Formulation of Image Recovery Problem into Eckstein's Algorithm

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1 Objective

Formulate the image recovery problem below into Eckstein's algorithm

$$\begin{aligned}\min_{x_1\in\mathbb{R}^N, x_2\in\mathbb{R}^N} \quad & \sum_{k=1}^{N}\phi_{1,k}(\langle x_1|e_{1,k}\rangle) + \sum_{k=1}^{N}\phi_{2,k}(\langle x_2|e_{2,k}\rangle) + \\ & \frac{1}{2\sigma_1^2}||\mathcal{L}_1x_1 - z_1||^2 + \frac{1}{2\sigma_2^2}||\mathcal{L}_2x_2 - z_2||^2 + \frac{\vartheta}{2}||x_1 - Dx_2||^2\end{aligned} \tag{1}$$

where $\phi_{i,k} = \mu_{i,k}|\cdot|$ and $e_{i,k}$ are symlet wavelet orthogonal bases for $i \in \{1,2\}$ and $k \in [N]$

2 Formulation

Let $I = \{1,2\}$ and $K = \{1,2,3\}$.
For an image of size mxn, let $N = m \cdot n$ and define $\mathscr{H}_1 = \mathscr{H}_2 = \mathscr{G}_1 = \mathscr{G}_2 = \mathscr{G}_3 = \mathbb{R}^N$
$\forall i \in I$ and $\forall k \in K$, let $f_i : \mathscr{H}_i \to]-\infty,+\infty]$ and $g_k : \mathscr{G}_k \to]-\infty,+\infty]$ as

$f_1 : x_1 \longmapsto \langle(\mu_{1,k})_{k\in N} \,|\, \mathrm{DWT}(x_1)\rangle$
$f_2 : x_2 \longmapsto \langle(\mu_{2,k})_{k\in N} \,|\, \mathrm{DWT}(x_2)\rangle$
$g_1 : y_1 \longmapsto \frac{1}{2\sigma_1^2}||y_1 - z_1||^2$
$g_2 : y_2 \longmapsto \frac{1}{2\sigma_2^2}||y_2 - z_2||^2$
$g_3 : y_3 \longmapsto \frac{\vartheta}{2}||y_3||^2$

where $y_k = \sum_{i\in I}(L_{ki} \cdot x_i)$

where L_{ki} are functions : $\mathscr{H}_i \to \mathscr{G}_k$

Thus,
$L_{11} = \mathcal{L}_1$, $L_{12} = 0$,
$L_{21} = 0$, $L_{22} = \mathcal{L}_2$,
$L_{31} = 1$, $L_{32} = -D$ with $\mathcal{L}_1, \mathcal{L}_2$ and D being the same as in (1) and $\mathcal{L}_1^* = \mathcal{L}_1$, $\mathcal{L}_2^* = \mathcal{L}_2$