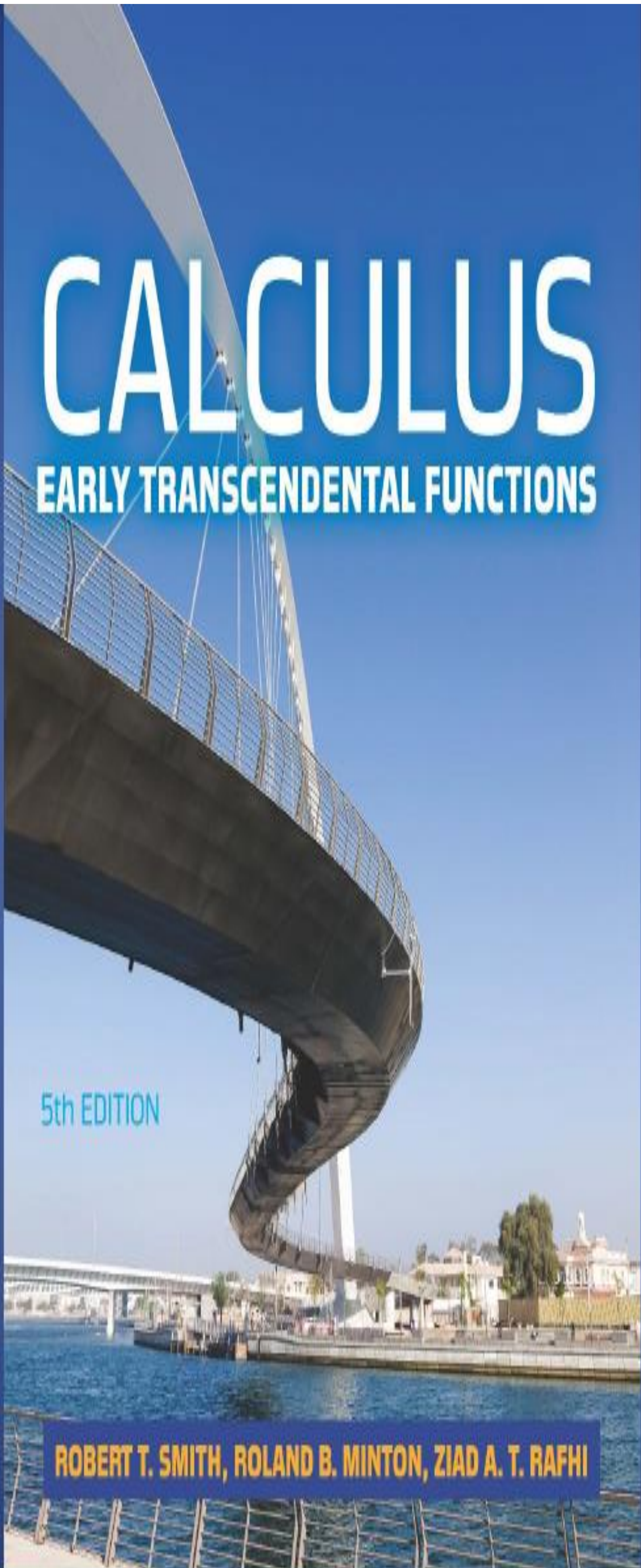
CALCULUS
EARLY TRANSCENDENTAL FUNCTIONS
5th EDITION
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Methods of Integration

1- Integration by Parts

2- Integration by substitution

i) Algebraic substitutions

ii) Trigonometric substitutions

a) ***For*** $\sqrt{a^2 - x^2}$

we use $x = a\ sin\theta$

$$dx = a\ cos\theta\ d\theta$$

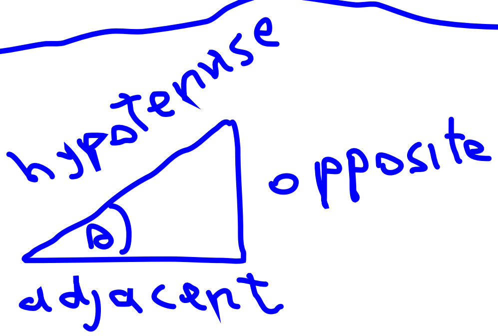
because $\sqrt{a^2 - x^2} =$

$$= \sqrt{a^2 - a^2 sin^2\theta}$$

$$= a\sqrt{1 - sin^2\theta} = a\ cos\theta$$

Remark

$sin\theta = \frac{opp}{hyp}$, $cos\theta = \frac{adj}{hyp}$, $tan\theta = \frac{opp}{adj}$



Example $$\int \frac{1}{x^2\sqrt{4-x^2}}\ dx$$

$$x = 2\ sin\theta$$

$$dx = 2\ cos\theta\ d\theta$$

$$\rightarrow \int \frac{1}{x^2\sqrt{4-x^2}}\ dx =$$

$$= \int \frac{1}{4sin^2\theta\sqrt{4-4sin^2\theta}}\ 2\ cos\theta\ d\theta$$

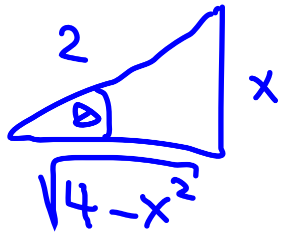
$$= \frac{1}{4}\int \frac{cos\theta}{sin^2\theta\sqrt{1-sin^2\theta}}\ d\theta$$

$$= \frac{1}{4}\int \frac{cos\theta}{sin^2\theta\ cos\theta}\ dx$$

$$= \frac{1}{4}\int csc^2\theta\ \ dx$$

$$= -\frac{1}{4}\ cot\theta + c$$

$$\sin\theta = \frac{x}{2}$$



$$\cot\theta = \frac{\sqrt{4-x^2}}{x}$$

$$\rightarrow \int \frac{1}{x^2\sqrt{4-x^2}}\,dx =$$

$$= -\frac{1}{4}\frac{\sqrt{4-x^2}}{x} + c$$

Remark:

$$\textbf{For} \qquad \sqrt{a^2 - x^2}$$

$$\textbf{we can use} \qquad x = a\cos\theta$$

$\boldsymbol{b})$ $\qquad \boldsymbol{For} \qquad \sqrt{a^2 + x^2}$

$\boldsymbol{we\ use} \qquad x = a\ tan\theta$

$$dx = a\ sec^2\theta\ d\theta$$

$\boldsymbol{because} \qquad \sqrt{a^2 + x^2} =$

$$= \sqrt{a^2 + a^2tan^2\theta}$$

$$= a\sqrt{1 + tan^2\theta}$$

$$= a\ sec\theta$$

Example $$\int \frac{1}{\sqrt{9+x^2}}\,dx$$

$$x = 3\,tan\theta$$

$$dx = 3\,sec^2\theta\,d\theta$$

$$\rightarrow \int \frac{1}{\sqrt{9+x^2}}\,dx =$$

$$= \int \frac{1}{\sqrt{9+9\,tan^2\theta}}\,3\,sec^2\theta\,d\theta$$

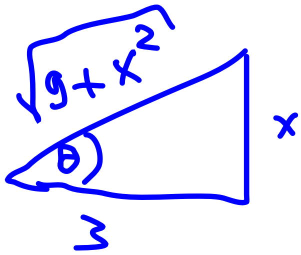
$$= \int \frac{sec^2\theta}{\sqrt{1+tan^2\theta}}\,d\theta$$

$$= \int \frac{sec^2\theta}{sec\theta}\,d\cancel{x}\theta$$

$$= \int sec\theta\,d\cancel{x}\theta$$

$$= ln|sec\theta + tan\theta| + c$$

$$, \quad tan\theta = \frac{x}{3}$$



$$sec\theta = \frac{\sqrt{9+x^2}}{3}$$

$$\rightarrow \qquad \int \frac{1}{\sqrt{9+x^2}}\, dx =$$

$$= ln\left|\frac{\sqrt{9+x^2}}{3} + \frac{x}{3}\right| + c$$

$\boldsymbol{c})$ $\boldsymbol{For}$ $\sqrt{x^2 - a^2}$

$\boldsymbol{we\ use}$ $x = a\ sec\theta$

$$dx = a\ sec\theta\ tan\theta\ d\theta$$

$\boldsymbol{because}$ $\sqrt{x^2 - a^2} =$

$$= \sqrt{a^2 sec^2\theta - a^2}$$

$$= a\sqrt{sec^2\theta - 1}$$

$$= a\ tan\theta$$

Example $$\int \frac{\sqrt{x^2-16}}{x}\,dx$$

$$x = 4\,sec\theta$$

$$dx = 4\,sec\theta\,tan\theta\,d\theta$$

$$\rightarrow \quad \int \frac{\sqrt{x^2-16}}{x}\,dx =$$

$$= \int \frac{\sqrt{16sec^2\theta - 16}}{4\,sec\theta} 4\,sec\theta\,tan\theta\,d\theta$$

$$= \int \sqrt{16sec^2\theta - 16}\,tan\theta\,d\theta$$

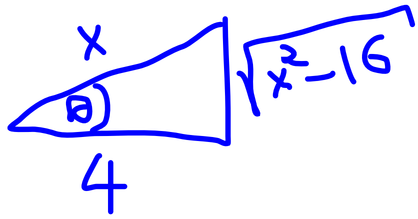
$$= 4\int tan^2\theta\,d\theta$$

$$= 4\int (sec^2\theta - 1)\,d\theta$$

$$= 4(tan\theta - \theta) + c$$

$$, \quad sec\theta = \frac{x}{4}, \quad \theta = sec^{-1}(\frac{x}{4})$$

$$, \quad cos\theta = \frac{4}{x}$$



$$, tan\theta = \frac{\sqrt{x^2 - 16}}{4}$$

$$\rightarrow \qquad \int \frac{\sqrt{x^2 - 16}}{x}\, dx =$$

$$= 4\left[\frac{\sqrt{x^2 - 16}}{4} - sec^{-1}\left(\frac{x}{4}\right)\right] + c$$

$$= \sqrt{x^2 - 16} - 4sec^{-1}\left(\frac{x}{4}\right) + c$$