Isometry, Unitary and Orthogonal

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1 Introduction

Isometry: iff preserves IP iff $U^*U = I$

- Norm-preserving

Unitary($U : X \to Y$):

- dimX = dimY
- $U^*U = I$
- Invertible

2 Isometry

An operator $U : X \to Y$ is an isometry, if it preserves the norm $||Ux|| = ||x||$, $\forall x \in X$.

3 Thm 6.1

An operator U is an isometry iff it preserves the inner product

$$(x, y) = (Ux, Uy), \forall x, y \in X$$

This is proved from the polarization identity, using the def of isometry

4 Lem 6.2

An operator $U : X \to Y$ is an isometry iff $U^*U = I$.

$$(x, x) = (U^*Ux, x) = (Ux, Ux), \forall x \in X$$

. Therefore $||x|| = ||Ux||$, and so U is an isometry.

This lemma implies that an isometry is always left invertible (U^* **being a left inverse).** Also note that a left invertible square matrix is invertible.

5 Unitary operator

An isometry $U : X \to Y$ is called an unitary operator if it is invertible.

6 Prep 6.3

An isometry U is an unitary iff $dimX = dimY$.

7 Unitary matrix

A square matrix U is unitary if $U^*U = I$
i.e, a unitary matrix is a matrix of a unitary operator acting in F^n.

8 Orthogonal matrix

A unitary matrix with real entries is called an orthogonal matrix.

9 Properties of unitary operators

1. $U^{-1} = U^*$
2. If U unitary, $U^* = U^{-1}$ also unitary
3. If U is an isometry, and v_i is an orthonormal basis, then Uv_i is an orthonormal system. Moreover, if U is unitary, Uv_i is an orthonormal basis. $(1 \le i \le n)$.
4. A product of unitary operators is a unitary operator as well.
5. $|detU| = 1$. In particular, for an orthogonal matrix $detU = \pm 1$
6. If λ is an eigenvalue of U, the $|\lambda| = 1$.

10 Unitarily equivalent

Operators A and b are unitarily equivalent if there exists a unitary operator U st $A = UBU^*$.
Funce for a unitary U we have $U^{-1} = U^*$, any two unitary equivalent matrices are similar as well.

11 Prep 6.5

A matrix A is unitarily equivalent to a diagonal one iff it has an orthogonal (orthonormal) basis of eigenvectors.