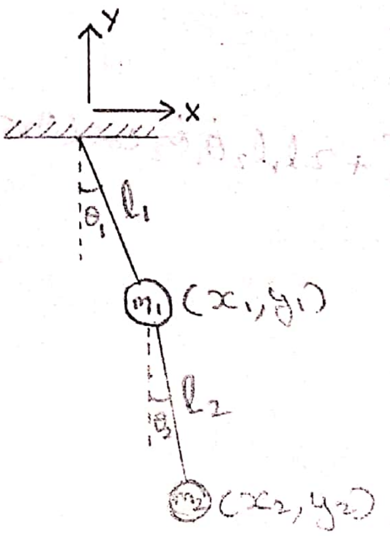
A Simple Double Pendulum.



Assumptions :

- The pendulum wires are considered to be massless and rigid
- The ends of the pendulum act as point masses

By trignomentry we get position equation

$x_1 = l_1 . \sin\theta_1$ — (1) $x_2 = l_1 . \sin\theta_1 + l_2 \sin\theta_2$ — (3)

$y_1 = -l_1 . \cos\theta_1$ — (2) $y_2 = -l_1 . \cos\theta_1 - l_2 \cos\theta_2$ — (4)

on differentiating these equations we obtain the velocities

$\dot{x}_1 = l_1 \dot{\theta}_1 \cos\theta_1$ — (5) $\dot{x}_2 = l_1 \dot{\theta}_1 \cos\theta_1 + l_2 \dot{\theta}_2 \cos\theta_2$ — (7)

$\dot{y}_1 = \pm l_1 . \dot{\theta}_1 \sin\theta_1$ — (6) $\dot{y}_2 = l_1 \dot{\theta}_1 \sin\theta_1 + l_2 \dot{\theta}_2 \sin\theta_2$ — (8)

Now, the Lagrangian function is

L = Kinetic energy − potential energy — (9)

Kinetic energy (k.e) $= \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$ — (10)

k.e $= \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$ — (11)

upon substituting equations (5), (6), (7), (8) in (11)

$$K.e = \frac{1}{2}m_1 l_1^2\dot{\theta}_1^2\cos^2\theta_1 + \frac{1}{2}m_1 l_1^2\dot{\theta}_1^2\sin^2\theta_1 + \frac{1}{2}m_2 l_1^2\dot{\theta}_1^2\cos^2\theta_1$$
$$+ \frac{1}{2}m_2 l_2^2\dot{\theta}_2^2\cos^2\theta_2 + \frac{1}{2}m_2 \cdot 2 l_1\dot{\theta}_1\cos\theta_1 l_2\dot{\theta}_2\cos\theta_2$$
$$+ \frac{1}{2}m_2 l_1^2\dot{\theta}_1^2\sin^2\theta_1 + \frac{1}{2}m_2 l_2^2\dot{\theta}_2^2\sin^2\theta_2$$
$$+ \frac{1}{2}m_2 \cdot 2 l_1\dot{\theta}_1\sin\theta_1 l_2\dot{\theta}_2\sin\theta_2$$

$$\Rightarrow K.e = \frac{1}{2}m_1 l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2\left(l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2)\right) \quad -(12)$$

Note :

$V_1 = \sqrt{\dot{x}_1^2 + \dot{y}_1^2}$

$\sin^2\theta + \cos^2\theta = 1$

$\cos(A-B) = \cos A\cos B + \sin A\sin B$

potential energy (p.e) $= m_1 g y_1 + m_2 g y_2$

$$p.e = m_1 g(-l_1\cos\theta_1) + m_2 g(-l_1\cos\theta_1 - l_2\cos\theta_2)$$
$$p.e = -(m_1+m_2) g l_1\cos\theta_1 - m_2 g l_2\cos\theta_2 \quad -(13)$$

Substituting equations (12) & (13) in equation (9)

$$L = k.e - p.e$$
$$\Rightarrow L = \frac{1}{2}m_1 l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2\left(l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2)\right)$$
$$+ (m_1+m_2) g l_1\cos\theta_1 + m_2 g l_2\cos\theta_2 \quad -(14)$$

Lagrange's equation : $\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i}$ $\quad -(15)$

where $q_i \rightarrow$ generalized coordinates

$Q_i \rightarrow$ generalized forces

θ_1 : $\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial \dot{\theta}_1} = m_1 l_1^2 \dot{\theta}_1^2 + m_2 l_1^2 \dot{\theta}_1^2 + m_2 l_1 l_2 \cos(\theta_1 - \theta_2)$ — (16)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = (m_1 + m_2) l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2)$$
$$- m_2 l_1 l_2 \dot{\theta}_2 \sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2) \quad -(17)$$

$$\frac{\partial L}{\partial \theta_1} = -l_1 g(m_1 + m_2)\sin\theta_1 - m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) \quad -(18)$$

Since the external forces is zero

$Q_i = 0$.

$$\Rightarrow (m_1 + m_2) l_1 \ddot{\theta}_1 + m_2 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) - m_2 l_2 \dot{\theta}_2 \sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2)$$
$$+ (m_1 + m_2) g \sin\theta_1 = 0 \quad -(19)$$

Similarly for θ_2

$$m_2 l_2 \ddot{\theta}_2 + m_2 l_1 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - m_2 l_1 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2)$$
$$+ mg \sin\theta_2 = 0 \quad -(20)$$

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