

# L-shaped Method for the Stochastic Vehicle Routing Problem

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**Abstract -** The number of cargos transported by the logistics industry is increasing every year due to the growth of sales on the internet. It is thus crucial to address issues such as long working hours for employees, fewer personnel, and increasing CO<sub>2</sub> emissions for an effective transportation system. However, it is possible to create a productive delivery plan by considering uncertainties in time fluctuations, such as delays in work and delays in transportation due to the traffic congestion. Previous studies have defined mathematical programming models to build delivery plans that satisfy customer demand. However, few studies consider time fluctuations with multiple vehicles. Therefore, this study proposes an effective solution for the stochastic vehicle routing problem (VRP) using the L-shaped method, demonstrating the effectiveness of the proposed model by evaluating the value of stochastic solution (VSS). Specifically, we develop a model for VRP using multiple vehicles, taking into consideration the fluctuations in service and travel time between customers.

**Keywords –** Vehicle routing problem, Logistics, Stochastic programming

## I. INTRODUCTION

The logistics industry has been facing problems such as long working hours and fewer delivery personnel due to the growth of sales on the internet. An effective delivery plan can address these problems by considering uncertainties such as road congestion and service time. Multiple studies have focused on the vehicle routing problem.

The single-dial a ride problem (S-DARP) was studied by Heilporn et al. [5]. The objective of S-DARP is to minimize total cost, considering delays due to service time. The time window constraints which is designated the time from earliest to latest service start times for each customer were considered. Furthermore, if a vehicle cannot arrive in the designated time, that is, from the earliest to the latest start time, it is necessary to pay a penalty cost and use an alternative vehicle to eliminate the delay. However, their study did not consider the fluctuation of service and travel time by only dealing with a single vehicle [4, 9].

The time-constrained traveling salesman problem (TCTSP) was studied by Teng et al. [8]. The objective of TCTSP is to maximize the profit gained from a customer visit by satisfying their demand. Their study also dealt with a single vehicle only and considered time fluctuations, such as service time for the customer and travel time between customers. Moreover, if a vehicle exceeds standard traveling hours, penalty was paid for delays because it is necessary to pay a penalty cost for these excessive hours.

However travel costs between customers were not considered as shown in [3].

There are few numerical experiments for the vehicle routing problem under uncertainty using multiple vehicles in previous studies. The aim of this study is to minimize the total costs related to travel and penalties due to time fluctuations by using multiple vehicles, as opposed to previous study. Moreover, we propose solutions for the vehicle routing problem by constructing a model via stochastic programming [1, 7] referring to the model framework of Teng et al. [8].

In summary, the purpose of this study is three-fold. First, it proposes a new model for the vehicle routing problem considering fluctuations in service and travel time between customers. Second, it provides efficient solutions to the vehicle routing problem under uncertainty. Finally, it verifies the effectiveness of the proposed model by comparing several solutions.

## II. FORMULATION

### A. Overview

The overview of the model is as follows. Let  $G = (V, A)$  be a complete directed graph, where  $V = \{0, 1, \dots, n\}$  is a set of vertices, and  $A = \{(i, j) \mid i, j \in V\}$  denotes a set of arcs. Vertex 0 represents a depot and vertex  $1, \dots, n$  denotes  $n$  customers. Arc  $(i, j)$  represents the distance traveled between customers  $i$  and  $j$ . The route must start from and go back to the depot. A vehicle can visit each customer only once and cannot visit customers that are visited by other vehicles. The capacity to which a vehicle can be loaded is limited. In addition, standard traveling hours are fixed for each vehicle. If a vehicle exceeds this limit, it is necessary to pay a penalty cost for exceeding the time limit. This study attempts to construct a delivery route that minimizes the sum of travel and penalty costs for time delays, considering fluctuations in service and travel time. We assume that the random variables involved in the problem follow a finite discrete distribution and support of distribution is denoted by  $\mathcal{E}$ .

### B. Notations

<Variables>

$x_{ij}^k$	0-1 decision variable (whether or not vehicle $k$ moves from customer $i$ to $j$ )
$w^{ks}$	Delay time of vehicle $k$ in scenario $s$
$\theta$	Penalty cost for delays
$\theta^s$	Penalty cost for delays in scenario $s$

< Sets >	
$K$	The set of vehicles
$V$	The set of customers and a depot
$V'$	$V \setminus \{0\}$
$\Xi$	Support of random variables

< Parameters >	
$c_{ij}$	Transportation cost from customer $i$ to $j$ ( $c_{ij} = c_{ji}$ )
$d_i^s$	Service time for the customer in scenario $s$ (stochastic)
$P^s$	Probability occurrence of scenario $s$
$Q$	Maximum loading capacity of vehicle
$q_i$	Customer $i$ 's demand
$r$	Penalty cost for unit delay time
$T$	Standard traveling hours
$t_{ij}^s$	Travel time between customers in scenario $s$ (stochastic)
$NCUT$	Total number of optimality cuts
$SCUT$	Total number of sub-tour elimination constraints

### C. Formulation

#### < Master problem >

$$\min \sum_{i \in V} \sum_{j \in V} \sum_{k \in K} c_{ij} x_{ij}^k + \theta \quad (1)$$

$$\text{s.t. } \sum_{k \in K} \sum_{j \in V'} x_{ij}^k = 1 \quad \forall i \in V' \quad (2)$$

$$\sum_{j \in N \setminus \{i\}} x_{ji}^k - \sum_{j \in N \setminus \{i\}} x_{ij}^k = 0 \quad \forall i \in V', \forall k \in K \quad (3)$$

$$\sum_{j \in V'} x_{0j}^k = 1 \quad \forall k \in K \quad (4)$$

$$\sum_{i \in V'} x_{i0}^k = 1 \quad \forall k \in K \quad (5)$$

$$\theta \geq \sum_{s \in \Xi} P^s \theta^s \quad (6)$$

$$\theta^s \geq \sum_{i \in N} \sum_{j \in N} \sum_{k \in K} \alpha_{ij}^{k,s,ncut} x_{ij}^k + \beta^{s,ncut} \quad \forall s \in \Xi, \forall ncut = 1, \dots, NCUT \quad (7)$$

$$\sum_{i \in N} \sum_{j \in N} q_i x_{ij}^k \leq Q \quad \forall k \in K \quad (8)$$

$$\sum_{i \in N} \sum_{j \in N \setminus \{i\}} \alpha_{ij}^{scut} x_{ij}^k \leq |S^{scut}| - 1 \quad \forall scut = 1, \dots, SCUT, |S^{scut}| \geq 1, \forall k \in K \quad (9)$$

$$x_{ij}^k \in \{0,1\} \quad \forall i \in N, \forall j \in N, \forall k \in K \quad (10)$$

$$\theta, \theta^s \geq 0 \quad \forall s \in \Xi \quad (11)$$

#### < Sub-problem >

$$\min \sum_{k \in K} r w^{ks} \quad (12)$$

$$\text{s.t. } w^{ks} \geq \sum_{i \in N} \sum_{j \in N} t_{ij}^s x_{ij}^k + \sum_{i \in N} \sum_{j \in N} d_i^s x_{ij}^k - T \quad \forall k \in K, \forall s \in \Xi \quad (13)$$

$$w^{ks} \geq 0 \quad \forall k \in K, \forall s \in \Xi \quad (14)$$

Problems are decomposed into master problem and sub-problem to solve efficiently. The objective function of the master problem (1) is to minimize the sum of travel and penalty costs. Constraints (2)–(5) ensure that the route must start and end in the same depot, constraints (6) and (7) are optimality cuts, constraint (8) indicates capacity of the vehicle, and constraint (9) is the sub-tour elimination constraint, which guarantees that no other sub-tour in set  $V'$  is allowed, except a sub-tour including the depot. Based on the solution of the master problem, the sub-problem is solved for each individual scenario. The objective function of sub-problem (12) is to minimize the penalty cost for the delay. This function represents the recourse function of scenario  $s$  and the penalty cost for decision  $x$ . Constraint (13) indicates the delay time that exceeds the standard traveling hours due to fluctuations in service and travel time.

### III. SOLUTION

The above problem is solved by using three solution procedures: sub-tour elimination constraints, L-shaped method, and solution restriction constraint. The algorithm used in this study is presented in the following. First, the route is checked using the solution obtained by solving the master problem. If the route is a sub-tour, it is eliminated. After solving the master problem, optimal solution  $(\hat{x}, \hat{\theta})$  is obtained. Convergence is judged as to whether  $\theta$  represents the correct penalty cost. If it converges, the algorithm is terminated. If not, the optimality cut is added to the master problem and solved again.

#### A. Sub-tour elimination constraints (SEC)

Because it is necessary to eliminate routes that do not include the depot, sub-tour elimination constraints are added when solving the vehicle routing problem. In the usual formulations, all sub-tours must be enumerated and eliminated. However, as the number of customers increases, it becomes difficult to enumerate all sub-tour elimination constraints. Therefore, the sub-tour is eliminated sequentially based on the solution obtained by solving the

master problem. It is possible to reduce the sub-tour elimination constraints by adding constraint (9).

### B. L-shaped method

If the constraints for all scenarios are explicitly described, the problem becomes large and calculation lengthy. To address this issue, the master problem is solved sequentially, including only some of the variables and constraints. This solution is based on Benders decomposition and is called the L-shaped method [6]. The algorithm of the L-shaped method is shown below.

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#### Step 0: Initial setting

Temporary objective function value  $\bar{z} = \infty$ ,  
Lower bound of objective function value  $\underline{z} = 0$ .

#### Step 1: Solve the master problem

Obtain the optimal solution  $(\hat{x}, \hat{\theta})$ .

#### Step 2: Update each value

If  $\sum_{i \in N} \sum_{j \in N} \sum_{k \in K} c_{ij} x_{ij}^k + \theta > \underline{z}$ ,  
then  $\sum_{i \in N} \sum_{j \in N} \sum_{k \in K} c_{ij} x_{ij}^k + \theta = \underline{z}$ .  
If  $\sum_{i \in N} \sum_{j \in N} \sum_{k \in K} c_{ij} \hat{x}_{ij}^k + Q(\hat{x}) > \bar{z}$ ,

then  $\sum_{i \in N} \sum_{j \in N} \sum_{k \in K} c_{ij} \hat{x}_{ij}^k + Q(\hat{x}) = \bar{z}$ .

#### Step 3: Convergence check

Check whether  $\bar{z} \leq (1 + \epsilon)\underline{z}$ .

If the inequality is satisfied, the algorithm is terminated.

If the inequality is not satisfied, go to Step 4.

#### Step 4: Add the optimality cut

If  $\hat{\theta}^s < Q^s(\hat{x})$ , ( $\xi^s \in \Xi$ ), then add optimality cut to the master problem and go to Step 1.

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Fig. 1. Algorithm of L-shaped method

Next, we show how to derive optimality cut (7). Let  $\hat{x}$  be the optimal solution of the master problems and  $\hat{\lambda}^k$  be the optimal solution of dual problem (15) and (16) of the subproblem. The dual problem of the sub-problem in scenario  $s$  is as follows.

$$\max \sum_{k \in K} \left( \sum_{i \in N} \sum_{j \in N} t_{ij}^s \hat{x} + \sum_{i \in N} \sum_{j \in N} d_i^s \hat{x} - T \right) \lambda^k \quad (15)$$

$$\text{s.t. } 0 \leq \lambda^k \leq r \quad \forall k \in K \quad (16)$$

Let  $\theta^s$  be the upper bound for penalty cost in scenario  $s$ ; then, the following inequality (17) must hold:

$$\theta^s \geq \sum_{i \in N} \sum_{j \in N} \sum_{k \in K} (t_{ij}^s \hat{\lambda}^k + d_i^s \hat{\lambda}^k) x_{ij}^k - \sum_{k \in K} \hat{\lambda}^k T \quad (17)$$

In constraint (7), let the coefficients of the optimality cut be  $\alpha$  and the constant  $\beta$ :

$$\alpha_{ij}^{k,s,ncut} = t_{ij}^s \hat{\lambda}^k + d_i^s \hat{\lambda}^k, \beta^{s,ncut} = - \sum_{k \in K} \hat{\lambda}^k T \quad (18)$$

The optimality cut is added sequentially. If the value of  $\theta$  is updated, the lower bound of the objective function value is also updated.

### C. Solution restriction constraint

The optimality cut is added when using the L-shaped method. However, as the iteration progresses, the number of optimality cuts increases, and the number of inefficient optimality cuts also increases. As a result, the time required for computation increases because extra iterations are performed before finding the optimal solution. It is necessary to reduce the computation time as shown in [2].

For example, assume that feasible solution  $x^{l-1}$  of the master problem at iteration  $l - 1$  can be obtained in the neighborhood of optimal solution  $x^*$ . At this point, feasible solution  $x^l$  of the master problem obtained at iteration  $l$  might move away from optimal solution  $x^*$ . In this case, a large number of steps are repeated to find a solution whose distance to the optimal solution  $x^*$  is closer than that of feasible solution  $x^{l-1}$ . To avoid an inefficient search for solutions, the regularized decomposition method [1] can be used. This method ensures the feasible solution obtained during each iteration does not move away from the temporary solutions.

Therefore, in this study, the concept of the regularized decomposition method is applied to the proposed model. The constraint that limits the difference in the obtained feasible solution in each iteration is added. As a result, inefficient optimality cuts can be reduced and the optimal solution obtained earlier.

$$H_{ij}^k \geq x_{ij}^{k,l} - x_{ij}^{k,l-1} \quad \forall i, j \in N, \forall k \in K \quad (19)$$

$$H_{ij}^k \geq x_{ij}^{k,l-1} - x_{ij}^{k,l} \quad \forall i, j \in N, \forall k \in K \quad (20)$$

$$\sum_{i \in N} \sum_{j \in N} \sum_{k \in K} H_{ij}^k \leq S \quad (21)$$

In constraints (19) and (20),  $H_{ij}^k$  indicates the difference between the solution of iteration  $l - 1$  and  $l$ . These constraints show whether the values of solutions  $x_{ij}^{k,l-1}$  and  $x_{ij}^{k,l}$  change. In constraint (21),  $S$  indicates the maximum amount of the difference in the obtained solution. This maximum amount is limited by the above-mentioned constraint, so it is possible to prevent inefficient searching for the solution.

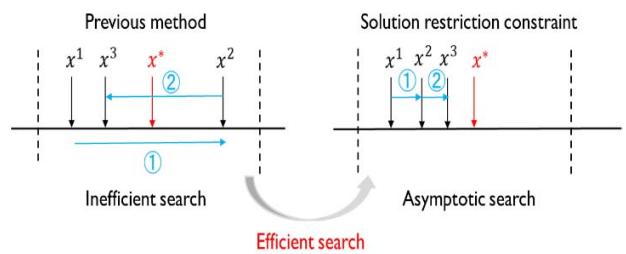


Fig. 2. How to search the solution

## IV. RESULTS

### A. Method of computational experiment

In this study, the model of the vehicle routing problem is considered the fluctuation of service time for the customer

and travel time between customers. Additionally, efficient solutions for the vehicle routing problem model were proposed. In this section, we present the data and the method of generating the scenario. The data are as follows.

Table 1. Parameter values

Mean	Notation	Value
Number of vehicles	$K$	3
Maximum loading capacity	$Q$	5000
Demand of customer	$q$	{200,400}
Penalty cost for the unit delay time	$r$	1200
Standard traveling hours	$T$	24

Customer demand is randomly generated at either 200 or 400. The position of the customer is also randomly generated in the range of  $100 \times 100 \text{ km}^2$  square. The position of the depot is the center [50, 50].

Next, the method of generating scenarios is as follows. The fluctuating time is the service time for the customer and the travel time between customers. Travel time  $t_{ij}^s$  depends on the traveling distance. Assuming that the normal speed of a vehicle is 50 km/h, the travel time between customer  $i$  and  $j$  is obtained by the following formula.

$$t_{ij}^s(h) = \frac{\text{the distance from customer } i \text{ to } j(\text{km})}{50 \text{ (km/h)}}$$

In addition, considering the fluctuation in travel time due to road congestion, the fluctuation is classified into three cases:

1. When a road is not crowded, travel time is reduced to 70% of the expected time.
2. The travel time becomes the expected time.
3. When a road is crowded, travel time is increased to 130% of the expected time.

The service time is classified into the following two cases, assuming no service finished earlier than the standard service time:

1. The service is completed at the standard service time.
2. The service time increases to 200% because of service delays.

Scenarios are generated based on the fluctuation of service and travel time. We set the number of patterns of service time to 2 and of travel time to 3. The customers are divided into 2 or 4 groups.

$$\mathcal{E} = 2^{(\text{number of customer groups})} \times 3$$

Thus, total number of scenarios  $|\mathcal{E}|$  is the power of the number of customer groups for the number of patterns of service time multiplied by the number of patterns of travel time. The number of scenarios increases or decreases by changing the number of customer groups. If we consider

the fluctuation of service time separately for all customers, there is the problem that the number of scenarios increases exponentially, meaning the scale of the problem becomes large. Therefore, the total number of scenarios has to be reduced by using this scenario generation. The probability of occurrence of the expected time is 0.6, and the probability of becoming later than expected or earlier than expected is 0.2 respectively. The computer used for this experiment had a 3.6 GHz Core i7-4790 (16.0 GB of memory) main processor and ran the IBM ILOG AMPL-CPLEX System 12.6.2.0 branch-and-bound solver.

### B. Results (evaluation of the solution)

The purpose of this study was to verify the effectiveness of the proposed model and the solution method. First, the solution method, using sub-tour elimination, L-shaped method, and solution restriction constraint, was verified by comparing them with the calculation time. Subsequently, the effectiveness of the proposed model was verified. The improvement of sub-tour elimination constraints (SEC) is shown as follows.

Table 2. Added number of SEC

Customers	Scenario	Optimal value	SEC	Iteration
10	12	12816.0	8	7
10	48	12816.0	8	7
15	12	15768.0	19	13
15	48	15768.0	19	12
20	12	17599.4	78	47
20	48	17416.4	68	37
25	12	19188.0	132	89
25	48	19019.0	127	76

If all sub-tour elimination constraints are enumerated, the number of constraints is  $2$  to the  $N$ th power ( $N$  is the number of customers). However, as shown in Table 2, the number of sub-tour elimination constraints added became small. As a result, it is possible to avoid the problem of becoming large-scale and reduce calculation time.

Next, the calculation time was compared for using the L-shaped method, and its effectiveness was verified.

Table 3. Comparing calculation time (L-shaped method)

Customer	Scenario	Optimal value	L-shaped (s)	Deterministic equivalent (s)
10	12	12816.0	1.0	1.0
10	48	12816.0	0.0	1.0
15	12	15768.0	2.0	1.0
15	48	15768.0	22.0	3.0
20	12	17599.4	59.0	63.0
20	48	17416.4	89.0	33.0
25	12	19188.0	1027.0	1804.0
25	48	19019.0	915.0	1585.0

A deterministic equivalent problem means a large-scale problem which is described constraints for all scenarios. By introducing the L-shaped method, as the number of customers' increases, calculation time is greatly reduced. It is presumed that this result is obtained because of solving

a small-scale problem instead of a large-scale one that considered all scenarios simultaneously. This result shows the effectiveness of using the L-shaped method.

In addition, calculation time was compared depending on whether to introduce the solution restriction constraint and its effectiveness was verified.

Table 4. Comparing calculation time (solution restrictions)

Customer	Scenario	Optimal value	Using (19)–(21) (s)	Without (19)–(21) (s)
10	12	12816.0	1.0	1.0
10	48	12816.0	1.0	1.0
15	12	15768.0	2.0	1.0
15	48	15768.0	12.0	3.0
20	12	17599.4	30.0	63.0
20	48	17416.4	66.0	33.0
25	12	19188.0	433.0	1804.0
25	48	19019.0	483.0	1585.0

Calculation time is greatly reduced as the number of customers increases by introducing solution restriction constraints (19)–(21). This is because the inefficient search was avoided in solving the master problem. From this result, the effectiveness by using the solution restriction constraint can be shown.

### C. Results (evaluation of proposed model)

The effectiveness of the proposed model was verified by comparing the case where the problem is solved by using stochastic programming with the one where the deterministic problem is solved. Let RP be the optimal objective function value obtained by solving the problem by using stochastic programming. However, after solving the problem using the expectation values in all scenarios, let EEV be the expectation value of the optimal objective function value obtained by using the deterministic solution. Moreover, the value of stochastic solution (VSS) is obtained by subtracting RP from EEV as in (22). VSS is the evaluation index of the effectiveness of the proposed model:

$$VSS = EEV - RP \quad (22)$$

$$\text{Improvement rate (\%)} = \frac{VSS * 100(\%)}{EEV} \quad (23)$$

Table 5. Evaluation of the solution

Customer	Scenario	EEV	RP	VSS	Rate (%)
10	12	12816.0	12816.0	0.0	0.00
10	48	12816.0	12816.0	0.0	0.00
15	12	15387.0	15768.0	69.0	0.44
15	48	15387.0	15768.0	69.0	0.44
20	12	19143.4	17599.4	1543.9	8.07
20	48	18581.3	17416.4	1164.9	6.27
25	12	20360.2	19188.0	1172.2	5.76
25	48	19298.2	19019.6	278.7	1.95

From this result, a better and more effective solution can be obtained when solving the proposed stochastic model than when solving the deterministic problem.

## V. CONCLUSIONS

There were three objectives for this study. The first was to propose a model of the vehicle routing problem,

considering the fluctuation of service time for the customer and travel time between customers by using multiple vehicles. The second was to propose efficient solutions. The third was to verify the effectiveness of the proposed model and the solution. It has been shown it is possible to decrease sub-tour elimination constraints and solve the problem efficiently by using the L-shaped method and the solution restriction constraint. Hence, the calculation time of the proposed model could be shortened significantly. In addition, the effectiveness of the stochastic programming model was shown for the proposed solution and its evaluation.

In future studies, we would like to consider time window constraint which is designated the time from the earliest start time to latest start time of service for each customer. This constraint is useful for the efficient vehicle transportation.

## REFERENCES

- [1] Birge. J. R, and Louveaux. F. V, “Introduction to Stochastic Programming”, Springer Series in Operations Research”, *Springer-Verlag*, 1997, pp. 137-152.
- [2] Campbell. A. M, and Thomas. B. W, “Runtime reduction techniques for the probabilistic traveling salesman problem with deadline”, *Computers & Operations Research* 36, 2009, pp. 1231-1248.
- [3] Campbell. A. M, and Thomas. B. W, “Probabilistic traveling salesman problem with deadlines”, *Transportation Science* 42, 2008, pp. 1-21.
- [4] Gendreau. M. G, Lapoete. G, and Seguin. R, “An exact algorithm for the vehicle routing problem with stochastic demands and customers”, *Transportation Science* 29, 1995, pp. 143-155.
- [5] Heilporn. G, Cordeau. J. F, and Laporte. G, “An integer L-shaped algorithm for the Dial-a-Ride Problem with stochastic customer delays”, *Discrete Applied Mathematics* 159, 2011, pp. 883-895.
- [6] Laporte. G, and Louveaux. F. V, “The integer L-shaped method for stochastic programs with complete recourse”, *Operations Research Letters* 13, 1992, pp. 133-142.
- [7] Shiina. T, “Stochastic programming” (In Japanese), *Asakura Syoten*, 2015, pp. 8 - 33, 64 - 71.
- [8] Teng. S. Y, Ong. H. L, and Huang. H. C, “An integer L-shaped algorithm for Time-Constrained traveling salesman problem with stochastic travel and service time”, *Asia-pacific Journal of Operational Research* 21, 2004, pp. 241 – 257.
- [9] Toth. P, and Vigo. D, “Vehicle Routing Problems, Methods, and Applications”, *MOS-SIAM Series on Optimization*, second edition, 2014.