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Vehicle routing with probabilistic capacity constraints

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ABSTRACT

In this paper, we study chance-constraint vehicle routing with stochastic demands. We propose a set-partitioning formulation for the underlying problem and solve it via a branch-and-price method. Our method is flexible in modeling different types of demand randomness while ensuring that the resulting problem is tractable. An extensive computational analysis, which includes simulation tests and a sensitivity analysis, is carried out to investigate the solution quality and computational efficiency. Some large instances of the underlying problems from the VRP library are solved to optimality for the first time. Our sensitivity analysis provides some useful insights about the impact of the probability of route failure on the decision variables, the expected cost and the route reliability.

1. Introduction

The vehicle routing problem (VRP) and its variants are among the most studied problems in combinatorial optimization, due to practical applications and theoretical challenges. In this study, we address an important variant of the VRP, known as *capacitated VRP with stochastic demands* (CVRPSD), in which the vehicle capacity is limited, and the customers' demands are not exactly known in advance and revealed only upon the arrival of a vehicle. Given stochastic demand, a vehicle may fail to serve a customer, and hence one possible recourse action is to return to the depot before completing its pre-planned route in order to empty its load. The CVRPSD arises in many applications such as bank deliveries, waste collection and grocery distribution (Heilporn, Cordeau, & Laporte, 2011).

The first result on the VRPSD, the CVRPSD without capacity constraint, dates back to the late 1960s with Tillman (1969). In the 1980s, the VRPSD received more attention with Stewart and Golden (1983) and Laporte, Louveaux, and Mercure (1989). Since then there has been considerable advancement in modeling and solving the VRPSD (Gounaris, Wiesemann, & Floudas, 2013). We briefly review the VRPSD research from three aspects: approaches to treat stochastic demand, formulations and exact solution methods. For more details, the reader is referred to some notable surveys: (Erera, Morales, & Savelsbergh, 2010; Gendreau, Jabali, & Rei, 2016; Pillac, Gendreau, Guéret, & Medaglia, 2013).

There have been three main approaches to dealing with stochastic demand in the VRPSD: stochastic dynamic programming, stochastic programming with recourse actions and stochastic programming without recourse actions. The first approach uses Markov chains and leads to a *re-optimization policy* in which replenishment and routing decisions are dynamic (Zhang, Lam, & Chen, 2016). Despite the fact that it is known to be the most promising approach (Dror, 2002), the resulting problem suffers a difficulty known as *the curse of dimensionality* (Iancu, Sharma, & Sviridenko, 2013). The second approach assumes that routes and replenishment decisions are static and include recourse actions and costs for route failures. This approach minimizes the total expected cost consisting of routing and recourse costs. Similar to the second approach, in the last approach routing and replenishment decisions are static. Probabilistic constraints are imposed on probability of route failure to guarantee a certain level of routing reliability. A more conservative approach is to enforce the routes validity against all possible demand realizations, i.e., to apply robust optimization to the VRPSD. Sungur, Ordóñez, and Dessouky (2008) study the application of robust optimization to the CVRPSD. In their study on the CVRPSD, Gounaris et al. (2013) provide an insight to the problem structure and its relationship with chance constraint models. In addition to the above popular approaches of modeling demand randomness, fuzzy theory is used to represent stochastic demands. For more on this subject, the reader is referred to Allahviranloo, Chow, and Recker (2014) and Kuo, Zulvia, and Suryadi (2012) and references therein.

In addition to demands, other parameters of a vehicle routing problem may also be subject to uncertainty. For instance, Adulyasak and Jaillet (2016) and Lee, Lee, and Park (2012) assume

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that the travel time is not exactly known in advance. In these studies, a deadline for visiting a customer is imposed.

Formulations for the VRPSD in the literature are mainly based on the flow formulation and the Miller-Tucker-Zemlin formulation (Miller, Tucker, & Zemlin, 1960). Under some specific settings, these formulations could lead to tractable models when demands are random. Laporte et al. (1989) show that chance constrained counterparts of the CVRPSD are equivalent to the deterministic VRP for a number of routing problems and stochasticity assumptions. Similarly, Gounaris et al. (2013) demonstrate that robust optimization counterparts of the CVRPSD can be reformulated by their deterministic equivalents.

In terms of exact solution methods, the VRPSD has received little attention compared to deterministic VRP. Stochastic integer programs (SIPs), which the VRPSD belongs to, are known to be very difficult to solve (Sherali & Zhu, 2009). Exploiting the structure of an SIP usually has a significant impact on efficiency of modeling and solution methods. In the literature, branch-and-cut techniques combined with decomposition algorithms are main methods for solving an SIP, particularly for the VRPSD. For instance, Laporte and Louveaux (1993) propose an integer L-Shaped method to solve stochastic CVRP with recourse costs. Novoa, Berger, Linderoth, and Storer (2006) and Christiansen and Lysgaard (2007) propose a set-partitioning formulation for specific settings of the CVRPSD with recourse costs. Noorizadegan (2013), for the first time, proposes set-partitioning formulations for the chance-constrained CVRPSD and a robust optimization model of the VRPSD. Dinh, Fukasawa, and Luedtke (2017) later extend the set partitioning formulation for the CVRPSD and provide more theoretical insights for the application of the chance-constrained VRPs.

Despite the effectiveness of branch-and-price based methods for deterministic integer programs, there are very few works on modeling and solving stochastic integer programs using these methods. This lack of research demonstrates an interesting research gap on efficiently formulating and solving stochastic integer programs using branch-and-price methods.

In this paper, we address this research gap and study set-partitioning formulations for two variants of the CVRPSD: a chance-constrained CVRPSD and a (distributionally) robust chance-constrained VRPSD. Our contribution can be categorized into two parts: modeling of the CVRPSD and computational analysis and enhancement. The contributions in the modeling part consist of (a) an efficient reformulation and search algorithm for the CVRPSD, (b) valid and effective dominance rules to ensure the optimality and feasibility conditions, and (c) the use of probability bounds in the pricing problem to limit search space. The pricing problem provides a flexible framework, capable of incorporating various settings and assumptions on random demands without increasing the model complexity.

On the computational analysis and enhancement, we demonstrate usefulness of our simulation experiment and sensitivity analysis. We provide some helpful practical insights for route planners regarding the quality of solutions, the impact of the user-specified reliability level and sensitivity analysis. The contribution of our computational analysis is threefold. (a) The proposed method enables us to solve several large standard instances of the underlying problems from the VRP library (branchandcut.org) to optimality for the first time. The largest instance (Dinh et al., 2017) solve contains 55 customers and 10 vehicles. We are able to solve several larger instances up to 60 customers and 15 vehicles and some very large instances up to 101 customers and 18 vehicles with relatively small integrality gaps. (b) We look at the solution quality on failures, particularly we use Monte-Carlo simulation and compare several performance measures for the deterministic, chance-constrained and distributionally robust chance-constrained models. In the literature of chance-constrained programming, the probability of failure is set and fixed to a small value. The chance-constrained formulation does not provide information on the violated routes, that is, measures such as failure costs are not investigated. Our computational analysis addresses this issue by computing and comparing the total expected routing cost, which consists of the cost of pre-planned routing decision and the cost of fulfilling demands for failed routes. (c) Moreover, small values of the probability of failure may result in unnecessary cost. We carry out a sensitivity analysis for route reliability level and study its impact on the routing and replenishment decisions and the objective values. The simulation experiment provides some useful and practical insights that help route planners to choose appropriate reliability levels for the CVRPSD, which result in the minimum total expected routing cost.

The remainder of the paper is organized as follows. Section 2 presents the set-partitioning formulation for the underlying CVRPSD. In Section 3, we introduce feasibility conditions, probability bounds and dominance rules for the pricing problem under some popular distribution functions. In Section 4, some optimality conditions are explained and general algorithmic steps of the proposed method are outlined. Section 5 is devoted to the computational analysis, where we assess the efficiency of the proposed method, the solution quality and the solution sensitivity with respect to variation of route reliability level. In Section 6 we provide some concluding remarks.

2. Model description

Let $G(N_0, A)$ be a complete graph, where $N_0 = N \cup \{0\}$ is the set of nodes and A is the set of arcs connecting the nodes. Node 0 is the depot and the other nodes form the set $N = \{1, \ldots, n\}$ of customers. There are m homogenous vehicles, with capacity Q each, available at the depot. Each customer is associated with a random demand q_i (such that $\mathbb{P}[0 \le q_i \le Q] = 1$), and each arc $a = (i, j)$ $(i, j \in N_0)$ is associated with a deterministic traveling cost c_a. A *route* r is denoted by the sequence of the nodes it goes through: $r = (r_0 = 0, r_1, ..., r_{n_r}, r_{n_r+1} = 0)$, where n_r is the number of different customers on the route and $N_r = \{r_1, \ldots, r_{n_r}\} \subseteq N$. A vehicle starts from the depot, serves a set of customers and returns to the depot. In the CVRPSD without recourse actions, if a vehicle *fails* to serve a customer (i.e., insufficient capacity left with the vehicle when it arrives at the customer point) on a pre-planned route, that customer and the remaining customers on the route remain unserved. Therefore, route planners intend to design routes that are *valid* (i.e., without failing to serve any customers on the route) with a high probability. A route is *feasible* if the following conditions are satisfied:

(a) It starts from and ends at the depot;
(b) It visits each node in N at most once;
(c) The total realized demand from all customers it visits is within its capacity with high probability $(1 - \epsilon)$. This condition will be specified in detail later.

Let z_r be a binary variable which takes a value of one if route r is chosen and, zero otherwise. For any route $r = (r_0 = 0, r_1, ..., r_{n_r}, r_{n_r+1} = 0)$, denote by $A(r) = \{(r_k, r_{k+1}) : k = 0, \ldots, n_r\}$ the set of all arcs it goes through. The traveling cost f_r of route r is the sum of costs of its arcs, i.e., $f_r = \sum_{a \in A(r)} c_a = \sum_{k=0}^{n_r} c_{r_k r_{k+1}}$. Let R and $R(i)$ be the sets of all feasible routes and feasible routes containing node i (i.e., $R(i) = \{r \in R : i \in N_r\}$), respectively. The set partitioning formulation of the underlying stochastic vehicle routing problem without recourse cost is as follows:

$$(\mathrm{P}): \quad \min \ \sum_{r \in R} f_r z_r \tag{1}$$

$$\text{s.t.} \ \sum_{r \in R} z_r \le m; \tag{2}$$

$$\textstyle\sum_{r\in R(i)} z_r = 1, \ \ \forall\ i \in N; \qquad (3)$$
$$z_r \in \{0,1\}, \ \ \forall\ r \in R.$$

First note that uncertain elements of our problem are implicitly included in the above formulation in terms of route feasibility condition (c), which will be explicitly dealt with separately in Section 3 due to its distinct importance in our study. In the above problem (P), the objective function computes the total routing cost for serving all customers. Constraint (2) makes sure that at most m routes are chosen, and constraints (3) guarantee each customer is assigned to exactly one route. Since it is a minimization problem and the cost of arcs satisfy the triangle inequality, we can replace the equality by "$\geq$".

It is impractical to include all feasible routes at the beginning of solving (P). We use the following approach, which was successfully used by Fukasawa et al. (2006) and Pessoa, de Aragao, and Uchoa (2007) for the deterministic CVRP. Problem (P) is initiated with a subset of feasible routes instead of the whole set of all feasible routes, which results in a problem called the *restricted master problem*. Feasible routes that improve the current solution are iteratively constructed and added to the master problem. The process that identifies feasible and improving routes is known as the *column generation subproblem*, which is formed on the basis of the dual problem to the LP relaxation of the master problem. Let $\alpha \leq 0$ and $\beta_i \geq 0$ be the dual variables corresponding to constraints (2) and (3), respectively. The column generation subproblem is then as follows:

$$(\mathrm{CG}): \quad y = \min\left\{\bar{f}_r \equiv f_r - \alpha - \textstyle\sum_{i\in N(r)}\beta_i : r \in R\right\}.$$

If $y = \bar{f}_r$ is negative for some route r, then route r will be added to the master problem, where $\bar{f}_r$ is called the *reduced cost* of a column or route r, which is the sum of the reduced costs of its arcs: $\bar{f}_r = \sum_{a\in A(r)} \bar{c}_a$, where the reduced cost of an arc $a = (i, j) \in A$ is defined by

$$\bar{c}_a = \begin{cases} c_a - (\beta_j + \alpha)/2, & \text{if } i = 0; \\ c_a - (\beta_i + \beta_j)/2, & \text{if } i, j \in N; \\ c_a - (\beta_i + \alpha)/2, & \text{if } j = 0. \end{cases}$$

To find the routes with negative reduced cost, we solve a shortest path problem on a graph with its arc weights as their reduced costs defined above. Due to the negativity of reduced costs of some arcs, negative cycles on the graph is inevitable. Therefore, we look for an *elementary route* (Christofides, Mingozzi, & Toth, 1981), which starts and finishes at the depot, and visits nodes in N at most once with a total (realized) demand at most Q up to a certain probability. Finding an elementary route on such a graph is known to be strongly NP-hard (Pessoa et al., 2007). We adopt a labeling search algorithm, in which feasibility and optimality conditions are imposed, the former enforcing the three conditions for a route to be feasible stated earlier in the section, while the latter ensuring that all feasible routes with negative reduced cost are identified.

3. Feasibility conditions

The feasibility conditions (a) and (b) are satisfied by the route construction procedure, which will be explained in the next section. Feasibility condition (c), also known as *the capacity constraint condition*, depends on the assumptions of the random demands and approaches used to treat the randomness. In the literature, chance-constrained programming (CCP) and distributionally robust chance-constrained programming (DRCCP) are among popular approaches without recourse actions. While DRCCP takes a conservative action and needs less information on the random demands, CCP is less conservative and requires information of the exact distribution function. Our proposed method is capable of formulating CCP and DRCCP as long as verifying the probabilistic constraint for the route feasibility is doable.

3.1. Probabilistic capacity constraint

When complete information of distribution functions of random demands is known, we can control the probability of route failure by imposing a probabilistic capacity constraint on the vehicle load as follows:

$$\mathbb{P}\left[\sum_{i\in N_r} q_i \leq Q\right] \geq 1 - \epsilon, \qquad (4)$$

where ϵ is the pre-specified probability of route failure or the route reliability level. In order to demonstrate the flexibility of the proposed method, we present the feasibility condition for three commonly used distribution functions in the literature on the CVRPSD: Normal distribution function, scenario-based representation of demands and Poisson distribution function. Our proposed method also can be used for several other continuous and discrete distribution functions for demands in particular for those that the sum of their random variables follows a known distribution.

In the first case, we assume that the demands follow normal distributions: $q_i \sim N(\mu_i, \sigma_i^2)$, then the probabilistic constraint of (4) is in the form of

$$\mathbb{P}\left[z = \frac{\sum_{i\in N_r}(q_i - \mu_i)}{\sqrt{\sum_{i\in N_r}\sigma_i^2}} \leq \frac{Q - \sum_{i\in N_r}\mu_i}{\sqrt{\sum_{i\in N_r}\sigma_i^2}}\right] \geq 1 - \epsilon.$$

This condition implies that if $\frac{Q-\sum_{i\in N_r}\mu_i}{\sqrt{\sum_{i\in N_r}\sigma_i^2}} < \Phi^{-1}(1-\epsilon)$, then the feasibility condition is violated, otherwise the route is feasible. Here $\Phi^{-1}(1-\epsilon)$ is the inverse of the Cumulative Distribution Function (CDF) of the standard normal distribution. As one can see, routes with correlated normally distributed demands can also be verified using the above probabilistic constraint.

In the second case, we consider a situation where the probability distribution has finite support with a finite number of possible realizations called scenarios. Scenario-based presentations are commonly used because firstly in real applications, determining the true distribution functions of random variables may not be easy, so samples of the random variables are collected, and secondly it is quite common in practice to approximate continuous distributions with discrete ones (Sherali & Fraticelli, 2002). Therefore, scenarios can be considered independent from each other (Linderoth, Shapiro, & Wright, 2006). Let us assume that stochastic demands is presented by a set of discrete scenarios indexed in set Ω where the probability of outcome ω is equal to p_ω. Thus, the feasibility condition is reformulated by

$$\sum_{\omega\in\Omega} I_\omega p_\omega \geq 1 - \epsilon,$$

where I_ω is an indicator function such that $I_\omega = 1$ if $\sum_{i\in N_r} q_i(\omega) \leq Q$, otherwise zero.

There are studies (such as Laporte, Louveaux, & van Hamme, 2002) that assume the demands follow the Poisson distribution, which has a discrete and non-negative domain with a probability density function similar to the normal distribution. These properties make Poisson distribution more realistic than the normal distribution.

In the third case, we assume that demands at nodes i follow the Poisson distribution with λ_i. The calculation of CDF for the sum of Poisson variables is computationally expensive. We use two bounds to reduce computation. First, we use a tail bound, the *Chernoff*

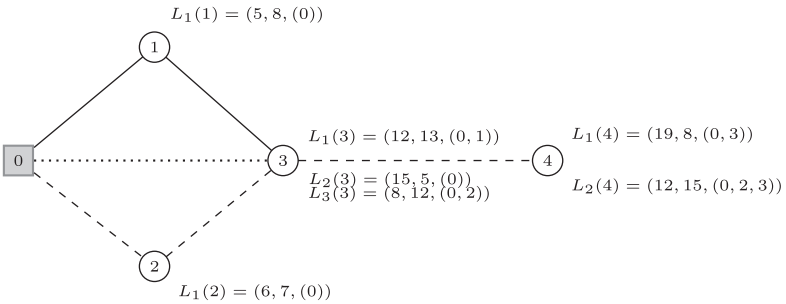
$L_1(1) = (5, 8, (0))$
1
0
3
$L_1(3) = (12, 13, (0, 1))$
$L_2(3) = (15, 5, (0))$
$L_3(3) = (8, 12, (0, 2))$
4
$L_1(4) = (19, 8, (0, 3))$
$L_2(4) = (12, 15, (0, 2, 3))$
2
$L_1(2) = (6, 7, (0))$

Fig. 1. Illustration for Rule 1–2 of Proposition 1.

bound, as follows (see page 64 in Mitzenmacher & Upfal., 2005). When X follows the Poisson distribution with parameter λ, then the following inequality is valid for any $x > \lambda$:

$$\mathbb{P}[X \geq x] \leq \frac{e^{-x}(e\lambda)^x}{x^x}.$$

The above inequality implies that if $e^{-Q}(e\lambda)^Q/Q^Q < \epsilon$, then $\mathbb{P}[X \geq Q] < \epsilon$. This means that the route is feasible and we do not need to compute the exact value of CDF. Second, we use bounds for the difference between the mean and median of Poisson variables to impose feasibility condition (c). From Theorem 2 of Chen and Rubin (1986), the bounds for the Poisson distribution is

$$\lambda - 1 \leq \text{med} \leq \lambda - 1/3.$$

Hence, if $\lambda - 1 > Q$, then $\mathbb{P}[X < Q] < 1 - \epsilon$ for $\epsilon < 0.5$. Routes that do not satisfy this condition, are fathomed in our search algorithm.

Dominance rules. As explained in the previous section and also shown in the next proposition, the demand randomness leads to a difficulty that we call *curse of dependency*, so that standard dynamic programming based on shortest path algorithms cannot be used. The reason is that, for instance, at a node if path 1 is shorter than path 2, we cannot eliminate path 2 because path 2 may be shorter than path 1 in the next stage. Therefore, the cost of a path or label is not a sufficient criterion for it to be eliminated. Other criteria that take into account properties of the demand distribution function, are required to be defined. Below we introduce some dominance rules for our proposed algorithm when demands follow the Poisson distribution.

Proposition 1. *When* $q_i \sim \text{Poisson}(\lambda_i)$, *routes are eliminated from the search space in Algorithm 1 according to the following two rules:*

Rule 1–1. *At each node, eliminate paths that violate the probabilistic capacity constraint,* $\mathbb{P}[\sum_{i \in N_r} q_i \leq Q] < 1 - \epsilon$,

Rule 1–2. $L_{l'}(i)$ *is dominated by* $L_l(i)$ *if* $\bar{\lambda}_l(i) < \bar{\lambda}_{l'}(i)$ *and* $\bar{c}_l(i) < \bar{c}_{l'}(i)$, *where* $\bar{\lambda}_l = \sum_{j \in N_l(i)} \lambda_j$ *and, as defined earlier,* $N_l(i)$ *is the set of nodes of path* l *staring from the depot ending at node* i.

Proof. Assume that at this stage Algorithm 1 saves all possible labels for each node and no label is eliminated or dominated. In order to prove the proposition, we need to show that eliminated paths could not lead to paths with smaller reduced cost. There exist two difficulties for identifying improving routes: probabilistic capacity constraints and negative cycles. First, let us assume that there are no negative cost cycles. By feasibility condition (c), if a path violates feasibility condition, it cannot be part of the optimal solution. Therefore, we do not need to keep it in any label set. We explain Rule 1–2 by an example illustrated in Fig. 1. Let $\lambda_1 = 8, \lambda_2 = 7, \lambda_3 = 5, \lambda_4 = 6$ and $\bar{c}_{01} = 5, \bar{c}_{02} = 6, \bar{c}_{03} = 15, \bar{c}_{13} = 7, \bar{c}_{23} = 2$ and $\bar{c}_{34} = 5$. The associated labels from the depot (node 0) to node 4 are computed and presented next to each node. Recall that each label consists of the reduced cost, the information of demand and the path to reach the node.

Algorithm 1: Search algorithm for the column generation subproblem.

```
begin
    $\mathcal{Q} = \emptyset$
    $L_1(0) = \{(0, 0, 0)\}$
    insert $L_1(0)$ into $\mathcal{Q}$
    for $i \in N$ do
        $L_1(i) = \{(\infty, \infty, (\emptyset))\}$
    end
    while $\mathcal{Q} \neq \emptyset$ do
        $\hat{l} \leftarrow$ the first label in $\mathcal{Q}$
        remove $\hat{l}$ from $\mathcal{Q}$
        $i \leftarrow$ node number of label $\hat{l}$
        for $j \in \mathcal{N}(i)$ do
            if extended label $\hat{l}$ to node $j$ does not hold
            feasibility conditions (a)–(c) then
                Continue
            else
                reducedcost $\leftarrow \bar{c}_{\hat{l}}(i) + \bar{c}_{ij}$
                update demand information
                if reducedcost $+ \bar{c}_{j0} < 0$ then
                    Stop
                else if new label is not dominated then
                    if any $L_l(j) \in L(j)$ is dominated then
                        remove the dominated labels
                    end
                    $l' \leftarrow$ proper index for the new label for node
                    $j$
                    insert $L_{l'}(j)$ into $\mathcal{Q}$ and sort $\mathcal{Q}$
                    insert $L_{l'}(j)$ into $L(j)$
                end
            end
        end
    end
end
```

Rule 1–2 implies that at node 3, $L_1(3)$ should be eliminated from the label set of node 3 as it is dominated by $L_3(3)$: For two random variables $X_l \sim \text{Poisson}(\lambda_l)$ and $X_{l'} \sim \text{Poisson}(\lambda_{l'})$ we know that if $\lambda_l(i) > \lambda_{l'}(i)$, then $\mathbb{P}[X_l > a] > \mathbb{P}[X_{l'} > a]$ for all $a \geq 0$. Therefore, $\mathbb{P}[q_1 + q_3 > Q] > \mathbb{P}[q_2 + q_3 > Q]$ and since $\bar{c}_1(3) > \bar{c}_3(3)$, extending $L_1(3)$ cannot lead to a better path than $L_3(3)$. On the other hand, we cannot eliminate $L_2(3)$ because $L_3(3)$ is more likely to lead to an infeasible route later on while $L_2(3)$ could be still feasible as $\mathbb{P}[q_2 + q_3 > Q] > \mathbb{P}[q_3 > Q]$.

Therefore, if $\bar{\lambda}_l(i) < \bar{\lambda}_{l'}(i)$ then given the CDF of Poisson distribution, $\mathbb{P}$[failure of l] < $\mathbb{P}$[failure of l'] and when $\bar{c}_k(i) < \bar{c}_{k'}(i)$, we can claim that $L_{l'}(i)$ is dominated by $L_l(i)$.

Similar to standard shortest path problems with negative cycles where no conditions are imposed to the problem, we can use s-cycle free policy to eliminate negative cycles. □

In the above proposition, if the rules are not satisfied for two paths, we will have to keep both paths in the label set of the node. In order to extend the second dominance rule for distribution functions with more than one parameter, we need to include all parameters in forming the dominance rules. The following proposition generalizes the dominance rules.

Proposition 2. *Let the accumulated demands of two paths/labels, starting from the depot and ending at node i, follow a distribution function $\mathcal{D}$ with multiple parameters, denoted by vectors $a_1(i)$ and $a_2(i)$, respectively, i.e., $\sum_{j\in N_1(i)} q_j \sim \mathcal{D}(a_1(i))$ and $\sum_{j\in N_2(i)} q_j \sim \mathcal{D}(a_2(i))$. The following elimination rules ensure the optimality of the proposed algorithm:*

Rule 2–1. *At each node, eliminate paths that violate the probabilistic capacity constraint, $\mathbb{P}[\sum_{i\in N_r} q_i \le Q] < 1-\epsilon$.*

Rule 2–2. *$L_{l'}(i)$ is dominated by $L_l(i)$ if for all parameters $a_{\iota,l}(i) < a_{\iota,l'}(i)$ and $c_l(i) < c_{l'}(i)$, where $a_{\iota,\,l}(i)$ is the ι-parameter of the distribution function of accumulated demands at node i.*

Proof. Rule 2–1 ensures the feasibility condition of the paths. Thus, if a path violates this condition, it must be eliminated. Following the proof of the previous proposition, Rule 2–2 compares the probability of failure for two feasible paths. If the conditions of Rule 2–2 hold, then $\mathbb{P}[\sum_{j\in N_1(i)} q_j > Q] < \mathbb{P}[\sum_{j\in N_2(i)} q_j > Q]$. This suggests that path 2 is dominated by path 1 and can be eliminated. □

Note that in Proposition 2, if demands follow the normal distribution, the second dominance rule will have three criteria: the cost, the mean and variance of the total demand for paths. Proposition 2 can be applied to the case where demand randomness is presented by a set of scenarios, by assuming each scenario as a parameter of the distribution. However, as the number of parameters of a distribution increases, fulfilling the dominance rules could become more difficult. Another factor that significantly affects the performance of the proposed method is the complexity of computation of violation probability. Discrete random variables are typically more difficult to work with than continuous random variables. As presented, one could incorporate appropriate inequalities in probability theory to improve the performance of the proposed method.

Also note that, in Proposition 2, we assume that the distribution functions associated with the accumulated demands of the two paths are perfectly known (i.e., distribution functions and parameter values). Such an assumption can be made in the case where the demands of customers are independent random variables. Dealing with dependent random demands is in general quite difficult. Many studies (such as Luedtke & Ahmed, 2008 and Luedtke, 2014) use discrete approximation (scenarios) to formulate chance-constrained models. We make a concluding remark at the end of Section 6 for a possible interesting extension to a more general case.

3.2. Distributionally robust probabilistic capacity constraint

In many cases, distribution functions of demands are not known precisely. One approach is to impose the probabilistic constraints for all distributions in a family $\mathcal{P}$ of distribution functions. Here, we assume that the family of distribution functions consists of all distribution functions that have the same known properties (such as the first and second moments) of the unknown true distribution function of the random parameters. Therefore, the probabilistic capacity constraint (4) will have to be robustly enforced for all the family distribution, i.e.,

$$\inf_{\mathbb{P}\in\mathcal{P}} \mathbb{P}\left[\sum_{i\in N_r} q_i \le Q\right] \ge 1-\epsilon, \tag{5}$$

where $\mathbb{P}$ is a distribution function which belongs to family $\mathcal{P}$. Given that the vehicle capacity is deterministic, the deterministic robust counterpart of the above constraint is formulated by the following proposition.

Proposition 3. *Let the demand vector of route r, denoted by $\boldsymbol{q}(r)$, follow an unknown distribution function with known mean vector, $\mu(r)$, and known covariance matrix, $\Lambda(r)$. For any $\epsilon \in (0,1)$, the distributionally robust probabilistic capacity constraint of route r, i.e., $\inf_{\mathbb{P}\in\mathcal{P}} \mathbb{P}\left[\sum_{i\in N_r} q_i \le Q\right] \ge 1-\epsilon$ is equivalent to*

$$\sum_{i\in N_r} \mu_i + \sqrt{\frac{1-\epsilon}{\epsilon}} \sqrt{\sum_{i\in N_r}\sum_{j\in N_r} \Lambda_{ij}} \le Q, \tag{6}$$

where Λ_{ij} is the covariance of q_i and q_j.

Proof. Let the demand vector $\boldsymbol{q}(r)$ be formulated by a zero-mean factor model, i.e.,

$$\boldsymbol{q}(r) = \mu(r) + \bar{\Lambda}(r)\upsilon,$$

where $\upsilon \in \Re^{|N_r|}$ is the vector of zero-mean factors such that $\mathbb{E}[\upsilon] = 0$ and $\mathrm{Var}[\upsilon] = I_{|N_r|\times|N_r|}$, and $\bar{\Lambda}$ is a full-rank factor matrix such that $\Lambda = \bar{\Lambda}^\top\bar{\Lambda}$. Note that for the sake of simplicity, index r is omitted in the notation in this proof, and also "$\top$" indicates the transpose of a matrix. According to the multivariate one-sided Chebyshev bound in (Bertsimas & Popescu, 2005), constraint (5) can be restated by

$$\sup_{\mathbb{P}\in\mathcal{P}(\mu,\Lambda)} \mathbb{P}\left[\sum_{i\in N_r} q_i > Q\right]$$
$$= \sup_{\mathbb{P}\in\mathcal{P}'(0,I)} \mathbb{P}\left[\sum_{i\in N_r} \upsilon_i \sum_{j\in N_r} \bar{\Lambda}_{ij} > Q - \sum_{i\in N_r} \mu_i\right] = \frac{1}{1+\theta^2}, \tag{7}$$

where $\mathcal{P}'(0, I)$ is a family of distribution functions whose mean is zero and covariance matrix is I, and θ^2 is computed as follows:

$$\theta^2 = \inf ||\upsilon||^2$$
$$\text{s.t.} \sum_{i\in N_r} \upsilon_i \sum_{j\in N_r} \bar{\Lambda}_{ij} > Q - \sum_{i\in N_r} \mu_i.$$

Following the proof of Theorem 3.1 in Calafiore and Ghaoui (2006), we have

$$\theta = \begin{cases} 0, & \text{if } \sum_{i\in N_r} \mu_i > Q; \\ \dfrac{\sum_{i\in N_r} \mu_i - Q}{\sum_{i\in N_r}\sum_{j\in N_r} \Lambda_{ij}}, & \text{if } \sum_{i\in N_r} \mu_i \le Q. \end{cases}$$

For the details of the proof, the reader is referred to Bertsimas and Popescu (2005) and Calafiore and Ghaoui (2006).

Given the restatement of constraint (5), we can have $\theta < \sqrt{(1-\epsilon)/\epsilon}$. Substituting θ in the latter inequality will lead to the final inequality (6). □

In order to apply Algorithm 1 to the distributionally robust probabilistic capacity constraint approach, we adapt the dominance rules in Proposition 2 alongside with the s-cycle free rule. Rule 2-1 is verified by Constraint (6). As argued in the proof of Proposition 2, we need to investigate the following inequality for two paths l and l': $\sup_{\mathbb{P}\in\mathcal{P}(\mu,\Lambda)} \mathbb{P}[\sum_{j\in N_l(i)} q_j > Q] <$

$\sup_{\mathbb{P}\in\mathcal{P}(\mu,\Lambda)} \mathbb{P}[\sum_{j\in N_{l'}(i)} q_j > Q]$. Given the definition of distributionally robust probabilistic capacity constraint, the failure probability of each path is computed using (7), which results in the following condition:

$$\theta_l^2 > \theta_{l'}^2 \;\Rightarrow\; \left|\frac{\sum_{i\in N_l}\mu_i - Q}{\sum_{i\in N_l}\sum_{j\in N_l}\Lambda_{ij}}\right| > \left|\frac{\sum_{i\in N_{l'}}\mu_i - Q}{\sum_{i\in N_{l'}}\sum_{j\in N_{l'}}\Lambda_{ij}}\right|.$$

For the case of independent demands, if the above condition holds and the reduced cost of path l is smaller than that of path l', then path l dominates path l'.

4. Optimality conditions

Labeling algorithms based on dynamic programming (such as the Bellman-Ford algorithm and Dijkstra's algorithm) have been used for the shortest path problems without additional conditions. However, Wang and Crowcroft (1996) prove that in general a shortest path problem subject to multiple constraints is NP-complete. The feasibility conditions described before, are interpreted as constraints which must be imposed to the shortest path problem. As discussed in the previous section, the pricing problem becomes even more complex due to randomness of customers' demands, i.e., as a result of random demands and the vehicle capacity constraint, the resulting pricing problem inherits a difficulty that longer paths among paths ending at a node cannot be eliminated.

We propose a search algorithm, which enumerates feasible routes. In the proposed algorithm, a set of labels are defined for each node. In order to manage the labels, some conditions and dominance rules based on the assumptions of the underlying problem are imposed, so that only useful labels are kept. Let $L(i) = \{L_1(i), L_2(i), \ldots\}$ be the label set for node i. Each label $L_l(i)$ is associated with a path to node i and consists of three components: $L_l(i) \leftarrow (\bar{c}_l(i), d_l(i), p_l(i))$, the total reduced cost $\bar{c}_l(i)$ of the path, the information $d_l(i)$ of the total demand on the path (e.g., mean and standard deviation) and the sequence $p_l(i)$ of nodes on the path. Depending on the assumption of the demand distribution function, the required information to be saved on a label may differ. The key point is to be able to determine the distribution function of the accumulated demands at a node using the information. For instance, if the demands follow the normal distributions, their mean and variance would be enough to compute the distribution function of the accumulated demands. We denote by $\mathcal{Q}$ the list of all labels in $\cup_{i\in N}L(i)$ arranged in a lexicographically ascending order based on the three label components.

The search algorithm starts from the depot 0 and extends the path to its neighborhood $\mathcal{N}(0)$. The extended path is added to the label set of node i and set $\mathcal{Q}$ if certain conditions and dominance rules are satisfied. The complexity and the exactness of the proposed method highly depend on the assumptions of stochastic demands. A general form of the proposed algorithm for the column generation subproblem is outlined in Algorithm 1.

5. Computational analysis and enhancement

In this section, we design computational experiments and report their results for our proposed method. Our computational experiments assess the efficiency and quality of the proposed method. Furthermore, we carry out a sensitivity analysis using a Monte Carlo simulation experiment in order to investigate the impact of probability of route failure on the decision variables.

We implement our proposed branch-and-price method in *scipoptsuite-3.2.0* (SCIP: solving constraint integer programs), which is a non-commercial mixed integer programming (MIP) solver available at http://scip.zib.de. All experiments are run on an iMac machine with a 3.1 GHz Intel Core i5 Processor and 8 GB RAM. Since SCIP does not provide parallel computing, we use only one thread out of available threads. We set a time limit of 7200 seconds.

The reminder of this section is organized as follows. First, we explain the branching strategy used for the branch-and-price method. Section 5.2 describes the data set used for our experiment and the required modification on some instances. In Section 5.3, we present the numerical results of the proposed method for the variants of stochastic vehicle routing problem we studied, and some important performance measures. In Section 5.4, we outline the tabu search algorithm for accelerating our solution to the CG subproblem. Section 5.5 presents the simulation results where the quality of solution based on for four key performance measures is examined. Section 5.6 is devoted to sensitivity analysis of the chance-constrained VRP with respect to probability of route failure.

5.1. *Branching strategy*

Branching strategy of a branch-and-price method is more complicated than that of a branch-and-cut method. For more details, the reader is referred to Achterberg (2007) and Lübbecke and Desrosiers (2005). One branching strategy is to branch on variables identified and added during the solution procedure. This strategy results in an unbalance branch-and-bound tree since unlike branch-and-cut methods, very few variables will take the value of one in the final solution. Moreover, when a variable is chosen for branching, on the zero branch, it is likely for the CG subproblem to identify the same variable again as an improving one, resulting in an indefinite loop in the solution procedure. In order to address this issue, Ryan-Foster's branching has been commonly used, where a cut is constructed at each node of the branch-and-bound tree to avoid any loop. For more details, the reader is referred to Barnhart, Johnson, Nemhauser, Savelsbergh, and Vance (1998). However, it requires keeping track of all identified variables and constructing branching constraints that affect the CG subproblem.

Another branching strategy is to include original variables, variables of the standard formulation, into the set-partitioning formulation and to branch on these variables. In this study, we use the latter strategy. We introduce binary variables x_{ij}, where $x_{ij} = 1$ if the arc from nodes i to j is selected in a route, and $x_{ij} = 0$ otherwise. The following constraints are therefore added to Problem (P): $\sum_{j\in N_0} x_{ij} = 1$ and $\sum_{j\in N_0} x_{ji} = 1$ for any node $i \in N$, which ensure that exactly one arc enters node i and exactly one arc leaves it. Constraints (3) are also modified to $\sum_{r\in R(i,j)} z_r - x_{ij} \leq 0$ for all pairs of (i, j), where R(i, j) is the set of routes that traverse arc (i, j). Although the CG subproblem slightly changes, the proposed solution algorithm remains almost the same.

5.2. *Data set*

We use standard instances available at http://branchandcut.org for our computational experiments. As the instances are originally designed for deterministic VRP, demands are modified according to the approaches described in Section 3. We focus on Poisson distribution functions as they are computationally more expensive than the other two classes of distribution functions mentioned before. Computational results for the two other classes can be found in Noorizadegan (2013). For the probabilistic capacity constraint, we assume that the demand presented in each instance is the mean of the Poisson distribution. For example, in instance E-n13-k4, the demand of the first customer is 1200 units. In our computational experiment, we assume that the demand of the first customer follows a Poisson distribution with mean equal to 1200.

In the approach of distributionally robust probabilistic capacity constraint, we study a case where the mean vector and the covariance matrix of demands are known but not distribution

Table 1
The solution quality and the efficiency of the proposed method when $\epsilon = 0.10$.

	Inst.	Det.		PCM					
		Obj.	# routes	Obj.	# routes	Sol. time (seconds)	Int. gap (percent)	# added variables	# nodes (BB tree)
1	A-n32-k5	784	5	857	5	6955	0	32315	3698
2	A-n33-k6	742	6	802	7	7200	0.81	25379	20173
3	A-n37-k6	949	6	1021	7	260	0	3641	307
4	A-n45-k7	1146	7	1285	8	7200	4.87	26962	3790
5	A-n55-k9	1073	9	1157	10	382	0	2950	165
6	A-n63-k10	1616	10	1435	11	7200	1.67	20784	4696
7	A-n80-k10	1763	10	2032	11	7200	7.16	13612	113
	Average	1153.3	7.6	1227.0	8.4	5199.4	2.1	17949	4706
8	E-n13-k4	247	4	277	4	0.12	0	40	3
9	E-n22-k4	375	4	424	5	11	0	944	158
10	E-n31-k7	379	7	416	8	186	0	1873	111
11	E-n51-k5	521	5	521	5	5152	0	4174	19
12	E-n76-k10	830	10	934	12	7200	8.2	4740	46
13	E-n76-k14	1021	14	1100	16	7200	1.1	11515	8589
14	E-n101-k14	1076	14	1248	18	7200	9.74	11017	247
	Average	635.6	8.3	702.9	9.7	3849.9	2.7	4900	1310
15	P-n16-k8†	450	8	460	8	0.49	0	108	79
16	P-n19-k2	212	2	212	2	500	0	5335	201
17	P-n20-k2	216	2	216	2	182	0	1712	16
18	P-n22-k8†	603	8	590	9	0.37	0	76	9
19	P-n23-k8	529	8	630	10	2.61	0	246	191
20	P-n40-k5	458	5	476	5	5936	0	18529	1907
21	P-n50-k10	696	10	751	12	5717	0	10522	25692
22	P-n55-k15	989	15	1071	19	347	0	1451	3384
23	P-n60-k10	744	10	795	11	7200	0.99	15236	11981
24	P-n60-k15	968	15	1078	17	694	0	2433	3213
25	P-n70-k10	827	19	909	12	7200	4.78	8501	434
	Average	608.4	8.5	653.5	9.7	2525.3	0.5	5831.7	4282

functions. Similar to the first approach, we choose the mean vector from the deterministic values given in the instances. We assume that the demands are uncorrelated, i.e., the covariance matrix is diagonal. The entry on the diagonal of the covariance matrix are the demand variance, which we set to μ_i.

The number of vehicles, which are listed for the standard CVRP instances, are the minimum number of vehicles required to serve all customers with deterministic demands. Given that we assume stochastic demands, these numbers of vehicles may not be sufficient to serve all customers and may result in infeasible solutions. Thus, we drop limitations of the number of vehicles and assume that an unlimited number of homogenous vehicles are available. Note that, for some ϵ and nodes, even such a route that consists of a single node, may be infeasible. For such cases in our experiment, we increase the vehicle capacity.

5.3. Numerical results

Our numerical results present solution efficiency of the proposed solution method. The following six performance measures are used to assess the solution quality: objective function value (Obj.) obtained from solving each instance, number of routes (# routes), solution time in seconds (Sol. time), integrality gap in percent (Int. gap), number of added variables (# added variables) and the number of nodes of the branch-and-bound (BB) tree.

Table 1 compares the solutions of the deterministic model (Det.) and those of the probabilistic constraint model (PCM), where the results for the deterministic model are taken from http://branchandcut.org. For two instances, there are no feasible solutions since the demand of some customers are larger than the capacity of vehicles, i.e., $\exists i \in N,\ \mathbb{P}[q_i > Q] > \epsilon$. These instances are marked by "†" in the table. In order to construct feasible routes for these instances, we increase the capacity of vehicles by 20 percent. It is worth mentioning that when the vehicle capacity is increased, the routing cost may decrease as vehicles can cover more customers in one trip.

We compare our results with those reported in Dinh et al. (2017) for some similar problems except that demands are assumed to follow the normal distributions. As mentioned in Section 3.1, the Poisson distributions are more realistic than the normal distributions to formulate demands. In addition, the Poisson distributions are computationally more expensive and challenging to work with than the normal distributions. In their work, they solve various cases, including independent demands with low and high variance, and also dependent demands that are formulated by the joint normal distributions. They solve 10 instances from the VRP library and the largest instance they are able to solve to optimality includes 55 customers and 10 vehicles.

As Table 1 reports, we apply the proposed method to 25 instances, among which 16 instances are solved to optimality within the time limit. The average integrality gap is very small, 1.6 percent. We solve several large instances to optimality, with the largest including 60 customers and 17 vehicles, for the first time. We are also able to find near optimal solutions for very large instances, with 101, 80 and 76 customers and 18, 11 and 16 vehicles. In addition to the difficulty of instances, the performance of the proposed solution depends on both numbers of customers and vehicles. The number of vehicles is related to the number of customers visited by a vehicle. As the ratio of the number of customers to the number of routes increases, the number of possible combinations of customers grouped for a vehicle increases. This leads to a larger search space and, as a result, longer solution time. This can be observed from instances in rows 12 and 13, and also rows 23 and 24. While the number of customers is the same for each pair of instances, the solution time and integrality gap decrease when the number of vehicles increases.

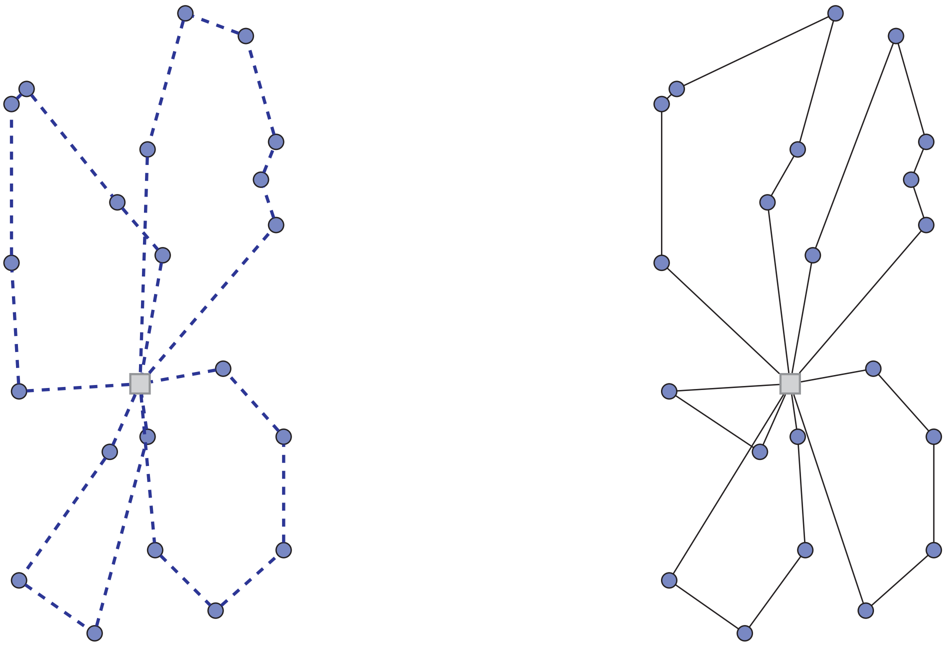

Fig. 2. The optimal routes of Instance E-n22-k4 for the deterministic (on the left) and probabilistic constrained (on the right) models.

If we ignore Instance E-n101-k14, which is very large, it seems that instances of class A are more difficult to solve. The main performance measures, including the solution time, the integrality gap, the number of added variables and the number of nodes of the branch-and-bound tree, for this class of instances are larger than those of the other two classes of instances.

As expected, the routing costs (i.e., objective function values) of most instances for the PCM are higher than those for the deterministic model. Three instances have the same routing costs for the deterministic and probabilistic models, which suggests that the routes obtained from the deterministic model satisfy the probabilistic constraint. The routing cost of Instance P-n22-k8 is smaller compared to the deterministic model since the vehicle capacity increases.

The numbers of routes and vehicles are a critical factor for route planners. A large increase in the number of vehicles usually leads to additional issues such as extra management and overhead costs. However, as reported in the table, in order to improve the routing reliability to 90%, we need to increase the number of vehicles by adding only about 1 vehicle on average to the fleet. Interestingly, the number of routes does not change in 7 instances, which suggests that increasing the fleet size is not always the only way of improving route reliability. In other words, more reliable routes may be achievable with a *better assignment* of customers to routes. Fig. 2 illustrates an example of how routes change for the deterministic model and the PCM.

Table 2 reports the results of the distributionally robust PCM (DRPCM) for 13 instances. The number of routes and the routing costs for this model dramatically increase. With respect to the deterministic model, the cost increases by more than 18% on average for the DRPCM, while the increase for the PCM was only 4%. Note that the percentage of increase is computed for the same instances for both the PCM and the DRPCM. The average number of routes for the corresponding instances for the deterministic, the PCM and the DRPCM are 8.4, 9.6 and 11.2, respectively. The average integrality gaps and solution time are almost equal for both models. Similar to the PCM, there are instances that include demands exceeding the vehicle capacity. We increase the vehicle capacity for these instances by 40% and mark them by ‡ in the table.

Table 2
Solution quality and efficiency of the proposed method for the DRPCM when $\epsilon = 0.10$

	Ins.	obj.	# routes	Sol. time (seconds)	Int. gap (percent)
1	A-n33-k6	898	8	5026	0
2	A-n55-k9	1356	12	7200	2.47
3	A-n63-k10	1666	13	7200	3.3
4	E-n13-k4	369	5	0.09	0
5	E-n22-k4	495	6	12.03	0
6	P-n16-k8‡	472	9	0.42	0
7	P-n19-k2	246	3	2322	0
8	P-n22-k8‡	649	10	4.38	0
9	P-n23-k8‡	568	9	1.62	0
10	P-n50-k10	850	14	7200	0.73
11	P-n55-k15	1237	23	6.52	0
12	P-n60-k10	884	13	3641	0
13	P-n60-k15	1241	21	97.23	0
	Average	841	11	2516	0.5

One could conclude that the RCM results in a better solution than the deterministic model and the DRPCM, because with a small change in the number of routes and in the routing cost, a significant route validity is achieved. As mentioned, for more detailed analysis, other important practical factors such as overhead cost for additional routes may be considered.

5.4. Solution acceleration

Tabu search algorithms have been commonly used for solving VRP variants. We use a tabu search algorithm proposed in Desaulniers, Lessard, and Hadjar (2008) in order to potentially accelerate the solution process for the CG subproblem in finding improving path(s). The tabu search algorithm is not exact. We first run the tabu search algorithm for the CG subproblem. If improving paths are identified, they are added to the restricted master problem. Otherwise, the proposed algorithm including the dominance rules (presented in Algorithm 1) is run to find improving routes if they exist.

Table 3
Efficiency of the proposed method when tabu search algorithm is incorporated in the column generation with $\epsilon = 0.10$

	Inst.	With Tabu Search			Without Tabu Search		
		# added variables	Int. gap (percent)	Sol. time (seconds)	# added variables	Int. gap (percent)	Sol. time (seconds)
1	A-n33-k6	20813	0.66	7200	25379	0.81	7200
2	A-n55-k9	3215	0	834	2950	0	381.5
3	A-n63-k10	11584	2.2	7200	20784	1.67	7200
4	E-n13-k4	116	0	0.23	40	0	0.12
5	E-n22-k4	1102	0	27.37	944	0	11.06
6	E-n51-k5	3890	16.09	7200	4174	0	5152
7	P-n19-k2	4202	0	1034	5335	0	500
8	P-n23-k8	262	0	2.97	246	0	2.61
9	P-n55-k15	1908	0	423	1451	0	347
10	P-n60-k10	10191	0.89	7200	15236	0.99	7200
	Average	5728	1.98	3112	7654	0.347	2799

We partially follow Desaulniers et al. (2008) in designing our tabu search algorithm. In our implementation, the main operation of constructing improving paths is to insert nodes in different positions of existing routes. A new path that is created by inserting only one node to the current path is called a *neighbor* of the current path. At each stage of the algorithm, neighbors of the current path are constructed. To reduce the size of the search space, we only allow feasible neighbors to form. Among all neighbors, the one with the least reduced cost replaces the current path, although the reduced cost of this neighbor may be higher. This allows to expand the search for feasible paths. Once an improving path is identified, the search stops and the path is added to the restricted master problem. We start the search with a set of multiple paths, which consists of all single-node paths, i.e., paths starting from the depot, visiting only one node and returning to the depot.

Here, we investigate the performance of the solution procedure with and without the tabu search algorithm. Table 3 reports the results on ten instances. The tabu search algorithm improves the results for only two instances. Also as the row “Average” suggests, the tabu search algorithm does not lead to a significant improvement, which suggests that Algorithm 1 together with the dominance rules outperforms the tabu search algorithm.

5.5. Simulation

In this section, we study the following performance measures for our simulation experiment: total expected cost[1] $\pi(x^*, \epsilon)$ for $\epsilon = 0.10$, standard deviation (std) of the total cost and 95% quartile (95% Q) of the distribution of the total cost and the expected failure cost $\mathbb{E}[f(x^*, 0.10)]$.

Table 4 summarizes the results of our simulation experiment with $\epsilon = 0.10$. The first column of each model reports the total expected cost. Here, the total expected cost is computed by $\pi(x^*, 0.10) = \text{Obj.} + \mathbb{E}[f(x^*, 0.10)]$, where “Obj.” is the optimal objective function value for $\epsilon = 0.10$, and $f(x^*, 0.10)$ is the recourse cost function. $f(x^*, 0.10)$ is computed with Monte Carlo simulation such that the solution of the instance (optimal solution if “Int. gap” is zero, and the final solution after 7200 seconds otherwise) will be evaluated against 10,000 different scenarios/realizations of random parameters. We assume that if a vehicle fails to serve a customer, it will have to revisit the depot for a replenishment and resume its pre-planned route. Therefore, the expected recourse cost is computed according to $\mathbb{E}[f(x^*, 0.10)] = \sum_{s\in S} p_s \sum_{r\in R^*} \sum_{i\in \text{Failed}_r(s)} (c_{i,0} + c_{0,i})$, where R^* is the set of optimal routes, $\text{Failed}_r(s)$ is the set of failed nodes for route r at realization s, and p_s is the probability of realization s. Note that i is the node that the vehicle at realization s fails to serve and requires to revisit the depot. The second and third columns of each model report the standard deviation and the 95% quartile of the distribution of the total cost, i.e., the total cost for scenario s is $\pi_s(x^*, 0.10) = \text{Obj.} + \sum_{r\in R^*} \sum_{i\in \text{Failed}_r(s)} (c_{i,0} + c_{0,i})$. The key assumption for the random demands in the DRPCM is that the mean of a demand is not exactly known. Therefore, we carry out two simulation experiments for the DRPCM: with known mean demand and with random mean demand. In the first case, similar to the other models, we use the demand value of each customer as its mean to generate scenarios and in the second case, we randomly choose the mean from the interval $[0.8\mu_i, 1.2\mu_i]$.

The results suggest that the PCM outperforms both the deterministic model and the DRPCM in terms of the expected cost and the 95% quartile. The reason behind it is that the deterministic model ignores the randomness of demand and the DRPCM is too conservative and risk averse. Although the DRPCM results in a very small failure cost (in both experiments), on the one hand, its total expected cost is higher than that of the PCM. On the other hand, it overprotects the route validity way behind the requirement. It suggests that even if we choose a very small value for ϵ, the solution of the DRPCM may stay unchanged. In other words, it seems the DRPCM is insensitive to ϵ due to the conservativeness of the approach. The impact of the conservativeness of the DRPCM has also been worsened by the integrality conditions of the decision variables.

As these two tables suggest, the PCM provides a good tradeoff between conflicting goals such as the number of vehicles, the objective value and the expected cost while it keeps the standard deviation relatively low. It is worth mentioning that the DRPCM is not entirely outperformed by the PCM and some decision makers may prefer the DRPCM due to its high reliability.

5.6. Sensitivity analysis

In the literature of chance-constrained programming, the probability (ϵ) of failure is usually assumed to be given and specified in advance. For a pre-specified ϵ, a probabilistic constraint may be violated when random parameters are realized and, therefore, an extra cost for recourse may have to be imposed to deal with or redeem failures. Hence, it is common to choose a very small ϵ to minimize failure costs. However, small ϵ may lead to unnecessary extra cost that does not have reasonable added value, particularly in stochastic integer programs. Choosing the right value for ϵ is a critical step that can have a significant impact on decision variables and objective function.

[1] When the integrality gap is not zero, the total expected cost is computed for the solution obtained after 7200 seconds.

Table 4
The Monte-Carlo simulation results for three models when $\epsilon = 0.10$

Instance	Det.				CCP				DRPCM				DRPCM-Random Mean			
	$\pi(x^*, 0.10)$	std	95% Q	$\mathbb{E}[f(x^*, 0.10)]$	$\pi(x^*, 0.10)$	std	95% Q	$\mathbb{E}[f(x^*, 0.10)]$	$\pi(x^*, 0.10)$	std	95% Q	$\mathbb{E}[f(x^*, 0.10)]$	$\pi(x^*, 0.10)$	std	95% Q	$\mathbb{E}[f(x^*, 0.10)]$
E-n13-k4	283	30	335	36	283	15	323	6	369	1.7	369	0.1	369	1.54	369	0.2
E-n22-k4	412	32	473	37	430	15	470	6	495	2	495	0.1	495	1.62	495	0.3
P-n16-k8†‡	513	45	592	63	468	18	504	8	472	3	472	0.2	472	3.66	472	0.8
P-n19-k2	230	18	264	18	235	20	277	20	246	1.1	246	0	246	0.98	246	0.1
P-n22-k8†‡	709	63	821	106	606	27	654	16	649	4.6	649	0.3	649	5.15	649	0.7
P-n23-k8‡	664	64	773	135	655	35	728	25	568	3.7	568	0.3	568	4.33	568	1.0
P-n50-k10	793	47	874	97	771	26	823	20	850	2.9	850	0.2	851	4.57	850	1.5
P-n55-k15	1192	64	1301	203	1101	36	1173	30	1238	5.9	1237	0.8	1238	7.60	1238	3.5
P-n60-k15	1133	66	1246	165	1108	34	1172	30	1242	4.8	1242	0.6	1242	6.55	1241	2.9
Average	659	47.67	742	96	629	25.11	680	18	681	3.30	681	0.29	681	4.00	681	1.20

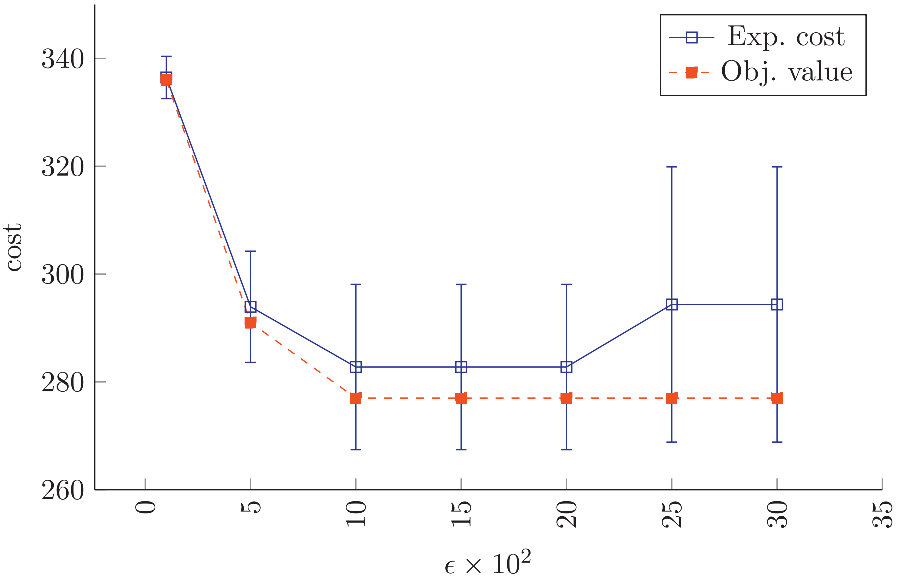

Fig. 3. Sensitivity analysis for Instance E-n13-k4.

One way of addressing this situation is to consider ϵ as a decision variable. However, due to difficulty of formulating and solving such problems, only few works (Rengarajan, Dimitrov, & Morton, 2013) and (Shen, 2014) under very limiting assumptions consider ϵ itself as a decision variable. A problem with such a setting becomes even more difficult when integrality conditions are imposed on some or all decision variables. Another approach is to use a sensitivity analysis and Monte Carlo simulation in order to investigate the impact of variation of ϵ and choose the right value for ϵ. In this study, we consider seven values for $\epsilon \in \{0.01, 0.05, 0.10, 0.15, 0.20, 0.25, 0.30\}$.

The procedure is as follows. First, each instance is solved for all the values of ϵ. Second, their solutions are evaluated against 10,000 demand realizations that are generated according to the associated Poisson distribution. Then, the total expected cost ($\pi(x^*, \epsilon)$), the expected failure cost ($\mathbb{E}[f(x^*, \epsilon)]$) and the standard deviation (std) similar to our simulation experiments in the previous section are computed. Fig. 3 presents the sensitivity analysis for Instance E-n13-k4 using an interval plot. The dashed line presents the objective function value and the solid line is associated with the total expected cost. The vertical intervals report the 95% interval of the total expected cost. As one can see, the objective function value increases when ϵ decreases, while the total expected cost shows a different behaviour. $\pi(x^*, \epsilon)$ achieves its minimum value when ϵ is equal to 0.10, 0.15 and 0.20. The standard deviations increase by ϵ, which means the optimal solution is more reliable and robust when ϵ is smaller.

We can observe that large ϵ (e.g., 0.25 and 0.30) may not be appropriate, as they result in large standard deviations and large total expected cost. On the other hand, very small ϵ (e.g., 0.01) may not be interesting, too, as there is a very sharp increase in total expected cost, while the route reliability does not change significantly. Also even the standard deviation may be reasonable in larger ϵ (e.g., 0.05). Note that the route reliability is implied by the total expected failure cost, which is the difference between the expected cost and the objective function value.

Table 5 provides some more details for the sensitivity analysis. As explained before, some instances do not have a feasible solution for small ϵ. Here, we do not change those instances and leave them unsolved. The last row of each instance, indicated by “Det.”, presents the results for the deterministic model.

We can observe that the number of routes does not always change. It means that in order to achieve higher reliability levels, we do not necessarily need to increase the fleet size. In other words, the reliability level can be increased by improving routing decisions and customers' assignment.

The standard deviation of the total cost and the expected failure cost decrease as ϵ decreases, while the number of routes and objective function value have opposite trends. The behaviour of the total expected cost is more complex. The recourse action is not explicitly invoked in the problem formulation, therefore, one would expect a non-convex behavior for the total expected cost. Also, the randomness of the simulation experiment may slightly affect the total expected cost.

Table 5
Sensitivity analysis on the impact of variation of the reliability level

Instance	ϵ	# routes	obj.	$\pi(x^*, \epsilon)$	std	$\mathbb{E}[f(x^*, \epsilon)]$
E-n13-k4	0.01	5	336	336	4	0
	0.05	4	291	294	10	3
	0.10*	4	277	283	15	6
	0.15*	4	277	283	15	6
	0.20*	4	277	283	15	6
	0.25	4	277	294	26	17
	0.30	4	277	294	26	17
	Det.	4	247	283	30	36
E-n22-k4	0.01	6	466	467	5	1
	0.05	5	443	446	10	3
	0.10	5	424	430	15	6
	0.15	5	412	421	20	9
	0.20	5	411	423	22	12
	0.25	5	401	420	26	19
	0.30	5	394	422	31	28
	Det.*	4	375	412	32	37
P-n16-k8	0.25	9	476	517	41	41
	0.30	9	472	517	42	45
	Det.*	8	450	513	45	63
P-n22-k8	0.15*	10	666	699	41	33
	0.20	10	649	705	55	56
	0.25	9	627	703	60	76
	0.30	9	627	703	60	76
	Det.	8	603	709	63	106
P-n23-k8	0.05	11	652	662	22	10
	0.10	10	630	655	35	25
	0.15	10	606	644	43	38
	0.20*	10	605	643	43	38
	0.25	9	586	647	50	61
	0.30	9	568	643	55	75
	Det.	8	529	664	64	135
P-n50-k10	0.01	13	802	804	9	2
	0.05	12	773	783	19	10
	0.10	12	751	771	26	20
	0.10	12	751	771	26	20
	0.15*	11	730	763	34	33
	0.20	11	730	771	36	41
	0.25	11	733	772	34	39
	0.30	11	725	775	41	50
	Det.	10	696	793	47	97
P-n55-k15	0.01	22	1191	1193	11	2
	0.05	20	1103	1119	25	16
	0.10	19	1071	1101	36	30
	0.15	18	1045	1094	44	49
	0.20	18	1030	1086	43	56
	0.25*	18	1000	1082	53	82
	0.30	17	997	1087	53	90
	Det.	15	989	1192	64	203
P-n60-k15	0.01	20	1193	1195	8	2
	0.05	18	1132	1144	21	12
	0.05	18	1132	1144	21	12
	0.15*	17	1062	1104	39	42
	0.20	16	1043	1109	44	66
	0.25	16	1025	1105	50	80
	0.30	16	1016	1107	51	91
	Det.	15	968	1133	66	165

Another important observation is that choosing very small value for ϵ is not always reasonable, as it may result in unnecessary extra cost and the addition of vehicles. As the experiment demonstrates, the minimum expected cost does not occur in smallest values of ϵ. In Table 5, the values of ϵ for which we have the minimum total expected cost, is indicated by "*". Depending on the situations and criteria in practice such as available fleet, decision makers may have to make a trade-off and choose different values of ϵ to achieve their goals.

6. Conclusions

In this study, we have presented a set-partitioning formulation for the vehicle routing problem with stochastic demands and homogenous vehicles. We have used a column generation method within a branch-and-bound framework to solve the underlying problem. The column generation subproblem is formulated with a constrained shortest path problem, where nodes have stochastic demands. Optimality and feasibility conditions are introduced and imposed to the shortest path problem. Chance-constrained programming and distributionally robust chance-constrained programming have been used to deal with stochastic demands. In the chance-constrained model, we have considered three different assumptions for the random demands: the Normal and Poisson distributions and the scenario-based presentation. As the computation of Poisson CDF is computationally expensive, upper and lower bounds such as Chernoff bound are used to speed up verifying the probabilistic constraints and construct feasible routes in the column generation subproblem. A customized shortest path algorithm has been developed to solve the underlying problem.

A comprehensive computational analysis has been carried out to test the proposed method and gain some practical insights. We have been able to solve to optimality some large standard instances that were not solved before. Monte-Carlo simulation is employed to investigate the quality of solutions. We observed that a chance-constrained model outperforms deterministic and distributionally robust chance-constraint models. Moreover, a sensitivity analysis has been performed to study the impact of the prespecified probability of failure on the optimal solutions. We observe that, to achieve a high reliability level, we do not need to always increase the fleet size. Also, we observe that very high reliability levels are not interesting on all occasions from practical point of view, because those cases may incur unnecessary extra costs, which do not have reasonable added value to the system.

The focus of this study is on independent random demands. An interesting and challenging line of research is to adapt the dominance rules for more practical settings such as correlated and/or conditional random demands, particularly when demands are represented by discrete random variables (instead of their continuous approximations). These settings are important and can be found in several applications, such as in waste and money collection problems. A possible interesting extension is to formulate random demands with a factor model in which demands are affected by a set of random factors, such as market indices considered in See and Sim (2010), which can incorporate correlated demands. In addition, our proposed method can be extended by solving the column generation subproblem more efficiently.

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