On vehicle routing problems with stochastic demands — Part I: Generic integer L-shaped formulations

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Abstract

We study a broad class of vehicle routing problems in which the cost of a route is allowed to be any nonnegative rational value computable in polynomial time in the input size. To address this class, we introduce a unifying framework that generalizes existing integer L-shaped (ILS) formulations developed for vehicle routing problems with stochastic demands (VRPSDs). This framework and subsequent analysis allow us to generalize previous ILS cuts and pinpoint which assumptions are needed to apply those generalizations to other problems. Using these tools, we develop the first algorithm for the VRPSD in the case where the demands are given by an empirical probability distribution of scenarios — a data-driven variant that tackles a significant challenge identified in the literature: dealing with correlations. Indeed, all previous ILS-based exact algorithms for the VRPSD assume either independence of customer demands or correlations through a single external factor. This shows the potential of this generic unifying framework to be applied to a multitude of different variants of the problem.

Keywords: integer programming, stochastic programming, vehicle routing problem.

1 Introduction

The *Capacitated Vehicle Routing Problem* (CVRP) is a fundamental combinatorial optimization problem in which one seeks minimum-cost routes to serve all customer demands while respecting vehicle capacity constraints. As a cornerstone problem in Operations Research, the CVRP has driven numerous theoretical and practical advances in combinatorial optimization and mathematical programming (Toth and Vigo, 2014). In this paper, we study the *Two-Stage Vehicle Routing Problem with Stochastic Demands* (VRPSD), a variant of the CVRP where routes are decided *a priori*, customer demands are random variables revealed upon vehicle arrival, and a *recourse cost* is incurred whenever a planned route exceeds vehicle capacity. This problem has been investigated for over 50 years (Gendreau et al., 2016; Tillman, 1969), with growing interest in the past decade (Louveaux and Salazar-González, 2018; Hoogendoorn and Spliet, 2023; Florio et al., 2022; Ota and Fukasawa, 2024; Parada et al., 2024; Salavati-Khoshghalb et al., 2019b,a; Legault et al., 2025; Florio et al., 2020).

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Despite this growing interest, most VRPSD studies have strong assumptions on the random variables, like independent probability distributions with a convolution property (Gendreau et al., 2016; Laporte et al., 2002; Jabali et al., 2014; Gauvin et al., 2014). The assumption of independence simplifies model tractability, but it is often unrealistic, as customer demands are frequently correlated in practice. The only exact approach that we are aware of that considers correlations is the recent work of Florio et al. (2022), which considers a single external factor. Indeed, handling correlations remains a recognized challenge (Ota and Fukasawa, 2024; Gendreau et al., 2016).

In contrast to this landscape, scenario-based approaches are standard and have been extensively studied in the general stochastic optimization literature. They approximate the true underlying distribution using empirical samples (Chen and Luedtke, 2022; Bertsimas et al., 2018; Verweij et al., 2003), allowing for correlations while sometimes offering theoretical guarantees via sample average approximation (Birge and Louveaux, 2011; Swamy and Shmoys, 2012; Luedtke and Ahmed, 2008).

To bridge this gap, we propose a two-part work on approaches for the VRPSD where the uncertainty is modeled using *demand scenarios*.

This paper constitutes the first part of this work and is devoted to studying integer L-shaped (ILS) formulations, which are the basis of the most successful branch-and-cut algorithms for the VRPSD to date (Hoogendoorn and Spliet, 2023; Parada et al., 2024; Legault et al., 2025). We emphasize that, for the VRPSD with scenarios, branch-and-price algorithms face intrinsic hardness issues (Ota and Fukasawa, 2024), further motivating our focus on branch-and-cut approaches. The second part of the work is presented in Ota and Fukasawa (2025) and uses some of the ideas presented in this first paper.

To better position our contributions, we review here the main ideas in ILS algorithms for the problem.

1.1 ILS approaches and our contributions

The basic idea in ILS approaches for the VRPSD is to formulate the problem as

$$\min\{c^\intercal x + \rho : \rho \geq \mathcal{Q}(x),\ x \in \mathcal{X} \cap \mathbb{Z}^E\}. \tag{1}$$

where $\mathcal{X} \cap \mathbb{Z}^E$ is the set of feasible first-stage decisions (routes that are decided a priori), and $\mathcal{Q}(x)$ is the *recourse function* representing the expected recourse cost incurred by taking those decisions.

ILS approaches replace the constraints on the recourse cost variable ρ in Formulation (1) with optimality cuts (or *lower bounding functionals*) defined by an *activation function*, which determines when the ILS cut is “active”, and a corresponding lower bound on the recourse cost that applies whenever the cut is “active”. These inequalities are then used to lower bound the recourse function $\mathcal{Q}(x)$ with linear expressions on x.

Additionally, recent ILS algorithms rely on a so-called *disaggregation* of the recourse cost into several smaller components (Hoogendoorn and Spliet, 2023; Parada et al., 2024, 2025; Côté et al., 2020; Legault et al., 2025). For instance, instead of having a single recourse cost variable ρ, the *disaggregated integer L-shaped (DL-shaped) method* (Parada et al., 2024, 2025; Legault et al., 2025) uses $\rho = \sum_v \theta_v$, where each θ_v-variable represents the part of the recourse cost paid at each customer. This enables the generation of ILS cuts based only on a subset of the θ_v-variables, enabling better approximations of the recourse cost at a given vertex or set of vertices.

However, these formulations are derived, as previously mentioned, under strong assumptions on the random variables and the set of feasible solutions $\mathcal{X} \cap \mathbb{Z}^E$. In addition, each of these works depends on specific assumptions about the *recourse policy*, which determine the behavior of the recourse function $\mathcal{Q}$. Due to all these particular situations, applying the same results directly to different variants of the problem — including ones where the demands are given by scenarios — is not directly possible. One particular hurdle that needs to be overcome is to identify among the previous results in the literature which parts depend on which assumptions of the studied problem.

In this article, we propose a way to formally address this gap. Specifically, our main contributions are:

- We introduce in Section 3 a unifying framework under which all known ILS formulations for VRPSDs fall, with very few assumptions on the set of feasible first-stage solutions $\mathcal{X} \cap \mathbb{Z}^E$, the recourse function $\mathcal{Q}$, and the chosen disaggregation;

- We extend the ILS cuts of Gendreau et al. (1995) (Section 4.1), the *route-split* and *partial route-split inequalities* of Hoogendoorn and Spliet (2023) (Section 4.2), and the *path* and *set cuts* from the DL-shaped method of Parada et al. (2024) (Section 4.3) to the more general setting established in our framework;

- We characterize which disaggregations lead to valid models for the generic Formulation (1) when combined with some of the generalized ILS cuts (Theorem 2). We also characterize when the *path cuts* of Parada et al. (2024) are valid for a reformulation of Formulation (1) obtained through our framework (Theorem 5), generalizing the previous result of Legault et al. (2025) to more general choices of $\mathcal{X} \cap \mathbb{Z}^E$;

- Applying our framework, we obtain in Section 5 the first branch-and-cut approach for the VRPSD with scenarios. In particular, the framework enables the combination of (generalized) *partial route inequalities* (Hoogendoorn and Spliet, 2023) and *set cuts* (Parada et al., 2024). The computational experiments in Section 6 show that this combination enables our algorithm to solve 17 more instances compared to a variant that does not combine the cuts. These results suggest that our generalizations could also benefit other vehicle routing problems.

While the results on this first part are of significant importance on their own, our framework will also serve as a foundation for Part II (Ota and Fukasawa, 2025), where we develop an approach specifically tailored to the VRPSD with scenarios.

Notation. We use $\mathbb{R}_+$ and $\mathbb{R}_{++}$ to denote the sets of nonnegative and positive real numbers, respectively. Similar notation applies to $\mathbb{Q}$ and $\mathbb{Z}$. Let a be an integer, then $[a] \coloneqq \{1, \ldots, a\}$ if a is positive, and $[a] = \emptyset$ otherwise. We write $\mathbb{I}(\cdot)$ to denote the indicator function.

For any undirected graph G, the notations $V(G)$ and $E(G)$ refer to the set of vertices and edges of G, respectively. If G is a directed graph (digraph), we use $A(G)$ to refer to the set of arcs of G. For ease of presentation, we sometimes abbreviate an edge $\{u, v\}$ or an arc (u, v) simply to uv. Given an undirected graph G and a set $S \subseteq V(G)$, the notation $\delta_G(S)$ denotes the set of edges in G with exactly one endpoint in S (we omit the subscript G, whenever it is clear from the context). If $S \subseteq V(G)$ is a singleton $\{v\}$, we may refer to S as simply v. We use $G' \subseteq G$ to indicate that G' is a subgraph of G.

If f is a vector and i is one of its coordinates, we write f_i and $f(i)$ interchangeably. For any function (respectively, vector) f and a subset H of its domain (respectively, coordinates), we use $f(H)$ as shorthand for $\sum_{i \in H} f(i)$. The notation $\mathbb{1}$ refers to the all-ones vector.

2 The setup

In this section, we describe the class of problems our framework addresses, the assumptions we make, and we present a Vehicle Routing Problem with Stochastic Demands (VRPSD) as a representative example.

2.1 Problem definition

From now on, we fix G to be a complete undirected graph with vertex set $V := V(G) = \{0\} \cup V_+$, edge set $E := E(G)$ and edge weights $c \in \mathbb{Q}_+^E$. The vertex 0 represents the *depot* and V_+ denotes the set of *customers*. We also fix $\mathcal{I}$ to be a tuple representing the input of the generic problem that we consider, and we assume that $\mathcal{I}$ contains the graph G and its edge weights. For example, an instance of the VRPSD with demands following independent normal distributions (Laporte et al., 2002; Jabali et al., 2014; Parada et al., 2024; Hoogendoorn and Spliet, 2023) can be represented as $\mathcal{I} = (G, c, k, C, \bar{d}, \sigma)$, where $k \in \mathbb{Z}_{++}$ is the number of vehicles, $C \in \mathbb{Q}_+$ is the vehicle capacity, $\bar{d} \in \mathbb{Q}_+^{V_+}$ denotes the vector of expected demands, and $\sigma \in \mathbb{Q}_+^{V+}$ denotes the vector of standard deviations.

Our goal is to find feasible routing plans that cover all customers in G with a collection of routes. Formally, we define a *route* $R \subseteq G$ as a simple undirected cycle that starts and ends at the depot, i.e., $V(R) = \{0, v_1, v_2, \ldots, v_\ell\}$ and $E(R) = \{\{0, v_1\}, \{v_1, v_2\}, \ldots, \{v_\ell, 0\}\}$, where all customers v_i are distinct. The notation $V_+(R)$ refers to the set of customers inside R, that is, $V_+(R) := \{v_1, \ldots, v_\ell\}$. If R is a route containing a single customer v, then $E(R)$ denotes a multiset that contains edge $\{0, v\}$ with multiplicity 2. (Since in this paper we do not refer to *nonelementary routes* (Irnich and Desaulniers, 2005), this is the only case where repeated edges may appear in $E(R)$.) For convenience, we use $c(R)$ as a shorthand to $c(E(R))$. Additionally, we often represent R with the tuple $(v_1, \ldots, v_\ell)$ and, in this context, we assume that $v_0 = v_{\ell+1} = 0$. A *routing plan* is a set of routes $\mathcal{R} = \{R_1, \ldots, R_t\}$ such that $\{V_+(R_i)\}_{i \in [t]}$ forms a partition of V_+.

Note that routes are undirected, so if $R = (v_1, \ldots, v_\ell)$ and $R' = (v_\ell, \ldots, v_1)$ are both routes, then $R = R'$. However, we sometimes have to refer explicitly to the different orientations of a route. To this end, we associate with route $R = (v_1, \ldots, v_\ell)$ two digraphs $\vec{R}$ and $\overleftarrow{R}$, which we call *directed routes*. Both $\vec{R}$ and $\overleftarrow{R}$ have the same vertex set as R, but the arcs are in opposite directions, that is, $A(\vec{R}) = \{(0, v_1) \ldots, (v_\ell, 0)\}$ and $A(\overleftarrow{R}) = \{(0, v_\ell) \ldots, (v_1, 0)\}$. Similarly to the routes, we write $\vec{R} = (v_1, \ldots, v_\ell)$ and $\overleftarrow{R} = (v_\ell, \ldots, v_1)$, and since these are directed graphs, $\vec{R}$ differs from $\overleftarrow{R}$ whenever $\ell \geq 2$. We need to clarify a detail here: strictly speaking, our notation is ambiguous, since if $R = (v_1, \ldots, v_\ell)$ and $R' = (v_\ell, \ldots, v_1)$ are both routes, then $R = R'$ and the notation $\vec{R}$ might refer to either $(v_1, \ldots, v_\ell)$ or $(v_\ell, \ldots, v_1)$. In such situations, we always assume that the arrows in the notation are according to how we first write the tuple for the underlying (undirected) route, so even though $R = R'$, we have that $\vec{R} \neq \vec{R'}$ and $\overleftarrow{R} = \vec{R'}$.

In the rest of this paper, we fix $\mathcal{Q}$ to denote a generic *recourse function* (Ota and Fukasawa, 2024), that is, $\mathcal{Q}$ is a function that takes the input $\mathcal{I}$ as a parameter and maps each route R to a value $\mathcal{Q}(R; \mathcal{I}) \in \mathbb{Q}_+$. Since the instance $\mathcal{I}$ is fixed, we write $\mathcal{Q}(R)$ instead of $\mathcal{Q}(R; \mathcal{I})$. While the correctness of the approaches proposed here depends only on $\mathcal{Q}$ returning nonnegative rational numbers, our algorithms evaluate $\mathcal{Q}$ at multiple routes. Therefore, in practice, we also assume access to an algorithm that, given a route R, computes $\mathcal{Q}(R; \mathcal{I})$ efficiently (say in polynomial or pseudo-polynomial time in the size of $\mathcal{I}$).

We next turn our attention to formalizing our assumptions on the set of feasible routing plans by means of an edge-based formulation for the problem. By using the classical *subtour elimination constraints* (SECs) (Cook et al., 2011; Toth and Vigo, 2014), we have a bijection between the set of all routing plans and the integer vectors inside the polytope

$$\mathcal{X}_{\text{SUB}} = \left\{ x \in [0,2]^E : \begin{array}{ll} x(\delta(v)) = 2, & \forall v \in V_+ \\ x(S) \leq |S| - 1, & \forall \emptyset \subsetneq S \subseteq V_+ \end{array} \right\}, \qquad (\mathcal{X}_{\text{SUB}})$$

where $x(S)$ is a shorthand for $\sum_{uv \in E: u,v \in S} x_{uv}$. With each vector $\bar{x} \in \mathcal{X}_{\text{SUB}} \cap \mathbb{Z}^E$ we denote its corresponding routing plan with the notation $\mathcal{R}(\bar{x})$. Moreover, for any vector $\bar{x} \in \mathbb{R}^E$, we use $G(\bar{x})$ to refer to its *support graph*, that is, $V(G(\bar{x})) = V$ and $E(G(\bar{x})) = \{e \in E : \bar{x}_e > 0\}$.

In many vehicle routing problems, additional problem-specific intra/inter-route constraints, are imposed to define what is a feasible routing plan, for example, bounds on the number of routes, capacity restrictions, and/or time windows (Toth and Vigo, 2014). These constraints are frequently handled well by existing formulations with additional inequalities and variables. To isolate the role of the recourse function $\mathcal{Q}$, we make the following assumption on the set of feasible routing plans.

Assumption 1. We are given a linear programming (LP) formulation of a polytope $\mathcal{X} \subseteq \mathcal{X}_{\text{SUB}}$ such that the set of feasible routing plans is given by $\{\mathcal{R}(x) : x \in \mathcal{X} \cap \mathbb{Z}^E\}$.

Note that $\mathcal{X}$ could potentially be given as the projection of a higher-dimensional polyhedron.

We can now formally define the class of problems that we address (for which $\mathcal{I}$ serves as input).

Definition 1. The *Vehicle Routing Problem with Recourse* (VRPR) with respect to $\mathcal{Q}$ and $\mathcal{X}$ seeks a routing plan $x \in \mathcal{X} \cap \mathbb{Z}^E$ that minimizes $\sum_{R \in \mathcal{R}(x)} [c(R) + \mathcal{Q}(R)]$.

Similarly to how we treat recourse functions, we define all the mathematical objects in this paper (such as functions and sets) relative to $\mathcal{I}$, without making this dependence explicit in the notation.

With a slight abuse of notation, we extend the definition of recourse functions by defining $\mathcal{Q}(\bar{x}) \coloneqq \sum_{R \in \mathcal{R}(\bar{x})} \mathcal{Q}(R)$, for every $\bar{x} \in \mathcal{X} \cap \mathbb{Z}^E$. Hence, as mentioned in Section 1.1, we express problem VRPR as

$$\min\{c^\intercal x + \rho : \rho \geq \mathcal{Q}(x),\ x \in \mathcal{X} \cap \mathbb{Z}^E\}. \qquad (\text{VRPR}(\mathcal{Q}, \mathcal{X}))$$

We denote the feasible region of $\text{VRPR}(\mathcal{Q}, \mathcal{X})$ by $\text{EPI}(\mathcal{Q}, \mathcal{X})$, since it can be interpreted as the epigraph of a function with domain $\mathbb{R}^E$ that equals $\mathcal{Q}(x)$ for $x \in \mathcal{X} \cap \mathbb{Z}^E$ and $+\infty$ otherwise.

2.2 Motivating example: the VRPSD with scenarios under the classical recourse policy

Besides the original graph $G = (V, E)$ and the edge costs vector $c \in \mathbb{Q}_+^E$, the VRPSD also receives as input the vehicle capacity $C \in \mathbb{Q}_{++}$ and the number of available vehicles $k \in \mathbb{Z}_{++}$. The input data related to the stochastic customer demands is described next.

Let d be a random vector following a probability distribution $\mathbb{P}$, where each component $d(v)$ indicates the random demand of customer $v \in V_+$. We assume that $\mathbb{P}$ is *given by scenarios*, meaning that we receive vectors $d^1, \ldots, d^N \in \mathbb{Q}_+^{V_+}$ in the input, each representing a certain scenario. We are also given probabilities $p_1, \dots, p_N \in \mathbb{Q}_+$ that sum up to one and such that $\mathbb{P}(d = d^\xi) = p_\xi$, for every $\xi \in [N]$. The expected demand vector is denoted $\bar{d} := \mathbb{E}[d]$ and every entry of $\bar{d}$ is assumed to be strictly positive. The input for the VRPSD with scenarios is represented by the tuple $\mathcal{I}_{\text{VRPSD}} = (G, c, k, C, N, d^1, \dots, d^N, p_1, \dots, p_N)$.

2.2.1 CVRP formulation

We say that a routing plan $\mathcal{R}$ is feasible for the VRPSD if it is feasible for the CVRP with respect to the demand vector $\bar{d}$, that is, $\bar{d}(R) := \sum_{v \in V_+(R)} \bar{d}(v) \leq C$, for every $R \in \mathcal{R}$.[1] Using the classical CVRP formulation of Laporte and Nobert (1983), we can model the set of routing plans feasible for the VRPSD as the integer vectors belonging to the polytope

$$\mathcal{X}_{\text{CVRP}} := \mathcal{X}_{\text{SUB}} \cap \left\{ x \in [0,2]^E : \begin{array}{l} x(\delta(0)) = 2k, \\ x(S) \leq |S| - \bar{k}(S), \quad \forall \emptyset \subsetneq S \subseteq V_+ \end{array} \right\}, \qquad (\mathcal{X}_{\text{CVRP}})$$

where $\bar{k}(S) := \lceil \bar{d}(S)/C \rceil$. Inequalities $x(S) \leq |S| - \bar{k}(S)$ are the well-known *rounded capacity inequalities* (RCIs) (in their "inside form") and they imply the SECs since $\bar{d}(S)$ is positive.

2.2.2 The classical recourse policy

When demands are *i.i.d.,* several recourse policies were proposed in the VRPSD literature (Dror et al., 1989; Yee and Golden, 1980; Salavati-Khoshghalb et al., 2019a,b). However, to our knowledge, no existing work explicitly addresses recourse policies for the VRPSD with scenarios. Still, the *classical recourse policy* can be easily applied to this setting.

Consider traversing a directed route $\vec{R} = (v_1, \dots, v_\ell)$ and suppose that the sum of the realized demands exceeds the vehicle capacity when we reach customer v_j. In this case, following standard conventions in the literature (see Oyola et al. (2018) for variants that consider exact stockouts and nonsplittable demands), the classical recourse policy prescribes that the vehicle executes a back-and-forth trip between the depot and v_j before continuing the route. Using the formula of Dror et al. (1989), the expected recourse cost of the directed route $\vec{R}$ under the classical recourse policy is computed as

$$\mathcal{Q}_C(\vec{R}) = \sum_{j=1}^{\ell} 2\, c_{0v_j} \sum_{t=1}^{\infty} \mathbb{P}\left(\sum_{i \in [j-1]} d(v_i) \leq tC < \sum_{i \in [j]} d(v_i) \right), \qquad (\mathcal{Q}_C)$$

and the recourse cost of the (undirected) route R is then set as $\mathcal{Q}_C(R) = \min \left\{ \mathcal{Q}_C(\vec{R}), \mathcal{Q}_C(\overleftarrow{R}) \right\}$.

Since we assume that $\mathbb{P}$ is given by scenarios, we define

$$\mathcal{Q}_C^\xi(\vec{R}) := \sum_{j \in [\ell]} 2\, c_{0v_j} \sum_{t=1}^{\infty} \mathbb{I}\left(\sum_{i \in [j-1]} d^\xi(v_i) \leq tC < \sum_{i \in [j]} d^\xi(v_i) \right), \qquad (\mathcal{Q}_C^\xi)$$

and we have that $\mathcal{Q}_C(\vec{R}) = \sum_{\xi=1}^{N} p_\xi \mathcal{Q}_C^\xi(\vec{R})$. Thus, by computing the accumulated demands along the directed routes for each scenario, we can evaluate $\mathcal{Q}_C(R)$ in polynomial

[1] Constraints of this type are used in most of the VRPSD literature (Laporte et al., 2002; Jabali et al., 2014; Hoogendoorn and Spliet, 2023; Parada et al., 2024; Florio et al., 2020), but Hoogendoorn and Spliet (2025) recently considered a version of the problem without it.

time on the number of scenarios N and the size of V_+ (see Section 5 for an explicit argument). The *VRPSD with scenarios under the classical recourse policy* can now be concisely expressed as $\textsc{vrpr}(\mathcal{Q}_C, \mathcal{X}_{\textsc{cvrp}})$.

3 The unifying framework

In this section, we propose a framework that, as we argue later in Section 4, unifies and generalizes several ILS-based formulations for VRPSDs. Furthermore, our framework enables the combination of (generalized) partial route inequalities (Hoogendoorn and Spliet, 2023; Jabali et al., 2014) and set cuts (Parada et al., 2024; Legault et al., 2025), which was not directly possible before. This latter point is indeed further illustrated in Section 5, where the framework is applied to solve problem $\textsc{vrpr}(\mathcal{Q}_C, \mathcal{X}_{\textsc{cvrp}})$.

3.1 Recourse disaggregation

A natural way to model $\textsc{vrpr}(\mathcal{Q}, \mathcal{X})$ as a mixed-integer linear program (MILP) is to under-approximate the epigraphical variable $\rho \in \mathbb{R}_+$ with ILS cuts (Laporte and Louveaux, 1993) that guarantee that every feasible tuple $(x, \rho) \in (\mathcal{X} \cap \mathbb{Z}^E) \times \mathbb{R}_+$ satisfies $\rho \geq \mathcal{Q}(x)$. In the context of the VRPSD, Parada et al. (2024) recently proposed the DL-shaped method, whose main idea is to disaggregate the recourse variable ρ along the set of customers V_+. Specifically, they write $\rho = \sum_{v \in V_+} \theta_v$, where, roughly speaking, θ_v represents the recourse cost incurred at customer v. Similar ideas were also explored in (Séguin, 1994; Côté et al., 2020; Hoogendoorn and Spliet, 2023; Parada et al., 2025; Legault et al., 2025). Rather than adding ILS cuts with respect to a single variable ρ, these methods add ILS cuts with respect to a subset of the θ-variables.

Motivated by these recent developments, we generalize the domain V_+ of the θ-variables to a generic finite set Ω, leading to the following definition.

Definition 2. Let Ω be a nonempty finite set and let $\hat{\mathcal{Q}}$ be a function that maps routes and elements in Ω to nonnegative rational values, i.e., $\hat{\mathcal{Q}}(R, \omega) \in \mathbb{Q}_+$, for every route R and $\omega \in \Omega$. We say that $\hat{\mathcal{Q}}$ is a *disaggregation of* $\mathcal{Q}$ *along* Ω if $\mathcal{Q}(R) = \sum_{\omega \in \Omega} \hat{\mathcal{Q}}(R, \omega)$, for every $\bar{x} \in \mathcal{X} \cap \mathbb{Z}^E$ and $R \in \mathcal{R}(\bar{x})$.

Given such a disaggregation $\hat{\mathcal{Q}}$, we shall later use $\Omega(R)$ to refer to the support of $\hat{\mathcal{Q}}$ with respect to route R, i.e., $\Omega(R) \coloneqq \{\omega \in \Omega : \hat{\mathcal{Q}}(R, \omega) > 0\}$.

Given a disaggregation $\hat{\mathcal{Q}}$ as in Definition 2, we define the feasible region

$$\mathcal{F}(\hat{\mathcal{Q}}, \mathcal{X}, \Omega) \coloneqq \left\{ (x, \theta) \in (\mathcal{X} \cap \mathbb{Z}^E) \times \mathbb{R}_+^\Omega : \theta_\omega \geq \sum_{R \in \mathcal{R}(x)} \hat{\mathcal{Q}}(R, \omega), \quad \forall \omega \in \Omega \right\}. \tag{$\mathcal{F}$}$$

As a validity check, we verify that the definition of $\mathcal{F}(\hat{\mathcal{Q}}, \mathcal{X}, \Omega)$ yields a formulation for $\textsc{vrpr}(\mathcal{Q}, \mathcal{X})$ in the (x, θ)-space. From now on, for any set $P \subseteq \mathbb{R}^E \times \mathbb{R}^\Omega$, we use $\textsc{proj}_{(x,\rho)}(P)$ to denote the projection $\{(x, \rho) : \rho = \mathbb{1}^\intercal \theta,\ (x, \theta) \in P\}$.

Claim 1. *Let* Ω *be a nonempty finite set and let* $\hat{\mathcal{Q}}$ *be a disaggregation of* $\mathcal{Q}$ *along* Ω*, then* $\textsc{epi}(\mathcal{Q}, \mathcal{X}) = \textsc{proj}_{(x,\rho)}(\mathcal{F}(\hat{\mathcal{Q}}, \mathcal{X}, \Omega))$.

Proof.

$$
\begin{aligned}
\text{EPI}(\mathcal{Q},\mathcal{X}) &= \left\{(x,\rho)\in x\in(\mathcal{X}\cap\mathbb{Z}^E)\times\mathbb{R}_+ : \rho\geq \sum_{R\in\mathcal{R}(x)}\sum_{\omega\in\Omega}\hat{\mathcal{Q}}(R,\omega)\right\}\\
&= \left\{(x,\rho)\in x\in(\mathcal{X}\cap\mathbb{Z}^E)\times\mathbb{R}_+ : \rho\geq \sum_{\omega\in\Omega}\theta_\omega,\ \theta_\omega=\sum_{R\in\mathcal{R}(x)}\hat{\mathcal{Q}}(R,\omega),\ \forall\omega\in\Omega\right\}\\
&= \text{PROJ}_{(x,\rho)}(\mathcal{F}(\hat{\mathcal{Q}},\mathcal{X},\Omega)). \qquad\square
\end{aligned}
$$

Remark 1. A disaggregation $\hat{\mathcal{Q}}$ of $\mathcal{Q}$ along Ω always exists, since we can choose an arbitrary element $\omega\in\Omega$ and set our disaggregation so that $\hat{\mathcal{Q}}(R,\omega)=\mathcal{Q}(R)$, for every route R. More interestingly, the structure of the recourse function $\mathcal{Q}$ often suggests alternative choices for the disaggregation. For example, consider the formula for the VRPSD recourse function $\mathcal{Q}_C$ and set $\Omega=V_+$. Let $R=(v_1,\dots,v_\ell)$ be a route and assume without loss of generality that $\mathcal{Q}_C(R)=\mathcal{Q}_C(\vec{R})$. In this case, we may set

$$
\hat{\mathcal{Q}}_C(R,v_j)=2\,c_{0v_j}\sum_{t=1}^{\infty}\mathbb{P}\left(\sum_{i\in[j-1]}d(v_i)\leq tC<\sum_{i\in[j]}d(v_i)\right),
$$

for each $j\in[\ell]$ (and consequently, $\hat{\mathcal{Q}}_C(R,v)=0$, for all $v\in V_+\setminus V_+(R)$). $\square$

For convenience, in the rest of this paper, we fix a nonempty finite set Ω and a corresponding disaggregation $\hat{\mathcal{Q}}$. With a slight abuse of notation, we then write $\mathcal{Q}(R,\omega)$ rather than $\hat{\mathcal{Q}}(R,\omega)$. In particular, this implies that the notation $\mathcal{F}(\mathcal{Q},\mathcal{X},\Omega)$ is well-defined, and for every route R, $\Omega(R)=\{\omega\in\Omega:\mathcal{Q}(R,\omega)>0\}$.

Why define $\mathcal{F}(\mathcal{Q},\mathcal{X},\Omega)$? Previous ILS-based algorithms that use recourse disaggregation (Hoogendoorn and Spliet, 2023; Parada et al., 2024, 2025; Côté et al., 2020; Legault et al., 2025), do not explicitly specify the choice of the recourse disaggregation (as in Definition 2) nor the corresponding feasible region $\mathcal{F}(\mathcal{Q},\mathcal{X},\Omega)$. Instead, they implicitly define a recourse disaggregation using a mathematical formulation in the (x,θ)-space (with $\Omega=V_+$), and then establish correctness by projecting feasible solutions of this formulation onto the (x,ρ)-space of $\text{EPI}(\mathcal{Q},\mathcal{X})$. Essentially, they show that feasible solutions $(\bar{x},\bar{\theta})\in(\mathcal{X}\cap\mathbb{Z}^E)\times\mathbb{Q}_+^{V_+}$ to their formulation satisfy $\mathbb{1}^\intercal\bar{\theta}\geq\mathcal{Q}(\bar{x})$, and equality can be achieved.

Although sufficient for proving correctness, the described approach may hide the fact that the validity of the ILS cuts depends on the choice of the recourse disaggregation. For example, suppose that $\hat{\mathcal{Q}}^1$ and $\hat{\mathcal{Q}}^2$ are two disaggregations of $\mathcal{Q}$ along the same set Ω. By Claim 1, both $\mathcal{F}(\hat{\mathcal{Q}}^1,\mathcal{X},\Omega)$ and $\mathcal{F}(\hat{\mathcal{Q}}^2,\mathcal{X},\Omega)$ are (nonlinear) *extended formulations* (Conforti et al., 2014) of $\text{EPI}(\mathcal{Q},\mathcal{X})$, that is,

$$
\text{PROJ}_{(x,\rho)}(\mathcal{F}(\hat{\mathcal{Q}}^1,\mathcal{X},\Omega))=\text{PROJ}_{(x,\rho)}(\mathcal{F}(\hat{\mathcal{Q}}^2,\mathcal{X},\Omega))=\text{EPI}(\mathcal{Q},\mathcal{X}).
$$

However, inequalities valid for $\mathcal{F}(\hat{\mathcal{Q}}^1,\mathcal{X},\Omega)$ may not be valid for $\mathcal{F}(\hat{\mathcal{Q}}^2,\mathcal{X},\Omega)$ (or vice versa), so one cannot simply combine inequalities derived under different disaggregations, i.e., it may be the case that

$$
\text{PROJ}_{(x,\rho)}\left(\mathcal{F}(\hat{\mathcal{Q}}^1,\mathcal{X},\Omega)\cap\mathcal{F}(\hat{\mathcal{Q}}^2,\mathcal{X},\Omega)\right)\subsetneq\text{EPI}(\mathcal{Q},\mathcal{X}).
$$

Indeed, as we later show in Section 5 (and Appendix H), cuts from Hoogendoorn and Spliet (2023) cannot be applied to the setting of Parada et al. (2024), since they are derived for different disaggregations. However, such a dependency on the disaggregation may not be evident at first glance from their work.

In contrast, our definition of $\mathcal{F}(\mathcal{Q}, \mathcal{X}, \Omega)$ makes this dependency explicit. More importantly, this explicit treatment allows us to combine the different (generalized) ILS cuts that we discuss in Section 4. This will be crucial when we apply our framework in Section 5, where we combine (generalizations of) partial route inequalities (Hoogendoorn and Spliet, 2023) and set cuts (Parada et al., 2024). The feasible region $\mathcal{F}(\mathcal{Q}, \mathcal{X}, \Omega)$ will also play a key role in Part II (Ota and Fukasawa, 2025), where the choice of the recourse disaggregation is further explored to derive a new class of valid inequalities for the VRPSD with scenarios.

3.2 Key definitions: activation functions, recourse lower bounds and ILS cuts

In what follows, we generalize the expositions in (Hoogendoorn and Spliet, 2023; Parada et al., 2024; Jabali et al., 2014), and present ILS cuts for $\textsc{vrpr}(\mathcal{Q}, \mathcal{X})$ using the concepts of *activation functions* and *recourse lower bounds*. We begin with the activation functions.

Definition 3. Let $\mathcal{X}' \subseteq \mathcal{X} \cap \mathbb{Z}^E$. A function $W(\,\cdot\,; \mathcal{X}') : \mathbb{R}^E \to \mathbb{R}$ is an *activation function* with respect to the set $\mathcal{X}'$ if $W(\,\cdot\,; \mathcal{X}')$ is affine and, for every $x \in \mathcal{X} \cap \mathbb{Z}^E$, we have that

$$W(x; \mathcal{X}') \begin{cases} = 1, & \text{if } x \in \mathcal{X}', \\ \leq 0, & \text{otherwise.} \end{cases}$$

Note that Definition 3 depends on the set $\mathcal{X}$ that we fixed in Assumption 1. To avoid repeating ourselves, whenever we say that $W(x; \mathcal{X}')$ is an activation function, it is implicit that it is with respect to $\mathcal{X}' \subseteq \mathcal{X} \cap \mathbb{Z}^E$.

Given an activation function $W(x; \mathcal{X}')$, consider the inequality $\theta(U) \geq \tilde{\mathcal{L}} \cdot W(x; \mathcal{X}')$, where $U \subseteq \Omega$ and $\tilde{\mathcal{L}} \in \mathbb{Q}_+$. This inequality enforces a lower bound of $\tilde{\mathcal{L}}$ on $\theta(U)$ whenever $x \in \mathcal{X}'$, so only the following values of $\tilde{\mathcal{L}}$ can be used.

Claim 2. *Let $W(x; \mathcal{X}')$ be an activation function, $U \subseteq \Omega$ and $\tilde{\mathcal{L}} \in \mathbb{Q}_+$. The inequality $\theta(U) \geq \tilde{\mathcal{L}} \cdot W(x; \mathcal{X}')$ is valid for $\mathcal{F}(\mathcal{Q}, \mathcal{X}, \Omega)$ if and only if $\sum_{\omega \in U} \sum_{R \in \mathcal{R}(\bar{x})} \mathcal{Q}(R, \omega) \geq \tilde{\mathcal{L}}$, for every $\bar{x} \in \mathcal{X}'$.*

Proof. By the definition of the set $\mathcal{F}(\mathcal{Q}, \mathcal{X}, \Omega)$, we have that $\theta(U) \geq \tilde{\mathcal{L}} \cdot W(x; \mathcal{X}')$ is a valid inequality for $\mathcal{F}(\mathcal{Q}, \mathcal{X}, \Omega)$ if and only if $\sum_{\omega \in U} \sum_{R \in \mathcal{R}(\bar{x})} \mathcal{Q}(R, \omega) \geq \tilde{\mathcal{L}} \cdot W(\bar{x}; \mathcal{X}')$, for every $\bar{x} \in \mathcal{X} \cap \mathbb{Z}^E$. By nonnegativity of $\sum_{\omega \in U} \sum_{R \in \mathcal{R}(\bar{x})} \mathcal{Q}(R, \omega)$ and $\tilde{\mathcal{L}}$, this last inequality is trivially satisfied for $\bar{x} \notin \mathcal{X}'$. Thus, we may assume that $\bar{x} \in \mathcal{X}'$ and $W(\bar{x}; \mathcal{X}') = 1$, which finishes the proof. □

Claim 2 leads to the following definition.

Definition 4. Let $U \subseteq \Omega$ and $\mathcal{X}' \subseteq \mathcal{X} \cap \mathbb{Z}^E$. We say that $\mathcal{L}(U, \mathcal{X}') \in \mathbb{Q}_+$ is a *recourse lower bound* with respect to U and $\mathcal{X}'$ if $\sum_{\omega \in U} \sum_{R \in \mathcal{R}(\bar{x})} \mathcal{Q}(R, \omega) \geq \mathcal{L}(U, \mathcal{X}')$, for all $\bar{x} \in \mathcal{X}'$.

Similarly to activation functions, recourse lower bounds depend on $\mathcal{X}$ and on the disaggregation of $\mathcal{Q}$ that we fixed earlier. Again, whenever we say that $\mathcal{L}(U, \mathcal{X}')$ is a recourse lower bound, it is implicit that it is with respect to U and $\mathcal{X}'$.

Remark 2. The term "activation function" was first introduced by Hoogendoorn and Spliet (2023), but the concept can also be seen in earlier works (Laporte et al., 2002; Jabali et al., 2014; Salavati-Khoshghalb et al., 2019a). In all of these cases, the functions are defined only for *partial routes* (a notion that we discuss in Section 4.2), and not for a general set $\mathcal{X}' \subseteq \mathcal{X} \cap \mathbb{Z}^E$. Thus, one may argue that Definition 3 is more of a notational convenience. In contrast, Definition 4 explicitly defines a recourse lower bound *relative to a chosen disaggregation of* $\mathcal{Q}$, which as we mentioned earlier, previous works do not specify. This definition is central to our framework, as it characterizes the lower bounds that can be used in an *ILS cut* (see Definition 5), and this characterization will be crucial for the developments in Sections 4 and 5. □

Having established Definitions 3 and 4, we henceforth reserve the name *ILS cuts* to the following.

Definition 5. An *ILS cut* is an inequality of the form $\theta(U) \geq \mathcal{L}(U, \mathcal{X}') \cdot W(x; \mathcal{X}')$, where $U \subseteq \Omega$, $\mathcal{X}' \subseteq \mathcal{X} \cap \mathbb{Z}^E$, $\mathcal{L}(U, \mathcal{X}')$ is a recourse lower bound and $W(x; \mathcal{X}')$ is an activation function.

Definition 6. A *family of ILS cuts* is a set $\mathcal{C}$ of tuples $(U, \mathcal{X}', \mathcal{L}(U, \mathcal{X}'), \alpha, \beta)$, where $U \subseteq \Omega$, $\mathcal{X}' \subseteq \mathcal{X} \cap \mathbb{Z}^E$, $(\alpha^\intercal x + \beta)$ is an activation function with respect to $\mathcal{X}'$, and $\mathcal{L}(U, \mathcal{X}')$ is a recourse lower bound.

With each family of ILS cuts $\mathcal{C}$, we associate the MILP formulation below.

$$
\begin{aligned}
\min \quad & c^\intercal x + \mathbb{1}^\intercal \theta, \\
\text{s.t.} \quad & \theta(U) \geq \mathcal{L}(U, \mathcal{X}') \cdot (\alpha^\intercal x + \beta), \qquad \forall (U, \mathcal{X}', \mathcal{L}(U, \mathcal{X}'), \alpha, \beta) \in \mathcal{C}, \\
& (x, \theta) \in (\mathcal{X} \cap \mathbb{Z}^E) \times \mathbb{R}_+^\Omega.
\end{aligned}
\tag{ILS($\mathcal{X}, \mathcal{C}, \Omega$)}
$$

Next, we prove that the feasible region $\mathcal{F}'$ of Formulation ILS($\mathcal{X}, \mathcal{C}, \Omega$) is a relaxation of $\mathcal{F}(\mathcal{Q}, \mathcal{X}, \Omega)$, and we characterize when the projection of $\mathcal{F}'$ onto the (x, ρ)-space is equivalent to the feasible region of VRPR($\mathcal{Q}, \mathcal{X}$) (i.e., EPI($\mathcal{Q}, \mathcal{X}$)).

Theorem 1. *Let $\mathcal{C}$ be a family of ILS cuts and let $\mathcal{F}'$ be the feasible region of Formulation ILS($\mathcal{X}, \mathcal{C}, \Omega$). Then the following holds:*

(i) $\mathcal{F}(\mathcal{Q}, \mathcal{X}, \Omega) \subseteq \mathcal{F}'$*; and*

(ii) $\text{EPI}(\mathcal{Q}, \mathcal{X}) = \text{PROJ}_{(x,\rho)}(\mathcal{F}')$ *if and only if* $\mathbb{1}^\intercal \bar{\theta} \geq \mathcal{Q}(\bar{x})$*, for all* $(\bar{x}, \bar{\theta}) \in \mathcal{F}'$.

Proof. Item (i) follows from Claim 2 and the definitions of ILS cuts. To prove item (ii), we write

$$
\text{EPI}(\mathcal{Q}, \mathcal{X}) \overset{\text{Claim 1}}{=} \text{PROJ}_{(x,\rho)}(\mathcal{F}(\mathcal{Q}, \mathcal{X}, \Omega)) \overset{\text{Item (i)}}{\subseteq} \text{PROJ}_{(x,\rho)}(\mathcal{F}').
$$

Hence, we are done by observing that, for any arbitrary point $(\bar{x}, \bar{\theta}) \in \mathcal{F}'$, we have that $(\bar{x}, \mathbb{1}^\intercal \bar{\theta}) \in \text{EPI}(\mathcal{Q}, \mathcal{X})$ if and only if $\mathbb{1}^\intercal \bar{\theta} \geq \mathcal{Q}(\bar{x})$. □

Note that since problem VRPR($\mathcal{Q}, \mathcal{X}$) can be written as a minimization problem over EPI($\mathcal{Q}, \mathcal{X}$), there is no need to guarantee that a candidate solution $(\bar{x}, \bar{\theta}) \in (\mathcal{X} \cap \mathbb{Z}^E) \times \mathbb{R}_+^{V_+}$ belongs to $\mathcal{F}(\mathcal{Q}, \mathcal{X}, \Omega)$. Instead, it suffices to check whether the projection of $(\bar{x}, \bar{\theta})$ under $\text{PROJ}_{(x,\rho)}(\cdot)$ lies inside EPI($\mathcal{Q}, \mathcal{X}$), which can be easily verified using item (ii) of Theorem 1.

We also mention in passing that in the original ILS paper, Laporte and Louveaux (1993) named as *valid optimality cuts* a family of inequalities in the (x, ρ)-space that implies $\rho \geq \mathcal{Q}(x)$, for all $x \in \mathcal{X} \cap \mathbb{Z}^E$ (adapting the notation to our context). This is similar to our statement in item (ii), except that here the family of ILS cuts $\mathcal{C}$ is defined in the extended space of the (x, θ)-variables.

3.3 Applying the framework

The first step in applying our framework is to choose Ω and a corresponding recourse disaggregation (Definition 2). Based on the ILS formulations used in previous works (Hoogendoorn and Spliet, 2023; Parada et al., 2024, 2025; Legault et al., 2025; Côté et al., 2020), a natural choice here is as follows (a formal justification for this assumption is later provided in Theorem 2).

Assumption 2. The disaggregation of $\mathcal{Q}$ is along $\Omega = V_+$ and, for every route R, $\Omega(R) \subseteq V_+(R)$.

Now let $\mathcal{C}$ denote a family of inequalities of the form $\theta(U) \geq \mathcal{L}(U, \mathcal{X}') \cdot W(x; \mathcal{X}')$ (with $\mathcal{L}(U, \mathcal{X}') \in \mathbb{Q}_+$ and $W(x; \mathcal{X}')$ affine in x). To express problem $\text{VRPR}(\mathcal{Q}, \mathcal{X})$ as an MILP in the form of Formulation $\text{ILS}(\mathcal{X}, \mathcal{C}, \Omega)$, we need to show that:

(1) every inequality $\theta(U) \geq \mathcal{L}(U, \mathcal{X}') \cdot W(x; \mathcal{X}')$ in the family $\mathcal{C}$ is an ILS cut, that is,

  (1a) $W(x; \mathcal{X}')$ is an activation function; and

  (1b) $\mathcal{L}(U, \mathcal{X}')$ is a recourse lower bound;

(2) every $(\bar{x}, \bar{\theta})$ feasible for Formulation $\text{ILS}(\mathcal{X}, \mathcal{C}, \Omega)$ satisfies $\mathbb{1}^\intercal \bar{\theta} \geq \mathcal{Q}(\bar{x})$.

In Section 4, we present activation functions that are valid for any $\mathcal{X} \subseteq \mathcal{X}_{\text{SUB}}$. Moreover, some of the ILS cuts associated with these activation functions use trivially valid recourse lower bounds (such as $\mathcal{Q}(R)$ or $\mathcal{Q}(\bar{x})$), and they already guarantee item (2). Although there may be exponentially many such inequalities, they can be separated easily for integer vectors $\bar{x} \in \mathcal{X} \cap \mathbb{Z}^E$. Therefore, without any extra work, this already yields a branch-and-cut algorithm for solving problem $\text{VRPR}(\mathcal{Q}, \mathcal{X})$

Although correct, the previously described algorithm may converge slowly. To improve the performance, one can derive problem-specific recourse lower bounds (item (1b)) for some of the ILS cuts presented in the next section. This is the approach we take when applying our framework to solve problem $\text{VRPR}(\mathcal{Q}_C, \mathcal{X}_{\text{CVRP}})$ in Section 5, where the main focus is on deriving strong recourse lower bounds.

4 Generalizing existing ILS cuts

In this section, we show that our framework unifies and generalizes existing VRPSD formulations (Gendreau et al., 1995; Laporte et al., 2002; Hoogendoorn and Spliet, 2023; Parada et al., 2024; Legault et al., 2025). In addition to extending previous ILS cuts, the framework also provides theoretical justifications for certain assumptions made in the literature (see Theorems 2 and 5). To set the stage, we begin by reviewing the first ILS-based algorithm for the VRPSD, proposed by Gendreau et al. (1995).

4.1 The ILS cuts of Gendreau et al. (1995)

In their work, Gendreau et al. (1995) assume that feasible routing plans use exactly $k \in \mathbb{Z}_{++}$ vehicles, so we have $\mathcal{X} \subseteq \{x \in \mathbb{R}^E : x(\delta(0)) = 2k\}$. For each $\bar{x} \in \mathcal{X} \cap \mathbb{Z}^E$, they consider the ILS cut

$$\mathbb{1}^\intercal \theta \geq \mathcal{Q}(\bar{x}) \cdot W_G^k(x; \{\bar{x}\}), \tag{2}$$

where

$$W_G^k(x; \{\bar{x}\}) \coloneqq 1 + x(E(G(\bar{x})) \setminus \delta(0)) - |V_+| + k, \tag{W_G^k}$$

and it is easy to check that $W_G^k(x; \{\bar{x}\})$ is indeed an activation function. The next result is a consequence of Theorem 1 and summarizes their formulation. (To recover the same formulation as that used by Gendreau et al. (1995), one applies Corollary 1 with Ω being a singleton.)

Corollary 1. *Let $k \in \mathbb{Z}_{++}$ and suppose that $\mathcal{X} \subseteq \{x \in \mathbb{R}^E : x(\delta(0)) = 2k\}$. Let*

$$\mathcal{C}_G^k = \{(\Omega, \{\bar{x}\}, \mathcal{Q}(\bar{x}), \alpha^{\bar{x}}, \beta^{\bar{x}})\}_{\bar{x} \in \mathcal{X} \cap \mathbb{Z}^E}$$

be a family of ILS cuts, where, for every $\bar{x} \in \mathcal{X} \cap \mathbb{Z}^E$, $W_G^k(x; \{\bar{x}\}) = (\alpha^{\bar{x}})^\intercal x + \beta^{\bar{x}}$. Let $\mathcal{F}_G^k$ be the feasible region of formulation ILS($\mathcal{X}, \mathcal{C}_G^k, \Omega$). Then $\textsc{epi}(\mathcal{Q}, \mathcal{X}) = \textsc{proj}_{(x,\rho)}(\mathcal{F}_G^k)$. □

Several ideas have been proposed to improve the basic approach of Gendreau et al. (1995). One such an idea, first explored by Laporte et al. (2002), derives *globally valid lower bounds* that enables a strengthening of the coefficients in the ILS cuts (see Appendix A for how this fits into our framework). A more relevant direction is to consider activation functions and recourse lower bounds for larger subsets $\mathcal{X}' \subseteq \mathcal{X} \cap \mathbb{Z}^E$, rather than just the singleton $\{\bar{x}\}$. Several works propose improvements along these lines by exploring the concept of *partial routes* (Hjorring and Holt, 1999; Laporte et al., 2002; Jabali et al., 2014; Hoogendoorn and Spliet, 2023), and among these, the recent work of Hoogendoorn and Spliet (2023) yields the best performing algorithm. We show in Section 4.2 that a formulation based on the *route cuts* of Hoogendoorn and Spliet (2023) gives a valid reformulation of problem $\textsc{vrpr}(\mathcal{Q}, \mathcal{X})$ if and only if the disaggregation of $\mathcal{Q}$ is *route-disjoint*. Consequently, an ILS algorithm based on route cuts is applicable to the VRPR with any generic recourse cost function.

Section 4.3 discusses the *DL-shaped method* (Parada et al., 2024; Legault et al., 2025), which considers subsets of solutions $\mathcal{X}' \subseteq \mathcal{X} \cap \mathbb{Z}^E$ that include certain paths or visit a set of customers using exactly the minimum required number of vehicles. This method achieved state-of-the-art performance on many benchmark instances of the VRPSD with independent probability distributions, but its correctness relies on some properties of the recourse cost function. Under certain assumptions on the set of feasible routes $\mathcal{X} \cap \mathbb{Z}^E$, Legault et al. (2025) recently established that *superadditivity* is the key property that characterizes when the DL-shaped method can be applied. Using our framework, we extend the characterization of Legault et al. (2025) to any choice of $\mathcal{X} \cap \mathbb{Z}^E$ (with $\mathcal{X} \subseteq \mathcal{X}_{\textsc{sub}}$). Moreover, we prove that the ILS cuts of Parada et al. (2024) dominate those of Gendreau et al. (1995), and we argue that the *set cuts* used in the DL-shaped method can be applied even when $\mathcal{Q}$ is not superadditive. These results were unknown prior to our framework.

4.2 Inequalities based on routes and partial routes

To improve upon the ILS cuts of Gendreau et al. (1995), we consider in Section 4.2.1 the *ILS route cuts*, which are inequalities that are tight at every solution containing a given route. Section 4.2.2 then extends this idea by presenting generalizations of the *partial route inequalities* of Hoogendoorn and Spliet (2023) which can be used on top of the ILS route cuts to further tighten the formulation. Lastly, Sections 4.2.3 and 4.2.4 address the activation functions and recourse lower bounds for partial route inequalities, respectively.

4.2.1 Route cuts

Let R be a route and define $\mathcal{X}(R) := \{x \in \mathcal{X} \cap \mathbb{Z}^E : R \in \mathcal{R}(x)\}$ as the set of solutions containing route R. An *ILS route cut* (or simply *route cut*) associated with R is of the form

$$\theta(\Omega(R)) \geq \mathcal{Q}(R) \cdot W(x; \mathcal{X}(R)), \tag{3}$$

where $W(x; \mathcal{X}(R))$ is an activation function (Section 4.2.3 contains explicit constructions of this function).

Observe that, for every $\bar{x} \in \mathcal{X}(R)$,

$$\sum_{R' \in \mathcal{R}(\bar{x})} \sum_{\omega \in \Omega(R)} \mathcal{Q}(R', \omega) \geq \sum_{\omega \in \Omega(R)} \mathcal{Q}(R, \omega) = \mathcal{Q}(R),$$

so $\mathcal{Q}(R)$ is a recourse lower bound with respect to $\Omega(R)$ and $\mathcal{X}(R)$. In other words, ILS route cuts (3) are ILS cuts (Definition 5), hence the name. Furthermore, it turns out that the following property exactly determines whether ILS route cuts can be used to reformulate the feasible region of $\text{VRPR}(\mathcal{Q}, \mathcal{X})$.

Definition 7. The disaggregation of $\mathcal{Q}$ is *route-disjoint* if, for every $\bar{x} \in \mathcal{X} \cap \mathbb{Z}^E$ and distinct routes $R, R' \in \mathcal{R}(\bar{x})$, it holds that $\Omega(R) \cap \Omega(R') = \emptyset$.

Theorem 2. *Consider the family of ILS cuts*

$$\mathcal{C}_R = \{(\Omega(R), \mathcal{X}(R), \mathcal{Q}(R), \alpha^R, \beta^R) : R \textit{ is a route}\},$$

where, for every route R, $W(x; \mathcal{X}(R)) = (\alpha^R)^\intercal x + \beta^R$ is an activation function. Let $\mathcal{F}_R$ be the feasible region of ILS($\mathcal{X}, \mathcal{C}_R, \Omega$). Then $\text{EPI}(\mathcal{Q}, \mathcal{X}) = \text{PROJ}_{(x,\rho)}(\mathcal{F}_R)$ if and only if the disaggregation of $\mathcal{Q}$ is route-disjoint.

Proof. Suppose first that the disaggregation of $\mathcal{Q}$ is route-disjoint and let $(\bar{x}, \bar{\theta}) \in \mathcal{F}_R$. Consider an ILS route cut $\theta(\Omega(R)) \geq \mathcal{Q}(R) \cdot W(x; \mathcal{X}(R))$ associated with a route $R \in \mathcal{R}(\bar{x})$. Since $W(\bar{x}; \mathcal{X}(R)) = 1$, we know that $\bar{\theta}(\Omega(R)) \geq \mathcal{Q}(R)$. Hence, by route-disjointness,

$$\mathbb{1}^\intercal \bar{\theta} \geq \sum_{R \in \mathcal{R}(\bar{x})} \bar{\theta}(\Omega(R)) \geq \sum_{R \in \mathcal{R}(\bar{x})} \mathcal{Q}(R) = \mathcal{Q}(\bar{x}).$$

Item (ii) of Theorem 1 then gives $\text{EPI}(\mathcal{Q}, \mathcal{X}) = \text{PROJ}_{(x,\rho)}(\mathcal{F}_R)$.

Assume now that the disaggregation of $\mathcal{Q}$ is not route-disjoint. In this case, there exists $\hat{x} \in \mathcal{X} \cap \mathbb{Z}^E$ such that, for some $\omega' \in \Omega$, we have $|\{R \in \mathcal{R}(\hat{x}) : \omega' \in \Omega(R)\}| \geq 2$. For each $\omega \in \Omega$, set $\hat{\theta}_\omega = \max_{R \in \mathcal{R}(\hat{x})}\{\mathcal{Q}(R, \omega)\}$. Since $\hat{\theta}_\omega \leq \sum_{R \in \mathcal{R}(\hat{x})} \mathcal{Q}(R, \omega)$, for all $\omega \in \Omega$, and $\hat{\theta}_{\omega'} < \sum_{R \in \mathcal{R}(\hat{x})} \mathcal{Q}(R, \omega')$, we know that $\mathbb{1}^\intercal \hat{\theta} < \mathcal{Q}(\hat{x})$. Therefore, by item (ii) of Theorem 1, it suffices to show that $(\hat{x}, \hat{\theta})$ belongs to $\mathcal{F}_R$. To do this, take an arbitrary route R'. If $R' \in \mathcal{R}(\hat{x})$, then $(\alpha^{R'})^\intercal \hat{x} + \beta^{R'} = 1$ and

$$\mathcal{Q}(R') \cdot ((\alpha^{R'})^\intercal \hat{x} + \beta^{R'}) = \sum_{\omega \in \Omega} \mathcal{Q}(R', \omega) \leq \sum_{\omega \in \Omega} \hat{\theta}_\omega,$$

where the last inequality follows from how we set $\hat{\theta}$. Otherwise, $(\alpha^{R'})^\intercal \hat{x} + \beta^{R'} \leq 0$ and we are done. □

Theorem 2 leads to a simple branch-and-cut procedure for solving $\textsc{vrpr}(\mathcal{Q}, \mathcal{X})$: at each integer solution $\bar{x} \in \mathcal{X} \cap \mathbb{Z}^E$, the routes in $\mathcal{R}(\bar{x})$ are examined, and violated ILS route cuts are added to the formulation. In the next subsection, we describe ILS cuts based on *partial routes*, which can be used in conjunction with the ILS route cuts to strengthen the relaxations that arise when solving Formulation $\text{ILS}(\mathcal{X}, \mathcal{C}_R, \Omega)$.

Remark 3. There is always a choice of Ω and a corresponding disaggregation of $\mathcal{Q}$ that is route-disjoint. For instance, we may set $\Omega = V_+$ (or $\Omega = E$) and define the disaggregation so that $\Omega(R) \subseteq V_+(R)$ (or $\Omega(R) \subseteq E(R)$) for every route R. This justifies the use of Assumption 2 and shows that the branch-and-cut algorithm suggested by Theorem 2 can, in principle, be applied to any recourse function $\mathcal{Q}$. $\square$

4.2.2 Partial route inequalities

We begin with the definition of partial routes (see Figure 1).

Definition 8. Let $H = (S_1, \dots, S_\ell)$ be a tuple of disjoint subsets of V_+. For each $i \in [\ell]$, we call S_i an *unstructured component* if $|S_i| > 1$, and denote by $V_+(H)$ the set $\cup_{i=1}^{\ell} S_i$. We say that H is a *partial route* if there exists no index $i \in [\ell - 1]$ such that both S_i and S_{i+1} are unstructured components.

For convenience, whenever $H = (S_1, \dots, S_\ell)$ is a partial route, it is implicit that $S_0 = S_{\ell+1} = \{0\}$.

Remark 4. Partial routes are sometimes described as generalizations of routes (Hoogendoorn and Spliet, 2023), since, given a route $R = (v_1, \dots, v_\ell)$, one can define a partial route $H = (S_1, \dots, S_\ell)$ where each S_i is a singleton $\{v_i\}$. In this special case, we say that H *corresponds* to the route R and, with a slight abuse of notation, we may write $H = R$. $\square$

Intuitively, a partial route H compactly represents all routes that can be obtained by replacing each unstructured set $S_i \in H$ with a permutation of its elements. We formalize this as follows.

Definition 9. Let $H = (S_1, \dots, S_\ell)$ be a partial route. For convenience, define $h_i := \sum_{j \in [i]} |S_j|$, for each $i \in [\ell]$. A directed route $\vec{R} = (v_1, \dots, v_t)$ *exactly adheres* to H if $t = h_\ell = |V_+(H)|$ and, for every $i \in [\ell]$, $S_i = \{v_{h_{i-1}+1}, \dots, v_{h_i}\}$. In other words,

$$\vec{R} = (\underbrace{v_1, \dots, v_{h_1}}_{S_1}, \dots, \underbrace{v_{h_{i-1}+1}, \dots, v_{h_i}}_{S_i}, \dots, \underbrace{v_{h_{\ell-1}+1}, \dots, v_{h_\ell}}_{S_\ell}).$$

An (undirected) route R *exactly adheres* to H if either $\vec{R}$ or $\overleftarrow{R}$ exactly adheres to H. Finally, a vector $\bar{x} \in \mathcal{X} \cap \mathbb{Z}^E$ *exactly adheres* to H if there exists $R \in \mathcal{R}(\bar{x})$ such that R exactly adheres to H.

The concept in Definition 9 was simply called *adherence* by Hoogendoorn and Spliet (2023). We adjusted the terminology slightly because we shall also discuss a related set of routes that possibly visit more customers than those in the partial route. To formalize this concept, let $R = (v_1, \dots, v_\ell)$ be a route. A *subroute* of R is a route R' that can be written as $R' = (v_i, \dots, v_j)$ with $j \in [\ell]$ and $i \in [j]$. We write $R' \subseteq R$ to indicate that R' is a subroute of R (even though R' is not necessarily a subgraph of R).

Definition 10. A route $R = (v_1, \dots, v_\ell)$ *adheres* to H if there exists $R' \subseteq R$ such that R' exactly adheres to H. A vector $\bar{x} \in \mathcal{X} \cap \mathbb{Z}^E$ *adheres* to H if there exists a route $R \in \mathcal{R}(\bar{x})$ such that R adheres to H.

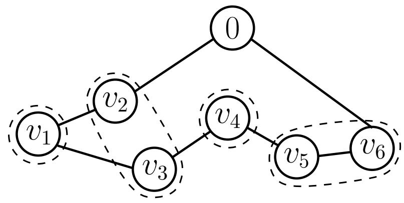


Figure 1: Illustration of a route $R = (v_2, v_1, v_3, v_4, v_5, v_6)$ that does not adhere to partial route $H = (\{v_1\}, \{v_2, v_3\}, \{v_4\}, \{v_5, v_6\})$. However, note that R does adhere (but not exactly) to partial route $H' = (\{v_4\}, \{v_5, v_6\})$, since the subroute $R' = (v_4, v_5, v_6) \subseteq R$ exactly adheres to H'.

Using Definitions 9 and 10, for every partial route H, we define

$$\mathcal{X}_{=}(H) := \left\{x \in \mathcal{X} \cap \mathbb{Z}^E :\ x \text{ exactly adheres to } H\right\} \text{ and} \tag{$\mathcal{X}_{=}(H)$}$$

$$\mathcal{X}_{\supseteq}(H) := \left\{x \in \mathcal{X} \cap \mathbb{Z}^E :\ x \text{ adheres to } H\right\}. \tag{$\mathcal{X}_{\supseteq}(H)$}$$

Given $U \subseteq \Omega$, a recourse lower bound $\mathcal{L}(U, \mathcal{X}_{=}(H))$ and an activation function $W(x; \mathcal{X}_{=}(H))$, we say that an *ILS partial route exact adherence (ILS PR-EA) cut* is an inequality of the form

$$\theta(U) \geq \mathcal{L}(U, \mathcal{X}_{=}(H)) \cdot W(x; \mathcal{X}_{=}(H)). \tag{4}$$

Note that ILS PR-EA cuts generalize the ILS route cuts introduced earlier, as the definition of $\mathcal{X}_{=}(H)$ coincides with our previous definition of $\mathcal{X}(R)$ whenever H corresponds to a route R (see Remark 4).

In a similar way, given $U \subseteq \Omega$, a recourse lower bound $\mathcal{L}(U, \mathcal{X}_{\supseteq}(H))$ and an activation function $W(x; \mathcal{X}_{\supseteq}(H))$, we say that an *ILS partial route adherence (ILS PR-A) cut* is an inequality of the form

$$\theta(U) \geq \mathcal{L}(U, \mathcal{X}_{\supseteq}(H)) \cdot W(x; \mathcal{X}_{\supseteq}(H)). \tag{5}$$

In line with the literature nomenclature, we sometimes refer to ILS PR-EA and ILS PR-A cuts as simply *partial route inequalities*.

In Section 4.2.3, we discuss activation functions for inequalities (4) and (5). Section 4.2.4 then comments on a simplification made by previous works that, in some cases, enables the use of efficient procedures to compute the recourse lower bound $\mathcal{L}(U, \mathcal{X}_{=}(H))$. The recourse lower bounds for ILS PR-A cuts are addressed later in Section 4.3.

Remark 5. Inequalities (4) generalize the *route-split* and *partial route-split inequalities* of Hoogendoorn and Spliet (2023). To see this, suppose that our fixed disaggregation of $\mathcal{Q}$ is along $\Omega = V_+$ and, for every route R, we have $\Omega(R) = \{v\}$, where v is the customer with the smallest index in R. In this case, and letting W_{HS} denote their activation functions for partial route inequalities, the ILS route cut (3) associated with route R reduces to $\theta_v \geq \mathcal{Q}(R) \cdot W_{HS}(x; \mathcal{X}_{=}(R))$, which is precisely their route-split inequality. More generally, consider an ILS PR-EA cut (4) associated with a partial route H. If we let U be the singleton corresponding to the customer with the smallest index in $V_+(H)$, then $\mathcal{L}(U, \mathcal{X}_{=}(H))$ can be set to any value that lower bounds $\mathcal{Q}(R)$, for every route R that exactly adheres to H. This shows that (4) also generalizes the partial route-split inequalities proposed by Hoogendoorn and Spliet (2023). Finally, we note that Hoogendoorn and Spliet (2023) also introduced the *multi-route-split inequalities*, but we do not address them in this work, since they are not ILS cuts (Definition 5). □

4.2.3 Partial route activation functions

Based on the analysis of Hoogendoorn and Spliet (2023), we present an activation function for PR-EA cuts that gives a slight improvement over theirs. Perhaps more importantly, our proof is considerably simpler and easier to verify than the four-page argument in their online appendix. This simplification might be particularly valuable given the history of partial route activation functions for VRPSDs: they were first proposed for the multiple vehicle case by Laporte et al. (2002), later shown to be incorrect by Jabali et al. (2014), whose own activation functions were also shown to be incorrect by Hoogendoorn and Spliet (2023). In Appendix E, we also provide a formal description of the separation heuristic proposed by Hoogendoorn and Spliet (2023).

Fix a partial route $H = (S_1, \ldots, S_\ell)$. We start by presenting an activation function that is active at solutions that adhere to H, in other words, an activation function with respect to $\mathcal{X}_\supseteq(H)$. To our knowledge, no such function has been developed previously. We emphasize that all the activation functions in this subsection make no assumptions on the set $\mathcal{X}$ (besides Assumption 1). In particular, our activation function for $\mathcal{X}_\supseteq(H)$ is in fact active (returns one) at any point belonging to the set $\{x \in \mathcal{X}_{\text{SUB}} \cap \mathbb{Z}^E : x \text{ adheres to } H\}$.

Henceforth, for any two disjoint sets $A, B \subseteq V$, we use $x(A, B)$ as a shorthand for $\sum_{uv \in E: u \in A, v \in B} x_{uv}$. Moreover, $x(H)$ denotes the sum $\sum_{i \in [\ell]} x(S_i) + \sum_{i \in [\ell-1]} x(S_i, S_{i+1})$ and $|H|$ corresponds to the number of customers in H. The proposed activation is as follows:

$$W_{OF}(x; \mathcal{X}_\supseteq(H)) \coloneqq 1 + (x(H) - |H| + 1) + \begin{cases} \mathbb{I}(|S_1| = 1 \text{ or } |S_3| = 1)(x(S_2) - |S_2| + 1), & \text{if } \ell = 3, \\ \mathbb{I}(|S_1| = 1)(x(S_2) - |S_2| + 1) \\ \quad + \mathbb{I}(|S_\ell| = 1)(x(S_{\ell-1}) - |S_{\ell-1}| + 1), & \text{if } \ell = 2 \text{ or } \ell \geq 4. \end{cases} \tag{$W_{OF}(x; \mathcal{X}_\supseteq(H))$}$$

To get an intuitive understanding for the expression defining $W_{OF}(x; \mathcal{X}_\supseteq(H))$, it is instructive to examine why the simpler expression $1 + (x(H) - |H| + 1)$ is generally *not* an activation function with respect to $\mathcal{X}_\supseteq(H)$. Suppose $H = (\{v_1\}, \{v_2, v_3\}, \{v_4\}, \{v_5, v_6\})$ and consider the solution $\bar{x} \in \mathcal{X} \cap \mathbb{Z}^E$ shown in Figure 1, meaning that $\mathcal{R}(\bar{x}) = \{R\}$ with $R = (v_2, v_1, v_3, v_4, v_5, v_6)$. In this example, even though $\bar{x}(H) = |H| - 1$, the solution $\bar{x}$ does not adhere to H, that is, $\bar{x} \notin \mathcal{X}_\supseteq(H)$. The additional term $(x(S_2) - |S_2| + 1)$ in $W_{OF}(x; \mathcal{X}_\supseteq(H))$ enforces that a solution $x' \in \mathcal{X} \cap \mathbb{Z}^E$ with $W_{OF}(x'; \mathcal{X}_\supseteq(H)) = 1$ induces a solution that enters/exits $S_2 = \{v_2, v_3\}$ exactly once, fixing this issue. In Appendix B, we formalize this intuition to prove the following result.

Theorem 3. *For every partial route $H = (S_1, \ldots, S_\ell)$, $W_{OF}(x; \mathcal{X}_\supseteq(H))$ is an activation function.*

With the activation function $W_{OF}(x; \mathcal{X}_\supseteq(H))$ in hand, we construct an activation function $W_{OF}(x; \mathcal{X}_=(H))$ by enforcing that, if $W_{OF}(\bar{x}; \mathcal{X}_=(H)) = 1$, then $\bar{x} \in \mathcal{X}_\supseteq(H)$ and $\mathcal{R}(\bar{x})$ contains a route beginning at S_1 and ending at S_ℓ (or vice versa). The proof of Theorem 4 is left to Appendix C.

$$W_{OF}(x; \mathcal{X}_=(H)) \coloneqq W_{OF}(x; \mathcal{X}_\supseteq(H)) + \begin{cases} (x(0, S_1) + 2x(S_1) - 2|S_1|), & \text{if } \ell = 1, \\ (x(0, S_1) + 2x(S_1) + x(S_1, S_2) - 2|S_1|) \\ (x(0, S_\ell) + 2x(S_\ell) + x(S_{\ell-1}, S_\ell) - 2|S_\ell|), & \text{if } \ell \geq 2. \end{cases} \tag{$W_{OF}(x; \mathcal{X}_=(H))$}$$

Theorem 4. *Let $H = (S_1, \ldots, S_\ell)$ be a partial route. Then $W_{OF}(x; \mathcal{X}_=(H))$ is an activation function.*

Finally, rearranging the expression in Hoogendoorn and Spliet (2023) (see Appendix D), one may check that the activation function $W_{HS}(x; \mathcal{X}_=(H))$ is equivalent to the activation function $W_{OF}(x; \mathcal{X}_=(H))$ with the indicator functions in $W_{OF}(x; \mathcal{X}_\supseteq(H))$ replaced by ones. Since, except for the leading one, every term defining $W_{OF}(x; \mathcal{X}_=(H))$ is nonpositive for $x \in \mathcal{X}_{\text{SUB}}$, we get the following result.

Claim 3. *Let H be a partial route. Then, for all $x \in \mathcal{X}_{\text{SUB}}$, $W_{OF}(x; \mathcal{X}_=(H)) \geq W_{HS}(x; \mathcal{X}_=(H))$.* $\square$

4.2.4 Partial route lower bounds

Following previous works using recourse disaggregation (Côté et al., 2020; Parada et al., 2024; Séguin, 1994; Hoogendoorn and Spliet, 2023; Parada et al., 2025; Legault et al., 2025), let us now assume that the disaggregation of $\mathcal{Q}$ satisfies Assumption 2. Let H be a partial route and consider the following special case of an ILS PR-EA cut (inequality (4)),

$$\theta(V_+(H)) \geq \mathcal{L}(V_+(H), \mathcal{X}_=(H)) \cdot W(x; \mathcal{X}_=(H)), \tag{6}$$

where $\mathcal{L}(V_+(H), \mathcal{X}_=(H))$ is a recourse lower bound. By Definition 4, for every route R that satisfies

(i) R exactly adheres to H, and

(ii) there exists $\bar{x} \in \mathcal{X} \cap \mathbb{Z}^E$ with $R \in \mathcal{R}(\bar{x})$,

we have $\sum_{v \in V_+(H)} \mathcal{Q}(R, v) = \mathcal{Q}(R) \geq \mathcal{L}(V_+(H), \mathcal{X}_=(H))$, where the equality follows from $V_+(R) = V_+(H)$ and Assumption 2.

However, to our knowledge, every method previously proposed for computing recourse lower bounds for partial routes (Laporte et al., 2002; Jabali et al., 2014; Hoogendoorn and Spliet, 2023; Salavati-Khoshghalb et al., 2019a) drops condition (ii) and computes a lower bound on the recourse of *every* route that exactly adheres to H, regardless if it appears in a feasible routing plan. This leads to the following definition.

Definition 11. Let H be a partial route. We say that $\mathcal{L}(H) \in \mathbb{Q}_+$ is a *partial route lower bound* if, for every route R that exactly adheres to H, we have that $\mathcal{Q}(R) \geq \mathcal{L}(H)$.

By focusing solely on adherence rather than feasibility, Definition 11 guides the derivation of efficiently computable recourse lower bounds. For example, Hjorring and Holt (1999); Laporte et al. (2002); Jabali et al. (2014); Hoogendoorn and Spliet (2023) compute a partial route recourse lower bound for the VRPSD under the classical recourse policy by contracting each unstructured set $S \in H$ into a single vertex v_S. In Section 5, we show that this approach extends naturally to the case where the probability distribution is given by scenarios. Definition 11 will also play a role in the next subsection, where we show that, as long as $\mathcal{Q}$ satisfies a certain monotonicity/superadditivity property, a partial route lower bound $\mathcal{L}(H)$ is a valid recourse lower bound for all solutions that adhere (not necessarily exactly) to H.

4.3 Inequalities based on paths, sets and the DL-shaped method

In the context of the VRPSD, Parada et al. (2024) recently introduced two new ILS cuts, named *path cuts* and *set cuts*, and incorporated them into a branch-and-cut algorithm which they called the *disaggregated integer L-shaped method* (DL-shaped method). Even more recently, Legault et al. (2025) have shown that, under additional assumptions on the set of feasible routing plans (e.g., unlimited number of available vehicles), the DL-shaped method is valid if and only if the recourse function $\mathcal{Q}$ has a certain superadditivity property.

Building on our framework, we obtain in Section 4.3.1 a weaker variant of superadditivity that characterizes the validity of path cuts without making any additional assumptions on the set of feasible routing plans. Furthermore, in Section 4.3.2 we prove the correctness of formulations based on path cuts by showing a dominance relationship between the path cuts and the inequalities of Gendreau et al. (1995) (see Corollary 1). Lastly, Section 4.3.3 highlights that if set cuts are expressed using the concept of recourse lower bounds introduced in Definition 4 (rather than using the lower bounds proposed in Parada et al. (2024) and Legault et al. (2025)), then they are always valid for $\mathcal{F}(\mathcal{Q}, \mathcal{X}, \Omega)$, regardless if $\mathcal{Q}$ (or its disaggregation) satisfies any of the the properties established in Section 4.3.1. Thus, while path cuts depend on specific properties of $\mathcal{Q}$, the set cuts can be applied much more broadly.

As in the previous subsection, we assume for most of our discussion that $\mathcal{Q}$ satisfies Assumption 2. Additionally, we use $\mathfrak{R}$ to denote the set of all routes appearing in a feasible routing plan, that is, $\mathfrak{R} \coloneqq \bigcup_{x \in \mathcal{X} \cap \mathbb{Z}^E} \mathcal{R}(x)$.

4.3.1 Monotonicity, superadditivity and the ILS path cuts

An *ILS path cut* (or simply a *path cut*) for a route R is an inequality of the form

$$\theta(V_+(R)) \geq \mathcal{Q}(R) \cdot W_{OF}(x; \mathcal{X}_{\supseteq}(R)), \tag{7}$$

where $W_{OF}(x; \mathcal{X}_{\supseteq}(R)) = 1 + \sum_{e \in E(R) \setminus \delta(0)} (x_e - 1)$ coincides with the activation function proposed by Parada et al. (2024).

Note that whenever $\mathcal{Q}(R)$ is a recourse lower bound with respect to $V_+(R)$ and $\mathcal{X}_{\supseteq}(R)$, the ILS path cut (7) is a special case of an ILS PR-A cut (5). Moreover, applying Assumption 2 and Claim 2, leads to the following definition and result.

Definition 12. Let $\hat{\mathcal{Q}}$ be a disaggregation of $\mathcal{Q}$ along $\Omega = V_+$ satisfying $\Omega(R) \subseteq V_+(R)$, for every route R (i.e., $\hat{\mathcal{Q}}$ satisfy the conditions in Assumption 2). We say that $\hat{\mathcal{Q}}$ is *monotone* if, for every $R \in \mathfrak{R}$ and $R' \subseteq R$, we have that $\sum_{v \in V_+(R')} \hat{\mathcal{Q}}(R, v) \geq \mathcal{Q}(R')$.

Fact 1. *Suppose Assumption 2 holds. Then the fixed disaggregation of $\mathcal{Q}$ is monotone if and only if, for every $R' \subseteq R$, with $R \in \mathfrak{R}$, the value $\mathcal{Q}(R')$ is a recourse lower bound with respect to $V_+(R')$ and $\mathcal{X}_{\supseteq}(R')$.*

Definition 12 is similar to the inequality used in Proposition 2 of Parada et al. (2024), with the following important differences: we explicitly account for the disaggregated recourse terms $\hat{\mathcal{Q}}(R, v)$; we only consider routes belonging to feasible routing plans (i.e., $R \in \mathfrak{R}$); and we only consider subroutes of R, rather than subsequences.[2] We remark, however, that Legault et al. (2025) has recently shown that Proposition 2

[2]In fact, Parada et al. (2024) also define a monotonicity property, but their property specifically concerns the probability distributions of the VRPSD. Their Proposition 2 is then presented as a consequence of this property.

of Parada et al. (2024) does not guarantee the validity of ILS path cuts. On the other hand, Fact 1 shows that Definition 12 is both necessary and sufficient for the validity of ILS path cuts with respect to $\mathcal{F}(\mathcal{Q}, \mathcal{X}, \Omega)$. In fact, as we show next, Definition 12 is also equivalent to a weaker variant of the superadditivity property in Legault et al. (2025).

To discuss this result, we need some notation. Given two routes $R_1 = (v_1, \ldots, v_a)$ and $R_2 = (v_{a+1}, \ldots, v_\ell)$ with $V_+(R_1) \cap V_+(R_2) = \emptyset$, we denote by $R_1 \oplus R_2$ the route obtained by concatenating R_1 and R_2, i.e., $R_1 \oplus R_2 = (v_1, \ldots, v_\ell)$. For convenience, whenever we write $R_1 \oplus R_2$, we assume that R_1 and R_2 are subroutes of $R_1 \oplus R_2$ visiting disjoint sets of customers and whose concatenation produces $R_1 \oplus R_2$. Adapting the result of Legault et al. (2025) to our more general setting, we define their superadditivity property as follows.

Definition 13. The recourse function $\mathcal{Q}$ is *superadditive* if, for every $R \in \mathfrak{R}$ and $R_1 \oplus R_2 \subseteq R$, we have that $\mathcal{Q}(R_1 \oplus R_2) \geq \mathcal{Q}(R_1) + \mathcal{Q}(R_2)$.

Suppose that $\mathcal{Q}$ is superadditive and let $\hat{\mathcal{Q}}$ be a disaggregation of $\mathcal{Q}$ along $\Omega = V_+$ such that, for every route $R = (v_1, \ldots, v_\ell)$ belonging to a feasible routing plan,

$$\hat{\mathcal{Q}}(R, v_i) = \begin{cases} \mathcal{Q}((v_1, \ldots, v_i)) - \mathcal{Q}((v_1, \ldots, v_{i-1})), & \text{if } i \in [\ell] \setminus \{1\}, \\ \mathcal{Q}((v_1)), & \text{otherwise.} \end{cases} \tag{8}$$

It is not hard to show that $\hat{\mathcal{Q}}$ is monotone. We prove this claim in Appendix F, where we also show that the converse might not hold, that is, monotonicity alone does not imply superadditivity. To further clarify the distinction between these two properties, we also demonstrate in Appendix F that the following notion of *weak superadditivity* precisely characterizes when a recourse function admits a monotone disaggregation (in the same appendix, we also argue that superadditivity implies weak superadditivity, hence the name).

Definition 14. The recourse function $\mathcal{Q}$ is *weakly superadditive* if, for every $R \in \mathfrak{R}$, we have that $\mathcal{Q}(R) \geq \mathcal{Q}(R_1) + \ldots + \mathcal{Q}(R_t)$, for all disjoint subroutes $R_1, \ldots, R_t \subseteq R$, that is, $V_+(R_i) \cap V_+(R_j) = \emptyset$, for every $i \in [t]$ and $j \in [i-1]$. (Recall that $[0] = \emptyset$.)

Theorem 5. *The following holds.*

(i) If the fixed disaggregation of $\mathcal{Q}$ satisfies Assumption 2 and is monotone, $\mathcal{Q}$ is weakly superadditive.

(ii) Conversely, if $\mathcal{Q}$ is weakly superadditive, then there exists a disaggregation of $\mathcal{Q}$ along $\Omega = V_+$ that satisfies the conditions in Assumption 2 and is monotone.

In Appendix F, we also prove that if $\mathfrak{R}$ *downward closed* — meaning that $R \in \mathfrak{R}$ implies $R' \in \mathfrak{R}$, for all $R' \subseteq R$ — then weak superadditivity implies superadditivity. This clarifies the connection between Theorem 5 and the result of Legault et al. (2025), since in their setup, $\mathfrak{R}$ is indeed downward closed.

We conclude this discussion by pointing out that the central idea of the DL-shaped method, as described by Parada et al. (2024), is to disaggregate the total recourse cost of a solution by customers, rather than by routes (or not disaggregating at all). Legault et al. (2025) showed that, under some assumptions on the set of feasible routing plans, the path and set cuts of the DL-shaped method can be applied if and only if $\mathcal{Q}$ is superadditive. Theorem 5 (and Fact 1) refines this result by stating that superadditivity concerns only the validity of the path cuts, that is, whether for every subroute R' of a route $R \in \mathfrak{R}$, the value $\mathcal{Q}(R')$ can be used as recourse lower bound with respect to $V_+(R')$ and $\mathcal{X}_{\supseteq}(R')$. On the other hand, as established by our framework, the principle of disaggregating the recourse (either along V_+ or along another choice of Ω) does not rely on monotonicity/superadditivity and can be applied more generally.

4.3.2 Correctness of ILS path cuts formulations

We now turn to proving the correctness of a VRPR formulation based on ILS path cuts. Rather than proving this directly as in Parada et al. (2024); Legault et al. (2025), we show that ILS path cuts dominate the inequalities from Gendreau et al. (1995) (discussed previously in Section 4.1). This not only proves correctness, but also reveals a previously unknown, yet simple, dominance relationship between the polyhedra associated with the LP relaxation of the two approaches.

Lemma 1. *Suppose that $\mathcal{Q}$ satisfies Assumption 2. Let $\bar{x} \in \mathcal{X} \cap \mathbb{Z}^E$ and set $k = |\mathcal{R}(\bar{x})|$. Then the ILS path cuts (7) (with respect to the routes in $\mathcal{R}(\bar{x})$) dominate inequality $\mathbb{1}^\intercal \theta \geq \mathcal{Q}(\bar{x}) \cdot W_G^k(x; \{\bar{x}\})$.*

Proof. Fix $\bar{x} \in \mathcal{X} \cap \mathbb{Z}^E$ and let $k = |\mathcal{R}(\bar{x})|$. Let $(x', \theta') \in \mathbb{R}^E \times \mathbb{R}_+^{V_+}$ be a point that satisfies the ILS path cuts (7) with respect to the routes in $\mathcal{R}(\bar{x})$. Since the fixed disaggregation of $\mathcal{Q}$ is route-disjoint, we obtain the desired result by summing the ILS path cuts:

$$\begin{aligned}
\mathbb{1}^\intercal \theta' &\geq \sum_{R \in \mathcal{R}(\bar{x})} \theta'(V_+(R)) \geq \sum_{R \in \mathcal{R}(\bar{x})} \left(\mathcal{Q}(R) \cdot W_{OF}(x; \mathcal{X}_{\supseteq}(R))\right) \\
&= \mathcal{Q}(\bar{x}) + \sum_{R \in \mathcal{R}(\bar{x})} \left(\mathcal{Q}(R) \cdot \sum_{e \in E(R) \setminus \delta(0)} (x'_e - 1) \right) \\
&\geq \mathcal{Q}(\bar{x}) \left(1 + \sum_{R \in \mathcal{R}(\bar{x})} \sum_{e \in E(R) \setminus \delta(0)} (x'_e - 1) \right) = \mathcal{Q}(\bar{x}) \cdot W_G^k(x'; \{\bar{x}\}).
\end{aligned}$$
□

Corollary 2. *Suppose that the fixed disaggregation of $\mathcal{Q}$ satisfies Assumption 2 and is monotone. Consider the family of ILS cuts*

$$\mathcal{C}_P = \{(V_+(R), \mathcal{Q}(R), \alpha^R, \beta^R) : R \in \mathfrak{R}\},$$

where $W_{OF}(x; \mathcal{X}_{\supseteq}(R)) = (\alpha^R)^\intercal x + \beta^R$, for every route $R \in \mathfrak{R}$. Let $\mathcal{F}_P$ be the feasible region of ILS$(\mathcal{X}, \mathcal{C}_P, \Omega)$. Then $\text{EPI}(\mathcal{Q}, \mathcal{X}) = \text{PROJ}_{(x,\rho)}(\mathcal{F}_P)$.

Proof. By Lemma 1, the family of ILS cuts $\mathcal{C}_P$ satisfies item (ii) of Theorem 1. □

Corollary 3. *Suppose that the fixed disaggregation of $\mathcal{Q}$ satisfies Assumption 2 and is monotone. Assume that there exists $k \in \mathbb{Z}_{++}$ such that $\mathcal{X} \subseteq \{x \in \mathbb{R}^E : x(\delta(0)) = 2k\}$. Consider the families of ILS cuts*

$$\begin{aligned}
\mathcal{C}_G^k &= \{(V_+, \{\bar{x}\}, \mathcal{Q}(\bar{x}), \alpha^{\bar{x}}, \beta^{\bar{x}})\}_{\bar{x} \in \mathcal{X} \cap \mathbb{Z}^E} \qquad \textit{and} \\
\mathcal{C}_P &= \{(V_+(R), \mathcal{X}_{\supseteq}(R), \mathcal{Q}(R), \alpha^R, \beta^R) : R \in \mathfrak{R}\},
\end{aligned}$$

where $W_G^k(x; \{\bar{x}\}) = (\alpha^{\bar{x}})^\intercal x + \beta^{\bar{x}}$, for every $\bar{x} \in \mathcal{X} \cap \mathbb{Z}^E$, and $W_{OF}(x; \mathcal{X}_{\supseteq}(R)) = (\alpha^R)^\intercal x + \beta^R$, for every route $R \in \mathfrak{R}$. Let $\mathcal{P}_G^k$ and $\mathcal{P}_P$ be polyhedra associated with the LP relaxations of ILS$(\mathcal{X}, \mathcal{C}_G^k, \Omega)$ and ILS$(\mathcal{X}, \mathcal{C}_P, \Omega)$, respectively. Then $\text{PROJ}_{(x,\rho)}(\mathcal{P}_P) \subseteq \text{PROJ}_{(x,\rho)}(\mathcal{P}_G^k)$.

Proof. By Lemma 1. □

4.3.3 Recourse lower bounds via monotonicity and ILS set cuts

In this subsection, we show that monotonicity can often be leveraged to derive recourse lower bounds for a set of customers by focusing only on routes that visit exactly this set. As a first example of this principle, we observe that, if $\mathcal{Q}$ is monotone (in the sense of Definition 12), then any partial route lower bound for H (Definition 11) yields valid recourse lower bounds for solutions that adhere (not necessarily exactly) to H.

Claim 4. *Suppose that the fixed disaggregation of $\mathcal{Q}$ satisfies Assumption 2 and is monotone. Let H be a partial route with corresponding partial route lower bound $\mathcal{L}(H)$. Then, $\sum_{v\in V_+(H)} \mathcal{Q}(R,v) \geq \mathcal{L}(H)$, for all $\bar{x} \in \mathcal{X}_{\supseteq}(H)$ with $R \in \mathcal{R}(\bar{x})$ adhering to H.*

Proof. Let $\bar{x} \in \mathcal{X}_{\supseteq}(H)$ and let $R \in \mathcal{R}(\bar{x})$ be such that R adheres to H. By Definition 10, there exists $R' \subseteq R$ such that R' exactly adheres to H. Since $V_+(H) = V_+(R')$,

$$\sum_{v\in V_+(H)} \mathcal{Q}(R,v) \overset{\text{Definition 12}}{\geq} \mathcal{Q}(R') \overset{\text{Definition 11}}{\geq} \mathcal{L}(H). \qquad \square$$

A consequence of Proposition 4 is that, whenever Assumption 2 holds and the fixed disaggregation of $\mathcal{Q}$ is monotone (which can be built with Theorem 5, if $\mathcal{Q}$ is weakly superadditive), the following inequality is valid for $\mathcal{F}(\mathcal{Q}, \mathcal{X}, \Omega)$,

$$\theta(V_+(H)) \geq \mathcal{L}(H) \cdot W(x; \mathcal{X}_{\supseteq}(H)), \tag{9}$$

where H is a partial route and $\mathcal{L}(H)$ is a corresponding partial route lower bound. Therefore, in this case, one may use the methods mentioned in Section 4.2.4 to compute recourse lower bounds for solutions that adhere, not necessarily exactly, to a partial route. To our knowledge, this has not been explored before, and we use it in Part II (Ota and Fukasawa, 2025). However, since the recourse function $\mathcal{Q}_C$ (see Section 2.2) is not weakly superadditive (see Appendix G), we do not apply inequalities (9) in the present work.

Monotonicity can also be exploited to derive recourse lower bounds for the set cuts first introduced in Parada et al. (2024). Let $S \subseteq V_+$ and suppose that $\tilde{k}$ is a lower bound on the number of routes that we need to attend the customers in S, meaning that the inequality $x(S) \leq |S| - \tilde{k}$ is valid for $\mathcal{X} \cap \mathbb{Z}^E$.

Define

$$\mathcal{X}(S,\tilde{k}) := \left\{ x \in \mathcal{X} \cap \mathbb{Z}^E : x(S) = |S| - \tilde{k} \right\} \quad \text{and} \tag{$\mathcal{X}(S,\tilde{k})$}$$

$$W_P(x; \mathcal{X}(S,\tilde{k})) := 1 + (x(S) - |S| + \tilde{k}). \tag{$W_P(x; \mathcal{X}(S,\tilde{k}))$}$$

It is easy to see that $W_P(x; \mathcal{X}(S,\tilde{k}))$ is an activation function with respect to $\mathcal{X}(S,\tilde{k})$. For any $U \subseteq V_+$, we say that an *ILS set cut* (or simply *set cut*) is an inequality of the form

$$\theta(U) \geq \mathcal{L}(U, \mathcal{X}(S,\tilde{k})) \cdot W_P(x; \mathcal{X}(S,\tilde{k})), \tag{10}$$

where $\mathcal{L}(U, \mathcal{X}(S,\tilde{k}))$ is a recourse lower bound.

Since $\Omega = V_+$ by Assumption 2, we shall apply inequality (10) solely with $U = S$. The single (but important) reason we claim that this inequality generalizes the set cuts in the DL-shaped method is that Parada et al. (2024) (and Legault et al. (2025)) only consider the following particular case for the recourse lower bound $\mathcal{L}(S, \mathcal{X}(S,\tilde{k}))$.

Claim 5. *Suppose that the fixed disaggregation of $\mathcal{Q}$ satisfies Assumption 2 and is monotone. Let $C' \in \mathbb{Q}_{++}$ and assume that there exists $\bar{d} \in \mathbb{Q}_{+}^{V_+}$ such that every route $R \in \mathfrak{R}$ satisfies $\bar{d}(R) \leq C'$. Let $S \subseteq V_+$ and suppose $\tilde{k}$ is such that the inequality $x(S) \leq |S| - \tilde{k}$ is valid for $\mathcal{X} \cap \mathbb{Z}^E$. Define $\tilde{\mathcal{L}}(S)$ so that $\tilde{\mathcal{L}}(S) \leq \sum_{i=1}^{\tilde{k}} \mathcal{Q}(R_i)$, for every set of routes $R_1, \ldots, R_{\tilde{k}}$, such that $\{V_+(R_i)\}_{i \in [\tilde{k}]}$ forms a partition of S and $\bar{d}(R_i) \leq C'$, for every $i \in [\tilde{k}]$. Then $\tilde{\mathcal{L}}(S)$ is a recourse lower bound with respect to S and $\mathcal{X}(S, \tilde{k})$.*

Proof. Let $\bar{x} \in \mathcal{X}(S, \tilde{k})$. The subgraph of $G(\bar{x})$ induced by S is made of $\tilde{k}$ disjoint paths $P_1, \ldots, P_{\tilde{k}}$, each P_i corresponding to a route R_i such that $P_i = R_i \setminus \{0\}$. Since each R_i is a subroute of a route in $\mathcal{R}(\bar{x})$, we have that $\bar{d}(R_i) \leq C'$. Applying Definition 12, $\sum_{R \in \mathcal{R}(\bar{x})} \sum_{v \in S} \mathcal{Q}(R, v) \geq \sum_{i=1}^{\tilde{k}} \mathcal{Q}(R_i) \geq \tilde{L}(S)$. □

Notice that the validity of the lower bound in Claim 5 also follows directly from monotonicity.

We can now state more precisely the simple observation made in the beginning of Section 4.3: although the value $\tilde{\mathcal{L}}(S)$ in Claim 5 may not always yield a recourse lower bound if $\mathcal{Q}$ has no monotonicity/weakly superadditive property, inequalities (10) are always valid for $\mathcal{F}(\mathcal{Q}, \mathcal{X}, \Omega)$ (since they are ILS cuts).

4.4 Summary

Table 1 summarizes the results from this section. For each derived cut, column RLB indicates the recourse lower bound used, column VALID refers to the conditions that the disaggregation or the set $\mathcal{X}$ must satisfy for the cut to be valid for $\mathcal{F}(\mathcal{Q}, \mathcal{X}, \Omega)$, and column FORMULATE indicates the conditions under which the inequalities are sufficient to describe $\text{EPI}(\mathcal{Q}, \mathcal{X})$ after projection (if such conditions are known). When RLB is listed as “Generic”, it means that no recourse lower bound has been specified (except that it satisfies Definition 4).

CUT	RLB	CONDITION	
		VALID	FORMULATE
(2)	$\mathcal{Q}(\bar{x})$	$\mathcal{X} \subseteq \{x : x(\delta(0)) = 2k\}$	$\mathcal{X} \subseteq \{x : x(\delta(0)) = 2k\}$
(3)	$\mathcal{Q}(R)$	Any	Route disjoint (Def. 7)
(4)	Generic	Any	-
(5)	Generic	Any	-
(6)	$\mathcal{L}(H)$ (Def. 11)	Assumption 2	-
(7)	$\mathcal{Q}(R)$	Assumption 2 and Def. 12	Assumption 2 and Def. 12
(9)	$\mathcal{L}(H)$ (Def. 11)	Assumption 2 and Def. 12	-
(10)	Generic	$\mathcal{X} \cap \mathbb{Z}^E \subseteq \{x : x(S) \leq \lvert S \rvert - \tilde{k}\}$	-

Table 1: Summary of the cuts from Section 4.

5 Application to the VRPSD with scenarios under the classical recourse policy

In this section, we apply our framework to the VRPSD with scenarios under the classical recourse policy, i.e., problem $\text{VRPR}(\mathcal{Q}_C, \mathcal{X}_{\text{CVRP}})$ defined in Section 2.2. As before, we assume that we have a disaggregation of $\mathcal{Q}_C$ that satisfies Assumption 2. Following the "recipe" in Section 3.3, we immediately obtain an algorithm for solving $\text{VRPR}(\mathcal{Q}_C, \mathcal{X}_{\text{CVRP}})$ using either the cuts of Gendreau et al. (1995) (Corollary 1) or the ILS route cuts (Theorem 2). The former approach is not expected to perform well, given the superior performance of recent algorithms relying on ILS cuts that use recourse disaggregation (Hoogendoorn and Spliet, 2023; Parada et al., 2024; Legault et al., 2025). Therefore, our branch-and-cut algorithm captures the recourse cost using the ILS route cuts, and we concentrate here on developing recourse lower bounds for the additional ILS cuts introduced in Section 4.

In fact, since the recourse function $\mathcal{Q}_C$ is not weakly superadditive (see Appendix G), the ILS path cuts (7) cannot be applied. Instead, Section 5.1 derives partial route lower bounds (Definition 11) and recourse lower bounds for the ILS set cuts (10). These bounds are then incorporated into our separation routine in Section 5.2.

5.1 Deriving recourse lower bounds by counting failures

The derivation of our recourse lower bounds depends on counting the number of times that a vehicle exceeds the demand capacity in each scenario. This motivates the next definition.

Definition 15. Let $\vec{R} = (v_1, \ldots, v_\ell)$ be a directed route and let $j \in [\ell]$. The *number of failures observed by customer v_j in $\vec{R}$* with respect to scenario $\xi \in [N]$ is given by the expression

$$\sum_{t=1}^{\infty} \mathbb{I}\left(\sum_{i\in[j-1]} d^\xi(v_i) \leq tC < \sum_{i\in[j]} d^\xi(v_i)\right).$$

When the scenario is clear from context, we simply say that the above expression gives the number of failures observed by v_j in $\vec{R}$.

Given a scenario $\xi \in [N]$, we extend Definition 15 to undirected routes and solutions in $\mathcal{X}_{\text{CVRP}} \cap \mathbb{Z}^E$ in a natural way: for an undirected route R, the *number of failures observed by $v \in V_+(R)$ in R* is the same as the number of failures observed by v in $\vec{R}$, where $\mathcal{Q}_C(R) = \mathcal{Q}_C(\vec{R})$ (ties are broken in an arbitrary but fixed way); similarly, the *number of failures observed by $v \in V_+$ in $\bar{x} \in \mathcal{X}_{\text{CVRP}} \cap \mathbb{Z}^E$* is the number of failures observed by v in $R \in \mathcal{R}(\bar{x})$, where R is the route in $\mathcal{R}(\bar{x})$ that contains v.

Our first step is to derive a closed-form expression for the number of failures that a set of customers may observe in a given scenario. In this sense, we define a function $\text{FAIL}_\xi(\alpha, S)$ that counts the number of failures observed at $S \subseteq V_+$ (with respect to scenario ξ) by a vehicle that has already loaded a certain demand value $\alpha \in \mathbb{R}_+$. Formally, for any $\alpha \in \mathbb{R}_+$, define $r(\alpha) := \alpha - C\lfloor \alpha/C \rfloor$ (note that $r(\alpha) \in [0, C)$), and for each $\xi \in [N]$ and $S \subseteq V_+$, define

$$\text{FAIL}_\xi(\alpha, S) := \begin{cases} \left(\left\lceil \frac{d^\xi(S)}{C} \right\rceil - 1\right)^+, & \text{if } \alpha = 0, \\ \left\lceil \frac{r(\alpha)+d^\xi(S)}{C} \right\rceil - \left\lceil \frac{r(\alpha)}{C} \right\rceil, & \text{otherwise,} \end{cases} \tag{FAIL_ξ}$$

where $(\,\cdot\,)^+ := \max\{0, \cdot\}$. We prove the following lemma in Appendix I.

Lemma 2. *Let $S = \{v_1, \dots, v_\ell\} \subseteq V_+$ and $\alpha \in \mathbb{R}_+$. For every $\xi \in [N]$,*

$$\sum_{j\in[\ell]} \sum_{t=1}^{\infty} \mathbb{I}\left(\alpha + \sum_{i\in[j-1]} d^\xi(v_i) \le tC < \alpha + \sum_{i\in[j]} d^\xi(v_i)\right) = \text{FAIL}_\xi\,(\alpha, S)\,. \tag{11}$$

It follows from Lemma 2 that, for every customer $v \in V_+$ and scenario $\xi \in [N]$, we may assume $d^\xi(v) \le C$, since otherwise v observes at least $\lceil d^\xi(v)/C \rceil - 1$ failures in any solution. So we can preprocess the instance by decrementing $d^\xi(v)$ by qC, with $q = \lceil d^\xi(v)/C \rceil - 1$, and adding a constant term of $q(2c_{0v}p_\xi)$ to the objective function. This motivates the next assumption.

Assumption 3. For every scenario $\xi \in [N]$ and customer $v \in V_+$, $d^\xi(v) \le C$.

Let $S = \{v_1, \dots, v_\ell\} \subseteq V_+$ and let $v'_1, \dots, v'_\ell$ be a reordering of its elements such that $c_{0v'_1} \le \cdots \le c_{0v'_\ell}$. We define the following function, which, as we shall demonstrate, represents a lower bound on the recourse cost of serving S with $\nu \in \mathbb{Z}_{++}$ vehicles, given that $\alpha \in \mathbb{R}_+$ demand has already been collected:

$$\mathcal{L}^\nu_\xi(\alpha, S) := \sum_{j\in[\text{FAIL}_\xi(\alpha,S)-\nu+1]} 2\, c_{0v'_j}. \tag{$\mathcal{L}^\nu_\xi$}$$

Since Assumption 3 implies that, in every scenario, any customer observes at most one failure, we apply Lemma 2 to prove the following lower bound.

Corollary 4. *Suppose that customer demands satisfy Assumption 3. For every route $R = (v_1, \dots, v_\ell)$, scenario $\xi \in [N]$ and $\alpha \in \mathbb{R}_+$,*

$$\sum_{j\in[\ell]} 2c_{0v_j} \sum_{t=1}^{\infty} \mathbb{I}\left(\alpha + \sum_{i\in[j-1]} d^\xi(v_i) \le tC < \alpha + \sum_{i\in[j]} d^\xi(v_i)\right) \ge \mathcal{L}^1_\xi\,(\alpha, V_+(R))\,. \qquad \square$$

Corollary 4 allows us to construct a partial route lower bound (Definition 11) in a similar way to the basic approach adopted in (Hjorring and Holt, 1999; Laporte et al., 2002; Jabali et al., 2014).

Proposition 1. *Suppose that customer demands satisfy Assumption 3. For every partial route $H = (S_1, \dots, S_\ell)$, define*

$$\vec{\mathcal{L}}_C(H) := \sum_{\xi=1}^{N} p_\xi \sum_{j\in[\ell]} \mathcal{L}^1_\xi\left(\sum_{i\in[j-1]} d^\xi(S_i), S_j\right),$$

and $\overleftarrow{\mathcal{L}}_C(H) := \vec{\mathcal{L}}_C(H')$, where $H' = (S_\ell, \dots, S_1)$. Then $\mathcal{L}_C(H) := \min\{\vec{\mathcal{L}}_C(H), \overleftarrow{\mathcal{L}}_C(H)\}$ is a partial route lower bound with respect to $\mathcal{Q}_C$.

Proof. Fix $\xi \in [N]$ and let $\vec{R}$ be a directed route that exactly adheres to H. Let $j \in [\ell]$ and define $h_j := \sum_{i=1}^{j} |S_i|$. By Definition 9, we may write $\vec{R} = (v_1, \dots, v_{h_\ell})$ where, for every $j \in [\ell]$, $S_j = \{v_{h_{j-1}+1}, \dots, v_{h_j}\}$. By Corollary 4, for each $j \in [\ell]$,

$$\sum_{q=h_{j-1}+1}^{h_j} 2c_{0v_q} \sum_{t=1}^{\infty} \mathbb{I}\left(\sum_{i\in[q-1]} d^\xi(v_i) \le tC < \sum_{i\in[q]} d^\xi(v_i)\right) \ge \mathcal{L}^1_\xi\left(\sum_{i\in[h_{j-1}]} d^\xi(v_i), S_j\right).$$

This shows $\vec{\mathcal{L}}_C(H) \le \mathcal{Q}(\vec{R})$. A symmetrical reasoning yields $\overleftarrow{\mathcal{L}}_C(H) \le \mathcal{Q}(\overleftarrow{R})$, concluding the proof. $\square$

Lastly, we show in Appendix J that, assuming that the disaggregation of $\mathcal{Q}_C$ is according to Remark 1, Lemma 2 can also be used to derive recourse lower bounds for the ILS set cuts presented in Section 4.3.

Proposition 2. *Suppose that customer demands satisfy Assumption 3 and assume that the disaggregation of $\mathcal{Q}_C$ is as described in Remark 1. Let $S \subseteq V_+$ and $\tilde{k} \in \mathbb{Z}_{++}$ be such that $x(S) \leq |S| - \tilde{k}$ is valid for $\mathcal{X} \cap \mathbb{Z}^E$. Then $\mathcal{L}_C(S, \tilde{k}) := \sum_{\xi=1}^{N} p_\xi \mathcal{L}_\xi^{\tilde{k}}(0, S)$ is a recourse lower bound with respect to S and $\mathcal{X}(S, \tilde{k})$, that is, for every $\bar{x} \in \mathcal{X}(S, \tilde{k})$,*

$$\sum_{R \in \mathcal{R}(\bar{x})} \sum_{v \in S} \mathcal{Q}_C(R, v) \geq \mathcal{L}_C(S, \tilde{k}).$$

5.2 Separation routine

Our branch-and-cut algorithm for problem $\mathrm{VRPR}(\mathcal{Q}_C, \mathcal{X}_{\mathrm{CVRP}})$ uses two families of ILS cuts: the ILS PR-EA cuts (4) and the ILS set cuts (10). Note that the ILS set cuts may not be valid if we use the same recourse disaggregation $\mathcal{Q}_C$ as in Hoogendoorn and Spliet (2023) (see Appendix H for an example), so we only apply these cuts when using the disaggregation described in Remark 1. To refer more easily to the different choices of recourse disaggregation, we introduce a parameter $D \in \{1, 2\}$, where $D = 1$ corresponds to the disaggregation of Hoogendoorn and Spliet (2023) (recall Remark 5), while $D = 2$ corresponds to the disaggregation described in Remark 1. Note that in both cases, the disaggregation is along $\Omega = V_+$.

Given a candidate solution $(\bar{x}, \bar{\theta}) \in \mathbb{R}_+^E \times \mathbb{Q}_+^{V_+}$, we separate valid inequalities for the feasible region $\mathcal{F}(\mathcal{Q}_C, \mathcal{X}_{\mathrm{CVRP}})$ as follows. We first call the CVRPSEP package (Lysgaard et al., 2004) to separate violated RCIs and get a corresponding family of customer sets $\mathcal{S} \subseteq 2^{V_+}$. If $\mathcal{S} \neq \emptyset$ and $D = 2$, we also consider adding the ILS set cuts $\theta(S) \geq \mathcal{L}_C(S, \tilde{k}) \cdot W_P(x; \mathcal{X}(S, \tilde{k}))$, for each $S \in \mathcal{S}$. Otherwise, we call Algorithm 1 to generate a collection of partial routes $\mathcal{H}$. For each partial route $H \in \mathcal{H}$, we try to separate valid inequalities in the following way: if $D = 2$, we test both the ILS set cut with $S = V_+(H)$ and the ILS PR-EA cut $\theta(V_+(H)) \geq \mathcal{L}_C(H) \cdot W(x; \mathcal{X}_=(H))$; if instead $D = 1$, then the ILS set cuts may not be valid, so we only separate the ILS PR-EA cut $\theta_v \geq \mathcal{L}_C(H) \cdot W(x; \mathcal{X}_=(H))$, where v is the customer in H with the smallest index.

Observe that when $D = 2$ and $H \in \mathcal{H}$ corresponds to a route R, we could instead have used the ILS route cut $\theta(\Omega(R)) \geq \mathcal{L}_C(H) \cdot W_{OF}(x; \mathcal{X}_=(H))$. However, preliminary experiments suggest no benefit in doing so. A detailed description of the separation routine is given in Appendix K.

6 Computational study

We conduct computational experiments on the VRPSD with scenarios under the classical recourse policy, that is, the problem $\mathrm{VRPR}(\mathcal{Q}_C, \mathcal{X}_{\mathrm{CVRP}})$ defined in Section 2.2 and further discussed in Section 5. All algorithms were executed in single-thread mode on a machine with an Intel(R) Xeon(R) Gold 6142 CPU @ 2.60GHz processor. We implemented the algorithms in C++, using Gurobi 12 as the LP/MIP solver and the Lemon graph library (Dezsho et al., 2011). The time limit for each run was set to 1800 seconds.

We consider two benchmark sets of instances. The first set is based on the 270 instances proposed by Jabali et al. (2014) for a VRPSD problem under the classical recourse policy and with demands following independent normal distributions. For each instance, we generated 200 scenarios by sampling from these distributions and rounding the demand values to the nearest integer in the interval $[0, C]$. The second benchmark set consists of the 20 instances originally proposed for the chance-constrained vehicle routing problem by Dinh et al. (2018). For each instance, they generated 200 demand scenarios by sampling from a multivariate normal distribution with correlations between customers proportional to their distances (see Appendix 2 of Dinh et al. (2018) for details).

The code, benchmark sets, and tables containing detailed computational results for each instance were submitted jointly with the paper.

6.1 Numerical results

Recall from Section 5.2 that $D = 1$ corresponds to the recourse disaggregation in Remark 5, while $D = 2$ corresponds to the recourse disaggregation in Remark 1. We designed our experiments to address two main questions:

Q1. Is there a computational advantage on using the improved activation functions from Section 4.2.3?

Q2. How do the recourse disaggregations $D = 1$ and $D = 2$ compare? In particular, since set cuts are only valid for $D = 2$, what is their impact on the overall execution time of the algorithm?

Activation functions and separation heuristic for partial route inequalities. To address Q1, we test two algorithms with $D = 1$ that differ only in the activation functions used in the ILS PR-EA cuts: algorithm PR+W_{HS} uses $W_{HS}(x, \mathcal{X}_=(H))$, while algorithm PR+$W_{OF}$ uses $W_{OF}(x; \mathcal{X}_=(H))$ (both do not use the ILS set cuts (10)). Their performance is very similar: PR+W_{HS} solves 252 out of 290 instances, while PR+W_{OF} solves 251. Examining the ILS-PR EA inequalities added by both algorithms, we noticed that, for most partial routes $H = (S_1, \ldots, S_\ell)$ with $\ell \geq 2$, both S_2 and $S_{\ell-1}$ are singletons, in which case the two activation functions $W_{HS}(x, \mathcal{X}_=(H))$ and $W_{OF}(x; \mathcal{X}_=(H))$ coincide. We thus conclude that using the theoretically stronger activation function $W_{OF}(x; \mathcal{X}_=(H))$ (see Claim 3) does not provide a practical computational benefit. Further results comparing both algorithms can be found in the tables submitted with the paper.

Different choices of recourse disaggregation. Given the previous results, we label PR*+$D1$ the best-performing variant for question Q1, that is, PR*+$D1$ = PR+W_{HS}. To address Q2, we then evaluate two other variants of PR*+$D1$:

- PR*+$D2$: identical to PR*+$D1$, but uses the disaggregation $D = 2$ (and do not use ILS set cuts);
- PR*+$D2$+SET: extends PR*+$D2$ with the ILS set cuts $\theta(S) \geq \mathcal{L}_C(S) \cdot W_P(x; \mathcal{X}(S, \tilde{k}))$ discussed in Section 5.2.

The results of our experiments are summarized in Figures 2a and 2b, which are similar to the standard performance profile graphs of Dolan and Moré (2002). Each of these figures have two parts: the left part shows the execution time, and the right part shows the final optimality gaps (with respect to the best primal solution found by all algorithms). In the left part, a point $(p_1, p_2) \in \mathbb{R}^2$ on the curve of an algorithm indicates that p_2% of all instances were solved by the algorithm within p_1 seconds. In the right part, a point (p'_1, p'_2) means that, for a given algorithm variant, p'_2% of the instances had a final optimality gap of at most p'_1% (this gap is always computed with respect to the best solution found by all algorithms). Additionally, Table 2 shows the number of instances solved by each algorithm variant in the time limit of 1800 seconds.

We first note that the ILS PR-EA cuts with $D = 1$ dominate those with $D = 2$. To see this, consider a partial route H and recall from Section 5.2 that with $D = 2$ we use the inequality

$$\theta(V_+(H)) \geq \mathcal{L}_C(H) \cdot W_{HS}(\bar{x}, \mathcal{X}_=(H)), \tag{12}$$

while with $D = 1$ we use

$$\theta_v \geq \mathcal{L}_C(H) \cdot W_{HS}(\bar{x}, \mathcal{X}_=(H)), \tag{13}$$

where v is the customer in H with the smallest index. Hence, any $(\bar{x}, \bar{\theta})$ that satisfies (13) also satisfies (12). However, Figure 2 (and Table 2) suggest that this dominance has limited impact on the overall algorithm performance.

In contrast, our results show that adding ILS set cuts significantly improves performance, indicating that it is better to select recourse disaggregations that allow additional valid inequalities — such as the ILS set cuts (10) — rather than using the previous dominance argument. This reinforces our discussion in Section 3.1 on the importance of explicitly defining the disaggregation and the feasibility region $\mathcal{F}(\mathcal{Q}, \mathcal{X}, \Omega)$. In doing so, we leveraged the results from Section 4 to safely combine the ILS PR-EA and the ILS set cuts. In particular, our approach allowed us to use ILS set cuts to solve the VRPR problem with a recourse function that is not weakly superadditive (Definition 14), which is something that has not been done before.

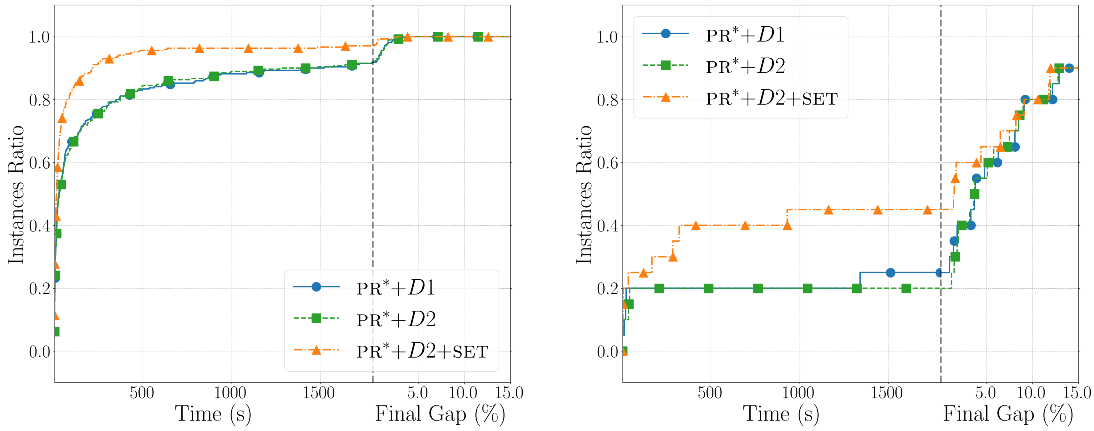
Table 2 confirms that PR*+$D$2+SET solves more instances to optimality on both benchmark sets. The previous ILS-based state-of-the-art algorithm for recourse functions that are not weakly superadditive is due to Hoogendoorn and Spliet (2023), which solves 246 instances from the benchmark set of Jabali et al. (2014) within one hour, assuming that demands are independent and normally distributed. While our experiments use different probability distributions, the scenarios were sampled from the same normal distributions, making our instances closely related to those used by Hoogendoorn and Spliet (2023). Moreover, our approach can approximate a much broader class of probability distributions, and we solve 263 of the scenario-based instances (adapted from Jabali et al. (2014)) within a shorter time limit. Finally, we remark that our approach is the only exact algorithm currently available that can handle the correlations present in the instances of Dinh et al. (2018).

Instance Set	# Instances	PR*+$D$1	PR*+$D$2	PR*+$D$2+SET
Jabali et al. (2014)	270	247	248	262
Dinh et al. (2018)	20	5	4	9

Table 2: Total number of instances solved by each algorithm within the time limit of 1800 seconds.

7 Concluding remarks

In this work, we proposed a framework for integer L-shaped (ILS) cuts that tries to explicitly identify common themes in previous ILS-based formulations. In particular, our results clarify the assumptions required to derive each component of specific ILS cuts. This generic perspective not only allowed us to generalize previous ILS cuts, but



(a) Results for Jabali et al. (2014) instances. (b) Results for Dinh et al. (2018) instances.

Figure 2: Comparison of the different ILS branch-and-cut algorithms.

also enabled capabilities that were not possible before. For instance, we can now apply simultaneously cuts that could not be combined without our generalization. In addition, our framework allowed us to design a new ILS-based branch-and-cut approach for the VRPSD with demands given by scenarios, overcoming a well-known difficulty in the field: dealing with correlations.

We believe that future research on ILS-based methods for the VRPSD should try to follow our framework as much as possible. Specifically, future work should, if possible, identify the recourse disaggregation, feasible region, activation functions, and recourse lower bounds. Doing so would make these approaches applicable to a wider variety of problem variations, thus broadening their benefits to the OR community.

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A Globally valid recourse lower bounds

Suppose that $\mathtt{LB} \in \mathbb{Q}_+^{\Omega}$ is a vector of *globally valid recourse lower bounds*, that is, for every $\omega \in \Omega$, we have $\sum_{R\in\mathcal{R}(\bar{x})} \mathcal{Q}(R,\omega) \geq \mathtt{LB}_\omega$, for all $\bar{x} \in \mathcal{X} \cap \mathbb{Z}^E$. Define the *translated feasible region*

$$\tilde{\mathcal{F}} := \mathcal{F}(\hat{\mathcal{Q}}, \mathcal{X}, \Omega) - \{\mathtt{LB}\} = \{(x, \theta - \mathtt{LB}) : (x,\theta) \in \mathcal{F}(\hat{\mathcal{Q}}, \mathcal{X}, \Omega)\}$$

and observe that

$$\min\{c^\intercal x + \mathbb{1}^\intercal \theta : (x,\theta) \in \mathcal{F}(\hat{\mathcal{Q}}, \mathcal{X}, \Omega)\} = \mathbb{1}^\intercal \mathtt{LB} + \min\{c^\intercal x + \mathbb{1}^\intercal \tilde{\theta} : (x,\tilde{\theta}) \in \tilde{\mathcal{F}}\}.$$

So we can solve the original problem $\textsc{vrpr}(\mathcal{Q}, \mathcal{X})$ by optimizing over $\tilde{\mathcal{F}}$.

The advantage of doing this translation step is that it may allow us to improve the coefficients in an ILS cut. Specifically, if we are given an ILS cut

$$\theta(U) \geq \mathcal{L}(U, \mathcal{X}') \cdot W(x; \mathcal{X}'), \tag{14}$$

then the inequality

$$\tilde{\theta}(U) \geq (\mathcal{L}(U, \mathcal{X}') - \mathtt{LB}(U)) \cdot W(x; \mathcal{X}') \tag{15}$$

is valid for $\tilde{\mathcal{F}}$. Moreover, if $(x', \tilde{\theta}')$ satisfies $W(x'; \mathcal{X}') \leq 1$ and (15), then $(x', \tilde{\theta}' + \texttt{LB})$ satisfies the ILS cut (14), since

$$\begin{aligned}\tilde{\theta}'(U) + \texttt{LB}(U) &\geq \texttt{LB}(U) + (\mathcal{L}(U, \mathcal{X}') - \texttt{LB}(U)) \cdot W(x'; \mathcal{X}') \\ &\geq (\texttt{LB}(U) + \mathcal{L}(U, \mathcal{X}') - \texttt{LB}(U)) \cdot W(x'; \mathcal{X}') \\ &= \mathcal{L}(U, \mathcal{X}') \cdot W(x'; \mathcal{X}').\end{aligned}$$

Therefore, given a family of ILS cuts $\mathcal{C}$, rather than using Formulation ILS($\mathcal{X}, \mathcal{C}, \Omega$), it might be beneficial to instead use the translated formulation:

$$\begin{aligned}\min \quad & c^\intercal x + \mathbb{1}^\intercal \tilde{\theta}, \\ \text{s.t.} \quad & x \in \mathcal{X} \cap \mathbb{Z}^E, \\ & \tilde{\theta}(U) \geq (\mathcal{L}(U, \mathcal{X}') - \texttt{LB}(U)) \cdot (\alpha^\intercal x + \beta), \qquad \forall (U, \mathcal{X}', \mathcal{L}(U, \mathcal{X}'), \alpha, \beta) \in \mathcal{C}, \\ & \tilde{\theta} \in \mathbb{R}^\Omega_+.\end{aligned}$$

We remark, however, that for VRPSDs, globally valid recourse lower bounds are usually only available when no disaggregation is used, that is, when $\Omega = \{\hat{\omega}\}$. In this setting, Laporte et al. (2002) (see also Jabali et al. (2014); Salavati-Khoshghalb et al. (2019a); Hoogendoorn and Spliet (2023)) replace the ILS cuts of Gendreau et al. (1995) with the inequalities

$$\tilde{\theta}_{\hat{\omega}} = \theta_{\hat{\omega}} - \texttt{LB}_{\hat{\omega}} \geq (\mathcal{Q}(\bar{x}) - \texttt{LB}_{\hat{\omega}}) \cdot W^k_G(x; \{\bar{x}\}).$$

On the other hand, when recourse disaggregation is used (e.g., $\Omega = V_+$), globally valid recourse lower bounds are typically not available. Still, ILS algorithms that use recourse disaggregation without globally valid recourse lower bounds tend to outperform algorithms that use such lower bounds but do not use recourse disaggregation (for instance, see the experiments in Hoogendoorn and Spliet (2023)).

B Proof of Theorem 3

To prove Theorem 3, we first show Lemma 3 below. Intuitively, item (i) of Lemma 3 states that, although $\bar{x}(H) = |H| - 1$ does not guarantee $\bar{x} \in \mathcal{X}_{\supseteq}(H)$, the singletons in H gives a strong ordering property on a directed route $\vec{R}'$ corresponding to a subroute $R' \subseteq R \in \mathcal{R}(\bar{x})$ that visits exactly the customers in H. Item (ii) of Lemma 3 then shows that, if $\vec{R}'$ starts by visiting all vertices in S_1 and ends by visiting all vertices in S_ℓ, then $\bar{x} \in \mathcal{X}_{\supseteq}(H)$.

Lemma 3. *Let $H = (S_1, \ldots, S_\ell)$ be a partial route and, for each $i \in [\ell]$, let $h_i = \sum_{j=1}^{i} |S_j|$. Let $\bar{x} \in \mathcal{X} \cap \mathbb{Z}^E$ be such that $\bar{x}(H) = |H| - 1$. Then there exists $R \in \mathcal{R}(\bar{x})$, $R' \subseteq R$ and a corresponding directed route $\vec{R}' = (v_1, \ldots, v_\ell)$ satisfying the following*

(i) $V_+(R') = V_+(H)$ and, for every $i \in [\ell] \setminus \{1, \ell\}$ and $|S_i| = 1$, we have

$$S_1 \cup \ldots \cup S_i = \{v_1, \ldots, v_{h_i}\} \text{ and } S_i = \{v_{h_i}\}; \text{ and}$$

(ii) if $S_1 = \{v_1, \ldots, v_{h_1}\}$, $S_\ell = \{v_{h_{\ell-1}+1}, \ldots, v_{h_\ell}\}$ and $\vec{R}'$ satisfies item (i), then $\vec{R}'$ exactly adheres to H.

Proof. We first observe that, since $\bar{x}$ satisfies the subtour elimination constraints, the condition $\bar{x}(H) = |H| - 1$ implies that the subgraph of $G(\bar{x})$ induced by the vertices in H is a tree spanning $V_+(H)$. Furthermore, by the degree constraints in $\mathcal{X}$, this subgraph must be a path P. Let R' denote the subroute corresponding to P, that is, $P = R' - \{0\}$. Since R' is a subroute of a route in $\mathcal{R}(\bar{x})$ and $V_+(R') = V_+(H)$, there is nothing else to show if $\ell \leq 2$ (for both parts (i) and (ii)). We thus assume in the rest of the proof that $\ell \geq 3$.

To prove (i), let i be the first index in $\{2, \ldots, \ell-1\}$ such that $|S_i| = 1$ (if no such index exists, then we are done). Let $v_{i'}$ be the unique vertex in S_i, where the subindex i' refers to the position of customer $v_{i'}$ in some orientation of the path P. By the definition of partial routes (Definition 8), $v_{i'}$ is a cut vertex in H separating $S_1 \cup \ldots \cup S_{i-1}$ and $S_{i+1} \cup \ldots \cup S_\ell$. Since P is a path in H covering all of its vertices, $v_{i'}$ is also a cut vertex in P separating $\{v_1, \ldots, v_{i'-1}\}$ and $\{v_{i'+1}, \ldots, v_\ell\}$. Therefore, as $V_+(H) = \{v_1, \ldots, v_{h_\ell}\}$, either $v_{i'} = v_{h_i}$ and $S_1 \cup \ldots \cup S_i = \{v_1, \ldots, v_{h_i}\}$, or $v_{i'} = v_{\ell-h_i+1}$ and $S_1 \cup \ldots \cup S_i = \{v_{\ell-h_i+1}, \ldots, v_\ell\}$. We thus choose $\vec{R}'$ so that $v_{i'} = v_{h_i}$ and $S_1 \cup \ldots \cup S_i = \{v_1, \ldots, v_{h_i}\}$. Now suppose that $j \in [\ell-1] \setminus [i]$ is another index satisfying the conditions in the statement and let $S_j = \{v_{j'}\}$. Again, $v_{j'}$ is a cut vertex in P separating $S_1 \cup \ldots \cup S_{j-1}$ and $S_{j+1} \cup \ldots \cup S_\ell$. Since $v_{i'} = v_{h_i}$, we now have that $\{v_1, \ldots, v_{h_i}\} \subseteq S_1 \cup \ldots \cup S_i \cup \ldots \cup S_{j-1}$, which implies that $v_{j'} = v_{h_j}$ and $S_1 \cup \ldots \cup S_j = \{v_1, \ldots, v_{h_j}\}$, concluding the proof of (i).

For item (ii), we assume that $S_1 = \{v_1, \ldots, v_{h_1}\}$ and $S_\ell = \{v_{h_{\ell-1}+1}, \ldots, v_{h_\ell}\}$, and we show that $S_i = \{v_{h_{i-1}+1}, \ldots, v_{h_i}\}$, for every $i \in \{2, \ldots, \ell-1\}$. Note that if $\ell = 3$, $V_+(H) = \{v_1, \ldots, v_\ell\}$ already yields $S_2 = V_+(H) \setminus (S_1 \cup S_3) = \{v_{h_1+1}, \ldots, v_{h_2}\}$, so we assume $\ell \geq 4$.

Let $i_1 < \ldots < i_t$ denote the indices $i \in \{2, \ldots, \ell-1\}$ such that $|S_i| = 1$ (since $\ell \geq 4$, at least one such index is guaranteed to exist). For convenience, set $i_0 = 1$. We first show that, for every $q \in [t]$, it holds that $S_i = \{v_{h_{i-1}+1}, \ldots, v_{h_i}\}$, for all $i \in [i_q] \setminus [i_{q-1}]$. By the definition of partial routes (and how we set i_0), $i_q \leq i_{q-1} + 2$. Item (i) implies that $S_{i_q} = \{v_{h_{i_q}}\}$, so we may assume that $i_q = i_{q-1} + 2$ and it suffices to prove that $S_{i_q-1} = \{v_{h_{(i_q-2)}+1}, \ldots, v_{h_{(i_q-1)}}\}$. To ease notation, set $i = i_{q-1}+1$, meaning that $i_q = i+1$ and $i_{q-1} = i-1$. Since either $i = 2$ or $|S_{i-1}| = 1$, item (i) (and the assumption that $S_1 = \{v_1, \ldots, v_{h_1}\}$) implies that $S_1 \cup \ldots \cup S_{i-1} = \{v_1, \ldots, v_{h_{i-1}}\}$. Additionally, as $S_{i+1} = \{v_{h_{i+1}}\}$, item (i) also gives $S_1 \cup \ldots \cup S_i = \{v_1, \ldots, v_{h_i}\}$. Hence, $S_i = (S_1 \cup \ldots \cup S_i) \setminus (S_1 \cup \ldots \cup S_{i-1}) = \{v_{h_{i-1}+1}, \ldots, v_{h_i}\}$.

The previous argument shows that $S_i = \{v_{h_{i-1}+1}, \ldots, v_{h_i}\}$, for all $i \in [i_t]$. Thus, we are done if $i_t = \ell - 1$. Otherwise, the definition of partial routes (and the fact that $\ell \geq 4$) implies that $i_t = \ell - 2$. Then $S_1 \cup \ldots \cup S_{\ell-2} = \{v_1, \ldots, v_{h_{\ell-2}}\}$ implies $S_{\ell-1} \cup S_\ell = V_+(R') \setminus (S_1 \cup \ldots \cup S_{\ell-2}) = \{v_{h_{\ell-2}+1}, \ldots, v_{h_\ell}\}$, and since $S_\ell = \{v_{h_{\ell-1}+1}, \ldots, v_{h_\ell}\}$, we get $S_{\ell-1} = (S_{\ell-1} \cup S_\ell) \setminus S_\ell = \{v_{h_{\ell-2}+1}, \ldots, v_{h_{\ell-1}}\}$. □

Theorem 3. *For every partial route $H = (S_1, \ldots, S_\ell)$, $W_{OF}(x; \mathcal{X}_{\supseteq}(H))$ is an activation function.*

Proof. Fix a partial route $H = (S_1, \ldots, S_\ell)$ and $\bar{x} \in \mathcal{X} \cap \mathbb{Z}^E$. We first show that if $\bar{x} \in \mathcal{X}_{\supseteq}(H)$, then $W_{OF}(\bar{x}; \mathcal{X}_{\supseteq}(H)) = 1$. Since $\bar{x}$ satisfies the SECs, we know that, except for the leading one, all the terms in $W_{OF}(\bar{x}; \mathcal{X}_{\supseteq}(H))$ are nonpositive. Since $G(\bar{x})$ contains a path spanning the vertices in H, we have that $\bar{x}(H) = |H| + 1$. In a similar way, if $\ell \geq 2$, then it follows from the definition of $\mathcal{X}_{\supseteq}(H)$ and the notion of adherence (Definition 10), that $G(\bar{x})$ also contains connected subgraphs spanning S_2 and $S_{\ell-1}$, so $\bar{x}(S_2) = |S_2| - 1$ and $\bar{x}(S_{\ell-1}) = |S_{\ell-1}| - 1$. This proves that $W_{OF}(\bar{x}; \mathcal{X}_{\supseteq}(H)) = 1$.

For the converse, we assume $W_{OF}(\bar{x}; \mathcal{X}_{\supseteq}(H)) = 1$ and we show that $\bar{x} \in \mathcal{X}_{\supseteq}(H)$. Observe that $W_{OF}(\bar{x}; \mathcal{X}_{\supseteq}(H)) = 1$ implies that, except for the leading one, every term in $W_{OF}(\bar{x}; \mathcal{X}_{\supseteq}(H))$ evaluates to zero. In particular, $\bar{x}(H) = |H| - 1$, so we choose $R' \subseteq R \in \mathcal{R}(\bar{x})$ and a corresponding directed route $\vec{R}' = (v_1, \ldots, v_{h_\ell})$ according to item (i) of Lemma 3. If both $\vec{R}'$ and $\overleftarrow{R}'$ satisfy item (i), we choose $\vec{R}'$ so that $|\{v_1\} \cap S_1|$ is maximum. Applying item (ii) of the same lemma it then suffices to show that $S_1 = \{v_1, \ldots, v_{h_1}\}$ and $S_\ell = \{v_{h_{\ell-1}+1}, \ldots, v_{h_\ell}\}$. We proceed by case analysis on $\ell \geq 2$ (the case $\ell = 1$ follows from $V_+(R') = V_+(H)$).

Case $\ell = 2$: By the definition of partial routes (Definition 8), we may assume without loss of generality that S_1 is a singleton. Note that $\bar{x}(H) = |H| - 1$ implies $\bar{x}(S_1, S_2) \geq 1$. Moreover, since $\bar{x}(S_2) = |S_2| - 1$ (by $W_{OF}(\bar{x}; \mathcal{X}_{\supseteq}(H)) = 1$), we apply the subcycle elimination constraints to $S_1 \cup S_2$ to learn that $\bar{x}(S_1, S_2) \leq 1$. Therefore, $\bar{x}(S_1, S_2) = 1$, meaning that the single vertex in S_1 is either v_1 of v_{h_ℓ}. Thus, by the choice of $\vec{R}'$, $S_1 = \{v_1\}$ and $S_2 = \{v_2, \ldots, v_{h_2}\}$.

Case $\ell \geq 3$: We only show $S_1 = \{v_1, \ldots, v_{h_1}\}$, as the argument for S_ℓ follows symmetrically. If S_2 is a singleton, item (i) of Lemma 3 gives $S_2 = \{v_{h_2}\}$ and $S_1 = \{v_1, \ldots, v_{h_1}\}$. Otherwise, by the definition of partial routes, we may write $S_1 = \{v_a\}$. Using the same argument as in the case $\ell = 2$, we learn that $\bar{x}(S_1, S_2) = 1$. Therefore, v_a has degree one in the path $P = R' - \{0\} \subseteq G(\bar{x})$, that is, v_a is either v_1 or v_{h_ℓ}. In fact, by the choice of $\vec{R}'$, we know that $v_a = v_1$. □

C Proof of Theorem 4

Theorem 4. *Let $H = (S_1, \ldots, S_\ell)$ be a partial route. Then $W_{OF}(x; \mathcal{X}_{=}(H)))$ is an activation function.*

Proof. We first show that if $\bar{x} \in \mathcal{X}_{=}(H)$, then $W_{OF}(\bar{x}; \mathcal{X}_{=}(H)) = 1$. Let $R = (v_1, \ldots, v_\ell)$ be the route in $\mathcal{R}(\bar{x})$ that exactly adheres to H. Since R also adheres to H, we know that $W_{OF}(\bar{x}; \mathcal{X}_{\supseteq}(H)) = 1$. When $\ell = 1$ it follows from the definition of exact adherence (Definition 9) that $\bar{x}(S_1) = |S_1| - 1$ and $\bar{x}(0, S_1) = 2$, meaning that $\bar{x}(0, S_1) + 2\bar{x}(S_1) - 2|S_1| = 0$. In a similar way, if $\ell \geq 2$ the definition of exact adherence implies that $\bar{x}(0, S_1) = \bar{x}(0, S_\ell) = \bar{x}(S_1, S_2) = \bar{x}(S_{\ell-1}, S_\ell) = 1$, $\bar{x}(S_1) = |S_1| - 1$ and $\bar{x}(S_\ell) = |S_\ell| - 1$, so the terms in $W_{OF}(\bar{x}; \mathcal{X}_{=}(H)) - W_{OF}(\bar{x}; \mathcal{X}_{\supseteq}(H))$ all evaluate to zero.

To prove the converse, we assume $W_{OF}(\bar{x}; \mathcal{X}_{=}(H)) = 1$ and we show that $\bar{x} \in \mathcal{X}_{=}(H)$. Since $W_{OF}(x; \mathcal{X}_{\supseteq}(H))$ is an activation function and all the terms in $W_{OF}(x; \mathcal{X}_{=}(H)) - W_{OF}(x; \mathcal{X}_{\supseteq}(H))$ are nonpositive for any $x \in \mathcal{X}$, we have that $1 = W_{OF}(\bar{x}; \mathcal{X}_{=}(H)) \leq W_{OF}(\bar{x}; \mathcal{X}_{\supseteq}(H)) \leq 1$, so $W_{OF}(\bar{x}; \mathcal{X}_{\supseteq}(H)) = 1$.

Furthermore, all the terms in $W_{OF}(\bar{x}; \mathcal{X}_{=}(H)) - W_{OF}(\bar{x}; \mathcal{X}_{\supseteq}(H))$ evaluate to zero. Theorem 3 then yields that $\bar{x} \in \mathcal{X}_{\supseteq}(H)$, i.e., there exists $R' = (v_1, \ldots, v_{|H|}) \subseteq R \in \mathcal{R}(\bar{x})$ such that R' exactly adheres to H. For convenience, assume that $v_1 \in S_1$ and $v_{|H|} \in S_\ell$. We show that $R' = R$, that is, both v_1 and $v_{|H|}$ are adjacent to the depot in R.

If $\ell = 1$, then $\bar{x}(S_1) = |S_1| - 1$ and $\bar{x}(0, S_1) + 2\bar{x}(S_1) = 2|S_1|$ imply that $\bar{x}(0, S_1) = 2$, so both v_1 and $v_{|H|}$ are adjacent to the depot in R. If $\ell \geq 2$, then the definition of exact adherence (Definition (9)) applied to R' gives that $\bar{x}(S_1) = |S_1| - 1$, $\bar{x}(S_2) = |S_2| - 1$ and $\bar{x}(S_1, S_2) \geq 1$. The SECs applied to the set $S_1 \cup S_2$ then implies that $\bar{x}(S_1, S_2) = 1$.

Hence, $\bar{x}(0, S_1) + 2\bar{x}(S_1) + \bar{x}(S_1, S_2) = 2|S_1|$ implies that $\bar{x}(0, S_1) = 1$, meaning that v_1 is adjacent to the depot in R. A symmetric argument shows that $v_{|H|}$ is also adjacent to the depot in R. □

D Comparison with the activation function of Hoogendoorn and Spliet (2023)

Let $H = (S_1, \dots, S_\ell)$ be a partial route (with $S_0 = \{0\}$ and $S_{\ell+1} = \{0\}$). Hoogendoorn and Spliet (2023) designed the following activation function.

$$W_{HS}(x; \mathcal{X}_=(H)) = \gamma + \sum_{i=1}^{\ell} \alpha_i \left(x(S_i) - (|S_i| - 1)\right) + \sum_{i=0}^{\ell} \beta_i \left(x(S_i, S_{i+1}) - 1\right), \tag{17}$$

where

$$(\alpha_1, \dots, \alpha_\ell) = \begin{cases} (3), & \text{if } \ell = 1, \\ (4, 4), & \text{if } \ell = 2, \\ (3, 2, 3), & \text{if } \ell = 3, \\ (3, 2, 1, \dots, 1, 2, 3), & \text{if } \ell \geq 4. \end{cases}$$

$$(\beta_0, \dots, \beta_\ell) = \begin{cases} (1, 0), & \text{if } \ell = 1, \\ (1, 3, 1), & \text{if } \ell = 2, \\ (1, 2, 1, \dots, 1, 2, 1), & \text{if } \ell \geq 3, \end{cases}$$

$$\gamma = \begin{cases} 0, & \text{if } \ell = 1, \\ 1, & \text{if } \ell \geq 2. \end{cases}$$

In their work, they show that, for all $x \in \mathcal{X} \cap \mathbb{Z}^E$, $W_{HS}(x; \mathcal{X}_=(H)) = 1$ if $x \in \mathcal{X}_=(H)$, and $W_{HS}(x; \mathcal{X}_=(H)) \leq 0$ otherwise.

To see that W_{HS} can be obtained from $W_{OF}(x; \mathcal{X}_=(H))$ by replacing the indicator functions in $W_{OF}(x; \mathcal{X}_\supseteq(H))$ with ones, we check the different possibilities for ℓ:

Case $\ell = 1$:

$$\begin{aligned} W_{HS}(x; \mathcal{X}_=(H)) &= (x(0, S_1) - 1) + 3(x(S_1) - (|S_1| - 1)) \\ &= 1 + (x(0, S_1) + 2\,x(S_1) - 2\,|S_1|) + (x(H) - |H| + 1) \end{aligned}$$

Case $\ell = 2$:

$$\begin{aligned} W_{HS}(x; \mathcal{X}_=(H)) &= 1 + (x(0, S_1) - 1) + 4(x(S_1) - (|S_1| - 1)) + 3(x(S_1, S_2) - 1) \\ &\quad + 4(x(S_2) - (|S_2| - 1)) + (x(0, S_2) - 1) \\ &= 1 + (x(0, S_1) + 2\,x(S_1) + x(S_1, S_2) - 2\,|S_1|) \\ &\quad + (x(0, S_2) + 2\,x(S_2) + x(S_1, S_2) - 2\,|S_2|) \\ &\quad + (x(S_1) - |S_1| + 1) + (x(S_2) - |S_2| + 1) \\ &\quad + (x(H) - |H| + 1) \end{aligned}$$

Case $\ell = 3$:

$$\begin{aligned} W_{HS}(x; \mathcal{X}_=(H)) &= 1 + (x(0, S_1) - 1) + 3(x(S_1) - (|S_1| - 1)) + 2(x(S_1, S_2) - 1) \\ &\quad + 2(x(S_2) - (|S_2| - 1)) + 2(x(S_2, S_3) - 1) \\ &\quad + 3(x(S_3) - (|S_3| - 1)) + (x(0, S_3) - 1) \\ &= 1 + (x(0, S_1) + 2\,x(S_1) + x(S_1, S_2) - 2\,|S_1|) \\ &\quad + (x(0, S_3) + 2\,x(S_3) + x(S_2, S_3) - 2\,|S_3|) \\ &\quad + (x(S_2) - |S_2| + 1) + (x(H) - |H| + 1) \end{aligned}$$

Case $\ell \geq 4$:

$$\begin{aligned}
W_{HS}(x; \mathcal{X}_{=}(H)) &= 1 + (x(0, S_1) - 1) + 3(x(S_1) - (|S_1| - 1)) + 2(x(S_1, S_2) - 1) \\
&\quad + 2(x(S_2) - (|S_2| - 1)) + (x(S_2, S_3) - 1) \\
&\quad + \sum_{i=3}^{\ell-2} (x(S_i) - |S_i| + 1) + (x(S_i, S_{i+1}) - 1) \\
&\quad + 2(x(S_{\ell-1}) - (|S_{\ell-1}| - 1)) + 2(x(S_{\ell-1}, S_\ell) - 1) \\
&\quad + 3(x(S_\ell) - (|S_\ell| - 1)) + (x(0, S_\ell) - 1) \\
&= 1 + (x(0, S_1) + 2\, x(S_1) + x(S_1, S_2) - 2\, |S_1|) \\
&\quad + (x(0, S_\ell) + 2\, x(S_\ell) + x(S_{\ell-1}, S_\ell) - 2\, |S_\ell|) \\
&\quad + (x(S_2) - |S_2| + 1) + (x(S_{\ell-1}) - |S_{\ell-1}| + 1) \\
&\quad + (x(H) - |H| + 1)
\end{aligned}$$

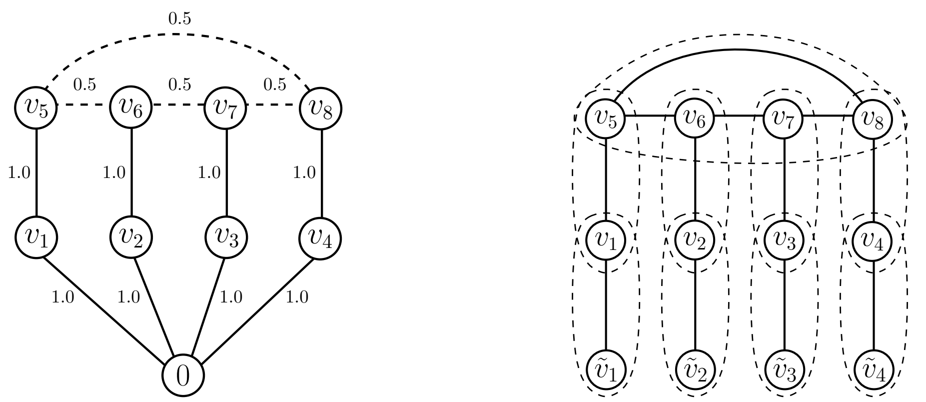
E Separating partial route inequalities

In this appendix section, we present a separation heuristic for partial route inequalities that also gives a slight (theoretical) improvement upon the one proposed by Hoogendoorn and Spliet (2023). Like in their heuristic, given a fractional solution $\bar{x} \in \mathcal{X}$, our algorithm uses the classical depth-first search procedure of Hopcroft and Tarjan (1973) to detect biconnected components and cut-vertices (or articulation points) of the support graph $G(\bar{x}) \setminus \{0\}$. However, Hoogendoorn and Spliet (2023) only consider the components of $G(\bar{x}) \setminus \{0\}$ whose total flow to the depot is exactly 2. Instead, we leverage the concept of *block-cut trees* (also known as *block graphs*, see Diestel (2005)) to generalize their approach and possibly separate more partial route inequalities.

In order to describe our heuristic, we need some graph theoretical concepts. For our purposes, given a connected graph G', we say that a *block* $B \subseteq V(G')$ of G' is the vertex set of a maximal biconnected subgraph of G', and a *cut-vertex* (or *articulation point*) is a vertex $v \in V(G')$ such that $G' - v$ is disconnected. Let $\mathcal{B}(G')$ (resp. $\mathcal{C}(G')$) denote the set of all blocks (resp. cut vertices) of G'. The *block-cut tree* of G' is a bipartite graph T with vertex set $\mathcal{B}(G') \cup \mathcal{C}(G')$ and edges $\{B, v\}$ for every block $B \in \mathcal{B}(G')$ and cut-vertex $v \in B$. It is easy to verify that T is a tree: if T contained a cycle, then that cycle would contain a vertex in $\mathcal{C}(G')$, contradicting the definition of a cut-vertex (see Diestel (2005) for more details). When G' is not connected, we have a block-cut tree for each connected component $G'_1, \ldots, G'_t$ of the graph G'. In this case, the notations $\mathcal{B}(G')$ and $\mathcal{C}(G')$ refer to the sets $\cup_{i \in [t]} \mathcal{B}(G'_i)$ and $\cup_{i \in [t]} \mathcal{C}(G'_i)$, respectively.

Let $(\bar{x}, \bar{\theta}) \in \mathcal{X} \times \mathbb{R}_+^{V_+}$ be a given candidate solution. We construct block-cut trees from $G(\bar{x})$ as follows. Let $\tilde{V} = \{\tilde{v} : \bar{x}_{0v} \geq 1, v \in V_+\}$ be a set of "dummy vertices" (the label $\tilde{v}$ indicates that $\tilde{v} \neq v$ and that $\tilde{v}$ maps to vertex $v \in V_+$ and vice versa) and let $\tilde{G}$ be an auxiliary graph with vertex set $V_+ \cup \tilde{V}$ and edge set $\tilde{E} = (E(G(\bar{x})) \setminus \delta(0)) \cup \{\{\tilde{v}, v\} : \tilde{v} \in \tilde{V}\}$. Note that, due to the introduction of the dummy vertices, every vertex $v \in V_+$ with $\bar{x}_{0v} \geq 1$ is a cut-vertex of $\tilde{G}$. Let $\tilde{G}_1, \ldots, \tilde{G}_t$ be the connected components of $\tilde{G}$. The *block-cut forest associated with* $G(\bar{x})$ is given by the set $\mathcal{F} = \{T_i\}_{i \in [t]}$, where, for each $i \in [t]$, T_i is a block-cut tree of $\tilde{G}_i$.

Using the notion of block-cut forests, we present our heuristic for separating partial route inequalities in Algorithm 1. In line 3, we build the graph $\tilde{G}$ and apply the algorithm of Hopcroft and Tarjan (1973) to construct the block-cut forest $\mathcal{F}$. Then, in lines 5-12, we generate a partial route H for each simple path $P \in \mathcal{P}$ connecting distinct leaves of $\mathcal{F}$



(a) A fractional solution $\bar{x} \in \mathcal{X}$ with support graph $G(\bar{x})$. Solid edges have $\bar{x}_e = 1$, while dashed edges have $\bar{x}_e = 0.5$.

(b) The auxiliary graph $\tilde{G}$ associated with $G(\bar{x})$. The dashed circled regions indicate the blocks.

Figure 3: Example of a solution $\bar{x} \in \mathcal{X}$ where $G(\bar{x}) \setminus \{0\}$ is made of a single component whose total flow to the depot is 4. The block–cut tree associated with $G(\bar{x})$ contains a branching vertex corresponding to the block $\{v_5, v_6, v_7, v_8\}$. In this case, our algorithm considers partial route $H = (\{v_1\}, \{v_5\}, \{v_6\}, \{v_2\})$ and we have that $W_{OF}(\bar{x}; \mathcal{X}_=(H)) = 0.5$.

(see Figure 3). In our implementation, we do not store the entire set of paths $\mathcal{P}$. Instead, we iteratively construct the paths by running depth-first searches from each leaf of $\mathcal{F}$. Additionally, to avoid examining symmetric partial routes, we assume that we have a total order $\prec$ on all the vertices of the block-cut forest $\mathcal{F}$.

We remark that, although we described Algorithm 1 as a separation heuristic, the algorithm itself is not finding violated inequalities. Instead, it generates a candidate set of partial routes $\mathcal{H}$, which can later be tested for violation depending on the form of the partial route inequalities and the choice of recourse lower bounds. For example, in Section 5 we define a partial route lower bound $\mathcal{L}_C(H)$ for the VRPSD with scenarios under the classical recourse policy. Then, given a partial route $H \in \mathcal{H}$, if we use the disaggregation from Remark 1, we check if $\bar{\theta}(V_+(H)) > \mathcal{L}_C(H) \cdot W_{OF}(\bar{x}; \mathcal{X}_=(H))$. If instead we use the disaggregation of Hoogendoorn and Spliet (2023) (Remark 5), then we check if $\bar{\theta}_v > \mathcal{L}_C(H) \cdot W_{OF}(\bar{x}; \mathcal{X}_=(H))$, where $v \in V_+(H)$ is the customer with the smallest index in H. More details on the separation routine are given in Section 5.2.

To examine the time complexity of Algorithm 1, we first note that line 3 of the algorithm can be executed in $\mathcal{O}(|E|)$ using the algorithm of Hopcroft and Tarjan (1973). Since G is complete, constructing the set $\mathcal{P}$ also takes $\mathcal{O}(|E|)$: for each leaf of $\mathcal{F}$, we run a depth-first search in a tree $T \in \mathcal{F}$ with $\mathcal{O}(|V|)$ nodes.[3] Therefore, since $|\mathcal{P}| \leq |E|$ and each iteration of the loop in lines 5-12 runs in $\mathcal{O}(|V|)$, the overall time complexity of Algorithm 1 is $\mathcal{O}(|V||E|)$.

In contrast, the separation heuristic proposed by Hoogendoorn and Spliet (2023) has time complexity of $\mathcal{O}(|E|)$. Their algorithm is equivalent to Algorithm 1 but only considers the components of $G(\bar{x}) \setminus \{0\}$ whose total flow to the depot is exactly 2.[4] More precisely,

[3]Given a graph G' with $|V(G')| \geq 2$, one can show that $|\mathcal{B}(G')| \leq |V(G')| - 1$ (see Proposition 1 of Li and Wu (2021)), so the number of nodes in the block-cut tree of G' is at most $2|V(G')| - 1$.

[4]In fact, they construct $\tilde{V}$ by only considering the customers v with $\bar{x}_{0v} = 1$, rather than $\bar{x}_{0v} \geq 1$, but this do not invalidate our argument in Proposition 3.

Algorithm 1 GetPartialRoutes

Input: Vectors $\bar{x} \in \mathcal{X}$ and $\bar{\theta} \in \mathbb{Q}_+^{V_+}$.

Output: A set of partial routes $\mathcal{H}$ for which we try to separate partial route inequalities.

procedure GetPartialRoutes($\bar{x}, \bar{\theta}$)
  $\mathcal{H} \leftarrow \emptyset$
  Let $\mathcal{F}$ be the block-cut forest associated with $G(\bar{x})$.
  Let $\mathcal{P}$ be the set of all simple paths $P = (S_1^{\mathcal{F}}, \dots, S_\ell^{\mathcal{F}}) \subseteq \mathcal{F}$ connecting leaves of $\mathcal{F}$ and such that either $\ell = 1$ or $S_1^{\mathcal{F}} \prec S_\ell^{\mathcal{F}}$.
  for $P = (S_1^{\mathcal{F}}, \dots, S_\ell^{\mathcal{F}}) \in \mathcal{P}$ **do**
    $H \leftarrow ()$
    for $j \in [\ell]$ **do**
      if $S_j^{\mathcal{F}}$ is a block and $(S_j^{\mathcal{F}} \cap V_+) \setminus \mathcal{C}(\tilde{G}) \neq \emptyset$ **then**
        Append $(S_j^{\mathcal{F}} \cap V_+) \setminus \mathcal{C}(\tilde{G})$ to the end of H.
      else if $S_j^{\mathcal{F}}$ is a cut-vertex v **then**
        Append $\{v\}$ to the end of H.
    $\mathcal{H} \leftarrow \mathcal{H} \cup \{H\}$
  return $\mathcal{H}$

they replace the block-cut forest $\mathcal{F}$ in Algorithm 1 with a forest $\mathcal{F}' \subseteq \mathcal{F}$ that contains all block-cut trees $T \in \mathcal{F}$ such that $\bar{x}(0, V_+(T)) = 2$, where $V_+(T) := (\cup_{S^{\mathcal{F}} \in V(T)} S^{\mathcal{F}}) \cap V_+$. In what follows, we demonstrate that $\mathcal{F}'$ is made of a disjoint set of paths, which indeed reduces the time complexity of Algorithm 1 to $\mathcal{O}(|E|)$ (since each vertex in $\mathcal{F}'$ is examined just once).

Proposition 3. *Let $\bar{x} \in \mathcal{X}$ and let $\mathcal{F}$ be the block-cut forest associated with $G(\bar{x})$. Then every tree $T \in \mathcal{F}$ with $\bar{x}(0, V_+(T)) = 2$ is a path.*

Proof. Take a tree $T \in \mathcal{F}$ with $\bar{x}(0, V_+(T)) = 2$. If $|V(T)| = 1$, we are done. Otherwise, we show that T has exactly two leaves. Since T has at least two leaves and $\bar{x}(0, V_+(T)) = 2$, it suffices to prove that, for each leaf $S^{\mathcal{F}}$ of T, either $S^{\mathcal{F}}$ is the block $\{\tilde{v}, v\}$ (meaning that $v \in \mathcal{C}(T)$ and $\bar{x}_{0v} \geq 1$) or $S^{\mathcal{F}} \subseteq V_+$ and $\bar{x}(0, S^{\mathcal{F}} \setminus \mathcal{C}(T)) \geq 1$.

Suppose by contradiction that $S^{\mathcal{F}} \subseteq V_+$ is a leaf with $\bar{x}(\{0\}, S) < 1$, where $S = S^{\mathcal{F}} \setminus \mathcal{C}(T)$ (note that $S \neq \emptyset$). Let $\tilde{S}^{\mathcal{F}}$ be the unique neighbor of $S^{\mathcal{F}}$. As $S^{\mathcal{F}}$ is a block, $\tilde{S}^{\mathcal{F}}$ is a cut-vertex v and $S^{\mathcal{F}} = S \cup \{v\}$. Since $\bar{x}(\{0\}, S) + \bar{x}(\{v\}, S) = \bar{x}(\delta(S)) \geq 2$ and, by assumption, $\bar{x}(\{0\}, S) < 1$, it follows that $\bar{x}(\{v\}, S) > 1$. Thus, by the degree constraint $\bar{x}(\delta(v)) = 2$, we have that $\bar{x}(\{v\}, V \setminus S) < 1$. But then $\bar{x}(\delta(S \cup \{v\})) = \bar{x}(\{0\}, S) + \bar{x}(\{v\}, V \setminus S) < 2$, contradicting the fact that $\bar{x}$ satisfies the subtour elimination constraints. □

Proposition 3 also provides a formal justification for the description given by Hoogendoorn and Spliet (2023) of their separation algorithm, where the authors state that, for each considered component, "In the appropriate order, a singleton corresponding to each articulation point of the connected component is added to the partial route, as well as the sets of vertices in between two articulation points that are added as an unstructured component". Since block-cut trees may contain branching vertices (see Figure 3), this description may appear somewhat imprecise, but Proposition 3 shows that, up to symmetry, the "appropriate order" is well-defined.

We conclude this appendix by mentioning that, although Algorithm 1 can (in principle) find more violated partial route inequalities than the separation routine proposed by Hoogendoorn and Spliet (2023), our preliminary experiments indicate no advantage in considering trees $T \in \mathcal{F}$ with $\bar{x}(0, V_+(T)) > 2$. Therefore, in practice, we only consider trees with $\bar{x}(0, V_+(T)) = 2$, which, as argued earlier, is equivalent to the algorithm of Hoogendoorn and Spliet (2023).

F On the equivalence of monotonicity and weak superadditivity

Let us start by proving that the disaggregation in Equation (8) is indeed monotone.

Claim 6. *Let $\hat{\mathcal{Q}}$ be a disaggregation of $\mathcal{Q}$ along V_+ satisfying Equation (8) for every route $R = (v_1, \ldots, v_\ell)$ belonging to a feasible routing plan. Then $\hat{\mathcal{Q}}$ is monotone.*

Proof. Take an arbitrary route $R = (v_1, \ldots, v_\ell) \in \mathfrak{R}$. By superadditivity, for every $i \in [\ell]$, $\hat{\mathcal{Q}}(R, v_i)$ is nonnegative. Moreover, $\sum_{i=1}^{\ell} \hat{\mathcal{Q}}(R, v_i)$ is a telescoping sum that evaluates to $\mathcal{Q}(R)$. Therefore, $\hat{\mathcal{Q}}$ is indeed a disaggregation of $\mathcal{Q}$ (i.e., it satisfies Definition 2).

Now let $R' = (v_a, \ldots, v_b) \subseteq R$. We need to show that $\sum_{i=a}^{b} \hat{\mathcal{Q}}(R, v_i) \geq \mathcal{Q}(R')$. If $a = 1$, we apply again the telescoping argument to learn that $\sum_{i=a}^{b} \hat{\mathcal{Q}}(R, v_i) = \mathcal{Q}((v_1, \ldots, v_b)) = \mathcal{Q}(R')$. In a similar way, if $a \geq 2$, then $\sum_{i=a}^{b} \hat{\mathcal{Q}}(R, v_i) = \mathcal{Q}((v_1, \ldots, v_b)) - \mathcal{Q}((v_1, \ldots, v_{a-1})) \geq \mathcal{Q}(R')$, where the last inequality follows by superadditivity, since $\mathcal{Q}((v_1, \ldots, v_b)) \geq \mathcal{Q}((v_1, \ldots, v_{a-1})) + \mathcal{Q}(R')$. $\square$

Next, we give an example showing that monotonicity does not necessarily imply superadditivity.

Example 1. Suppose that the only feasible routing plan consists of the single route $R = (v_1, v_2, v_3)$, that is, $\mathfrak{R} = \{R\}$. Assume that $\mathcal{Q}(R) = 3$, $\mathcal{Q}((v_1)) = \mathcal{Q}((v_2)) = \mathcal{Q}((v_3)) = \mathcal{Q}((v_1, v_2)) = 1$ and $\mathcal{Q}((v_2, v_3)) = 0$. Since $0 = \mathcal{Q}((v_2, v_3)) < \mathcal{Q}((v_2)) + \mathcal{Q}((v_3)) = 2$, the recourse function $\mathcal{Q}$ is not superadditive. Let $\hat{\mathcal{Q}}$ be a disaggregation of $\mathcal{Q}$ such that $\hat{\mathcal{Q}}(R, v_1) = \hat{\mathcal{Q}}(R, v_2) = \hat{\mathcal{Q}}(R, v_3) = 1$. For every subroute $R' \subseteq R$, we have that $\sum_{v \in V_+(R')} \mathcal{Q}(R, v) \geq \mathcal{Q}(R')$, so $\hat{\mathcal{Q}}$ is monotone. $\square$

As mentioned in Section 4.3.1, it can be easily shown that superadditivity implies weak superadditivity. (However, the reader may verify that the function $\mathcal{Q}$ in Example 1 is weakly superadditive but not superadditive.)

Claim 7. *Suppose that $\mathcal{Q}$ is superadditive, then $\mathcal{Q}$ is also weakly superadditive.*

Proof. First note that superadditivity implies that, if $R \in \mathfrak{R}$ and $R' \subseteq R$, then $\mathcal{Q}(R') \leq \mathcal{Q}(R)$. Now take an arbitrary $R \in \mathfrak{R}$ and let $R_1, \ldots, R_t$ be disjoint subroutes of R. There exists disjoint subroutes $R'_1, \ldots, R'_t \subseteq R$ such that $R = R'_1 \oplus \ldots \oplus R'_t$ and, for every $i \in [t]$, $R_i \subseteq R'_i$. Thus, by superadditivity,

$$\mathcal{Q}(R) \geq \mathcal{Q}(R'_1) + \ldots + \mathcal{Q}(R'_t) \geq \mathcal{Q}(R_1) + \ldots + \mathcal{Q}(R_t).$$

$\square$

We now turn to the main proof in this appendix.

Theorem 5. *The following holds.*

(i) If the fixed disaggregation of $\mathcal{Q}$ satisfies Assumption 2 and is monotone, $\mathcal{Q}$ is weakly superadditive.

(ii) Conversely, if $\mathcal{Q}$ is weakly superadditive, then there exists a disaggregation of $\mathcal{Q}$ along $\Omega = V_+$ that satisfies the conditions in Assumption 2 and is monotone.

Proof. For (i), let $R \in \mathfrak{R}$ and $R_1, \ldots, R_t$ be disjoint subroutes of R. Then

$$\mathcal{Q}(R) = \sum_{v \in V_+(R)} \mathcal{Q}(R, v) \geq \sum_{i=1}^{t} \sum_{v \in V_+(R_i)} \mathcal{Q}(R, v) \geq \sum_{i=1}^{t} \mathcal{Q}(R_i),$$

where the last inequality follows from monotonicity of the fixed disaggregation of $\mathcal{Q}$.

Proving item (ii) requires more work, as we need to build a disaggregation $\hat{\mathcal{Q}}$ of $\mathcal{Q}$ satisfying the conditions in Assumption 2 using weak superadditivity alone. Let $R = (v_1, \ldots, v_\ell)$ be a route. If R does not belong to a feasible routing plan (meaning that $R \notin \mathfrak{R}$), we can simply set $\hat{\mathcal{Q}}(R, v) = \mathcal{Q}(R)$, for an arbitrarily chosen vertex $v \in V_+(R)$. So we may safely assume that $R \in \mathfrak{R}$. In the remainder of the proof, we show that Algorithm 2 finds values $\hat{\mathcal{Q}}(R, v_1), \ldots, \hat{\mathcal{Q}}(R, v_\ell)$ such that $\mathcal{Q}(R) = \sum_{i=1}^{\ell} \hat{\mathcal{Q}}(R, v_i)$ and, for every $R' \subseteq R$, it holds that $\sum_{v \in V_+(R')} \hat{\mathcal{Q}}(R, v) \geq \mathcal{Q}(R')$.

Algorithm 2 GetDisaggregation

Input: A route $R = (v_1, \ldots, v_\ell)$.

Output: The disaggregated recourse values $\hat{\mathcal{Q}}(R, v_1), \ldots, \hat{\mathcal{Q}}(R, v_\ell)$.

1: **procedure** GetDisaggregation$(R = (v_1, \ldots, v_\ell))$
2: $\quad \mathcal{A}_0 \leftarrow \emptyset$
3: $\quad$ **for** $b = 1, \ldots, \ell$ **do**
4: $\quad\quad \hat{\mathcal{Q}}(R, v_b) \leftarrow 0$
5: $\quad\quad a \leftarrow \arg\max_{a' \in [b]} \{\mathcal{Q}((v_{a'}, \ldots, v_b)) - \sum_{i=a'}^{b} \hat{\mathcal{Q}}(R, v_i)\}$
6: $\quad\quad \Delta_b \leftarrow \mathcal{Q}((v_a, \ldots, v_b)) - \sum_{i=a}^{b} \hat{\mathcal{Q}}(R, v_i)$
7: $\quad\quad$ **if** $\Delta_b > 0$ **then**
8: $\quad\quad\quad \mathcal{A}_b \leftarrow \mathcal{A}_{a-1} \cup \{(v_a, \ldots, v_b)\}$
9: $\quad\quad\quad \hat{\mathcal{Q}}(R, v_b) \leftarrow \Delta_b$
10: $\quad\quad$ **else**
11: $\quad\quad\quad \mathcal{A}_b \leftarrow \mathcal{A}_{b-1}$
12: $\quad \Delta_R \leftarrow \mathcal{Q}(R) - \sum_{i=1}^{\ell} \hat{\mathcal{Q}}(R, v_i)$
13: $\quad$ **if** $\Delta_R > 0$ **then**
14: $\quad\quad \hat{\mathcal{Q}}(R, v_1) \leftarrow \hat{\mathcal{Q}}(R, v_1) + \Delta_R$
15: $\quad$ **return** $\hat{\mathcal{Q}}(R, v_1), \ldots, \hat{\mathcal{Q}}(R, v_\ell)$

Let $b \in [\ell]$ and consider the corresponding iteration of the loop in lines 3–11. For now, we focus on the variables a in line 5 and Δ_b in line 6, the purpose of the collections of routes $\mathcal{A}_b$ will be explained later. By the choice of a, for every subroute $R_{a'} = (v_{a'}, \ldots, v_b)$, with $a' \in [b]$, we have that $\Delta_b \geq \mathcal{Q}(R_{a'}) - \sum_{v \in V_+(R_{a'}) \setminus \{b\}} \hat{\mathcal{Q}}(R, v)$. Since by the end of this iteration we set $\hat{\mathcal{Q}}(R, v_b) = (\Delta_b)^+$, it follows that $\sum_{v \in V_+(R_{a'})} \hat{\mathcal{Q}}(R, v) \geq \mathcal{Q}(R_{a'})$. Repeating this algorithm for every iteration of $b \in [\ell]$, we learn that, by the end of Algorithm 2, for every subroute $R' \subseteq R$, the returned values satisfy $\sum_{v \in V_+(R')} \hat{\mathcal{Q}}(R, v) \geq \mathcal{Q}(R')$.

It remains to show that $\mathcal{Q}(R) = \sum_{v \in V_+(R)} \hat{\mathcal{Q}}(R, v)$, or equivalently, that the value of Δ_R in line 12 is always nonnegative. To prove this, we use induction to show that, $\sum_{i \in [b]} \hat{\mathcal{Q}}(R, v_i) =$

$\sum_{R' \in \mathcal{A}_b} \mathcal{Q}(R')$, for all $b \in \{0, \ldots, \ell\}$. Since the set $\mathcal{A}_\ell$ is made of disjoint subroutes of R, weak superadditivity then implies that $\mathcal{Q}(R) \geq \sum_{R' \in \mathcal{A}_\ell} \mathcal{Q}(R') = \sum_{i=1}^{\ell} \hat{\mathcal{Q}}(R, v_i)$, as desired.

For the base case of $b = 0$, we have that $\mathcal{A}_0 = [0] = \emptyset$ and we are done. Assume that $b \geq 1$. If $\Delta_b \leq 0$, then $\hat{\mathcal{Q}}(R, v_b) = 0$ and $\mathcal{A}_b = \mathcal{A}_{b-1}$, so by the induction hypothesis,

$$\sum_{i \in [b]} \hat{\mathcal{Q}}(R, v_i) = \sum_{i \in [b-1]} \hat{\mathcal{Q}}(R, v_i) = \sum_{R' \in \mathcal{A}_{b-1}} \mathcal{Q}(R') = \sum_{R' \in \mathcal{A}_b} \mathcal{Q}(R').$$

If $\Delta_b > 0$, then $\hat{\mathcal{Q}}(R, v_b) = \Delta_b$ and by the choice of a in line 5,

$$\sum_{i=a}^{b} \hat{\mathcal{Q}}(R, v_i) = \mathcal{Q}((v_a, \ldots, v_b)). \tag{18}$$

Furthermore, since $a \geq 1$, we know by the induction hypothesis that

$$\sum_{i \in [a-1]} \hat{\mathcal{Q}}(R, v_i) = \sum_{R' \in \mathcal{A}_{a-1}} \mathcal{Q}(R'). \tag{19}$$

Using $\mathcal{A}_b = \mathcal{A}_{a-1} \cup \{(v_a, \ldots, v_b)\}$, we sum Equations (18) and (19) to obtain $\sum_{i=1}^{b} \hat{\mathcal{Q}}(R, v_i) = \sum_{R' \in \mathcal{A}_b} \mathcal{Q}(R')$. □

Finally, recall that $\mathfrak{R}$ is *downward closed* if, for every $R \in \mathfrak{R}$ and $R' \subseteq R$, we have that $R' \in \mathfrak{R}$.

Claim 8. *Suppose that $\mathcal{Q}$ is weakly superadditive. If $\mathfrak{R}$ is downward closed, then $\mathcal{Q}$ is also superadditive. Furthermore, if $\mathfrak{R}$ is not downward closed, then there exists an instance of problem* $\text{VRPR}(\mathcal{Q}, \mathcal{X})$ *where $\mathcal{Q}$ is not superadditive.*

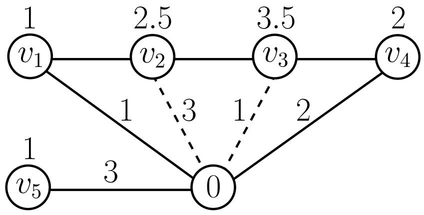
Proof. We first show that, under the assumption that $\mathcal{Q}$ is weakly superadditive, downward closedness of $\mathfrak{R}$ implies superadditivity. Let $R \in \mathfrak{R}$ and $R_1 \oplus R_2 \subseteq R$. By downward closedness of $\mathfrak{R}$, we know that $R_1 \oplus R_2 \in \mathfrak{R}$, so by weak superadditivity, $\mathcal{Q}(R_1 \oplus R_2) \geq \mathcal{Q}(R_1) + \mathcal{Q}(R_2)$.

For the second part of the statement, consider the instance in Example 1. □

G Illustrative example on different disaggregations

We now show that problem $\text{VRPR}(\mathcal{Q}_C, \mathcal{X}_{\text{CVRP}})$ may fail to satisfy the monotonicity/weak superadditivity properties in Section 4.3.1. For convenience, we assume that the disaggregation of $\mathcal{Q}_C$ is according to Remark 1. Our construction, illustrated in Figure 4, is essentially the same as the one used in Proposition 9 of Parada et al. (2024), except that here we assume that $\mathbb{P}$ is given by scenarios and we also have to consider the feasibility of the routing plans.

Consider the instance in Figure 4 and the routes $R = (v_1, v_2, v_3, v_4)$ and $R' = (v_2, v_3, v_4)$. Consider also the routing plans $\{R, (v_5)\}$ and $\{R', (v_5, v_1)\}$. Note that both routing plans are feasible, so there exist $\bar{x}, x' \in \mathcal{X}_{\text{CVRP}} \cap \mathbb{Z}^E$ such that $\mathcal{R}(\bar{x}) = \{R, (v_5)\}$ and $\mathcal{R}(x') = \{R', (v_5, v_1)\}$. Since $\vec{R} = (v_1, v_2, v_3, v_4)$ and $\vec{R}' = (v_2, v_3, v_4)$, one may check that $\mathcal{Q}_C(R) = \mathcal{Q}_C(\vec{R})$ and $\mathcal{Q}_C(R') = \mathcal{Q}_C(\vec{R}')$. Moreover, $\mathcal{Q}_C(R) = \mathcal{Q}_C(R, v_3) + \mathcal{Q}_C(R, v_4) = (2 \cdot c_{0v_3} + 2 \cdot c_{0v_4})/4 = 3/2$ and $\mathcal{Q}_C(R') = \mathcal{Q}_C(R', v_4) = (2 \cdot 2 \cdot c_{0v_3})/4 = 2$. Therefore, $3/2 = \mathcal{Q}_C(R) = \sum_{v \in V_+(R')} \mathcal{Q}_C(R, v) < \mathcal{Q}_C(R') = 2$. This shows not only that the disaggregation of $\mathcal{Q}_C$ in Remark 1 is not monotone (Definition 12), but also that $\mathcal{Q}_C$ is not weakly superadditive (Definition 14). Hence, by Theorem 5, there is no disaggregation of $\mathcal{Q}_C$ that is monotone.



	$d^\xi(v_1)$	$d^\xi(v_2)$	$d^\xi(v_3)$	$d^\xi(v_4)$	$d^\xi(v_5)$
$\xi = 1$	1	2	0	2	1
$\xi = 2$	1	2	7	2	1
$\xi = 3$	1	3	0	2	1
$\xi = 4$	1	3	7	2	1

Figure 4: Instance with $V_+ = \{v_1, v_2, v_3, v_4, v_5\}$, $k = 2$ and $C = 10$ that does not satisfy the monotonicity property. The numbers next to the edges correspond to the cost of the edges incident to the depot, while the numbers on top of the vertices refer to the vector of expected demands $\bar{d} \in \mathbb{Q}_{++}^{V_+}$. This instance has $N = 4$ scenarios, each with a realization probability of $\frac{1}{4}$. The table on the right show the scenario demands vectors $d^\xi \in \mathbb{Q}_{+}^{V_+}$. Note that failures may occur only in scenarios $\xi = 2$ and $\xi = 4$.

H Validity of ILS set cuts and different recourse disaggregations

Let $\hat{Q}_1$ and $\hat{Q}_2$ be recourse disaggregations of $\mathcal{Q}_C$ according to Remarks 5 and 1, respectively. Using the same instance as in Figure 4, we illustrate that ILS set cuts of the form $\theta(S) \geq \mathcal{L}_C(S) \cdot W_P(x; \mathcal{X}(S, \tilde{k}))$ are valid for $\mathcal{F}(\hat{\mathcal{Q}}_2, \mathcal{X}_{\text{CVRP}})$ (by Proposition 2) but may not be valid for $\mathcal{F}(\hat{\mathcal{Q}}_1, \mathcal{X}_{\text{CVRP}})$.

Let $\bar{x}$ be as specified in Figure 4. Note that $\mathcal{Q}_C(R) = \hat{Q}_1(R, v_1) = 3/2$, so we have that $(\bar{x}, \bar{\theta}^1) \in \mathcal{F}(\hat{\mathcal{Q}}_1, \mathcal{X}_{\text{CVRP}})$, where $\bar{\theta}^1 = [3/2, 0, 0, 0, 0]^\intercal$. Similarly, $\mathcal{Q}_C(R) = \hat{Q}_2(R, v_3) + \hat{Q}_2(R, v_4)$ and we have that $(\bar{x}, \bar{\theta}^2) \in \mathcal{F}(\hat{\mathcal{Q}}_2, \mathcal{X}_{\text{CVRP}})$, where $\bar{\theta}^2 = [0, 0, (2c_{0v_3})/4, (2c_{0v_4})/4, 0]^\intercal = [0, 0, 1/2, 1, 0]^\intercal$.

Now set $S = \{v_2, v_3, v_4\}$ and $\tilde{k} = 1$. Observe that $W_P(\bar{x}; \mathcal{X}(S, 1)) = 1$. Moreover, $\lceil d^2(S)/C \rceil - 1 = \lceil d^4(S)/C \rceil - 1 = 1$, so we have that $\mathcal{L}_C(S, 1) = (2c_{0v_3})/4 = 1/2$. Hence,

$$0 = \bar{\theta}^1(S) < \mathcal{L}_C(S, 1) \cdot W_P(\bar{x}; \mathcal{X}(S, 1)) = \frac{1}{2},$$

while, as predicted by Proposition 2,

$$\frac{3}{2} = \bar{\theta}^2(S) \geq \mathcal{L}_C(S, 1) \cdot W_P(\bar{x}; \mathcal{X}(S, 1)) = \frac{1}{2}.$$

I Proof of Lemma 2

Lemma 2. *Let $S = \{v_1, \dots, v_\ell\} \subseteq V_+$ and $\alpha \in \mathbb{R}_+$. For every $\xi \in [N]$,*

$$\sum_{j \in [\ell]} \sum_{t=1}^{\infty} \mathbb{I}\left(\alpha + \sum_{i \in [j-1]} d^\xi(v_i) \leq tC < \alpha + \sum_{i \in [j]} d^\xi(v_i)\right) = \text{FAIL}_\xi\left(\alpha, S\right). \tag{11}$$

Proof. For every $t \in \mathbb{Z}_{++}$, we have that

$$\sum_{j \in [\ell]} \mathbb{I}\left(\alpha + \sum_{i \in [j-1]} d^\xi(v_i) \leq tC < \alpha + \sum_{i \in [j]} d^\xi(v_i)\right) \leq 1$$

and equality holds if and only if $\alpha \leq tC < \alpha + d^\xi(S)$. Therefore, if we let β be the expression in the LHS of (11), we can rewrite β as follows

$$\beta = \sum_{t=1}^{\infty} \mathbb{I}(\alpha \leq tC < \alpha + d^\xi(S))$$

$$= \sum_{t=1}^{\infty} \mathbb{I}(tC < \alpha + d^\xi(S)) - \sum_{t=1}^{\infty} \mathbb{I}(tC < \alpha). \tag{20}$$

To finish the proof, we simplify (20) using the following simple fact.

Fact 2. *For every $\gamma \in \mathbb{R}_+$, $\sum_{t=1}^{\infty} \mathbb{I}(t < \gamma) = (\lceil \gamma \rceil - 1)^+$.*

Hence, if $\alpha = 0$, then $\beta = \sum_{t=1}^{\infty} \mathbb{I}(tC < \alpha + d^\xi(S)) = \left(\left\lceil \frac{d^\xi(S)}{C} \right\rceil - 1\right)^+$. Alternatively, when $\alpha = qC + r(\alpha) > 0$, with $q = \lfloor \alpha/C \rfloor$,

$$\beta = \left(\left\lceil \frac{\alpha + d^\xi(S)}{C} \right\rceil - 1\right) - \left(\left\lceil \frac{\alpha}{C} \right\rceil - 1\right)$$

$$= \left\lceil q + \frac{r(\alpha) + d^\xi(S)}{C} \right\rceil - \left\lceil q + \frac{r(\alpha)}{C} \right\rceil$$

$$= \left\lceil \frac{r(\alpha) + d^\xi(S)}{C} \right\rceil - \left\lceil \frac{r(\alpha)}{C} \right\rceil. \qquad \square$$

J Proof of Proposition 2

Proposition 2. *Suppose that customer demands satisfy Assumption 3 and assume that the disaggregation of $\mathcal{Q}_C$ is as described in Remark 1. Let $S \subseteq V_+$ and $\tilde{k} \in \mathbb{Z}_{++}$ be such that $x(S) \leq |S| - \tilde{k}$ is valid for $\mathcal{X} \cap \mathbb{Z}^E$. Then $\mathcal{L}_C(S, \tilde{k}) := \sum_{\xi=1}^{N} p_\xi \mathcal{L}_\xi^{\tilde{k}}(0, S)$ is a recourse lower bound with respect to S and $\mathcal{X}(S, \tilde{k})$, that is, for every $\bar{x} \in \mathcal{X}(S, \tilde{k})$,*

$$\sum_{R \in \mathcal{R}(\bar{x})} \sum_{v \in S} \mathcal{Q}_C(R, v) \geq \mathcal{L}_C(S, \tilde{k}).$$

Proof. Fix a scenario $\xi \in [N]$ and let $v_1', \ldots v_\ell'$ be a reordering of the customers in S such that $c_{0v_1'} \leq \ldots \leq c_{0v_\ell'}$. Expanding the formula for ($\mathcal{L}_\xi^\nu$) yields (note that if $d^\xi(S) = 0$, then $\mathcal{L}_\xi^{\tilde{k}}(0, S) = 0$):

$$\mathcal{L}_\xi^{\tilde{k}}(0, S) = \sum_{j \in [1 - \tilde{k} + (\lceil d^\xi(S)/C \rceil - 1)^+]} 2\, c_{0v_j'} = \sum_{j \in [\lceil d^\xi(S)/C \rceil - \tilde{k}]} 2\, c_{0v_j'}.$$

Hence, without loss of generality, we assume that $\tilde{k} < \lceil d^\xi(S)/C \rceil$. Let $\bar{x} \in \mathcal{X}(S, \tilde{k})$. By Assumption 3, it suffices to show that $\bar{x}$ observes at least $\lceil d^\xi(S)/C \rceil - \tilde{k}$ failures in S. As $\bar{x}(S) = |S| - \tilde{k}$, the subgraph of $G(\bar{x})$ induced by S is made of $\tilde{k}$ disjoint paths. These paths correspond to customer-disjoint routes $R_1', \ldots, R_{\tilde{k}}'$ such that $S = \cup_{i \in [\tilde{k}]} V_+(R_i')$ and, for every $i \in [\tilde{k}]$, R_i' is a subroute of a route in $\mathcal{R}(\bar{x})$.

Consider one of these subroutes, say $R_1' = (v_a, \ldots, v_b)$, and let $R = (v_1, \ldots, v_\ell) \in \mathcal{R}(\bar{x})$ be such that $R_1' \subseteq R$. Lemma 2 implies that R observes at least

$$\text{FAIL}_\xi\left(\sum_{i \in [a-1]} d^\xi(v_i), V_+(R_1')\right) \geq \left\lceil \frac{d^\xi(R_1')}{C} \right\rceil - 1$$

failures in R'_1. Applying the same reasoning for all subroutes $R'_1, \dots, R'_{\tilde{k}}$, we learn that $\bar{x}$ observes at least

$$\sum_{i \in [\tilde{k}]} \left(\left\lceil \frac{d^{\xi}(R'_i)}{C} \right\rceil - 1 \right) \geq \left\lceil \frac{d^{\xi}(S)}{C} \right\rceil - \tilde{k}$$

failures in S. □

K Separation algorithm for the VRPSD with scenarios under the classical recourse policy

Algorithm 3 SeparationVRPSD

Input: A candidate solution $(\bar{x}, \bar{\theta}) \in \mathbb{R}^E \times \mathbb{Q}_+^{V_+}$ and an integer $D \in \{1, 2\}$, which indicates if we use the disaggregation from Hoogendoorn and Spliet (2023) ($D = 1$) or the disaggregation in Remark 1 ($D = 2$).

procedure SeparationVRPSD($\bar{x}, \bar{\theta}, D$)
    Call CVRPSEP to get a family of customer sets $\mathcal{S} \subseteq 2^{V_+}$.
    for $S \in \mathcal{S}$ **do**
        Add RCI $x(S) \leq |S| - \bar{k}(S)$.
        if $D = 2$ and $\bar{\theta}(S) < \mathcal{L}_C(S, \bar{k}(S)) \cdot (\bar{x}(S) - |S| + \bar{k}(S))$ **then**
            Add ILS set cut $\theta(S) \geq \mathcal{L}_C(S, \bar{k}(S)) \cdot (x(S) - |S| + \bar{k}(S))$.
    if $\mathcal{S} \neq \emptyset$ **then**
        return
    $\mathcal{H} \leftarrow$ GetPartialRoutes($\bar{x}, \bar{\theta}$)
    for $H \in \mathcal{H}$ **do**
        if $D = 1$ **then**
            $v \leftarrow$ customer in $V_+(H)$ with the smallest index.
            if $\bar{\theta}_v < \mathcal{L}_C(H) \cdot W_{OF}(\bar{x}; \mathcal{X}_=(H))$ **then**
                Add ILS PR-EA cut $\theta_v \geq \mathcal{L}_C(H) \cdot W_{OF}(x; \mathcal{X}_=(H))$.
        else
            $S \leftarrow V_+(H)$
            if $\bar{\theta}(S) < \mathcal{L}_C(S, \bar{k}(S)) \cdot (\bar{x}(S) - |S| + \bar{k}(S))$ **then**
                Add ILS set cut $\theta(S) \geq \mathcal{L}_C(S, \bar{k}(S)) \cdot (x(S) - |S| + \bar{k}(S))$.
            else
                if $\bar{\theta}(S) < \mathcal{L}_C(H) \cdot W_{OF}(\bar{x}; \mathcal{X}_=(H))$ **then**
                    Add ILS PR-EA cut $\theta(S) \geq \mathcal{L}_C(H) \cdot W_{OF}(x; \mathcal{X}_=(H))$.