CALCULUS
EARLY TRANSCENDENTAL FUNCTIONS

5th EDITION

ROBERT T. SMITH, ROLAND B. MINTON, ZIAD A. T. RAFHI

Mc Graw Hill Education

Integration by completing squares

1-

$$\int \frac{f'}{\sqrt{a^2 - f^2}} \, dx = sin^{-1}\frac{f}{a} + c$$

Example
$$\int \frac{1}{\sqrt{1 - x^2}} \, dx = sin^{-1}x + c$$

Example
$$\int \frac{1}{\sqrt{9 - x^2}} \, dx = sin^{-1}\frac{x}{3} + c$$

Example
$$\int \frac{1}{\sqrt{9 - (x + 1)^2}} \, dx =$$

$$= sin^{-1}\frac{x + 1}{3} + c$$

$$\textbf{\textit{Example}} \quad \int \frac{1}{\sqrt{-5+6x-x^2}}\ dx$$

$$-5+6x-x^2 = -(x^2-6x+5)$$

$$= -(x^2-6x+9-9+5)$$

$$= -((x-3)^2-4)$$

$$= 4-(x-3)^2$$

$$\rightarrow \int \frac{1}{\sqrt{-5+6x-x^2}}\ dx =$$

$$= \int \frac{1}{\sqrt{4-(x-3)^2}}\ dx$$

$$= sin^{-1}\frac{x-3}{2}+c$$

2-

$$\int \frac{f'}{\sqrt{a^2 + f^2}} \, dx = \ln \left| \frac{f}{a} + \frac{\sqrt{a^2 + f^2}}{a} \right| + c$$

Example $\quad \int \frac{1}{\sqrt{1 + x^2}} \, dx$

$$= \ln \left| x + \sqrt{1 + x^2} \right| + c$$

Example $\quad \int \frac{1}{\sqrt{9 + x^2}} \, dx$

$$= \ln \left| \frac{x}{3} + \frac{\sqrt{9 + x^2}}{3} \right| + c$$

Example $\quad \int \frac{1}{\sqrt{9 + (x + 1)^2}} \, dx =$

$$= \ln \left| \frac{x + 1}{3} + \frac{\sqrt{9 + (x + 1)^2}}{3} \right| + c$$

$$\textit{Example} \qquad \int \frac{1}{\sqrt{x^2 + 6x + 13}} \, dx$$

$$x^2 + 6x + 13 = x^2 + 6x + 9 - 9 + 13$$

$$= (x + 3)^2 + 4$$

$$\rightarrow \int \frac{1}{\sqrt{x^2 + 6x + 13}} \, dx =$$

$$= \int \frac{1}{\sqrt{4 + (x + 3)^2}} \, dx$$

$$= ln \left| \frac{x + 3}{2} + \frac{\sqrt{4 + (x + 3)^2}}{2} \right| + c$$

3-

$$\int \frac{f'}{\sqrt{f^2 - a^2}} \, dx = ln \left| \frac{f}{a} + \frac{\sqrt{f^2 - a^2}}{a} \right| + c$$

Example $\qquad \int \frac{1}{\sqrt{x^2 - 1}} \, dx$

$$= ln \left| x + \sqrt{x^2 - 1} \right| + c$$

Example $\qquad \int \frac{1}{\sqrt{x^2 - 9}} \, dx$

$$= ln \left| \frac{x}{3} + \frac{\sqrt{x^2 - 9}}{3} \right| + c$$

Example $\qquad \int \frac{1}{\sqrt{(x + 1)^2 - 9}} \, dx =$

$$= ln \left| \frac{x + 1}{3} + \frac{\sqrt{(x + 1)^2 - 9}}{3} \right| + c$$

Example

$$\int \frac{1}{\sqrt{x^2 + 6x + 5}} \, dx$$

$$x^2 + 6x + 5 = x^2 + 6x + 9 - 9 + 5$$

$$= (x + 3)^2 - 4$$

$$\rightarrow \int \frac{1}{\sqrt{x^2 + 6x + 5}} \, dx =$$

$$= \int \frac{1}{\sqrt{(x + 3)^2 - 4}} \, dx$$

$$= ln \left| \frac{x + 3}{2} + \frac{\sqrt{(x + 3)^2 - 4}}{2} \right| + c$$

4-

$$\int \frac{f'}{a^2 + f^2} \, dx = \frac{1}{a} tan^{-1} \frac{f}{a} + c$$

Example $\quad \int \frac{1}{1 + x^2} \, dx = tan^{-1}x + c$

Example $\quad \int \frac{1}{9 + x^2} \, dx = \frac{1}{3} tan^{-1} \frac{x}{3} + c$

Example $\quad \int \frac{1}{9 + (x + 1)^2} \, dx =$

$$= \frac{1}{3} tan^{-1} \frac{x + 1}{3} + c$$

$$\textbf{\textit{Example}} \qquad \int \frac{1}{x^2 + 2x + 5}\, dx$$

$$x^2 + 2x + 5 = x^2 + 2x + 1 - 1 + 5$$

$$= (x + 1)^2 + 4$$

$$\rightarrow \int \frac{1}{x^2 + 2x + 5}\, dx =$$

$$= \int \frac{1}{(x + 1)^2 + 4}\, dx$$

$$= \frac{1}{2} \tan^{-1} \frac{x + 1}{2} + c$$

5-

$$\int \frac{f'}{a^2 - f^2} \, dx = \frac{1}{2a} \ln \left| \frac{a + f}{a - f} \right| + c$$

Example $\quad \int \frac{1}{1 - x^2} \, dx = \frac{1}{2} \ln \left| \frac{1 + x}{1 - x} \right| + c$

Example $\quad \int \frac{1}{9 - x^2} \, dx = \frac{1}{6} \ln \left| \frac{3 + x}{3 - x} \right| + c$

Example $\quad \int \frac{1}{9 - (x + 1)^2} \, dx =$

$$= \frac{1}{6} \ln \left| \frac{3 + (x + 1)}{3 - (x + 1)} \right| + c$$

$$= \frac{1}{6} \ln \left| \frac{4 + x}{2 - x} \right| + c$$

Example $\quad \int \dfrac{1}{-x^2 - 2x + 3} \, dx$

$$-x^2 - 2x + 3 = -(x^2 + 2x - 3)$$

$$= -(x^2 + 2x + 1 - 1 - 3)$$

$$= -((x + 1)^2 - 4)$$

$$= 4 - (x + 1)^2$$

$$\rightarrow \int \dfrac{1}{-x^2 - 2x + 3} \, dx =$$

$$= \int \dfrac{1}{4 - (x + 1)^2} \, dx$$

$$= \dfrac{1}{4} \ln \left| \dfrac{2 + (x + 1)}{2 - (x + 1)} \right| + c$$

$$= \dfrac{1}{4} \ln \left| \dfrac{3 + x}{1 - x} \right| + c$$

6-

$$\int \frac{f'}{f^2 - a^2} \, dx = \frac{1}{2a} \ln \left| \frac{a - f}{a + f} \right| + c$$

Example $\quad \displaystyle\int \frac{1}{x^2 - 1} \, dx = \frac{1}{2} \ln \left| \frac{1 - x}{1 + x} \right| + c$

Example $\quad \displaystyle\int \frac{1}{x^2 - 9} \, dx = \frac{1}{6} \ln \left| \frac{3 - x}{3 + x} \right| + c$

Example $\quad \displaystyle\int \frac{1}{(x + 1)^2 - 9} \, dx =$

$$= \frac{1}{6} \ln \left| \frac{3 - (x + 1)}{3 + (x + 1)} \right| + c$$

$$= \frac{1}{6} \ln \left| \frac{2 - x}{4 + x} \right| + c$$

Example $\displaystyle\int \frac{1}{x^2 + 2x - 3}\, dx$

$$x^2 + 2x - 3 = x^2 + 2x + 1 - 1 - 3$$

$$= (x + 1)^2 - 4$$

$$\rightarrow \int \frac{1}{x^2 + 2x - 3}\, dx =$$

$$= \int \frac{1}{(x + 1)^2 - 4}\, dx$$

$$= \frac{1}{4} \ln \left| \frac{2 - (x + 1)}{2 + (x + 1)} \right| + c$$

$$= \frac{1}{4} \ln \left| \frac{1 - x}{3 + x} \right| + c$$