UCL Programming Principles, Languages, Logic, and Verification

Reading group

Stefan Zetzsche

Towards a Mathematical Operational Semantics

Daniele Turi Gordon Plotkin

$\sim$ 1997

Gordon Plotkin

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Gordon Plotkin

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TITLE	CITED BY	YEAR
A structural approach to operational semantics GD Plotkin Computer Science Department, Aarhus University	4562	1981
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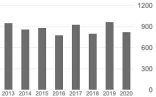

Co-authors   VIEW ALL

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- John Power — Reader in Computer Science, Un...
- Glynn Winskel — University of Cambridge
- Mogens Nielsen — Professor of Computer Science, ...
- Luca Cardelli — University of Oxford
- Marcelo Fiore — Professor in Mathematical Found...
- Martin Hyland — Professor of Mathematical Logic,...

Daniele Turi

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Semantics　Semantic web　Personalisation　Machine Learning

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TITLE	CITED BY	YEAR
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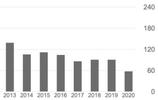

Semantics

$$\text{Syntax} \quad \xrightarrow{\texttt{denotational, operational}} \quad \text{Domain}$$

Example 1: Regular Expressions (not in the paper)

$$\mathrm{RExp}_A{}^{1} \quad \xrightarrow{\texttt{denotational, operational}} \quad \mathcal{P}(A^*)$$

[1] $e, f \in \mathrm{RExp}_A ::= a \in A \mid 0 \mid 1 \mid e \cdot f \mid e + f \mid e^*$

Example 1: Regular Expressions (denotational)

For $L_1, L_2 \subseteq A^*$ define

$$\begin{aligned}
L_1 \cdot L_2 &\stackrel{\text{def}}{=} \{w_1 \cdot w_2 \mid w_1 \in L_1, w_2 \in L_2\} \\
L_1 + L_2 &\stackrel{\text{def}}{=} L_1 \cup L_2 \\
L^* &\stackrel{\text{def}}{=} \bigcup_{n \geq 0} L^n = \{w_1 \cdot \ldots \cdot w_n \mid n \geq 0, w_i \in L, 1 \leq i \leq n\}
\end{aligned}$$

where $L^0 \stackrel{\text{def}}{=} \{\varepsilon\}$ and $L^{n+1} \stackrel{\text{def}}{=} L \cdot L^n$.

The *standard interpretation* is given by the unique homomorphism

$$\mathtt{de} : (\mathrm{RExp}_A, a_{a \in A}, 0, 1, \cdot, +, {}^*) \to (\mathcal{P}(A^*), \{a\}_{a \in A}, \emptyset, \{\varepsilon\}, \cdot, +, {}^*).$$

Example 1: Regular Expressions (operational)

Define

$$E : \mathrm{RExp}_A \to 2 \qquad D_a : \mathrm{RExp}_A \to \mathrm{RExp}_A, \;\; a \in A$$

as follows:

$$\begin{aligned} E(b) &\stackrel{\text{def}}{=} 0, \;\; b \in A & \qquad D_a(b) &\stackrel{\text{def}}{=} [a = b], \;\; b \in A \\ E(0) &\stackrel{\text{def}}{=} 0 & D_a(0) &\stackrel{\text{def}}{=} 0 \\ E(1) &\stackrel{\text{def}}{=} 1 & D_a(1) &\stackrel{\text{def}}{=} 0 \\ E(e \cdot f) &\stackrel{\text{def}}{=} E(e) \cdot E(f) & D_a(e \cdot f) &\stackrel{\text{def}}{=} D_a(e) \cdot f + E(e) \cdot D_a(f) \\ E(e + f) &\stackrel{\text{def}}{=} E(e) + E(f) & D_a(e + f) &\stackrel{\text{def}}{=} D_a(e) + D_a(f) \\ E(e^*) &\stackrel{\text{def}}{=} 1 & D_a(e^*) &\stackrel{\text{def}}{=} D_a(e) \cdot e^*. \end{aligned}$$

The *language interpretation* is given by the unique homomorphism

$$\mathrm{op} : (\mathrm{RExp}_A, E, (D_a)_{a \in A}) \to (\mathcal{P}(A^*), \varepsilon, (\delta_a)_{a \in A})^2.$$

---

[2] $\varepsilon(L) = [\varepsilon \in L], \;\; \delta_a(L) = \{w \mid a \cdot w \in L\}, \;\; a \in A$

Example 1: Regular Expressions

We have established

$$\mathtt{op} : (\mathsf{RExp}_A, E, (D_a)_{a\in A}) \to (\mathcal{P}(A^*), \varepsilon, (\delta_a)_{a\in A})$$
$$\mathtt{de} : (\mathsf{RExp}_A, a_{a\in A}, 0, 1, \cdot, +, \ ^*) \to (\mathcal{P}(A^*), \{a\}_{a\in A}, \emptyset, \{\varepsilon\}, \cdot, +, \ ^*).$$

Some questions:

- Is the equality $\mathtt{op} = \mathtt{de}$ given?
- Are $\mathtt{op}$ and $\mathtt{de}$ also homomorphisms of the respective other type?
- In general, how can we ensure such properties?

Main idea

To derive an *adequate* semantics:

Relate syntax and semantics by a distributive law.

Example 1: Regular Expressions

Let $\Sigma, B : \mathsf{Set} \to \mathsf{Set}$ satisfy

$$\Sigma X = A + 1 + 1 + X^2 + X^2 + X \qquad BX = X^A \times 2.$$

Then one observes

$$\begin{aligned} (\mathrm{RExp}_A, a_{a\in A}, 0, 1, \cdot, +, {}^*) &: \text{initial } \Sigma\text{-algebra} \\ (\mathcal{P}(A^*), \varepsilon, (\delta_a)_{a\in A}) &: \text{final } B\text{-coalgebra.} \end{aligned}$$

We defined morphisms and obtained

$$\begin{tikzcd} \Sigma\mathrm{RExp}_A \arrow[r, dashed, "\Sigma\mathrm{de}"] \arrow[d] & \Sigma\mathcal{P}(A^*) \arrow[d] \\ \mathrm{RExp}_A \arrow[r, dashed, "\mathrm{de}"] & \mathcal{P}(A^*) \end{tikzcd} \qquad \begin{tikzcd} \mathrm{RExp}_A \arrow[r, dashed, "\mathrm{op}"] \arrow[d] & \mathcal{P}(A^*) \arrow[d] \\ B\mathrm{RExp}_A \arrow[r, dashed, "B\mathrm{op}"] & B\mathcal{P}(A^*) \end{tikzcd} .$$

Instead of *defining* above morphisms, we can *deduce* them from $\lambda : \Sigma B \Rightarrow B\Sigma$.

Full picture (simple)

Let $\Sigma, B : \mathsf{C} \to \mathsf{C}$ be endofunctors.

Let $(\mu\Sigma, h)$ denote the initial Σ-algebra, and let $(\nu B, k)$ denote the final B-coalgebra.

Let $\lambda : \Sigma B \to B\Sigma$ be a natural transformation.

By initiality and finality there are unique morphisms

$$
\begin{tikzcd}
\Sigma(\mu\Sigma) \arrow[r, dashed, "\Sigma!"] \arrow[d, "h"'] & \Sigma B(\mu\Sigma) \arrow[d, "Bh \circ \lambda_{\mu\Sigma}"] \\
\mu\Sigma \arrow[r, dashed, "!"] & B(\mu\Sigma)
\end{tikzcd}
\qquad
\begin{tikzcd}
\Sigma(\nu B) \arrow[r, dashed, "!"] \arrow[d, "\lambda_{\nu B} \circ \Sigma k"'] & \nu B \arrow[d, "k"] \\
B\Sigma(\nu B) \arrow[r, dashed, "B!"] & B(\nu B).
\end{tikzcd}
$$

Full picture (simple)

One observes that

$$\langle \Sigma(\mu\Sigma) \xrightarrow{h} \mu\Sigma \xrightarrow{!} B(\mu\Sigma)\rangle \qquad \langle \Sigma(\nu B) \xrightarrow{!} \nu B \xrightarrow{k} B(\nu B)\rangle$$

are initial and final, respectively, along tuples of algebra and coalgebra morphisms related by λ.

Thus we find

$$\begin{tikzcd}
\Sigma(\mu\Sigma) \arrow[r, dashed, "\Sigma\mathtt{op}=\Sigma\mathtt{de}"] \arrow[d, "h"'] & \Sigma(\nu B) \arrow[d, "!"] \\
\mu\Sigma \arrow[r, dashed, "\mathtt{op}=\mathtt{de}"] \arrow[d, "!"'] & \nu B \arrow[d, "k"] \\
B(\mu\Sigma) \arrow[r, dashed, "B\mathtt{op}=B\mathtt{de}"] & B(\nu B)
\end{tikzcd} .$$

Full picture (hard)

The picture complicates if one allows variations in the sense of

$$e \in \mathrm{RExp}_A X ::= x \in X \mid e \in \mathrm{RExp}_A.$$

Mathematically, we move from endofunctors to monads and comonads.

Every endofunctor freely generates a monad and a comonad.

Freely generated monads/comonads

Let $\Sigma, B : \mathsf{C} \to \mathsf{C}$ be endofunctors.

The following definitions induce a monad T_Σ and a comonad D_B on C,

$$T_\Sigma X = \mu(\lambda Y.X + \Sigma Y) \qquad D_B X = \nu(\lambda Y.X \times BY).$$

There exist equivalences

$$\mathsf{Alg}(\Sigma) \simeq \mathsf{Alg}(T_\Sigma) \qquad \mathsf{Coalg}(B) \simeq \mathsf{Coalg}(D_B)$$

satisfying

$$(\mu\Sigma, h) \mapsto (T_\Sigma 0, \mu_0) \qquad (\nu B, k) \mapsto (D_B 1, \delta_1).$$

Distributive laws

A *distributive law* between a monad $\langle T, \eta, \mu \rangle$ and a comonad $\langle D, \varepsilon, \delta \rangle$ is a natural transformation

$$\lambda : TD \Rightarrow DT$$

satisfying the laws

$$\lambda \circ \eta_D = D\eta \qquad \lambda\mu_D = D\mu \circ \lambda_T \circ T\lambda$$

and their dual

$$T\varepsilon = \varepsilon_T \circ \lambda \qquad D\lambda \circ \lambda_D \circ T\delta = \delta_T \circ \lambda.$$

Distributive laws

There exist liftings T_λ and D_λ

$$\begin{tikzcd} \mathrm{Coalg}(D) \arrow[r, "T_\lambda"] \arrow[d, "U_D"'] & \mathrm{Coalg}(D) \arrow[d, "U_D"] \\ \mathrm{C} \arrow[r, "T"] & \mathrm{C} \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Alg}(T) \arrow[r, "D_\lambda"] \arrow[d, "U_T"'] & \mathrm{Alg}(T) \arrow[d, "U_T"] \\ \mathrm{C} \arrow[r, "D"] & \mathrm{C} \end{tikzcd}$$

satisfying

$$T_\lambda(X \stackrel{k}{\to} DX) = TX \stackrel{\lambda_X \circ Tk}{\to} DTX$$
$$D_\lambda(TX \stackrel{h}{\to} X) = TDX \stackrel{Dh \circ \lambda_X}{\to} DX.$$

In fact, liftings of T to $\mathrm{Coalg}(D)$, liftings of D to $\mathrm{Alg}(T)$, and distributive laws coincide.

Bialgebras

A λ-*bialgebra* is an object with both a T-algebra and a D-coalgebra structure

$$\langle TX \xrightarrow{h} X \xrightarrow{k} DX \rangle,$$

satisfying

$$\begin{tikzcd}
TX \arrow[r, "Tk"] \arrow[dd, "h"'] & TDX \arrow[d, "\lambda_X"] \\
 & DTX \arrow[d, "Dh"] \\
X \arrow[r, "k"] & DX
\end{tikzcd} .$$

There exist the following equivalences of categories

$$\mathrm{Alg}(T_\lambda) \simeq \mathrm{Bialg}(\lambda) \simeq \mathrm{Coalg}(D_\lambda).$$

Bialgebras

There are free-forgetful adjunctions $F^\lambda \dashv U^\lambda$ and $U_\lambda \dashv G_\lambda$

$$\begin{tikzcd}
 & \mathrm{Bialg}(\lambda) \arrow[dl, bend left, "U_\lambda"] \arrow[dr, bend left, "U^\lambda"] & \\
\mathrm{Alg}(T) \arrow[ur, bend left, "G_\lambda"] & & \mathrm{Coalg}(D) \arrow[ul, bend left, "F^\lambda"]
\end{tikzcd}$$

where

$$G_\lambda(TX \xrightarrow{h} X) = \langle TDX \xrightarrow{D_\lambda(h)} DX \xrightarrow{\delta_X} D^2X \rangle$$
$$F^\lambda(X \xrightarrow{k} DX) = \langle T^2X \xrightarrow{\mu_X} TX \xrightarrow{T_\lambda(k)} DTX \rangle.$$

$F^\lambda(0 \xrightarrow{!_{D0}} D0)$ is initial, and $G_\lambda(T1 \xrightarrow{!_{T1}} 1)$ is final.

Adequate semantics

By initiality of $F^\lambda(0 \overset{!_{D0}}{\to} D0)$ and finality of $G_\lambda(T1 \overset{!_{T1}}{\to} 1)$:

$$\begin{tikzcd}
T^2 0 \arrow[r, dashed, "T\mathtt{op}=T\mathtt{de}"] \arrow[d, "\mu_0"'] & TD1 \arrow[d, "D_\lambda(!_{T1})"] \\
T0 \arrow[r, dashed, "\mathtt{op}=\mathtt{de}"] \arrow[d, "T_\lambda(!_{D0})"'] & D1 \arrow[d, "\delta_1"] \\
DT0 \arrow[r, dashed, "D\mathtt{op}=D\mathtt{de}"] & D^2 1.
\end{tikzcd}$$

In particular, note

$$T_\Sigma 0 = \mu\Sigma \qquad D_B 1 = \nu B,$$

and compare above with the simpler picture before.

Example 2: GSOS

Given a finite set A, consider the following language

$$e \in T_\Sigma X ::= x \in X \mid 0 \mid a.e \mid e \mid\mid e$$

with operational semantics

$$\mathcal{R} = \{ \frac{}{a.e \xrightarrow{a} e} \quad , \quad \frac{e \xrightarrow{a} f}{e \mid\mid g \xrightarrow{a} f \mid\mid g} \quad , \quad \frac{e \xrightarrow{a} f}{g \mid\mid e \xrightarrow{a} g \mid\mid f} \}.$$

Above rules generate the smallest relation $R \subseteq T_\Sigma X \times A \times T_\Sigma X$ satisfying

$$(e, a, f) \in R \Leftrightarrow \exists \text{ Proof} : \frac{\cdots}{e \xrightarrow{a} f}.$$

Example 2: GSOS

In other words, if

$$\Sigma X = 1 + A \times X + X \times X \qquad BX = (\mathcal{P}_{\mathrm{fi}} X)^A = (1 + \mathcal{P}_f X)^A$$

then $(T_\Sigma X, R)$ is a B-coalgebra.

Same question as before:

How do we ensure an adequate semantics?

Typically answered by:

Constraints on the format of $\mathcal{R}$, such as GSOS.

Example 2: GSOS $\Rightarrow$ DL (very short)

$$
\begin{tikzcd}
\mathcal{R} \arrow[d] \\
\rho : \Sigma(\mathsf{Id} \times B) \Rightarrow BT_\Sigma \arrow[d] \\
\varrho : \Sigma(T_\Sigma \times BT_\Sigma) \Rightarrow BT_\Sigma \arrow[d] \\
(T_\Sigma)^\varrho : \mathsf{Coalg}(B) \to \mathsf{Coalg}(B) \arrow[d] \\
\lambda : T_\Sigma D_B \Rightarrow D_B T_\Sigma .
\end{tikzcd}
$$