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AN INTEGER *L*-SHAPED ALGORITHM FOR THE CAPACITATED VEHICLE ROUTING PROBLEM WITH STOCHASTIC DEMANDS

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The classical Vehicle Routing Problem consists of determining optimal routes for m identical vehicles, starting and leaving at the depot, such that every customer is visited exactly once. In the capacitated version (CVRP) the total demand collected along a route cannot exceed the vehicle capacity. This article considers the situation where some of the demands are stochastic. This implies that the level of demand at each customer is not known before arriving at the customer. In some cases, the vehicle may thus be unable to load the customer's demand, even if the expected demand along the route does not exceed the vehicle capacity. Such a situation is referred to as a failure. The capacitated vehicle routing problem with stochastic demands (SVRP) then consists of minimizing the total cost of the planned routes and of expected failures. Here, penalties for failures correspond to return trips to the depot. The vehicle first returns to the depot to unload, then resumes its trip as originally planned. This article studies an implementation of the Integer L-shaped method for the exact solution of the SVRP. It develops new lower bounds on the expected penalty for failures. In addition, it provides variants of the optimality cuts for the SVRP that also hold at fractional solutions. Numerical experiments indicate that some instances involving up to 100 customers and few vehicles can be solved to optimality within a relatively short computing time.

The classical, or deterministic, capacited *Vehicle Routing Problem* (VRP) is defined on an undirected graph $G = (V, E)$ where $V = \{v_1, \ldots, v_n\}$ is a vertex set, and $E = \{(v_i, v_j) : v_i, v_j \in V, i < j\}$ is an edge set. Vertex v_1 is a depot at which are based m identical vehicles of capacity D, while the remaining vertices are customers. A symmetric travel cost matrix $C = (c_{ij})$ is defined on E. With each customer v_i is associated a nonnegative demand d_i to be collected or delivered, but not both. Without loss of generality, we consider the first case in this article. The VRP consists of designing m vehicle routes (i) each starting and ending at the depot, (ii) such that every customer is visited only once by one vehicle, (iii) the total demand of any route does not exceed D, and (iv) the total routing cost is minimized. The VRP is NP-hard and no exact algorithm can consistently solve instances in excess of 50 customers (Toth and Vigo 1998), although some isolated instances involving more than 100 customers have been solved optimally (Fisher 1994, Augerat et al. 1995).

In several real-life situations, customers have *stochastic demands* ξ_i, which give rise to the stochastic VRP (or SVRP). A direct consequence of stochastic demands is that a planned vehicle route may fail at a given customer location whenever the accumulated demand exceeds D. In such a case, *failure* is said to occur and a *recourse action* generating extra costs must be implemented. As indicated by Dror et al. (1989), what is meant by solving a stochastic VRP depends on a number of assumptions about the problem and several solution concepts are possible. Here we assume that customer demands are independent and that demand becomes known only when the vehicle arrives at the customer location.

One solution concept is called *a priori optimization* (Bertsimas et al. 1990). This consists of modeling the problem into two stages. In the first stage, a planned or a priori

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solution is constructed. In our context, such a solution consists of m vehicle routes. The routes are then followed as planned and customer demands are gradually revealed. The second stage solution is the sequence of visits actually followed by the vehicle. We use the same recourse policy as in the "new recourse model" introduced by Dror et al. (1989); namely if the vehicle cannot fully satisfy a customer's demand, it returns to the depot to unload and resumes its collections at the point of failure in the same order as in the a priori solution. The a priori solution is constructed so as to minimize the sum of the planned route cost and the expected cost of recourse. At the other extreme, one could consider a *reoptimization strategy* consisting of optimally resequencing the unvisited customers whenever a vehicle arrives at a new customer location. This leads to a formulation of the SVRP as a Markov decision-process model (Dror et al. 1989, Dror 1993, Secomandi 1998). Intermediate strategies are possible, such as using a priori routes, or planning preventive returns to the depot at strategic points along the vehicle routes (see, e.g., Bertsimas et al. 1995, Yang et al. 2000). While a priori optimization is dominated in terms of solution quality by a reoptimization strategy, it is a preferable approach from a computational point of view since it entails solving only one instance of an NP-hard problem. It also produces a more stable and practically predictable solution since the customer sequence of each route is predetermined (Bertsimas et al. 1990). From a solution quality standpoint, a priori optimization is, however, preferable to solving a deterministic VRP with customer demands taken as expected values $E(\xi_i)$ (Louveaux 1998).

Less research has been done on the SVRP than on its deterministic counterpart. Early heuristics were proposed by Golden and Stewart (1978) and Dror and Trudeau (1986), while properties and bounds were derived by Bertsimas (1992). A priori models and exact algorithms are described by Gendreau et al. (1995) for problems in which the presence of customers is also stochastic. These authors have solved instances with $m = 2$. The most difficult cases arise when f, the expected filling rate of vehicles, is large. Thus, when $f = 1.0$, their method can rarely solve instances with $n > 10$; in contrast, when $f = 0.3$, sizes attaining 70 customers can be solved to optimality. Hjorring and Holt (1999) considered the particular case where $m = 1$ and solved instances with $n = 90$ and $0.95 \leqslant f \leqslant 1.05$. The case where vehicle routes are reoptimized at each customer visit was formulated by Dror et al. (1989) and Bastian and Rinnooy Kan (1992) for $m = 1$, and solved to optimality for $n \leqslant 12$ by Secomandi (1998), using dynamic programming combined with a state space decomposition of a stochastic shortest path formulation of the problem. For recent reviews and general references on the SVRP, see the survey articles of Bertsimas et al. (1990), Gendreau et al. (1996), Bertsimas and Simchi-Levi (1996), and Laporte and Louveaux (1998).

This literature review demonstrates that the SVRP is very hard to solve optimally, particularly if very general reoptimization recourse policies and demand distributions are considered. Rather than solving the problem in its full generality and thus achieving limited results, we show that reasonably large instances can be solved for important classes of demand distributions provided two simplifications are made. First, the recourse policy only consists of return trips to the depot in case of failure. Second, we impose the constraint that the expected total demand of a route does not exceed the vehicle capacity. These two assumptions make managerial sense. The first one yields an easy to implement recourse policy. The second is one way among others to obtain a more realistic model, i.e., achieving a solution with balanced routes. This modeling choice also plays an important role from an optimization perspective since it yields a stronger relaxation. The algorithm we propose for the SVRP is based on the *Integer L-Shaped Method* (Laporte and Louveaux 1993), an extension to the stochastic integer case of Benders' decomposition (1962). (For a recent introduction to stochastic programming and related decomposition methods, see Birge and Louveaux 1997). The implementation of this branch-and-cut method is problem dependent. Its success relies on the availability of good lower bounds on the cost of recourse, and on the generation of lower bounding functionals. The relative success of the Hjorring and Holt (1999) algorithm (applicable to the case where $m = 1$) with respect to the earlier implementation of Gendreau et al. (1995), is largely explained by the introduction of such features. We develop a new method for computing better lower bounds for any value of m and for important classes of demand distributions (e.g., Poisson and normal). This is achieved under a simple yet realistic assumption about the expected demand of any vehicle route. We show how to exploit the shape of the demand distribution to sharpen the lower bound. We also develop new lower bounding functionals for the case where $m \geqslant 2$, as well as an efficient heuristic for their generation. As a result we are capable of optimally solving much larger instances than in the past.

The remainder of this article is organized as follows. In §1, we formulate the problem and provide the computation of the expected cost of recourse. The Integer L-Shaped Method is summarized in §2. New lower bounds on the expected cost of recourse are investigated in §3, while lower bounding functionals are developed in §4. Computational results are presented in §5, followed by the conclusions in §6.

1. THE MODEL

In order for the model to be realistic, it makes sense to impose a restriction on the total expected demand of a route for otherwise some routes will systematically fail while on others vehicles will be highly underutilized. A sensible way to implement this restriction is to require that the expected demand of a route $(v_{i_1} = v_1, v_{i_2}, \dots, v_{i_{t+1}} = v_1)$ does not exceed the vehicle capacity, i.e.,

$$\sum_{j=2}^{t} E(\xi_{i_j}) \leqslant D. \tag{1}$$

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The SVRP can now be formulated as a two-index stochastic program as follows. Let x_{ij} be an integer variable equal to the number of times edge (v_i, v_j) appears in the planned (or first-stage) solution. If $i, j > 1$, then x_{ij} can only take the values 0 or 1; if $i = 1$, x_{ij} can also be equal to 2 if a vehicle makes a return trip between the depot and v_j. Denote by $Q(x)$ the expected cost of recourse. The model is then

$$\text{(SVRP)} \quad \text{Minimize } \sum_{i<j} c_{ij}x_{ij} + Q(x) \tag{2}$$

subject to

$$\sum_{j=2}^{n} x_{1j} = 2m, \tag{3}$$

$$\sum_{i<k} x_{ik} + \sum_{j>k} x_{kj} = 2 \qquad (k = 2, \ldots, n), \tag{4}$$

$$\sum_{v_i, v_j \in S} x_{ij} \leqslant |S| - \left\lceil \sum_{v_i \in S} E(\xi_i)/D \right\rceil \qquad (S \subset V \setminus \{v_1\}; 2 \leqslant |S| \leqslant n-2), \tag{5}$$

$$0 \leqslant x_{ij} \leqslant 1 \qquad (2 \leqslant i < j < n), \tag{6}$$

$$0 \leqslant x_{1j} \leqslant 2 \qquad (j = 2, \ldots, n), \tag{7}$$

$$x = (x_{ij}) \qquad \text{integer.} \tag{8}$$

Apart from $Q(x)$, this model is that of a deterministic capacitated VRP (see, e.g., Laporte et al. 1985) in which customer demands are $E(\xi_i)$. Constraints (3) and (4) specify the degree of each vertex. Constraints (5) ensure that the solution contains no subtour and that the expected demand of any route does not exceed the vehicle capacity (see constraint (1)).

There is no simple way to formulate the computation of $Q(x)$ in terms of decision variables and linear relationships. However, given an a priori solution x, the expected cost of recourse, $Q(x)$, can easily be computed under some assumptions. The computation is separable in the routes, and for each route the expected cost must be computed for each of its two orientations:

$$Q(x) = \sum_{k=1}^{m} \min\{Q^{k,1}, Q^{k,2}\}, \tag{9}$$

where $Q^{k,\delta}$ denotes the expected cost of recourse corresponding to route k and orientation $\delta = 1$ or 2. The computation of $Q^{k,\delta}$ depends on the recourse strategy and on a number of characteristics of the problem. It is worth observing that the model defined by (2)–(8) does not capture route orientation, and this is indeed irrelevant in a deterministic setting where both orientations are equivalent when the travel cost matrix is symmetric. In a stochastic context, however, orientation matters, and because demands are not known until the customer is reached the orientations must be selected a priori. It is an attractive feature of the model that this orientation decision can be captured in a simple fashion by (9).

As a first example, consider the case where the goods to be collected are divisible. This case includes situations where goods are fluid as in normally distributed demands, or where they are divisible to unit size as in Poisson demands. Our approach can effectively handle both situations. For a given route k defined by the vector $V_k = (v_{i_1} = v_1, v_{i_2}, \ldots, v_{i_{t+1}} = v_1)$ the expected cost for the first orientation is

$$Q^{k,1} = 2\sum_{j=2}^{t}\sum_{\ell=1}^{\infty} P\left(\sum_{s=2}^{j-1} \xi_{i_s} \leqslant \ell D < \sum_{s=2}^{j} \xi_{i_s}\right) c_{1i_j}. \tag{10}$$

The probability term corresponds to having the ℓth failure at customer i_j. As in Dror et al. (1989) (see also Teodorović and Paković 1992), (10) can be rewritten as

$$Q^{k,1} = 2\sum_{j=2}^{t}\sum_{\ell=1}^{\infty} [F^{j-1}(\ell D) - F^{j}(\ell D)] c_{1i_j}, \tag{11}$$

where $F^j(\ell D) = P\left(\sum_{s=2}^{j} \xi_{i_s} \leqslant \ell D\right)$. If $\xi_i \leqslant D$ for all i with probability 1, then the upper limit of the second summation in (10) and (11) can be brought down to $j-1$. The computation of (10) and (11) rests on the hypothesis that the only available recourse is a return trip to the depot in case of failure. In other words, no exceptional policy is implemented for the case where the cumulative demand equals the vehicle capacity. This is not limiting in the case of continuous demands since such an event has measure zero. In the discrete case, this hypothesis may be limiting, but not necessarily when customer demand can be zero, as in the case of the Poisson distribution considered later in this paper. The computation of $Q^{k,2}$ is obtained by replacing the index of ξ by i_{t+2-s} in (10) and in the definition of $F^j(\ell D)$.

The case where the goods are not divisible is more difficult and workable results can only be achieved by making stronger assumptions on the demand distribution. Alternative computations of $Q(x)$, under different assumptions, are provided in Dror and Trudeau (1986), Bertsimas (1992), Gendreau et al. (1995), Secomandi (1998), and Hjorring and Holt (1999). It is important to observe that the algorithmic solution we propose, in particular the new method for computing a lower bound L on $Q(x)$, is independent of these specific assumptions.

2. THE INTEGER *L*-SHAPED METHOD

The model is solved by means of an Integer L-Shaped Method, a branch-and-cut algorithm that operates on a so-called "current problem" (CP) at each node of the search tree. Initially, CP is defined by relaxing (SVRP) in three ways: (i) integrality constraints (8) are relaxed; (ii) subtour elimination and vehicle capacity constraints (5) are relaxed; and (iii) $Q(x)$ is replaced by a lower bound θ on the objective function. As in standard branch-and-cut methods, CP is dynamically modified by gradually introducing integrality conditions through the branching process, and by generating constraints (5) as they are found to be violated. In

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addition, a lower bounding constraint on θ, called "optimality cut," is introduced into CP at integer solutions. In other words, contrary to what is done in deterministic branch-and-cut, a cut may be introduced at a feasible integer solution as θ may be strictly less than $Q(x)$ and therefore, the objective value is not yet known and better solutions may exist down the current branch. In addition, valid inequalities are introduced at (fractional) optimal solutions.

We now outline the Integer L-Shaped Method for the SVRP. This method assumes that given a feasible solution x, the value of $Q(x)$ can be computed. Moreover, a finite lower bound L on $Q(x)$ is assumed to be available. In §3, we show how to compute such a bound.

STEP 0. Set the iteration count $\nu := 0$ and introduce the bounding constraint $\theta \geqslant L$ into the initial CP. Set the value $\bar{z}$ of the best known solution equal to ∞. The only pendent node corresponds to the initial current problem.

STEP 1. Select a pendent node from the list. If none exists stop.

STEP 2. Set $\nu := \nu + 1$ and solve CP. Let (x^ν, θ^ν) be an optimal solution.

STEP 3. Check for any violations of constraints (1) or (5) and introduce at least one violated constraint. At this stage, valid inequalities or lower bounding functionals may also be generated. Return to Step 2. Otherwise, if $cx^\nu + \theta^\nu \geqslant \bar{z}$, fathom the current node and return to Step 1.

STEP 4. If the solution is not integer, branch on a fractional variable. Append the corresponding subproblems to the list of pendent nodes and return to Step 1.

STEP 5. Compute $Q(x^\nu)$ and set $z^\nu := cx^\nu + Q(x^\nu)$. If $z^\nu < \bar{z}$, set $\bar{z} := z^\nu$.

STEP 6. If $\theta^\nu \geqslant Q(x^\nu)$, then fathom the current node and return to Step 1. Otherwise, impose the optimality cut

$$\sum_{\substack{1<i<j \\ x_{ij}^\nu = 1}} x_{ij} \leqslant \sum_{1<i<j} x_{ij}^\nu - 1 \tag{12}$$

into CP and return to Step 2.

In Step 6, constraints (12) force the algorithm to move to a solution different from the current point (x^ν, θ^ν). In our application, these constraints can be strengthened by replacing "1" by "2" in the right-hand side since in any such solution, at least two edges not incident to the depot will have to be replaced. Since all coefficients of (12) are equal to 1, this cut is more stable from a numerical point of view than the optimality cut originally proposed by Laporte and Louveaux (1993) and used in Gendreau et al. (1995).

3. COMPUTATION OF A LOWER BOUND L ON $Q(x)$

We propose a new lower bound L on $Q(x)$ based on the probability of failure on each route taken separately, rather than working with the number of failures on the entire solution as in Laporte and Louveaux (1998). Relabel all customers in nondecreasing order of their distance to the depot. Denote by X_k the random demand on route k $(k = 1, \ldots, m)$, and let $F_k(\cdot)$ be its distribution function. Let also X_T be the total random demand and $F_T(\cdot)$ its distribution function.

PROPOSITION 1. *Let $\phi(F_k, D)$ be a lower bound on the probability of having at least one failure on a route whose demand is defined by F_k. A valid lower bound on $Q(x)$ is given by*

$$L = \inf_{(F_1, \ldots, F_m)} 2 \sum_{k=1}^{m} c_{1,k+1} \phi(F_k, D) \tag{13}$$

subject to

$$\int \cdots \int_{x_1 + \cdots + x_m \leqslant x} dF_1(x_1) \cdots dF_m(x_m) = F_T(x) \qquad \textit{for all } x, \tag{14}$$

$$F_k \in \mathscr{F}_k \qquad (k = 1, \ldots, m), \tag{15}$$

where (15) *specify that F_k is a distribution function belonging to a family that can possibly be restricted. In particular, we impose $0 \leqslant E(X_k) \leqslant D$.*

PROOF. As indicated in §2, the expected cost $Q^{k,\delta}$ of route k with orientation δ is obtained by computing the cost of having the ℓth failure at the jth customer, and then summing up over all ℓ and j. All these terms contribute to $Q^{k,\delta}$ by a nonnegative amount. We thus obtain a valid lower bound by only considering the terms related to the first return failure $(\ell = 1)$. Using the definition of $\phi(F_k, D)$ and letting γ_k be a lower bound on the distance of any customer of route k to the depot, we obtain $Q(x) \geqslant 2 \sum_{k=1}^{m} \gamma_k \phi(F_k, D)$. Since index k is immaterial to this reasoning, it suffices to replace the m lower bounds γ_k by the distances of the customers closest to the depot to obtain the desired result. □

We now show how to solve (13)–(15) for Poisson and normally distributed demands. These two distributions are rich and of wide applicability. Since they contain only one or two parameters, they are relatively easy to estimate and simpler to handle than general distributions. Given the recourse policy, $\phi(F_k, D)$ is simply $1 - F_k(D)$. We use the fact that $F_k(D)$ can be expressed in terms of $E(X_k)$ and $\mathrm{Var}(X_k)$. We also use (15) and the additivity of expected values and variances resulting from the assumption of independence.

3.1. Poisson Demands

Since a Poisson variable is uniquely determined by its mean, define decision variables $x_k = E(X_k)$ and $x = (x_1, \ldots, x_m)$, so that (13)–(15) reduces to

$$L = \min_{x} 2 \sum_{k=1}^{m} c_{1,k+1} \left(1 - \sum_{j=0}^{D} e^{-x_k} x_k^j / j!\right) \tag{16}$$

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subject to

$$\sum_{k=1}^{m} x_k = \mu_T, \tag{17}$$

$$0 \leqslant x_k \leqslant D \qquad (k = 1, \dots, m), \tag{18}$$

where $\mu_T = E(X_T)$. The objective defined by (16) is convex in x. Indeed, it is separable in the x_k variables and each term can be shown to be convex since

$$\frac{d^2}{dx_k^2}\left(\sum_{j=0}^{D} e^{-x_k} x_k^j / j!\right) = e^{-x_k} x_k^{D-1}(x_k/D-1)/(D-1)!$$

and $x_k \leqslant D$ by (18). The system defined by (16)–(18) can therefore be solved by applying the Karush-Kuhn-Tucker conditions. Noting that the case $x_k = 0$ can be excluded since m vehicles must be used in the solution, two cases remain to be considered.

If $x_k < D$, then x_k is the solution of $2c_{1,k+1} e^{-x_k} x_k^D / D! = -\gamma$, where γ is the multiplier of (17). Letting $b = \ln(-\gamma D!/2)$, we obtain

$$-x_k + D \ln x_k = b - \ln c_{1,k+1}. \tag{19}$$

Equation (19) effectively yields $x_k < D$ whenever

$$-D + D \ln D > b - \ln c_{1,k+1}. \tag{20}$$

The case $x_k = D$ applies whenever (20) does not hold.

We observe that the left-hand side of (19) is a strictly increasing concave function of x_k. Thus, the unknown values of b and of x_k $(k = 1, \dots, m)$ can be obtained recursively until (17) holds within some tolerance. Starting from some initial tentative value of b, the x_k values are obtained by (19) whenever (20) holds, and by $x_k = D$ otherwise. Since the left-hand side of (19) is monotone in x_k and $\ln c_{1,k+1}$ is a constant, if the sum of the x_k values obtained in this manner is smaller (larger) than μ_T, the next tentative value of b must be larger (smaller).

Initial bounds on b can be obtained as follows. First observe that

$$x_0 = \max\left\{\mu_T - (m-1)D, \min_{i=2,\dots,n} E(\xi_i)\right\} \tag{21}$$

is a valid positive lower bound on x_k for all k (and thus on x_m) since $(m-1)D$ represents the maximum total demand on all routes but k. Suppose that $x_m < D$, so that (19) holds. Since $-x + D \ln x$ is increasing in x, $x_0 \leqslant x_m$ and (19) imply that $-x_0 + D \ln x_0 \leqslant -x_m + D \ln x_m = b - \ln c_{1,m+1}$, or $-x_0 + D \ln x_0 + \ln c_{1,m+1} \leqslant b$. Still, assuming that $x_m < D$, then (20) implies that $b < -D + \ln D + \ln c_{1,m+1}$. Another speed-up can obtained by observing that for a given b, the right-hand side of (19) decreases with k. Therefore for a given b, the solution in x_k to (19) is an upper bound on the solution x_{k+1} to (19).

3.2. Normal Demands

Since a normal variable is determined by its mean and variance, define decision variables $x_k = E(X_k)$ and $y_k = \mathrm{Var}(X_k)$, and let $x = (x_1, \dots, x_m)$, $y = (y_1, \dots, y_m)$, so that (13)–(15) reduces to

$$L = \min_{x,y} 2 \sum_{k=1}^{m} c_{1,k+1}\left[1 - \Phi\left(\frac{D - x_k}{\sqrt{y_k}}\right)\right] \tag{22}$$

subject to (17), (18) and

$$\sum_{k=1}^{m} y_k = \sigma_T^2, \tag{23}$$

$$y_k \geqslant 0 \qquad (k = 1, \dots, m), \tag{24}$$

where $\sigma_T^2 = \mathrm{Var}(X_T)$ and $\Phi(t) = P(Z \leqslant t)$, $Z \sim N(0,1)$. Contrary to the previous case, the objective function is neither convex nor concave in (x, y). However, a lower bound on L can be obtained by dropping (23)–(24) and by making use of a lower bound y_0 on y_k, obtained as follows. Using the lower bound on x_k provided by (21), the value of y_0 can be obtained by solving the knapsack problem

$$y_0 = \min_z \left\{ \sum_{i=2}^{n} \mathrm{Var}(\xi_i) z_i, \quad \text{subject to } \sum_{i=2}^{n} E(\xi_i) z_i \geqslant x_0, z_i \in \{0,1\} \right\} \tag{25}$$

or its linear relaxation, where $z = (z_2, \dots, z_n)$. The problem of determining L therefore reduces to

$$L = \min_{x} 2 \sum_{k=1}^{m} c_{1,k+1}\left[1 - \Phi\left(\frac{D - x_k}{\sqrt{y_0}}\right)\right] \tag{26}$$

subject to (17), (18). This objective function is convex in x_k for $x_k \leqslant D$, and the problem can once more be solved by applying the Karush-Kuhn-Tucker conditions. Letting again γ be the multiplier of (17) and defining $b = -\gamma\sqrt{\pi y_0/2}$, we obtain

$$x_k = \begin{cases} D & \text{if } c_{1,k+1} \leqslant b, \\ D - \sqrt{2y_0 \ln c_{1,k+1}/b} & \text{if } c_{1,k+1} \geqslant b. \end{cases} \tag{27}$$

Again, x_k is a nondecreasing function of b and can be determined recursively.

4. GENERATION OF LOWER BOUNDING FUNCTIONALS ON $Q(x)$

We now compute lower bounding functionals on $Q(x)$ that can be introduced at an integer or at a noninteger solution. These are based on the concept of "partial route" originally introduced by Hjorring and Holt (1999) for the single vehicle case. A partial route h is specified by two ordered vertex sets $S_h = \{v_1, \dots, v_{s_h}\}$, $T_h = \{v_1, \dots, v_{t_h}\}$ satisfying $S_h \cap T_h = \{v_1\}$, and a third set U_h satisfying $S_h \cap U_h = \{v_{s_h}\}$ and $T_h \cap U_h = \{v_{t_h}\}$. For simplicity, we write $(v_i, v_j) \in S_h$

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or T_h if v_i and v_j are consecutive in S_h or T_h. The partial route h induced by these sets is made up of the two chains $(v_1, \ldots, v_{s_h})$, $(v_1, \ldots, v_{t_h})$ and of some unstructured vertex set U_h. Let $R_h = S_h \cup T_h \cup U_h$, $x = (x_{ij})$ and

$$W_h(x) = \sum_{(v_i, v_j) \in S_h} x_{ij} + \sum_{(v_i, v_j) \in T_h} x_{ij} + \sum_{v_i, v_j \in U_h} x_{ij} - |R_h| + 1. \tag{28}$$

Assume that a lower bound P_h on the cost of recourse associated with partial route h is available. If r $(\leqslant m)$ such partial routes exist (with disjoint customer sets), let P_{r+1} be a lower bound on the cost of recourse for the $m-r$ routes on the customers of $V \setminus \bigcup_{h=1}^{r} R_h$, with $P_{m+1} = 0$. Let $P = \sum_{h=1}^{r+1} P_h$ and let L be a lower bound on $Q(x)$, as defined in §2.

PROPOSITION 2. *The constraint*

$$\theta \geqslant L + (P - L)\left(\sum_{h=1}^{r} W_h(x) - r + 1\right) \tag{29}$$

is a valid inequality for (*SVRP*).

PROOF. Since $W_h \leqslant 1$, it follows that $\sum_{h=1}^{r} W_h(x) - r + 1$ can be at most 1, and this bound is reached if $W_h = 1$ for all h. In this case, (29) becomes $\theta \geqslant P$. Otherwise (29) is redundant. □

Observe that L is obtained by considering m routes whose potential failures occur at the m customers closest to the depot and whose expected demands are only constrained by (17) and (18). In comparison P is obtained by restricting r of the m routes to be partial routes. For each route, the expected demand is now fixed to the sum of the expected demands of its customers, while (19) and (18) still apply. Moreover, potential failures may now occur at customers other than the m closest to the depot. Therefore $P \geqslant L$. To compute the lower bounds P_h $(h \leqslant r)$, we proceed as in Hjorring and Holt (1999). First create an artificial customer v_0 with demand

$$\xi_0 = \sum_{v_i \in U_h \setminus \{v_{s_h}, v_{t_h}\}} \xi_i \quad \text{and} \quad c_{1,0} = \min_{v_i \in U_h \setminus \{v_{s_h}, v_{t_h}\}} \{c_{1i}\}.$$

Then construct route k equal to $(v_1, \ldots, v_{s_h}, v_0, v_{t_h}, \ldots, v_1)$ and compute $Q^{k,\delta}$ as in (9), for the two orientations $\delta = 1$ and 2. Then

$$P_h = \min\{Q^{k,1}, Q^{k,2}\}. \tag{30}$$

The computation of P_{r+1} $(r \leqslant m-1)$ follows from that of L described in §3, restricted to the customer set $V \setminus \bigcup_{h=1}^{r} R_h$ and $m-r$ vehicles.

The separation procedure developed for constraints (29) is a heuristic that determines in a first phase a set R_h, and then sets S_h and T_h. The set R_h is determined in a greedy fashion, by initially selecting the vertex v_i connected to the depot and for which $(x_{1i} + \max_{j \neq 1}\{x_{ij}\})$ is maximized. At subsequent steps, the next vertex v_j to be included in R_h is a vertex maximizing $\sum_{v_\ell \in R_h} x_{j\ell}$, and such that $\sum_{v_\ell \in R_h \cup \{v_j\}} E(\xi_\ell) \leqslant D$. The procedure stops either when $v_j = v_1$ or when no vertex v_j can be found. The construction of S_h and T_h is initialized as follows: for S_h, the first vertex v_i included in R_h is selected; for T_h, we select $\operatorname{argmax}_{v_j \in R_h \setminus \{v_i\}}\{x_{1j}\}$. These two sets are then extended in a greedy fashion, according to maximal x_{ij} values, until desired cardinalities are reached. As a rule, increasing the cardinality of S_h and T_h yields a larger value of P_h but a smaller value of W_h. As is commonly done in branch-and-cut, only those combinations of partial routes yielding sufficient violations of (29) are retained.

5. IMPLEMENTATION AND COMPUTATIONAL RESULTS

The method we have just described to solve (SVRP) was coded in C^{++}, and the branch-and-cut mechanism was implemented by means of the OMP mixed-integer solver (De Decker et al. 1987), appropriately modified to encompass the Integer L-Shaped rules. The following procedures were used for the generation of violated constraints. Subtour elimination and capacity constraints (5) and constraints related to the total expected demand of a route (1) were generated by means of the shrinking procedure described by Padberg and Grötschel (1985). To increase the value of the initial linear relaxation, 2-matching inequalities were generated as in Padberg and Rinaldi (1990). All blocks of the subgraph induced by the fractional x_{ij} variables were identified and each block was then considered as a potential handle for the comb; a check for violations of the 2-matching inequality was then made by examining all edges with one end in the handle and one end outside. Constraints (1) and (5) were generated throughout the branch-and-cut tree whereas 2-matching inequalities were only generated at the root node. Optimality cuts (12) were generated at each integer solution. To generate the lower bounding functionals described in §4, we restricted our attention to sets S_h and T_h inducing at most one edge not previously fixed at 1 by the branching process. When several such sets were available a search was made over all possible combinations until the first sufficient violation of the cut was identified. Violated lower bounding functionals can be generated both at fractional or integer solutions, but preliminary computational experiments have suggested that it is sufficient to restrict their generation to the first case. Finally, to generate the value of L (§3), we use (16) or (26), depending on the demand distribution. Our tests show that as expected, this bound is much stronger that the previous bound used in Laporte and Louveaux (1998).

Several sets of randomly generated instances were used to assess the performance of the algorithm and several experiments and tests were executed. For each instance, we first generated n vertices in the $[0, 100]^2$ square according to a continuous uniform distribution. To increase the level of realism of the instances, we also generated five rectangular obstacles O_k in $[20, 80]^2$, each having a base

Table 1. Computational results on instances with Poisson demands ($f = 0.9$).

n	m	SubCap	2-Match	OptCut	LBF	LB/OPT	Nodes	Seconds
25	2	27.4	0.0	5.4	1.2	0.991	15.8	0.2
	3	231.2	0.0	42.0	191.4	0.967	431.4	10.0
	4	1,459.2	5.8	144.6	5,803.6	0.963	5,029.8	117.4
50	2	772.6	7.4	26.0	69.2	0.984	958.6	34.4
	3	1,324.8	4.8	238.2	655.8	0.983	3,431.0	174.8
75	2	3,030.4	5.6	165.6	147.2	0.987	5,414.8	296.4
100	2	4,397.5	9.8	138.0	197.5	0.987	6,520.0	537.3

of 4 and a height of 25. Thus the obstacles cover 5% of the entire area. Denoting by (p_k, q_k) the center of O_k, we ensured that $|p_k - p_\ell| \leqslant 12.5$ for $k \neq \ell$. Any vertex included within an obstacle was then shifted horizontally to the nearest vertical side of the obstacle. The distance between any two vertices v_i and v_j both having an ordinate lying in $[q_k - 12.5, q_k + 12.5]$ for some O_k were then artificially increased by $\alpha\Delta$, where

$$\Delta = \min(d(v_i, w'_k) + d(v_j, w'_k), d(v_i, w''_k) + d(v_j, w''_k)), \tag{31}$$

d denotes the Euclidean distance between two points, $w'_k = (p_k, q_k + 12.5)$, $w''_k = (p_k, q_k - 12.5)$, and α is randomly generated in $[1.1, 1.25]$. All other intervertex distances were deemed to be unaffected by the presence of obstacles.

In a first series of tests, we generated Poisson but divisible demands for several combinations of n and m, where the expected demand of any customer is randomly generated according to a continuous uniform distribution in $[1, 10]$. With these values, the probability of route failure is significant and therefore the instances generated through this process are nontrivial. In these tests, we used a "filling coefficient" $f = 0.9$, defined as the total expected demand divided by the total vehicle capacity, i.e., $f = \sum_{i=2}^{n} E(\xi_i)/mD$. The results of these experiments are presented in Table 1. The table headings are as follows:

n	:	number of vertices;
m	:	number of vehicles;
SubCap	:	number of subtour elimination and capacity constraints (5) and capacity constraints (1) generated;
2-Match	:	number of 2-matching constraints generated;
OptCut	:	number of optimality cuts (12) generated;
LBF	:	number of lower bounding functionals (29) generated;
LB/OPT	:	value of the linear relaxation of CP at the root of the search tree (after generation of all cuts) divided by the optimal solution value;
Nodes	:	number of nodes in the branch-and-cut tree;
Seconds	:	number of CPU seconds on a DEC Alpha 2100 server 4/275, excluding input and output time.

In each case, five instances were generated and a maximum of 1,800 seconds were allowed for the solution of any instance. All reported statistics are averages on the number of instances that could be solved within this time limit. This number is five, except for $n = 100$ where it is equal to four.

Results reported in Table 1 indicate that the proposed algorithm can successfully solve stochastic VRPs to optimality for the selected values of n, m, and f, within reasonable computing times. Problem difficulty and computational efforts increase sharply with m. Our results compare very favourably with previous best results obtained by Gendreau et al. (1995) and by Hjorring and Holt (1999). First we were able to solve, for the first time, instances with m taking values up to 4. In Gendreau et al., the largest instances solved for $f = 0.9$ and $m = 2$ contained 10 customers. In Hjorring and Holt, instances involving up to 90 customers were solved for values of the filling coefficient f in the range $[0.95, 1.05]$, but only for $m = 1$. Both in Gendreau et al. and in Hjorring and Holt, demands have a smaller variance than in our experiments, which yields easier instances. The success of our algorithm with respect to that of previous methods can be partly attributed to the use of stronger constraints and lower bounds, and of lower bounding functionals that apply when $m \geqslant 2$. The separation heuristic for lower bounding functionals proved to be particularly effective since it generated several cuts. The efficiency of our approach translates into very large LB/OPT ratios at the root of the search tree. However, even if this ratio is large, the number of nodes in the search tree

Table 2. Effect of the filling coefficient f ($n = 50$, $m = 3$).

f	SubCap	2-Match	OptCut	LBF	LB/OPT	Nodes	Seconds
0.80	391.6	4.8	129.4	680.2	0.987	1,012.0	35.0
0.85	403.6	5.4	66.6	699.8	0.985	1,071.8	56.0
0.90	1,324.8	4.8	238.2	655.8	0.983	3,431.0	174.8
0.95	6,668.0	6.5	331.0	2,581.5	0.973	11,866.0	1,180.5

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Table 3. Instances with normal demand distributions ($n = 50$, $m = 3$, $f = 0.9$).

CV	SubCap	2-Match	OptCut	LBF	LB/OPT	Nodes	Seconds
0.25	1,468.2	19.0	115.2	1,797.6	0.981	2,886.6	136.6
0.33	4,503.4	4.2	304.6	6,483.0	0.977	9,184.2	482.0
0.40	7,437.3	170.3	485.5	13,377.0	0.972	18,227.8	956.0
[0.25, 0.40]	4,919.8	116.4	281.2	8,496.7	0.978	11,033.0	629.4

can still be quite high. This is so because several good but often very dissimilar solutions exist in a close vicinity of the optimum.

In a second series of tests, we have investigated the effect of f on instances generated as above, with $n = 50$ and $m = 3$. We summarize in Table 2 average results obtained on five randomly generated instances. The number of successful instances is five for $f = 0.80, 0.85, 0.90$, and two for $f = 0.95$. As expected, problem difficulty increases with f, and instances become very easy for small values of f. Very small values of f are without interest since they yield instances equivalent to *m–Traveling Salesman Problems*. Values of f very close to 1 tend to generate solutions for which the probability that no route fails is very small.

Finally, in a third series of tests, we set $n = 50$, $m = 3$, $f = 0.9$, and we generated demands according to normal distributions with the same means as for the Poisson demands, and with coefficients of variation $CV = \sigma/\mu = 0.25, 0.33, 0.40$, and randomly generated in $[0.25, 0.40]$ according to a continuous uniform distribution. The results are summarized in Table 3. Five instances were attempted in each case. All instances were successful, except for $CV = 0.40$ where only four successes were obtained. On the whole these instances are more difficult to solve than in the Poisson case. This is due to the fact that the lower bound L given by (26) is weaker than its counterpart (16). Indeed, in the case of normal demands we use a lower bound on the variance of each route, as opposed to the Poisson case where individual variances are equal to the means.

6. CONCLUSION

In this article, we have proposed an Integer L-Shaped algorithm for the vehicle routing problem with stochastic demands, where the objective is to minimize the expected solution cost under the restriction that the expected demand of a route never exceeds the vehicle capacity. Instances involving between 25 and 100 vertices and between 2 and 4 vehicles were solved to optimality for the case where demands follow a Poisson or a normal distribution. The success of the proposed approach is largely dependent on the computation of stronger lower bounds and on the generation of more efficient lower bounding functionals. The combination of these two features results in high lower bounds at the root of the branch-and-cut tree and in relatively little branching, considering the difficulty of the problem. Our approach throughout this research has been that given the considerable difficulty of the stochastic VRP, it is preferable to achieve concrete results on medium size instances under realistic hypotheses rather than hoping to solve the problem in its full generality. Future research efforts should concentrate on gradually relaxing some of the underlying assumptions of our model.

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