Lecture 5

Special Discrete distributions

The Bernoulli Distribution

We use the Bernoulli distribution when we have an experiment which can result in one of two outcomes. One outcome is labeled "success," and the other outcome is labeled "failure."

- The probability of a success is denoted by p.
- The probability of a failure is then 1 − p.

Such a trial is called a **Bernoulli trial** with success probability p.

Examples 1 and 2

1. The simplest Bernoulli trial is the toss of a coin. The two outcomes are heads and tails. If we define heads to be the success outcome, then p is the probability that the coin comes up heads. For a fair coin, $p = 0.5$.

2. Another Bernoulli trial is a selection of a component from a population of components, some of which are defective. If we define "success" to be a defective component, then p is the proportion of defective components in the population.

Mean and Variance

For any Bernoulli trial, we define a random variable X as follows: If the experiment results in success, then $X = 1$. Otherwise $X = 0$. It follows that X is a discrete random variable, with probability mass function $p(x)$ defined by

$$p(0) = P(X = 0) = 1 - p$$

$$p(1) = P(X = 1) = p$$

$$p(x) = 0 \text{ for any value of } x \text{ other than } 0 \text{ or } 1$$

Summary

If $X \sim \text{Bernoulli}(p)$, then

$$\mu_X = p \tag{4.1}$$

$$\sigma_X^2 = p(1 - p) \tag{4.2}$$

The Binomial Distribution

If a total of *n* Bernoulli trials are conducted, and,

- **The trials are independent.**
- **Each trial has the same success probability *p*.**
- ***X* is the number of successes in the *n* trials.**

then *X* has the **binomial distribution** with parameters *n* and *p*, denoted $X \sim \text{Bin}(n, p)$.

- If $X \sim \text{Bin}(n, p)$, the probability mass function of X is

$$p(x) = P(X = x) = \begin{cases} \dfrac{n!}{x!(n-x)!} p^x (1-p)^{n-x}, x = 0,1,...,n \\ \\ 0, \quad \text{otherwise} \end{cases}$$

- Mean: $\mu_X = np$

- Variance: $\sigma_X^2 = np(1-p)$

Find the probability mass function of the random variable X if $X \sim \text{Bin}(10, 0.4)$. Find $P(X = 5)$.

Solution

with $n = 10$ and $p = 0.4$. The probability mass function is

$$
p(x) = \begin{cases} \dfrac{10!}{x!(10 - x)!}(0.4)^x(0.6)^{10-x} & x = 0, 1, \ldots, 10 \\[2em] 0 & \text{otherwise} \end{cases}
$$

$$
P(X = 5) = p(5) = \frac{10!}{5!(10 - 5)!}(0.4)^5(0.6)^{10-5}
$$

$$
= 0.2007
$$

A fair die is rolled eight times. Find the probability that no more than 2 sixes come up.

Solution

Each roll of the die is a Bernoulli trial with success probability 1/6. Let X denote the number of sixes in 8 rolls. Then $X \sim \text{Bin}(8, 1/6)$. We need to find $P(X \leq 2)$. Using the probability mass function,

$$P(X \leq 2) = P(X = 0 \text{ or } X = 1 \text{ or } X = 2)$$

$$= P(X = 0) + P(X = 1) + P(X = 2)$$

$$= \frac{8!}{0!(8-0)!}\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^{8-0} + \frac{8!}{1!(8-1)!}\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^{8-1}$$

$$+ \frac{8!}{2!(8-2)!}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^{8-2}$$

$$= 0.2326 + 0.3721 + 0.2605$$

$$= 0.8652$$

Thank You