Tutorial 9

Exercise 1

1. In a certain community, levels of air pollution may exceed federal standards for ozone or for particulate matter on some days. For a particular summer season, let X be the number of days on which the ozone standard is exceeded and let Y be the number of days on which the particulate matter standard is exceeded. Assume that the joint probability mass function of X and Y is given in the following table:

	y		
x	**0**	**1**	**2**
0	0.10	0.11	0.05
1	0.17	0.23	0.08
2	0.06	0.14	0.06

a. Find $P(X = 1 \text{ and } Y = 0)$.

b. Find $P(X \geq 1 \text{ and } Y < 2)$.

c. Find $P(X < 1)$.

d. Find $P(Y \geq 1)$.

e. Find the probability that the standard for ozone is exceeded at least once.

f. Find the probability that the standard for particulate matter is never exceeded.

g. Find the probability that neither standard is ever exceeded.

Exercise 1 Solution

(a) 0.17

(b) $P(X \geq 1 \text{ and } Y < 2) = P(1, 0) + P(1, 1) + P(2, 0) + P(2, 1) = 0.17 + 0.23 + 0.06 + 0.14 = 0.60$

(c) $P(X < 1) = P(X = 0) = P(0, 0) + P(0, 1) + P(0, 2) = 0.10 + 0.11 + 0.05 = 0.26$

(d) $P(Y \geq 1) = 1 - P(Y = 0) = 1 - P(0, 0) - P(1, 0) - P(2, 0) = 1 - 0.10 - 0.17 - 0.06 = 0.67$

(e) $P(X \geq 1) = 1 - P(X = 0) = 1 - P(0, 0) - P(0, 1) - P(0, 2) = 1 - 0.10 - 0.11 - 0.05 = 0.74$

(f) $P(Y = 0) = P(0, 0) + P(1, 0) + P(2, 0) = 0.10 + 0.17 + 0.06 = 0.33$

(g) $P(X = 0 \text{ and } Y = 0) = 0.10$

Exercise 2

x	y: 0	y: 1	y: 2
0	0.10	0.11	0.05
1	0.17	0.23	0.08
2	0.06	0.14	0.06

2. Refer to Exercise 1.

a. Find the marginal probability mass function $p_X(x)$.

b. Find the marginal probability mass function $p_Y(y)$.

c. Find μ_X.

d. Find μ_Y.

e. Find σ_X.

f. Find σ_Y.

g. Find $\mathrm{Cov}(X, Y)$.

h. Find $\rho_{X,Y}$.

Exercise 2 solution

(a) The marginal probability mass function $p_X(x)$ is found by summing along the rows of the joint probability mass function.

x \ y	0	1	2	$p_X(x)$
0	0.10	0.11	0.05	0.26
1	0.17	0.23	0.08	0.48
2	0.06	0.14	0.06	0.26
$p_Y(y)$	0.33	0.48	0.19	

$p_X(0) = 0.26$, $p_X(1) = 0.48$, $p_X(2) = 0.26$, $p_X(x) = 0$ if $x \neq 0, 1,$ or 2

(b) The marginal probability mass function $p_Y(y)$ is found by summing down the columns of the joint probability mass function. So $p_Y(0) = 0.33$, $p_Y(1) = 0.48$, $p_Y(2) = 0.19$, $p_Y(y) = 0$ if $y \neq 0, 1,$ or 2

(c) $\mu_X = 0p_X(0) + 1p_X(1) + 2p_X(2) = 0(0.26) + 1(0.48) + 2(0.26) = 1.00$

(d) $\mu_Y = 0p_Y(0) + 1p_Y(1) + 2p_Y(2) = 0(0.33) + 1(0.48) + 2(0.19) = 0.86$

Exercise 2 solution

(e) $\sigma_X^2 = 0^2 p_X(0) = 1^2 p_X(1) + 2^2 p_X(2) - \mu_X^2 = 0^2(0.26) + 1^2(0.48) + 2^2(0.26) - 1.00^2 = 0.5200$

$\sigma_X = \sqrt{0.5200} = 0.7211$

(f) $\sigma_Y^2 = 0^2 p_Y(0) = 1^2 p_Y(1) + 2^2 p_Y(2) - \mu_Y^2 = 0^2(0.33) + 1^2(0.48) + 2^2(0.19) - 0.86^2 = 0.5004$

$\sigma_Y = \sqrt{0.5004} = 0.7074$

(g) $\mathrm{Cov}(X, Y) = \mu_{XY} - \mu_X \mu_Y$

$$\begin{aligned}
\mu_{XY} &= (0)(0)p_{X,Y}(0,0) + (0)(1)p_{X,Y}(0,1) + (0)(2)p_{X,Y}(0,2) + (1)(0)p_{X,Y}(1,0) + (1)(1)p_{X,Y}(1,1) \\
&\quad + (1)(2)p_{X,Y}(1,2) + (2)(0)p_{X,Y}(2,0) + (2)(1)p_{X,Y}(2,1) + (2)(2)p_{X,Y}(2,2) \\
&= (0)(0)(0.10) + (0)(1)(0.11) + (0)(2)(0.05) + (1)(0)(0.17) + (1)(1)(0.23) \\
&\quad + (1)(2)(0.08) + (2)(0)(0.06) + (2)(1)(0.14) + (2)(2)(0.06) \\
&= 0.91
\end{aligned}$$

$\mu_X = 1.00, \mu_Y = 0.86$

$\mathrm{Cov}(X, Y) = 0.91 - (1.00)(0.86) = 0.0500$

(h) $\rho_{X,Y} = \dfrac{\mathrm{Cov}(X,Y)}{\sigma_X \sigma_Y} = \dfrac{0.0500}{(0.7211)(0.7074)} = 0.0980$

Exercise 3

In a piston assembly, the specifications for the clearance between piston rings and the cylinder wall are very tight. In a lot of assemblies, let X be the number with too little clearance and let Y be the number with too much clearance. The joint probability mass function of X and Y is given in the table below:

	y			
x	**0**	**1**	**2**	**3**
0	0.15	0.12	0.11	0.10
1	0.09	0.07	0.05	0.04
2	0.06	0.05	0.04	0.02
3	0.04	0.03	0.02	0.01

a. Find the marginal probability mass function of X.

b. Find the marginal probability mass function of Y.

c. Are X and Y independent? Explain.

d. Find μ_X and μ_Y.

e. Find σ_X and σ_Y.

f. Find $\mathrm{Cov}(X, Y)$.

g. Find $\rho(X, Y)$.

Exercise 3 solution

(a) The marginal probability mass function $p_X(x)$ is found by summing along the rows of the joint probability mass function.

	y				
x	0	1	2	3	$p_X(x)$
0	0.15	0.12	0.11	0.10	0.48
1	0.09	0.07	0.05	0.04	0.25
2	0.06	0.05	0.04	0.02	0.17
3	0.04	0.03	0.02	0.01	0.10
$p_X(y)$	0.34	0.27	0.22	0.17	

$p_X(0) = 0.48$, $p_X(1) = 0.25$, $p_X(2) = 0.17$, $p_X(3) = 0.10$, $p_X(x) = 0$ if $X \neq 0, 1, 2,$ or 3

(b) The marginal probability mass function $p_Y(y)$ is found by summing down the columns of the joint probability mass function. So $p_Y(0) = 0.34$, $p_Y(1) = 0.27$, $p_Y(2) = 0.22$, $p_Y(3) = 0.17$, $p_Y(y) = 0$ if $y \neq 0, 1, 2,$ or 3

(c) No, X and Y are not independent. For example, $P(X = 0 \text{ and } Y = 0) = 0.15$, but $P(X = 0)P(Y = 0) = (0.48)(0.34) = 0.1632 \neq 0.15$.

(d) $\mu_X = 0p_X(0) + 1p_X(1) + 2p_X(2) + 3p_X(3) = 0(0.48) + 1(0.25) + 2(0.17) + 3(0.10) = 0.89$
$\mu_Y = 0p_Y(0) + 1p_Y(1) + 2p_Y(2) + 3p_Y(3) = 0(0.34) + 1(0.27) + 2(0.22) + 3(0.17) = 1.22$

Exercise 3 solution

(e) $\sigma_X^2 = 0^2 p_X(0) + 1^2 p_X(1) + 2^2 p_X(2) + 3^2 p_X(3) - \mu_X^2$

$= 0^2(0.48) + 1^2(0.25) + 2^2(0.17) + 3^2(0.10) - 0.89^2 = 1.0379$

$\sigma_X = \sqrt{1.0379} = 1.0188$

$\sigma_Y^2 = 0^2 p_Y(0) + 1^2 p_Y(1) + 2^2 p_Y(2) + 3^2 p_Y(3) - \mu_Y^2$

$= 0^2(0.34) + 1^2(0.27) + 2^2(0.22) + 3^2(0.17) - 1.22^2 = 1.1916$

$\sigma_Y = \sqrt{1.1916} = 1.0916$

(f) $\mathrm{Cov}(X,Y) = \mu_{XY} - \mu_X \mu_Y$

μ_{XY} = $(0)(0)p_{X,Y}(0, 0) + (0)(1)p_{X,Y}(0, 1) + (0)(2)p_{X,Y}(0, 2) + (0)(3)p_{X,Y}(0, 3) + (1)(0)p_{X,Y}(1, 0)$
$+ (1)(1)p_{X,Y}(1, 1) + (1)(2)p_{X,Y}(1, 2) + (1)(3)p_{X,Y}(1, 2) + (2)(0)p_{X,Y}(2, 0) + (2)(1)p_{X,Y}(2, 1)$
$+ (2)(2)p_{X,Y}(2, 2) + (2)(3)p_{X,Y}(2, 2) + (3)(0)p_{X,Y}(3, 0) + (3)(1)p_{X,Y}(3,1) + (3)(2)p_{X,Y}(3, 3)$
$+ (3)(3)p_{X,Y}(3, 3)$

= $(0)(0)(0.15) + (0)(1)(0.12) + (0)(2)(0.11) + (0)(3)(0.10)$
$+ (1)(0)(0.09) + (1)(1)(0.07) + (1)(2)(0.05) + (1)(3)(0.04)$
$+ (2)(0)(0.06) + (2)(1)(0.05) + (2)(2)(0.04) + (2)(3)(0.02)$
$+ (3)(0)(0.04) + (3)(1)(0.03) + (3)(2)(0.02) + (4)(3)(0.01)$

= 0.97

$\mu_X = 0.89$, $\mu_Y = 1.22$

$\mathrm{Cov}(X, Y) = 0.97 - (0.89)(1.22) = -0.1158$

(g) $\rho_{X,Y} = \dfrac{\mathrm{Cov}(X,Y)}{\sigma_X \sigma_Y} = \dfrac{-0.1158}{(1.0188)(1.0196)} = -0.1041$

Homework

Automobile engines and transmissions are produced on assembly lines, and are inspected for defects after they come off their assembly lines. Those with defects are repaired. Let X represent the number of engines, and Y the number of transmissions that require repairs in a one-hour time interval. The joint probability mass function of X and Y is as follows:

	y			
x	**0**	**1**	**2**	**3**
0	0.13	0.10	0.07	0.03
1	0.12	0.16	0.08	0.04
2	0.02	0.06	0.08	0.04
3	0.01	0.02	0.02	0.02

a. Find the marginal probability mass function $p_X(x)$.

b. Find the marginal probability mass function $p_Y(y)$.

c. Find μ_X.

d. Find μ_Y.

e. Find σ_X.

f. Find σ_Y.

g. Find $\mathrm{Cov}(X, Y)$.

h. Find $\rho_{X,Y}$.

Homework

The General Social Survey asked a sample of adults how many siblings (brothers and sisters) they had *(X)* and also how many children they had *(Y)*. We show results for those who had no more than 4 children and no more than 4 siblings. Assume that the joint probability mass function is given in the following contingency table.

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	y				
x	**0**	**1**	**2**	**3**	**4**
0	0.03	0.01	0.02	0.01	0.01
1	0.09	0.05	0.08	0.03	0.01
2	0.09	0.05	0.07	0.04	0.02
3	0.06	0.04	0.07	0.04	0.02
4	0.04	0.03	0.04	0.03	0.02

a. Find the marginal probability mass function of X.

b. Find the marginal probability mass function of Y.

c. Are X and Y independent? Explain.

d. Find μ_X and μ_Y.

e. Find σ_x and σ_Y.

f. Find $\mathrm{Cov}(X, Y)$.

g. Find $\rho(X, Y)$.

Thank You