORE: ______________

SECTION: ___________DATE: _______________ PROF: __________________________________

Using the appropriate method of differentiation, find y'.

1. $x = \dfrac{4y - 3y^2}{2 + y^2}$

2. $y = w(w^2 + 4)^4; \quad x = \dfrac{w-3}{w+3}$

3. $y = \sqrt{t^2 + 1}; \quad t = \dfrac{1}{x-1}$

4. $y = \dfrac{2}{3m-1}; m = \dfrac{1}{2x^2 + 3}$

5. $y = \dfrac{v^2 - 4}{v^2 + 4}; \quad x = \dfrac{1}{v+1}$

ASAQ14
ACTIVITY 3.11 – K

NAME: ____ SCORE: ______________

SECTION: ___________DATE: _______________ PROF: __________________________________

Using the appropriate method of differentiation, find y'.

1. $x = \dfrac{4y - 3y^2}{2 + y^2}$ $\quad Answer: \dfrac{-4(y^2 + 3y - 2)}{(2 + y^2)^2}$

2. $y = w(w^2 + 4)^4; \quad x = \dfrac{w - 3}{w + 3}$ $\quad Answer: (w^2 + 4)^3(9w^2 + 4)$

3. $y = \sqrt{t^2 + 1}; \quad t = \dfrac{1}{x - 1}$ $\quad Answer: \dfrac{-1}{(x - 1)^2\sqrt{1 + (x - 1)^2}}$

4. $y = \dfrac{2}{3m - 1}; m = \dfrac{1}{2x^2 + 3}$ $\quad Answer: \dfrac{6}{x^3}$

5. $y = \dfrac{v^2 - 4}{v^2 + 4}; \quad x = \dfrac{1}{v + 1}$ $\quad Answer: \dfrac{-16v(v + 1)^2}{(v^2 + 4)^2}$

ACTIVITY 3.11 – L

NAME: ____ SCORE: _______________

SECTION: ____________DATE: ________________ PROF: _____________________________________

I. Using the appropriate method of differentiation, find y'.

1. $y = \frac{1}{(t+1)^2}$; $t = x^2 + 4$

2. $x = t + 2$; $y = \left(\frac{t}{t-1}\right)^2$

3. $x = \frac{1}{y} - \frac{2}{y^2}$

4. $y = w + 1$; $w = \frac{1}{(v-1)^2}$; $v = \sqrt{x-1}$

II. A curve is given parametrically by the equations $x = (1+t)^2$, $y = (1-t)^2$.Find the equation of the tangent to the curve at the point where $x = y$. Hint: Use the chain-rule of differentiation to find the derivative. To find the point of intersection, use substitution method. Furthermore, find the value of t using the given condition $x = y$, then, compute the corresponding value of x and y.

MODULE 12

MAXIMUM AND MINIMUM VALUE OF A FUNCTION

Specific Objectives:

At the end of the module, students must be able to:

1. Define and determine critical value and inflection point.
2. Learn concavity of a curve at the maximum and minimum value of x.
3. Apply the first derivative and second derivative test to determine whether a critical point is maximum or minimum.
4. Draw the graph of a given function after having gathered some of its properties using concept of derivative.

MAXIMUM AND MINIMUM FUNCTION VALUE

Sketching the graph of function is better facilitated using the geometrical interpretation of the derivative of a function as the slope of the tangent line at a point to the graph of the function. The derivative serves as a great tool in determining at what point on the curve is the tangent line horizontal; that is, where the slope of the tangent line or y' equals zero.

Definition: The function f is said to have a relative maximum value at c if there exists an open interval that contains c, on which f is defined such that $f(c) \geq f(x)$ for all x in this interval. Figures A and B below each exhibit a sketch of a part of the graph of the function that has a relative maximum value at c.

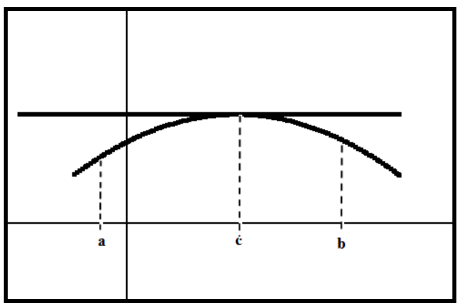


Figure A
Relative Maximum: $f'(c) = 0$

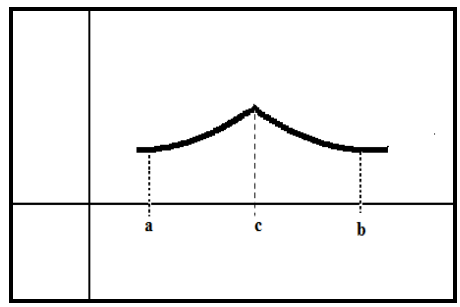


Figure B
Relative Maximum: $f'(c)$ is infinite (a cusp)

Definition: The function f is said to have a relative minimum value at c if there exists an open interval that contains c, on which f is defined such that $f(c) \leq f(x)$ for all x in this interval. Figures C and D shown below each exhibit a sketch of a part of the graph of the function that has a relative minimum value at c.

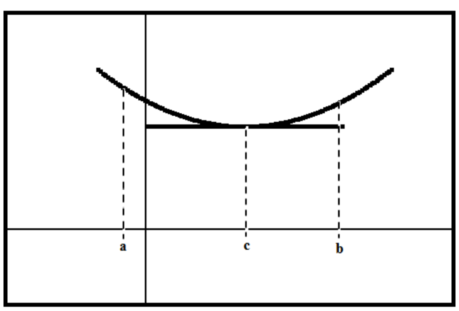


Figure C
Relative Maximum: $f'(c) = 0$

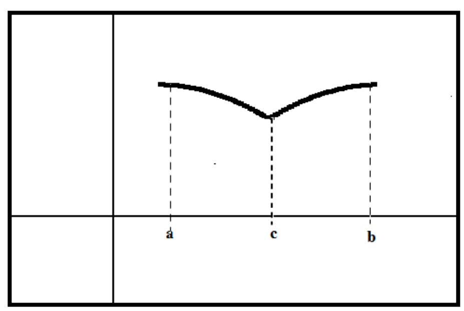


Figure D
Relative Maximum: $f'(c)$ is infinite (a cusp)

Definition: A critical value of a function f is a value of x where $f' = 0$. A critical point of a function f is the point $(x, f(x))$ on the graph that corresponds to the critical value x.

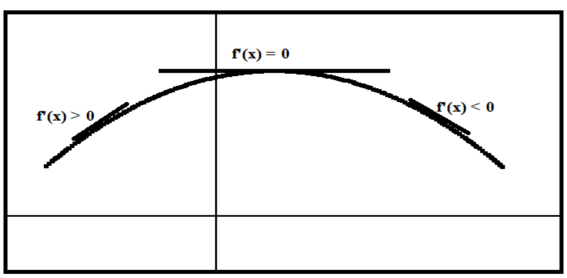
In Figures A and C on the preceding page, $f'(c)=0$ since tangent at is horizontal, while in Figures B and D, $f'(c)$ is infinite since tangent line is vertical. In the succeeding discussions, relative maximum (minimum) value will simply be termed maximum (minimum) value.

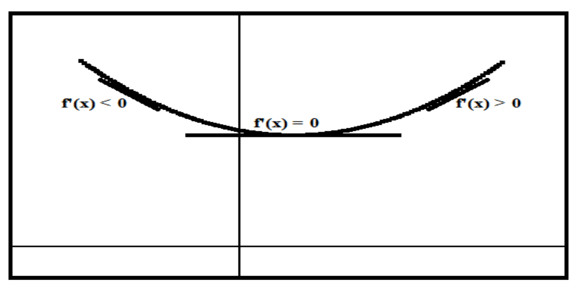
CONCAVITY TEST

(a). First-Derivative Test

This test is used to classify whether a critical point is a maximum, minimum point or neither maximum nor minimum.

1. Get $f'(x)$ and equate it to zero to find the critical values.
2. Locate the critical values on the rectangular coordinate system to establish a number of intervals.
3. Determine the sign of $f'(x)$ on each interval.
4. If $x = x_o$ is a critical value, assume increasing value of x through $x = x_o$. If $f'(x)$
   - Changes from + to –, then, $x = x_o$ is a relative maximum value and concavity of the curve is downward.
   - Changes from – to +, then, $x = x_o$ is a relative minimum value and concavity of the curve is upward.
   - Does not change in sign, then, $x = x_o$ is neither a relative maximum nor a minimum value.





(b). Second-Derivative Test

1. Set $f'(x)=0$ and solve for the critical values.
2. Critical value $x = x_0$ is
   - A maximum value if $f''(x_0) < 0$.
   - A minimum value if $f''(x_0) > 0$.

Note: If $f''(x_0)=0$ or becomes infinite, the second-derivative test fails. In this case, the first-derivative test must be used.

Definition: A point (x_0, y_0)on a curve is a point of inflection if $f''(x_o)=0$ at this point and if the curve changes its direction of concavity from upward to downward, or vice versa.

Example 26.Determine and classify using the First-Derivative Test the critical point of the curve represented by equation $y=\dfrac{x^{\frac{2}{3}}}{x+2}$.

Solution: Find $y'(x)$and equate to zero. $y'(x)=\dfrac{(x+2)\left(\frac{2}{3}x^{-\frac{1}{3}}\right)-x^{\frac{2}{3}}}{(x+2)^2}$

$$y'(x)=\frac{\frac{2(x+2)}{3x^{\frac{1}{3}}}-x^{\frac{2}{3}}}{(x+2)^2}$$

$$y'(x)=\frac{\frac{2x+4-3x}{3x^{\frac{1}{3}}}}{(x+2)^2}$$

$$y'(x)=\frac{4-x}{3x^{\frac{1}{3}}(x+2)^2}$$

At the critical points: $y'(x)=0=\dfrac{4-x}{3x^{\frac{1}{3}}(x+2)^2}$

$$0=4-x$$

$$x=4$$

$$y=0.42$$

Using the First-Derivative Test to classify point$(4,0.42)$:

When $x<4: x=3.9, y'(3.9)=\frac{+}{+}=+$

When $x>4: x=4.1, 'y'(4.1)=\frac{-}{+}=-$

Since $y'(x)$changes from $+$to$-$as x increases through$x=4$, therefore, $(4,0.42)$is a maximum point. Take note also that the line $x=-2$is a vertical asymptote of the given curve. This is because when $x\to -2, y\to \pm\infty$.

Example 27.Examine the given curve for relative maxima and minima, concavity and point of inflection and roughly sketch the curve.

(a). $y=x^4-4x^3-2x^2+12x-8$

Solution: Find $y'(x)$ and equate it to zero.

$y'(x) = 4x^3 - 12x^2 - 4x + 12 = 0$ which is factorable by grouping.

$$y'(x) = 4(x^3 - 3x^2 - x + 3) = 0$$
$$y'x) = x^3 - 3x^2 - x + 3 = 0$$
$$y'(x) = x^2(x-3) - (x-3) = 0$$
$$y'(x) = (x-3)(x^2 - 1) = 0$$
$$(x-3)(x+1)(x - y'(x) = 1) = 0$$

$$x = 3 \qquad x = -1 \qquad x = 1$$
$$y = -17 \qquad y = -17 \qquad y = -1$$

To classify the critical points $(3,-17), (-1,-17),$ and $(1,-1)$, we may perform either of the following tests:

(a) First-Derivative Test:

$(3,-17)$: $\quad x < 3:\ x = 2.9,\ y'(2.9) = +(-)(+)(+) = -$
$\qquad\qquad x > 3:\ x = 3.1,\ y'(3.1) = +(+)(+)(+) = +$

Since $y'(x)$ changes from $-$ to $+$ as x increases through $x = 3$, therefore, $(3,-17)$ is a minimum point.

$(-1,-17)$: $\quad x < -1:\ x = -1.1,\ y'(-1.1) = +(-)(-)(-) = -$
$\qquad\qquad x > -1:\ x = -0.9,\ y'(-0.9) = +(-)(+)(-) = +$

Since $y'(x)$ changes from $-$ to $+$ as x increases through $x = -1$, thus, $(-1,-17)$ is a minimum point.

$(1,-1)$: $\quad x < 1:\ x = 0.9,\ y'(0.9) = +(-)(+)(-) = +$
$\qquad\qquad x > 1:\ x = 1.1,\ y'(1.1) = +(-)(+)(+) = -$

With $y'(x)$ changing from + to – as x increases through $x = 1$, hence, $(1, -1)$ is a maximum point of the curve.

(b) Second-Derivative Test

$$y''(x) = 12x^2 - 24x - 4$$
$$y''(x) = 4(3x^2 - 6x - 1)$$

$(3, -17)$: $\quad y''(3) = +(+) = +,\ \ (3, -17$ is a minimum point, concavity is upward)

$(-1, -17)$: $\quad y''(-1) = +(+) = +,\ \ (-1, -17$ is a minimum point, concavity is upward)
$(1, -1)$: $\quad y''(1) = +(-) = -,\ \ (1, -1)$ is a maximum point, concavity is downward)

Find the points of inflection by setting $y''(x)=0$.

$$y''(x)=4\left(3x^2-6x-1\right)=0$$
$$3x^2-6x-1=0$$

The trinomial above is not factorable. Solve for x by using quadratic formula.

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

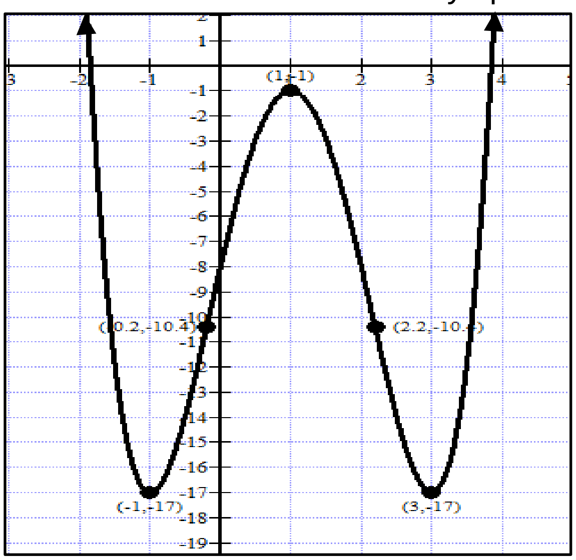
$$x=\frac{-(-6)\pm\sqrt{(-6)^2-4(3)(-1)}}{2(3)}$$

$$x=\frac{6\pm\sqrt{48}}{6}$$

$$x=2.2 \qquad x=-0.2$$
$$y=-10.4 \qquad y=-10.4$$

Thus, the curve has $(2.2,-10.4)$ and $(-0.2,-10.4)$ as points of inflection.

Note: In some cases, finding the x-intercepts and the y-intercepts helps in roughly sketching the graph of the given function. Likewise, knowing extra points on the curve is sometimes necessary. Hence, assume extra points, when needed. Furthermore, it is will be helpful if you determine horizontal and vertical asymptotes, if any.



(b). $y=\dfrac{2x}{x^2+1}$

Solution: Find the critical points by setting $y'(x)=0$

$$y'(x)=\frac{(x^2+1)(2)-2x(2x)}{(x^2+1)^2}$$

$$0=\frac{2x^2+2-4x^2}{(x^2+1)^2}$$

$$2-2x^2=0$$

$$2(1-x^2)=(1-x)(1+x)=0$$

$$x=1 \qquad x=-1$$

$$y=1 \qquad y=-1$$

Use the Second-derivative test to classify the critical points $(1,1)$ and $(-1,-1)$.

$$y''(x)=\frac{(x^2+1)^2(-4x)-(2-2x^2)(2)(x^2+1)(2x)}{(x^2+1)^4}$$

$$y''(x)=\frac{-4x(x^2+1)[(x^2+1)+(2-2x^2)]}{(x^2+1)^2}$$

$$y''(x)=\frac{-4x(x^2+1)(3-x^2)}{(x^2+1)^4}$$

$$y''(x)=\frac{-4x(3-x^2)}{(x^2+1)^3}$$

$(1,1)$: $\quad y''(1)=\frac{-(+)(+)}{+}=-$, hence, $(1,1)$ is a maximum point, concavity is downward.

$(-1,-1)$: $y''(x)=\frac{-(-)(+)}{+}=+$, therefore, $(-1,-1)$ is a minimum point, concavity is upward.

To find the points of inflection, set $y''(x)=0$

$$0=\frac{4x(3-x^2)}{(x^2+1)^3}$$

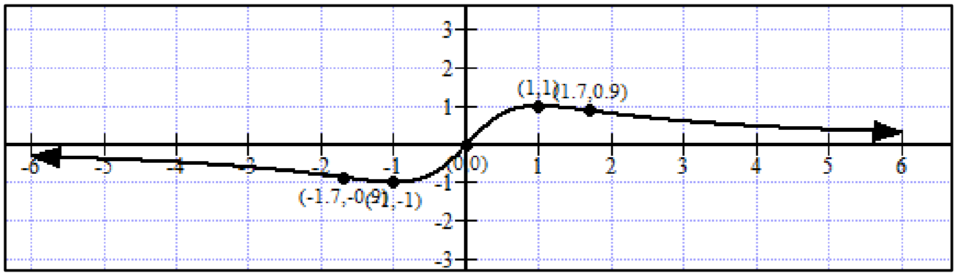
$$4x(3-x^2)=0$$

$$x=0 \qquad x=-\sqrt{3}=-1.7 \qquad x=\sqrt{3}=1.7$$

$$y=0 \qquad y=-0.9 \qquad y=0.9$$

Thus, the points inflection are $(0,0)$, $(1.7,0.9)$, $(-1.7,-0.9)$

As to the possible asymptotes of the curve, observe that when we equate the denominator to zero, corresponding values of x are imaginary numbers. Thus, the curve has no vertical asymptote. However, dividing both numerator and denominator of the given function by the highest power of x which is x^2 yields equation $y = 0$. Thus, $y = 0$ or the x-axis, is a horizontal asymptote.



(c). $y = x^2 + \dfrac{1}{x^2}$

Solution: Find $y'(x)$ and $y''(x)$. $y = x^2 + x^{-2}$

$$y'(x) = 2x - 2x^{-3}$$

$$y'(x) = 2\left(x - \frac{1}{x^3}\right)$$

$$0 = 2\left(\frac{x^4 - 1}{x^3}\right)$$

$$0 = x^4 - 1$$

$$0 = (x^2 - 1)(x^2 + 1)$$

$$0 = (x + 1)(x - 1)(x^2 + 1)$$

$$x = 1 \qquad x = -1$$

$$y = 2 \qquad y = 2$$

The critical points are $(1,2)$ and $(-1,2)$.

Use the Second-Derivative Test to classify the critical points.

$$(1,2): y''(1) = +(+) = +$$

$$(-1,2): \quad y''(-1) + (+) = +$$

Therefore, both $(1,2)$ and $(-1,2)$ are minimum points.

Find the points of inflection, if any. Set $y'' = 0$.

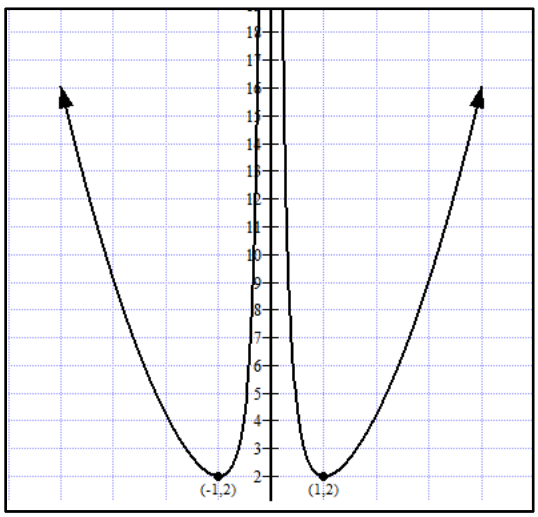
$$y''(x) = 2 + 6x^{-4} = 2\left(1 + \frac{3}{x^4}\right)$$

$$0 = 2\left(\frac{x^4 + 3}{x^4}\right)$$

$$0 = x^4 + 3$$

Therefore, no points of inflection since x-values are imaginary.

Equating the denominator to zero shows $x = 0$ (the y − axis)is a vertical asymptote. Analysis will show the curve has no x and y intercepts, thus, it does not cross the coordinate axes.



a, b, c, d Problems

Example 28. Find a, b and c so that the curve $y = ax^3 + bx + c$ will have a critical point at $\left(\frac{1}{2}, -11\right)$and pass through$(0, -7)$.

Condition (1): Points $(0, -7)$and $\left(\frac{1}{2}, -11\right)$satisfy equation of the curve$y = ax^3 + bx + c$.

$(0,7)$: $\quad c = -7$

$\left(\frac{1}{2}, -11\right)$: $\quad -11 = \frac{1}{8}a + \frac{1}{2}b + c$

$$-88 = a + 4b + 8c$$

Substitute $c = -7$. $\quad -88 = a + 4b + 8(-7)$

$$-32 = a + 4b \quad \text{-------- Equation (1)}$$

Condition (2): At $\left(\frac{1}{2}, -11\right)$, y’ = 0.

Differentiate y with respect to x. $y' = 3ax^2 + b$

$$0 = 3a\left(\frac{1}{4}\right) + b$$

$0 = 3a + 4b$ ---------- Equation (2)

Subtract Equation (1) to Equation (2). $-32 = -2a$ $\quad a = 16$

Substitute in Equation (2). $0 = 3(16) + 4b$ $\quad b = -12$

Therefore, for the curve to have a critical point at $\left(\frac{1}{2}, -11\right)$and pass through$(0, -7)$, $a = 16, b = -12$ and $c = -7$.

Example 29. Find a, b, c and d so that the curve $y = ax^3 + bx^2 + cx + d$ will pass through$(0,1)$, have critical point at $(1, -2)$ and a point of inflection where $x = \frac{1}{3}$.

Differentiate. $y' = 3ax^2 + 2bx + c$
$y'' = 6ax + 2b$

Condition (1): At $x = \frac{1}{3}$, , y"=0.

$$0 = 6a\left(\frac{1}{3}\right) + 2b$$

$$0 = 2a + 2b$$

$a = -b$ ---------------- Equation (1)

Condition (2): At $(1, -2)$, $y' = 0$.

$0 = 3a + 2b + c$ ---------- Equation (2)

Condition (3). Points $(1, -2)$ and $(0, 1)$ satisfy given equation $y = ax^3 + bx^2 + cx + d$.

$(1, -2)$: $-2 = a + b + c + d$ ------------- Equation (3)

$(0,1)$: $1 = d$

Substitute $d = 1$ into Equation (3). $-2 = a + b + c + 1$

$a + b + c = -3$--------------Equation (4)

Subtract Equation (2) to Equation (4). $3 = 2a + b$ ---- Equation (5)

Substitute Equation (1) into Equation (5). $3 = 2(-b) + b$

$b = -3$

Substitute into Equation (1) $\quad a = -(-3) = 3$.

Substitute into Equation (2). $0 = 9 - 6 + c$

$c = -3$

Therefore, so that the curve $y = ax^3 + bx^2 + cx + d$ will pass through$(0,1)$, will have critical point at $(1, -2)$ and have a point of inflection where $x = \frac{1}{3}$, $a = 3$, $b = -3$,

$c = -3$ and $d = 1$.

Example 30. Make the curve $y = ax^3 + bx^2 + cx + d$ pass through $(4{,}8)$ and have tangent line $2y - 6x + 3 = 0$ at the inflection point $(2{,}4)$. Find equation of the curve.

Solution: The tangent line to the curve at its point of inflection is called an *inflectional tangent*.

Differentiate to get y' and $y"$.

$$y' = 3ax^2 + 2bx + c$$

$$y" = 6ax + 2b$$

To get slope of the tangent line, reduce its equation to $y = 3x - \frac{3}{2}$. Thus, at $(2, 4)$, slope of tangent is given by $y' = 3$.

Condition (1). Points $(4, 8)$ and $(2{,}4)$ satisfy $y = ax^3 + bx^2 + cx + d$.

$(4, 8)$: $\quad 8 = a(4)^3 + b(4)^2 + c(4) + d$

$8 = 64a + 16b + 4c + d$-------------- Equation (1)

$(2, 4)$: $\quad 4 = a(2)^3 + b(2)^2 + c(2) + d$

$4 = 8a + 4b + 2c + d$ -----------------Equation (2)

Condition (2). At $(2, 4)$, $y" = 0$. $\quad 0 = 6a(2) + 2b$

$0 = 6a + b$

$b = -6a$ --------------------------- Equation (3)

Condition (3). At $(2, 4)$, $y' = 3$. $\quad 3 = 3a(2)^2 + 2b(2) + c$

$3 = 12a + 4b + c$ ---------------Equation (4)

Subtract Equation (1) to Equation (2). $\quad 4 = 56a + 12b + 2c$

$2 = 28a + 6b + c$----------------Equation (5)

Subtract Equation (4) to Equation (5). $1 = -16a - 2b$ --- Equation (6)

Substitute Equation (3) into Equation (6). $\quad 1 = -16a - 2(-6a)$

$1 = -4a$

$a = -\frac{1}{4}$

Hence, $b = -6\left(-\frac{1}{4}\right) = \frac{3}{2}$

Substitute into Equation (4). $3 = 12\left(-\frac{1}{4}\right) + 4\left(\frac{3}{2}\right) + c$

$$3 = -3 + 6 + c$$

$$c = 0$$

Substitute into Equation (2). $4 = 8\left(-\frac{1}{4}\right) + 4\left(\frac{3}{2}\right) + 0 + d$

$$4 = -2 + 6 + d$$

$$d = 0$$

Equation of the curve, $y = -\frac{1}{4}x^3 + \frac{3}{2}x^2$

$$4y = -x^3 + 6x^2$$

SAQ15
ACTIVITY 3.12 – M

NAME: ____ SCORE: ________________

SECTION: ____________DATE: _________________ PROF: _______________________________________

I. Find the intercepts, the critical points and point of inflection for each of the following curves. Determine the vertical and horizontal asymptotes, if any. Roughly sketch the curve.

1. $y = \frac{1+x}{1-x}$

2. $y = \frac{2x+5}{x^2-4}$

3. $y = x^4 - 5x^2$

II. Find value of the arbitrary constants on the given equations.

1. If $y = ax^3 + bx^2$ has a point of inflection at $(2, 8)$, find values of a and b.

2. Determine a, b and c so that the curve $y = ax^2 + bx + c$ will pass through point $(-1, 6)$ and have tangent line $y = 4x - 2$ to the curve at point $(1,2)$.

3. Determine a, b and c so that the curve $y = ax^2 + bx + c$ will pass through $(0, -2)$ and will have a critical point at $\left(-\frac{2}{5}, -\frac{14}{5}\right)$.

ASAQ15
ACTIVITY 3.12 – M

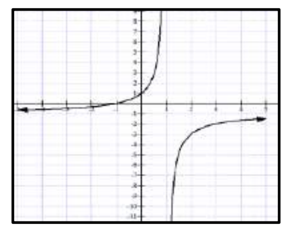
NAME: ____ SCORE: ________________

SECTION: ____________DATE: _________________ PROF: _______________________________________

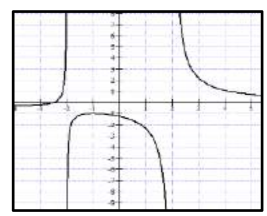
I. Find the intercepts, the critical points and point of inflection for each of the following curves. Determine the vertical and horizontal asymptotes, if any. Roughly sketch the curve.

1. $y = \frac{1+x}{1-x}$

$Answer:$

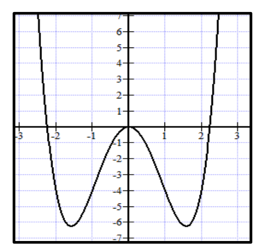


$Answer:$



2. $y = \dfrac{2x+5}{x^2-4}$

$Answer:$



3. $y = x^4 - 5x^2$

II. Find value of the arbitrary constants on the given equations.

1. If $y = ax^3 + bx^2$ has a point of inflection at $(2, 8)$, find values of a and b. $Answer\colon a = -\frac{1}{2},\ \ b = 3$

2. Determine a, b and c so that the curve $y = ax^2 + bx + c$ will pass through point $(-1, 6)$ and have tangent line $y = 4x - 2$ to the curve at point $(1,2)$. $Answer\colon a = 3, \qquad b = -2, \qquad c = 1$

3. Determine a, b and c so that the curve $y = ax^2 + bx + c$ will pass through $(0, -2)$ and will have a critical point at $\left(-\frac{2}{5}, -\frac{14}{5}\right)$. $Answer\colon a = 5, \qquad b = 4, \qquad c = -2$

ACTIVITY 3.12 – N

NAME: ____ SCORE: ______________

SECTION: ___________DATE: _______________ PROF: ___________________________________

I. Find the intercepts, the critical points and point of inflection for each of the following curves. Determine the vertical and horizontal asymptotes, if any. Roughly sketch the curve.

1. $y = \frac{x^3}{2}(x-2)^2$

2. $y = \frac{4x}{x^2+1}$

1.

II. Find value of the arbitrary constants on the given equations.

1. Make the curve $y = ax^3 + bx^2 + cx + d$ pass through $(0,0)$, $(-1,-1)$ and have critical point at $(3,7)$.

2. Find a, b, c and d so that the curve $y = ax^3 + bx^2 + cx + d$ will pass through points $(0,12)$and $(-1, 6)$ and have inflection point at $(2, -6)$.

MODULE 13

OPTIMIZATION PROBLEMS

Specific Objectives:

At the end of the module, students must be able to:

1. Understand the concept of and steps in solving an optimization problem.
2. Differentiate a constraint equation from optimization equation and be able to set—up both equations.
3. Solve a given optimization problem.

MAXIMA AND MINIMA APPLICATIONS

If a contractor wants to construct a stainless steel cylindrical water tank with a specified volume at the least possible cost, he is faced with the problem of finding a minimum. Making the biggest window of constant perimeter and in the shape of a rectangle surmounted by a semi-circle presents a problem of finding a maximum. Words like biggest, largest, most, smallest, least, best, and others can be translated into mathematical language in terms of maxima and minima.

One of the most important applications of derivative is illustrated on maximum/minimum optimization problems. Many students find this application intimidating because they are "word" problems, and no fixed pattern of solution exists to these problems. However, with their patience, they can minimize their anxiety and maximize their success with these problems by following the guidelines listed below:

1. Read the problem slowly and carefully. It is imperative to know exactly what the problem is asking. If appropriate, draw a sketch or diagram of the problem to be solved. Pictures are a great help in organizing and sorting out ones thought.
2. Identify the constant quantity in the given problem. Define variables to be used and carefully label the picture or diagram with these variables. This step is very important because it leads directly or indirectly to the creation of mathematical equations.
3. Identify the quantity to be maximized or minimized and if it shall consist of more than one variable, express it in terms of one variable (if possible and practical) using the given conditions in the problem. Experience shows that most optimization problems begin with two equations. One equation is a "constraint" equation and the other is the "optimization" equation. The "constraint" equation is used to solve for one of the variables. This is then substituted into the "optimization" equation before differentiation occurs. Some problems may have no constraint equation. Some problems may have two or more constraint equations.
4. Then differentiate using the well-known rules of differentiation.
5. Verify that your result is a maximum or minimum value using the first or second derivative test for extrema.

Example 31. What positive number added to its reciprocal gives the minimum sum?

Let : S be the minimum sum

x = the required positive number

$\frac{1}{x}$ = the reciprocal of the number

Optimization equation: $S = x + \frac{1}{x} = x - x^{-1} = f(x)$

$$\frac{dS}{dx} = 1 - \frac{1}{x^2}$$

For S to be a minimum, $\frac{dS}{dx} = 0$.

$$0 = 1 - \frac{1}{x^2}$$

$$0 = \frac{x^2 - 1}{x^2}$$

$$0 = (x+1)(x-1)$$

$x = 1$ (Answer) $\qquad x = -1$(Reject)

Verification of the critical value using the second-derivative test:

$$S''(x) = 1 + x^{-2} = 1 + \frac{1}{x^2}$$

$$S''(1) = 1 + 1 = 2$$

Since S'' is positive or greater than zero when $x = 1$, then, the sum S is minimum at $x = 1$.,

Example 32. Find two positive integers having a sum of 132 and the sum of their cubes has the minimum value.

Let: S be the minimum sum of their cubes

x be one positive number

Optimization equation: $S = x^3 + y^3$ -------------------- Equation (1)
Constraint equation: $x + y = 132$ ----------------------- Equation (2)
Therefore $y = 32 - x$ be the other number
Substitute Equation (2) into Equation (1).

$$\frac{dS}{dx} = 3x^2 + 3(132 - x)^2(-1)$$

$$0 = 3\left[x^2 - (132 - x)^2\right]$$

$$0 = x^2 - (17424 - 264x + x^2)$$

$$0 = -17424 + 264x$$

$$264x = 17424$$

$$x = 66$$

Therefore, $y = 132 - x = 132 - 66 = 66$

Verification of the critical value using the second-derivative test:

$$y'(x) = 3x^2 - 3(132 - x)^2$$
$$y"(x) = 6x + 6(132 - x)$$
$$y"(66) = 6(66) + 6(132 - 66) = 792$$

Hence, the sum S is minimum at $x = 66$ since $y"(66)$ is greater than zero.

Example 33. If the product of the square of one number by the cube of the other is to be the greatest, find the two numbers if there sum equals 20.

Let x be one number

Z be the product of the square of one number by the cube of the other number

Optimization equation: $Z = x^2(20 - x)^3$
Constraint equation: $y + x = 20$

Therefore, expression $(20 - x)$ is the other number. Now, differentiate Z with respect to x.

$$\frac{dZ}{dx} = x^2 \frac{d}{dx}(20 - x)^3 + (20 - x)^3 \frac{d}{dx}(x^2)$$
$$\frac{dZ}{dx} = x^2(3)(20 - x)^2(-1) + (20 - x)^3(2x)$$
$$\frac{dZ}{dx} = x(20 - x)^2[-3x + 2(20 - x)]$$

For Z to be the greatest, $\frac{dZ}{dx} = 0 = x(20 - x)^2(40 - 5x)$

Take note x cannot be both zero and 20; otherwise, the product Z will have zero value also. Hence, values $x = 0$ and x= 20 are rejected. Hence, $40 - 5x = 0$ and $x = 8$ and the other number $(20 - x) = 12$.

Therefore, the numbers are 8 and 12.

MAXIMA-MINIMA PROBLEM USING IMPLICIT DIFFERENTIATION

Example 34. Resolve Example 33 by using implicit differentiation.
Let x and y be the numbers

Constraint Equation: $x + y = 20$ ------------------- Equation (1)

Optimization Equation: $Z = x^2y^3$ --------------------- Equation (2)

No need to express Z in terms of one variable. Rather, to get $\frac{dZ}{dx}$, we use implicit differentiation and equate the derivative to zero.

$$\frac{dZ}{dx} = x^2\left(3y^2 \frac{dy}{dx}\right) + y^3(2x)$$

$$\frac{dZ}{dx} = 3x^2y^2\frac{dy}{dx} + 2xy^3$$
$$0 = xy^2\left(3x\frac{dy}{dx} + 2y\right)$$

Solve for $\frac{dy}{dx}$. $\quad \frac{dy}{dx} = \frac{-2y}{3x}$ ------------------------ Equation (3)

Differentiate Equation (1) implicitly with respect to x.

$$1 + \frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = -1$$ ----------------- Equation (4)

Substitute Equation (4) into Equation (3).

$$-1 = \frac{-2y}{3x}$$
$$y = \frac{3x}{2}$$ ------------------ Equation (5)

Substitute Equation (5) into Equation (1) to get an equation in terms of one variable. Solve the resulting equation for the value of the variable.

$$x + \frac{3x}{2} = 20$$
$$2x + 3x = 40$$
$$5x = 40$$
$$x = 8$$

Substitution into Equation (1) or Equation (5) will give the corresponding value of y.

Hence, $\quad y = 20 - 8 = 12$

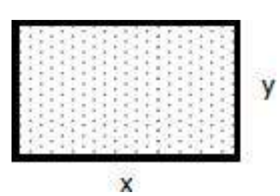
Example 35.A rectangular lot of area 150 m^2**.** What should be the shape of the lot if it is to be enclosed by the least amount of fencing?

Let x and y be the dimensions of derivative the rectangular lot
P be the perimeter of the lot; the least amount, therefore, its derivative equals zero.
A be the area of the lot; $A = 150\ m^2$, a constant

Method (1). $\quad A = xy$

Constraint equation: $150 = xy$----------------- Equation (A)

$$y = \frac{150}{x}$$ ----------------- Equation (1)



Optimization equation: $P = 2x + 2y$ -------- Equation (2)

Substitute Equation (1) into equation (2):

$$P = 2x + 2\left(\frac{150}{x}\right)$$
$$P = 2x + 300x^{-1}$$

Differentiate P with respect to variable x and equate its derivative to zero for P to be the least.

$$\frac{dP}{dx} = 2 - 300x^{-2}$$
$$0 = 2 - \frac{300}{x^2}$$
$$x^2 = 150$$
$$x = \sqrt{150} = 5\sqrt{6}$$

Substitute into Equation (1): $y = \frac{150}{\sqrt{150}} = \sqrt{150} = 5\sqrt{6}$

Therefore, the rectangular lot is a square measuring $5\sqrt{6}$ by $5\sqrt{6}$.

Method (2). Use implicit differentiation.

Differentiate Equation (A) with respect to x.

$$0 = x\frac{dy}{dx} + y$$

$\frac{dy}{dx} = -\frac{y}{x}$ -------------- Equation (3)

Differentiate Equation (2) with respect to x, substitute Equation (3) into the resulting equation and, then, equate the derivative of P to zero.

$\frac{dP}{dx} = 2 + 2\frac{dy}{dx}$ -------- Equation (4)

$$\frac{dP}{dx} = 2 + 2\left(-\frac{y}{x}\right)$$
$$0 = 2 - \frac{2y}{x}$$

$y = x$ ------------------ Equation (5)

Substitute Equation (5) into Equation (1).

$$x = \frac{150}{x}$$

$$x^2 = 150x = y = \sqrt{150} = 5\sqrt{6}$$

Example 36.A rectangular lot is to be fenced off along a highway. If the fence on the highway costs a pesos per meter, on the other sides$b$ pesos per meter, find the area of the largest lot that can be fenced off for T pesos.

Let T be the total cost (in pesos) to fence the rectangular lot; a constant.

A be the area ((in m^2)of the lot; it has to be the largest.

Constraint equation: $T = a(x) + b(x + 2y)$

$T = ax + bx + 2by = (a + b)x + 2by$ ------------ Equation (A)

$y = \frac{T - x(a+b)}{2b}$---------------------Equation (1)

Optimization equation: $A = xy$ ----------Equation (2)

Method (1). Express the quantity area A which is to be maximized in terms of a single variable. Substitute Equation (1) into Equation (2), differentiate the resulting equation, set the derivative to zero and solve for the value of the variable.

$$A = x\left[\frac{T - x(a+b)}{2b}\right] = \frac{1}{2b}[Tx - (a + b)x^2]$$

$$\frac{dA}{dx} = \frac{1}{2b}[T - (a + b)(2x)]$$

$$0 = \frac{1}{2b}[T - (a + b)(2x)]$$

$$0 = T - 2(a + b)x$$

$x = \frac{T}{2(a+b)}$ ------------ Equation (3)

Substitute the Equation (3) onto Equation (1).

$y = \frac{T - \frac{T}{2(a+b)}(a+b)}{2b} = \frac{\frac{T}{2}}{2b} = \frac{T}{4b}$-------------- Equation (4)

Substitute Equation (3) and Equation (4) into Equation (2).

$$A = \frac{T}{2(a+b)}\left[\frac{T}{4b}\right]$$

$$A = \frac{T^2}{8b(a+b)}m^2$$

Method (2). Implicitly differentiate Equation (A) and Equation (2) with respect to x. Since T is a constant, its derivative with respect to x is zero.

$$\frac{dT}{dx} = (a + b) + 2b\frac{dy}{dx}$$

$$0 = (a + b) + 2b\frac{dy}{dx}$$

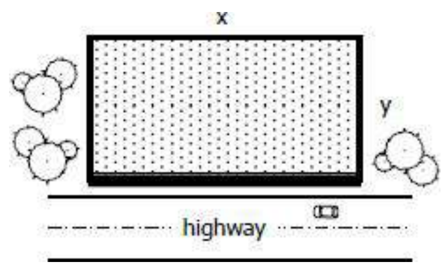
$\frac{dy}{dx} = -\frac{a+b}{2b}$ --------------- Equation (5)

Likewise, $\frac{dA}{dx} = x\frac{dy}{dx} + y$ -------------Equation (6)

Substitute Equation (5) into Equation (6).Since A is to be the greatest, its derivative equals zero.

$$0 = x\left(-\frac{a+b}{2b}\right) + y$$

$y = x\left(\frac{a+b}{2b}\right)$ ---------Equation (7)



Substitute Equation (7) into Equation (1).

$$\frac{T-x(a+b)}{2b} = x\left(\frac{a+b}{2b}\right)$$

$$2x(a + b) = T$$

$$x = \frac{T}{2(a+b)}$$

Correspondingly, after substitution into Equation (7),

$$y = \frac{T}{2(a+b)}\left[\frac{a+b}{2b}\right] = \frac{T}{4b}$$

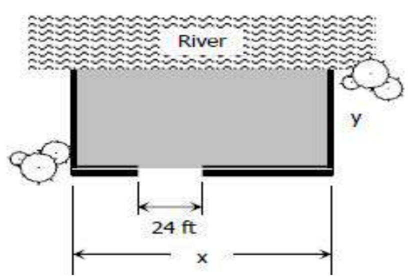
$$A = \frac{T}{2(a+b)}\left[\frac{T}{4b}\right] = \frac{T^2}{8b(a+b)}$$

Example 37.A rectangular lot is bounded at the back by a river. It is to be fenced, however, no fence is needed along the river and there is to be 24-ft opening in front. If the fence along the front costs $\$8$ per foot, along the sides $\$5$ per foot, find the dimensions of the largest lot which can be thus fenced in for $2208.

Constraint equation:Total cost C $=2208 = 8(x - 24) + 5(2y)$ ---------- Equation (1)

$$\frac{dC}{dx} = 0 = 8(1) + 10\frac{dy}{dx}$$

$$\frac{dy}{dx} = -\frac{8}{10} = -\frac{4}{5}$$



Optimization equation: $A = xy$ ----------- Equation (2)

$$\frac{dA}{dx} = x\frac{dy}{dx} + y(1)$$

Area A is to be the greatest; $\frac{dA}{dx}$=0.

$$0 = x\left(-\frac{4}{5}\right) + y$$

$y = \frac{4}{5}x$ -------- Equation (3)

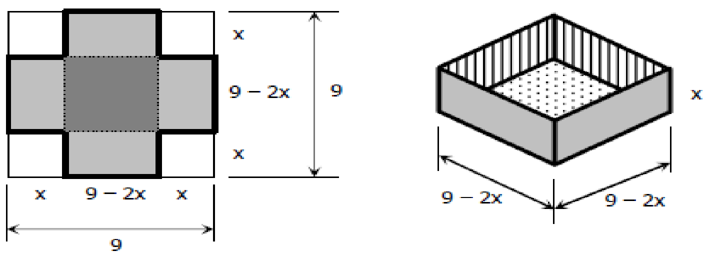
Substitute equation (3) into equation (1).

$$2208 = 8(x - 24) + 10\left(\frac{4}{5}x\right)$$
$$2208 = 8x - 192 + 8x$$
$$16x = 2400$$
$$x = 150 \text{ feet}$$

Substitute $x = 150$ feet into Equation (3).$y = \frac{4}{5}(150) = 120$ feet

Example 38.A box is to be made of a piece of cardboard 9 inches square by cutting equal squares out of the corners and turning up the sides. Find the volume of the largest box that can be made in this way.

Volume of the box V is to be the largest; hence, its derivative equals zero.



Optimization equation: $V = LWH = x(9-2x)(9-2x)$

$$V = x(9-2x)^2$$
$$\frac{dV}{dx} = x(2)(9-2x)(-2) + (9-2x)^2(1)$$
$$0 = (9-2x)(-4x + 9 - 2x)$$
$$0 = (9-2x)(9-6x)$$

Equate each factor to zero and solve for the value of x.

$$9 - 2x = 0 \qquad\qquad 9 - 6x = 0$$

The value $x = \frac{9}{2}$ is rejected since $9 - 2x = 0$ when $x = \left(\frac{9}{2}\right)$

Therefore, $x = \frac{9}{6} = \frac{3}{2}$ inches.
Take note that this problem has no constraint equation.

Therefore, the volume of the largest box that can be made as described above is

$$V = \frac{3}{2}\left[9 - 2\left(\frac{3}{2}\right)\right]^2 = \frac{3}{2}(9-3)^2 = \frac{3}{2}(36) = 54 \text{ inches}^3$$

Example 39. The strength of a rectangular beam is proportional to the breadth and the square of the depth. Find the shape of the strongest beam that can be cut from a log of diameter 24 inches.

Let W be the breadth of the beam and L be its depth .

Optimization equation: Strength of the beam $S = WL^2$; S is to be a maximum, $\frac{dS}{dw} = 0$

$$\frac{dS}{dW} = W\left(2L\frac{dL}{dW}\right) + L^2(1)$$

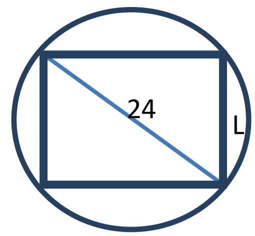
$$0 = 2LW\frac{dL}{dW} + L^2 \text{------------- Equation (1)}$$

By Pythagorean Theorem, $(24)^2 = W^2 + L^2$ ------------- Equation (2)

$$0 = 2W + 2L\frac{dL}{dW}$$

$$\frac{dL}{dW} = -\frac{W}{L} \text{----------------------- Equation (3)}$$

Substitute Equation (3) into Equation (1).



$$0 = 2LW\left(-\frac{W}{L}\right) + L^2$$

$$0 = -2W^2 + L^2$$

$$L = \sqrt{2}W$$

For the beam to be the strongest one, the depth L must be $\sqrt{2}$ times the breadth W.

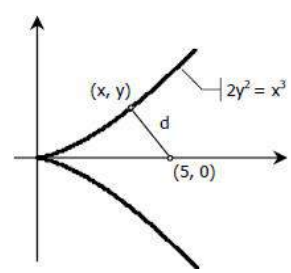
Example 40.Find the shortest distance from the point (5, 0) to the curve $2y^2 = x^3$.

Let (x, y) be the point on the curve $2y^2 = x^3$ nearest to point $(5, 0)$

D be the shortest distance$\text{between}(x, y)$and $(5, 0)$.

Method (1).Use distance formula between two points, differentiate implicitly with respect to x.

Optimization equation: $D = \sqrt{(x-5)^2 + y^2}$-------- Equation (A)



$$\frac{dD}{dx} = \frac{1}{2\sqrt{(x-5)^2+y^2}}[2(x-5) + 2yy']$$

$$0 = 2(x-5) + 2yy'$$

$$y' = -\frac{(x-5)}{y} \text{ -------------- Equation (1)}$$

Constraint equation: $2y^2 = x^3$

Differentiate implicitly the given equation of the curve.

$$4yy' = 3x^2 \text{ -------------- Equation (2)}$$

Substitute Equation (1) into Equation (2).

$$4y\left[-\frac{(x-5)}{y}\right] = 3x^2$$

$-4x + 20 = 3x^2$

$3x^2 + 4x - 20 = 0$

$(3x + 10)(x - 2) = 0$

$3x + 10 = 0$ x – 2 = 0

$x = -\frac{10}{3}$ $x = 2$

We reject $x = -\frac{10}{3}$ since corresponding y-value is imaginary. Thus, we use $x = 2$ and substitute it on the given equation of the curve to get the corresponding y-value.

$2y^2 = (2)^3 y^2 = 4 y = \pm 2$

The shortest distance $D = \sqrt{(2-5)^2 + 4} = \sqrt{9+4} = \sqrt{13}$

Method (2). From the given equation of the curve, $y^2 = \frac{x^3}{2}$.

Substitute on Equation (A), differentiate D with respect to x and equate the derivative to zero.

$$D = \sqrt{(x-5)^2 + \frac{x^3}{2}}$$

$$D = \sqrt{2(x-5)^2 + x^3}$$

$$\frac{dD}{dx} = \frac{1}{2\sqrt{2(x-5)^2 + x^3}}[4(x-5)(1) + 3x^2]$$

$$0 = -20 + 4x + 3x^2$$

$$(3x + 10)(x - 2) = 0$$

$3x + 10 = 0$ x – 2 = 0

$x = -\frac{10}{3}$ $x = 2$

Compare the results of Method (1) with those of Method (2). Thus, the shortest distance $D = \sqrt{13}$.

Example 41.A cylindrical can is to hold $20\pi\text{m}^3$.If the material for the top and bottom costs $\$10/m^2$ and the material for the side costs $\$8/m^2$, find the radius r and height h of the most economical dimension.

The most economical dimensions refer to the dimensions of a cylindrical can that will entail the minimum cost of manufacturing it.

Let *r* be the radius of the circular base and variable *h* the height of the cylinder.
C be the least cost of manufacturing a cylindrical can

V be the volume of the cylindrical can; $V = \pi r^2 h$

Constraint equation: $20\pi = \pi r^2$h $h = \frac{20}{r^2}$ ---------- Equation (1)

Optimization equation: Total Cost C= $20\pi r^2 + 16\pi rh$ ------------------ Equation (2)

Substitute Equation (1) into Equation (2). $C = 20\pi r^2 + 16\pi r\left(\frac{20}{r^2}\right) = 20\pi r^2 + 320\pi r^{-1}$

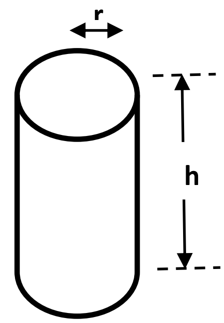
For C to be the least cost,$\frac{dC}{dr} = 0$.$\frac{dC}{dr} = \pi[20(2)r + 320(-1)r^{-2}]$

$$0 = 40r - \frac{320}{r^2}$$

$$\frac{320}{r^2} = 40r$$

$$r^3 = 8$$

$$r = 2\,m$$



Substitute $r = 2\,m$ into Equation (1). $h = \frac{20}{2^2} = 5\,m$

The least cost $C = 20\pi(2)^2 + \frac{320}{2}\pi = 80\pi + 160\pi = \$240\pi$ to manufacture a cylindrical can.

Example 42. Find the dimensions of a rectangle of maximum perimeter that can be inscribed in a circle of diameter 10 cm.

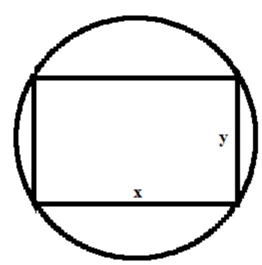
From the figure at the right, $\sqrt{100 - x^2} = y$.

Square both sides of the equation. $100 - x^2 = x^2$

$$2x^2 = 100$$

$$x^2 = 50 = 25(2)$$

$$x = 5\sqrt{2}$$

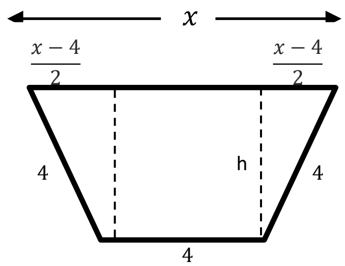


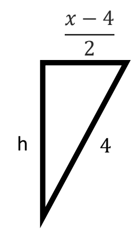
Solve for the length y. $y = \sqrt{100 - 50} = \sqrt{50} = 5\sqrt{2}$

Therefore, the rectangle of maximum perimeter is a square of dimensions $5\sqrt{2}$ x $5\sqrt{2}$.

Example 43.A trapezoidal gutter is to be made from a strip of tin by bending up the edges. If the cross-section of the gutter is shown below, with the lower base 4 inches, what width x across the top gives maximum carrying capacity?

Let A be the area of the trapezoidal cross-section





The carrying capacity is maximum if the area is maximum.

Optimization equation: $A = \frac{1}{2}(x+4)h$------------ Equation (1)

Constraint equation: $h = \sqrt{(4)^2 - \left(\frac{x-4}{2}\right)^2} = \sqrt{(4)^2 - \frac{[x^2-8x+16]}{4}}$

$$h = \frac{1}{2}\sqrt{48+8x-x^2}$$ --------------Equation (2)

Substitute Equation (2) into Equation (1). $A = \frac{1}{2}(x+4)\left[\frac{1}{2}\sqrt{48+8x-x^2}\right] = \frac{1}{4}(4+x)\sqrt{48+8x-x^2}$

$$\frac{dA}{dx} = \frac{1}{4}\left[(4+x)\left(\frac{1}{2\sqrt{48+8x-x^2}}\right)(8-2x) + \sqrt{48+8x-x^2}\right]$$

$$0 = \frac{1}{4}\left[\frac{(4+x)(4-x)}{\sqrt{48+8x-x^2}} + \sqrt{48+8x-x^2} = \frac{1}{4}\left[\frac{(16-x^2)+(48+8x-x^2)}{\sqrt{48+8x-x^2}}\right]\right]$$

$$0 = 64 + 8x - 2x^2$$

$$0 = 32 + 4x - x^2$$

$$0 = (8-x)(4+x)$$

$$x = 8 \quad x = -4(\text{Rejected})$$

Therefore, the width across the top that will give the maximum carrying capacity of the gutter is 8 inches.

Example 44.A piece of wire 40cm long is to be cut into two pieces. One piece will be bent to form a circle; the other will be bent to form a square. Find the lengths of the two pieces that cause the sum of the area of the circle and square to be a minimum.

Let $x$be the perimeter of the square of edge s.

$(40-x)$be the circumference of the circle of radius r.

Therefore, $= 4s, \quad s = \frac{x}{4}$

Likewise, $\quad 40 - x = 2\pi r, \qquad r = \frac{40-x}{2\pi}$

Hence, $\quad A = s^2 + \pi r^2 = \left(\frac{x}{4}\right)^2 + \pi\left(\frac{40-x}{2\pi}\right)^2 = \frac{x^2}{16} + \frac{1}{4\pi}(40-x)^2$

Differentiate. $\quad \frac{dA}{dx} = \frac{1}{16}(2x) + \frac{1}{4\pi}(2)(40-x)(-1)$

$$\frac{dA}{dx} = \frac{x}{8} - \frac{1}{2\pi}(40-x)$$

$$0 = \frac{\pi x - 4(40-x)}{8\pi}$$

$$0 = \frac{\pi x - 160 + 4x}{8\pi}$$

$$0 = \frac{x(\pi + 4) - 160}{8\pi}$$

$$0 = x(\pi + 4) - 160$$

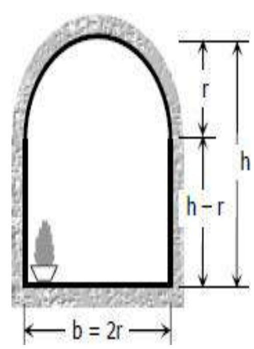
$$x = \frac{160}{\pi+4} = 22.4 \text{ cm}$$

And, $\quad 40 - x = 40 - 22.4 = 17.6$ cm

Therefore, the lengths of the wire that will give the minimum combined area of the circle and the square are 22.4 cm and 17.6 cm.

Example 45.A Norman window consists of a rectangle surmounted by a semicircle. What shape admits the greatest amount of light if perimeter of the window is 30 feet?

The greatest amount of light is admitted by the window if its area is a maximum. Furthermore, the diameter of the semi-circular part of the window is equal to the width of the rectangular portion.



Constraint equation: $\quad 30 = 2r + 2(h - r) + \frac{1}{2}(2\pi r)$

$$30 = 2r + 2h - 2r + \pi r$$

$$30 = 2h + \pi r$$

$h = \frac{30 - \pi r}{2}$ -------------------------- Equation (1)

Optimization equation: $A = 2r(h - r) + \frac{1}{2}\pi r^2$

$A = 2rh - 2r^2 + \frac{1}{2}\pi r^2$ ----------- Equation (2)

Substitute Equation (1) into Equation (2).

$$A = 2r\left(\frac{30-\pi r}{2}\right) - 2r^2 + \frac{1}{2}\pi r^2 = 30r - \pi r^2 + r^2\left(\frac{\pi}{2} - 2\right)$$
$$A = 30r - \pi r^2 + \frac{\pi-4}{2}(r^2)$$
$$A = \frac{60r - 2\pi r^2 + \pi r^2 - 4r^2}{2} = \frac{60r - 4r^2 - \pi r^2}{2}$$
$$A = \frac{60r - (4+\pi)r^2}{2}$$

Differentiate. $\frac{dA}{dx} = \frac{1}{2}[60 - (4+\pi)(2r)]$

$$0 = 60 - 2r(4+\pi)$$
$$r = \frac{30}{4+\pi}$$

Hence, $2r = 2\left(\frac{30}{4+\pi}\right) = \frac{60}{4+\pi}$

Substitute into Equation (1). $h = \frac{30 - \pi\left(\frac{30}{4+\pi}\right)}{2}$

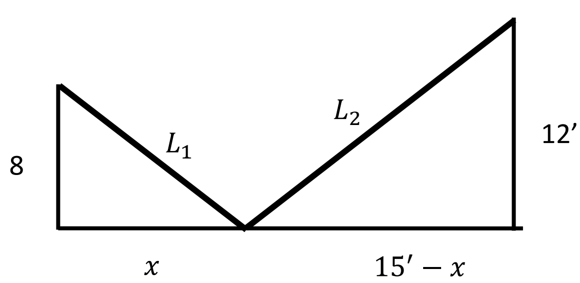
$$h = \frac{120 - 30\pi - 30\pi}{2(4+\pi)}$$
$$h = \frac{60}{4+\pi}$$

Taking the ratio of the height h to the width $2r$, $\frac{h}{2r} = \frac{\frac{60}{4+\pi}}{\frac{60}{4+\pi}} = 1h = 2r$

Therefore, for the maximum amount of light be admitted, the width of the window equals its height.

Example 46. Two posts, one 8 feet high and the other 12 feet high, stand 15 feet apart. They are to be supported by wires attached to a single stake at ground level. Where should the stake be placed so that the least amount of wire is used?

Let x be the distance of the stake from the shorter pole.



From the figure at the right, $L_1 = \sqrt{(8)^2 + x^2}$

Likewise, $L_2 = \sqrt{(12)^2 + (15-x)^2}$

Thus, total length of wires used $= L = L_1 + L_2$

$$L = \sqrt{(8)^2 + x^2} + \sqrt{(12)^2 + (15-x)^2}$$

Differentiate. $\frac{dL}{dx} = \frac{1}{2\sqrt{(8)^2+x^2}}(2x) + \frac{1}{2\sqrt{(12)^2+(15-x)^2}}(2)(15-x)(-1)$

$$0 = \frac{x}{\sqrt{(8)^2+x^2}} - \frac{15-x}{\sqrt{(12)^2+(15-x)^2}}$$

$$0 = \frac{x\sqrt{144+ (15-x)^2} - (15-x)\left(\sqrt{64+ x^2}\right)}{\sqrt{64+ x^2}\left(\sqrt{144\mp(15-x)^2}\right)}$$

$$0 = x\left(\sqrt{144+(15-x)^2}\right) - (15-x)\sqrt{64+x^2}$$

Solve the radical equation. $(15-x)\sqrt{64+x^2} = x\sqrt{144+(15-x)^2}$

$$(15-x)^2(64+x^2) = x^2[144+(225-30x+x^2)]$$

$$64(225-30x+x^2)+x^2(225-30x+x^2) = x^2(369-30x+x^2)$$

$$1400-1920x+64x^2+225x^2-30x^3+x^4 = 369x^2-30x^3+x^4$$

$$0 = 80x^2+1920x-14400$$

$$0 = x^2+24x-180$$

$$0 = (x-6)(x+30)$$

$x = 6$ $\qquad$ $x = -30$ (Rejected)

Thus, the stake should be positioned 6 m from the shorter post or 9 m from the longer post.

SAQ16
ACTIVITY 3.13 – O

NAME: ____ SCORE: _______________

SECTION: ____________DATE: ________________ PROF: _____________________________________

Solve the following maxima-minima problems using algebraic function.

1. Find two numbers having a sum of 9 if the product of one number by the square of the other is a maximum.

2. What is the smallest sum of two numbers if their product is 16?

3. A rectangular field of area 800ft^2 is to be fenced off along a straight river. If no fencing is required along river, find the dimensions of the lot that will require the least amount of fencing.

4. Find the area of the largest rectangular garden that can be made so that one side of the house serves as the natural boundary and 10 m of fencing material is required for the remaining three sides.

5. Find the coordinates of the point or points on the curve $x^2 - y^2 = 16$ which are nearest to the point$(0, 6)$.

6. A manufacturer makes aluminum cups of volume $16\ cm^3$ and in the form of right circular cylinder open at the top. What dimensions of the cup will require the least amount of material?

7. A closed right circular cylinder (that means including the top and the bottom) has a surface area of 100cm^2. What should the radius and altitude be in order to provide the largest possible volume?

ASAQ16
ACTIVITY 3.13 – O

NAME: ____ SCORE: ______________

SECTION: ___________DATE: _______________ PROF: __________________________________

Solve the following maxima-minima problems using algebraic function.

1. Find two numbers having a sum of 9 if the product of one number by the square of the other is a maximum. $Answer: 3\ and\ 6$

2. What is the smallest sum of two numbers if their product is 16? $Answer: 4\ and\ 4$

3. A rectangular field of area 800ft^2 is to be fenced off along a straight river. If no fencing is required along river, find the dimensions of the lot that will require the least amount of fencing. $Answer: 40'x\ and\ 20'$

4. Find the area of the largest rectangular garden that can be made so that one side of the house serves as the natural boundary and 10 m of fencing material is required for the remaining three sides. $Answer: 5\ m\ x\ \frac{5}{2}m$

5. Find the coordinates of the point or points on the curve $x^2 - y^2 = 16$ which are nearest to the point$(0, 6)$. $Answer: (5, 3), (-5, 3)$

6. A manufacturer makes aluminum cups of volume $16\ cm^3$ and in the form of right circular cylinder open at the top. What dimensions of the cup will require the least amount of material? $Answer: r = 1.42\ cm, h = 2.52 cm$

7. A closed right circular cylinder (that means including the top and the bottom) has a surface area of 100cm^2. What should the radius and altitude be in order to provide the largest possible volume? $Answer: r = 2.3\ cm, h = 4.6\ cm$

ACTIVITY 3.13 – P

NAME: ____ SCORE: _______________

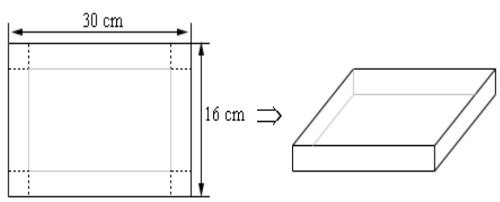
SECTION: ____________DATE: ________________ PROF: ____________________________________

Solve the following optimization problems.

1. Find two positive numbers such that their product is 192 and the sum of the first plus three times the second is a minimum.

2. An open box whose base is a square is to be made and will enclose $100\ m^3$. Find the dimensions of the box that will minimize the material needed to construct the box.

3. A piece of cardboard measures 30 cm by 16 cm, corners cut-out and sides folded up to form a box. Find the height of the box that will give the maximum volume.

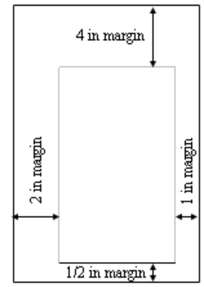


4. The combined perimeter of an equilateral triangle and a square is 10. Find the dimensions of the triangle and square that produce a minimum total area.

5. Four feet of wire is to be used to form a square and a circle. How much of the wire should be bent into a square and how much bent for the circle to enclose the maximum total area?

6. A tank with a rectangular base and rectangular sides is open at the top. It is to be constructed so that its width is 4 meters and its volume 36 cubic meters. If building the tank costs $10/sq. m. for the base and $5/sq. m. for the sides, what is the cost of the least expensive tank and what are its dimensions?

7. A poster having a total area of 125 in^2 is to be printed in a way so that there will be $\frac{1}{2}$ inch margin on the bottom, 1 inch margin on the right, 2 inch margin on the left and 4 inch margin on the top. What dimensions of the poster will give the largest printed area?



8. Which point on the graph of y = x is closest to the point (5, 0)?

9. A rectangular page is to contain 24 square inches of print. The margins at the top and bottom of the page are $1\frac{1}{2}$ inches. The margins of each side are 1 inch. What should be the dimensions of the page so that the least amount of paper is used?

MODULE 14

TIME~RATES

Specific Objectives:

At the end of the module, students must be able to:

1. Understand the concept of and steps in solving a time-rate problem.
2. Identify the quantities in the problem that change with time..
3. Solve a given time-rate problem.

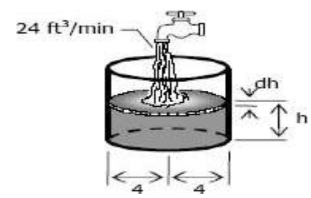
TIME – RATES

If the value of a variable y depends on the time t, then, $\frac{dy}{dt}$ is called its time-rate or rate of change with respect to time. When two or more quantities, all functions of t, are related by an equation, the relation between their time-rates may be attained by differentiating both sides of the equation with respect to time t. Basic time-rates are velocity= $\frac{ds}{dt}$, acceleration $a = \frac{dv}{dt}$, discharge $Q = \frac{dV}{dt}$ and angular speed $\omega = \frac{d\theta}{dt}$. If the time-rate $\frac{dy}{dt}$ is positive, it means the quantity y is increasing with time.

Steps in Solving Time Rates Problem

1. Identify what quantities are changing and what are fixed with time.
2. Assign variables to those that are changing and appropriate value (constant) to those that are fixed.
3. Find an equation relating all the variables and constants in Step 2.
4. Differentiate the equation with respect to time.

Example 47.Water is flowing into a vertical cylindrical tank at the rate of $24\ ft^3/min$. If the radius of the tank is 4 feet, how fast is the surface rising?



Solution: The quantities that are changing with time are the depth h and the volume V of the water in the cylindrical tank. The radius of the tank remains constant with time at a value of 4 feet. The required quantity in this problem is the time-rate of h or $\frac{dh}{dt}$. The time-rate of volume $\frac{dV}{dt} = +24\frac{ft^3}{min}$; it is positive, just like time-rate of depth h, since the volume of water in the tank increases with time.

At any time t, volume of water in the cylindrical can is given by equation

$$V = \pi r^2 h = \pi(4)^2 h = 16\pi h$$

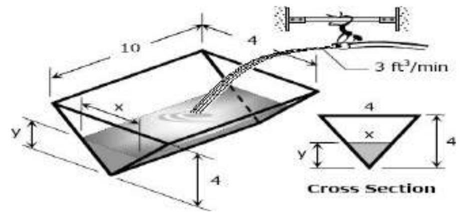
Differentiate both sides of the equation above with respect to time.

$$\frac{dV}{dt} = 16\pi\frac{dh}{dt}$$
$$+24 = 16\pi\ \frac{dh}{dt}$$
$$\frac{dh}{dt} = \frac{3}{2\pi}\ ft/min$$

Example 48.A triangular trough 10 ft long is 4 ft across the top, and 4 ft deep. Water flows in at the rate of $3\ ft^3/min$. How fast is the surface rising when the water is 6 in depth?

Solution:

The volume of water in the triangular trough at any time equals the volume of a triangular prism whose altitude is $10\ ft,$ a constant. The variables with time are volume V of water and its depth h. The other given constants are $\frac{dV}{dt} = +3ft/min$; same as the dimensions of the trough $4ft\ x\ 4ft\ x\ 10ft$. Required quantity is $\frac{dy}{dt}$ when $y = 6\text{ inches or }\frac{1}{2}\text{ft.}$



Volume of water at any time t. $\quad V = Bh = \left[\frac{1}{2}xy\right](10)$

$V = 5xy$ -------------- Equation (1)

By similar triangles: $\frac{x}{y} = \frac{4}{4} x = y$ ----------- Equation (2)

Method (1). Substitute Equation (2) into Equation (1), then, differentiate the resulting equation with respect to time. $V = 5(y)y = 5y^2$

$$\frac{dV}{dt} = 5(2y)\frac{dy}{dt}$$
$$3 = 10\left(\frac{1}{2}\right)\frac{dy}{dt} = 5\frac{dy}{dt}$$
$$\frac{dy}{dt} = \frac{3}{5}\text{ft/min}$$

Method (2). Differentiate Equation (1) with respect to time.

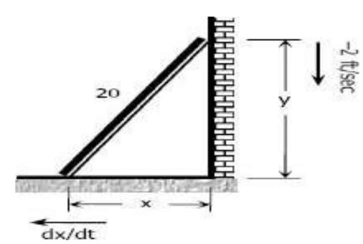
$\frac{dV}{dt} = 5\left[x\frac{dy}{dt} + y\frac{dx}{dt}\right]$ ----------- Equation (3)

Differentiate Equation (2) with respect to time. $\quad \frac{dx}{dt} = \frac{dy}{dt}$ ----------- Equation (4)

Substitute Equation (2) and Equation (4) into Equation (3). $\quad \frac{dV}{dt} = 5\left[y\frac{dy}{dt} + y\frac{dy}{dt}\right]$

$$3 = 5\left[\frac{1}{2}\frac{dy}{dt} + \frac{1}{2}\frac{dy}{dt}\right] = 5\left[\frac{dy}{dt}\right]$$
$$\frac{dy}{dt} = \frac{3}{5}\text{ft/min}$$

Example 49. A ladder 20 feet long leans against a vertical wall. If the top slides downward at the rate of 2 ft/sec, find how fast the lower end is moving when it is 16 feet from the wall.



Solution: The quantity that remains fixed with time is the length of the ladder which is equalto20 feet. The varying quantities with time are the distance x of the foot or lower end of the ladder from the wall and the distance y of the top of the ladder from the ground. Required quantity is time-rate of x or $\frac{dx}{dt}$ when $x = 16$ feet, with the time-rate of y or $\frac{dy}{dt} = -2$ ft/sec. It is negative since y decreases with time.

The equation that relates quantities dependent on time is obtained by Pythagorean Theorem. $y^2 + x^2 = (20)^2$ ----------- Equation (1)

Differentiate with respect to time. $2y\frac{dy}{dt} + 2x\frac{dx}{dt} = 0$

$y\frac{dy}{dt} + x\frac{dx}{dt} = 0$ ------------ Equation (2)

Substitute $x = 16$ into Equation (1) to get the corresponding value of y.

$$y^2 + (16)^2 = (20)^2$$
$$y^2 = 400 - 256 = 144$$
$$y = 12 \text{ feet}$$

Substitute into Equation (2). $12(-2) + 16\frac{dx}{dt} = 0$

$$\frac{dx}{dt} = \frac{24}{16} = \frac{3}{2}\text{ft/sec}$$

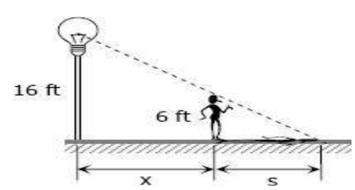
Take note that the computed $\frac{dx}{dt}$ is positive since x is increasing with time.

Example 50.A man 6 feet tall walks away from a lamp post 16 feet high at the rate of 5 miles per hour. How fast does his shadow lengthen?

Solution: Let the length of the man's shadow at any time be s and his distance from the lamp post be x; the given time-rate is $\frac{dx}{dx} = +5\frac{\text{miles}}{\text{hr}}$; the required time-rate is $\frac{ds}{dt}$.

By similar triangles: $\frac{6}{s} = \frac{16}{s+x}$

$$6(s + x) = 16s$$
$$6s + 6x = 16s$$
$$10s = 6x$$
$$s = \frac{3}{5}x$$



Differentiate with respect to time t. $\frac{ds}{dt} = \frac{3}{5}\frac{dx}{dt} = \frac{3}{5}(+5) = 3\ \frac{miles}{hr}$

Hence, the shadow of the man is lengthening at the rate of 3 $\frac{miles}{hr}$.

SAQ17
ACTIVITY 3.14 – Q

NAME: ____ SCORE: _______________

SECTION: ___________DATE: ________________ PROF: ____________________________________

Solve the following time-rate problems.

1. A conical cistern is 10 feet across the top and 12 feet deep. Water runs into the cistern at the rate of $1\ \text{ft}^3/sec$. How fast is the surface rising when the water is 8 feet deep?

2. A point moves along the upper half of parabola$y^2 = 2x + 1$ in such a way that $dx/dt = \sqrt{2x+1}$. Find $dy/dt$when $x = 4$.

3. The two equal sides of an isosceles triangle are each equal to 13 inches. If the third side is increasing at$24\ \text{in/min}$, at what is the altitude drawn to this side changing when the altitude is 12 inches?

4. Ohm's law for a certain electrical circuit states that $E = IR$, where E is the voltage in volts, I the current in amperes and R the resistance in ohms. If the circuit heats up and the voltage is kept constant, the resistance increases at the rate of $\frac{1}{2}\mathrm{ohm/sec}$. Find the rate at which the current decreases when $I = 2\ \mathrm{amperes}$ and E is kept constant at 10 volts.

5. The adiabatic law (no gain or heat loss) for the expansion of air is $PV^{1.4} = C$, where P is the pressure in$\mathrm{lb/in^2}$, V is the volume in cubic inches, and C is a constant. At a specific instant, the pressure is $40\,\mathrm{lb/in^2}$an is increasing at the rate of $8\,\mathrm{lb/in^2}$each second. If $C = \frac{5}{16}$, what is the rate of change of the volume at this instant?

6. The side of an equilateral triangle is 10 inches and is lengthening at the rate of $2\ \mathrm{in/sec}$. How fast is the area increasing?

ASAQ17
ACTIVITY 3.14 – Q

NAME: ____ SCORE: _______________

SECTION: ____________DATE: ________________ PROF: _____________________________________

Solve the following time-rate problems.

1. A conical cistern is 10 feet across the top and 12 feet deep. Water runs into the cistern at the rate of $1\ \mathrm{ft}^3/\mathrm{sec}$. How fast is the surface rising when the water is 8 feet deep?
   $Answer: 20.6\ \frac{in}{min}$

2. A point moves along the upper half of parabola $y^2 = 2x + 1$ in such a way that $dx/dt = \sqrt{2x+1}$. Find dy/dt when $x = 4$. $Answer: 1$

3. The two equal sides of an isosceles triangle are each equal to 13 inches. If the third side is increasing at $24\ \mathrm{in/min}$, at what is the altitude drawn to this side changing when the altitude is 12 inches?
   $Answer: -\frac{1}{2}\ \frac{in}{min}$

4. Ohm's law for a certain electrical circuit states that $E = IR$, where E is the voltage in volts, I the current in amperes and R the resistance in ohms. If the circuit heats up and the voltage is kept constant, the resistance increases at the rate of $\frac{1}{2}\mathrm{ohm/sec}$. Find the rate at which the current decreases when $I = 2$ amperes and E is kept constant at 10 volts.

   $Answer: -\frac{1}{5}\ \frac{amp}{sec}$

5. The adiabatic law (no gain or heat loss) for the expansion of air is $PV^{1.4} = C$, where P is the pressure in$\mathrm{lb/in^2}$, V is the volume in cubic inches, and C is a constant. At a specific instant, the pressure is $40\,\mathrm{lb/in^2}$an is increasing at the rate of $8\,\mathrm{lb/in^2}$each second. If $C = \frac{5}{16}$, what is the rate of change of the volume at this instant?

   $Answer: -\frac{32}{7}\ \frac{in^3}{sec}$

6. The side of an equilateral triangle is 10 inches and is lengthening at the rate of $2\ \mathrm{in/sec}$. How fast is the area increasing?

   $Answer: 10\sqrt{3}\ \frac{in^2}{sec}$

ACTIVITY 3.14 – R

NAME: ____ SCORE: _____________

SECTION: ___________DATE: _______________ PROF: _________________________________

Solve the following time-rate problems.

1. A kite is 24 feet high with 25 feet cord out. If the kite moves horizontally at 4 miles per hour directly away from the boy flying it, how fast is the cord being paid out?

2. A rectangular through is 9 feet long and 4 feet wide. How fast does the surface rise if water flows at the rate of $10\ \frac{ft^3}{min}$?

3. . A triangular through is 10 feet long, 6 feet wide across the top and 3 feet deep. If water flows at the rate of $12\ \frac{ft^3}{min}$, find how fast is the surface rising when the water is 6 inches deep.

4. A rock is dropped into a pool of water, causing ripples to form in expanding outward circles. The radius *r* of a ripple is increasing at a rate of $\frac{3}{2}$ feet per second. When the radius is 8 feet, at what rate is the area *A* of the water inside the ripple changing?

5. A rock is dropped into a pool of water, creating ripples which move outward from it. One of these ripples creates a circle with an area increasing at a rate of 28 ft^2/sec. When the area is 49π square feet, at what rate is the radius expanding?

6. A tank of water in the shape of a cone has a base radius of 4 feet and a height of 12 feet. If water is leaking at the bottom at a constant rate of 2 ft^3/hour, at what rate is the depth of the water in the tank changing when the depth of the water is 4 feet. At what rate is the radius of the top of the water changing when the depth of water is 8 feet?

MODULE 15

DERIVATIVES OF TRIGONOMETRIC FUNCTION AND THEIR APPLICATIONS

Specific Objectives:

At the end of the module, students must be able to:

1. Know and use correctly the differentiation formulas for trigonometric function.
2. Apply the differentiation formulas for trigonometric function in solving problems on slope of tangent and normal, rectilinear motion, angle between curves, optimization problems and time-rates.

TRANSCENDENTAL FUNCTIONS

Transcendental function is a function that is not an algebraic function. Such a function cannot be expressed as a solution of a polynomial equation whose coefficients are themselves polynomials with rational coefficients. Transcendental functions include trigonometric functions, inverse trigonometric functions, exponential and logarithmic functions.

DERIVATIVE OF TRIGONOMETRIC FUNCTION

Let u be a differentiable function of x.

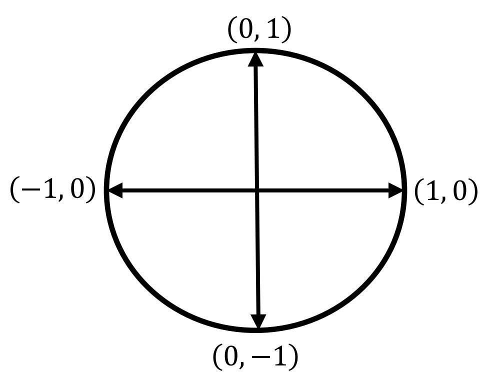
1. $\frac{d}{dx}\sin u = \cos u\frac{du}{dx}$	4. $\frac{d}{dx}\csc u = -\csc u \cot u\frac{du}{dx}$
2. $\frac{d}{dx}\cos u = -\sin u\frac{du}{dx}$	5. $\frac{d}{dx}\sec u = secu \tan u\frac{du}{dx}$
3. $\frac{d}{dx}\tan u = sec^2 u\frac{du}{dx}$	6. $\frac{d}{dx}\cot u = -csc^2 u\frac{du}{dx}$

It is but wise to review some basic concepts and the trigonometric relations to facilitate the differentiation and simplification of derivatives of trigonometric functions.

TRIGONOMETRIC FUNCTIONS OF ANGLE A WITH POINT $P(x, y)$ ON ITS TERMINAL SIDE

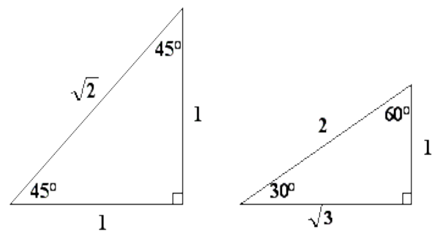
1. $\sin A = \frac{y}{r}$	4. $\csc A = \frac{r}{y}$
2. $\cos A = \frac{x}{r}$	5. $\sec A = \frac{r}{x}$
3. $\tan A = \frac{y}{x}$	6. $\cot A = \frac{x}{y}$

THE UNIT CIRCLE AND TRIGONOMETRIC FUNCTIONS OF QUADRANTAL ANGLES



TRIGONOMETRIC FUNCTIONS OF SPECIAL ANGLES

Use the SOH-CAH-TOA definitions of the trigonometric functions of the special acute acute angles.



BASIC RELATIONS FOR TRIGONOMETRIC FUNCTIONS

1. $\tan A = \frac{\sin A}{\cos A}$	5. $\tan A \cot A = 1$
2. $\cot A = \frac{\cos A}{\sin A}$	6. $\sin^2 A + \cos^2 A = 1$
3. $\sec A = \frac{1}{\cos A}$	7. $1 + \tan^2 A = \sec^2 A$
4. $\csc A = \frac{1}{\sin A}$	8. $1 + \cot^2 A = \csc^2 A$

NEGATIVE ANGLE FORMULAS

1. $\sin(-A) = -\sin A$	2. $\cos(-A) = \cos A$	3. $\tan(-A) = -\tan A$

SUM AND DIFFERENCE OF TWO ANGLES IDENTITIES

1. $\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$	3. $\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$
2. $\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$	4. $\cot(A \pm B) = \frac{\cot A\cot B \mp 1}{\cot B \pm \cot A}$

DOUBLE ANGLE IDENTITIES

1. $\sin 2A = 2\sin A cos\ A$	4. $\cos 2A = 1 - 2sin^2 A$
2. $\cos 2A = cos^2 A - sin^2 A$	5. $\tan 2A = \frac{2\tan A}{1-tan^2 A}$
3. $\cos 2A = 2cos^2 A - 1$	6. $\cot 2A = \frac{cot^{2A-1}}{2\cot A}$

HALF - ANGLE IDENTITIES

1. $\sin\frac{1}{2}A = \pm\sqrt{\frac{1-\cos A}{2}}$	4. $\tan\frac{1}{2}A = \frac{1-\cos A}{\sin A}$
2. $\cos\frac{1}{2}A = \pm\sqrt{\frac{1+\cos A}{2}}$	5. $\tan\frac{1}{2}A = \csc A - \cot A$
3. $\tan\frac{1}{2}A = \frac{\sin A}{1+\cos A}$	6. $\cot\frac{1}{2}A = \frac{1+\cos A}{\sin A} = \frac{\sin A}{1-\cos A} = \csc A + \cot A$

SUM TO PRODUCT IDENTITIES

1. $sin\ A + sin\ B = 2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)$
2. $sin\ A - sin\ B = 2\cos\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)$
3. $cos\ A + cos\ B = 2\cos\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)$
4. $cos\ A - cos\ B = -2\sin\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)$

Find the $\frac{dy}{dx}$, given the following trigonometric functions.

Example1. $y = \sin(x^2+4)^3$

Solution: Use $\frac{d}{dx}\sin u = \cos u\frac{du}{dx}$

With: $u = (x^2+4)^3$, $\frac{du}{dx} = 3(x^2+4)^2(2x) = 6x(x^2+4)^2$

$$y' = \cos(x^2+4)^3[6x(x^2+4)^2]$$
$$y' = 6x(x^2+4)^2\cos(x^2+4)^3$$

Example 2. $y = \csc\sqrt{x}$

Solution: Use $\frac{d}{dx}\csc u = -\csc u\cot u\frac{d}{dx}u$

With $u = \sqrt{x}$, $\frac{du}{dx} = \frac{1}{2\sqrt{x}}$

$$y' = -\csc\sqrt{x}\cot\sqrt{x}\left[\frac{1}{2\sqrt{x}}\right]$$
$$y' = \frac{-1}{2\sqrt{x}}\csc\sqrt{x}\cot\sqrt{x}$$

Example 3. $y = \frac{1}{2}\tan x\sin 2x$

Solution: Use the product formula $\frac{d}{dx}uv = u\frac{dv}{dx} + v\frac{du}{dx}$

With $u = \tan x$, $\frac{du}{dx} = \sec^2 x$

$v = \sin 2x$, $\frac{dv}{dx} = (\cos 2x)(2) = 2\cos 2x$

$$y' = \frac{1}{2}[\tan x(2\cos 2x) + \sin 2x(\sec^2 x)]$$
$$y' = \tan x \cos 2x + \tfrac{1}{2}\sin 2x \sec^2 x$$
$$y' = \left[\frac{\sin x}{\cos x}(2cos^2 x - 1)\right] + \frac{1}{2}(2\sin x\cos x)\left(\frac{1}{cos^2 x}\right)$$
$$y' = 2\sin x\cos x - \tan x + \frac{\sin x}{\cos x}$$
$$y' = 2\sin x\cos x - \tan x + \tan x$$
$$y' = 2\sin x\cos x = \sin 2x$$

Example 4. $y = x^3\tan(2x^2)$

Solution: Use again the product formula $\frac{d}{dx}uv = u\frac{dv}{dx} + v\frac{du}{dx}$

With $u = x^3$ $\quad v = \tan(2x^2)$

$\frac{du}{dx} = 3x^2$ $\quad \frac{dv}{dx} = (4x)[\sec^2(2x^2)]$

$$y' = x^3(4x)[\sec^2(2x^2)] + 3x^2\tan(2x^2)$$
$$y' = x^2[4x^2\sec^2(2x^2) + 3\tan(2x^2)]$$

Example 5. $y = \frac{1-\cos 2x}{1+\cos 2x}$

Solution: Use the quotient formula $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

With $u = 1 - \cos 2x$ $\quad v = 1 + \cos 2x$

$\frac{du}{dx} = -(-\sin 2x)(2) = 2\sin 2x$ $\quad \frac{dv}{dx} = -(\sin 2x)(2) = -2\sin 2x$

$$y' = \frac{(1+\cos 2x)(2\sin 2x) - (1-\cos 2x)(-2\sin 2x)}{(1+\cos 2x)^2}$$
$$y' = \frac{2\sin 2x\,[1+\cos 2x + 1 - \cos 2x]}{(1+\cos 2x)^2}$$
$$y' = \frac{2(\sin 2x)(2)}{(1+\cos 2x)^2}$$
$$y' = \frac{4\sin 2x}{(1+\cos 2x)^2}$$

Example 6. $y = \sec^4\left(\frac{1}{x^3}\right)$

Solution: Use the power formula $\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}$ $\quad$ with $u = \sec\left(\frac{1}{x^3}\right)$; $\;n = 4$

$$\frac{du}{dx} = \sec\left(\frac{1}{x^3}\right)\tan\left(\frac{1}{x^3}\right)\frac{d}{dx}\left(\frac{1}{x^3}\right)$$
$$\frac{du}{dx} = \sec\left(\frac{1}{x^3}\right)\tan\left(\frac{1}{x^3}\right)(-3x^{-4})$$
$$\frac{du}{dx} = \frac{-3}{x^4}\sec\left(\frac{1}{x^3}\right)\tan\left(\frac{1}{x^3}\right)$$
$$y\ ' = 4\sec^3\left(\frac{1}{x^3}\right)\left[\frac{-3}{x^4}\sec\left(\frac{1}{x^3}\right)\tan\left(\frac{1}{x^3}\right)\right]$$

$$y' = \frac{-12}{x^4}\sec^4\left(\frac{1}{x^3}\right)\tan\left(\frac{1}{x^3}\right)$$

However, $\sec^4\left(\frac{1}{x^3}\right) = y$. Therefore, $y' = \frac{-12y\tan\left(\frac{1}{x^3}\right)}{x^4}$.

Example 7. $y = \left(\frac{\cot x}{1+\csc x}\right)^{\frac{1}{3}}$

Solution: Again, use the power formula $\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}$

$$u = \frac{\cot x}{1+\csc x};\quad n = \frac{1}{3};\quad n-1 = \frac{1}{3} - 1 = \frac{2}{3}$$

$$\frac{du}{dx} = \frac{(1+\csc x)(-\csc^2 x) - \cot x[-\csc x\cot x]}{(1+\csc x)^2}$$

$$\frac{du}{dx} = \frac{\csc x\,[-\csc x(1+\csc x)+\cot^2 x]}{(1+\csc x)^2}$$

$$\frac{du}{dx} = \frac{\csc x[-\csc x - \csc^2 x + \cot^2 x]}{(1+\csc x)^2}$$

But, from Pythagorean Relation: $1 + \cot^2 A = \csc^2 A$

Therefore, it follows that $-\csc^2 x + \cot^2 x = -1$

Substitution yields, $\frac{du}{dx} = \frac{\csc x\,(-\csc x - 1)}{(1+\csc x)^2} = \frac{\csc x\,(-1)(1+\csc x)}{(1+\csc x)^2} = \frac{-\csc x}{1+\csc x}$

Thus,
$$y' = \frac{1}{3}\left(\frac{\cot x}{1+\csc x}\right)^{-\frac{2}{3}}\left(\frac{-\csc x}{1+\csc x}\right)$$

$$y' = \frac{1}{3}\left(\frac{1+\csc x}{\cot x}\right)^{\frac{2}{3}}\left(\frac{-\csc x}{1+\csc x}\right)$$

$$y' = \frac{-\csc x}{3(\cot x)^{\frac{2}{3}}(1+\csc x)^{\frac{1}{3}}}$$

$$y' = \frac{-\csc x}{3\sqrt[3]{\cot^2 x(1+\csc x)}}$$

Example 8. $y = \sin(x+y)$

Solution: Use Implicit differentiation.

$$y' = \cos(x+y)(1+y')$$
$$y' = \cos(x+y) + y'\cos(x+y)$$
$$y' - y'\cos(x+y) = \cos(x+y)$$
$$y'[1-\cos(x+y)]$$
$$y' = \frac{\cos(x+y)}{1-\cos(x+y)}$$

Example 9. $x\cos y + y\cos x = 2$

Solution: Use implicit differentiation. $x\left[-\sin y\frac{dy}{dx}\right] + \cos y + y[-\sin x] + \cos x\left(\frac{dy}{dx}\right) = 0$

$$-x\sin y\frac{dy}{dx} + \cos x\frac{dy}{dx} = y\sin x - \cos y$$

$$\frac{dy}{dx}(-x\sin y + \cos x) = y\sin x - \cos y$$

$$\frac{dy}{dx} = \frac{y\sin x - \cos y}{\cos x - x\sin y}$$

DERIVATIVE AS SLOPE OF THE TANGENT LINE

Example 10. Find equation of the tangent line to the curve $y = x\cos x$ at the point where $x = \pi$.

Solution: To get the equation of the tangent line, we need to know its slope and the point of tangency.

Find the slope the tangent by differentiating the given function. Hence,

$$\frac{dy}{dx} = x(-\sin x) + \cos x = -x\sin x + \cos x$$

At $x = \pi$: $$\frac{dy}{dx} = -\pi(\sin\pi) + \cos\pi = -\pi(0) + (-1) = -1$$

Substitute $x = \pi$ to the given equation of the curve to get the corresponding value of y.
$y = \pi\cos\pi = \pi(-1) = -\pi$

Thus, point of tangency is at $(\pi, -\pi)$
Use the point-slope form to get equation of the tangent line at$(\pi, -\pi)$.

$$y - (-\pi) = -1(x - \pi)$$
$$y + \pi = -x + \pi$$
$$y = -x$$

Therefore, at $x = \pi$, equation of the tangent line to curve $y = x\cos x$ is $y = -x$.

Example 11. At what point does the tangent line to the curve $y = 4\tan 2x$ parallel to the line $y - 8x + 3 = 0$?

Solution: First, differentiate the given function and equate the resulting derivative to the slope of the given line. They are parallel lines, so their slopes are equal. Solve for the value of x and the corresponding $y -$ value.

$$y' = 4(\sec^2 2x)(2) = 8\sec^2 2x$$

At the unknown point, $y' = m_{TL} = 8$.

$$8 = 8\sec^2 2x$$
$$\sec^2 2x = 1$$
$$(\sec 2x + 1)(\sec 2x - 1) = 0$$

$\sec 2x = 1$	$\sec 2x = -1$
$2x = 0,\ 2\pi, 4\pi$	$2x = \pi,\ 3\pi$
$x = 0, \pi, 2\pi$	$x = \frac{\pi}{2},\ \frac{3\pi}{2}$

Substitute the x-value on equation $\mathrm{y} = 4\tan 2\mathrm{x}$.

When $x = 0$, $\quad y = \tan 0 = 0$.
When $x = \pi$, $\quad y = \tan \pi = 0$.
When $x = \frac{\pi}{2}$, $\quad y = \tan \pi = 0$.
When $x = \frac{3\pi}{2}$, $y = \tan 3\pi = 0$.

Therefore, the four points on the curve $y = 4\tan 2x$ where the tangent line is parallel to line $y - 8x + 3 = 0$ are $(0,0)$, $\left(\frac{\pi}{2}, 0\right)$, $(\pi, 0)$and $\left(\frac{3\pi}{2}, 0\right)$.

MAXIMA-MINIMA APPLICATIONS USING TRIGONOMETRIC FUNCTIONS

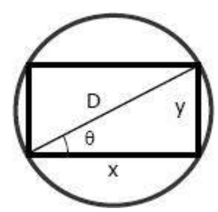
The use of trigonometric functions facilitates solution to many maxima-minima applications.To do it, identify the constant terms and the quantity/variable to be maximized (or minimized), differentiate that quantity/variable, equate the derivative to zero and then, solve for the value of the variable left on the resulting equation.

Example 12. Find the shape of the rectangle of maximum perimeter inscribed in a circle of diameter D.

Let x and y be the breadth and length of the rectangle whose perimeter P needs to be the maximum;
θ be the acute angle the diameter makes with the breadth of the rectangle, as shown on the accompanying figure.

The diameter D of the circle is a constant quantity in this problem.

Optimization equation: $\mathrm{P} = 2\mathrm{x} + 2\mathrm{y}$ ------------ Equation (1)



Constraint equations: $\cos\theta = \frac{\mathrm{x}}{\mathrm{D}}$

$\mathrm{x} = \mathrm{D}\cos\theta$ ----------------- Equation (2)

Likewise, $\sin\theta = \frac{\mathrm{y}}{\mathrm{D}}$

$\mathrm{y} = \mathrm{D}\sin\theta$------------------ Equation (3)

Method (1).Substitute Equation (2) and Equation (3) into Equation (1) to express perimeter P, the quantity to be maximized, in terms of variable θ.

$$\mathrm{P} = 2\mathrm{D}\cos\theta + 2\mathrm{D}\sin\theta$$
$$\mathrm{P} = 2\mathrm{D}(\cos\theta + \sin\theta)$$

Differentiate P and set its derivative to zero. $\frac{\mathrm{dP}}{\mathrm{d}\theta} = 2\mathrm{D}[-\sin\theta + \cos\theta]$

$$0 = 2\mathrm{D}[-\sin\theta + \cos\theta]$$
$$0 = -\sin\theta + \cos\theta$$

Solve the above trigonometric equation for θ. $0 = -\sin\theta + \cos\theta$

$$\sin\theta = \cos\theta$$

Divide the above equation by $\cos\theta$. $\frac{\sin\theta}{\cos\theta} = 1$

$$\tan\theta = 1$$

$$\theta = 45^{o} = \frac{\pi}{4}$$

Solve for x and y by substituting into Equation (2) and Equation (3).

$$x = D\cos 45^{o} = \frac{\sqrt{2}}{2}D$$

$$y = D\sin 45^{o} = \frac{\sqrt{2}}{2}D$$

Therefore the rectangle of biggest perimeter that can be cut from a circle of radius D has its breadth equals its length; or, the rectangle is a square.

Method (2). Differentiate Equation (1) with respect to one of the variables, say x, set $\frac{dP}{dx} = 0$ and solve for $\frac{dy}{dx}$. $0 = 2 + 2\frac{dy}{dx}$

$$\frac{dy}{dx} = -1 \text{-------------- Equation (A)}$$

Differentiate Equation (2) with respect to x.

$$1 = D(-\sin\theta)\frac{d\theta}{dx}$$

$$\frac{d\theta}{dx} = \frac{-1}{D\sin\theta} \text{------------ Equation (B)}$$

Differentiate Equation (3) with respect to x.

$$\frac{dy}{dx} = D\cos\theta\frac{d\theta}{dx} \text{-------- Equation (C)}$$

Substitute Equation (A) and Equation (B) into Equation (C).

$$-1 = D\cos\theta\left(\frac{-1}{D\sin\theta}\right)$$

$$\frac{\sin\theta}{\cos\theta} = 1$$

$$\tan\theta = 1$$

$$\theta = 45^{o} = \frac{\pi}{4}$$

Therefore, we get same results as Method 1.

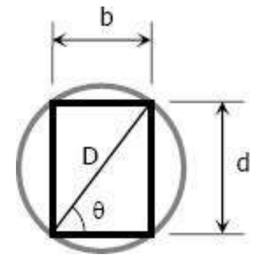
$$x = D\cos 45^{o} = \frac{\sqrt{2}}{2}D$$

$$y = D\sin 45^{o} = \frac{\sqrt{2}}{2}D$$

Example 13.The strength of rectangular beam is proportional to the product of the breadth and the square of the depth. Find the shape of the strongest beam that can be cut from a log of given size.

Let S be the strength of the rectangular beam
b be its breadth;
d be its depth, and
θ be the acute angle the given diameter D of the log makes with the diameter d of the beam.

Optimization equation: $S = bd^2$------------ Equation (1)



Constraint equations: $\cos\theta = \frac{b}{D}$

$b = D\cos\theta$ ----------- Equation (2)

And, $sin\theta = \frac{d}{D}$

$d = D\sin\theta$ ----------- Equation (3)

Substitute Equation (2) and Equation (3) into Equation (1).

$$S = D\cos\theta(D\sin\theta)^2$$
$$S = D^3\cos\theta sin^2\theta$$

Differentiate S with respect to θ and set $\frac{dS}{d\theta} = 0$.

$$\frac{dS}{d\theta} = D^3[\cos\theta(2\sin\theta)(\cos\theta) + sin^2\theta(-\sin\theta)]$$
$$0 = D^3[\sin\theta(2cos^2\theta - sin^2\theta)]$$
$$0 = \sin\theta[2cos^2\theta - (1 - cos^2\theta)]$$
$$0 = \sin\theta(3cos^2\theta - 1)$$

$\sin\theta = 0\ \theta = 0^o$ (Rejected)

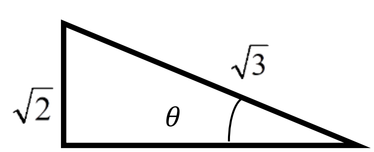
Likewise, $3cos^2\theta - 1 = 0$

$cos^2\theta = \frac{1}{3}$

$\cos\theta = \frac{1}{\sqrt{3}}$

Take note that $\cos\theta = \frac{-1}{\sqrt{3}}$ is rejected since θ is an acute angle.

To avoid using calculator to get the value of θ from $\cos\theta = \frac{1}{\sqrt{3}}$, draw a right triangle having θas an acute angle.



Therefore, $b = D\left(\frac{1}{\sqrt{3}}\right) = \frac{D}{\sqrt{3}}$

And, $h = D\left(\frac{\sqrt{2}}{\sqrt{3}}\right) = \frac{\sqrt{2}}{\sqrt{3}}D = \sqrt{2}\left(\frac{D}{\sqrt{3}}\right)$

Thus, the height is equal to $\sqrt{2}$ times the breadth.

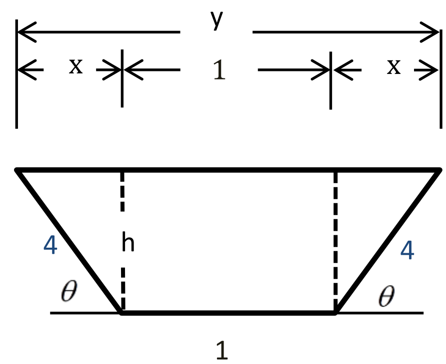
Example 14. A trapezoidal gutter is to be made, from a strip of metal 22 inches wide by bending up the edges. If the base is 14 inches wide, what width across the top gives the greatest carrying capacity?

Let θ be the acute angle the leg of the trapezoidal cross-section makes with the horizontal

The gutter having the maximum cross-sectional area A will have the greatest carrying capacity. Thus, optimization equation: $A = 2A_{triangle} + A_{rectangle}$

$$A = 2\left[\frac{1}{2}xh\right] + 14(h) = xh + 14h \text{-------------- Equation (1)}$$

From the given figure, constraint equations:



$$\cos\theta = \frac{x}{4}$$

$$x = 4\cos\theta \text{ ------------ Equation (2)}$$

And $$\sin\theta = \frac{h}{4}$$

$$h = 4\sin\theta \text{ ------------ Equation (3)}$$

Substitute Equation (2) and Equation (3) into Equation (1).

$$A = (4\cos\theta)(4\sin\theta) + 14(4\sin\theta)$$
$$A = 16\sin\theta\cos\theta + 56\sin\theta$$

Differentiate the above equation with respect to θ.

$$\frac{dA}{d\theta} = 16[\sin\theta(-\sin\theta) + \cos\theta(\cos\theta)] + 56\cos\theta$$

Set $\frac{dA}{d\theta} = 0$, express the resulting equation in terms of a single trigonometric function and solve for value of variable θ.

$$0 = 16[-sin^2\theta + cos^2\theta] + 56\cos\theta$$
$$0 = 16[(cos^2\theta - 1) + cos^2\theta] + 56\cos\theta$$
$$0 - 2[2cos^2\theta - 1] + 7\cos\theta$$
$$0 = 4cos^2\theta + 7\cos\theta - 2$$
$$0 = (4\cos\theta - 1)(\cos\theta + 2)$$
$$0 = 4\cos\theta - 1 \qquad \cos\theta + 2 = 0$$
$$\cos\theta = \frac{1}{4}$$

However, $\cos\theta = -2$ is rejected since $-1 \leq \cos\theta \leq 1$.

Substitute into Equation (2).$x = 4\left(\frac{1}{4}\right) = 1$

Width across the top that will give the gutter the maximum carrying capacity:

$$y = 1 + 14 + 1 = 16 \text{ inches}$$

Example 15. A pole 24 feet long is carried horizontally along a corridor 8 feet wide and into a second corridor at right angles to the first. How wide must the second corridor be?

Let θ be the acute angle which the pole makes with the second corridor
w be the minimum width of the second corridor that will permit the passage of the pole from the first corridor.

Constraint equation: $\cos\theta = \frac{8}{a}$ $\quad a = \frac{8}{\cos\theta} = 8\sec\theta$------------- Equation (1)

$\sin\theta = \frac{w}{b}\, b = \frac{w}{\sin\theta} = wcsc\,\theta$--------- Equation (2)

Optimization equation: $24 = a + b$

$24 = 8sec\theta + wcsc\,\theta$ ------------------ Equation (3)

Substitute Equation (1) and Equation (2) into equation (3).

$$24 = 8\sec\theta + wcsc\,\theta$$

$$w = \frac{24 - 8\sec\theta}{\csc\theta} = \frac{24}{\csc\theta} - \frac{8\sec\theta}{\csc\theta}$$

$$w = 24\sin\theta - \frac{8\left(\frac{1}{\cos\theta}\right)}{\frac{1}{\sin\theta}}$$

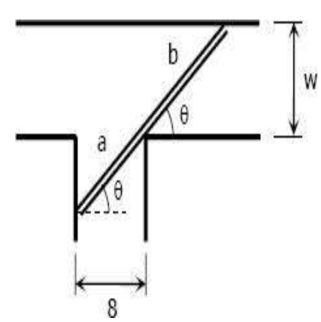
$$w = 24\sin\theta - \frac{8\sin\theta}{\cos\theta}$$

$w = 24\ \sin\theta - 8\tan\theta$----Equation (4)

Differentiate Equation (4) with respect to θ.

$$\frac{dw}{d\theta} = 24\cos\theta - 8sec^2\theta$$

Set $\frac{dw}{d\theta} = 0$ and solve for value of variable θ.



$$0 = 8\left(3\cos\theta - \frac{1}{cos^2\theta}\right)$$

$$0 = \frac{3cos^3\theta - 1}{cos^2\theta}$$

$$3cos^3\theta = 1$$

$$cos^3\theta = \frac{1}{3}$$

$$\cos\theta = \sqrt[3]{\frac{1}{3}} = 0.693361$$

$$\theta = 46.10^o$$

Substitute $\theta = 46.10^o$ into Equation (4).

$$w = 24\sin 46.10^o - 8\tan 46.10^o$$

$$w = 8.98 \text{ feet}$$

Therefore, the second corridor needs to be at least 8.98 feet to permit the passage of the 27-foot pole from the first corridor to the second corridor.

TIME-RATE PROBLEM INVOLVING TRIGONOMETRIC FUNCTION

Example 16. A ladder 20 feet long leans against a vertical wall. If the top slides downward at the rate of 2 ft/sec, find

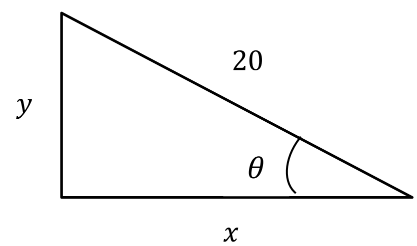
a). how fast is the inclination of the ladder changing when the lower end is 16 feet from the wall

b). the rate by which the foot of the ladder moves away from the wall

Solution:

Let θ be the inclination of the ladder at anytime
y be the distance of the top of the ladder from the ground at anytime
x be the distance of the foot of the ladder from the wall at anytime

Given quantities are the lengths of the ladder equal to 20 ft, $\frac{dy}{dt} = 2\ ft/sec$ and the required are $\frac{d\theta}{dt}$ and $\frac{dx}{dt}$.



a). We choose now a trigonometric function that involves θ and y since time rate of y $\frac{dy}{dt}$ is known.

$$\sin\theta = \frac{y}{20}$$
$$y = 20\sin\theta$$

Differentiate above equation with respect to time t.

$$\frac{dy}{dt} = 20\cos\theta\frac{d\theta}{dt} \text{------- Equation 1}$$

Quantity $\frac{\mathrm{dy}}{\mathrm{dt}} = -2\mathrm{ft}/sec$. Observe that a negative sign is assigned to it since y is decreasing with time t while $\frac{d\theta}{dt}$ expectedly is negative since it also decreases with time. From the triangle, when $x = 16$, $\cos\theta = \frac{16}{20} = \frac{4}{5}$. Substituting all known values into Equation 1 yields

$$-2 = 20\left(\frac{4}{5}\right)\frac{d\theta}{dt}$$
$$\frac{d\theta}{dt} = \frac{-2(5)}{80} = -\frac{1}{8}\ \frac{rad}{sec}$$

Thus, the inclination θ decreases by $\frac{1}{8}$ rad per second at that particular time the foot of the ladder is 16 feet away from the wall.

b). To find $\frac{dx}{dt}$, the rate by which the foot of the ladder moves horizontally away from the wall, we use Pythagorean theorem to define relation between x and y. The rate $\frac{dx}{dt}$ is expected to be a positive quantity since the distance of the foot of the ladder from the wall increases with time.

$$20^2 = x^2 + y^2 \text{---------- Equation 2}$$

Differentiate with respect to time. $0 = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$

$$0 = x\frac{dx}{dt} + y\frac{dy}{dt} \text{--------- Equation 3}$$

Use Equation 2 to find the value of y at that particular time when $x = 16$.

$$400 = (16)^2 + y^2$$
$$y = \sqrt{400 - 256} = 12$$

Substitute known quantities into Equation 3.

$$0 = 16\frac{dx}{dt} + 12(-2)$$
$$0 = 16\frac{dx}{dt} - 24$$
$$\frac{dx}{dt} = \frac{24}{16} = \frac{3}{2}\frac{feet}{sec}$$

Therefore, we say that the foot of the ladder moves horizontally away from the wall at the rate of $\frac{3}{2}$ feet per second.

SAQ18
ACTIVITY 4.15 – A

NAME: ____ SCORE: ______________
SECTION: ___________DATE: _______________ PROF: ___________________________________

Differentiate each of the following using the appropriate differentiation formula.

1. $y = \tan\frac{1}{2}x^3$	6. $y = x^3 \sin\frac{1}{3}x^3$
2. $y = 3\cos(2x^2 - 3x - 1)$	7. $w = csc^3 2x$
3. $t = \csc\sqrt{1 + 2\theta}$	8. $\mathrm{y} = \sin^2\sqrt{4 - \mathrm{x}}$
4. $v = \cos 2\theta\ cot2\theta$	9. $\mathrm{y} = \frac{1+\cos\mathrm{x}}{\sin x}$
5. $\mathrm{y} = \sin x\ cos^2 x$	10. $\mathrm{y} = \frac{1+\tan\mathrm{x}}{1-\tan x}$

SAQ18
ACTIVITY 4.15 – A

NAME: ____ SCORE: ______________
SECTION: ___________DATE: _______________ PROF: ___________________________________

Differentiate each of the following using the appropriate differentiation formula.

1. $y = \tan\frac{1}{2}x^3$	6. $y = x^3 \sin\frac{1}{3}x^3$
2. $y = 3\cos(2x^2 - 3x - 1)$	7. $w = csc^3 2x$
3. $t = \csc\sqrt{1 + 2\theta}$	8. $\mathrm{y} = \sin^2\sqrt{4 - \mathrm{x}}$
4. $v = \cos 2\theta\ cot2\theta$	9. $\mathrm{y} = \frac{1+\cos \mathrm{x}}{\sin x}$
5. $\mathrm{y} = \sin x\ cos^2 x$	10. $\mathrm{y} = \frac{1+\tan \mathrm{x}}{1-\tan x}$

II. Find $\frac{d^2y}{dx^2}$ of the given functions.

1. $y = x \sin x - \cos x$

2. $y = x - \frac{1}{2} \sin x$

III. Differentiate the given functions using the appropriate differentiation method.

1. $x = y + \tan y$

2. $x \csc y = 3$

3. $\sin(x - y) + \sin(x + y) = 4$ [Hint: Use sum to product identity to simplify $\frac{dy}{dx}$.]

4. $x = 2 \sec t; \;\; y = 4 \tan t$

5. $y = 4 - 2v + v^2; \; v = 2 - \sec x$

IV. Solve for the unknown.

1. Find equation of the tangent to the graph of $f(x) = \cot x$ at $x = \frac{\pi}{4}$.

2. Find the values of x on the graph of $f(x) = x - 2\cos x$ for $0 < x < 2\pi$ where the tangent line has a slope of 2.

3. A particle moves along a coordinate axis in such a way that its position at time t is given by $s(t) = \sqrt{3}t + 2\cos t$ for $0 \le t \le 2\pi$. At what time is the particle at rest?

4. An observer watches a rocket launched from a distance of 3 km. The angle of elevation θ is increasing at $4^o/sec$ at the instant $\theta = 45^0$. How fast is the rocket climbing at that instant?

5. If a ladder of length 30 feet that is leaning against a wall has its upper edge sliding down the wall at the rate of $\frac{1}{2}$ ft/sec, what is the rate of change of the measure of the acute angle made by the ladder with the ground when the upper end is 18 feet above the ground?

ASAQ18
ACTIVITY 4.15 – A

NAME: ____ SCORE: ______________
SECTION: ___________DATE: _______________ PROF: ___________________________________

Differentiate each of the following using the appropriate differentiation formula.

1. $y = \tan\frac{1}{2}x^3$ $Answer: \dfrac{3x^2}{2cos^2\frac{1}{2}x^3}$	6. $y = x^3 \sin\frac{1}{3}x^3$ $Answer: x^2\left(x^3\cos\frac{1}{3}x^3 + 3\sin\frac{1}{3}x^3\right)$
2. $y = 3\cos(2x^2 - 3x - 1)$ $Answer: 3(3 - 4x)\sin(2x^2 - 3x - 1)$	7. $w = csc^3 2x$ $Answer: -6csc^3 2x \cot 2x$
3. $t = \csc\sqrt{1 + 2\theta}$ $Answer: \dfrac{\sqrt{1+2\theta}\cot\sqrt{1+2\theta}}{2(1+2\theta)\sin\sqrt{1+2\theta}}$	8. $\mathrm{y} = \sin^2\sqrt{4 - \mathrm{x}}$ $Answer: -\dfrac{\sin 2\sqrt{4-\mathrm{x}}}{2\sqrt{4-\mathrm{x}}}$
4. $v = \cos 2\theta\ cot2\theta$ $Answer: -2(\cot 2\theta)(\sin 2\theta + \csc 2\theta)$	9. $\mathrm{y} = \frac{1+\cos \mathrm{x}}{\sin x}$ $Answer: -\dfrac{1+\cos x}{sin^2 x}$
5. $\mathrm{y} = \sin x\ cos^2 x$ $Answer: \cos x\,(3cos^2 x - 2)$	10. $\mathrm{y} = \frac{1+\tan \mathrm{x}}{1-\tan x}$ $Answer: \dfrac{2sec^2 x}{(1-\tan x)^2}$

II. Find $\frac{d^2y}{dx^2}$ of the given functions.

1. $y = x\sin x - \cos x \quad Answer: 3\cos x - x\sin x$

2. $y = x - \frac{1}{2}\sin x \quad Answer: \ \frac{1}{2}\sin x$

III. Differentiate the given functions using the appropriate differentiation method.

1. $x = y + \tan y \quad Answer: \ \frac{1}{1 + sec^2 x}$

2. $x\csc y = 3 \quad Answer: \ \frac{\tan y}{x}$

3. $\sin(x - y) + \sin(x + y) = 4$ [Hint: Use sum to product identity to simplify$\frac{dy}{dx}$.]

$Answer: \ \cot x \cot y$

4. $x = 2\sec t; \quad y = 4\tan t \quad Answer: 2\csc t$

5. $y = 4 - 2v + v^2; \ v = 2 - \sec x \quad Answer: \ -2(\sec x \tan x)(1 - \sec x)$

IV. Solve for the unknown.

1. Find equation of the tangent to the graph of $f(x) = \cot x$ at $x = \frac{\pi}{4}$. $Answer: x + 2y = 2 + \pi$

2. Find the values of x on the graph of $f(x) = x - 2\cos x$ for $0 < x < 2\pi$ where the tangent line has a slope of 2. $Answer: \frac{\pi}{6}; \frac{5\pi}{6}$

3. A particle moves along a coordinate axis in such a way that its position at time t is given by $s(t) = \sqrt{3}t + 2\cos t$ for $0 \leq t \leq 2\pi$. At what time is the particle at rest? $Answer: \frac{\pi}{3}; \frac{2\pi}{3}$

4. An observer watches a rocket launched from a distance of 3 km. The angle of elevation θ is increasing at $4^o/sec$ at the instant $\theta = 45^0$. How fast is the rocket climbing at that instant?

$Answer: \frac{\pi}{15} \frac{km}{sec}$

5. If a ladder of length 30 feet that is leaning against a wall has its upper edge sliding down the wall at the rate of $\frac{1}{2}$ ft/sec, what is the rate of change of the measure of the acute angle made by the ladder with the ground when the upper end is 18 feet above the ground? $Answer: -\frac{1}{48} \frac{rad}{sec}$

II. Find $\frac{d^2y}{dx^2}$ of the given functions.

1. $y = x\sin x - \cos x$

2. $y = x - \frac{1}{2}\sin x$

III. Differentiate the given functions using the appropriate differentiation method.

1. $x = y + \tan y$

2. $x\csc y = 3$

3. $\sin(x - y) + \sin(x + y) = 4$ [Hint: Use sum to product identity to simplify $\frac{dy}{dx}$.]

4. $x = 2\sec t;\;\; y = 4\tan t$

5. $y = 4 - 2v + v^2;\; v = 2 - \sec x$

IV. Solve for the unknown.

1. Find equation of the tangent to the graph of $f(x) = \cot x$ at $x = \frac{\pi}{4}$.

2. Find the values of x on the graph of $f(x) = x - 2\cos x$ for $0 < x < 2\pi$ where the tangent line has a slope of 2.

3. A particle moves along a coordinate axis in such a way that its position at time t is given by $s(t) = \sqrt{3}t + 2\cos t$ for $0 \leq t \leq 2\pi$. At what time is the particle at rest?

4. An observer watches a rocket launched from a distance of 3 km. The angle of elevation θ is increasing at $4^o/sec$ at the instant $\theta = 45^0$. How fast is the rocket climbing at that instant?

5. If a ladder of length 30 feet that is leaning against a wall has its upper edge sliding down the wall at the rate of $\frac{1}{2}$ ft/sec, what is the rate of change of the measure of the acute angle made by the ladder with the ground when the upper end is 18 feet above the ground?

ACTIVITY 4.15 – B

NAME: ____ SCORE: _______________
SECTION: ____________DATE: ________________ PROF: ____________________________________

I. Differentiate each of the following using the appropriate differentiation formula.

1. $y = \sin^4 3x - \cos^4 3x$

2. $y = \cot^2 \frac{1}{4}x - \csc^2 \frac{1}{4}x$

3. $y = \sqrt{\frac{1-\sin 3x}{1+\sin 3x}}$ (Hint: Use $1 - \sin^2 A = \cos^2 A$)

4. $y = \sec(2x + y)$

5. $xy = \sin y + 1$

6. $x = 4 - \csc t; \quad y = t + \sec t$

II. Find the equation of the tangent line to the given curve at the indicated value of x.

a. $y = \sqrt{\sin x + \cos x}$, $x = \frac{1}{4}\pi$

b. $y = \tan^2 2x \sec x$, $x = \frac{\pi}{6}$

III. Find the relative maxima and minima of given the curve for $0 \leq x \leq 2\pi$ and roughly sketch the graph of the given equations.

a. $y = \sin x + \cos x$

b. $\mathrm{y} = 4\cos^2\mathrm{x} - 8\sin\mathrm{x}$, $0 < x < 180^o$

IV. Solve the given time-rate problems.

1. Given that $y = cos^2x + \sin 3x$ and that x is changing at a rate of $\frac{1}{5}$ unit/sec, find the rate of change of y when x=$\pi/4$.

2. The base of an isosceles triangle remains constant at 10 feet with its base angles decreasing at a rate of $2^{\mathrm{o}}\mathrm{per\ second}$. Find the rate by which its area changes when its base angles measure $45^{\mathrm{o}}\mathrm{each}$.

3. The measure of one of the acute angles of a right triangle is decreasing at the rate of $\frac{1}{36}\pi\ \mathrm{rad}/sec$. If the length of the hypotenuse remains a constant at 40 cm, find how fast the area is changing when the measure of the acute angle is $\frac{1}{6}\pi$.

4. Each of the two sides of a triangle are increasing at the rate of $\frac{1}{2}\ \mathrm{foot/sec}$ with their included angle decreasing at the rate of $\frac{\pi}{90}\mathrm{radian}/sec$. Find the rate of change of area when the sides and the included angle are respectively 5 feet, 8 feet and 60^o?

5. If a ladder of length 30 feet that is leaning against a wall has its upper end sliding down the wall at the rate of $\frac{1}{2}\,\mathrm{ft/sec}$, what is the rate of change of the measure of the acute angle made by the ladder with the ground when the upper end is 18 feet above the ground?

MODULE 16

DERIVATIVES OF INVERSE TRIGONOMETRIC FUNCTION AND THEIR APPLICATIONS

Specific Objectives:

At the end of the module, students must be able to:

1. Know and use correctly the differentiation formulas for inverse trigonometric function.
2. Apply the differentiation formulas for inverse trigonometric function in solving problems on slope of tangent and normal, rectilinear motion, angle between curves, optimization problems and time-rates.

DERIVATIVES OF INVERSE TRIGONOMETRIC FUNCTIONS

Let u be a differentiable function of x.

1. $\frac{d}{dx}(\text{Arcsin}\, u) = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$	4. $\frac{d}{dx}(\text{Arccot}\, u) = -\frac{1}{1+u^2}\frac{du}{dx}$
2. $\frac{d}{dx}(\text{Arccos}\, u) = -\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$	5. $\frac{d}{dx}(\text{Arcsec}\, u) = \frac{1}{u\sqrt{u^2-1}}\frac{du}{dx}$
3. $\frac{d}{dx}(\text{Arctan}\, u) = \frac{1}{1+u^2}\frac{du}{dx}$	6. $\frac{d}{dx}(\text{Arccsc}\, u) = -\frac{1}{u\sqrt{u^2-1}}\frac{du}{dx}$

PRINCIPAL VALUES OF INVERSE TRIGONOMETRIC FUNCTIONS

Let us have a brief review of the principal values of an angle. The table below shows the principal value of the inverse trigonometric functions in their domain and the corresponding range.

Function	Domain	Range
$\text{Sin}^{-1}x$	[–1, 1]	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\text{Cos}^{-1}x$	[–1, 1]	$[0, \pi]$
$\text{Tan}^{-1}x$	(–∞, ∞)	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$\text{Cot}^{-1}x$	(–∞, ∞)	$(0, \pi)$
$\text{Sec}^{-1}x$	$(-\infty, -1] \cup [1, \infty)$	$\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$
$\text{Csc}^{-1}x$	$(-\infty, -1] \cup [1, \infty)$	$\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$

In Calculus, Sin^{-1} x, Cos^{-1} x and Tan^{-1} x are the most important inverse trigonometric functions. However, there is often disagreement in the choice of principal value for inverse secant and inverse cosecant. Some authors define the value as between $-\pi$ and $-\frac{\pi}{2}$ for negative value of trigonometric functions secant and cosecant. Relative to this and in as much that they are seldom used, these inverse trigonometric functions may be conveniently avoided.

Example 17. Find $\frac{dy}{dx}$ using the indicated value of x, if given.

1. $y = \text{Arcsin}(2-3x)$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(2-3x)^2}}\frac{d}{dx}(2-3x)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(4-12x+9x^2)}}(-3)$$

$$\frac{dy}{dx} = \frac{-3}{\sqrt{-3+12x-9x^2}}$$

2. $y = Arcsec\sqrt{1+2x}$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1+2x}\sqrt{\left(\sqrt{1+2x}\right)^2-1}}\frac{d}{dx}\left(\sqrt{1+2x}\right)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1+2x}\sqrt{(1+2x)-1}}\left[\frac{d}{dx}\sqrt{1+2x}\right]$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1+2x}\sqrt{2x}}\left[\frac{1}{2\sqrt{1+2x}}(2)\right]$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{2x}(1+2x)}$$

$$\frac{dy}{dx} = \frac{1}{(1+2x)\sqrt{2x}}$$

3. $y = \text{Arctan}\left(\frac{2-x}{2+x}\right)$

$$\frac{dy}{dx} = \frac{1}{1+\left(\frac{2-x}{2+x}\right)^2}\frac{d}{dx}\left(\frac{2-x}{2+x}\right)$$

$$\frac{dy}{dx} = \frac{1}{1+\frac{(2-x)^2}{(2+x)^2}}\left[\frac{(2+x)\frac{d}{dx}(2-x)-(2-x)\frac{d}{dx}(2+x)}{(2+x)^2}\right]$$

$$\frac{dy}{dx} = \frac{(2+x)^2}{(2+x)^2+(2-x)^2}\left[\frac{(2+x)(-1)-(2-x)}{(2+x)^2}\right]$$

$$\frac{dy}{dx} = \frac{-2-x-2+x}{4+4x+x^2+4-4x+x^2}$$

$$\frac{dy}{dx} = \frac{-2-x-2+x}{4+4x+x^2+4-4x+x^2}$$

$$\frac{dy}{dx} = \frac{-2-x-2+x}{4+4x+x^2+4-4x+x^2}$$

$$\frac{dy}{dx} = \frac{-4}{8+2x^2}$$

$$\frac{dy}{dx}=\frac{-4}{2(4+x^2)}$$
$$\frac{dy}{dx}=\frac{-2}{4+x^2}$$

4. $y = xArccos\ x$; when $x=\frac{\sqrt{3}}{2}$

$$\frac{dy}{dx}=x\left[\frac{-1}{\sqrt{1-x^2}}\right]+(Arccos\ x)$$

When $x=\frac{\sqrt{3}}{2}$, $\quad \frac{dy}{dx}=\frac{-\frac{\sqrt{3}}{2}}{\sqrt{1-\left(\frac{\sqrt{3}}{2}\right)^2}}+\frac{\pi}{6}$

$$\frac{dy}{dx}=\frac{-\frac{\sqrt{3}}{2}}{\sqrt{1-\frac{3}{4}}}+\frac{\pi}{6}=\frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}}+\frac{\pi}{6}$$
$$\frac{dy}{dx}=-\sqrt{3}+\frac{\pi}{6}$$

5. $x^2y+\sin y=\text{Arccos}\,x$

$$x^2y'+y(2x)+(\cos y)y'=-\frac{1}{\sqrt{1-x^2}}\frac{d}{dx}(x)$$
$$x^2y'+y(2x)+(\cos y)y'=-\frac{1}{\sqrt{1-x^2}}$$
$$y'(x^2+\cos y)=-2xy-\frac{1}{\sqrt{1-x^2}}$$
$$y'=\frac{-2xy\sqrt{1-x^2}-1}{(x^2+\cos y)\sqrt{1-x^2}}$$

6. $Arctan\ \frac{x}{y}=x-y$

$$\frac{1}{1+\left(\frac{x}{y}\right)^2}\frac{d}{dx}\left(\frac{x}{y}\right)=1-y'$$
$$\frac{1}{\frac{y^2+x^2}{y^2}}\left[\frac{y-xy'}{y^2}\right]=1-y'$$
$$\frac{y-xy'}{y^2+x^2}=1-y'$$
$$y-xy'=y^2+x^2-(y^2+x^2)y'$$
$$y'(y^2+x^2-x)=y^2+x^2-y$$

$$y' = \frac{y^2 + x^2 - y}{y^2 + x^2 - x}$$

7. $y = \mathrm{x}\mathrm{Arctan}\frac{1}{2}\mathrm{x}$

$$\frac{dy}{dx} = (x)\frac{d}{dx}\left(\mathrm{Arctan}\ \frac{1}{2}\ x\right) + \left(\mathrm{Arctan}\frac{1}{2}x\right)\frac{d}{dx}(x)$$

$$\frac{dy}{dx} = (x)\left[\frac{1}{1 + \left(\frac{1}{2}x\right)^2}\right]\frac{d}{dx}\left(\frac{1}{2}x\right) + \left(\mathrm{Arctan}\frac{1}{2}x\right)(1)$$

$$\frac{dy}{dx} = (x)\left[\frac{1}{1 + \frac{x^2}{4}}\right]\left(\frac{1}{2}\right) + \left(\mathrm{Arctan}\ \frac{1}{2}x\right)$$

$$\frac{dy}{dx} = (x)\left[\frac{1}{\frac{4 + x^2}{4}}\right]\left(\frac{1}{2}\right) + \mathrm{Arctan}\ \frac{1}{2}x$$

$$\frac{dy}{dx} = \frac{2x}{4 + x^2} + \mathrm{Arctan}\ \frac{1}{2}x$$

Example 18. Find the equation of the tangent line to the curve $y = \frac{1}{x}\mathrm{Arctan}\frac{1}{x}$ at point $\left(1, \frac{\pi}{4}\right)$.

Slope of tangent line at any point on the curve is

$$\frac{dy}{dx} = \frac{1}{x}\left[\frac{1}{1+\frac{1}{x^2}}\left(-\frac{1}{x^2}\right)\right] + \left[\mathrm{Arctan}\left(\frac{1}{x}\right)\right]\left(\frac{-1}{x^2}\right)$$

$$\frac{dy}{dx} = -\frac{1}{x^3}\left[\frac{x^2}{1 + x^2}\right] + \left[\mathrm{Arctan}\left(\frac{1}{x}\right)\right]\left(\frac{-1}{x^2}\right)$$

$$\frac{dy}{dx} = -\frac{1}{x}\left[\frac{1}{1 + x^2}\right] + \left[\mathrm{Arctan}\left(\frac{1}{x}\right)\right]\left(\frac{-1}{x^2}\right)$$

$$\frac{dy}{dx} = \frac{1}{x(1+x^2)} - \left(\frac{1}{x}\right)\mathrm{Arctan}\frac{1}{x}$$

At point $\left(1, \frac{\pi}{4}\right), \frac{dy}{dx} = -\frac{1}{2} - \frac{\pi}{4} = \frac{-2-\pi}{4}$.

Use the point-slope form to get equation of the tangent line.

$$y - \frac{\pi}{4} = \frac{-2-\pi}{4}(x - 1)$$

$$\frac{4y-\pi}{4} = \frac{(-2-\pi)x+2+\pi}{4}$$

$$4y + \pi + (2 + \pi)x - 2 - \pi = 0$$

$$4y + (2 + \pi)x - 2 - 2\pi = 0$$

$$4y + (2 + \pi)x - 2(1 + \pi) = 0$$

Example 19. A ladder 15 feet long leans against a vertical wall. If the top slides down at 2 feet/sec, how fast is the angle of elevation of the top of the ladder decreasing as observed from its foot, when the lower end is 12 feet from the wall? Use Inverse Trigonometric Function.

Let x be the distance of the foot of the ladder from the wall at any time t

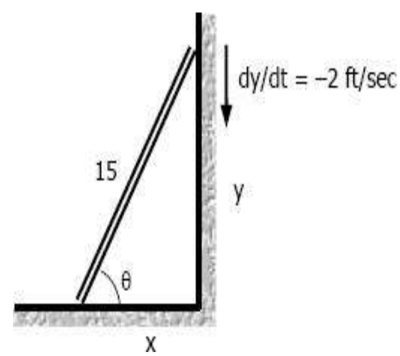
y be the distance of the top of the ladder from the ground at any time t.

θ be the angle of elevation of the top of the ladder as observed from its foot at any time t.

Find the equation that relates variables θ and y. We use the definition of either sin θ or csc θ. Let us choose csc θ.

$$\csc\theta = \frac{15}{y}$$

$$\theta = Arccsc\frac{15}{y}$$



Differentiate the above equation with respect to t.

$$\frac{d\theta}{dt} = -\frac{1}{\left(\frac{15}{y}\right)\sqrt{\left(\frac{15}{y}\right)^2 - 1}}\frac{d}{dt}\left(\frac{15}{y}\right)$$

$$\frac{d\theta}{dt} = -\frac{1}{\left(\frac{15}{y}\right)\sqrt{\frac{225 - y^2}{y^2}}}\left[-\frac{15}{y^2}\left(\frac{dy}{dt}\right)\right]$$

$$\frac{d\theta}{dt} = -\frac{1}{\frac{15}{y^2}\sqrt{225 - y^2}}\left[-\frac{15}{y^2}(-2)\right]$$

$$\frac{\mathrm{d}\theta}{\mathrm{dt}} = \frac{\mathrm{y}^2}{\sqrt{225 - \mathrm{y}^2}}\left[-\frac{2}{\mathrm{y}^2}\right] = \frac{-2}{\sqrt{225 - \mathrm{y}^2}}$$

When $x = 12\,, y = \sqrt{(15)^2 - (12)^2} = \sqrt{225 - 144} = \sqrt{81} = 9$

Therefore, $\frac{d\theta}{dt} = \frac{-2}{\sqrt{225-81}} = \frac{-2}{\sqrt{144}} = \frac{-2}{(12)} = -\frac{1}{6}\frac{\text{rad}}{\text{sec}}$

The negative sign of $\frac{d\theta}{dt}$, time rate of the angle of elevation of the top of the ladder, means that as the top of the ladder slides down the wall θ is decreasing with time.

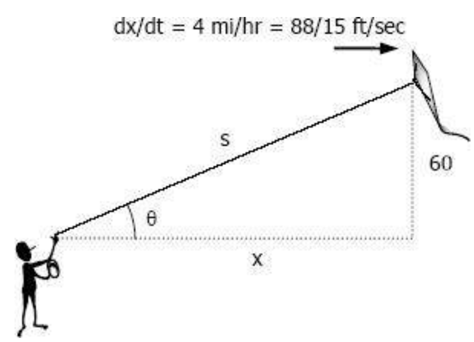
Example 20. A kite is 60 feet high with 100 feet of cord out. If the kite is moving horizontally 4 mi/hr directly away from the boy flying it, find the rate of change of the angle of elevation of the cord. Solve using an inverse trigonometric function.

The variables in the problem are the length of the cord s that is out; the horizontal distance of the kite x from the boy and the angle of elevation of the cord θ at anytime t. The height of the kite is fixed at 60 feet.

The equation relating the variables is

$$\tan\theta = \frac{60}{x}$$

$$\theta = \text{Arctan}\left(\frac{60}{x}\right)$$



$$\frac{d\theta}{dt} = \frac{1}{1+\left(\frac{60}{x}\right)^2}\left(-\frac{60}{x^2}\right)\frac{dx}{dt}$$

$$\frac{d\theta}{dt} = \frac{1}{\frac{x^2+3600}{x^2}}\left(-\frac{60}{x^2}\right)\frac{dx}{dt}$$

$$\frac{d\theta}{dt} = -\frac{60}{x^2+3600}\left(\frac{dx}{dt}\right)$$

When $s = 100$ ft, $x = \sqrt{s^2-(60)^2} = \sqrt{(100)^2-(60)^2}$

$$x = \sqrt{10{,}000-3600} = \sqrt{6400} = 80\text{feet}$$

$$\frac{d\theta}{dt} = -\frac{60}{(80)^2+3600}\left(\frac{88}{15}\right) = \frac{-4(88)}{6400+3600} = \frac{-4(88)}{10{,}000} = \frac{-88}{2500} = \frac{-22}{625}\frac{rad}{sec}$$

The negative sign of $\frac{d\theta}{dt}$ implies that as the cord of the kite lengthens, angle θ is decreasing with time.

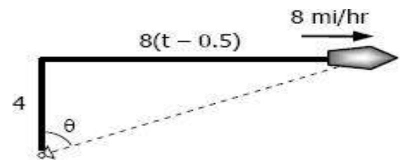
Example 21. A ship, moving 8 mi/hr, sails north for 30 min, then turns east. If a searchlight at the point of departure follows the ship, how fast is the light rotating 2 hr after the start.

Let t be the total number of hours sailed by the ship

θ be the acute angle the searchlight makes with the vertical.

The distance travelled when it sailed north for 30 minutes (0.5 hour) = 8 $\frac{mi}{hr}(0.5\ hr) = 4\ mi$. The distance travelled east for$(t-0.5)\ hr$ = 8 $\frac{mi}{hr}(t-0.5)hr = 8(t-0.5)mi$

$$\tan\theta = \frac{8\,(t-0.5)}{4} = 2(t-0.5)$$



$$\theta = \text{Arctan}\ 2(t-0.5)$$

$$\frac{d\theta}{dt} = \frac{1}{1+\ [2(t-0.5)]^2}\frac{d}{dt}2(t-0.5)$$

$$\frac{d\theta}{dt} = \frac{1}{1+4\left[\frac{2t-1}{2}\right]^2} = \frac{2}{1+(2t-1)^2}$$

Two hours after leaving the point of departure, $\frac{d\theta}{dt} = \frac{2}{1+(3)^2} = \frac{2}{10} = \frac{1}{5}\frac{rad}{hr}$.

Therefore, the searchlight following the ship from the point of departure is rotating at the rate of $\frac{1}{5}\frac{rad}{hr}$.

Example 22. The lower edge of the picture is 5 feet, the upper edge is 12 feet, above the eye of an observer. At what horizontal distance should he stand, if the angle subtended by the picture is to be the greatest?

Let x be the horizontal distance the observer needs to stand so that the angle subtended by the picture θ is the greatest.

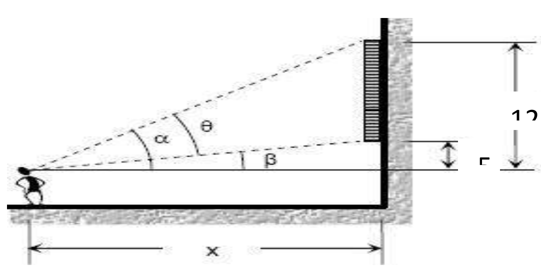
On the Figure below, $\theta = \alpha - \beta$ ---------------- Equation (1)

But, $\tan\beta = \frac{5}{x}$ $\beta = Arctan\ \frac{5}{x}$ ---------- Equation (2)

$\tan\alpha = \frac{12}{x}$ $\alpha = Arctan\ \frac{12}{x}$ ------Equation (3)

Substitute Equation (2) and Equation (3) into Equation (1).

$$\theta = Arctan\ \frac{12}{x} - Arctan\ \frac{5}{x}$$



Differentiate with respect to x. $\frac{d\theta}{dx} = \frac{1}{1+\left(\frac{12}{x}\right)^2}\frac{d}{dx}\left(\frac{12}{x}\right) - \frac{1}{1+\left(\frac{5}{x}\right)^2}\frac{d}{dx}\left(\frac{5}{x}\right)$

$$\frac{d\theta}{dx} = \frac{1}{1+\left(\frac{144}{x^2}\right)}\left[-\frac{12}{x^2}\right] - \frac{1}{1+\left(\frac{25}{x^2}\right)}\left[-\frac{5}{x^2}\right]$$

For the subtended angle to be the greatest, $\frac{d\theta}{dx} = 0$.

$$0 = \frac{x^2}{x^2+144}\left[-\frac{12}{x^2}\right] + \frac{x^2}{x^2+25}\left[\frac{5}{x^2}\right]$$

$$0 = \frac{-12}{x^2+144} + \frac{5}{x^2+25} = \frac{-12(x^2+25)+5(x^2+144)}{(x^2+144)(x^2+25)}$$

$$0 = -12x^2 - 300 + 5x^2 + 720 = -7x^2 + 420$$

$$x^2 = \frac{420}{7} = 60x = \sqrt{4(15)} = 2\sqrt{15} \text{ feet}$$

Therefore, for the subtended angle by the picture to be the greatest, the horizontal distance of the observer from the picture should be $2\sqrt{15}$ feet.

SAQ19
ACTIVITY 4.16 – C

NAME: ____ SCORE: _______________
SECTION: ____________DATE: ________________ PROF: ____________________________________

I. Find $\frac{dy}{dx}$ of the given functions.

1. $y = \frac{1}{2}\arctan\frac{x}{2}$

2. $y = \arcsin\sqrt{1 - x^2}$

3. $y = \arccos\frac{1-x^2}{1+x^2}$

4. $y = \operatorname{arccot}\frac{1}{x} - \arctan x$

5. $y = \sqrt{x^2 - 4} + \frac{1}{2}\operatorname{arcsec}\frac{x}{2}$

6. $xy = Arctan\,(x + y)$

7. $Arccos\,\sqrt{y} = x - y$

II. Evaluate $\frac{dy}{dx}$ at the indicated value of x.

1. $y = xArcsin\,x + \sqrt{1 - x^2};\ x = -\frac{1}{2}$

2. $y = x\sqrt{4 - x^2} + 4Arcsin\,\frac{x}{2};\ \ x = 0$

3. $y = 2x + 10Arccot\,x;\ x = 0$

4. $y = xArccot\,2x;\ x = 1$

III. Find the equation of the tangent line to the given graph of the given function at the indicated value of x.

1. $y = (1 + x^2)\,\text{Arctan}\,x; \quad x = 1$

2. $y = \text{Sec}^{-1}(2x + 1); \quad x = \frac{1}{2}$

IV. Find the angle of intersection between curves $y = Arcsin\left(\frac{x}{2}\right)$ and $y = Arctan\,x$.

V. A movie screen on the front wall of your classroom is 16 ft high and positioned 9 ft above your eye level. How far away from the front of the room should you sit in order to have the “best view”?

VI. The lower edge of a mural 12 feet high, is 6 feet above an observer's eye. If the most favourable view is obtained when the angle subtended by the mural at the observer is a maximum, how far should the observer stand from the wall?

VII. A ladder 25 feet long is leaning against a vertical wall. If the bottom of the ladder is pulled horizontally away from the wall so that the top of the ladder is sliding down at3 ft/sec, how fast is the measure of the measure of the acute angle between the ladder and the ground changing when the bottom of the ladder is 15 feet from the wall?

VIII. The position of a particle at time t is given by $s(t) = Tan^{-1}\left(\frac{1}{t}\right)$ for $t \geq \frac{1}{2}$. Find the velocity when $t = 1$.

ASAQ19
ACTIVITY 4.16 – C

NAME: ____ SCORE: ________________
SECTION: ____________DATE: _________________ PROF: ____

I. Find $\frac{dy}{dx}$ of the given functions.

1. $y = \frac{1}{2}\arctan\frac{x}{2}$ $\quad Answer: \frac{1}{4+x^2}$

2. $y = \arcsin\sqrt{1-x^2}$ $\quad Answer: -\frac{1}{\sqrt{1-x^2}}$

3. $y = \arccos\frac{1-x^2}{1+x^2}$ $\quad Answer: \frac{1}{1+x^2}$

4. $y = \operatorname{arccot}\frac{1}{x} - \arctan x$ $\quad Answer: 0$

5. $y = \sqrt{x^2-4} + \frac{1}{2}\operatorname{arcsec}\frac{x}{2}$ $\quad Answer: \frac{1+x^2}{x\sqrt{x^2-4}}$

6. $xy = Arctan\ (x + y)$ $\quad Answer: \frac{1-y-y(x+y)^2}{x+x(x+y)^2-1}$

7. $Arccos\sqrt{y} = x - y$ $\quad Answer: \frac{2\sqrt{y(1-y^2)}}{2\sqrt{y(1-y^2)}-1}$

II. Evaluate $\frac{dy}{dx}$ at the indicated value of x.

1.$y = xArcsin\, x + \sqrt{1-x^2}$; $x = -\frac{1}{2}$ $\quad Answer: -\frac{\pi}{6}$

2. $y = x\sqrt{4-x^2} + 4Arcsin\ \frac{x}{2}$; $x = 0$ $\quad Answer: 4$

3. $y = 2x + 10Arccot\ x$; $x = 0$ $\quad Answer: -8$

4. $y = xArccot\ 2x$; $x = 1$ $\quad Answer: \frac{\pi-2}{4}$

III. Find the equation of the tangent line to the given graph of the given function at the indicated value of x.

1. $y = (1 + x^2)\,\text{Arctan}\,x; \quad x = 1 \quad Answer: (2+\pi)x - 2y - 2 = 0$

2. $y = \text{Sec}^{-1}(2x+1); \quad x = \frac{1}{2} \quad Answer: 2\sqrt{3}x - 6y + 2\pi - \sqrt{3} = 0$

IV. Find the angle of intersection between curves $y = Arcsin\left(\frac{x}{2}\right)$ and $y = Arctan\,x$.

$Answer: \emptyset = 0.32\,rad, 0.54\,rad$

V. A movie screen on the front wall of your classroom is 16 ft high and positioned 9 ft above your eye level. How far away from the front of the room should you sit in order to have the "best view"?

$Answer: 15\,ft$

VI. The lower edge of a mural 12 feet high, is 6 feet above an observer's eye. If the most favourable view is obtained when the angle subtended by the mural at the observer is a maximum, how far should the observer stand from the wall? $Answer: 6\sqrt{3}\,ft$

VII. A ladder 25 feet long is leaning against a vertical wall. If the bottom of the ladder is pulled horizontally away from the wall so that the top of the ladder is sliding down at3 ft/sec, how fast is the measure of the measure of the acute angle between the ladder and the ground changing when the bottom of the ladder is 15 feet from the wall? $Answer$: $-\frac{1}{5}\ rad/sec$

VIII. The position of a particle at time t is given by $s(t) = Tan^{-1}\left(\frac{1}{t}\right)$ for $t \geq \frac{1}{2}$. Find the velocity when $t = 1$. $Answer$: $-\frac{1}{2}$

ACTIVITY 4.16 – D

NAME: ____ SCORE: _______________
SECTION: ____________DATE: ________________ PROF: _____________________________________

Solve the following problems using derivative of inverse trigonometric functions.

1. If $y = x\text{Arcsin } x + \sqrt{1 - x^2}$, at what value of x is the slope of the curve equal to $\frac{\pi}{6}$?

2. Find the equation of the tangent line to the curve represented by $y = 3Arcsin\ 2x$ at the point where $x = \frac{1}{4}$.

3. A picture 40 cm high is placed on a wall with its base 30 cm above the level of the eye of the observer. If the observer is approaching the wall at the rate of 40 cm/sec, how fast is the measure of the angle subtended at the observer's eye by the picture changing when the observer is 1 meter from the wall?

4. Two vertical poles respectively 1 meter and 9 meters high are 6 meters apart. How far from the foot of the shorter pole where the line segment joining the tops of the poles subtends the greatest angle?

MODULE 17

DERIVATIVES OF LOGARITHMIC AND EXPONENTIAL FUNCTION AND THEIR APPLICATIONS

Specific Objectives:

At the end of the module, students must be able to:

1. Know and use correctly the differentiation formulas for logarithmic and exponential function.
2. Apply the differentiation formulas for logarithmic and exponential function in solving problems on slope of tangent and normal, rectilinear motion, angle between curves, optimization problems and time-rates.

DERIVATIVES OF LOGARITHMIC AND EXPONENTIAL FUNCTIONS

Let u is a differentiable function of $x, a > 0, a \neq 1$.

1. $\frac{d}{dx}(\log_a u) = \frac{1}{u}\log_a e \frac{du}{dx}$	3. $\frac{d}{dx}(a^u) = a^u \ln a \frac{du}{dx}$
2. $\frac{d}{dx}(\log_a u) = \frac{1}{u}\log_a e \frac{du}{dx}$	4. $\frac{d}{dx}(e^u) = e^u \frac{du}{dx}$

SYSTEMS OF LOGARITHM

The most common bases of logarithms are the base 10 and the base e.

a. Logarithms with a base 10 are called common logarithms. It is also known as the decadic logarithm and also as the decimal logarithm, named after its base, or Briggsian logarithm, after Henry Briggs, an English mathematician who pioneered its use. When the base is 10, we can leave off the 10 in the notation.

b. Logarithms with a base e (Napierian constant **e** = 2.718281…) are called natural logarithm or Napierian logarithm first defined by John Napier.

PROPERTIES OF LOGARITHMS

1. $\log_b 1 = 0$
2. $\log_b b = 1$(i.e., $\ln e = 1, \quad \log 10 = 1$)
3. $\log_b MN = \log_b M + \log_b N$
4. $\log_b \frac{M}{N} = \log_b M - \log_b N$
5. $\log_b M^n = n \log_b M$
6. $\log_b \sqrt[n]{M} = \frac{1}{n}\log_b M$
7. $b^{\log_b M} = M$
8. $\log_b M = \frac{\log M}{\log b} = \frac{\ln M}{\ln b}$ (This is known as the Change of Base Formula)

Example 23. Find $\frac{dy}{dx}$ and simplify whenever possible.

a. $y = \log\sqrt{2x - 7}$

$$\frac{dy}{dx} = \frac{1}{\sqrt{2x-7}}\log e \frac{d}{dx}\sqrt{2x-7}$$

$$\frac{dy}{dx} = \frac{\log e}{\sqrt{2x-7}}\left[\frac{1}{2\sqrt{2x-7}}\frac{d}{dx}(2x-7)\right]$$

$$\frac{dy}{dx} = \frac{\log e}{2(2x-7)}(2) = \frac{\log e}{2x-7}$$

b. $y = \log \cos^2 3x$

$$\frac{dy}{dx} - \frac{1}{\cos^2 3x} \log e \frac{d}{dx}(\cos^2 3x)$$

$$\frac{dy}{dx} = \frac{\log e}{\cos^2 3x}\left[2\cos 3x \frac{d}{dx}\cos 3x\right]$$

$$\frac{dy}{dx} = \frac{2\log e}{\cos 3x}[(-\sin 3x)(3)]$$

$$\frac{dy}{dx} = -\frac{6(\log e)(\sin 3x)}{\cos 3x} = -6\log e \tan 3x$$

c. $y = x^3(3\ln x - 1)$

$$\frac{dy}{dx} = x^3\left[(3)\left(\frac{1}{x}\right)\right] + (3\ln x - 1)(3x^2)$$

$$\frac{dy}{dx} = 3x^2 + (3\ln x - 1)(3x^2)$$

$$\frac{dy}{dx} = 3x^2(1 + 3\ln x - 1)$$

$$\frac{dy}{dx} = 3x^2(1 + 3\ln x - 1) = 9x^2 \ln x$$

d. $y = \ln^2 \sin x$

$$\frac{dy}{dx} = 2\ln\sin x \frac{d}{dx}\ln \sin x$$

$$\frac{dy}{dx} = 2\ln\sin x\left[\frac{1}{\sin x}\frac{d}{dx}\sin x\right]$$

$$\frac{dy}{dx} = 2\ln\sin x\left[\frac{1}{\sin x}(\cos x)\right]$$

$$\frac{dy}{dx} = 2\cot x \; \ln\sin x$$

e. $y = \frac{1}{2}x^2 e^{-2x}$

$$\frac{dy}{dx} = \frac{1}{2}\left[x^2\frac{d}{dx}e^{-2x} + e^{-2x}\frac{d}{dx}(x^2)\right]$$

$$\frac{dy}{dx} = \frac{1}{2}\left[x^2(e^{-2x})\frac{d}{dx}(-2x) + e^{-2x}(2x)\right]$$

$$\frac{dy}{dx} = \frac{1}{2}[x^2(e^{-2x})(-2) + 2xe^{-2x}]$$

$$\frac{dy}{dx} = \frac{1}{2}e^{-2x}[-2x^2 + 2x]$$
$$\frac{dy}{dx} = \frac{1}{2}e^{-2x}(2x)(-x+1)$$
$$\frac{dy}{dx} = x(1-x)e^{-2x}$$

f. $u = e^{-2y}\ln y$

$$\frac{du}{dy} = e^{-2y}\left(\frac{1}{y}\right) + \ln y\,(e^{-2y})\frac{d}{dy}(-2y)$$
$$\frac{du}{dy} = \frac{e^{-2y}}{y} + \ln y(e^{-2y})(-2)$$
$$\frac{du}{dy} = e^{-2y}\left(\frac{1}{y} - 2\ln y\right)$$
$$\frac{du}{dy} = \frac{e^{-2y}}{y}(1 - 2y\ln y)$$

g. $x = \ln\ln(1 + e^{-t})$

$$\frac{dx}{dt} = \frac{1}{\ln(1+e^{-t})}\frac{d}{dt}(\ln(1+e^{-t}))$$
$$\frac{dx}{dt} = \frac{1}{\ln(1-e^{-t})}\left[\frac{1}{1+e^{-t}}(e^{-t})(-1)\right]$$
$$\frac{dx}{dt} = \frac{-e^{-t}}{(1+e^{-t})\ln(1-e^{-t})}$$

h. $B = (1 - e^{-2x})^{\frac{3}{2}}$

$$\frac{dB}{dx} = \frac{3}{2}(1-e^{-2x})^{\frac{1}{2}}\frac{d}{dx}(1-e^{-2x})$$
$$\frac{dB}{dx} = \frac{3\sqrt{1-e^{-2x}}}{2}(-e^{-2x})(-2)$$
$$\frac{dB}{dx} = 3e^{-2x}\sqrt{1-e^{-2x}}$$

j. $\alpha = 5^{-3\beta}$

$$\frac{d\alpha}{d\beta} = 5^{-3\beta}\ln 5\frac{d}{d\beta}(-3\beta)$$
$$\frac{d\alpha}{d\beta} = 5^{-3\beta}(\ln 5)(-3)$$

$$\frac{d\alpha}{d\beta} = -3(\ln 5)5^{-3\beta}$$

j. $r = \ln\dfrac{1+e^{-2\theta}}{1-\ e^{-2\theta}}$

$$\frac{dr}{d\theta} = \frac{1}{\dfrac{1\ +\ e^{-2\theta}}{1\ -\ e^{-2\theta}}}\frac{d}{d\theta}\frac{1+e^{-2\theta}}{1-e^{-2\theta}}$$

$$\frac{dr}{d\theta} = \frac{1-e^{-2\theta}}{1+e^{-2\theta}}\left[\frac{\left(1-e^{-2\theta}\right)\frac{d}{d\theta}\left(1+e^{-2\theta}\right)-\left(1+e^{-2\theta}\right)\frac{d}{d\theta}\left(1-e^{-2\theta}\right)}{(1\ -\ e^{-2\theta})^2}\right]$$

$$\frac{dr}{d\theta} = \frac{\left(1-\ e^{-2\theta}\right)\left(e^{-2\theta}\right)(-2)-\ \left(1+\ e^{-2\theta}\right)\left(-e^{-2\theta}\right)(-2)}{\left(1+e^{-2\theta}\right)\left(1-e^{-2\theta}\right)}$$

$$\frac{dr}{d\theta} = \frac{-2e^{-2\theta}\left(1-e^{-2\theta}+1+e^{-2\theta}\right)}{1-\ e^{-4\theta}}$$

$$\frac{dr}{d\theta} = \frac{-4e^{-2\theta}}{1-\ e^{-4\theta}}$$

k. $y = \text{Arctan}\ \ln x$

$$\frac{dy}{dx} = \frac{1}{1+ln^2x}\frac{d}{dx}(\ln x)$$

$$\frac{dy}{dx} = \frac{1}{1\ +\ \ ln^2x}\left(\frac{1}{x}\right)$$

$$\frac{dy}{dx} = \frac{1}{x(1+\ ln^2x)}$$

l. $V = e^x(1-e^{-2x})^{-\frac{1}{2}}$

$$\frac{dV}{dx} = e^x\frac{d}{dx}(1-e^{-2x})^{-\frac{1}{2}} + (1-e^{-2x})^{-\frac{1}{2}}\frac{d}{dx}e^x$$

$$\frac{dV}{dx} = e^x\left(-\frac{1}{2}\right)(1-e^{-2x})^{-\frac{3}{2}}\frac{d}{dx}(1-e^{-2x})+(1-e^{-2x})^{-\frac{1}{2}}(e^x)$$

$$\frac{dV}{dx} = \frac{-e^x(-e^{-2x})(-2)}{2(1-e^{-2x})^{\frac{3}{2}}} + \frac{e^x}{(1-e^{-2x})^{\frac{1}{2}}}$$

$$\frac{dV}{dx} = \frac{-e^{-x}+e^x(1-e^{-2x})}{(1-e^{-2x})^{\frac{3}{2}}}$$

$$\frac{dV}{dx} = \frac{-e^{-x}+e^x-e^{-x}}{(1-e^{-2x})\sqrt{1-e^{-2x}}}$$

$$\frac{dV}{dx} = \frac{e^x-2e^{-x}}{(1-e^{-2x})\sqrt{1-e^{-2x}}}$$

m. $x = \ln \frac{t^2}{(1-t^2)^3}$

$$\frac{dx}{dt} = \frac{1}{\frac{t^2}{(1-t^2)^3}} \frac{d}{dt}\left[\frac{t^2}{(1-t^2)^3}\right]$$

$$\frac{dx}{dt} = \frac{(1-t^2)^3}{t^2}\left[\frac{(1-t^2)^3(2t) - t^2(3)(1-t^2)^2(-2t)}{(1-t^2)^6}\right]$$

$$\frac{dx}{dt} = \frac{(1-t^2)^3}{t^2}\left[\frac{2t(1-t^2)^3 + 6t^3(1-t^2)^2}{(1-t^2)^6}\right]$$

$$\frac{dx}{dt} == \frac{2t(1-t^2)^2[1-t^2+3t^2]}{t^2(1-\ t^2)^3}$$

$$\frac{dx}{dt} = \frac{2(1+2t^2)}{t(1-t^2)}$$

n. $g = \ln\left(1 + e^{-3\theta}\right)^2$

$$\frac{dg}{d\theta} = \frac{1}{(1+\ e^{-3\theta})^2}\left[2\left(1+e^{-3\theta}\right)\frac{d}{d\theta}\left(1+e^{-3\theta}\right)\right]$$

$$\frac{dg}{d\theta} = \frac{2\left(1+e^{-3\theta}\right)\left(e^{-3\theta}\right)(-3)}{(1+e^{-3\theta})^2}$$

$$\frac{dg}{d\theta} = \frac{-6e^{-3\theta}}{1+e^{-3\theta}}$$

o. $y = e^{-2x}(1+\ e^{2x})^{-\frac{1}{2}}$

$$\frac{dy}{dx} = e^{-2x}\frac{d}{dx}(1+e^{2x})^{-\frac{1}{2}} + (1+e^{2x})^{-\frac{1}{2}}\frac{d}{dx}e^{-2x}$$

$$\frac{dy}{dx} = e^{-2x}\left(-\frac{1}{2}\right)(1+\ e^{2x})^{-\frac{3}{2}}\frac{d}{dx}(1+\ e^{2x}) + (1+\ e^{2x})^{-\frac{1}{2}}(e^{-2x})(-2)$$

$$\frac{dy}{dx} = \frac{-e^{-2x}}{2(1+\ e^{2x})^{\frac{3}{2}}}(e^{2x})(2) - \frac{2e^{-2x}}{(1+\ e^{2x})^{\frac{1}{2}}}$$

$$\frac{dy}{dx} = \frac{-1}{(1+e^{2x})^{\frac{3}{2}}} - \frac{2e^{-2x}}{(1+\ e^{2x})^{\frac{1}{2}}}$$

$$\frac{dy}{dx} = \frac{-1-2e^{-2x}(1+\ e^{2x})}{(1+\ e^{2x})^{\frac{3}{2}}}\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{-1-2e^{-2x}-2}{(1+\ e^{2x})^{\frac{3}{2}}}\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{-3-2e^{-2x}}{(1+\ e^{2x})\sqrt{1+\ e^{2x}}}$$

Example 24. Given that $\ln\left(\frac{y}{2x-1}\right) = 0$, find the equation of the normal line when $x = 1$.

Solution: $\ln\left(\frac{y}{2x-1}\right) = 0$

Transform the given logarithmic equation to equivalent exponential equation.

$$e^0 = \frac{y}{2x-1}$$

$$1 = \frac{y}{2x-1}$$

$y = 2x - 1$ ------------ Equation (1)

$$\frac{dy}{dx} = 2 = \text{slope of the tangent line to the curve}$$

$\text{Therefore, slope of the normal line} = -\frac{1}{2}.$

When $= 1, y = 2(1) - 1 = 1.$

Equation of the normal line to the curve at $(1,1)$:

$$y - 1 = -\frac{1}{2}(x - 1)$$

$$2y - 2 = -x + 1$$

$$x + 2y - 3 = 0$$

Alternative Solution: This alternative solution is to derive Equation (1).

$$\ln\left(\frac{y}{2x-1}\right) = 0$$

$$\ln y - \ln(2x - 1) = 0$$

$$\ln y = \ln(2x - 1)$$

$\text{These are two equal logaritms with the same base, thus, the numbers of the logarithms are equal.}$

Hence, $y = 2x - 1$ (Same as Equation (1))

Example 25. Show that the curve $y = xe^{-\frac{x}{2}}$ has its maximum point at $\left(2, \frac{2}{e}\right)$.

Solution: $y = xe^{-\frac{x}{2}}$

Differentiate. $y' = x\frac{d}{dx}e^{-\frac{x}{2}} + e^{-\frac{x}{2}}\frac{d}{dx}(x)$

$$y' = xe^{-\frac{x}{2}}\left(-\frac{1}{2}\right) + e^{-\frac{x}{2}}$$

$$y' = e^{-\frac{x}{2}}\left(-\frac{x}{2} + 1\right)$$

$$y' = e^{-\frac{x}{2}}\left[\frac{-x+2}{2}\right]$$

Set each factor to zero. $\frac{-x+2}{2} = 0e^{-\frac{x}{2}} = 0$

$x = 2 - \frac{x}{2}\ln e = \ln 0$ (Rejected since $\ln 0$ has no value)

When $x = 2$, $y = 2e^{-\frac{2}{2}} = 2e^{-1} = \frac{2}{e}$

Perform the second derivative test. $y" = e^{-\frac{x}{2}}\left(-\frac{1}{2}\right) + \left[\frac{-x+2}{2}\right]\left(e^{-\frac{x}{2}}\right)\left(-\frac{1}{2}\right)$

$$y" = -\frac{1}{2}e^{-\frac{x}{2}}\left[1 + \left(\frac{-x+2}{2}\right)\right]$$

$$y" = -\frac{1}{4}e^{-\frac{x}{2}}(4 - x)$$

When $x = 2$, $y" = -(+)(+) = - < 0$

Therefore, point $\left(2, \frac{2}{e}\right)$is a relative maximum point of the curve $y = xe^{-\frac{x}{2}}$.

Example 26. If $y = \ln\left(\frac{1+x}{1-x}\right)$ and if the rate of change of y is 4 units per second, find the rate of change of x when $y = \ln 3$.

Solution: $y = \ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$

$$\frac{dy}{dt} = \frac{1}{1+x}\frac{dx}{dt} - \frac{1}{1-x}\left(-\frac{dx}{dt}\right)$$

$$\frac{dy}{dt} = \left(\frac{dx}{dt}\right)\left[\frac{1}{1+x} + \frac{1}{1-x}\right]$$

When$y = \ln 3$, substitute on the given equation of the curve.

$$\ln 3 = \ln\left(\frac{1+x}{1-x}\right)$$

$$3 = \frac{1 + x}{1 - x}$$

$$3 - 3x = 1 + x$$

$$4x = 2$$

$$x = \frac{1}{2}$$

Therefore, when $x = \frac{1}{2}$and $\frac{dy}{dt} = 4$ units per second,

$$4 = \left(\frac{dx}{dt}\right)\left[\frac{1}{1++\frac{1}{2}} + \frac{1}{1-\frac{1}{2}}\right] = \left(\frac{dx}{dt}\right)\left[\frac{1}{\frac{3}{2}} + \frac{1}{\frac{1}{2}}\right]$$

$$4 = \left(\frac{dx}{dt}\right)\left[\frac{2}{3} + 2\right] = \left(\frac{dx}{dt}\right)\left(\frac{8}{3}\right)$$

$\frac{dx}{dt} = \frac{12}{8} = \frac{3}{2}$units/second

Example 27. Find the critical points and roughly sketch the curve$y = x\ln x$.

$$\frac{dy}{dx} = x\left(\frac{1}{x}\right) + \ln x$$

$$\ln x\frac{dy}{dx} = 1 + \ln x$$

At the critical point, $\frac{dy}{dx} = 0 = 1 + \ln x$

Solve the logarithmic equation. $\ln x = -1$

$$x = e^{-1} = \frac{1}{e}$$

Solve the corresponding value of y. $y = \frac{1}{e}\ln e^{-1} = \frac{1}{e}(-1) = -\frac{1}{e}$

Hence, the critical point is$\left(\frac{1}{e}, -\frac{1}{e}\right)$.

Use the second derivative test to classify the critical point. $y"(x) = \frac{1}{x}$.

When $x = \frac{1}{e}$, $y"\left(\frac{1}{e}\right) = +(> 0)$.

Therefore, the point $\left(\frac{1}{e}, -\frac{1}{e}\right)$ is a minimum point.

Find the point of inflection by setting$y" = 0$. $y" = 0 = \frac{1}{x}$

$$x = \frac{1}{0} = \text{undefined}$$

Therefore, the curve has no point of inflection.

To help roughly sketch the curve, find the intercepts. When $x = 0$, $y_i = 0(\ln 0) = \text{no value}$

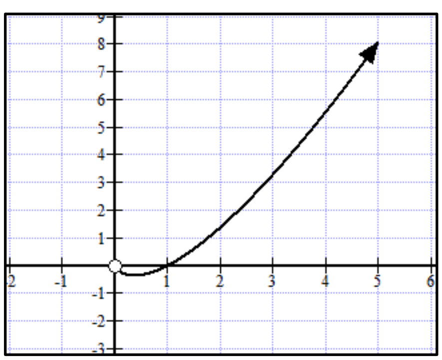
When $y = 0$, $x_i = ?$ $0 = x\ln x$

Equate each factor to zero. $x_i = 0$. This value is rejected since$\ln 0$ has no value.

Likewise, $0 = \ln x$

$$x_i = e^0 = 1$$

Hence, the curve passes through point$(1, 0)$.



SAQ20
ACTIVITY 4.17 – E

NAME: ____ SCORE: ______________
SECTION: ___________DATE: _______________ PROF: __________________________________

I. Differentiate the following functions.

1. $y = \ln(4x - 7)$

2. $y = \ln(2x^2 + 3x - 6)^3$

3. $\mathrm{y = xln^2x - 3x}$

4. $y = \ln(\sec x + \tan x)$

5. $y = \frac{\ln x^2}{x^2}$

6. $\mathrm{y = e^{-x} \cos x}$

7. $\mathrm{y = tan^2 e^{3x}}$

8. $y = Arctan(e)^{-x^2}$

9. $x \ln y + y \ln x = 1$

10. $e^x + e^y = e^{xy}$

11. $x = \frac{1}{e^t - 1}, \quad y = \frac{1}{e^t + 1}$

II. Find the y’’’ of function $y = x^2 e^x$.

III. If $y = e^{-2x}(\sin 2x + \cos 2x)$, show $y" + 4y' + 8y = 0$.

IV. Find the equation of the tangent line to the given curve $y = \ tan\ (\ln x)$ at $x = \frac{\pi}{4}$.

V. For the curve $\mathrm{y} = \mathrm{x} \ln \mathrm{x}$, find the equation of tangent line parallel to the line $3\mathrm{x} - \mathrm{y} = 5$.

VI. Find the acute angle of intersection between the given pair of curves.

1. $y = e^{2x}\ ;\ y = e^{x}$

2. $y = e^{1+x}; \quad y = e^{2x}$

3. $y = \ln(\ 1 + x)$; $y = \ln(3 - x^2)$

VII. Find and classify the critical point of the given function.

1. $y = xe^{\frac{1}{x}}$.

2. $y = x^2 e^{-x}$.

VIII. Let $y = Tan^{-1}(\ln x)$ for $x > 0$. If x is increasing at a constant rate of 2 units/sec, find the rate at which y is changing when $x = e$.

ASAQ20
ACTIVITY 4.17 – E

NAME: ____ SCORE: ______________
SECTION: ___________DATE: _______________ PROF: ___________________________________

I. Differentiate the following functions.

1. $y = \ln(4x - 7)$ $Answer: \frac{4}{4x-7}$

2. $y = \ln(2x^2 + 3x - 6)^3$ $Answer: \frac{3(4x+3)}{2x^2+3x-6}$

3. $\mathrm{y = xln^2x - 3x}$ $Answer: (3 + \ln x)(\ln x - 1)$

4. $y = \ln(\sec x + \tan x)$ $Answer: \sec x$

5. $y = \frac{\ln x^2}{x^2}$ $Answer: \frac{2(1-2\ln x)}{x^3}$

6. $\mathrm{y = e^{-x} \cos x}$ $Answer: -e^{-x}(\sin x + \cos x)$

7. $\mathrm{y = tan^2 e^{3x}}$ $Answer: 6e^{3x}(\tan e^{3x})(sec^2 e^{3x})$

8. $y = Arctan(e)^{-x^2}$ $Answer:\ -\frac{2xe^{-x^2}}{1+e^{-2x^2}}$

9. $x\ln y + y\ln x = 1$ $Answer: -\frac{y(x\ln y+y)}{x(x+y\ln x)}$

10. $e^x + e^y = e^{xy}$ $Answer:\ \frac{ye^{xy}-e^x}{e^y-xe^{xy}}$

11. $x = \frac{1}{e^t-1},\quad y = \frac{1}{e^t+1}$ $Answer:\ -\frac{(e^t-1)^2}{1+e^t}$

II. Find the y''' of function $y = x^2e^x$. $Answer: e^x(x^2+6x+6)$

III. If $y = e^{-2x}(\sin 2x + \cos 2x)$, show $y" + 4y' + 8y = 0$.

IV. Find the equation of the tangent line to the given curve $y = \; tan\,(\ln x)$ at $x = \frac{\pi}{4}$.
$Answer: 4x - 2y - \pi = 0$

V. For the curve $\mathrm{y = x \ln x}$, find the equation of tangent line parallel to the line $\mathrm{3x - y = 5}$.
$Answer: 3x - y - e^2 = 0$

VI. Find the acute angle of intersection between the given pair of curves.

3. $y = e^{2x}\,;\; y = e^{x}$ $\quad Answer: 18.4^0$

4. $y = e^{1+x};\quad y = e^{2x}$ $\quad Answer: 3.8^o$

3. $y = \ln(\ 1 + x);\quad y = \ln(3 - x^2)\ \ Answer: 71.6^o$

VII. Find and classify the critical point of the given function.

1. $y = xe^{\frac{1}{x}}$. $Answer$: $(1, e)$, minimum point

2. $y = x^2e^{-x}$. $Answer$: $(0, 0)$, maximim point; $(2, 4e^{-2})$, maximum point

VIII. Let $y = Tan^{-1}(\ln x)$ for$x > 0$. If x is increasing at a constant rate of 2 units/sec, find the rate at which y is changing when $x = e$. $Answer$: e^{-1} units/sec

ACTIVITY 4.17 – F

NAME: ____ SCORE: _______________
SECTION: ____________DATE: ________________ PROF: _____________________________________

I. Differentiate using the appropriate formula or method.

1. $\mathrm{y} = \mathrm{x}\ln(4+\mathrm{x}^2) + 4\mathrm{Arctan}\,\frac{1}{2}\mathrm{x} - 2\mathrm{x}$

2. $y = \ln\frac{1+2\tan x}{2+\tan x}$

3. $\ln(x^2+y^2) + 2Arctan\frac{y}{x} = 0$

4. $y = \ln\frac{e^x}{e^x-1}$

5. $y = e^{2x+\ln 2x}$

II. Find equation of the tangent line to the curve $y = \ln(x^2 + 4) - xArctan\frac{x}{2}$ at $x = 2$.

III. Find the angle of intersection between the given pair of curves.

1. $y = xe^x$ and $y = x^2e^x$

2. $y = \log x$; $y = \log(x^2)$

IV. Find the point of inflection of the curve $xy = 4\ln\frac{1}{2}x$.

V. Find and classify the critical points, find point of inflection and sketch the curve $y = x^2e^{-x^2}$.

MODULE 18

DERIVATIVE OF A VARIABLE RAISED TO ANOTHER VARIABLE

Specific Objectives:

At the end of the module, students must be able to:

1. Understand the derivation of the formula to differentiate a variable raised to a variable.
2. Apply the differentiation formula for derivative of a variable with a variable exponent

DERIVATIVE OF A VARIABLE WITH A VARIABLE EXPONENT

Let $y = u^v$ where both u and v are functions of x. To derive the differentiation formula for $y = u^v$, let us take the logarithm of both sides of the given equation, then, differentiate implicitly with respect to x.

Differentiate using the power formula. $\ln y = \ln u^v = v \ln u$

Differentiate using the product formula. $\frac{d}{dx}\ln y = v\frac{d}{dx}\ln u + (\ln u)\frac{d}{dx}v$

Apply the formula $\frac{d}{dx}\ln u$. $\frac{1}{y}\frac{dy}{dx} = v\left(\frac{1}{u}\frac{du}{dx}\right) + (\ln u)\frac{dv}{dx}$

Multiply both sides of the equation above, then, replace y by u^v.

$$\frac{dy}{dx} = y\left[\frac{v}{u}\frac{du}{dx} + (\ln u)\frac{dv}{dx}\right]$$

$$\frac{dy}{dx} = u^v\left[\frac{v}{u}\frac{du}{dx} + (\ln u)\frac{dv}{dx}\right]$$

$$\boxed{\frac{dy}{dx} = vu^{v-1}\frac{du}{dx} + u^v \ln u\frac{dv}{dx}}$$

The boxed-formula above is best to use to differentiate a variable raised to another variable.

Example28. Find $\frac{dy}{dx}$, given $y = (sinx)^x$

Solution: To differentiate, one may opt to use the formula above or do the step by step process as shown on the derivation of the above formula. Let me show both ways.
Method 1. Using the formula,

$u = sinx$	$\frac{du}{dx} = \cos x$		$v = x$	$\frac{dv}{dx} = 1$

Apply the derived differentiation formula.

$$\frac{dy}{dx} = x(\sin x)^{x-1}\cos x + (\sin x)^x \ln \sin x$$

Method 2. Follow the step by step process as shown on the derivation of the formula.

$$y = (sinx)^x$$

Take the logarithm of both sides of the given equation. $\ln y = \ln(sinx)^x$

Apply $\log A^n = n \log A$. $\ln y = x \ln \sin x$

Differentiate implicitly. $\frac{1}{y}\frac{dy}{dx} = x\left[\frac{1}{\sin x}(\cos x)\right] + (\ln \sin x)(1)$

Multiply both sides of equation by y. $\frac{dy}{dx} = y\left[\frac{x\cos x}{\sin x} + \ln \sin x\right]$

But, $y = (sinx)^x$. $\frac{dy}{dx} = (\sin x)^x \left[\frac{x\cos x}{\sin x} + \ln \sin x\right] = x\cos x$

Apply distributive law. $\frac{dy}{dx} = \frac{(\cos x)(\sin x)^x}{\sin x} + (\sin x)^x \ln \sin x$

Use $\frac{a^m}{a^n} = a^{m-n}$. $\frac{dy}{dx} = (\text{x}\cos x)(\sin x)^{x-1} + (\sin x)^x \ln \sin x$

Apply commutative law. $\frac{dy}{dx} = x(sinx)^{\,x-1}\cos x + (\sin x)^x \ln \sin x$

Observe that both methods of differentiation yield same $\frac{dy}{dx}$.

Example 29. Find y'(x), given $y = (x)^{\ln x}$

Method 1.

$u = x$	$\frac{du}{dx} = 1$		$v = \ln x$	$\frac{dv}{dx} = \frac{1}{x}$

Use the formula.

$$\frac{dy}{dx} = vu^{v-1}\frac{du}{dx} + u^v \ln u \frac{dv}{dx}$$
$$\frac{dy}{dx} = (\ln x)(x)^{\ln x - 1}(1) + (x)^{\ln x}(\ln x)\left(\frac{1}{x}\right)$$
$$\frac{dy}{dx} = (\ln x)(x)^{\ln x - 1} + (x)^{\ln x - 1}(\ln x)$$

Add similar terms. $\frac{dy}{dx} = 2(x)^{\ln x - 1}(\ln x)$

Method 2. $y = (x)^{\ln x}$

$$\ln y = (\ln x)(\ln x) = (\ln x)^2$$
$$\frac{1}{y}\frac{dy}{dx} = 2(\ln x)\frac{d}{dx}(\ln x)$$
$$\frac{1}{y}\frac{dy}{dx} = 2(\ln x)\left(\frac{1}{x}\right)$$
$$\frac{dy}{dx} = \frac{2y\ln x}{x}$$

But $y = (x)^{\ln x}$. $\frac{dy}{dx} = \frac{2(x)^{\ln x}(\ln x)}{x} = 2(x)^{\ln x - 1}\ln x$

Example 3. Find $f'(x)$, given $y = (\tan x)^{e^x}$

Use the formula.

$u = \tan x$	$\frac{du}{dx} = sec^2 x$		$v = e^x$	$\frac{dv}{dx} = e^x$

$$\frac{dy}{dx} = vu^{v-1}\frac{du}{dx} + u^v \ln u \frac{dv}{dx}$$

$$\frac{dy}{dx} = (e^x)(\tan x)^{e^x - 1}(sec^2 x) + (\tan x)^{e^x}(\ln \tan x)(e^x)$$

Simplify. $\frac{dy}{dx} = e^x(\tan x)^{e^x}[(\tan x)^{-1}(sec^2 x) + \ln \tan x]$

But, $(\tan x)^{-1} = \frac{1}{\tan x} = \cot x$. $\frac{dy}{dx} = e^x(\tan x)^{e^x}[\cot x sec^2 x + \ln \tan x]$

Method 2. $y = (\tan x)^{e^x}$

$$\ln y = \ln(\tan x)^{e^x} = (e^x)\ln \tan x$$

$$\frac{1}{y}\frac{dy}{dx} = e^x\left[\frac{1}{\tan x}(sec^2 x)\right] + (\ln \tan x)(e^x)$$

$$\frac{dy}{dx} = y\left[\frac{1}{\tan x}(sec^2 x)e^x + (\ln \tan x)(e^x)\right]$$

$$\frac{dy}{dx} = (\tan x)^{e^x}[\cot x\,(sec^2 x)e^x + (\ln \tan x)(e^x)]$$

$$\frac{dy}{dx} = e^x(\tan x)^{e^x}[\cot x\,(sec^2 x) + \ln \tan x]$$

Example 30. What positive number when raised to itself will give a minimum value?

Solution: Let x be the positive number and y be theminimum value of the number raised to itself. That is, $y = x^x$

$$\ln y = x \ln x$$

$$\frac{1}{y}\frac{dy}{dx} = x\left(\frac{1}{x}\right)(1) + \ln x$$

$$\frac{dy}{dx} = y(1 + \ln x)$$

$$\frac{dy}{dx} = x^x(1 + \ln x)$$

For the value of y to be a minimum, $\frac{dy}{dx} = 0 = x^x(1 + \ln x)$

Equate each factor to zero. $1 + \ln x = 0$ $\quad$ $x^x = 0$

$\ln x = -1$ $\quad$ $x \ln x = \ln 0$ (Rejected since $\ln 0$ has no value)

$$x = e^{-1}$$

$$x = \frac{1}{e}$$

SAQ21
ACTIVITY 4.18 – G

NAME: ____ SCORE: _______________

SECTION: ____________DATE: ________________ PROF: ____________________________________

Find $\frac{dy}{dx}$.

1. $y = (e^x)^{\cos x}$

2. $y = \left(\frac{1}{x}\right)^x$

3. $y = (x)^{x^2}$

ASAQ21
ACTIVITY 4.18 – G

NAME: ____ SCORE: ________________

SECTION: ____________DATE: _________________ PROF: _______________________________________

Find $\frac{dy}{dx}$.

1. $y = (e^x)^{\cos x}$ $Answer: e^{x\cos x}(\cos x - x\sin x)$

2. $y = \left(\frac{1}{x}\right)^x$ $Answer: -\left(\frac{1}{x}\right)^x(1 + \ln x)$

3. $y = (x)^{x^2}$ $Answer: (x)^{1+x^2}(1 + 2\ln x)$

MODULE 19

CONCEPT OF INDETERMINATE FORMS

Specific Objectives:

At the end of the module, students must be able to:

1. Recognize when to apply L' Hospital's Rule.
2. Identify indeterminate forms produced by quotient, products, difference, powers and apply L' Hospital's Rule.

CONCEPT OF INDETERMINATE FORMS

In Chapter 1, we experienced getting limit of expression $\frac{f(x)}{g(x)}$ equal to $\frac{0}{0}$ or $\frac{\infty}{\infty}$as x approaches a certain value, say a. This limit is a meaningless symbol and is called indeterminate form. In Calculus, evaluating limit that involves indeterminate form is better facilitated using derivatives. Application (or repeated application) of L' Hospital's Rule, (pronounced: lopiˈtal) named after 17th-century French mathematician Guillaume de l'Hôpital (also written L'Hopital) often converts an indeterminate form to a determinate form, hence, allowing easy evaluation of limits. Other indeterminate forms are $\infty - \infty, 0 \cdot \infty, 0^{0}, \infty^{0}$ and 1^{∞}.

L' Hospital's Rule: If $f(x)$ and $g(x)$ are both continuous and differentiable functions in an open interval containing $x = a$, except possibly at $a,$ and if $f(x)$and $g(x)$ are both 0 when $x = a,$ (or when both approach ∞ $w$hen $x \to a$), provided $g'(x) \neq 0,$ then:

$$\boxed{\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} = L}$$

provided L is finite or $+\infty$ or $-\infty$. Moreover, this statement is also true in the case of a limit as $x \to a^{+}, x \to a^{-}, x \to +\infty$, $or\ as\ x \to -\infty$.

We will discuss limit evaluation resulting to any of the indeterminate forms $\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty - \infty$, $0 \cdot \infty$; and, 0^0, ∞^0, 1^∞.

A. Indeterminate Forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$

When limit of a given function takes the indeterminate form $\frac{0}{0}$ and $\frac{\infty}{\infty}$, apply L'Hospital's Rule repeatedly until limit becomes a finite number. Let us see how to do it from the illustrative examples.

I. Evaluate the limit, if it exists.

$$\lim_{x \to 0} \frac{x - Sin^{-1}x}{x^3} = \frac{0-0}{0} = \frac{0}{0}$$

Differentiate separately the numerator and denominator since the limit is $\frac{0}{0}$

$$= \lim_{x \to 0} \frac{1 - \frac{1}{\sqrt{1-x^2}}}{3x^2} = \frac{1-1}{0} = \frac{0}{0}$$

Reapply L' Hospital's Rule since limit is again $\frac{0}{0}$

Reapply L' Hospital's Rule. $$= \lim_{x \to 0} \frac{0 - \left(\frac{-1}{1-x^2} \cdot \frac{1}{2\sqrt{1-x^2}} \cdot (-2x) \right)}{6x} = \frac{1-1}{0} = \frac{0}{0}$$

Simplify.
$$= \lim_{x\to 0} \frac{-\frac{x}{(1-x^2)\sqrt{1-x^2}}}{6x} = \lim_{x\to 0} \frac{-1}{6(1-x^2)^{\frac{3}{2}}} = -\frac{1}{6}$$

The finite value of $\lim_{x\to 0} \frac{x - Sin^{-1}x}{x^3}$

2. $$\lim_{x\to\frac{1}{4}\pi} \frac{sec^2x - 2\tan x}{1+\cos 4x} = \frac{\left(\sqrt{2}\right)^2 - 2(1)}{1+(-1)} = \frac{0}{0}$$

$$= \lim_{x\to\frac{1}{4}\pi} \frac{2\sec x(\sec x\tan x) - 2sec^2x}{-4\sin 4x} = \frac{2sec^2x(\tan x - 1)}{-4\sin 4x} = \frac{sec^2x(\tan x-1)}{\sin 4x} - \frac{\left(\sqrt{2}\right)^2(1-1)}{0} = \frac{0}{0}$$

Use of L' Hospital's Rule yields $\frac{0}{0}$

$$= \lim_{x\to\frac{1}{4}\pi} -\frac{1}{2}\left[\frac{sec^2x(sec^2x) + (\tan x - 1)(2sec^2x\tan x)}{4\cos 4x}\right]$$

$$= -\frac{1}{8}\left[\frac{2(2) + (1-1)(2)(2)(1)}{-1}\right] = \frac{1}{2}$$

Reapply L' Hospital's Rule

3. $$\lim_{x\to\pi} \frac{\ln\cos 2x}{(\pi - x)^2} = \frac{\ln\cos 2\pi}{\left((\pi-\pi)\right)^2} = \frac{\ln 1}{0} = \frac{0}{0}$$

Result of using L' Hospital's

$$= \lim_{x\to\pi} \frac{\frac{1}{\cos 2x}\cdot(-\sin 2x)(2)}{2(\pi - x)(-1)} = \lim_{x\to\pi} \frac{\tan 2x}{\pi - x} = \frac{0}{0}$$

$$= \lim_{x\to\pi} \frac{(sec^2 2x)(2)}{-1} = \frac{2(1)^2}{-1} = -2$$

Reapplying L' Hospital's Rule yields finite limit of value -2

4. $$\lim_{x\to 0} \frac{x\ln(1+x)}{1-\cos x} = \frac{0(\ln 1)}{1-1} = \frac{0(0)}{0} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{x\left(\frac{1}{1+x}\right)(1) + \ln(1+x)(1)}{-(-\sin x)} = \lim_{x\to 0} \frac{\frac{x}{1+x} + \ln(1+x)}{\sin x} = \frac{\frac{0}{1+0} + \ln 1}{\sin 0} = \frac{0+0}{0} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{x + (1+x)\ln(1+x)}{(1+x)\sin x} = \lim_{x\to 0} \frac{1 + (1+x)\left(\frac{1}{1+x}\right)(1) + \ln(1+x)}{(1+x)\cos x + \sin x}$$

$$= \lim_{x\to 0} \frac{1 + 1 + \ln(1+x)}{(1+x)\cos x + \sin x} = \lim_{x\to 0} \frac{2 + \ln(1+x)}{(1+x)\cos x + \sin x} = \frac{2 + \ln 1}{1(1) + 0} = 2$$

5. $$\lim_{x\to 0} \frac{(1-e^x)^2}{x\sin x} = \frac{(1-1)^2}{0} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{2(1-e^x)(-e^x)}{x\cos x + \sin x} = \lim_{x\to 0} \frac{-2(1-e^x)e^x}{x\cos x + \sin x} = \frac{-2(1-1)(1)}{0+0} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{-2[e^x(-e^x) + (1-e^x)(e^x)]}{x(-\sin x) + \cos x + \cos x} = \lim_{x\to 0} \frac{-2e^x(-e^x + 1 - e^{-x})}{-x\sin x + 2\cos x} = = \frac{-2[1(1-2)]}{0+2(1)} = 1$$

6. $\lim\limits_{x\to 0} \dfrac{e^x - e^{-x}}{\ln(x+1)} = \dfrac{1-1}{0} = \dfrac{0}{0}$

$$= \lim_{x\to 0} \frac{e^x - e^{-x}(-1)}{\frac{1}{x+1}} \lim_{x\to 0} (e^x + e^{-x})(x+1) = (1+1)(0+1) = 2$$

7. $\lim\limits_{x\to\infty} \dfrac{\ln x^{10}}{x} = \dfrac{\ln\infty}{\infty} = \dfrac{\infty}{\infty}$

$$= \lim_{x\to\infty} \frac{\frac{1}{x^{10}}(10x^9)}{1} \lim_{x\to\infty} \frac{10}{x} = \frac{10}{\infty} = 0$$

8. $\lim\limits_{x\to\infty} \dfrac{2x^2+1}{4x^2+x} = \dfrac{\infty}{\infty}$

$$= \lim_{x\to\infty} \frac{4x}{8x+1} = \frac{\infty}{\infty}$$

$$= \lim_{x\to\infty} \frac{4}{8} = \frac{1}{2}$$

9. $\lim\limits_{x\to\infty} \dfrac{e^{2x}}{x^3} = \dfrac{\infty}{\infty}$

$$= \lim_{x\to\infty} \frac{2e^{2x}}{3x^2} = \frac{\infty}{\infty}$$

$$= \lim_{x\to\infty} \frac{4e^{2x}}{6x} = \frac{\infty}{\infty}$$

$$= \lim_{x\to\infty} \frac{8e^{2x}}{6} = \frac{8(\infty)}{6} = \frac{\infty}{6} = \infty \text{ (Limit does not exist)}$$

10. $\lim\limits_{x\to 0} \dfrac{1-\ln x}{e^{\frac{1}{x}}} = \dfrac{1-\infty}{e^\infty} = -\left(\dfrac{\infty}{\infty}\right)$

$$= \lim_{x\to 0} \frac{-\frac{1}{x}}{e^{\frac{1}{x}}\left(-\frac{1}{x^2}\right)} = \lim_{x\to 0} \frac{x}{e^{\frac{1}{x}}} = \frac{0}{e^\infty} = \frac{0}{\infty} = 0$$

B. Indeterminate Form $\infty - \infty$

If functions $f(x) \to \infty$ and $g(x) \to \infty$ as $x \to a$, the difference $f(x) - g(x)$ is said to assume the indeterminate form $\infty - \infty$. The technique to evaluate $\lim\limits_{x \to a}[f(x) - g(x)]$ is to **rewrite the given difference into a quotient**. This is done through subtraction of fractions, that is $\frac{a}{b} - \frac{c}{d} = \frac{ad-cb}{bd}$, and on the result, apply L'Hospital's Rule.

II. Evaluate the limit, if it exists,

Since limit is $\infty - \infty$, transform $\left(\frac{1}{\ln x} - \frac{1}{x-1}\right)$ to quotient form to allow the use of L' Hospital's Rule

1. $$\lim_{x \to 1}\left(\frac{1}{\ln x} - \frac{1}{x-1}\right) = \frac{1}{0} - \frac{1}{0} = \infty - \infty$$

$$= \lim_{x \to 1} \frac{x - 1 - \ln x}{(x-1)\ln x} = \frac{1 - 1 - 0}{(1-1)(0)} = \frac{0}{0}$$

$$= \lim_{x \to 1} \frac{1 - \frac{1}{x}}{(x-1)\left(\frac{1}{x}\right) + \ln x} = \lim_{x \to 1} \frac{\frac{x-1}{x}}{\frac{x - 1 + x\ln x}{x}}$$

Simplify the expression resulting from the use of L' Hospital's Rule

$$= \lim_{x \to 1} \frac{x - 1}{x - 1 + x\ln x} = \frac{0}{0}$$

$$= \lim_{x \to 1} \frac{1}{1 + x\left(\frac{1}{x}\right) + \ln x} = \lim_{x \to 1} \frac{1}{1 + 1 + \ln x} = \frac{1}{1 + 1 + 0} = \frac{1}{2}$$

Apply L' Hospital's Rule the 2nd time. Result is finite limit of $\frac{1}{2}$.

2. $$\lim_{x \to 2}\left(\frac{5}{x^2 + x - 6} - \frac{1}{x-2}\right) = \frac{5}{0} - \frac{1}{0} = \infty - \infty$$

$$= \lim_{x \to 2} \frac{5(x-2) - (x^2 + x - 6)}{(x-2)(x^2 + x - 6)} = \lim_{x \to 2} \frac{-4 + 4x - x^2}{(x-2)(x^2 + x - 6)} = \frac{0}{0}$$

$$= \lim_{x \to 2} \frac{4 - 2x}{(x-2)(2x+1) + (x^2 + x - 6)(1)} = \lim_{x \to 2} \frac{4 - 2x}{3x^2 - 2x - 8} = \quad = \frac{0}{0}$$

$$= \lim_{x \to 2} \frac{-2}{6 - 2x} = \frac{-2}{10} = -\frac{1}{5}$$

3. $$\lim_{x \to \frac{\pi}{2}} (\pi \sec x - 2x \tan x) = \pi\left(\frac{1}{0}\right) - 2(0)\left(\frac{1}{0}\right) = \infty - \infty$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{0 - 2(x\cos x + \sin x)}{-\sin x} = \frac{-2\left[\left(\frac{\pi}{2}\right)(0) + 1\right]}{-1} = \frac{-2}{-1} = 2$$

4. $\lim_{x\to 0}\left(\frac{2}{sin^2x}-\frac{1}{1-\cos x}\right)=\frac{2}{0}-\frac{1}{0}=\infty-\infty$

$$=\lim_{x\to 0}\left(\frac{2}{sin^2x}-\frac{1}{1-\cos x}\right)=\frac{2}{0}-\frac{1}{0}=\infty-\infty$$

$$=\lim_{x\to 0}\left[\frac{2(1-\cos x)-sin^2x}{sin^2x(1-\cos x)}\right]=\frac{2(1-1)-0}{0(1-1)}=\frac{0}{0}$$

$$=\lim_{x\to 0}\left[\frac{2(\sin x)-2\sin x\cos x}{sin^2x(\sin x)+(1-\cos x)(2\sin x\cos x)}\right]$$

$$=\lim_{x\to 0}\frac{2\sin x(1-\cos x)}{\sin x(sin^2x+2\cos x-2cos^2x)}$$

$$=\lim_{x\to 0}\left[\frac{2(1-\cos x)}{sin^2x+2\cos x-2cos^2x}\right]$$

$$=\lim_{x\to 0}\frac{2\sin x}{2\sin x\cos x-2\sin x-4\cos x\,(-\sin x)}$$

$$=\lim_{x\to 0}\left[\frac{2\sin x}{-2\sin x+6\sin x\cos x}\right]=\lim_{x\to 0}\frac{2\sin x}{2\sin x(-1+3\cos x)}$$

$$=\frac{1}{-1+3(1)}=\frac{1}{2}$$

C. Indeterminate Form $0\cdot\infty$

If functions $f(x)\to 0$ and $g(x)$ increases without limit, that is $g(x)\to\infty$ as $x\to a$, then, the product $f(x)\cdot g(x)$ assumes the indeterminate form $0\cdot\infty$. Again, transformation needs to be done on the **product $f(x)\cdot g(x)$ in order to bring it to a quotient**. This can be done in either of the two ways below.

$$\boldsymbol{f(x)\cdot g(x)=\frac{f(x)}{\frac{1}{g(x)}}}\quad \textbf{or}\quad \boldsymbol{f(x)\cdot g(x)=\frac{g(x)}{\frac{1}{f(x)}}}$$

Then, apply L' Hospital's Rule.

III. Evaluate the limit, if it exists.

1. $\lim_{x\to\infty}x\sin\frac{3}{x}=\infty\cdot\sin\left(\frac{3}{\infty}\right)=\infty\cdot\sin 0=\infty\cdot 0$

$$=\lim_{x\to\infty}\frac{\sin\frac{3}{x}}{\frac{1}{x}}=\frac{\sin 0}{\frac{1}{\infty}}=\frac{0}{0}$$

Since limit is $0\cdot\infty$, transform $\left(x\sin\frac{3}{x}\right)$ to quotient form to allow the use of L' Hospital's Rule

$$= \lim_{x\to\infty} \frac{\left(\cos\frac{3}{x}\right)\left(\frac{-3}{x^2}\right)}{-\frac{1}{x^2}} = \lim_{x\to\infty} 3\cos\frac{3}{x} = 3\cos 0 = 3(1) = 3$$

Result of using L' Hospital's Rule

2. $\lim_{x\to 0^+} x\ln x = 0 \cdot \ln 0^+ = 0 \cdot \infty$

$$= \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} -x = 0$$

3. $\lim_{x\to\infty} xe^{-x} = \infty \cdot e^{-\infty} = \infty \cdot \left(\frac{1}{e^{\infty}}\right) = \infty \cdot \left(\frac{1}{\infty}\right) = \infty \cdot 0$

$$= \lim_{x\to\infty} \frac{x}{\frac{1}{e^x}} = \lim_{x\to\infty} \frac{x}{e^x} = \frac{\infty}{\infty}$$

$$= \lim_{x\to\infty} \frac{1}{e^x} = \frac{1}{e^{\infty}} = \frac{1}{\infty} = 0$$

Since limit is $0 \cdot \infty$, transform $(x^3 \ln x)$ to quotient form to allow the use of L' Hospital's Rule

4. $\lim_{x\to 0} x^3 \ln x = 0 \cdot \infty$

$$= \lim_{x\to 0} \frac{\ln x}{\frac{1}{x^3}} = \lim_{x\to 0} \frac{\frac{1}{x}}{-\frac{3}{x^4}} = \lim_{x\to 0} -\frac{x^3}{3} = -\frac{0}{3} = 0$$

5. $\lim_{x\to\frac{\pi}{4}} (1 - \tan x)\sec 2x = 0 \cdot \left(\frac{1}{0}\right) = 0 \cdot \infty$

$$= \lim_{x\to\frac{\pi}{4}} \frac{1 - \tan x}{\cos 2x} = \frac{1-1}{0} = \frac{0}{0}$$

$$= \lim_{x\to\frac{\pi}{4}} \frac{-sec^2 x}{(-\sin 2x)(2)} = \frac{\left(\sqrt{2}\right)^2}{2(1)} = 1$$

6. $\lim_{x\to 1} \csc \pi x \ln x = \frac{1}{0}(1) = \infty \cdot 0$

$$= \lim_{x\to 1} \frac{\ln x}{\sin \pi x} = \lim_{x\to 1} \frac{\frac{1}{x}}{(\cos \pi x)(\pi)} = \frac{1}{\pi(-1)} = -\frac{1}{\pi}$$

D. Indeterminate Forms $0^0, \infty^0, 1^{\infty}$

Limits of the form $\lim_{x\to a} [f(x)]^{g(x)}$ or $\lim_{x\to\infty} [f(x)]^{g(x)}$ frequently give rise to indeterminate of the types $0^0, \infty^0, 1^{\infty}$. Indeterminate forms of these types can be evaluated as follows:

(1). Let $y = [f(x)]^{g(x)}$. ------------- Equation (1)

(2). Take the logarithm of both sides of the Equation in (1).

Thus, $\ln y = \ln[f(x)]^{g(x)} = g(x)\ln f(x)$.

(3). Take $\lim\limits_{x\to a}\ln y = \lim\limits_{x\to a} g(x)\ln f(x)$. The limit on the right-hand side of the equation will usually be an indeterminate limit of the type $0\cdot\infty$. Evaluate this limit using the technique previously described. Assume that $\lim\limits_{x\to a} g(x)\ln f(x) = L$.

(4). Finally, if $\lim\limits_{x\to a}\ln y = L$, then, $\lim\limits_{x\to a}\ [f(x)]^{g(x)} = e^L$.

IV. Solve the limit, if it exists.

1. $\lim\limits_{x\to 0}(\cos x)^{\frac{1}{x}} = (1)^{\frac{1}{0}} = (1)^{\infty}$

Step (1)

$$\text{Let } y = (\cos x)^{\frac{1}{x}}$$

Step (2)

$$\ln y = \ln(\cos x)^{\frac{1}{x}} = \frac{1}{x}\ln\cos x$$

Step (3)

$$\lim_{x\to 0}\ln y = \lim_{x\to 0}\frac{1}{x}\ln\cos x = \infty\cdot 0$$

$$L = \lim_{x\to 0}\ln y = \lim_{x\to 0}\frac{\ln\cos x}{x} = \frac{0}{0}$$

$$L = \lim_{x\to 0}\frac{\frac{1}{\cos x}(-\sin x)}{1} = \frac{0}{1} = 0$$

Step (4)

Therefore,

$$\lim_{x\to 0}(\cos x)^{\frac{1}{x}} = e^L = e^0 = 1.$$

2. $\lim\limits_{x\to 0}(e^x + 3x)^{\frac{1}{x}} = (1)^{\frac{1}{0}} = (1)^{\infty} + 3$

$$\text{Let } y = (e^x + 3x)^{\frac{1}{x}}$$

$$\ln y = \ln(e^x + 3x)^{\frac{1}{x}} = \frac{1}{x}\ln(e^x + 3x)$$

$$L = \lim_{x\to 0}\ln y = \lim_{x\to 0}\frac{\ln(e^x + 3x)}{x} = \frac{\ln 1}{0} = \frac{0}{0}$$

$$L = \lim_{x\to 0}\frac{\frac{1}{e^x + 3x}(e^x + 3x)}{1} = \frac{e^0 + 3}{e^0} = \frac{1+3}{1} = 4$$

Therefore, $\lim\limits_{x\to 0}(e^x + 3x)^{\frac{1}{x}} = e^L = e^4$.

3. $\lim\limits_{x\to+\infty}\left(1+\frac{1}{x}\right)^x=\left(1+\frac{1}{\infty}\right)^\infty=(1+0)^\infty=1^\infty$

$$\text{Let } y=\left(1+\frac{1}{x}\right)^x$$

$$\ln y=\ln\left(1+\frac{1}{x}\right)^x=x\ln\left(1+\frac{1}{x}\right)$$

$$L=\lim_{x\to+\infty}\ln y=\lim_{x\to+\infty}x\ln\left(1+\frac{1}{x}\right)=+\infty\ln\left(1+\frac{1}{\infty}\right)=+\infty\ln(1+0)=+\infty\cdot 0$$

$$L=\lim_{x\to\infty}\ln y=\lim_{x\to+\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}=\frac{0}{0}$$

$$L=\lim_{x\to+\infty}\frac{\frac{1}{1+\frac{1}{x}}\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}}=\frac{1}{1+\frac{1}{\infty}}=\frac{1}{1+0}=1$$

Therefore, $\lim\limits_{x\to+\infty}\left(1+\frac{1}{x}\right)^x=e^L=e^1=e$

4. $\lim\limits_{x\to1}x^{\tan\frac{\pi}{2}x}=(1)^{\frac{1}{0}}=1^\infty$

$$\text{Let } y=x^{\tan\frac{\pi}{2}x}$$

$$\ln y=\ln x^{\tan\frac{\pi}{2}x}=\tan\frac{\pi}{2}x\ln x$$

$$L=\lim_{x\to1}\ln y=\lim_{x\to1}\left(\tan\frac{\pi}{2}x\right)\ln x=\infty\cdot 0$$

$$L=\lim_{x\to}\frac{\ln x}{\cot\frac{\pi}{2}x}=\lim_{x\to1}\frac{\frac{1}{x}}{\left(-csc^2\frac{\pi}{2}x\right)\left(\frac{\pi}{2}\right)}=\frac{\frac{1}{1}}{-\frac{\pi}{2}(1)}=-\frac{2}{\pi}$$

Therefore, $\lim\limits_{x\to1}x^{\tan\frac{\pi}{2}x}=e^L=e^{-\frac{2}{\pi}}$

5. $\lim\limits_{x\to\frac{\pi}{2}^-}(\tan x)^{\cos x}=\left(\frac{1}{0}\right)^0=\infty^0$

$$\text{Let } y=(\tan x)^{\cos x}$$

$$\ln y=\ln(\tan x)^{\cos x}=\cos x\ln(\tan x)$$

$$L=\lim_{x\to\frac{\pi}{2}^-}\ln y=\lim_{x\to\frac{\pi}{2}^-}\cos x\ln(\tan x)=0\cdot\ln\frac{1}{0}=0\cdot\ln\infty=0\cdot\infty$$

$$L = \lim_{x \to \frac{\pi}{2}^-} \frac{\ln \tan x}{\sec x} = \lim_{x \to \frac{\pi}{2}^-} \frac{\frac{1}{\tan x}(sec^2 x)}{\sec x \tan x} = \lim_{x \to \frac{\pi}{2}^-} \frac{\sec x}{tan^2 x} = \frac{\infty}{\infty}$$

$$L = \lim_{x \to \frac{\pi}{2}^-} \frac{\sec x \tan x}{2 \tan x sec^2 x} = \lim_{x \to \frac{\pi}{2}^-} \frac{1}{2\sec x} = \frac{1}{2(\infty)} = 0$$

Therefore, $\lim_{x \to \frac{\pi}{2}^-} (\tan x)^{\cos x} = e^L = e^0 = 1$

SAQ22
ACTIVITY 5.19 – A

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Evaluate the limit, if it exists.

1 $\lim_{x\to\infty} \dfrac{x^5+x^4+x^3+x^2+x+1}{2x^5+x^4+x^3+x+1}$

2. $\lim_{x\to -2} \dfrac{x+2}{\ln(x+3)}$

3. $\lim_{x\to 0} \dfrac{x^3}{x-\sin x}$

4. $\lim_{x\to\infty} \dfrac{x^2}{e^x}$

5 $\lim_{x\to 0} \dfrac{\ln \sec x}{x^2}$

6. $\lim_{x\to\infty} \frac{x^4}{e^x}$

7. $\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{\sin x}\right)$

8. $\lim_{x\to 2}\left(\frac{4}{x^2-4}-\frac{1}{x-2}\right)$

9 $\lim_{x\to 0}(e^x-1)\cot x$

10. $\lim_{x\to \frac{1}{2}^+}(2x-1)\tan \pi x$

11. $\lim_{x \to \infty} (\ln x)^{\frac{1}{x}}$

12. $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^{x}$

13. $\lim_{x \to 0^{+}} (\sin x)^{\tan x}$

14. $\lim_{x \to 0} (x)^{(x^{2})}$

15. $\lim_{x \to 0^{+}} (1 + x)^{\frac{1}{x}}$

ASAQ22
ACTIVITY 5.19 – A

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Evaluate the limit, if it exists.

1 $\lim_{x\to\infty} \dfrac{x^5 + x^4 + x^3 + x^2 + x + 1}{2x^5 + x^4 + x^3 + x + 1}$ $\quad Answer: \frac{1}{2}$

2. $\lim_{x\to -2} \dfrac{x + 2}{\ln(x + 3)}$ $\quad Answer: 1$

3. $\lim_{x\to 0} \dfrac{x^3}{x - \sin x}$ $\quad Answer: 6$

4. $\lim_{x\to\infty} \dfrac{x^2}{e^x}$ $\quad Answer: 0$

5. $\lim_{x\to 0} \dfrac{\ln \sec x}{x^2}$ $\quad Answer: \frac{1}{2}$

6. $\lim_{x\to\infty} \dfrac{x^4}{e^x}$ $\quad Answer: 0$

7. $\lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{\sin x}\right)$ $\quad Answer: 0$

8. $\lim_{x\to 2} \left(\dfrac{4}{x^2-4} - \dfrac{1}{x-2}\right)$ $\quad Answer:\ -\frac{1}{4}$

9 $\lim_{x\to 0}(e^x - 1)\cot x$ $\quad Answer: 1$

10. $\lim_{x\to \frac{1}{2}^+} (2x-1)\tan \pi x$ $\quad Answer:\ -\frac{2}{\pi}$

11. $\lim_{x\to\infty} (\ln x)^{\frac{1}{x}}$ $\quad Answer: 1$

12. $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^{x}$ $\quad Answer: e$

13. $\lim_{x\to 0^+} (\sin x)^{\tan x}$ $\quad Answer: e^{-2}$

14. $\lim_{x\to 0} (x)^{(x^2)}$ $\quad Answer: 1$

15. $\lim_{x\to 0^+} (1+x)^{\frac{1}{x}}$ $\quad Answer: e$

ACTIVITY 5.19 – B

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Evaluate the limit, if it exists.

1. $\lim\limits_{x \to 0} \dfrac{5x - \tan 5x}{x^3}$

2. $\lim\limits_{x \to \infty} \dfrac{\frac{1}{x^2}}{\sin\left(\frac{1}{x}\right)}$

3. $\lim\limits_{x \to 0} \dfrac{e^x - e^{-x}}{\ln(x+1)}$

4. $\lim\limits_{x \to \infty} \dfrac{\sqrt{4+x} - \sqrt{4-x}}{x}$

5. $\lim\limits_{x \to 0^+} \left(\dfrac{1}{4x} - \dfrac{1}{e^{4x} - 1}\right)$

6. $\lim_{x \to -\infty} x^2 e^{3x}$

7. $\lim_{x \to 2} \left[\frac{2}{\ln(x^2 - 3)} - \frac{x}{\ln(x^2 - 3)} \right]$

8. $\lim_{x \to \infty} \left(1 + \frac{4}{x}\right)^x$

9. $\lim_{x \to 0^+} (1 + x)^{\frac{1}{x}}$

10. $\lim_{x \to 0} (\cot x)^x$

MODULE 20

CONCEPT OF PARTIAL DIFFERENTIATION

Specific Objectives:

At the end of the module, students must be able to:

1. Recognize when to differentiate partially.
2. Find the higher order derivative of a function of more than one independent variable.

PARTIAL DIFFERENTIATION

From the previous chapters, we only dealt with derivatives of the functions with respect to only one independent variable. However, not all functions are dependent on only one variable; there are functions in which a variable may be dependent to more than one independent variable. In finding the derivative of such function, we will still be differentiating it with respect to one of the independent variables while holding the other independent variables as constants. Such differentiation is called a partial differentiation. Suppose a function $z = f(x,y)$ exists where the independent variables are x and y and z being the dependent variable, then, the partial derivative of a function $z = f(x,y)$ with respect to variable x is given by $\frac{\partial z}{\partial x}$ and the partial derivative of the same function with respect to variable y is given by $\frac{\partial z}{\partial y}$. In the derivative $\frac{\partial z}{\partial x}$, all variable y's in the function of z are held constant and, in the derivative $\frac{\partial z}{\partial y}$, all variable x's in the function of z are held constant.

Example 1. In the functions given below, perform the corresponding partial derivatives with respect to all its independent variable.

a. $f(x,y) = 4x^2y - 12xy^3 + x^2y^7$
b. $f(x,y) = 12x^3 \cos y - 3y^4e^{2x}$
c. $f(x,y,z) = 12x^2y^5 + 5y^3z^4 - 8x^2z$

Solution:
a. $f(x,y) = 4x^2y - 12xy^3 + xyz^2$
Differentiating f with respect to x means holding all variable y's constant.

$$\frac{\partial f}{\partial x} = 4y\frac{d}{dx}(x^2) - 12y^3\frac{d}{dx}(x) + \ y^7\frac{d}{dx}(x^2)$$
$$\frac{\partial f}{\partial x} = 8xy - 12y^3 + 2xy^7$$

Notice below that when we do partial differentiation with respect to y, all variable x's are held constant.

$$\frac{\partial f}{\partial y} = 4x^2\frac{d}{dy}(y) - 12x\frac{d}{dy}(y^3) + x^2\frac{d}{dy}(y^7)$$
$$\frac{\partial f}{\partial y} = 4x^2 - 36xy^2 + 7x^2y^6$$

b. $f(x,y) = 12x^3 \cos y - 3y^4e^{2x}$

$$\frac{\partial f}{\partial x} = 12\cos y\frac{d}{dx}(x^3) - 3y^4\frac{d}{dx}(e^{2x})$$
$$\frac{\partial f}{\partial x} = 36x^2 \cos y - 6y^4e^{2x}$$

Again, observe above that when we do partial differentiation with respect to x, all variable y's are held constant.

$$\frac{\partial f}{\partial y} = 12x^3 \frac{d}{dy}(\cos y) - 3e^{2x} \frac{d}{dy}(y^4)$$
$$\frac{\partial f}{\partial y} = -12x^3 \sin y - 12y^3 e^{2x}$$

In partially differentiating f with respect to y, all variable x's are held constant as observed above.

c. $f(x,y,z) = 12x^2y^5 + 5y^3z^4 - 8x^2z$

$$\frac{\partial f}{\partial x} = 12y^5 \frac{d}{dx}(x^2) - 8z \frac{d}{dx}(x^2)$$

Note that when we get the partial derivative of $5y^3z^4$ with respect to x, the variables y and z are considered constants, so then, the derivative of $5y^3z^4$ is 0.

$$\frac{\partial f}{\partial x} = 24xy^5 - 16xz$$

Notice that when we do partial differentiation with respect to y, all variable x's and z's are held constant. Thus, the partial derivative of $-8x^2z$ with respect to y is 0.

$$\frac{\partial f}{\partial y} = 12x^2 \frac{d}{dy}(y^5) + 5z^4 \frac{d}{dy}(y^3)$$
$$\frac{\partial f}{\partial y} = 60x^2y^4 + 15y^2z^4$$

Observe that the partial derivative of $12x^2y^5$ with respect to z, x and y are considered constants, thus, the derivative of $12x^2y^5$ is 0.

$$\frac{\partial f}{\partial z} = 5y^3 \frac{d}{dz}(z^4) - 8x^2 \frac{d}{dz}(z)$$
$$\frac{\partial f}{\partial z} = 20y^3z^3 - 8x^2$$

HIGHER ORDER PARTIAL DIFFERENTIATION

Now that we have learned the basics of single partial derivative, we can study its higher ordered differentiations.

Example 2. Find $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$, given $f(x,y) = \sin(xy) + 3xy$.

Differentiate f with respect to x by holding y as a constant.

$$\frac{\partial f}{\partial x} = [\cos xy](y)(1) + 3y = y\cos xy + 3y$$

Differentiate partially $\frac{\partial f}{\partial x}$ with respect to x holding y a constant to find $\frac{\partial^2 f}{\partial x^2}$.

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(ycos\ xy + 3y)$$
$$\frac{\partial^2 f}{\partial x^2} = y[-\sin(xy)\cdot(y)] + 0$$
$$\frac{\partial^2 f}{\partial x^2} = -y^2\sin(xy)$$

Differentiate f with respect to y by holding x as a constant.

$$\frac{\partial f}{\partial y} = [\cos xy](x)(1) + 3x = x\cos(xy) + 3x$$

Differentiate partially $\frac{\partial f}{\partial y}$ with respect to y holding x a constant to find $\frac{\partial^2 f}{\partial y^2}$.

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}[x\cos(xy) + 3x]$$
$$\frac{\partial^2 f}{\partial y^2} = x[-\sin(xy)\cdot x + 0] = -x^2\sin(xy)$$

Differentiate $\frac{\partial f}{\partial x}$ with respect to y to get $\frac{\partial^2 f}{\partial y \partial x}$, holding x a constant.

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}[ycos\ xy + 3y] = y[-\sin(xy)\cdot x] + \cos(xy)\cdot(1) + 3$$
$$\frac{\partial^2 f}{\partial y \partial x} = -xy\sin(xy) + y\cos(xy) + 3$$

Differentiate $\frac{\partial f}{\partial y}$ with respect to x to get $\frac{\partial^2 f}{\partial x \partial y}$, holding y a constant.

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}[x\cos(xy) + 3x] = x[-\sin(xy)\cdot y] + \cos(xy)\cdot(1) + 3$$
$$\frac{\partial^2 f}{\partial x \partial y} = -xy\sin(xy) + \cos(xy) + 3$$

Notice that $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$. It is because that the partial derivatives are continuous otherwise if the partial derivatives are discontinuous, $\frac{\partial^2 z}{\partial x \partial y} \neq \frac{\partial^2 z}{\partial y \partial x}$.

Example 3. In the function $z = f(x,y) = 4x^2y^8 - 11x^8y^2$, determine the following partial differentiations: $\frac{\partial^2 z}{\partial x \partial y}, \frac{\partial^2 z}{\partial y \partial x}$.

Given: $f(x,y) = 4x^2y^8 - 11x^8y^2$,

For $\frac{\partial^2 z}{\partial x \partial y}$, $\quad \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left[4x^2\frac{d}{dy}(y^8) - 11x^8\frac{d}{dy}(y^2)\right]$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}[32x^2y^7 - 22x^8y]$$

$$\frac{\partial^2 z}{\partial x \partial y} = 32y^7 \frac{d}{dx}(x^2) - 22y \frac{d}{dx}(x^8)$$

Therefore, $\frac{\partial^2 z}{\partial x \partial y} = 64xy^7 - 176x^7y$

For $\frac{\partial^2 z}{\partial y \partial x}$, $\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}\left[4y^8 \frac{d}{dx}(x^2) - 11y^2 \frac{d}{dx}(x^8)\right]$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}[8xy^8 - 88x^7y^2]$$

$$\frac{\partial^2 z}{\partial y \partial x} = 8x \frac{d}{dy}(y^8) - 88x^7 \frac{d}{dy}(y^2)$$

Therefore, $\frac{\partial^2 z}{\partial y \partial x} = 64xy^7 - 176x^7y$

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In the functions shown below, determine the partial derivatives with respect to each independent variable.

1. $f(x,y) = x^2y^4 - 5x^4y^2 + 12xy^3$

2. $f(x,y) = \sec x \tan^2 y - \cos^2 x \sin^3 y$

3. $f(x,y,z) = 30x^4y\sqrt{z} - 2\sqrt{xyz}$

4. $f(x,y,z) = yz^2e^{-4x} - xze^{-5y} + xy^2e^{7x}$

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In the functions shown below, determine the partial derivatives with respect to each independent variable.

1. $f(x,y) = x^2y^4 - 5x^4y^2 + 12xy^3$

$Answer$: $\frac{\partial f}{\partial x} = 2xy^4 - 20x^3y^2 + 12y^3$; $\frac{\partial f}{\partial y} = 4x^3y^3 - 10x^4y + 36xy^2$

2. $f(x,y) = \sec x \tan^2 y - \cos^2 x \sin^3 y$

$Answer$: $\frac{\partial f}{\partial x} = \sec x \tan x tan^2 y + \sin 2x sin^3 y$; $\frac{\partial f}{\partial y} = 2 \sec x \tan y sec^2 y - 3cos^2 x \cos y sin^2 y$

3. $f(x,y,z) = 30x^4y\sqrt{z} - 2\sqrt{xyz}$

$Answer$: $\frac{\partial f}{\partial x} = 120x^3y\sqrt{z} - \frac{\sqrt{xyz}}{x}$; $\frac{\partial f}{\partial y} = 30x^4\sqrt{z} - \frac{\sqrt{xyz}}{y}$; $\frac{\partial f}{\partial z} = 15\frac{x^4y\sqrt{z}}{x} - \frac{\sqrt{xyz}}{z}$

4. $f(x,y,z) = yz^2e^{-4x} - xze^{-5y} + xy^2e^{7x}$

$Answer$: $\frac{\partial f}{\partial x} = -4yz^2e^{-4x} - ze^{-5y} + y^{2e^{7z}}$;

$\frac{\partial f}{\partial y} = z^2e^{4x} + 5xze^{-5y} + 2xye^{7z}$;

$\frac{\partial f}{\partial z} = 2yze^{-4x} - xe^{-5y} + 7xy^2e^{7z}$

ACTIVITY 6.20 – B

NAME: ____ SCORE: _______________

SECTION: ____________DATE: ________________ PROF: _____________________________________

In the functions shown below, determine the corresponding partial differentiations.

1. $f(x,y,z) = yz^4e^{-4x} - x^3ze^{-5y} + xy^5e^{7x};\quad \frac{\partial^2 f}{\partial x \partial z}; \frac{\partial^2 f}{\partial x \partial y}; \frac{\partial^2 f}{\partial z \partial y}$

2. $f(x,y) = e^{-3x}\cos 377y + e^{2x}\sin 314y; \frac{\partial^2 f}{\partial x \partial y}; \frac{\partial^2 f}{\partial y \partial x}$

3. $f(x,y) = 2e^{-x+6y} + 4^{8x}\ln y; \frac{\partial^2 f}{\partial x \partial y}; \frac{\partial^2 f}{\partial y \partial x}; \frac{\partial^2 f}{\partial y^2}; \frac{\partial^2 f}{\partial x^2}$

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