

A “crash course” in nonsmooth convex optimization

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1 Introduction

These notes are supplementary material to an unofficial “crash course” I am teaching in May of 2023. The topic is *proximity operators and nonsmooth convex optimization*. These notes are not meant to be used as a standalone resource. Please cite peer-reviewed material. As a general reference text, I suggest *Convex Analysis and Monotone Operator Theory*, 2nd ed., by Bauschke and Combettes, published by Springer. Virtually all of the result in these notes also apply to real Hilbert spaces; for proofs in full-generality, read the book. If unspecified, \mathcal{H} is a real finite-dimensional vector space (e.g., \mathbb{R}^n is fine).

1.1 Optimization terminology and the extended real line

Notation 1.1 We will work with the **extended real line**, i.e., $[-\infty, +\infty] := \mathbb{R} \cup \{-\infty, +\infty\}$. Algebra on this field follows most “natural” rules one would expect (e.g., for $x \in \mathbb{R}$, $x + \infty = \infty$). However, the following quantities are **undefined**:

- Any subtraction of infinities: “ $+\infty - (+\infty)$ ”
- Zero times infinity: “ $0 \cdot (\pm\infty)$ ”
- Any quotient of infinities: “ $\pm\infty / \pm\infty$, $\pm\infty / \mp\infty$, \dots ”

As a result, if we work with extended-real-valued functions, we must be sure to avoid anything which is undefined (e.g., the objective function $f(x) + g(x)$ could be undefined if there exists z such that $g(z) = -\infty$ and $f(z) = \infty$.)

Definition 1.2 Given a real vector space \mathcal{H} , a function $f: \mathcal{H} \rightarrow [-\infty, +\infty]$, and a set $C \subset \mathcal{H}$, consider the following optimization problem.

$$\underset{x \in C}{\text{minimize}} \quad f(x) \tag{1}$$

*Please report typos/errors found in these notes. Homework solutions should be handed in to my office ZIB 3107.

We call f the **objective function**. We call C a **constraint**. For any $x \in C$, we say x is **feasible**. Otherwise, for $x \in \mathbb{R}^n \setminus C$, x is infeasible. If a point $x^* \in C$ satisfies

$$(\forall x \in C) \quad f(x^*) \leq f(x), \quad (2)$$

we call x^* a **solution** to the optimization problem (1).

For this class, we consider minimization; to maximize f , just use the objective function $-f$.

Definition 1.3 For $I \subset [-\infty, +\infty]$, $a \in [-\infty, +\infty]$ is a **lower bound (upper bound)** if, for every $\xi \in I$, $a \leq \xi$ ($a \geq \xi$). The **greatest lower bound**, or **infimum**, of the set I is denoted $\inf I$. Analogously, the **least upper bound**, or **supremum**, of the set I is denoted $\sup I$. In general, $\inf I, \sup I \in [-\infty, +\infty]$. If, additionally, $\inf I \in I$ ($\sup I \in I$), we call it the **minimum (maximum)**, and denote it $\min I$ ($\max I$). In these cases, we say the infimum (supremum) is *attained*.

A few things to mention:

- (i) For $I \neq \emptyset$, we have $\inf I \leq \sup I$. For the empty set, $\inf \emptyset = +\infty$ and $\sup \emptyset = -\infty$.
- (ii) While the \inf and \sup are always defined, \max and \min may not exist (e.g., consider $I = (0, 1)$ has $\inf I = 0$ and $\sup I = 1$. However, since $0, 1 \notin I$, neither $\max I$ nor $\min I$ exist.)
- (iii) Let $f: \mathbb{R}^n \rightarrow [-\infty, +\infty]$. We adopt the notation that $\inf_{x \in C} f(x) = \inf\{f(x) \mid x \in C\}$.
- (iv) It is common in optimization literature to abuse notation, and use

$$\min_{x \in C} f(x) \quad (3)$$

to describe the optimization problem (1). Technically, $\min_{x \in C} f(x)$ is not an optimization problem – it is the optimal value of the objective function at a solution, which may or may not exist.¹

Definition 1.4 Let $f: \mathcal{H} \rightarrow [-\infty, +\infty]$. We will use the following terms.

- (i) The **domain** of f is

$$\text{dom } f = \{x \in \mathcal{H} \mid f(x) < +\infty\} \quad (4)$$

- (ii) The **epigraph** of f is

$$\text{epi } f = \{(x, \xi) \in \mathcal{H} \times \mathbb{R} \mid f(x) \leq \xi\} \quad (5)$$

- (iii) The function f is **proper** if $\text{dom } f \neq \emptyset$ and it never outputs the value $-\infty$ (i.e., $-\infty \notin f(\mathcal{H})$).

¹The *Weierstraß Theorem*, loosely stated, guarantees that a solution to (1) exists if C is compact and f is lower-semicontinuous. For unconstrained functions, analytic notions of “coercivity” and “recession cones” can also yield existence results; however, they are not included in this class.

- (iv) The function f is **lower semicontinuous** (sometimes abbreviated “lsc”) at $x \in \mathcal{H}$ if, for every sequence $(x_n)_{n \in \mathbb{N}}$ satisfying $x_n \rightarrow x$, we have $f(x) \leq \liminf f(x_n)$

For this class, we will predominantly consider proper and lsc functions. A few things to note about the lsc assumption: (1) every continuous function is lsc, and (2) lower semicontinuity basically allows for a jump-discontinuity to occur at $x \in \mathcal{H}$, but requires that f takes the lowest possible limiting value at x (cf. the figures drawn in class, or [here](#)).

1.2 Inner product and norms

Definition 1.5 Let \mathcal{H} be a real finite-dimensional vector space. A **scalar product** (sometimes called **inner product**) is a function $\langle \cdot | \cdot \rangle: \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{R}$ which satisfies the following properties.

- (i) $(\forall x \in \mathcal{H} \setminus \{0\}) \quad \langle x | x \rangle > 0$
- (ii) $(\forall x, y \in \mathcal{H}) \quad \langle x | y \rangle = \langle y | x \rangle$
- (iii) $(\forall x, y, z \in \mathcal{H})(\forall \alpha \in \mathbb{R}) \quad \langle \alpha x + y | z \rangle = \alpha \langle x | z \rangle + \langle y | z \rangle$

Exercise 1.6 Let $0 \in \mathcal{H}$ be the zero element of \mathcal{H} . Show that, for every $x \in \mathcal{H}$, $\langle 0 | x \rangle = 0$.

Exercise 1.7 Consider $\mathcal{H} = \mathbb{R}^n$. For two vectors $x, y \in \mathbb{R}^n$, the *dot product* is given by $\langle x | y \rangle = x^\top y$. Show that the dot product on \mathbb{R}^n is a scalar product.

Exercise 1.8 Consider the vector space of matrices $\mathbb{R}^{n \times n}$. For two matrices $A = (a_{i,j})_{1 \leq i,j \leq n}$ and $B = (b_{i,j})_{1 \leq i,j \leq n}$, the *Frobenius inner product* is given by

$$\langle A | B \rangle = \sum_{i=1}^n \sum_{j=1}^n a_{i,j} b_{i,j} \quad (6)$$

Show (6) is an inner product.

Proposition 1.9 (Cauchy-Schwarz) For every $x, y \in \mathcal{H}$,

$$\langle x | y \rangle^2 \leq \langle x | x \rangle \langle y | y \rangle. \quad (7)$$

Proof. If $y = 0$, (C-S) holds. Now suppose that $y \neq 0$. By Definition 1.5, $\langle y | y \rangle > 0$. Set $\alpha = \langle x | y \rangle / \langle y | y \rangle$. First, we find

$$0 \leq \langle x - \alpha y | x - \alpha y \rangle \quad (8)$$

$$= \langle x | x \rangle - 2\alpha \langle x | y \rangle + \alpha^2 \langle y | y \rangle \quad (9)$$

$$= \langle x | x \rangle - 2\alpha \langle x | y \rangle + \alpha \langle x | y \rangle \quad (10)$$

$$= \langle x | x \rangle - \alpha \langle x | y \rangle. \quad (11)$$

Rearranging the inequality, we find that

$$\frac{\langle x | y \rangle^2}{\langle y | y \rangle} = \alpha \langle x | y \rangle \leq \langle x | x \rangle \quad (12)$$

$$\Leftrightarrow \langle x | y \rangle^2 \leq \langle y | y \rangle \langle x | x \rangle. \quad (13)$$

□

Definition 1.10 Let \mathcal{H} be a real finite-dimensional vector space. A function $\|\cdot\|: \mathcal{H} \rightarrow \mathbb{R}$ is a **norm** if the following hold.

- (i) $(\forall x \in \mathcal{H}) \quad \|x\| = 0 \Rightarrow x = 0$
- (ii) $(\forall x, y \in \mathcal{H}) \quad \|x + y\| \leq \|x\| + \|y\|$
- (iii) $(\forall x \in \mathcal{H})(\forall \alpha \in \mathbb{R}) \quad \|\alpha x\| = |\alpha| \|x\|$

A norm is a way to measure magnitude of vectors, or the distance from one vector to another $\|x - y\|$.

Exercise 1.11 Let \mathcal{H} be a real finite-dimensional vector space, and let $\langle \cdot | \cdot \rangle$ be a scalar product on \mathcal{H} . Show that the norm defined by

$$\|\cdot\|: \mathcal{H} \rightarrow \mathbb{R}: x \mapsto \sqrt{\langle x | x \rangle} \quad (14)$$

satisfies the properties in Definition 1.10.

The **Euclidean norm** on \mathbb{R}^n , given by $(\xi_1, \dots, \xi_n) \mapsto \sqrt{\xi_1^2 + \dots + \xi_n^2}$, arises from the dot product. Exercise 1.11 yields the following formulation of the Cauchy-Schwarz inequality

$$(\forall x, y \in \mathcal{H}) \quad \langle x | y \rangle \leq \|x\| \|y\|. \quad (\text{C-S})$$

Exercise 1.12 Let $(x_1, x_2, x_3) \in \mathbb{R}^3$. Show that

$$2x_1 - x_2^4 + 6x_3 \leq 4\sqrt{x_1^2 + x_2^8 + 9x_3^2}. \quad (15)$$

Can the coefficient 4 in (15) be reduced?