Orthogonal project and Gram-Schmidt orthogonalization

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August 2021

1 Orthogonal projection

For a vector v its orthogonal projection $P_E v$ onto the subspace E is a vector w st

1. $w \in E$
2. $v - w \perp E$

We denote with $w = P_E v$ for the orthogonal projection.

- Existence
- Uniqueness (Thm3.2)
- Ways to find (Prep3.3, find $P_E v$ with an orthogonal basis)

2 Thm 3.2

The orthogonal projection $w = P_E v$ minimizes the distance from v to E. i.e, $\forall x \in E$,

$||v - w|| \leq ||v - x||$.

Moreover, if for some $x \in E$

$||v - w|| = ||w - x||$,

then x = w.

3 Prep 3.3

Let $v_1, ..., v_r$ be an orthogonal basis in E. Then the orthogonal projection $P_E v$ of a vector v is given by

$P_E v = \sum_{k=1}^{r} \alpha_k v_k$, where $\alpha_k = \frac{(v, v_k)}{||v_k||^2}$.

In other words,

$P_E v = \sum_{k=1}^{r} \frac{(v, v_k)}{||v_k||^2} v_k$.

This formula applied to an orthogonal system (not a basis) gives us a projection onto its span.
It's also easy to see that P_E is a **linear transformation** by seeing the linearity of P_E from the def and uniqueness of orthogonal projection. (i.e, easy to check that for any x and y the vector $\alpha x + \beta y - (\alpha P_E x - \beta P_E y)$ is orthogonal any vector in E, so by def $P_E(\alpha x + \beta y) = \alpha P_E x + \beta P_E y)$.
The matrix of P_E where E is in $\mathbb{C}^n$ or $\mathbb{R}^n$ is given by
$P_E = \sum_{k=1}^{r} \frac{1}{||v_k||^2} v_k v_k^*$

4 Gram-Schmidt orthogonalization algorithm

Gram-Schmidt constructs from a linearly independent system $x1, ..., x_n$ an orthogonal system $v_1, ..., v_n$ st $span\{x_1, x_2, ..., x_n\} = span\{v_1, v_2, ..., v_n\}$.
Moreover, for all $r \leq n$, we have
$span\{x_1, ..., x_r\} = span\{v_1, ..., v_r\}$.
Step1: Define $v_1 = x_1$.
Define $E_1 : span\{x_1\} = span\{v_1\}$.
Step2: Define v_2 by
$v_2 = x_2 - P_{E_1} x_2 = x_2 - \frac{(x_2, v_1)}{||v_1||^2} v_1$.
Define $E_2 = span\{v_1, v_2\}$. (note that $span\{x_1, x_2\} = E_2$)
Step3: Define v_3 by
$v_3 = x_3 - P_{E_2} x_3 = x_3 - \frac{(x3, v_1)}{||v1||^2} v_1 - \frac{(x_3, v_2)}{||v_2||^2} v_2$.
Define $E_3 = span\{v_1, v_2, v_3\}$.
Step $r+1$: Define $v_{r+1} = x_{r+1} - P_{E_r} x_{r+1} = x_{r+1} - \sum_{k=1}^{r} \frac{(x_{r+1}, v_k)}{||v_k||^2} v_k$.

5 Orthogonal Complement

For a subspace E its orthogonal complement $E^\perp$ is the set of all vectors orthogonal to E.

$$E^\perp = \{x : x \perp E\}$$

If $x, y \perp E$ then for any linear combination $\alpha x + \beta y \perp E$, therefore $E^\perp$ is a subspace.

By def of orthogonal projection, any vector in an IPS V admits a unique representation $v = v_1 + v_2$, where $v_1 \in E, v_2 \in E^\perp$.

6 Prep 3.6

For a subspace E,

$$(E^\perp)^\perp = E$$

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