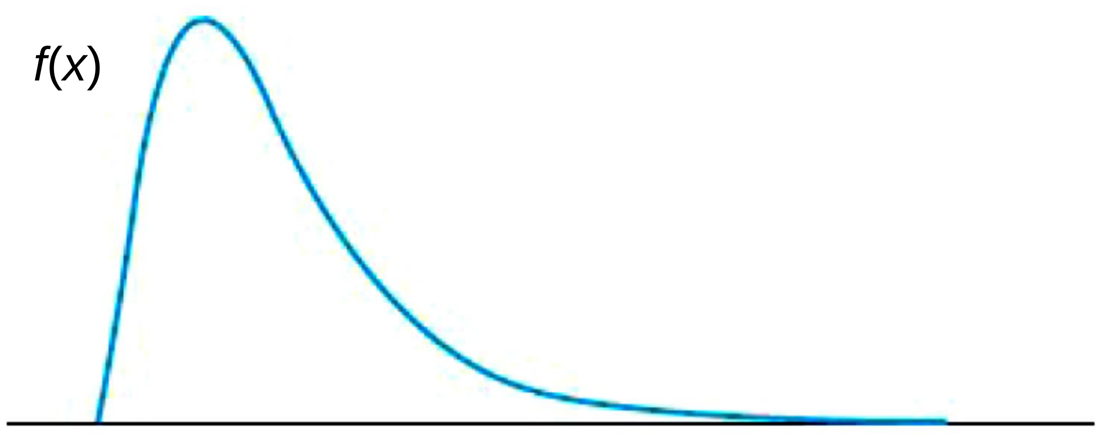
New Mansoura University

Lecture 7

Continuous Random Variables

Continuous Random Variables

- A random variable is **continuous** if its probabilities are given by areas under a curve.



- The curve is called a **probability density function** (pdf) $f(x)$.

Computing Probabilities

Let *X* be a continuous random variable with probability density function *f*(*x*). Let *a* and *b* be any two numbers, with *a* < *b*. Then

$$P(a \le X \le b) = P(a \le X < b) = P(a < X \le b) = \int_a^b f(x)dx.$$

In addition,

$$P(X \le a) = P(X < a) = \int_{-\infty}^{a} f(x)dx$$

$$P(X \ge a) = P(X > a) = \int_{a}^{\infty} f(x)dx.$$

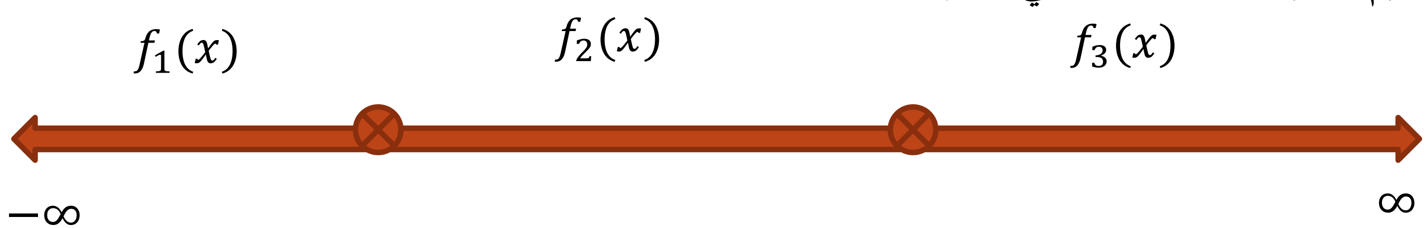
Cumulative function

- Let X be a continuous random variable with probability density function $f(x)$. The **cumulative distribution function** of X is the function

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)dt.$$

How to compute Cumulative function

1. تقسيم الفترات كما معطاه في السؤال



2. حساب الداله $F(x)$ في كل جزء عن طريق العلاقه التاليه

$$\boldsymbol{F(x) = F(\text{بداية الفتره}) + \int_{\text{بداية الفتره}}^{x} f(x)\,dx}$$

Mean and Variance

- The mean of *X* is given by

$$\mu_X = \int_{-\infty}^{\infty} x f(x) dx.$$

- The variance of *X* is given by

$$\sigma_X^2 = \int_{-\infty}^{\infty} (x - \mu_X)^2 f(x) dx$$
$$= \int_{-\infty}^{\infty} x^2 f(x) dx - \mu_X^2 .$$

- The variance of *X* may also be denoted by $V(X)$ or by σ^2.
- The standard deviation is the square root of the variance: $\sigma_X = \sqrt{\sigma_X^2}$.

Example

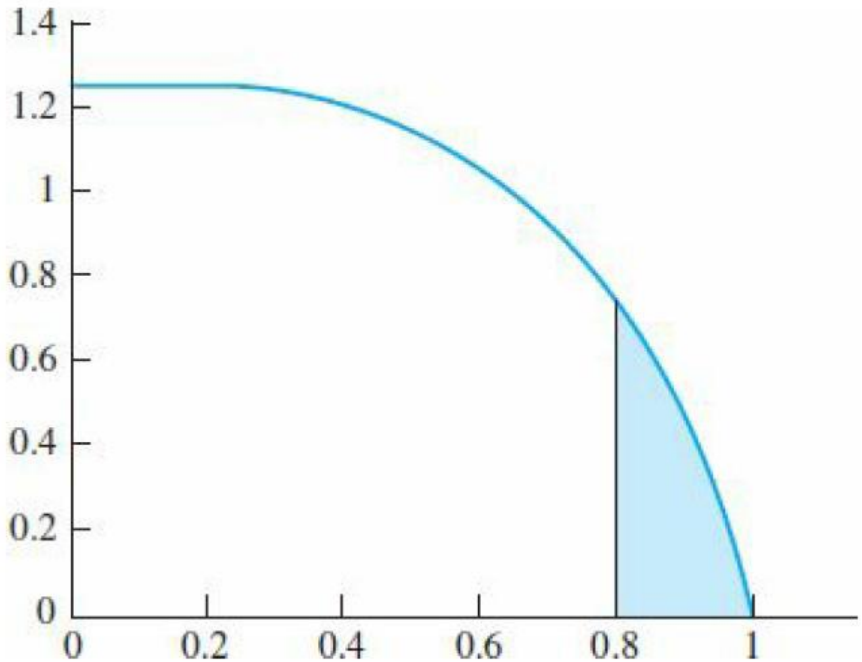
A hole is drilled in a sheet-metal component, and then a shaft is inserted through the hole. The shaft clearance is equal to the difference between the radius of the hole and the radius of the shaft. Let the random variable *X* denote the clearance, in millimeters. The probability density function of *X* is

$$f(x) = \begin{cases} 1.25(1 - x^4), & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

1. Components with clearances larger than 0.8 mm must be scrapped. What proportion of components are scrapped?
2. Find the cumulative distribution function *F*(*x*).

Example

The proportion of components that must be scrapped is P(X > 0.8), which is equal to the area under the probability density function to the right of 0.8.



Example

This area is given by

$$P(X > 0.8) = \int_{0.8}^{\infty} f(x)\,dx$$

$$= \int_{0.8}^{1} 1.25(1 - x^4)\,dx$$

$$= 1.25\left(x - \frac{x^5}{5}\right)\Bigg|_{0.8}^{1}$$

$$= 0.0819$$

Example

$f(t) = 0$ $\quad$ $f(t) = 1.25(1 - t^4)$ $\quad$ $f(t) = 0$

$$F(x) = F(\text{بداية الفتره}) + \int_{\text{بداية الفتره}}^{x} f(x)dx$$

Example

Interval 1 $-\infty < x < 0$ $\ldots\ldots\ldots\ldots\ldots\ldots. > f(x) = 0$

$$F(x) = F(-\infty) + \int_{-\infty}^{x} 0 \, dx$$

$$F(x) = 0$$

Example

Interval 2: $0 < x < 1 \; \dots\dots\dots\dots\dots\dots . > f(x) = 1.25(1 - t^4)$

$$F(x) = F(0) + \int_0^x 1.25(1 - t^4)\; dx$$

من الفتره السابقه
نعوض في الناتج
ب 0

$$F(x) = 0 + 1.25 \int_0^x (1 - t^4)\; dx$$

$$F(x) = 0 + 1.25 \left[t - \frac{t^5}{5} \right]_{t=0}^{t=x}$$

$$F(x) = 1.25(x - \frac{x^5}{5})$$

Example

Interval 3 $1 < x < \infty$ ……………….> $f(x) = 0$

$$F(x) = F(1) + \int_{1}^{x} 0\, dx$$

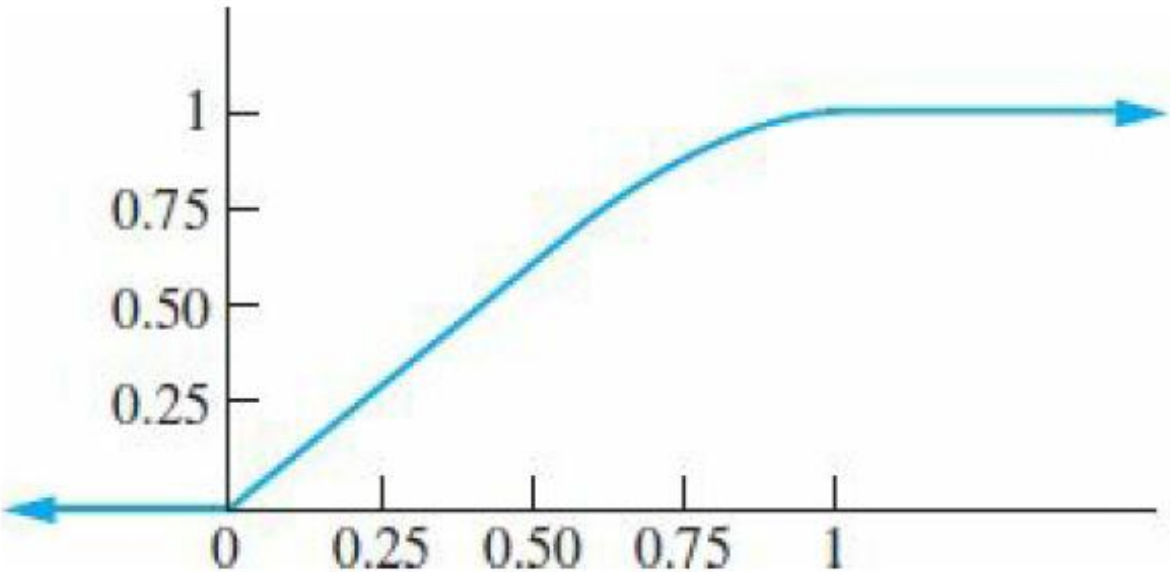
Note $F(1)$ from previous interval, substitute $x = 1$

$$F(1) = \left[1.25\left(x - \frac{x^5}{5}\right)\right]_{x=1} = 1$$

$$F(x) = 1 + 0 = 1$$

Example

$$F(x) = \begin{cases} 0 & x \le 0 \\ 1.25\left(x - \dfrac{x^5}{5}\right) & 0 < x < 1 \\ 1 & x \ge 1 \end{cases}$$



Example

Use the cumulative distribution function to find the probability that the shaft clearance is less than 0.5 mm.

Let X denote the shaft clearance. We need to find $P(X \leq 0.5)$.

This is equivalent to finding F(0.5), where F(x) is the cumulative distribution function.

$F(0.5) = 1.25(0.5 - 0.5^5/5) = 0.617.$

Example

$$\mu_X = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$= \int_0^1 x[1.25(1 - x^4)]\, dx$$

$$= 1.25 \left(\frac{x^2}{2} - \frac{x^6}{6} \right) \Bigg|_0^1$$

$$= 0.4167$$

$$\sigma_X^2 = \int_{-\infty}^{\infty} x^2 f(x)\, dx - \mu_X^2$$

$$= \int_0^1 x^2[1.25(1 - x^4)]\, dx - (0.4167)^2$$

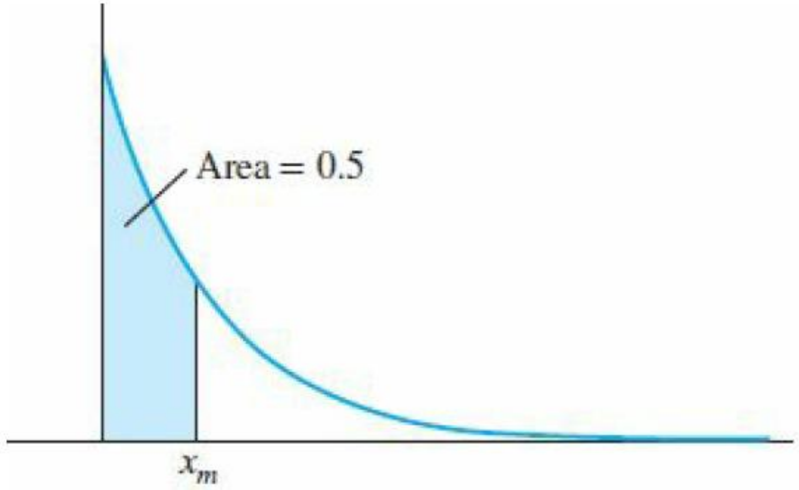
$$= 1.25 \left(\frac{x^3}{3} - \frac{x^7}{7} \right) \Bigg|_0^1 - (0.4167)^2$$

$$= 0.0645$$

Median for a Continuous Random Variable

- Let *X* be a continuous random variable with probability mass function *f*(*x*) and cumulative distribution function *F*(*x*).
- The median of *X* is the point x_m that solves the equation.

$$F(x_m) = P(X \le x_m) = \int_{-\infty}^{x_m} f(x)dx = 0.5.$$

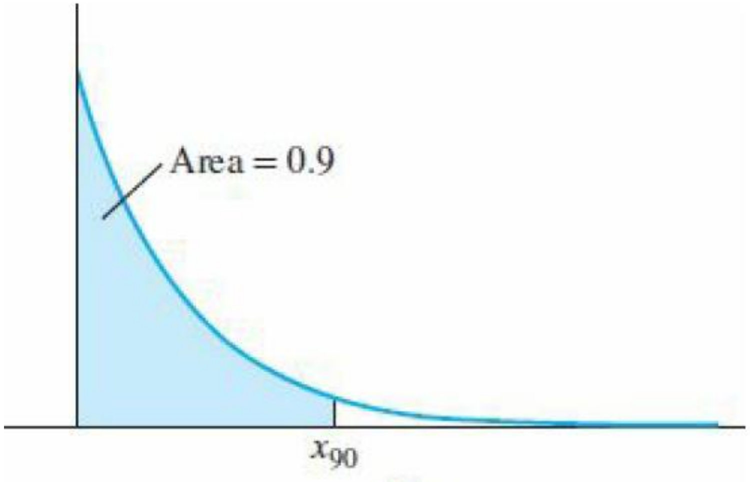


Percentiles

- If p is any number between 0 and 100, the pth percentile is the point x_p that solves the equation

$$\boxed{F(x_p) = P(X \le x_p) = \int_{-\infty}^{x_p} f(x)dx = p/100.}$$

- Note: the median is the 50th percentile.



Example 2

A certain radioactive mass emits alpha particles from time to time. The time between emissions, in seconds, is random, with probability density function

$$f(x) = \begin{cases} 0.1e^{-0.1x} & x > 0 \\ 0 & x \le 0 \end{cases}$$

Find the median time between emissions. Find the 60th percentile of the times.

Solution

The median x_m is the solution to

$$\int_{-\infty}^{x_m} f(x)\,dx = 0.5.$$

$$\int_0^{x_m} 0.1e^{-0.1x}\,dx = 0.5$$

$$-e^{-0.1x}\Big|_0^{x_m} = 0.5$$

$$1 - e^{-0.1x_m} = 0.5$$

$$e^{-0.1x_m} = 0.5$$

$$-0.1x_m = \ln 0.5$$

$$0.1x_m = 0.6931$$

$$x_m = 6.931$$

The 60th percentile x_{60} is the solution

$$\int_{-\infty}^{x_{60}} f(x)\,dx = 0.6.$$

$$1 - e^{-0.1x_{60}} = 0.6$$

$$e^{-0.1x_{60}} = 0.4$$

$$-0.1x_{60} = \ln 0.4$$

$$0.1x_{60} = 0.9163$$

$$x_{60} = 9.163$$

Thank You