Construction of Functions from Nonlinear Transformations

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Ph. D. defense, May 17, 2021

Acknowledgment: National Science Foundation Grant DGE-1746939

Outline: articles produced by this thesis

- P. L. Combettes and ZCW, A fixed point framework for recovering signals from nonlinear transformations,

  2020 Proc. Eur. Signal Process. Soc., pp. 2120–2124. Amsterdam, The Netherlands, Jan. 18–22, 2021.

- P. L. Combettes and ZCW, Reconstruction of functions from prescribed proximal points,

  J. Approx. Theory, resubmitted with minor revisions.

- P. L. Combettes and ZCW, A variational inequality model for the construction of signals from inconsistent nonlinear equations,

  SIAM J. Imaging Sci., submitted.

- M. N. Bùi, P. L. Combettes, and ZCW, Block-activated algorithms for multicomponent fully nonsmooth minimization,

  2021 Proc. Eur. Signal Process. Soc., submitted. Expanded version in preparation.

Outline

- Setting and history
- Firmly nonexpansive equations (Chapters 2, 3, and 4)
- Chapter 2 (EUSIPCO '20): Feasibility
- Chapter 3 (J. Approx. Theory): Best approximation
- Chapter 4 (SIAM J. Imaging Sci.): Inconsistent feasibility
- Chapter 5 (EUSIPCO '21): Case study of recent block-iterative minimization algorithms
- Future work

Motivation: the linear setting

Youla's Model, 1978

Let U_1 and U_2 be closed vector subspaces. Given $p \in U_2$,

$$\text{find } \ x \in U_1 \ \text{ such that } \ \mathrm{proj}_{U_2} x = p.$$

This can be solved using projection methods.

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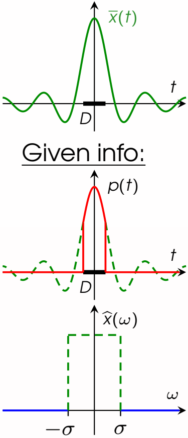
Example: Bandlimited extrapolation (Papoulis, 1975)

Let $\sigma > 0$, $D \subset \mathbb{R}$, and $p = \overline{x}|_D$.
Goal: find x such that

$$\begin{cases} p = x|_D \ a.e. \\ \widehat{x} = 0 \quad \text{outside of} \quad [-\sigma, \sigma] \ a.e. \end{cases}$$

Original signal:

Given info:



Extension of the linear setting

Combettes & Reyes, 2010

Let K be a finite set. For every $k \in K$, let U_k be a closed vector subspace of $\mathcal{H}$, and let $p_k \in U_k$. The goal is to

$$\text{find} \quad x \in \mathcal{H} \quad \text{such that} \quad (\forall k \in K) \quad \operatorname{proj}_{U_k} x = p_k.$$

- Projection methods are available for finding solutions.
- This model captures linear a priori constraints, since for any vector subspace $U \subset \mathcal{H}$, $x \in U \Leftrightarrow \operatorname{proj}_{U^\perp} x = 0$.

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However, there are many applications in which we seek to solve

$$(\forall k \in K) \quad F_k x = p_k,$$

where $(F_k)_{k \in K}$ are nonlinear operators on a real Hilbert space $\mathcal{H}$.

Our setting

Let $\mathcal{H}$ be a real Hilbert space. The operator $F\colon \mathcal{H} \to \mathcal{H}$ is firmly nonexpansive if

$$(\forall (x,y) \in \mathcal{H}^2) \quad \|Fx - Fy\|^2 \leqslant \|x - y\|^2 - \|(\mathrm{Id} - F)x - (\mathrm{Id} - F)y\|^2.$$

- General enough to capture many applications.
- Sufficiently structured to yield tractable, efficient algorithms which converge to a solution from any initial point.
- Special case: Projections onto closed convex sets.

These operators will appear in various contexts in Chapters 2, 3, and 4.

Roadblocks

Let $F\colon \mathcal{H} \to \mathcal{H}$ be firmly nonexpansive.

$$\text{How do we enforce that } Fx = p?$$

Difficulties:

- $\|F\cdot - p\|$ is typically nonconvex.
    - Convex minimization tools cannot be used.
    - Guarantees of convergence to a solution are rare.

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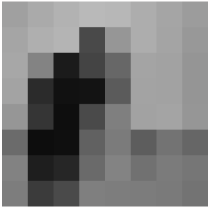
- $\|F \cdot - p\|$ is typically nonconvex.
  - Convex minimization tools cannot be used.
  - Guarantees of convergence to a solution are rare.
- In general, projecting onto $F^{-1}(\{p\})$ is not possible.
  - Cannot be solved using projection methods.

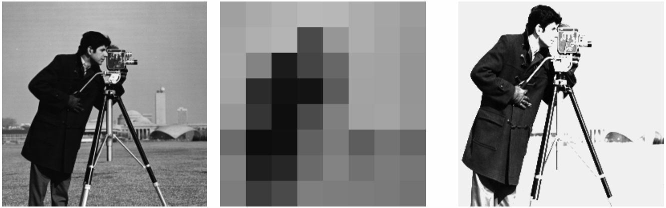
Examples: projections

- Dimension reduction



Examples: projections

- Dimension reduction and saturation



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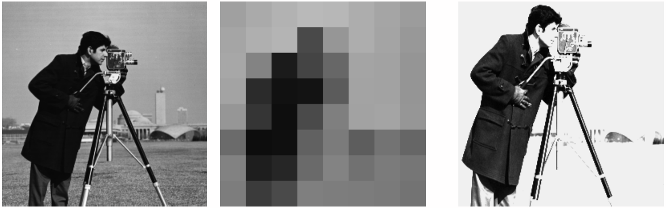
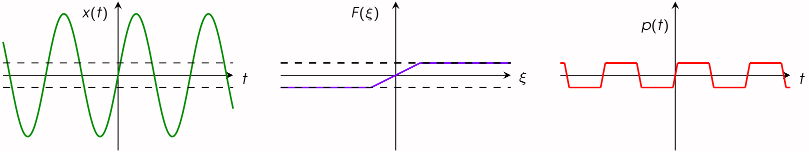


- Hard clipping



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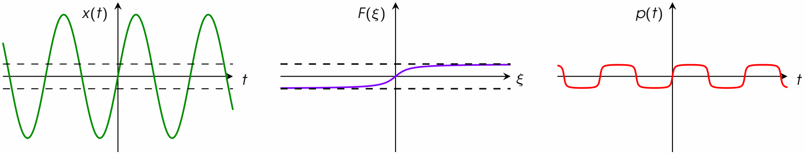

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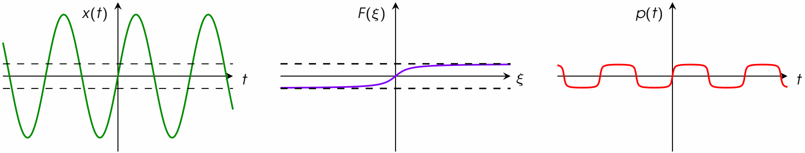

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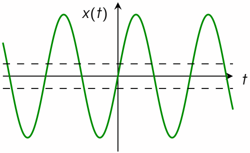

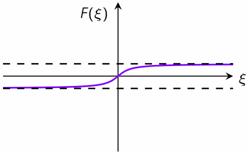

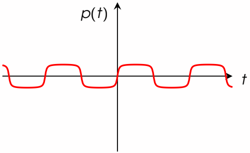

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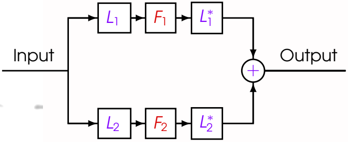

Examples: "proxification"

Definition: Given $Q\colon \mathcal{H} \to \mathcal{H}$ and $q \in \operatorname{ran} Q$, (Q, q) is proxifiable if there exists $F\colon \mathcal{H} \to \mathcal{H}$ which is firmly nonexpansive and $p \in \operatorname{ran} F$ such that

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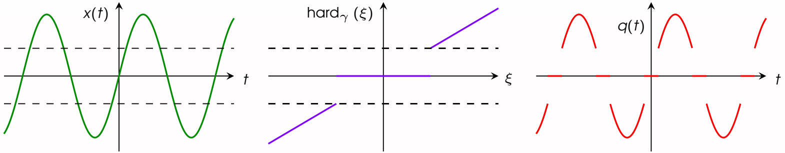
Example: Hard thresholding at level $\gamma > 0$

$$\operatorname{hard}_\gamma : \xi \mapsto \begin{cases} \xi, & \text{if } |\xi| > \gamma; \\ 0, & \text{if } |\xi| \leqslant \gamma, \end{cases} \tag{1}$$



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Example: Hard thresholding at level $\gamma > 0$ and soft thresholding

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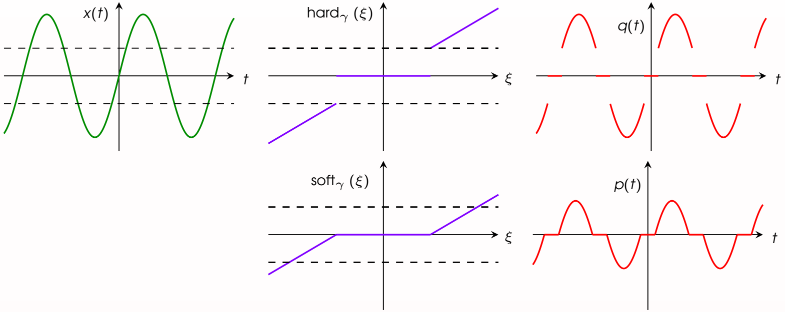

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Let $\mathcal{H} = \mathbb{R}^{N \times M}$, set $s = \min\{N, M\}$, let $\gamma > 0$, and denote the singular value decomposition of $x \in \mathcal{H}$ by

$$x = U_x \ \mathrm{diag}\left(\sigma_1(x), \dots, \sigma_s(x)\right) V_x^\top . \tag{2}$$

A **low rank approximation** q of x is

$$U_x \ \mathrm{diag}\Big(\ \mathrm{hard}_\gamma \left(\sigma_1(x)\right), \dots, \ \mathrm{hard}_\gamma \left(\sigma_s(x)\right) \Big) V_x^\top . \tag{3}$$

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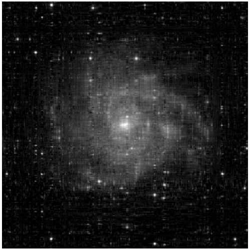

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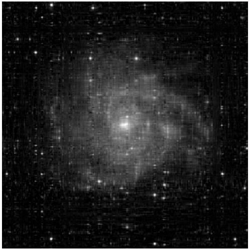
$F \colon \mathcal{H} \to \mathcal{H} \colon x \mapsto$

$U_x \ \operatorname{diag}\left(\ \operatorname{soft}_\gamma \left(\sigma_1(x)\right), \ldots, \ \operatorname{soft}_\gamma \left(\sigma_s(x)\right)\right) V_x^\top$,

and construct p by shifting the nonzero singular values of q by $-\gamma$.



Chapter 2

We consider the task of recovering a signal $\overline{x}$ in a real Hilbert space $\mathcal{H}$ from

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Problem 1

$$\text{find} \quad \underbrace{x \in \bigcap_{j\in J} C_j}_{\text{Prior information}} \quad \text{such that} \quad (\forall k \in K) \quad \underbrace{F_k x = p_k}_{\text{Transformations}},$$

assuming at least one solution exists.

Foregoing minimization: directly to fixed-points.

Main ingredients:

- For every $k \in K$, set $T_k = \mathrm{Id} - F_k + p_k$.

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- $\overline{x}$ solves the main problem if and only if

$$\overline{x} \in \left(\bigcap_{j\in J} \mathrm{Fix}\, \mathrm{proj}_{C_j}\right) \cap \left(\bigcap_{k\in K} \mathrm{Fix}\, T_k\right),$$

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- Adapting an algorithm from

  P. L. Combettes, Quasi-Fejérian analysis of some optimization algorithms, in: *Inherently Parallel Algorithms in Feasibility and Optimization and Their Applications*, pp. 115–152, New York, NY: Elsevier, 2001.

  We can now solve our problem.

Algorithm

Let $x_0 \in \mathcal{H}$, let $0 < \varepsilon < 1/\mathrm{card}(J \cup K)$, and

for $n = 0, 1, \ldots$
- $\varnothing \neq I_n \subset J \cup K$
- $\{\omega_{i,n}\}_{i \in I_n} \subset [\varepsilon, 1]$, $\sum_{i \in I_n} \omega_{i,n} = 1$
- for every $i \in I_n$
  - if $i \in J$
    - $y_{i,n} = \mathrm{proj}_{C_i} x_n - x_n$
  - else $i \in K$
    - $y_{i,n} = p_i - F_i x_n$
  - $\nu_{i,n} = \|y_{i,n}\|$
- $\nu_n = \sum_{i \in I_n} \omega_{i,n} \nu_{i,n}^2$
- if $\nu_n = 0$
  - $x_{n+1} = x_n$
- else
  - $y_n = \sum_{i \in I_n} \omega_{i,n} y_{i,n}$
  - $\Lambda_n = \nu_n / \|y_n\|^2$
  - $\lambda_n \in [\varepsilon, (2 - \varepsilon)\Lambda_n]$
  - $x_{n+1} = x_n + \lambda_n y_n$.

- Block iterative
- Extrapolated
- the projector onto C_i can be approximated.

Theorem

Under a mild condition on the blocks $(I_n)_{n \in \mathbb{N}}$, weak convergence to a solution is guaranteed.

Numerics: audio processing

Setting: $\overline{x} \in \mathbb{R}^N$ ($N = 312,346$) is sampled at 44, 100 Hz (7.1 seconds).
Constraint: $x \in [-1, 1]^N$

Observations:

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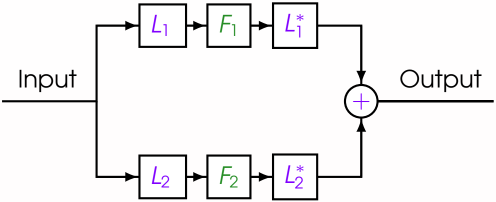
- p_2: Superposition of distorted recordings with echo and bandlimiting (400–3400 Hz).



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Problem 2 (Best approximation)

In the context of Problem 1, let $x_0 \in \mathcal{H}$ and let J and K be at most countable. The goal is to

$$\underset{x \in \mathcal{H}}{\text{minimize}} \ \ \|x - x_0\| \ \ \text{subject to} \begin{cases} x \in \bigcap_{j \in J} C_j; \\ (\forall k \in K) \quad F_k x = p_k. \end{cases} \tag{4}$$

- Examples of firmly nonexpansive operators appearing in signal and image processing.
- Constructive techniques for proxification.

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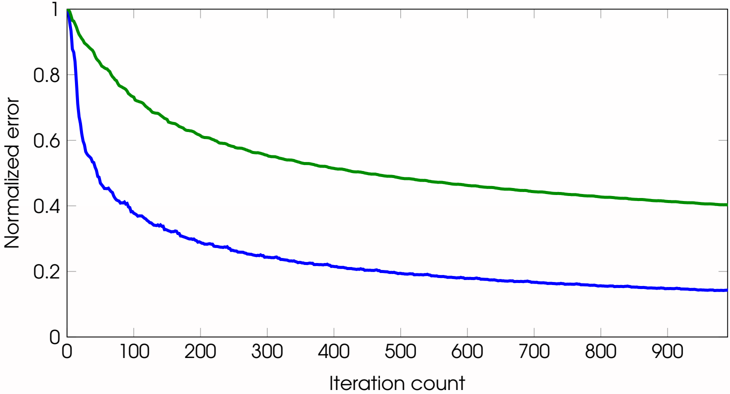
Solution to Problem 2

We present a new algorithm which is block-iterative and extrapolated. We prove that under a mild assumption, this algorithm converges strongly to the solution of (4).

An improved extrapolation scheme

In the presence of an affine constraint in Problem 2, the new algorithm has an improved extrapolation scheme.



Normalized error $\|x_n - x_\infty\|/\|x_0 - x_\infty\|$ versus iteration count n for new extrapolation versus old extrapolation.

Chapter 4

Let $C \subset \mathcal{H}$ be nonempty closed and convex and let I be finite. For every $i \in I$, let $\mathcal{G}_i$ be a real Hilbert space, let $p_i \in \mathcal{G}_i$, let $L_i \colon \mathcal{H} \to \mathcal{G}_i$ be a nonzero bounded linear operator, and let $F_i \colon \mathcal{G}_i \to \mathcal{G}_i$ be a firmly nonexpansive operator. The goal is to

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Let $(\omega_i)_{i\in I}$ be real numbers in $]0,1]$ such that $\sum_{i\in I} \omega_i = 1$.

$$\text{find } \; x \in C \; \text{ such that } \; (\forall y \in C) \;\; \sum_{i\in I} \omega_i \langle L_i(y-x) \mid F_i(L_i x) - p_i \rangle \geqslant 0.$$

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- If (5) has a solution, then it is equivalent to Problem 3.
- Problem 3 is guaranteed to possess solutions under mild conditions.

Intuition: relaxed problem

Example of Problem 3

Let $\beta > 0$ and let $f \colon \mathcal{H} \to \mathbb{R}$ be convex with a β^{-1}-Lipschitzian gradient. Set $F_1 = \beta \nabla f$, $p_1 = 0$, and $L_1 = \mathrm{Id}$. Then (5) is equivalent to

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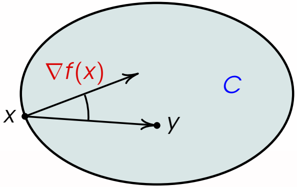
$$\text{find}\ \ x \in C\ \text{ such that }\ (\forall y \in C)\ \ \langle y - x \mid \nabla f(x)\rangle \geqslant 0,$$

i.e.,

$$\underset{x \in C}{\text{minimize}}\ \ f(x).$$



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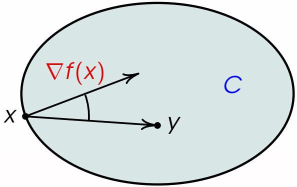

Solutions are guaranteed to exist when, e.g., C is bounded or f is coercive.

Existence results

Notation: N_C is the **normal cone** operator of C.

Proposition

Problem 3 admits a solution in each of the following instances.

1. $\sum_{i\in I}\omega_i L_i^* p_i \in ran(N_C + \sum_{i\in I}\omega_i L_i^* \circ F_i \circ L_i)$.
2. *C is bounded.*
3. $ranN_C + \sum_{i\in I}\omega_i L_i^*(ranF_i) = \mathcal{H}$.
4. *For some $i \in I$, L_i^* is surjective and one of the following holds:*
   1. $L_i^*(ranF_i) = \mathcal{H}$.
   2. *F_i is surjective.*
   3. $\|F_i(y)\| \to +\infty$ *as* $\|y\| \to +\infty$.
   4. *$ran(Id - F_i)$ is bounded.*
   5. *There exists a continuous convex function $g_i\colon \mathcal{G}_i \to \mathbb{R}$ such that $F_i = \mathrm{prox}_{g_i}$.*

Existence results

Proof idea: Problem 3 has a solution if and only if

$$\sum_{i\in I}\omega_i L_i^* p_i \in \mathrm{ran}\Big(\underbrace{N_C + \sum_{i\in I}\omega_i L_i^*\circ F_i\circ L_i}_{\text{maximally monotone}}\Big). \tag{6}$$

In this setting, we can apply a new Brézis-Haraux type theorem, which yields

$$\mathrm{int}\Big(\mathrm{ran}\Big(N_C + \sum_{i\in I}\omega_i L_i^*\circ F_i\circ L_i\Big)\Big) = \mathrm{int}\Big(\mathrm{ran}N_C + \sum_{i\in I}\omega_i L_i^*(\mathrm{ran}F_i)\Big). \tag{7}$$

The rest follows from surjectivity arguments.

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The rest follows from surjectivity arguments. For instance, if C is bounded, then $\mathrm{ran} N_C = \mathcal{H}$, so the operator in (6) is surjective.

Algorithm

Adapting an algorithm from

P. L. Combettes and L. E. Glaudin, Solving composite fixed point problems with block updates

Adv. Nonlinear Anal., vol. 10, pp. 1154–1177, 2021.

we arrive at a block-iterative solution method.

Let $x_0 \in \mathcal{H}$, let $\gamma \in]0, 2[$, and, for every $i \in I$, let $t_{i,-1} \in \mathcal{H}$ and set $\gamma_i = \gamma / \|L_i\|^2$. Iterate

$$
\begin{array}{l}
\text{for } n = 0, 1, \ldots \\
\left\lfloor
\begin{array}{l}
\varnothing \neq I_n \subset I \\
\text{for every } i \in I_n \\
\left\lfloor\; t_{i,n} = x_n - \gamma_i L_i^* \big(F_i(L_i x_n) - p_i \big) \right. \\
\text{for every } i \in I \smallsetminus I_n \\
\left\lfloor\; t_{i,n} = t_{i,n-1} \right. \\
x_{n+1} = \mathrm{proj}_C \left(\displaystyle\sum_{i=1}^{m} \omega_i t_{i,n} \right).
\end{array}
\right.
\end{array}
$$

Then under a mild condition on $(I_n)_{n \in \mathbb{N}}$, $(x_n)_{n \in \mathbb{N}}$ converges weakly to a solution to Problem 3.

Numerics: inconsistent image recovery

Experiment: $C = [0, 255]^N$ ($N = 256^2$), given noisy estimates of:

- Mean pixel value
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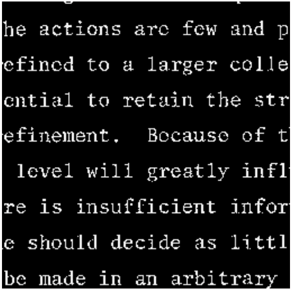
Numerics: promoting sparsity

Experiment: Given $C = [0, 255]^N$ ($N = 256$) and

- A low rank approximation
- $\bar{x}$ must be sparse.



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Experiment: Given $C = [0, 255]^N$ $(N = 256)$ and

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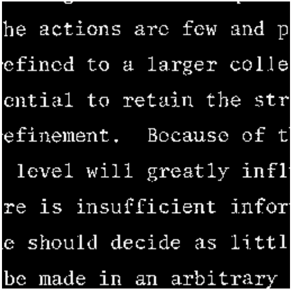
- $\overline{x}$ must be sparse. So, we set $\gamma = 1.5$, $F_2 = \mathrm{Id} - \mathrm{prox}_{\gamma\|\cdot\|_1} = \mathrm{proj}_{B_\infty(0;\gamma)}$ and $p_2 = 0$.

Motivation:
$F_2 x = p_2 \Leftrightarrow x \in \operatorname{argmin} \|\cdot\|_1$.



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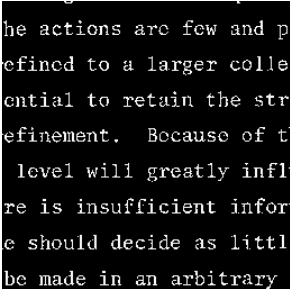
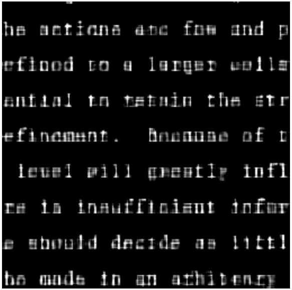
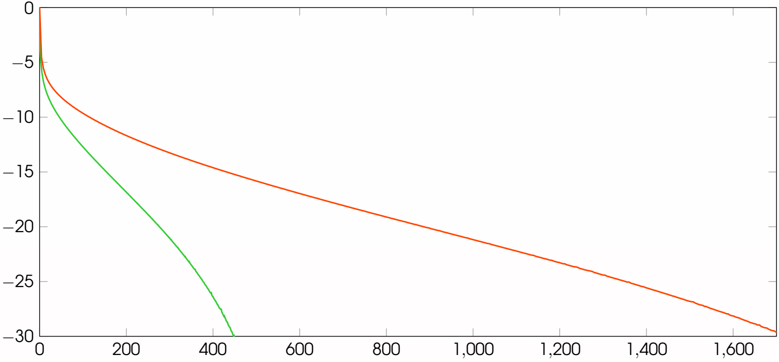


Numerics: promoting sparsity

F_1 is expensive to compute.



Relative error (dB) versus execution time (seconds) for full-activation, i.e., $I_n = I$ versus block activation, i.e.,

$$(\forall n \in \mathbb{N}) \quad I_n = \begin{cases} \{1,2\}, & \text{if } \; n \equiv 0 \mod 5; \\ \{2\}, & \text{if } \; n \not\equiv 0 \mod 5. \end{cases}$$

Chapter 5

Problem 4

Let $(\mathcal{H}_i)_{1\leqslant i\leqslant m}$ and $(\mathcal{G}_k)_{1\leqslant k\leqslant p}$ be real Hilbert spaces. For every $i \in \{1,\ldots,m\}$ and every $k \in \{1,\ldots,p\}$, let $f_i\colon \mathcal{H}_i \to \,]-\infty,+\infty]$ and $g_k\colon \mathcal{G}_k \to \,]-\infty,+\infty]$ be proper lower semicontinuous convex functions, and let $L_{k,i}\colon \mathcal{H}_i \to \mathcal{G}_k$ be a linear operator. The objective is to

$$\underset{x_1\in\mathcal{H}_1,\ldots,x_m\in\mathcal{H}_m}{\text{minimize}} \quad \underbrace{\sum_{i=1}^{m} f_i(x_i)}_{\text{separable term}} + \sum_{k=1}^{p} \underbrace{g_k\Bigg(\sum_{i=1}^{m} L_{k,i}x_i\Bigg)}_{k\text{th coupling term}}. \tag{8}$$

We analyze algorithms which solve the fully nonsmooth Problem 4 with the following features: splitting, block activation, global convergence, and no knowledge of the norms of the linear operators.

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Ch. 1: supplemental numerics

Original
$\overline{x}$

A priori information:

- $\overline{x} \in [0, 255]^{N \times N}$, where $N = 256$
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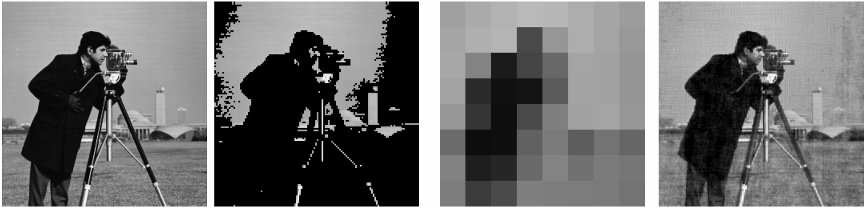
Ch. 1: supplemental numerics

Original $\overline{x}$ | Observations | Recovery



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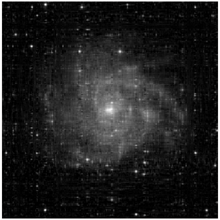
Ch. 4: supplemental numerics

Goal: Separate the background of stars $\overline{x}_1$ from the galaxy $\overline{x}_2$, given $C = [0, 255]^N$ $(N = 600^2)$ and

- A low rank approximation of the superposition $\overline{x}_1 + \overline{x}_2$
- $\overline{x}_1$ is sparse and $\overline{x}_2$ is sparse under the discrete cosine transform $L\colon \mathbb{R}^N \to \mathbb{R}^N$. We set $L_2\colon (x_1, x_2) \mapsto (x_1, Lx_2)$, $p_2 = 0$, and $F_2\colon (y_1, y_2) \mapsto (\mathrm{proj}_{B_\infty(0;10)} y_1, \mathrm{proj}_{B_\infty(0;45)} y_2)$.



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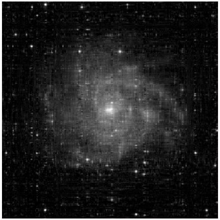

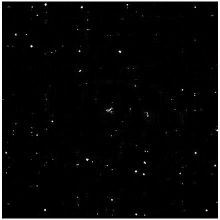

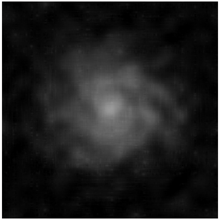

Let $A\colon \mathcal{H} \to 2^{\mathcal{H}}$. A is monotone if

$$(\forall (x,x^*) \in \operatorname{gra} A)\,(\forall (y,y^*) \in \operatorname{gra} A) \quad \langle x-y \mid x^*-y^* \rangle \geqslant 0, \tag{9}$$

and maximally monotone if, for every $(x,x^*) \in \mathcal{H} \times \mathcal{H}$,

$$(x,x^*) \in \operatorname{gra} A \quad \Leftrightarrow \quad (\forall (y,y^*) \in \operatorname{gra} A) \;\; \langle x-y \mid x^*-y^* \rangle \geqslant 0. \tag{10}$$

If A is monotone and satisfies

$$(\forall (x,x^*) \in \operatorname{dom} A \times \operatorname{ran} A) \;\; \sup\big\{\langle x-y \mid y^*-x^* \rangle \;\big|\; (y,y^*) \in \operatorname{gra} A\big\} < +\infty, \tag{11}$$

then it is 3^* monotone.
Let $f\colon \mathcal{H} \to \mathbb{R}$ be convex, continuous, and proper. Let $s(x)$ be a selection of the subgradient ∂f, and let $\xi \in \mathbb{R}$ be such that $C = \big\{x \in \mathcal{H} \;\big|\; f(x) \leqslant \xi\big\} \neq \varnothing$. The subgradient projector onto C associated with (f,ξ,s) is

$$Gx = \begin{cases} x + \dfrac{\xi - f(x)}{\|s(x)\|^2}\, s(x) & \text{if } \; f(x) > \xi \\ x & \text{otherwise.} \end{cases} \tag{12}$$