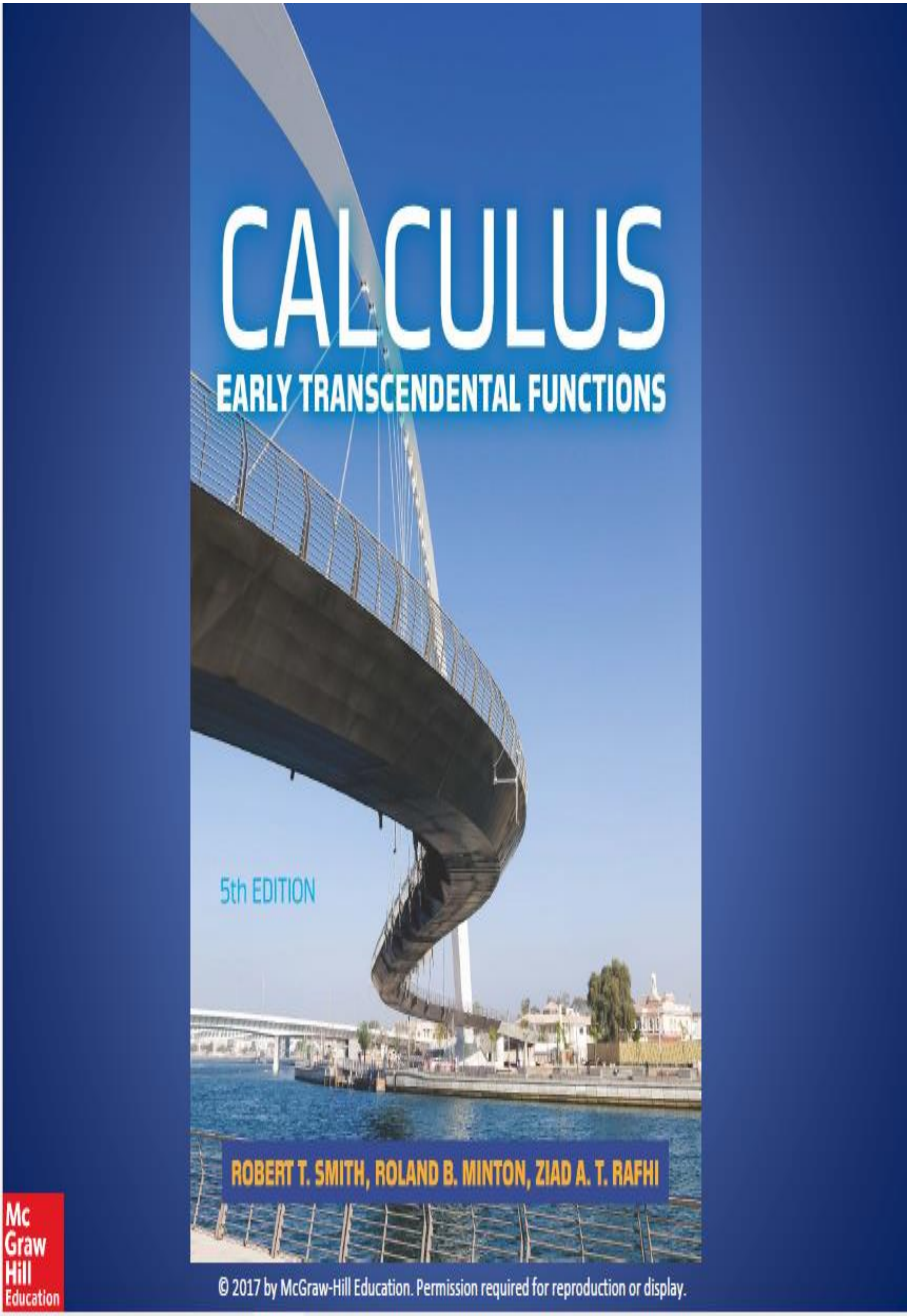
CALCULUS
EARLY TRANSCENDENTAL FUNCTIONS
5th EDITION
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Mc Graw Hill Education
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Improper Integral:

The integral is called improper in two cases

Case 1: If the integrated function is undefined at a point in the integral interval

Example:

The integral $\int_0^1 \frac{1}{x}\,dx$ is improper due to $x = 0$

Example:

The integral $\int_0^1 \frac{1}{x-1}\,dx$ is improper due to $x = 1$

Example:

The integral $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ is improper due to $x = 0$

Example:

The integral $\int_0^1 \frac{1}{\sqrt{1-x}}\,dx$ is improper due to $x = 1$

Example:

The integral $\int_0^3 \frac{1}{x^2}\, dx$ is improper due to $x = 0$

Example:

The integral $\int_0^3 \frac{1}{(x-3)^2}\, dx$ is improper due to $x = 3$

Example:

The integral $\int_{-1}^2 \frac{1}{x^2}\, dx$ is improper due to $x = 0$

Case 2: If the integral interval contains ∞, $-\infty$, or both

Example:

The integral $\int_0^\infty \frac{1}{1+x^2}\, dx$ is improper due to ∞

Example:

The integral $\int_{-\infty}^0 \frac{1}{1+x^2}\, dx$ is improper due to $-\infty$

Example:

The integral $\int_{-\infty}^\infty \frac{1}{1+x^2}\, dx$ is improper due to $-\infty, \infty$

Example:

The integral $\int_1^\infty \frac{1}{x^2}\, dx$ is improper due to ∞

Example:

The integral $\int_1^\infty \frac{1}{x}\, dx$ is improper due to ∞

Example:

The integral $\int_1^\infty \frac{1}{\sqrt{x}}\, dx$ is improper due to ∞

How to calculate an improper integral

Example $$\int_0^1 \frac{1}{\sqrt{x}}\, dx$$

$$\int_0^1 \frac{1}{\sqrt{x}}\, dx = \lim_{R\to 0} \int_R^1 \frac{1}{\sqrt{x}}\, dx$$

$$= \lim_{R\to 0} 2\sqrt{x} \;\Big|_R^1$$

$$= \lim_{R\to 0} 2\sqrt{1} - 2\sqrt{R}$$

$$= 2\sqrt{1} - 2\sqrt{0}$$

$$= 2 - 0 = 2$$

So, the integral $\int_0^1 \frac{1}{\sqrt{x}}\, dx$ is convergent to 2

Example: $$\int_0^1 \frac{1}{x}\,dx$$

$$\int_0^1 \frac{1}{x}\,dx = \lim_{R\to 0} \int_R^1 \frac{1}{x}\,dx$$

$$= \lim_{R\to 0} \; ln|x| \;\Big|_R^1$$

$$= \lim_{R\to 0} \; ln|1| - \; ln|R|$$

$$= ln|1| - \; ln|0|$$

$\ln 0 = -\infty$

$\ln 1 = 0$

$$= 0 + \infty$$

$$= \infty$$

So, the integral $\int_0^1 \frac{1}{x}\,dx$ is divergent

Example: $$\int_{-\infty}^{1} e^x \, dx$$

$$\int_{-\infty}^{1} e^x \, dx = \lim_{R \to -\infty} \int_{R}^{1} e^x \, dx$$

$$= \lim_{R \to -\infty} e^x \,\Big|_R^1$$

$$= \lim_{R \to -\infty} e - e^R$$

$$e^{-\infty} = 0$$

$$= e - e^{-\infty}$$

$$e^{\infty} = \infty$$

$$= e - 0 = e$$

So, the integral $\int_{-\infty}^{1} e^x \, dx$ is convergent to e

Example: $$\int_0^{\infty} \frac{1}{1+x^2}\,dx$$

$$\int_0^{\infty} \frac{1}{1+x^2}\,dx = \lim_{R\to\infty} \int_0^{R} \frac{1}{1+x^2}\,dx$$

$$= \lim_{R\to\infty} tan^{-1}x \,\Big|_0^R$$

$$= \lim_{R\to\infty} tan^{-1}R - tan^{-1}0$$

$$= tan^{-1}\infty - tan^{-1}0$$

$$= \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

$tan^{-1}\infty = \frac{\pi}{2}$

$tan^{-1}0 = 0$

So, the integral $\int_0^{\infty} \frac{1}{1+x^2}\,dx$ is convergent to $\frac{\pi}{2}$

Example: $$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx$$

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx = 2\int_{0}^{\infty} \frac{1}{1+x^2}\, dx$$

$$= 2\lim_{R\to\infty} \int_{0}^{R} \frac{1}{1+x^2}\, dx$$

$$= 2\lim_{R\to\infty}\ tan^{-1}x\ |_0^R$$

$$= 2\lim_{R\to\infty}\ tan^{-1}R - tan^{-1}0$$

$$= 2(tan^{-1}\infty - tan^{-1}0)$$

$$= 2(\frac{\pi}{2} - 0) = \pi$$

So, the integral $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx$ is convergent to π

Example: $$\int_{-1}^{2} \frac{1}{x^2}\, dx$$

$$\int_{-1}^{2} \frac{1}{x^2}\, dx = \int_{-1}^{0} \frac{1}{x^2}\, dx + \int_{0}^{2} \frac{1}{x^2}\, dx$$

The first integral

$$\int_{-1}^{0} \frac{1}{x^2}\, dx = \lim_{R\to 0} \int_{-1}^{R} \frac{1}{x^2}\, dx$$

$$= \lim_{R\to 0} \frac{-1}{x} \Big|_{-1}^{R}$$

$$= \lim_{R\to 0} \frac{-1}{R} - \frac{-1}{-1}$$

$$= \lim_{R\to 0} \frac{-1}{R} - 1$$

$\frac{-1}{0} = \infty$

So, the integral $\int_{-1}^{0} \frac{1}{x^2}\, dx$ is divergent

So, the integral $\int_{-1}^{2} \frac{1}{x^2}\, dx$ is divergent

Remark:

a) If $p \leq 1$, $\int_1^\infty \frac{1}{x^p}\,dx$ is divergent

b)If $p > 1$, $\int_1^\infty \frac{1}{x^p}\,dx$ is convergent to $\frac{1}{p-1}$

Example: $\int_1^\infty \frac{1}{\sqrt[3]{x}}\,dx$

$$\int_1^\infty \frac{1}{\sqrt[3]{x}}\,dx = \int_1^\infty \frac{1}{x^{\frac{1}{3}}}\,dx$$

Since $p = \frac{1}{3} \leq 1$, the integral $\int_1^\infty \frac{1}{\sqrt[3]{x}}\,dx$ is divergent

Example: $\int_1^{\infty} \frac{1}{x^5}\, dx$

Since $p = 5 > 1$, the integral $\int_1^{\infty} \frac{1}{\sqrt[3]{x}}\, dx$ is convergent to $\frac{1}{5-1} = \frac{1}{4}$