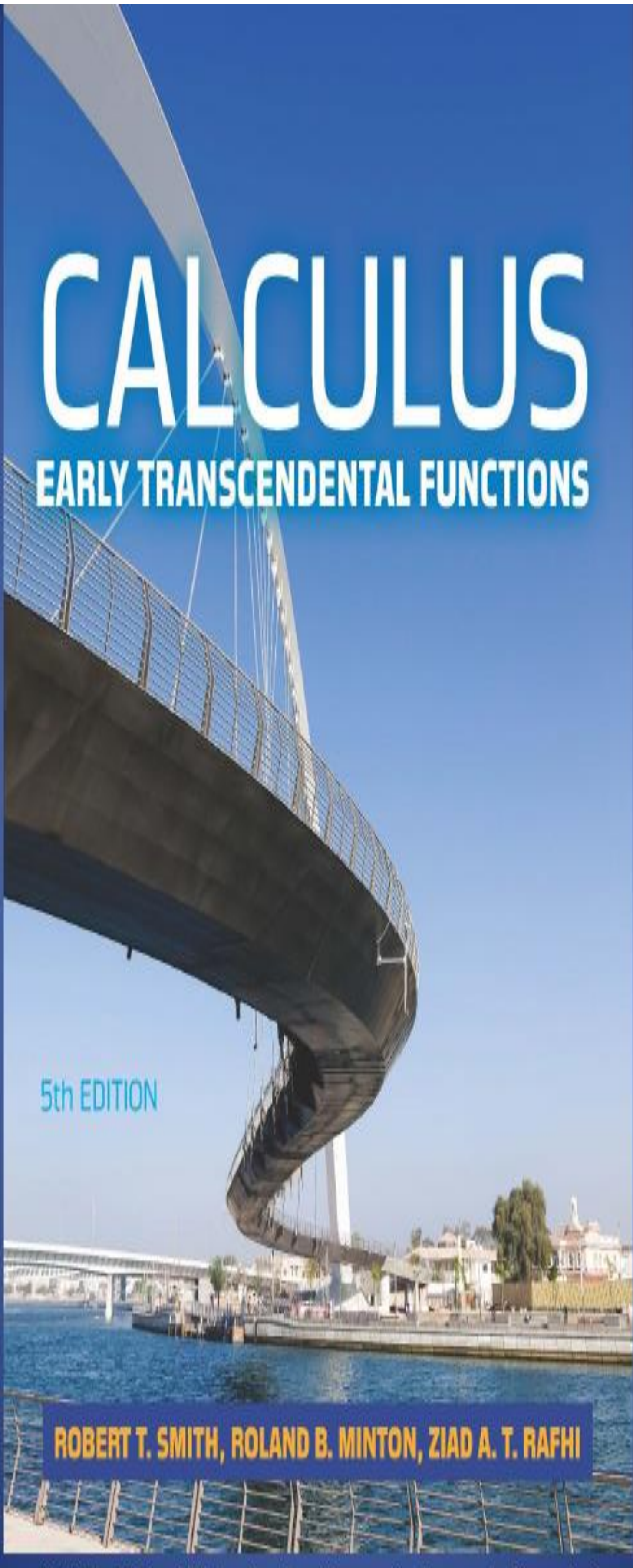
CALCULUS
EARLY TRANSCENDENTAL FUNCTIONS
5th EDITION
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Applications of Definite Integral:

1) Arc length

The arc length of the curve $y = f(x)$ for $a \leq x \leq b$ is given by

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

Example:

Find the length of the curve $y = \sqrt{3}\, x + 12$ for $0 \leq x \leq \frac{1}{2}$

$$\frac{dy}{dx} = \sqrt{3}$$

$$L = \int_0^{\frac{1}{2}} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

$$= \int_0^{\frac{1}{2}} \sqrt{1 + \left(\sqrt{3}\right)^2}\, dx$$

$$= \int_0^{\frac{1}{2}} \sqrt{1 + 3}\, dx$$

$$= 2\int_0^{\frac{1}{2}} dx$$

$$= 2x \ \ \Big|_0^{\frac{1}{2}}$$

$$= 1 - 0 = 1$$

Example:

Find the length of the curve $y = \frac{2}{3}\, x^{\frac{3}{2}} + 12$ for $0 \leq x \leq 3$

$$\frac{dy}{dx} = \frac{2}{3}\,\frac{3}{2}\, x^{\frac{3}{2}-1} = x^{\frac{1}{2}}$$

$$L = \int_0^3 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

$$= \int_0^3 \sqrt{1 + \left(x^{\frac{1}{2}}\right)^2}\ dx$$

$$= \int_0^3 \sqrt{1+x}\, dx$$

$$= \int_0^3 (1+x)^{\frac{1}{2}}\, dx$$

$$= \frac{(1+x)^{\frac{3}{2}}}{\frac{3}{2}}$$

$$= \frac{2}{3}(1+x)^{\frac{3}{2}} \quad \Big|_0^3$$

$$= \left(\frac{2}{3}(1+3)^{\frac{3}{2}}\right) - \left(\frac{2}{3}(1+0)^{\frac{3}{2}}\right)$$

$$= \frac{16}{3} - \frac{2}{3} = \frac{14}{3}$$

2) Area of region

The area enclosed by the curves

$$y = f(x), \;\; y = g(x), \;\; f(x) \geq g(x)$$

on the interval $a \leq x \leq b$ is given by

$$A = \int_a^b (f(x) - g(x))\, dx$$

Example

Find the area enclosed by the curves

$$y = x, \;\; y = x^3, \;\; 0 \leq x \leq 1$$

$$A = \int_0^1 (x - x^3)\, dx$$

$$= \frac{x^2}{2} - \frac{x^4}{4} \quad |_0^1$$

$$= \left(\frac{1^2}{2} - \frac{1^4}{4}\right) - \left(\frac{0^2}{2} - \frac{0^4}{4}\right)$$

$$= \frac{1}{4} - 0 = \frac{1}{4}$$

Example

Find the area enclosed by the curves

$$y = \sqrt{x}, \;\; x - axis, \;\; 0 \le x \le 1$$

$$A = \int_0^1 (\sqrt{x} - 0)\, dx$$

$$= \int_0^1 \sqrt{x}\, dx = \int_0^1 x^{\frac{1}{2}}\, dx$$

$$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}}$$

$$= \frac{2}{3}\, x^{\frac{3}{2}} \;\Big|_0^1$$

$$= \left(\frac{2}{3}\; 1^{\frac{3}{2}}\right) - \left(\frac{2}{3}\; 0^{\frac{3}{2}}\right)$$

$$= \frac{2}{3}$$

Example

Find the area enclosed by the curves

$$y = cosx, \;\; y = sinx, \;\; x = 0, \;\; x = \frac{\pi}{4}$$

$$A = \int_0^{\frac{\pi}{4}} (cosx - sinx)\, dx$$

$$= sinx + cosx \;\Big|_0^{\frac{\pi}{4}}$$

$$= \left(sin\frac{\pi}{4} + cos\frac{\pi}{4}\right) - (sin0 + cos0)$$

$$= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) - (0 + 1)$$

$$= \frac{2}{\sqrt{2}} - 1 = \sqrt{2} - 1$$

Remark

The area enclosed by the curves

$$x = f(y), \;\; x = g(y), \;\; f(y) \geq g(y)$$

on the interval $a \leq y \leq b$ is given by

$$A = \int_a^b (f(y) - g(y))\, dy$$

Example

Find the area enclosed by the curves

$$x = \sqrt{y}, \;\; y - axis, \;\; 0 \leq y \leq 1$$

$$A = \int_0^1 (\sqrt{y} - 0)\, dy$$

$$= \int_0^1 \sqrt{y}\, dy$$

$$= \int_0^1 y^{\frac{1}{2}}\, dy$$

$$= \frac{y^{\frac{3}{2}}}{\frac{3}{2}}$$

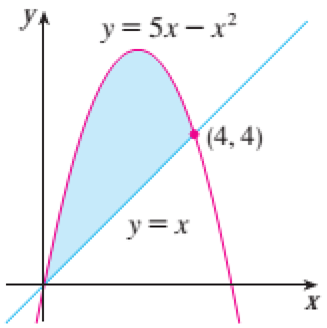
$$= \frac{2}{3}\, y^{\frac{3}{2}} \;\; |_0^1$$

$$= \left(\frac{2}{3}\, 1^{\frac{3}{2}}\right) - \left(\frac{2}{3}\, 0^{\frac{3}{2}}\right)$$

$$= \frac{2}{3}$$

Example

Find the area of the shaded region



$$A = \int_0^4 (5x - x^2 - x)\, dx$$

$$= \int_0^4 (4x - x^2)\, dx$$

$$= \frac{4x^2}{2} - \frac{x^3}{3}$$

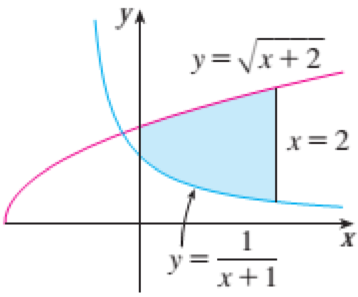
$$= 2x^2 - \frac{x^3}{3} \ \Big|_0^4$$

$$= \left(32 - \frac{64}{3}\right) - (0)$$

$$= \frac{32}{3}$$

Example

Find the area of the shaded region



$$A = \int_0^2 (\sqrt{x+2} - \frac{1}{x+1})\, dx$$

$$= \frac{(x+2)^{\frac{3}{2}}}{\frac{3}{2}} - ln|x+1| \ \Big|_0^2$$

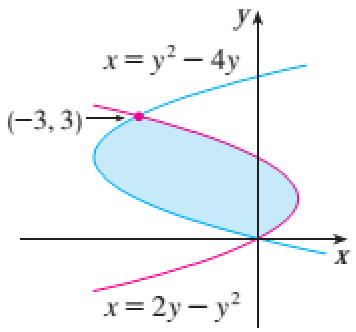
$$= \frac{2}{3}(x+2)^{\frac{3}{2}} - ln|x+1| \ |_0^2$$

$$= \left(\frac{2}{3}\ (4)^{\frac{3}{2}}\ - ln3\right) - \left(\frac{2}{3}\ (2)^{\frac{3}{2}}\ - ln1\right)$$

$$= \frac{16}{3}\ - ln3 - \frac{4\sqrt{2}}{3}$$

Example

Find the area of the shaded region



$$A = \int_0^3 ((2y\ - y^2) - (y^2 - 4y))\ dy$$

$$= \int_0^3 (2y\ - y^2\ - y^2 + 4y)\ dy$$

$$= \int_0^3 (6y\ - 2y^2\ )\ dy$$

$$= \frac{6y^2}{2} - \frac{2y^3}{3}$$

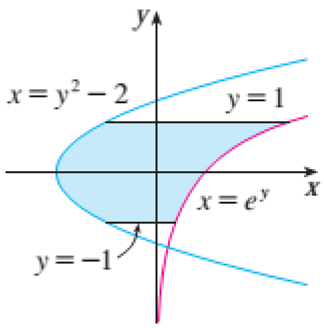
$$= 3y^2 - \frac{2y^3}{3} \quad |_0^3$$

$$= \left(27 - \frac{54}{3}\right) - (0)$$

$$= 27 - 18 = 9$$

Example

Find the area of the shaded region



$$A = \int_{-1}^{1} ((e^y) - (y^2 - 2))\, dy$$

$$= \int_{-1}^{1} (e^y - y^2 + 2)\, dy$$

$$= e^y - \frac{y^3}{3} + 2y \ \Big|_{-1}^{1}$$

$$= \left(e - \frac{1}{3} + 2\right) - \left(e^{-1} - \frac{-1}{3} - 2\right)$$

$$= e - \frac{1}{3} + 2 - e^{-1} + \frac{-1}{3} + 2$$

$$= e - \frac{1}{e} + \frac{10}{3}$$