Lecture 9

Joint probability distribution

Jointly Distributed Random Variables

If *X* and *Y* are jointly discrete random variables:

- The **joint probability mass function** of *X* and *Y* is the function

$$p(x, y) = P(X = x \text{ and } Y = y)$$

- The joint probability mass function has the property that

$$\sum_x \sum_y p(x, y) = 1$$

where the sum is taken over all the possible values of *X* and *Y*.

Jointly Distributed Random Variables

The following two-way table Joint probability distribution

$P(x, y)$	$x = 0$	$x = 1$	$x = 2$	row sum
$y = 0$	0.32	0.03	0.01	**0.36**
$y = 1$	0.06	0.24	0.02	**0.32**
$y = 2$	0.02	0.03	0.27	**0.32**
col sum	**0.40**	**0.30**	**0.30**	checksum = 1.0

Marginal Probability Mass Functions

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- The **marginal probability mass functions** of *X* and *Y* can be obtained from the joint probability mass function as follows:

$$p_X(x) = P(X = x) = \sum_y p(x, y)$$

$$p_Y(y) = P(Y = y) = \sum_x p(x, y)$$

where the sums are taken over all the possible values of *Y* and of *X*, respectively.

Independence for Two Random Variables

Two random variables X and Y are independent, provided that:

- If X and Y are jointly discrete, the joint probability mass function is equal to the product of the marginals:

$$p(x, y) = p_X(x)\, p_Y(y).$$

Covariance

- Let *X* and *Y* be random variables with means μ_X and μ_Y.
- The **covariance** of *X* and *Y* is

$$\text{Cov}(X,Y) = \mu_{XY} - \mu_X \mu_Y.$$

$$\mu_{XY} = \sum_{i=1}^{n} \sum_{j=1}^{m} x_i y_j p(x_i, y_j)$$

Correlation

- Let *X* and *Y* be jointly distributed random variables with standard deviations σ_X and σ_Y.
- The **correlation** between *X* and *Y* is denoted $\rho_{X,Y}$ and is given by

$$\rho_{X,Y} = \frac{\mathrm{Cov}(X,Y)}{\sigma_X \sigma_Y}.$$

- For any two random variables *X* and *Y*,

$$-1 \le p_{X,Y} \le 1.$$

Quality-control checks on wood paneling involve counting the number of surface flaws on each panel. On a given 2×8 ft panel, let X be the number of surface flaws due to uneven application of the final coat of finishing material, and let Y be the number of surface flaws due to inclusions of foreign particles in the finish. The joint probability mass function $p(x, y)$ of X and Y is presented in the following table. The marginal probability mass functions are presented as well, in the margins of the table. Find the covariance of X and Y.

	y			
x	0	1	2	$P_X(x)$
0	0.05	0.10	0.20	0.35
1	0.05	0.15	0.05	0.25
2	0.25	0.10	0.05	0.40
$P_Y(y)$	0.35	0.35	0.30	

Example

Solution

We will use the formula Cov(X,Y) = $\mu_{XY} - \mu_X\mu_Y$ (Equation 2.71). First we compute μ_{XY}:

$$\begin{aligned}\mu_{XY} &= \sum_{x=0}^{2}\sum_{y=0}^{2} xy\,p(x,y)\\ &= (1)(1)(0.15) + (1)(2)(0.05) + (2)(1)(0.10) + (2)(2)(0.05)\\ &= 0.65 \qquad \text{(omitting terms equal to 0)}\end{aligned}$$

We use the marginals to compute μ_X and μ_Y:

Page 146

$$\mu_X = (0)(0.35) + (1)(0.25) + (2)(0.40) = 1.05$$

$$\mu_Y = (0)(0.35) + (1)(0.35) + (2)(0.30) = 0.95$$

It follows that Cov(X,Y) = 0.65 − (1.05)(0.95) = −0.3475.

Example

In Example 2.68, we computed $\mathrm{Cov}(X,Y) = -0.3475$, $\mu_X = 1.05$, and $\mu_Y = 0.95$. We now must compute σ_X and σ_Y. To do this we use the marginal densities of X and of Y, which were presented in the table in Example 2.68. We obtain

$$\begin{aligned}\sigma_X^2 &= \sum_{x=0}^{2} x^2 p_X(x) - \mu_X^2 \\ &= (0^2)(0.35) + (1^2)(0.25) + (2^2)(0.40) - 1.05^2 \\ &= 0.7475\end{aligned}$$

$$\begin{aligned}\sigma_Y^2 &= \sum_{y=0}^{2} y^2 p_Y(y) - \mu_Y^2 \\ &= (0^2)(0.35) + (1^2)(0.35) + (2^2)(0.30) - 0.95^2 \\ &= 0.6475\end{aligned}$$

It follows that

$$\rho_{X,Y} = \frac{-0.3475}{\sqrt{(0.7475)(0.6475)}} = -0.499$$

Thank You