New Mansoura University

Lecture 3

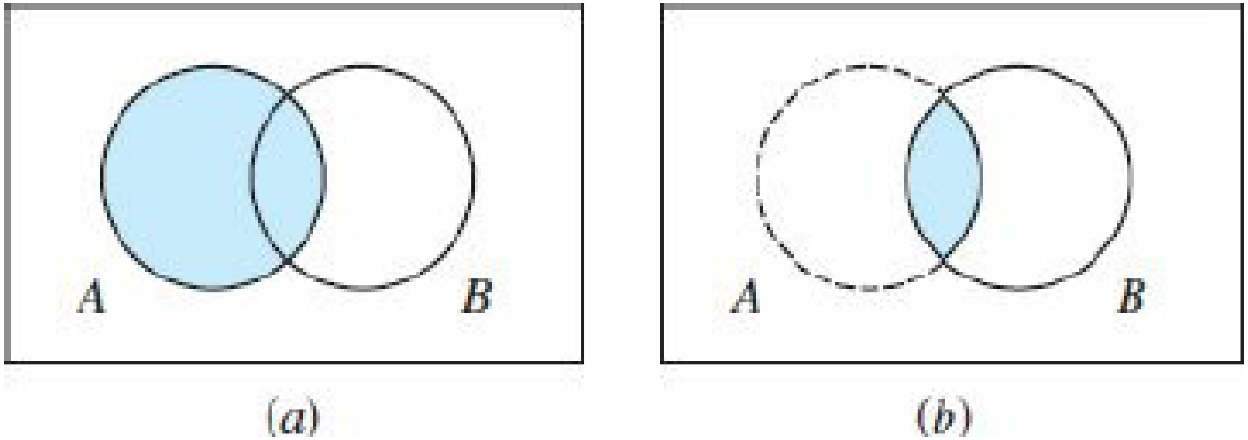
Bayes' theorem and Random variable

Conditional Probability and Independence

- Definition: A probability that is based on part of the sample space is called a **conditional probability**.
- Let *A* and *B* be events with $P(B) \neq 0.$ The conditional probability of *A* given *B* is

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Venn diagrams



(a) The diagram represents the unconditional probability P(A). P(A) is illustrated by considering the event A in proportion to the entire sample space, which is represented by the rectangle.

(b)The diagram represents the conditional probability P(A|B). Since the event B is known to occur, the event B now becomes the sample space. For the event A to occur, the outcome must be in the intersection A ∩ B.The conditional probability P(A|B) is therefore illustrated by considering the intersection A ∩ B in proportion to the entire event B.

Example

Ex 25:- In a certain college 25% of students failed in Mathematics , 15% failed in chemistry , 10% failed both a student is selected randomly.

a) If he failed in mathematics , what is the probability that he failed in chemistry ?

b) If failed chemistry , what is the probability that failed in mathematics ?

Sol

$P(M) = 0.25$ $P(C) = 0.15$

$P(M \cap C) = 0.1$

a) $P(C / M) = \dfrac{P(C \cap M)}{P(M)} = \dfrac{0.1}{0.25} = \dfrac{2}{5}$

b) $P(M / C) = \dfrac{P(M \cap C)}{P(C)} = \dfrac{0.1}{0.15} = \dfrac{2}{3}$

25% math

15% chem.

10% math and chem.

The Multiplication Rule

- If *A* and *B* are two events and $P(B) \neq 0,$ then $P(A \cap B) = P(B)P(A \mid B).$
- If *A* and *B* are two events and $P(A) \neq 0,$ then $P(A \cap B) = P(A)P(B \mid A).$

- When two events are independent, then $P(A \mid B) = P(A)$ and $P(B \mid A) = P(B),$ so the multiplication rule simplifies:

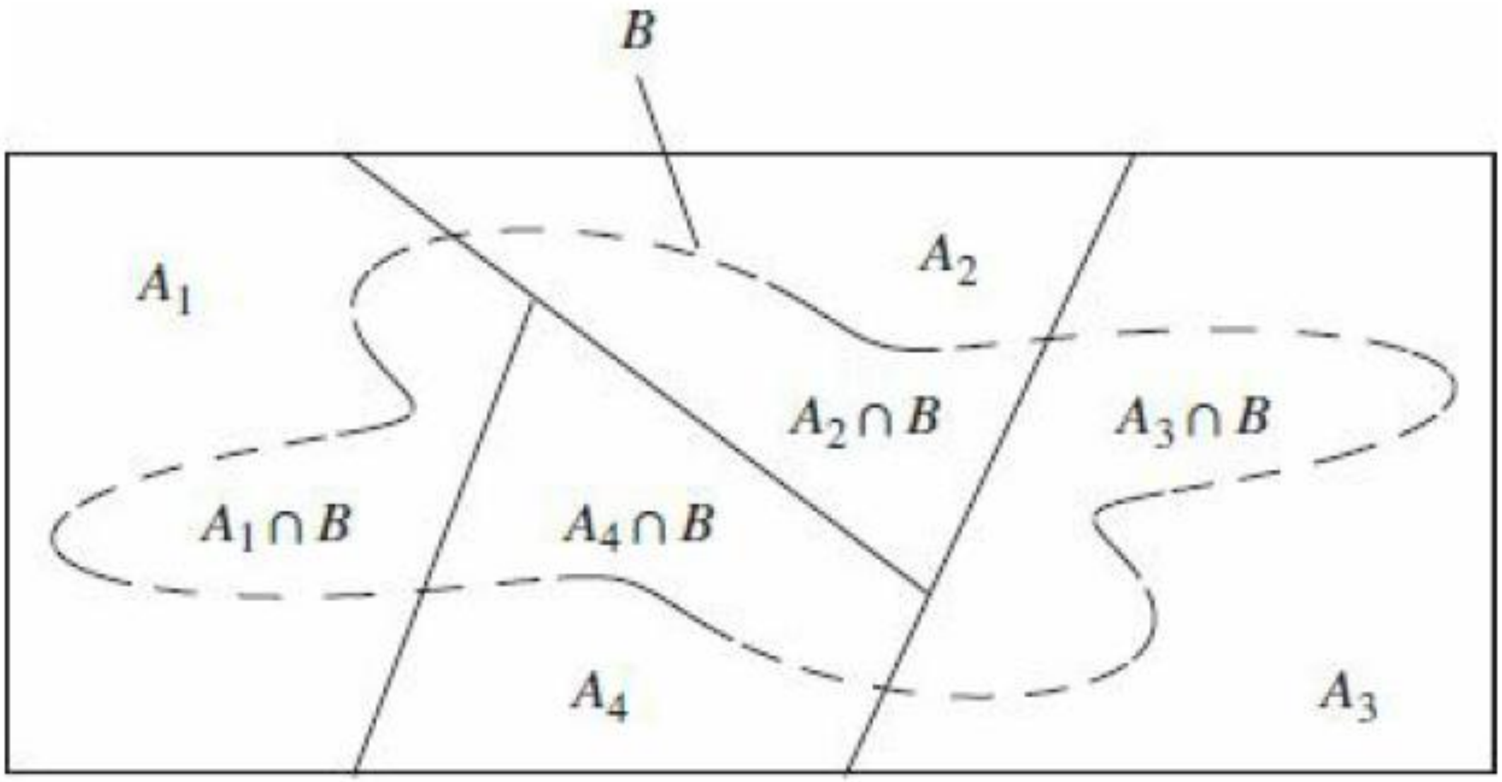
Law of Total Probability

- Law of Total Probability:
  If $A_1, \ldots, A_n$ are mutually exclusive and exhaustive events, and *B* is any event, then

$$P(B) = P(A_1 \cap B) + \cdots + P(A_n \cap B)$$

- Equivalently, if $P\left(A_i\right) \neq 0$ for each A_i,

$$P(B) = P(B \mid A_1)P(A_1) + \cdots + P(B \mid A_n)P(A_n)$$


B
A_1
A_2
$A_2 \cap B$
$A_3 \cap B$
$A_1 \cap B$
$A_4 \cap B$
A_4
A_3

Example

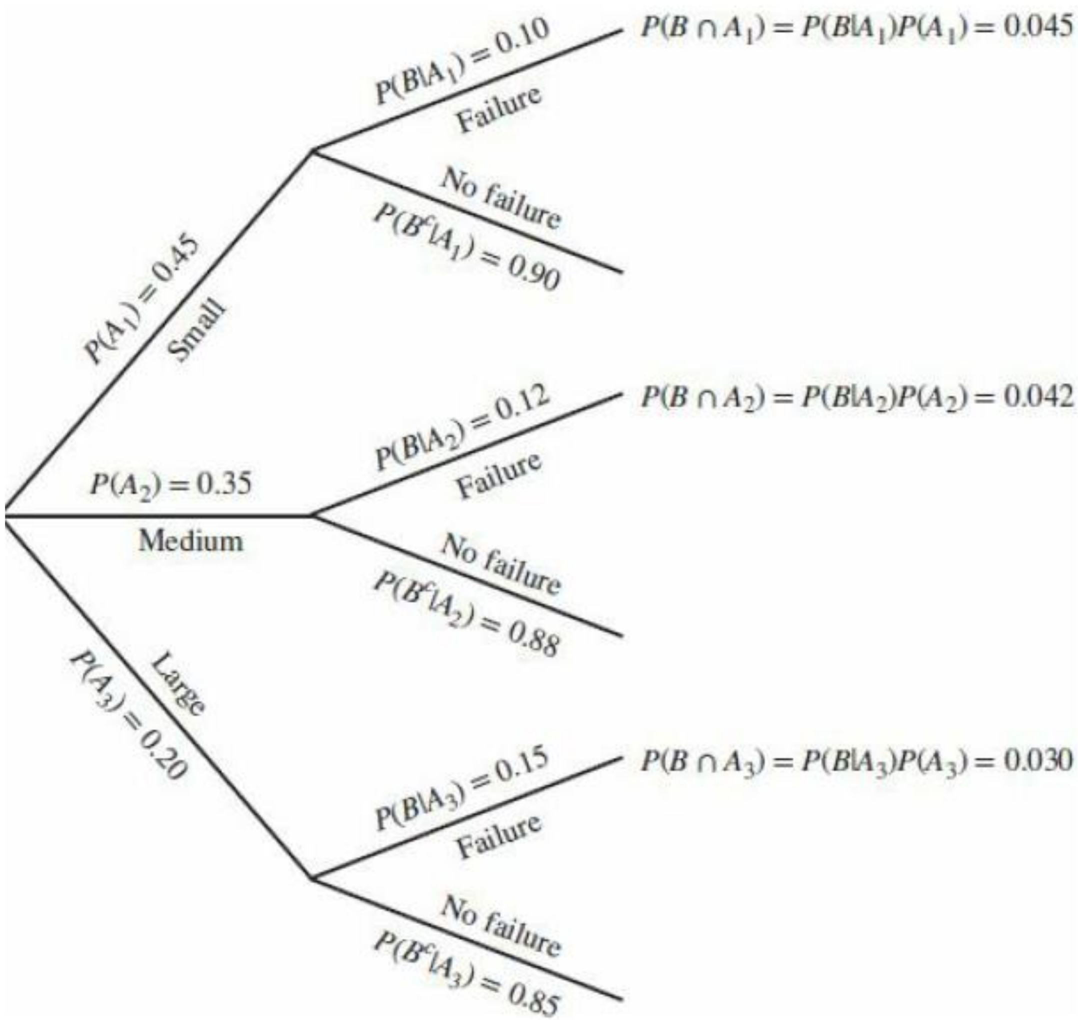
Ex1: Customers who purchase a certain make of car can order an engine in any of three sizes. Of all cars sold, 45% have the smallest engine, 35% have the medium-size one, and 20% have the largest. Of cars with the smallest engine, 10% fail an emissions test within two years of purchase, while 12% of the those with the medium size and 15% of those with the largest engine fail. What is the probability that a randomly chosen car will fail an emissions test within two years of purchase?

Solution

Let B denote the event that a car fails an emissions test within two years. Let A_1 denote the event that a car has a small engine, A_2 the event that a car has a medium-size engine, and A_3 the event that a car has a large engine. Then

$$P(A_1) = 0.45 \qquad P(A_2) = 0.35 \qquad P(A_3) = 0.20$$

Example



P(A1) = 0.45
Small
P(A2) = 0.35
Medium
P(A3) = 0.20
Large
P(B|A1) = 0.10
Failure
No failure
P(B^c|A1) = 0.90
P(B|A2) = 0.12
Failure
No failure
P(B^c|A2) = 0.88
P(B|A3) = 0.15
Failure
No failure
P(B^c|A3) = 0.85
P(B ∩ A1) = P(B|A1)P(A1) = 0.045
P(B ∩ A2) = P(B|A2)P(A2) = 0.042
P(B ∩ A3) = P(B|A3)P(A3) = 0.030

Example

The probability that a car will fail a test, given that it has a small engine, is 0.10. That is, $P(B|A_1) = 0.10$. Similarly, $P(B|A_2) = 0.12$, and $P(B|A_3) = 0.15$. By the law of total probability (Equation 2.24),

$$\begin{aligned} P(B) &= P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + P(B|A_3)P(A_3) \\ &= (0.10)(0.45) + (0.12)(0.35) + (0.15)(0.20) \\ &= 0.117 \end{aligned}$$

Bayes' Rule

Bayes' Rule

Special Case: Let A and B be events with $P(A) \neq 0$, $P(A^c) \neq 0$, and $P(B) \neq 0$. Then

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)} \tag{2.27}$$

General Case: Let $A_1,\ldots,A_n$ be mutually exclusive and exhaustive events with $P(A) \neq 0$ for each A_i. Let B be any event with $P(B) \neq 0$. Then

$$P(A_k|B) = \frac{P(B|A_k)P(A_k)}{\sum_{i=1}^{n} P(B|A_i)P(A_i)} \tag{2.28}$$

Example

A factory has five machines; the numbers of pieces per day by these machines are 1000, 1200, 1800, 2000 and 3000 respectively, the first machine produces on the average 1% defective pieces, the second 0.5%, the third 0.5%, the fourth 1% and the fifth 2%. If a piece selected at random is defective, what is the probability that it is produced by the fourth machine?

Example

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$$p(A_4 / A) = \frac{p(A_4)\,p(A / A_4)}{\sum_{k=1}^{5} p(A_k)\,p(A / A_k)}$$

$$p(A_1 / A) = \frac{\left(\frac{2}{9}\right)\left(\frac{1}{100}\right)}{\left(\frac{1}{9}\right)\left(\frac{1}{100}\right) + \ldots + \left(\frac{2}{9}\right)\left(\frac{1}{100}\right) + \left(\frac{1}{3}\right)\left(\frac{2}{100}\right)} = 0.19$$

Thank You