Asynchronous Non-Smooth Optimisation via Projective Splitting

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Outline

Introduction

In our project, we minimize a **sum of separable nonsmooth convex functions** with **nonsmooth convex coupling terms** which utilise a linear mixture of the components.

We utilise the **Projective Splitting Algorithm**[1] for solving our problem which uses the benefits of Asynchronous Programming, Iterative Block-Activation approach, nonsmooth functions and Splitting.

Our goal involves the implementation of the algorithm and investigation of its nature with different Hyperparameters, application to problems such as **Image Recovery** and **Group-Sparse Binary Classification**.

[1] Asynchronous Block-Iterative Primal-Dual Decomposition Methods for Monotone Inclusions - Patrick L. Combettes and Jonathan Eckstein, 2018

Definitions

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(e.g. $f(x) = x^2$, ReLU, Hinge loss, $\|Ax + b\|$). [3]

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The **direct sum**, sometimes called a product space, is denoted as $\bigoplus_{i=1}^{m} \mathcal{H}_i = \mathcal{H}_1 \times \ldots \times \mathcal{H}_m$, where $\mathcal{H}_i$ represents the i^{th} Hilbert space.

The inner product on the direct sum is defined as follows:

$$\langle (x_1, \ldots, x_m), (y_1, \ldots, y_m) \rangle = \sum_{i=1}^{m} \langle x_i, y_i \rangle_{\mathcal{H}_i}$$

where $(x_1, \ldots, x_m)$, $(y_1, \ldots, y_m) \in \bigoplus_{i=1}^{m} \mathcal{H}_i$.

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The **indicator function** of set C, denoted as $\iota_C : \mathcal{H} \to [-\infty, +\infty]$ is defined as:

$$\iota_C(x) = \begin{cases} 0 & \text{if } x \in C \\ +\infty & \text{if } x \notin C \end{cases} .$$

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Intuitively - proxes provide a way to minimise the function value of f over its domain, without staying too far from x (penalised by the norm term).

Proxes are especially useful because iteratively applying the Proximal Operator on x_n ensures that we reach a global minimizer *(Martinet, 1970)*.

This gives rise to iterative algorithms that converge to a global minimizer for functions in $\Gamma_0(\mathcal{H})$.

Specifically, we implement and analyse one such algorithm - Algorithm 4 mentioned in [2]. We mention the advantages of the specific algorithm later.

Problem Statement

$$\underset{(x_i)_{i\in I}\in\bigoplus \mathcal{H}_i}{\text{minimize:}} \quad \sum_{i\in I} f_i(x_i) + \sum_{k\in K} g_k\Big(\sum_{i\in I}(L_{ki}\cdot x_i)\Big) \qquad (\ddot\smile)$$

where $(\mathcal{H}_i)_{i\in I}$ and $(\mathcal{G}_k)_{k\in K}$ are real Hilbert spaces with
$I = \{1, .., m\}$ and $K = \{1, .., p\}$
$L_{ki} : \mathcal{H}_i \to \mathcal{G}_k$ are linear
$f_i \in \Gamma_0(\mathcal{H}_i)$, $g_k \in \Gamma_0(\mathcal{G}_k)$

This objective function above is minimised along with its dual problem.

Formulating some well known examples

- **Convex Feasibility Problem** can be stated as follows: Given non-empty and closed convex sets $C_1, C_2, \ldots, C_n$ in a vector space $\mathcal{H}$, find a point x such that:

$$x \in C_1 \cap C_2 \cap \ldots \cap C_n.$$

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We can formulate this in terms of our problem (☺) as :
$m = 1$
$f_1 = \mathbf{1}_{C_1}$
$p = n - 1$
$g_k = \mathbf{1}_{C_{k+1}}$
$\forall k \in K, i \in I \;\; L_{ki} = \mathbf{I}d$ (Identity)
which gives us $\underset{x_1 \in \mathcal{H}}{\text{minimize}} \quad \sum_{i \in I} \iota_{C_i}(x_1)$

- **Lasso Problem**

$$\underset{\beta \in \mathbb{R}^n}{\text{minimize}} \ \frac{1}{2n} \|y - X\beta\|_2^2 + \lambda \|\beta\|_1 \ .$$

- **Linear SVM Training**

$$\underset{x \in \mathcal{H}}{\text{minimize}} \ \sum_{d \in \mathcal{D}_1} \max\{0, 1 - d \cdot |x|\} + \sum_{d \in \mathcal{D}_2} \max\{0, 1 + h \cdot |x|\} + \lambda \cdot \|x\|_1.$$

- **Spectroscopic Image Recovery** (spoiler alert) : As we will see later, we can construct a loss function for degraded images if we know the degradation operator and minimise it to recover images.
  These can all, also be formulated as our problem (☺).

Solution Methods

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The algorithm solves the problem without adding restrictions such as smoothness, differentiability or strong convexity.

The algorithm offers a number of advantages over pre-existing algorithms which are as follows.

Advantage over pre-existing algorithms

Our proximal algorithm offers the following features which can not be found simultaneously in the pre-existing algorithms:

1. **Splitting:** The algorithm allows us to find the minimizer without calculating the prox of $\sum_{i \in I} f_i(x_i) + \sum_{k \in K} g_k(y_k)$ when the individual prox for f_i and g_k are known using splitting, since the proximity operator of composite functions are typically not known.

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2. **Block-Activation:** Using Block-Activation, we activate the prox of only a block of functions $f_1, \ldots f_m, g_1, \ldots g_p$ at each iteration. This is highly efficient in contrast to algorithms which activate all the functions in every single iteration.

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3. **Asynchrony:** The algorithm is asynchronous i.e., it is implementable on parallel architectures. This enables us to move to the calculation of next set of prox on a core without waiting for the other cores to finish the calculation.

Algorithm Overview

The overall idea is to iteratively refine current half-space by generating points in the graphs of monotone operators (In this case our subdifferentials of functions) until it successfully encapsulates the Kuhn-Tucker solution set.
The coordination step of the method is to project the current iterate onto the recently constructed half-space.
The advantages of this approach are as follows:

1. Does not require $(\|L_{ki}\|)_{i,k}$.
2. Does not involve the inversion of linear operators.
3. No additional assumptions are imposed on the operators present.

Motivation

- The Combettes-Eckstein algorithm has been there for a long time but no one has ever implemented it in the full setting.
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Our goals are as follows :

- Implement the algorithm and benchmark the performance.
- Open Source the algorithm as there are *very* few pre-existing algorithms of this class.
- Compare the convergence behaviour of our algorithm with different hyperparameter settings and find out the best initial conditions.

Implementation

- The algorithm is implemented in **Julia**. The **dependencies** include Julia packages - `LinearAlgebra`, `Wavelets`, `Images`, `Random`, `FileIO`, `ImageView` and `ProximalOperators`.
- Features in our Implementation:
  - **Flexible block selection strategies** that can be input by the users. Some of the block selection strategies are as follows:
    1. **Cyclic Strategy**
    2. **Random with Pooling**
    3. **Greedy Block Selection** based on expensiveness of Proxes.
  - Linear Operator L_{ki} (as mentioned in (☺)), can be **input as a function** (along with its adjoint operator) instead of matrices. This allows faster computations and less storage space.
  - Works correctly for vector product spaces where **each Hilbert Space can be of different dimensions**.
  - **Flexible γ_n, μ_n strategies** that the user can choose from.
  - **Asynchronous** in nature.

- Time taken Asynchronously for independent tasks vs Sync.

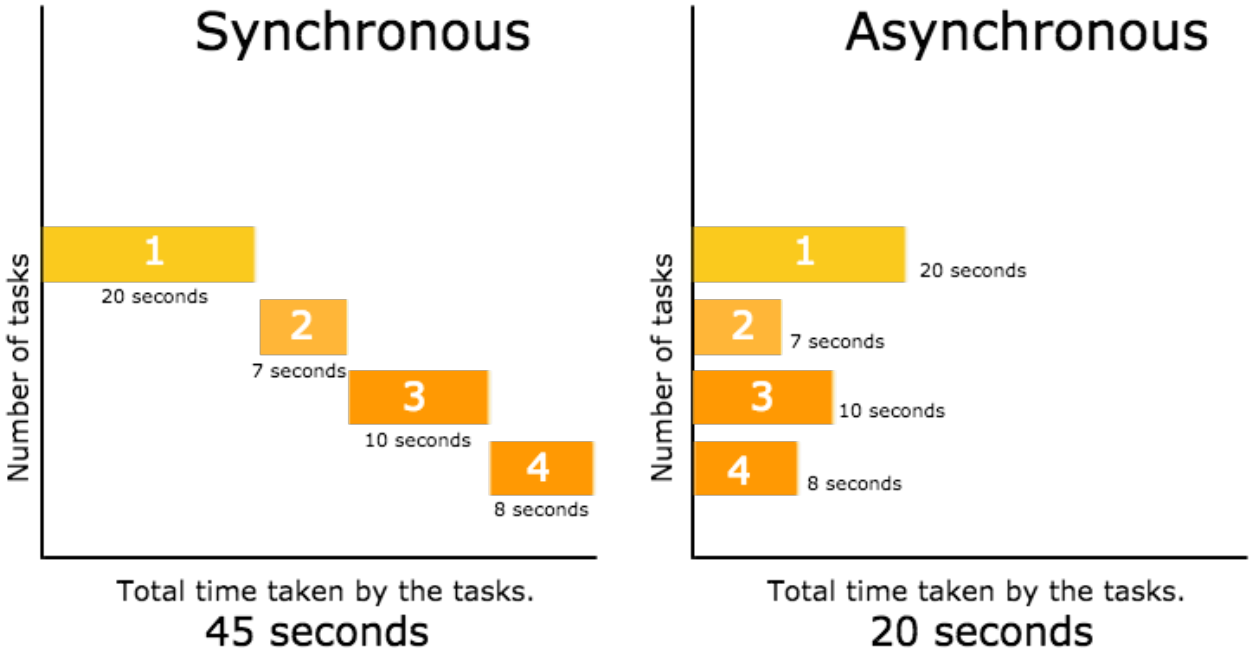
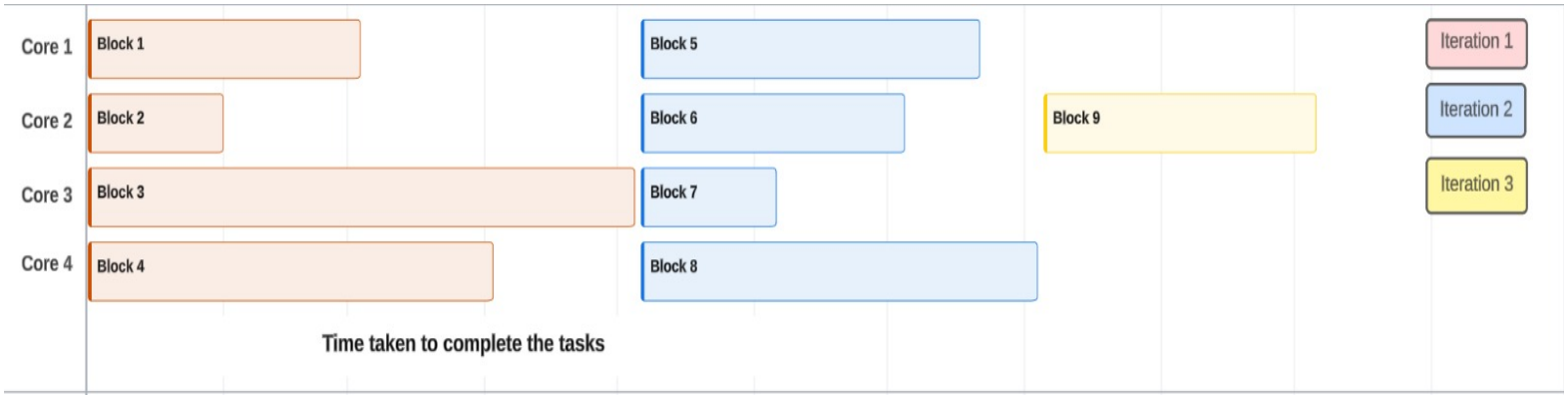
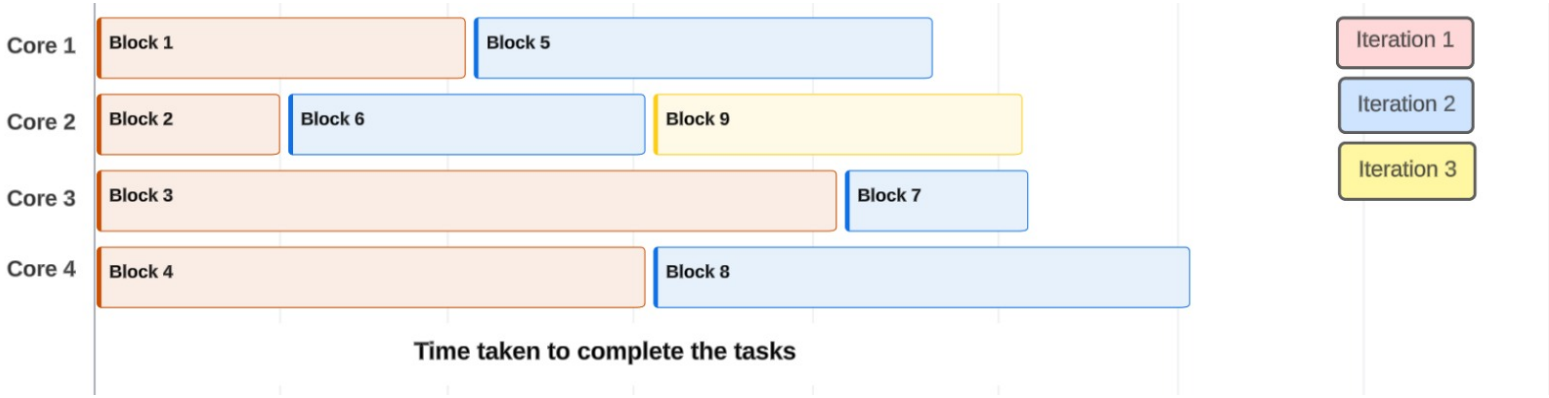


Diagram by Adrian Mejia

- The graphic below depicts the difference between block synchronous and block asynchronous algorithm



Block Synchronous Algorithm



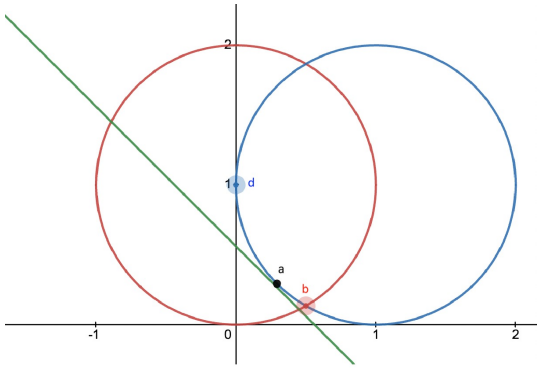
Block Asynchronous Algorithm

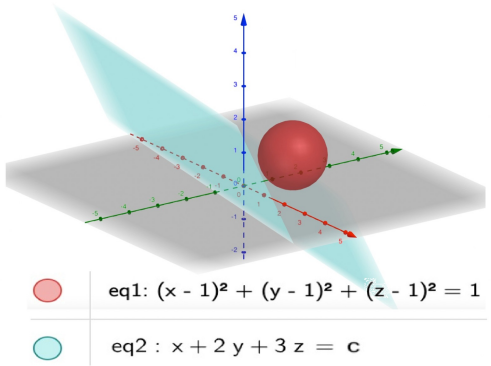
Examples Run

1. Let S_1, S_2 be a circles of radii 1 centered at $(0, 1)$ and $(1, 1)$ respectively in $\mathbb{R}^2$. Let S_3 be a sphere centered at $(1, 1, 1)$ in $\mathbb{R}^3$.
Problem:

$$\underset{\substack{x_1, x_2, x_3 \\ x_1, x_2 \in S_1 \cap S_2, x_3 \in S_3}}{\text{minimize}} \quad \langle [1, 0] \mid x_1 \rangle + \langle [0, 1] \mid x_2 \rangle + \langle [1, 2, 3] \mid x_3 \rangle + \iota_{(x_1 = x_2)}$$

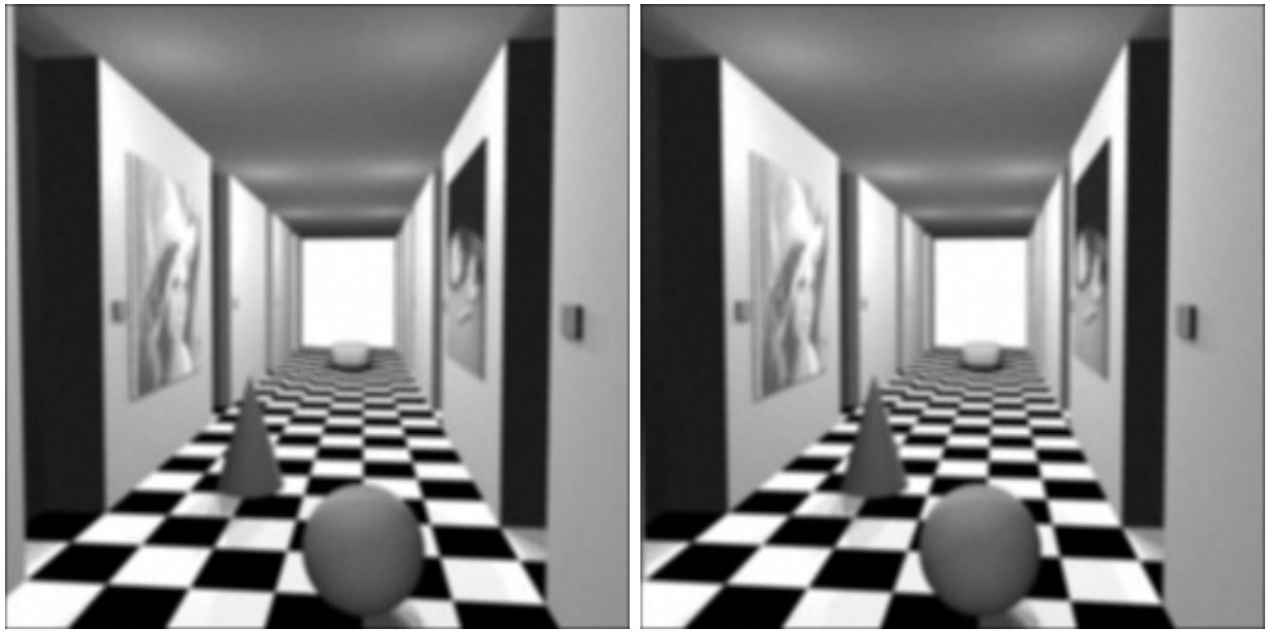
We get two slightly different problems depending on whether we include the indicator function term or not.





- Problem-1 is a combination of two problems (diagrams above) in different dimensions ($\mathbb{R}^2$ and $\mathbb{R}^3$) solved simultaneously.
- The answer is accurate within $\|x - x^*\| < 10^{-11}$ for both problems.
- True Values:
  1. Without $\iota_{(x_1=x_2)}$: $x_1 = [0, 1], x_2 = [\frac{1}{2}, 1 - \sqrt{\frac{3}{4}}]$, $x_3 = [1 - \frac{1}{\sqrt{14}}, 1 - \frac{2}{\sqrt{14}}, 1 - \frac{3}{\sqrt{14}}]$
  2. With $\iota_{(x_1=x_2)}$: $x_1 = x_2 = [1 - \sqrt{\frac{1}{2}}, 1 - \sqrt{\frac{1}{2}}]$, $x_3 = [1 - \frac{1}{\sqrt{14}}, 1 - \frac{2}{\sqrt{14}}, 1 - \frac{3}{\sqrt{14}}]$
- We reach the values within 10^{-11} error in < 1200 iterations

2. Consider the problem of restoring a pair of N-pixel Stereoscopic images $\overline{x_1}, \overline{x_2} \in \mathbb{R}^N$
The observations consist of degraded images
$z_1 = \mathcal{L}_1 \overline{x_1} + w_1$ and $z_2 = \mathcal{L}_2 \overline{x_2} + w_1$.
Where $\mathcal{L}_1$ and $\mathcal{L}_2$ are degradation operators (Gaussian Blurring in our case)



z_1 and z_2 ,i.e., the blurred images

We use the objective function mentioned in [1] to recover our images

$$\min_{x_1 \in \mathbb{R}^N, x_2 \in \mathbb{R}^N} \quad \sum_{k=1}^{N} \phi_{1,k}(\langle x_1 | e_{1,k} \rangle) + \sum_{k=1}^{N} \phi_{2,k}(\langle x_2 | e_{2,k} \rangle) + \frac{1}{2\sigma_1^2} \|\mathcal{L}_1 x_1 - z_1\|^2 + \frac{1}{2\sigma_2^2} \|\mathcal{L}_2 x_2 - z_2\|^2 + \frac{\vartheta}{2} \|x_1 - D x_2\|^2 \tag{1}$$

where $\phi_{i,k} = \mu_{i,k} |\cdot|$ for $i \in \{1, 2\}$ and $k \in [N]$

This can be formulated as (☺) by setting:

$f_1 : x_1 \longmapsto \langle (\mu_{1,k})_{k \in N} \,|\, \mathrm{DWT}(x_1)^2 \rangle$

$f_2 : x_2 \longmapsto \langle (\mu_{2,k})_{k \in N} \,|\, \mathrm{DWT}(x_2) \rangle$

$g_1 : y_1 \longmapsto \frac{1}{2\sigma_1^2} \|y_1 - z_1\|^2$

$g_2 : y_2 \longmapsto \frac{1}{2\sigma_2^2} \|y_2 - z_2\|^2$

$g_3 : y_3 \longmapsto \frac{\vartheta}{2} \|y_3\|^2$

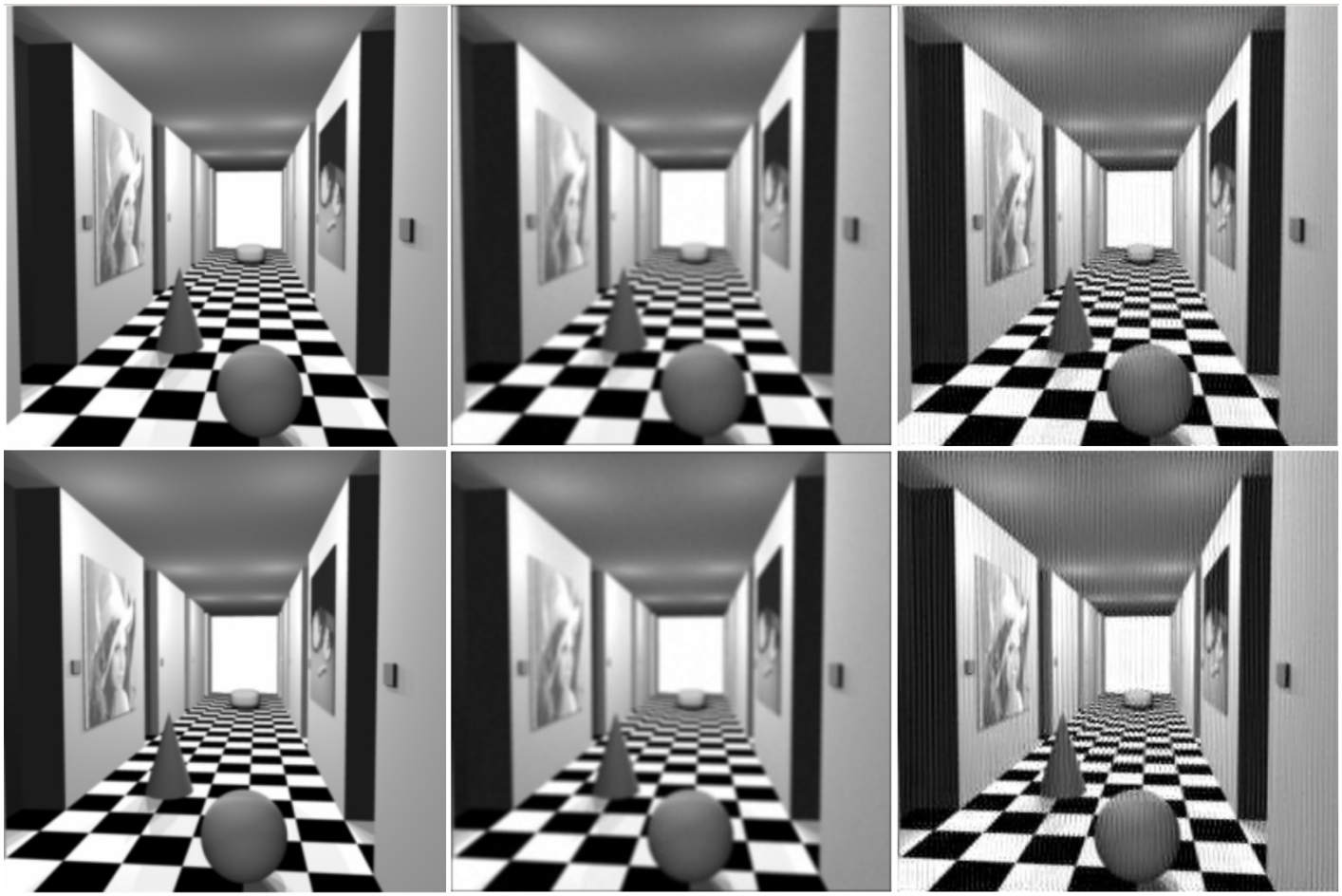
where $y_k = \sum_{i \in I} (L_{ki} \cdot x_i)$

where L_{ki} are functions : $H_i \to G_k$

$$\mathrm{L} = \begin{pmatrix} \mathcal{L}_1 & 0 \\ 0 & \mathcal{L}_2 \\ \mathbf{I_d} & -D \end{pmatrix}$$

[2] *DWT* refers to discrete wavelet transform in the symmlet bases with 4 vaishing moments

Results



Images in order (top to bottom and left to right) : Original Left and Right images , Degraded(Blurred) left and right , Recovered left and right

$\mu = 0.1$; $\theta = 0.1$; $\sigma_{noise} = 0.01$; $\sigma_{kernel} = 3.0$

Results

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Results

The following 3x3 strategies were tested for the above example - 1

Hyperparameter Strategies			**Errors**[3]		
No.	**γ Strategy**	**μ strategy**	**x_1**	**x_2**	**x_3**
1	constant 1	constant 1	9.7e-15	1.4e-12	1.6e-11
2	constant 1	random constant	0.0	1.4e-12	1.6e-11
3	constant 1	decreasing seq.	2.2e-16	0.00065	0.0297
4	random constant	constant 1	0.0482	0.01236	1.1e-7
5	random constant	random constant	2.2e-14	1.4e-12	1.6e-11
6	random constant	decreasing seq.	0.0136	0.0018	0.0178
7	**decreasing seq.**	constant 1	2.2e-16	1.4e-12	1.6e-11
8	**decreasing seq.**	random constant	2.8e-14	1.4e-12	1.6e-11
9	**decreasing seq.**	decreasing seq.	4.1e-10	3.7e-7	2.0e-11

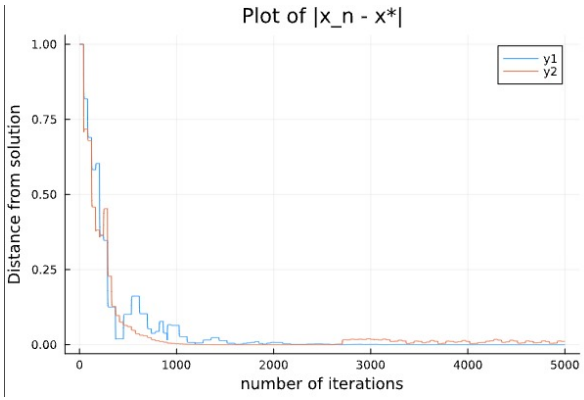
[3] Errors are the L2-Norm of the absolute difference between true solution and the solution obtained with the strategies given

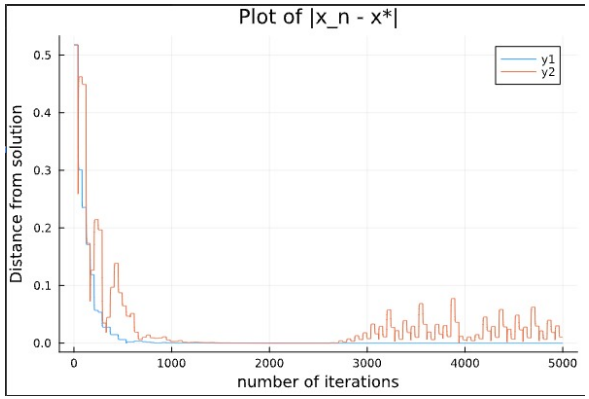
Results

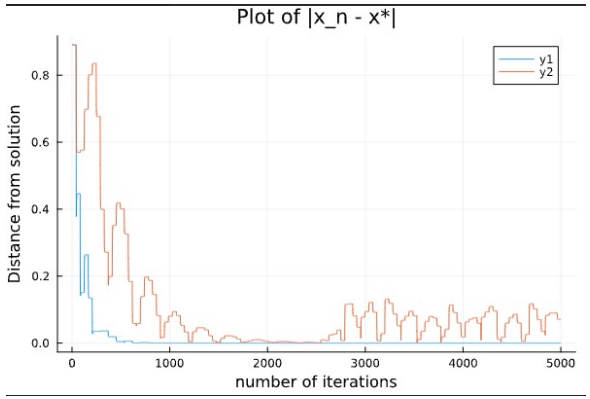
Note:

1. **Constant 1:** In this strategy, gamma or mu for all the functions equals 1
2. **Random Constant:** In this strategy, gamma or mu is a random constant in the range $[\epsilon, \frac{1}{\epsilon}]$ and is different for different functions
3. **Decreasing sequence:** In this strategy, gamma and mu form a decreasing sequence where the values start from $\frac{1}{\epsilon}$ and decrease with a common difference of 0.1 until it reaches the value, ϵ from where it becomes constant

Results







Plots for x_1, x_2, x_3 respectively where the distance from the minima is plotted at each iteration

y_1 : γ: Decreasing Sequence, μ: Constant 1

y_2 : γ: Random Constant, μ: Decreasing Sequence .

Thank you for your time!

References

L.M. Briceno-Arias, P.L. Combettes, J.-C. Pesquet, and N. Pustelnik.
Proximal algorithms for multicomponent image recovery problems.
Journal of Mathematical Imaging and Vision, 2011.

Jonathan Eckstein Patrick L. Combettes.
Asynchronous block-iterative primal-dual decomposition methods for monotone inclusions.
2018.

Zev Woodstock.
A crash course on proximity operators and nonsmooth convex optimization.
May, 2023.

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