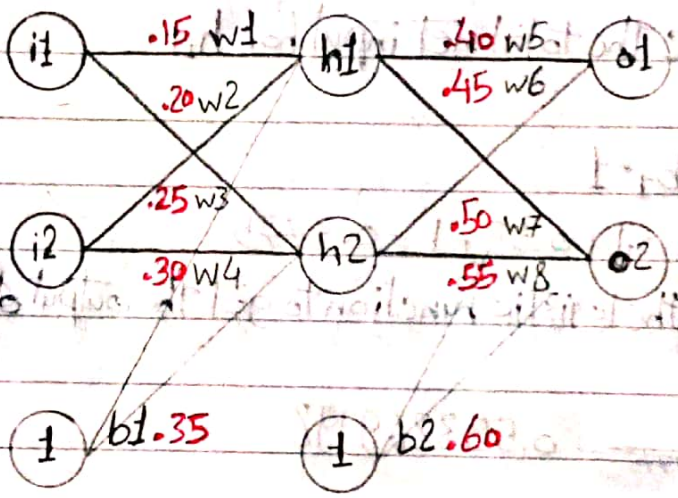
Sheet (1)

Zeyad Emad Mohamed Taha Neural Network, FCI-FU

-Overview

*We're going to use a neural network with two inputs, two hidden neurons, two output neurons. Additionally, the hidden and output neurons will include a bias.

-We have some numbers to work with, here are the Initial weights, the biases, and traing inputs/outputs:



-The goal of backpropagation is to optimize the weights so that the neural network can learn how to correctly map arbitrary inputs to outputs.

-For the rest of this sheet, we're going to work with a single training set: given inputs 0.05 and 0.10, we want the neural network to outputs 0.01 and 0.99.

The forward Pass

- To begin, lets see what the neural network currently predicts given the weights and biases above and inputs of 0.05 and 0.10. To do this, we'll feed those inputs forward though the network.

- We figure out the total net input to each hidden layer neuron, squash the total net input using an activation function (here we use the logistic function), then repeat the process with the output layer neurons.

- Here's how we calculate the total net input for h_1:

$net_{h1} = w_1 * i_1 + w_2 * i_2 + b_1 * 1$

$= 0.15 * 0.05 + 0.2 * 0.1 + 0.35 * 1 = 0.3775$

We then squash it using the logistic function to get the output of h_1:

$Out_{h1} = \frac{1}{1+e^{-net_{h1}}} = \frac{1}{1+e^{-0.3775}} = 0.593269992$

Carrying out the same process for h_2 we get:

$Out_{h2} = 0.596884378$

- We repeat this process for the output layer neurons, using the output from the hidden layer neurons as inputs.

Here's the output for O_1:

$net_{o1} = w_5 * Out_{h1} + w_6 * Out_{h2} + b_2 * 1$

$= 0.4 * 0.593269992 + 0.45 * 0.596884378 + 0.6 * 1 = 1.105905967$

$$out_{o1} = \frac{1}{1+e^{-net_{o1}}} = \frac{1}{1+e^{-1.105905967}} = 0.75136507$$

And carrying out the same process for O_2 we get:

$$out_{o2} = 0.772928465$$

Calculating the Total Error

We can now calculate the error for each output neuron using the squared error function and sum them to get the total error:

$$E_{total} = \sum \frac{1}{2}(target - output)^2$$

The $(\frac{1}{2})$ is included so that exponent is cancelled when we differentiate later on. The result is eventually multiplied by a learning rate anyway so it doesn't matter that we introduce a constant. For example, the target output for O_1 is 0.01 but the neural network output **0.75136507**, therefore its error is:

$$E_{o1} = \frac{1}{2}(target_{o1} - out_{o1})^2 = \frac{1}{2}(0.01 - 0.75136507)^2 = 0.274811083$$

Repeating this process for O_2 (target is 0.99) we get:

$$E_{o2} = 0.023560026$$

The total error for the neural network is the sum of these errors:

$$E_{total} = E_{o1} + E_{o2} = 0.274811083 + 0.023560026 = 0.298371109$$

The Backward Pass

- Our goal with backpropagation is to update each of the weights in the network so that they cause the actual output to be closer the target output, thereby minimizing the error for each output neuron and the network as a whole.

Output Layer:

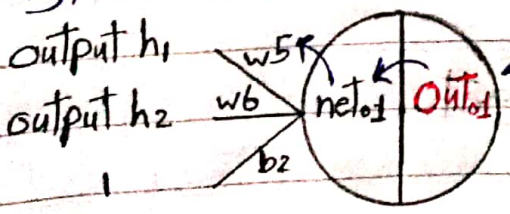
- Consider w_5. We want to know how much a change in w_5 affects the total error, aka $\frac{\partial E_{total}}{\partial w_5}$.

$\frac{\partial E_{total}}{\partial w_5}$ is read as "the partial derivative of E_{total} with respect to w_5. You can also say "the gradient with respect to w_5".

- By applying the chain rule we know that:

$$\frac{\partial E_{total}}{\partial w_5} = \frac{\partial E_{total}}{\partial out_{o1}} * \frac{\partial out_{o1}}{\partial net_{o1}} * \frac{\partial net_{o1}}{\partial w_5}$$

- Visually, here's what we're doing:



$$E_{o1} = \frac{1}{2}(target_{o1} - out_{o1})^2$$
$$E_{total} = E_{o1} + E_{o2}$$

- We need to figure out each piece in this equation.

- First, how much does the total error change with respect to the output?

$$E_{total} = \frac{1}{2}(target_{o1} - out_{o1})^2 + \frac{1}{2}(target_{o2} - out_{o2})^2$$

$$\frac{\partial E_{total}}{\partial out_{o1}} = 2 * \frac{1}{2}(target_{o1} - out_{o1})^{2-1} * -1 + 0$$
$$= -(target_{o1} - out_{o1}) = -(0.01 - 0.75136507)$$
$$= 0.74136507$$

Next, how much does the output of o_1 change with respect to its total net input?

The partial derivative of the logistic function is the output multiplied by 1 minus the output.

$$out_{o1} = \frac{1}{1+e^{-net_{o1}}}$$

$$\frac{\partial out_{o1}}{\partial net_{o1}} = out_{o1}(1 - out_{o1}) = 0.75136507(1 - 0.75136507) = 0.186815602$$

Finally, how much does the total net input of o_1 change with respect to w_5?

$$net_{o1} = w_5 * out_{h1} + w_6 * out_{h2} + b_2 * 1$$

$$\frac{\partial net_{o1}}{\partial w_5} = 1 * out_{h1} * w_5^{(1-1)} + 0 + 0 = out_{h1} = 0.593269992$$

Putting it all together:

$$\frac{\partial E_{total}}{\partial w_5} = \frac{\partial E_{total}}{\partial out_{o1}} * \frac{\partial out_{o1}}{\partial net_{o1}} * \frac{\partial net_{o1}}{\partial w_5}$$
$$= 0.74136507 * 0.186815602 * 0.593269992 = 0.082167041$$

To decrease the error, we then subtract this value from the current weight (optionally multiplied by some learning rate, eta, which we'll set to 0.5)

$$w_5^+ = w_5 - \eta * \frac{\partial E_{total}}{\partial w_5} = 0.4 - 0.5 * 0.082167041 = 0.35891648$$

- We can repeat this process to get the new weights w_6, w_7 and w_8.

$w_6^+ = 0.408666186$
$w_7^+ = 0.511301270$
$w_8^+ = 0.561370121$

- We perform the actual updates in the neural network after we have the new weights leading into the hidden layer neurons.

Hidden Layer

- Next, we'll continue the backwards pass by calculating new values for w_1, w_2, w_3 and w_4.

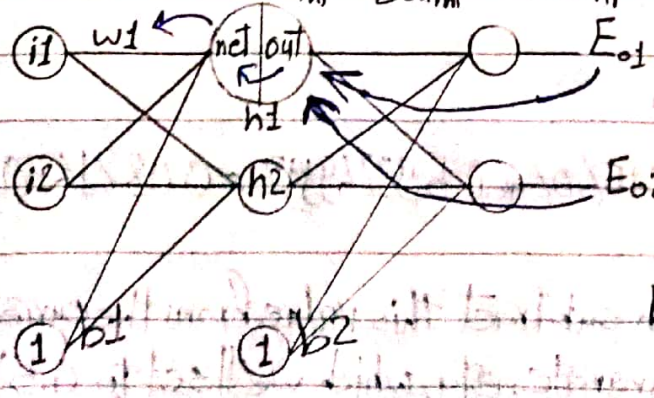
Big picture, here's what we need to figure out:

$$\frac{\partial E_{total}}{\partial w_1} = \frac{\partial E_{total}}{\partial out_{h1}} * \frac{\partial out_{h1}}{\partial net_{h1}} * \frac{\partial net_{h1}}{\partial w_1}$$

- Visually:

$$\frac{\partial E_{total}}{\partial w_1} = \frac{\partial E_{total}}{\partial out_{h1}} * \frac{\partial out_{h1}}{\partial net_{h1}} * \frac{\partial net_{h1}}{\partial w_1}$$

$$\frac{\partial E_{total}}{\partial out_{h1}} = \frac{\partial E_{o1}}{\partial out_{h1}} + \frac{\partial E_{o2}}{\partial out_{h1}}$$



$$E_{total} = E_{o1} + E_{o2}$$

We're going to use a similar process as we did for the output layer, but slightly different to account for the fact that the output of each hidden layer neuron contributes to the output (and therefore error) of multiple output neurons.

We

We Know that out_{h1} affects both out_{o1} and out_{o2} therefore the $\frac{\partial E_{total}}{\partial out_{h}}$ needs to take into consideration its effect on the both output neurons:

$$\frac{\partial E_{total}}{\partial out_{h1}} = \frac{\partial E_{o1}}{\partial out_{h1}} + \frac{\partial E_{o2}}{\partial out_{h1}}$$

Starting with $\frac{\partial E_{o1}}{\partial out_{h1}}$: $\frac{\partial E_{o1}}{\partial out_{h1}} = \frac{\partial E_{o1}}{\partial net_{o1}} * \frac{\partial net_{o1}}{\partial out_{h1}}$

We can calculate $\frac{\partial E_{o1}}{\partial net_{o1}}$ using values we calculated earlier:

$$\frac{\partial E_{o1}}{\partial net_{o1}} = 0.74136507 * 0.186815602 = 0.138498562$$

And $\frac{\partial net_{o1}}{\partial out_{h1}}$ is equal to w_5: $net_{o1} = w_5 * out_{h1} + w_6 * out_{h2} + b_2 * 1$

$$\frac{\partial net_{o1}}{\partial out_{h1}} = w_5 = 0.40$$

Plugging them in: $\frac{\partial E_{o1}}{\partial out_{h1}} = 0.138498562 * 0.40 = 0.055399425$

Following the same process for $\frac{\partial E_{o2}}{\partial out_{h1}}$ we get: $\frac{\partial E_{o2}}{\partial out_{h1}} = -0.019049119$

Therefore: $\frac{\partial E_{total}}{\partial out_{h1}} = \frac{\partial E_{o1}}{\partial out_{h1}} + \frac{\partial E_{o2}}{\partial out_{h1}} = 0.055399425 + -0.019049119 = 0.036350306$

Now that we have $\frac{\partial E_{total}}{\partial out_{h1}}$, we need to figure out $\frac{\partial out_{h1}}{\partial net_{h1}}$ and $\frac{\partial net_{h1}}{\partial w_1}$ for each weight:

$$out_{h1} = \frac{1}{1+e^{-net_{h1}}}$$

$$\frac{\partial out_{h1}}{\partial net_{h1}} = out_{h1}(1 - out_{h1}) = 0.59326999(1 - 0.59326999) = 0.241300709$$

We calculate the partial derivate of the total net input to h_1 with respect to w_1 the same as we did for the output neuron:

$net_{h_1} = w_1 * i_1 + w_3 * i_2 + b_1 * 1$

$\frac{\partial net_{h_1}}{\partial w_1} = i_1 = 0.05$

Putting all together: $\frac{\partial E_{total}}{\partial w_1} = 0.036350306 * 0.241300709 * 0.05$
$= 0.000438568$

We can update w_1 now:

$w_1^+ = w_1 - \eta * \frac{\partial E_{total}}{\partial w_1} = 0.15 - 0.5 * 0.000438568 = 0.149780716$

repeating this for w_2, w_3 and w_4:

$w_2^+ = 0.19956143$
$w_3^+ = 0.24975114$
$w_4^+ = 0.29950229$

- Finally, we've updated all of our weights! When we fed forward the 0.05 and 0.1 inputs originally, the error on the network was 0.298371109. After this first round of backpropagation, the total error is now down to 0.291027924. It might not seem like much, but after repeating this process 10000 times, for example, the error plummets to 0.0000351085. at this point when we feed forward 0.05 and 0.1, the two output neurons generate 0.015912196 (vs. 0.01 target) and 0.984065734 (vs 0.99 target).