Adjoint of a linear transformation and Fundamental subspaces

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1 Introduction

- Thm 5.1: Relation between fundamental subspaces
- "Essential part" of an operator: $B : Ran(A^*) \to RanA$.
- Complex rank theorem: Complex conjugation does not change the rank of a matrix

2 Properties of adjoint matrix

1. **Main property:** $(Ax, y) = (x, A^*y)$ for all $x \in \mathbb{C}^n$, $y \in \mathbb{C}^m$.
2. $(AB)^* = B^*A^*$
3. $(A^*)^* = A$
4. $(A + B)^* = A^* + B^*$
5. $(\alpha A^*) = \alpha A^*$
6. $(y, Ax) = (A^*y, x)$
7. Uniqueness: If a matrix B satisfies $(Ax, y) = (x, By)$ for all x, y then $B = A^*$.
8. $[A^*]_{AB} = ([A]_{BA})^*$. where the subscript A and B are orthogonal bases $A = v_1, ..., v_n$ in V and $B = w_1, ..., w_m$ in W, respectively.

3 Thm 5.1

Let $A : V \to W$ be an operator acting from one IPS to another. Then

1. $KerA^* = (RanA)^\perp$
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3. $RanA = (KerA^*)^\perp$
4. $RanA^* = (KerA)^\perp$

It follows from the theorem that the operator A can be represented as a composition of orthogonal projection onto A^* and as isomorphism from $RanA^*$ to $RanA$. Indeed, $A = BP_{RanA^*}$, where $B : RanA^* \to RanA$ is the restriction of A to the domain $RanA^*$ and the target space $RanA$ st $Bx = Ax$ for all $x \in RanA^*$.

Note also that $B : RanA^* \to RanA$ is an invertible transformation.

4 Complex rank theorem

$rankA = rankA^*$
$rankA = rank\overline{A}$