Least square solution and Formula for orthogonal projection

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August 2021

1 Introduction

Situations when we want to solve an equation that does not have a solution can appear naturally, we can solve the equation with the following methods

- Computation: LSS
- Ways of finding $P_{RanA}b$:
  1. Inner product (GA section)
  2. normal equation (OP section)

2 Least square solution

To write down the error $||Ax - b||$ and try to find x minimizing it. So if we can find x st the error is 0, the system is consistent and we have exact solution. Otherwise, we get the least-square solution.

$$||Ax - b||^2 = \sum_{k=1}^{m} |(Ax)_k - b_k|^2 = \sum_{k=1}^{m} |\sum_{j=1}^{n} A_{k,j}x_j - b_k|^2$$

If we are in $\mathbb{R}^n$ and everything is real, we can abandon abs values.

3 Geometric approach

The value of $||Ax - b||$ is minimal iff $Ax = P_{RanA}b$, which is the orthogonal projection of b onto the column space $RanA$.
So, to find the least square solution we simply need to solve

$$Ax = P_{RanA}b$$

If we know an orthogonal basis v_i in $RanA$, we can find vector $P_{RanA}b$by the formula

$$P_{RanA}b = \sum_{k=1}^{n} \frac{(b, v_k)}{||v_k||^2} v_k$$

If we only know a basis in $RanA$, we need to Gram-Schmidt it.

4 Normal equation

A solution of the **normal equation** $A^*Ax = A^*b$ will give the LSS of $Ax = b$. The LSS is unique iff A^*A is invertible.

5 Orthogonal projection

By the above (geometric approach), if x is a solution of the normal equation (a LSS of $Ax = b$), then $Ax = P_{RanA}b$.
Deriving from this, we have

$$P_{RanA}b = A(A^*A)^{-1}A^*b$$

And since this is true for all b,

$$P_{RanA} = A(A^*A)^{-1}A^*$$

is the formula for the matrix of orthogonal projection onto $RanA$.

6 Thm 4.1

For an $m \times n$ matrix A

$$KerA = Ker(A^*A)$$

This thm implies that for an $m \times n$ matrix A the matrix A^*A is invertible iff $rankA = n$.