

A “crash course” in nonsmooth convex optimization

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1 Introduction

These notes are supplementary material to a “crash course” I am teaching in May of 2023. The topic is *proximity operators and nonsmooth convex optimization*. These notes are not meant to be used as a standalone resource. Please cite peer-reviewed material. As a general reference text, I suggest *Convex Analysis and Monotone Operator Theory*, 2nd ed., by Bauschke and Combettes, published by Springer. Virtually all of the results in these notes also apply to real Hilbert spaces; for proofs in full-generality, read the book. If unspecified, \mathcal{H} is a real finite-dimensional vector space in Section 1 and a real finite-dimensional Hilbert space from Sections 2 onward (e.g., \mathbb{R}^n with the Euclidean inner product is fine).

1.1 Optimization terminology and the extended real line

Notation 1.1 We will work with the **extended real line**, i.e., $[-\infty, +\infty] := \mathbb{R} \cup \{-\infty, +\infty\}$. Algebra on this field follows most “natural” rules one could expect (e.g., for $x \in \mathbb{R}$, $x + \infty = \infty$). However, the following quantities are **undefined**:

- Any subtraction of infinities: “ $+\infty - (+\infty)$ ”
- Zero times infinity: “ $0 \cdot (\pm\infty)$ ”
- Any quotient of infinities: “ $\pm\infty / \pm\infty$, $\pm\infty / \mp\infty$, ...”

As a result, if we work with extended-real-valued functions, we must be sure to avoid anything which is undefined (e.g., the objective function $f(x) + g(x)$ could be undefined if there exists z such that $g(z) = -\infty$ and $f(z) = \infty$.)

*Please report typos/errors found in these notes. Homework solutions should be handed in to my office ZIB 3107.

Definition 1.2 Given a real vector space \mathcal{H} , a function $f: \mathcal{H} \rightarrow [-\infty, +\infty]$, and a set $C \subset \mathcal{H}$, consider the following optimization problem.

$$\underset{x \in C}{\text{minimize}} \quad f(x) \tag{1}$$

We call f the **objective function**. We call C a **constraint**. For any $x \in C$, we say x is **feasible**. Otherwise, for $x \in \mathbb{R}^n \setminus C$, x is infeasible. If a point $x^* \in C$ satisfies

$$(\forall x \in C) \quad f(x^*) \leq f(x), \tag{2}$$

we call x^* a **solution** to the optimization problem (1).

For this class, we consider minimization; to maximize f , just use the objective function $-f$.

Definition 1.3 For $I \subset [-\infty, +\infty]$, $a \in [-\infty, +\infty]$ is a **lower bound (upper bound)** if, for every $\xi \in I$, $a \leq \xi$ ($a \geq \xi$). The **greatest lower bound**, or **infimum**, of the set I is denoted $\inf I$. Analogously, the **least upper bound**, or **supremum**, of the set I is denoted $\sup I$. In general, $\inf I, \sup I \in [-\infty, +\infty]$. If, additionally, $\inf I \in I$ ($\sup I \in I$), we call it the **minimum (maximum)**, and denote it $\min I$ ($\max I$). In these cases, we say the infimum (supremum) is *attained*.

A few things to mention:

- (i) For $I \neq \emptyset$, we have $\inf I \leq \sup I$. For the empty set, $\inf \emptyset = +\infty$ and $\sup \emptyset = -\infty$.
- (ii) While the \inf and \sup are always defined, \max and \min may not exist (e.g., consider $I = (0, 1)$ has $\inf I = 0$ and $\sup I = 1$. However, since $0, 1 \notin I$, neither $\max I$ nor $\min I$ exist.)
- (iii) Let $f: \mathbb{R}^n \rightarrow [-\infty, +\infty]$. We adopt the notation that $\inf_{x \in C} f(x) = \inf \{f(x) \mid x \in C\}$.
- (iv) It is common in optimization literature to abuse notation, and use

$$\min_{x \in C} f(x) \tag{3}$$

to describe the optimization problem (1). Technically, $\min_{x \in C} f(x)$ is not an optimization problem – it is the optimal value of the objective function at a solution, which may or may not exist.¹

¹The *Weierstraß Theorem* (cf. Theorem 1.13), loosely stated, guarantees that a solution to (1) exists if C is compact and f is lower-semicontinuous. For unconstrained functions, analytic notions of “coercivity” and “recession cones” can also yield existence results; however, they are not included in this class.

Definition 1.4 Let $f: \mathcal{H} \rightarrow [-\infty, +\infty]$. We will use the following terms.

(i) The **domain** of f is

$$\text{dom } f = \{x \in \mathcal{H} \mid f(x) < +\infty\} \quad (4)$$

(ii) The **epigraph** of f is

$$\text{epi } f = \{(x, \xi) \in \mathcal{H} \times \mathbb{R} \mid f(x) \leq \xi\} \quad (5)$$

(iii) The function f is **proper** if $\text{dom } f \neq \emptyset$ and it never outputs the value $-\infty$ (i.e., $-\infty \notin f(\mathcal{H})$).

(iv) The function f is **lower semicontinuous** (sometimes abbreviated “lsc”) at $x \in \mathcal{H}$ if, for every sequence $(x_n)_{n \in \mathbb{N}}$ satisfying $x_n \rightarrow x$, we have $f(x) \leq \liminf f(x_n)$

For this class, we will predominantly consider proper and lsc functions. A few things to note about the lsc assumption: (1) every continuous function is lsc, and (2) lower semicontinuity basically allows for a jump-discontinuity to occur at $x \in \mathcal{H}$, but requires that f takes the lowest possible limiting value at x (cf. the figures drawn in class, or [here](#)).

1.2 Inner product and norms

Definition 1.5 Let \mathcal{H} be a real finite-dimensional vector space. A **scalar product** (sometimes called **inner product**) is a function $\langle \cdot \mid \cdot \rangle: \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{R}$ which satisfies the following properties.

$$(i) \ (\forall x \in \mathcal{H} \setminus \{0\}) \quad \langle x \mid x \rangle > 0$$

$$(ii) \ (\forall x, y \in \mathcal{H}) \quad \langle x \mid y \rangle = \langle y \mid x \rangle$$

$$(iii) \ (\forall x, y, z \in \mathcal{H})(\forall \alpha \in \mathbb{R}) \quad \langle \alpha x + y \mid z \rangle = \alpha \langle x \mid z \rangle + \langle y \mid z \rangle$$

Exercise 1.6 Let $0 \in \mathcal{H}$ be the zero element of \mathcal{H} . Show that, for every $x \in \mathcal{H}$, $\langle 0 \mid x \rangle = 0$.

Exercise 1.7 Consider $\mathcal{H} = \mathbb{R}^n$. For two vectors $x, y \in \mathbb{R}^n$, the *dot product* is given by $\langle x \mid y \rangle = x^\top y$. Show that the dot product on \mathbb{R}^n is a scalar product.

Exercise 1.8 Consider the vector space of matrices $\mathbb{R}^{n \times n}$. For two matrices $A = (a_{i,j})_{1 \leq i,j \leq n}$ and $B = (b_{i,j})_{1 \leq i,j \leq n}$, the *Frobenius inner product* is given by

$$\langle A \mid B \rangle = \sum_{i=1}^n \sum_{j=1}^n a_{i,j} b_{i,j} \quad (6)$$

Show (6) is an inner product.

Proposition 1.9 (Cauchy-Schwarz) For every $x, y \in \mathcal{H}$,

$$\langle x | y \rangle^2 \leq \langle x | x \rangle \langle y | y \rangle. \quad (7)$$

Proof. If $y = 0$, (7) holds. Now suppose that $y \neq 0$. By Definition 1.5, $\langle y | y \rangle > 0$. Set $\alpha = \langle x | y \rangle / \langle y | y \rangle$. First, we find

$$0 \leq \langle x - \alpha y | x - \alpha y \rangle \quad (8)$$

$$= \langle x | x \rangle - 2\alpha \langle x | y \rangle + \alpha^2 \langle y | y \rangle \quad (9)$$

$$= \langle x | x \rangle - 2\alpha \langle x | y \rangle + \alpha \langle x | y \rangle \quad (10)$$

$$= \langle x | x \rangle - \alpha \langle x | y \rangle. \quad (11)$$

Rearranging the inequality, we find that

$$\frac{\langle x | y \rangle^2}{\langle y | y \rangle} = \alpha \langle x | y \rangle \leq \langle x | x \rangle \quad (12)$$

$$\Leftrightarrow \langle x | y \rangle^2 \leq \langle y | y \rangle \langle x | x \rangle. \quad (13)$$

□

Definition 1.10 Let \mathcal{H} be a real finite-dimensional vector space. A function $\|\cdot\|: \mathcal{H} \rightarrow \mathbb{R}$ is a **norm** if the following hold.

- (i) $(\forall x \in \mathcal{H}) \quad \|x\| = 0 \Rightarrow x = 0$
- (ii) $(\forall x, y \in \mathcal{H}) \quad \|x + y\| \leq \|x\| + \|y\|$
- (iii) $(\forall x \in \mathcal{H})(\forall \alpha \in \mathbb{R}) \quad \|\alpha x\| = |\alpha| \|x\|$

A norm is a way to measure magnitude of vectors, or the distance from one vector to another $\|x - y\|$.

Exercise 1.11 Let \mathcal{H} be a real finite-dimensional vector space, and let $\langle \cdot | \cdot \rangle$ be a scalar product on \mathcal{H} . Show that the norm defined by

$$\|\cdot\|: \mathcal{H} \rightarrow \mathbb{R}: x \mapsto \sqrt{\langle x | x \rangle} \quad (14)$$

satisfies the properties in Definition 1.10.

The **Euclidean norm** on \mathbb{R}^n , given by $(\xi_1, \dots, \xi_n) \mapsto \sqrt{\xi_1^2 + \dots + \xi_n^2}$, arises from the dot product. Exercise 1.11 yields the following formulation of the Cauchy-Schwarz inequality

$$(\forall x, y \in \mathcal{H}) \quad \langle x | y \rangle \leq \|x\| \|y\|. \quad (\text{C-S})$$

While the actual definition can get quite technical, for our class, when we say “**Hilbert space**”, we are referring to the finite-dimensional vector space \mathcal{H} , equipped with a scalar product $\langle \cdot | \cdot \rangle$ and a norm who arises from the scalar product via $\|\cdot\| = \sqrt{\langle \cdot | \cdot \rangle}$. Some examples are \mathbb{R}^n under the Euclidean inner product, or the space of real $n \times m$ matrices under the Frobenius inner product.

Exercise 1.12 Let $(x_1, x_2, x_3) \in \mathbb{R}^3$. Show that

$$2x_1 - x_2^4 + 6x_3 \leq 4\sqrt{x_1^2 + x_2^8 + 9x_3^2}. \quad (15)$$

Can the coefficient 4 in (15) be reduced?

The following theorem is referenced a few times in the notes, so I will provide its statement here. Regrettably, this class does not have enough time to detail the topics of compact/closed/lsc. The following theorem is often used as a tool to ensure that a solution to an optimization problem exists.

Theorem 1.13 (Weierstraß) Let $f: \mathcal{H} \rightarrow [-\infty, +\infty]$ be lower semicontinuous and let C be a compact subset of \mathcal{H} . Suppose that $C \cap \text{dom } f \neq \emptyset$. Then f achieves its infimum over C .

2 Convexity

Definition 2.1 A set $C \subset \mathcal{H}$ is **convex** if, for every $x, y \in C$

$$(\forall \alpha \in]0, 1[) \quad \alpha x + (1 - \alpha)y \in C. \quad (16)$$

A function f is **convex** if $\text{epi } f$ is convex.

Proposition 2.2 $f: \mathcal{H} \rightarrow [-\infty, +\infty]$ is convex if and only if

$$(\forall x, y \in \text{dom } f) \quad (\forall \alpha \in]0, 1[) \quad f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y). \quad (17)$$

Proof. First, we note that if f is identically $+\infty$, then $\text{dom } f = \emptyset$ if and only if $\text{epi } f = \emptyset$, so (17) is vacuously true. Now assume that $\text{dom } f \neq \emptyset$. Let (x, ξ) and (y, η) be in $\text{epi } f$ and let $\alpha \in]0, 1[$.
 (\Rightarrow) Assume that $\text{epi } f$ is convex. Then

$$\alpha(x, \xi) + (1 - \alpha)(y, \eta) = (\alpha x + (1 - \alpha)y, \alpha \xi + (1 - \alpha)\eta) \in \text{epi } f. \quad (18)$$

Therefore, $f(\alpha x + (1 - \alpha)y) \leq \alpha \xi + (1 - \alpha)\eta$. Taking the limit as $\xi \searrow f(x)$ and $\eta \searrow f(y)$ yields (17).
 (\Leftarrow) Assume that (17) holds. By definition, $f(x) \leq \xi$ and $f(y) \leq \eta$. So, by (17),

$$f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y) \quad (19)$$

$$\leq \alpha \xi + (1 - \alpha)\eta. \quad (20)$$

Therefore, $(\alpha x + (1 - \alpha)y, \alpha \xi + (1 - \alpha)\eta) \in \text{epi } f$ which completes the proof. \square

Definition 2.3 Let $\rho > 0$ and let $x \in \mathcal{H}$. A **closed ball** of radius ρ is $B(x; \rho) = \{z \in \mathcal{H} \mid \|x - z\| \leq \rho\}$.

Definition 2.4 Let $f: \mathcal{H} \rightarrow [-\infty, +\infty]$ and let $x \in \mathcal{H}$. x is a **local minimizer** of f if there exists $\rho > 0$ such that

$$(\forall z \in \mathcal{H} \cap B(x; \rho)) \quad f(x) \leq f(z). \quad (21)$$

x is a **global minimizer** of f if

$$(\forall z \in \mathcal{H}) \quad f(x) \leq f(z). \quad (22)$$

Fact 2.5 Let f be a convex and proper function. Then every local minimizer is a global minimizer.

Proof. This is left as an exercise (easier to prove after we learn about convex subdifferentials). \square

Definition 2.6 Let $C \subset \mathcal{H}$ be nonempty.

(i) The **indicator function** of C is

$$\iota_C: \mathcal{H} \rightarrow [-\infty, +\infty] : x \mapsto \begin{cases} 0 & \text{if } x \in C \\ +\infty & \text{if } x \notin C. \end{cases} \quad (23)$$

(ii) Suppose that C is also closed. A **projection** of $x \in \mathcal{H}$ onto C is a solution to the minimization problem

$$\underset{z \in C}{\text{minimize}} \quad \|x - z\|. \quad (24)$$

A solution to (24) is a “closest” point to x which resides in C .

Fact 2.7 Let $C \subset \mathcal{H}$ and let $x \in \mathcal{H}$.

(i) Without loss of generality, constrained optimization can be rephrased as unconstrained optimization via changing the objective function:

$$\inf_{x \in C} f(x) = \inf_{x \in \mathcal{H}} f(x) + \iota_C(x). \quad (25)$$

The objective function $f + \iota_C$ on the righthand side, although a bit fancier, allows us to rephrase the constraint on the lefthand side.

(ii) C is convex if and only if its indicator function ι_C is convex.

(iii) C is closed if and only if its indicator function ι_C is lsc.

(iv) Suppose that C is closed. Then a solution to (24) exists.

(v) Suppose that C is convex. If a solution to (24) exists, it is guaranteed to be unique.

The proofs of (ii) and (iii) follow from the fact that $\text{epi } C = C \times [0, +\infty[$. Loosely speaking, the proof of (iv) follows from the Weierstraß theorem (compactness is achieved by intersecting C with $\{y \in \mathcal{H} \mid \|x - y\| \leq \eta\}$ for $\eta > 0$) and (v) follows from the fact that the norm is *strictly convex* – (a notion we have not yet defined, but the interested student could research).

Definition 2.8 Let $C \subset \mathcal{H}$ be nonempty, closed, and convex. In view of Fact 2.7(iv)–(v), for every $x \in \mathcal{H}$ there is a unique point, $\text{Proj}_C(x) \in C$, which solves (24). This implicitly defines the **projection operator** of C .

$$\text{Proj}_C: \mathcal{H} \rightarrow \mathcal{H}: x \mapsto \text{Proj}_C(x) \quad (\text{solution to (24)}) \quad (26)$$

Note: if $x \in C$, then $\text{Proj}_C x = x$.

For all of the algorithms in this course, we will focus on functions from the following class

$$\Gamma_0(\mathcal{H}) = \{f: \mathcal{H} \rightarrow]-\infty, +\infty] \mid f \text{ is proper, lower semicontinuous, and convex}\}. \quad (27)$$

The following functions live in $\Gamma_0(\mathcal{H})$:

- (i) Exponentials: e^x
- (ii) Log-barriers $f(x) = \begin{cases} -\ln(x) & \text{if } x > 0 \\ +\infty & \text{otherwise.} \end{cases}$
- (iii) Any norm: $\|\cdot\|$ (e.g., $\|\cdot\|_1$ which promotes sparsity, $\|\cdot\|_{\text{nuclear}}$ which promotes low-rank)
- (iv) Hinge-Loss, ReLU, KL-Divergence, ...
- (v) Given a collection of functions $(f_i)_{i \in I}$ in $\Gamma_0(\mathcal{H})$, we can remain in $\Gamma_0(\mathcal{H})$ via the following operations.
 - (a) $\max\{f_1, \dots, f_m\}$
 - (b) Positive linear combinations: $\lambda_1 f_1 + \dots + \lambda_m f_m$, where $\{\lambda_i\}_{i=1}^m$ are positive.
 - (c) Let \mathcal{H}_1 and \mathcal{H}_2 be two finite-dimensional real vector spaces. Let $b \in \mathcal{H}_2$ and let $A: \mathcal{H}_1 \rightarrow \mathcal{H}_2$ be a linear operator (e.g., a matrix from \mathbb{R}^n to \mathbb{R}^m). If $f_1 \in \Gamma_0(\mathcal{H}_2)$, then $g(x) = f_1(Ax + b) \in \Gamma_0(\mathcal{H}_1)$.

Exercise 2.9 The **Minkowski sum** of two subsets A, B of \mathcal{H} is given by

$$A + B = \{a + b \mid a \in A \text{ and } b \in B\}. \quad (28)$$

Assume that A and B are convex. Prove that $A + B$ is convex.

Exercise 2.10 Show that the norm $\|\cdot\|$ is convex using Definition 1.10.

3 What is Differentiability?

There are a lot of ML engineers who brush off the mathematical details of what it means for a function to be differentiable. Algorithmic differentiation (sometimes misleadingly-called “automatic” differentiation) is only guaranteed to work when certain theoretical conditions about the *existence* of a gradient hold. This part of the class is dedicated to explaining that differentiability is not a freebie.

To start our discussion on differentiability, we will begin with a few preliminaries from analysis.

Definition 3.1 Let $A: \mathcal{H}_1 \rightarrow \mathcal{H}_2$. Then A is **linear** if, for every $\alpha \in \mathbb{R}$ and every $x, y \in \mathcal{H}_1$,

$$A(\lambda x) = \lambda A(x) \quad \text{and} \quad A(x + y) = A(x) + A(y). \quad (29)$$

Theorem 3.2 (Riesz-Fréchet representation) Let $A: \mathcal{H} \rightarrow \mathbb{R}$ be linear. Then there exists a unique vector $u \in \mathcal{H}$ such that, for every $x \in \mathcal{H}$, $A(x) = \langle u | x \rangle$.

Although at first-glance it looks unrelated, Theorem 3.2 is a central notion for defining the gradient. A necessary (albeit insufficient) condition for the existence of a gradient is the existence of a directional derivative, defined below.

Definition 3.3 Let $f: \mathcal{H} \rightarrow]-\infty, +\infty]$ be proper. The **directional derivative** of f at $x \in \text{dom } f$ in the direction $y \in \mathcal{H}$ is

$$f'(x; y) = \lim_{\alpha \searrow 0} \frac{f(x + \alpha y) - f(x)}{\alpha}. \quad (30)$$

From Definition 3.3, we point out a few things.

- (i) The limit in (30) might not exist.
- (ii) If f is convex, then $f'(x; y) \in [-\infty, +\infty]$.
- (iii) Even if a directional derivative exists, it might not exist in \mathbb{R} (since it could be $+\infty$ or $-\infty$).

Definition 3.4 Let $x \in \text{dom } f$. If $f'(x; \cdot)$ is linear, we say f is **differentiable at x** . In this case, the unique vector provided by Theorem 3.2 is called the **gradient** of f at x and denoted $\nabla f(x)$.

$$f'(x; \cdot) = \lim_{\alpha \searrow 0} \frac{f(x + \alpha \cdot) - f(x)}{\alpha} = \langle \nabla f(x) | \cdot \rangle \quad (31)$$

If f is differentiable at every $x \in \text{dom } f$, we say that f is **differentiable**.

Exercise 3.5 Verify that $\nabla(\frac{1}{2}\|\cdot\|^2)(x) = x$.

All of the properties we know and love about differentiability (chain rule, product rule, etc.) have to be proven. Here is an example below.

Proposition 3.6 Let $A: \mathcal{H}_1 \rightarrow \mathcal{H}_2$ be a linear operator (with adjoint denoted A^*), let $b \in \mathcal{H}_2$, and let $f: \mathcal{H} \rightarrow \mathbb{R}$ be proper and differentiable. Set $g = f(Ax + b)$. Then g is differentiable and

$$\nabla g = A^*(\nabla f(A \cdot + b)). \quad (32)$$

Proof. Since $\text{dom } f = \mathcal{H}_2$, $\text{dom } g \neq \emptyset$ so we let $x \in \text{dom } g$. By definition,

$$g'(x; y) = \lim_{\alpha \searrow 0} \frac{g(x + \alpha y) - g(x)}{\alpha} \quad (33)$$

$$= \lim_{\alpha \searrow 0} \frac{f(A(x + \alpha y) + b) - f(Ax + b)}{\alpha} \quad (34)$$

$$= \lim_{\alpha \searrow 0} \frac{f(Ax + b + \alpha Ay) - f(Ax + b)}{\alpha} \quad (35)$$

$$= f'(Ax + b; Ay). \quad (36)$$

So the directional derivative of g exists. Now, since f is differentiable,

$$g'(x; y) = f'(Ax + b; Ay) = \langle \nabla f(Ax + b) \mid Ay \rangle = \langle A^*(\nabla f(Ax + b)) \mid y \rangle. \quad (37)$$

Hence the directional derivative of g is linear and g is differentiable. The specific form of the gradient is constructed in (37) \square

Algorithmic differentiation tools use results like Proposition 3.6 to approximate a gradient of a function by reading its machine code. However, these subroutines do not check the theoretical conditions required for their theorems (e.g., *f must be differentiable*) – this must be done (and is oftentimes unjustly ignored) by the user.

Definition 3.7 Let f be proper and differentiable. f is **smooth** (“ L -smooth”) if there exists $L > 0$ such that

$$(\forall x, y \in \mathcal{H}) \quad \|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\|. \quad (38)$$

Exercise 3.8 Construct a function which is differentiable and nonsmooth.

Proposition 3.9 Let $f: \mathcal{H} \rightarrow]-\infty, +\infty]$ be proper and convex. Then,

$$(\forall x \in \text{dom } f)(\forall y \in \mathcal{H}) \quad f'(x; y - x) + f(x) \leq f(y). \quad (39)$$

Proof. By Proposition 2.2, for every $\alpha \in]0, 1[$,

$$f(x + \alpha(y - x)) - f(x) = f((1 - \alpha)x + \alpha y) - f(x) \quad (40)$$

$$\leq (1 - \alpha)f(x) + \alpha f(y) - f(x) \quad (41)$$

$$= \alpha(f(y) - f(x)). \quad (42)$$

Therefore,

$$\frac{f(x + \alpha(y - x)) - f(x)}{\alpha} \leq f(y) - f(x). \quad (43)$$

Taking the limit as $\alpha \searrow 0$ implies $f'(x; y) \leq f(y) - f(x)$, which in turn yields (39). \square

Corollary 3.10 Let $f: \mathcal{H} \rightarrow]-\infty, +\infty]$ be proper and convex. If f is differentiable at an interior point x of its domain, then

$$(\forall y \in \mathcal{H}) \quad \langle y - x \mid \nabla f(x) \rangle + f(x) \leq f(y). \quad (44)$$

When the lefthand side of (44) is viewed as a function of y , we see it is the first-order Taylor series approximation of f . Therefore, it follows from (39) that a convex differentiable function always remains above its first-order Taylor approximation! This is the motivating idea in defining a (convex) subgradient²

Definition 3.11 Let $f: \mathcal{H} \rightarrow]-\infty, +\infty]$. A vector g is a **subgradient** of f at $x \in \mathcal{H}$ if

$$(\forall y \in \mathcal{H}) \quad \langle y - x \mid g \rangle + f(x) \leq f(y). \quad (45)$$

The **subdifferential** of f at x is the set of all subgradients, denoted $\partial f(x)$.

Example 3.12 As shown in class,

$$\partial(|\cdot|)(x) = \begin{cases} -1 & \text{if } x < 0 \\ [-1, 1] & \text{if } x = 0 \\ 1 & \text{if } x > 0. \end{cases} \quad (46)$$

This leads to the following fundamental theorem for optimization.

Theorem 3.13 (Fermat's Rule) Let $f: \mathcal{H} \rightarrow]-\infty, +\infty]$ be proper. Then x is a minimizer of f if and only if $0 \in \partial f(x)$.

Proof. By definition,

$$0 \in \partial f(x) \Leftrightarrow (\forall y \in \mathcal{H}) \quad \langle 0 \mid y - x \rangle + f(x) \leq f(y) \quad (47)$$

$$\Leftrightarrow (\forall y \in \mathcal{H}) \quad f(x) \leq f(y). \quad (48)$$

□

Unlike differentiable functions, there are technical conditions we must check in order to get the “standard” rules one would hope for.

²There are more general notions of subgradients (e.g., Clarke or Mordukhovich subdifferentials). For functions on $\Gamma_0(\mathcal{H})$, these notions are usually all equivalent.

Theorem 3.14 (Sum rule) Let $f, g \in \Gamma_0(\mathcal{H})$ and suppose that one of the following holds:

- (i) The interior of $\text{dom } g$ intersects with $\text{dom } f$
- (ii) $\text{dom } g = \mathcal{H}$
- (iii) The relative interiors of $\text{dom } f$ and $\text{dom } g$ intersect.

Then $\partial(f + g) = \partial f + \partial g$.

Remark 3.15 If f is convex and differentiable at $x \in \mathcal{H}$, then $\partial f(x) = \{\nabla f(x)\}$.

4 Proximity Operators

We will refer to \mathcal{H} as a “finite dimensional real **Hilbert space**.” When we say “Hilbert space,” we are referring to both a vector space and an inner product $(\mathcal{H}, \langle \cdot | \cdot \rangle)$, which also gives rise to a norm via $\| \cdot \| = \sqrt{\langle \cdot | \cdot \rangle}$ (see Exercise 1.11).

Definition 4.1 Let $f: \mathcal{H} \rightarrow]-\infty, +\infty]$. Then f is **strictly convex** if, for *distinct* points $x, y \in \mathcal{H}$ ($x \neq y$), we have

$$(\forall \alpha \in]0, 1[) \quad f(\alpha x + (1 - \alpha)y) < \alpha f(x) + (1 - \alpha)f(y). \quad (49)$$

Exercise 4.2 Let $f: \mathcal{H} \rightarrow]-\infty, +\infty]$ be strictly convex and suppose that at least one minimizer exists. Prove that the minimizer is unique.

Proposition 4.3 Let $f \in \Gamma_0(\mathcal{H})$ and let $x \in \mathcal{H}$. The following optimization problem has a unique solution

$$\underset{z \in \mathcal{H}}{\text{minimize}} \quad f(z) + \frac{1}{2} \|x - z\|^2 \quad (*)$$

The proof idea of Proposition 4.3 comes from the fact that the norm is strictly convex, which implies that the objective function in $(*)$ is convex. =.

Definition 4.4 Let $f \in \Gamma_0(\mathcal{H})$ and let $x \in \mathcal{H}$. The solution to $(*)$ is the **proximal point** of f at x , denoted $\text{Prox}_f(x) \in \mathcal{H}$. This defines an operator $\text{Prox}_f: \mathcal{H} \rightarrow \mathcal{H}$ where x maps to $\text{Prox}_f(x)$.

Example 4.5 Let C be a nonempty closed convex subset of \mathcal{H} . Then, for $x \in \mathcal{H}$, we have

$$\min_{z \in \mathcal{H}} \iota_C(z) + \frac{1}{2} \|x - z\|^2 = \min_{z \in C} \frac{1}{2} \|x - z\|^2. \quad (50)$$

However, the minimizer of (50) will be the same as in (24). Therefore, the proximity operator of ι_C is actually just the projection operator!

$$\text{Prox}_{\iota_C} = \text{Proj}_C. \quad (51)$$

Exercise 4.6 Let $f \in \Gamma_0(\mathcal{H})$ and let $x, p \in \mathcal{H}$. Show that

$$p = \text{Prox}_f x \quad \Leftrightarrow \quad x - p \in \partial f(p). \quad (52)$$

Definition 4.7 Let $\mathcal{H}_1, \dots, \mathcal{H}_m$ be real Hilbert spaces with inner products denoted $\langle \cdot | \cdot \rangle_{\mathcal{H}_1}, \dots, \langle \cdot | \cdot \rangle_{\mathcal{H}_m}$. We can construct another Hilbert space, called the **direct sum** (sometimes also called a *product space*) as follows. Our vector space is denoted $\bigoplus_{i=1}^m \mathcal{H}_i = \mathcal{H}_1 \times \dots \times \mathcal{H}_m$, and the inner product on the direct sum is given by

$$\langle \cdot | \cdot \rangle_{\bigoplus_{i=1}^m \mathcal{H}_i} : \bigoplus_{i=1}^m \mathcal{H}_i \times \bigoplus_{i=1}^m \mathcal{H}_i \rightarrow \mathbb{R} : ((x_i)_{i=1}^m, (y_i)_{i=1}^m) \mapsto \sum_{i=1}^m \langle x_i | y_i \rangle_{\mathcal{H}_i}. \quad (53)$$

If, for every $i \in I$, $f_i : \mathcal{H}_i \rightarrow]-\infty, +\infty]$, then the function

$$f : \bigoplus_{i=1}^m \mathcal{H}_i \rightarrow]-\infty, +\infty] : (x_i)_{i=1}^m \mapsto \sum_{i=1}^m f_i(x_i) \quad (54)$$

is called a **separable sum**.

Proposition 4.8 (Proximity Operator of a Separable Sum) Let $\mathcal{H}_1, \dots, \mathcal{H}_m$ be real Hilbert spaces, and for every $i \in \{1, \dots, m\}$, let $f_i \in \Gamma_0(\mathcal{H}_i)$. Let f be the separable sum of $(f_i)_{i=1}^m$ in the form (54). Then

$$\left(\forall (x_i)_{i=1}^m \in \bigoplus_{i=1}^m \mathcal{H}_i \right) \quad \text{Prox}_f((x_i)_{i=1}^m) = (\text{Prox}_{f_1}(x_1), \text{Prox}_{f_2}(x_2), \dots, \text{Prox}_{f_m}(x_m)). \quad (55)$$

Proof. For notational convenience, set $\mathcal{H} = \bigoplus_{i=1}^m \mathcal{H}_i$. Let $x = (x_i)_{i=1}^m \in \mathcal{H}$. Then

$$\min_{z \in \mathcal{H}} f(z) + \frac{1}{2} \|x - z\|_{\mathcal{H}}^2 = \min_{\substack{z_1 \in \mathcal{H}_1 \\ \vdots \\ z_m \in \mathcal{H}_m}} \sum_{i=1}^m f_i(z_i) + \frac{1}{2} \|x_i - z_i\|_{\mathcal{H}_i}^2 \quad (56)$$

$$= \sum_{i=1}^m \min_{z_i \in \mathcal{H}_i} f_i(z_i) + \frac{1}{2} \|x_i - z_i\|_{\mathcal{H}_i}^2. \quad (57)$$

Since all of the summands in (56) are only concerned with a single optimization variable z_i (as opposed to the entire vector $z \in \mathcal{H}$), we actually have a sum of independent optimization problems! This observation permits us to commute the sum and the minimization between (56) and (57). Finally, we see that the solution in each of the subproblems in (57) is precisely the proximity operator of f_i at x_i . Since the solution in each component is $\text{Prox}_{f_i}(x_i)$, we have are done since we have just shown (55). \square

Example 4.9 Let $\gamma > 0$. Let's compute the proximity operator of my favorite nonsmooth function $\gamma|\cdot|$. Let $x, p \in \mathbb{R}$. Combining Example 3.12 and the characterization in Exercise 4.6, we find

$$p = \text{Prox}_{\gamma|\cdot|} x \Leftrightarrow x - p \in \partial(\gamma|\cdot|)(p) \quad (58)$$

$$\Leftrightarrow x - p \in \partial(|\cdot|)(p) = \begin{cases} -\gamma & \text{if } p < 0 \\ [-\gamma, \gamma] & \text{if } p = 0 \\ \gamma & \text{if } p > 0. \end{cases} \quad (59)$$

Now, (59) looks a bit funny, since we don't know what p is to begin with. However, if we cobble together the case analysis, we will be able to formally invert this inclusion. Let's consider the first case: If $p < 0$, then

$$x - p \in \{-\gamma\} \Leftrightarrow p = x + \gamma, \quad (60)$$

which means $p < 0$ if and only if $x < -\gamma$. Note we have translated a condition on p to a condition on x . Similarly, for $p > 0$

$$x - p \in \{\gamma\} \Leftrightarrow p = x - \gamma, \quad (61)$$

and $p > 0$ if and only if $x > \gamma$. Finally, if $p = 0$

$$x - p \in [-\gamma, \gamma] \Leftrightarrow x \in [-\gamma, \gamma]. \quad (62)$$

Note that for each of the three cases, we were able to translate the condition on p in (59) to a condition on x ! Combining these three observations, we find that

$$p = \text{Prox}_{\gamma|\cdot|}(x) \begin{cases} x + \gamma & \text{if } x < -\gamma \\ 0 & \text{if } -\gamma \leq x \leq \gamma \\ x - \gamma & \text{if } \gamma \leq x. \end{cases} \quad (63)$$

This operation in (63) is known as the **soft thresholder**.

Exercise 4.10 Compute the proximity operator of the **one norm**

$$\|\cdot\|_1: \mathbb{R}^n \rightarrow \mathbb{R}: (x_i)_{i=1}^m \mapsto \sum_{i=1}^m |x_i| \quad (64)$$

hint: Proposition 4.8

Definition 4.11 Let $T: \mathcal{H} \rightarrow \mathcal{H}$ be an operator and let $x \in \mathcal{H}$. If $Tx = x$, then x is a **fixed point** of T .

Proposition 4.12 Let $f \in \Gamma_0(\mathcal{H})$ and let $x \in \mathcal{H}$. Then

$$\text{Prox}_f(x) = x \Leftrightarrow x \text{ minimizes } f. \quad (65)$$

Proof. First, note that $\text{dom}(\frac{1}{2}\|\cdot - x\|^2) = \mathcal{H}$, so by the Sum Rule (Theorem 3.14(ii)), we know that

$$\partial \left(f + \frac{1}{2}\|\cdot - x\|^2 \right) = \partial f + \partial \left(\frac{1}{2}\|\cdot - x\|^2 \right).$$

In a matter which proceeds similarly to Exercise 3.5, we observe from Remark 3.15 that, for every $z \in \mathcal{H}$, $\partial \left(\frac{1}{2}\|\cdot - x\|^2 \right)(z) = \{z - x\}$. Combining these facts, along with using Fermat's rule (Theorem 3.13) twice, we find

$$\text{Prox}_f(x) = x \Leftrightarrow x \text{ minimizes } f(\cdot) + \frac{1}{2}\|\cdot - x\|^2 \quad (66)$$

$$\Leftrightarrow 0 \in \partial \left(f(\cdot) + \frac{1}{2}\|\cdot - x\|^2 \right)(x) \quad (67)$$

$$\Leftrightarrow 0 \in \partial f(x) + \partial \left(\frac{1}{2}\|\cdot - x\|^2 \right) \quad (68)$$

$$\Leftrightarrow 0 \in \partial f(x) + \{x - x\} \quad (69)$$

$$\Leftrightarrow x \text{ minimizes } f. \quad (70)$$

□

Proposition 4.12 tells us that the fixed-points of a proximity operator are precisely the minimizers of our function. Spoiler alert: this motivates a fixed-point algorithm! It turns out that, if we just repeatedly apply the proximity operator, we will converge to a solution of our problem. This will be detailed in Theorem 4.14.

There are a **lot** of awesome properties of the proximity operator. However, in the interest of time (and practicability), we now shift focus towards algorithms.

4.1 The Proximal Point Algorithm

In optimization, there are a variety of methods to quantify “convergence” of an algorithm. One method – *primal convergence* shows that the function value of the objective (1) approaches the optimal value, i.e., $f(x_n) - \inf_{x \in C} f(x) \rightarrow 0$. However, this could be misleading.

Exercise 4.13 (*extra credit*) Construct a convex function f with at least one minimizer and a sequence $(x_n)_{n \in \mathbb{N}}$ such that $f(x_n) - \inf_{x \in \mathcal{H}} f(x) \rightarrow 0$, and for every minimizer z , $\|x_n - z\| \not\rightarrow 0$.

There are solutions to Exercise 4.13 (e.g., in Bauschke/Combettes' book mentioned in the introduction) which demonstrate that, no matter how low the objective value can be, our iterates could still be arbitrarily far away from the actual minimizers. As a result, some like to instead show that that, for a minimizer $z \in C$, the sequence of iterates actually approaches the minimizer, i.e., $\|x_n - z\| \rightarrow 0$. For the proximal point algorithm, we have both.

Theorem 4.14 (Proximal Point Algorithm (Martinet, 1970)) Let $f \in \Gamma_0(\mathcal{H})$ have at least one minimizer. Let $(\gamma_n)_{n \in \mathbb{N}}$ be a sequence in $]0, +\infty[$ such that $\sum_{n \in \mathbb{N}} \gamma_n = +\infty$, and let $x_0 \in \mathcal{H}$. Set

$$(\forall n \in \mathbb{N}) \quad x_{n+1} = \text{Prox}_{\gamma_n f}(x_n). \quad (71)$$

Then the following hold.

- (i) $f(x_n)_{n \in \mathbb{N}}$ converges monotonically to $\min f(\mathcal{H})$.
- (ii) $(x_n)_{n \in \mathbb{N}}$ converges to a minimizer of f .

Remark 4.15 (Comment on convergence proofs for prox-based algorithms) The proximal point algorithm (and many of its relatives) can be proven to converge using the following template³.

- (i) Show that the algorithm is **Fejér monotone**, i.e., for a solution to your problem $z \in \mathcal{H}$, we have

$$\|x_{n+1} - z\| \leq \|x_n - z\| \quad (72)$$

Oftentimes, the properties of $(T_n)_{n \in \mathbb{N}}$ actually reveal a strictly negative term being added to $\|x_n - z\|$ on the upper bound in (72). This can sometimes be used to obtain specific rates of convergence.

- (ii) Assume that a cluster point of the algorithm exists. Show that it is a solution of our problem.

Using existing theory about Fejér monotonicity, we can conclude $(x_n)_{n \in \mathbb{N}}$ converges to a solution of our problem. Note that we never actually had to prove the algorithm converges.

The Proximal Point Algorithm tells us that, as long as we can compute the proximity operator of our objective function, we can declare victory. That's it, right?

Well, it turns out that the story is not quite as simple. In practice, one can typically compute the proximity operator of each summand in an optimization problem. However, computing the prox of their sum is much trickier. Tune in next time, for an introduction to *splitting algorithms*!

³unfortunately we do not have time to detail the full convergence proofs