Inner product space

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1 Inner Product

The **inner product** (x, y) of two vectors $x = (x_1, x_2, ..., x_n)^T, y = (y_1, y_2, ...y_n)^T$ by $(x, y) = x_1y_1 + ... + x_ny_n = y^T x = x^T y$. So $||x|| = \sqrt{(x, x)}$.

Standard inner product in $\mathbb{C}$ is given by $(z, w) = z_1\overline{w_1} + ... + z_n\overline{w_n} = \sum_{k=1}^{n} z_k\overline{w}_k$

2 Hermitian adjoint

Define $A^* = \overline{A}^T$, meaning that we take the transpose of the matrix, then take the complex conjugate of each entry. For a real matrix, $A^* = A^T$.

Using the notion of A^*, one can write the standard inner product in $\mathbb{C}^n$ as $(z, w) = w^*z = z^*w$.

3 Inner product properties

The inner product for $\mathbf{R}^n$ and $\mathbb{C}^n$ satisfies the following:
1. **(Conjugate) Symmetry**: $(x, y) = \overline{(y, x)}$, note that for a real space, $(x, y) = (y, x)$, which is a symmetry.
2. **Linearity**: $(\alpha x + \beta y, z) = \alpha(x, z) + \beta(y, z)$ for all vectors x, y, z and all scalars α, β.
   Adv.Linearity: $(x, \alpha y + \beta z) = \overline{\alpha}(x, y) + \overline{\beta}(x, z)$.
   (0,x) = (x,0) = 0.
3. **Non-negativity**: $(x, x) \geq 0$ for all x.
4. **Non-degeneracy**: $(x, x) = 0$ iff $x = 0$.

Note that for a real space V we assume that (x, y) is always real, and for a complex space the inner product (x, y) can be complex.

Given an inner product space IPS, its norm is defined by $||x|| = \sqrt{(x, x)}$.

4 Lem 1.4

Let x be a vector in an IPS V, then $x = 0$ iff $(x, y) = 0$, for all $y \in V$.

Applying this lemma to the difference $x - y$ we get the following Corollary:

5 Cor 1.5

Let x, y be vectors in an IPS V. The equality $x = y$ holds iff $(x, z) = (y, z)$, for all $z \in V$.

6 Cor 1.6

Suppose two operators $A, B : X \to Y$ satisfy
$(Ax, y) = (Bx, y)$ for all $x \in X, y \in Y$.

Then $A = B$.

7 Thm 1.7: Cauchy-Schwarz inequality

$|(x, y)| \leq ||x|| \cdot ||y||$.

8 Cor 1.8: Triangle inequality

For any vectors x, y in an IPS, $||x + y|| \leq ||x|| + ||y||$.

9 Lem 1.9: Polarization identities

For $x, y \in V$,
if V is a real IPS, then
$(x, y) = \frac{1}{4}(||x + y||^2 - ||x - y||^2)$
if V is a complex IPS, then
$(x, y) = \frac{1}{4}\sum_{\alpha=\pm 1, \pm i}(\alpha||x + \alpha y||^2)$.

10 Lem 1.10: Parallelogram identity

For any vectors u, v,
$||u + v||^2 + ||u - v||^2 = 2(||u||^2 + ||v||^2)$.

11 Norm and Normed spaces properties

1. **Homogeneity**: $||\alpha v|| = |\alpha| \cdot ||v||$ for all vectors v and scalar α.
2. **Triangle inequality**: $||u + v|| \leq ||u|| + ||v||$.
3. **Non-negativity**: $||v|| \geq 0$ for all vectors v.
4. **Non-degeneracy**: $||v|| = 0$ iff $v = 0$.

Suppose that in a vectors space V we assign to each vector v a number $||v||$ st the above properties are satisfied, then we say that the function $v \mapsto ||v||$ is a norm.

A vector space equipped with a norm is called a normed space.

12 Thm 1.11

A norm in a NS is obtained from some inner product iff it satisfied the parallelogram identity.