1. Establish the following inequality by using the **Triangle Inequality**.
$||a| - |b|| \leq |a - b|$ (hint: add 0)

Proof. By Triangle Inequality,

$$|a| = |a - b + b| \leq |a - b| + |b| \text{ and } |b| = |b - a + a| \leq |b - a| + |a|$$

Then adding the inequalities above implies

$$|a| - |b| \leq |a - b| \text{ and } - |a - b| \leq |a| - |b|$$

Combining those, we get

$$-|a - b| \leq |a| - |b| \leq |a - b|$$

Thus, by definition of absolute value function, $||a| - |b|| \leq |a - b|$. □

2. Fill in the blanks.

(a) Let A be a nonempty set. A real number a is the **infimum** of A, written $a = inf(A)$ if and only if:
a is a lower bound of A. If b is any lower bound of B, then $b \leq a$.

(b) Let A be a nonempty set. A real number a is the **sumpremum** of A, written $x = sup(A)$ if and only if:
x is a upper bound of A. If b is any upper bound of A, then $x \leq b$.

(c) State the Archimedian Property: If x is a real number, then either x is an integer or there exists an integer n such that
$n - 1 \leq x < n$

3. Let $A = (1, 3)$. State the supremum of A and prove it.

Proof. $sup(A) = 3$.
By definition of open interval, $A = \{x \in \mathbb{R} : 1 < x < 3\}$. Then 3 is the upper bound of A. Suppose a is any upper bound of A and $a < 3$. By axiom 1, since $a, 3 \in \mathbb{R}$ and $a < 8$, then there is a number b between them such that $a < b < 3$ contradicting that a is any upper bound. Thus, by contradiction, if a is any upper bound of A, then $3 \leq A$. Therefore, by definition of supremum, $sup(A) = 3$. □

4. Let M be a set and suppose that x is a lower bound for M. Prove that $x = inf(M)$ if and only if for every $\epsilon > 0$, there exists $y \in M$ such that $y < x + \epsilon$.

Proof. $\Rightarrow$ Suppose $x = inf(M)$ and $\epsilon > 0$. Then $x + \epsilon > x$. Thus, $x + \epsilon$ is not a lower bound for M and by definition of lower bound, there exists $y \in M$ such that $y < x + \epsilon$. $\Leftarrow$ Suppose x is a lower bound for M and for every $\epsilon > 0$ there exists $y \in M$ such that $y < x + \epsilon$. Suppose b is a lower bound and $x < b$. Then equivalently $y < b$. This contradicts the assumption that b is a lower bound. Thus, $x = inf(M)$. □

5. Let A be bounded below, $B = \{b \in \mathbb{R} :$ b is a lower bound of A $\}$. Show that $sup(B) = inf(A)$.

Proof. By definition of B, for any $b \in B$, b is a lower bound of A. By definition of infimum, $inf(A) \in B$ and $b \leq inf(A)$. Then $inf(A)$ is an upper bound of B. Let $c \in \mathbb{R}$ be any upper bound of B such that $c < inf(A)$. Since $inf(A) \in B$ contradicting that c is an upper bound. Thus, $c \geq inf(A)$ and by definition of supremum, $sup(B) = inf(A)$. □

6. Fill in the blanks.

   (a) Set $A \subset \mathbb{R}$ is **bounded** if and only if
   there exists a number C such that $|a| \leqslant C$ for all $a \in A$.

   (b) A sequence (a_n) is **increasing** if and only if
   $a_n \leqslant a_{n+1}$ for all $n \in \mathbb{N}$.

   (c) A sequence (a_n) is **monotone** if
   it is either increasing or decreasing.

7. Prove that if a sequence is monotone and bounded, then it converges.
hint1: considering two cases, increasing and decreasing.
hint2: how to use question 3's result?

Proof. Suppose a sequence $A = \{a_n | n \in \mathbb{N}\}$ that is monotone and bounded. Since A has an upper-bound, by axiom of completeness, there exists $s = sup(A)$. By Ex13, for some $N \in \mathbb{N}$,

$$a_N > s - \epsilon$$

Assuming that (a_n) is increasing, for all $n \geqslant N$,

$$a_n \geqslant a_N$$

for all $a_n \in A$, since s is an upperbound of a_n,

$$s \geqslant a_n \text{ implies } s + \epsilon > s \geqslant a_n \geqslant a_N > s - \epsilon$$

Thus, $|a_n - s| < \epsilon$ for all $n \leqslant N$, by def of convergence, $A = \{a_n | n \in \mathbb{N}\}$ converges when A is increasing. Similarly, when A is decreasing, consider $i = inf(A)$, by property of infimum, definition of decreasing, and question 4, for all $n \geqslant N$,

$$i - \epsilon < s \leqslant a_n \leqslant a_N < i + \epsilon$$

Thus, $|a_n - i| < \epsilon$ for all $n \leqslant N$, by def of convergence, $A = \{a_n | n \in \mathbb{N}\}$ converges when A is decreasing. Therefore, combining both cases, $A = \{a_n | n \in \mathbb{N}\}$ converges. □

8. Prove that Cauchy sequences are bounded.

Proof. Let (a_n) be a cauchy sequence. Let $\epsilon = 1$, by definition of cauchy, there exists N such that for every $m, n > N$, $|a_n - a_m| < \epsilon = 1$. Let m be some Mth term in the sequence. Then by triangle inequality,

$$|a_n| = |a_n - a_m + a_m| \leq |a_n - a_m| + |a_m| < 1 + |a_M|$$

Then, $|a_n| \leq max\{|a_1|, |a_2|, ..., |a_{M-1}|, 1 + |a_M|\}$ for all n. Thus, we can find a constant c in $max\{|a_1|, |a_2|, ..., |a_{M-1}|, 1 + |a_M|\}$ such that $|a_n| \leq c$, and therefore, the cauchy sequences are bounded. □

9. State the definition of **continuity** and **uniform continuity**. Briefly explain what is the difference between the definitions and proofs.

A function f is continuous at the point $x = c$ if for every sequence $(a_n)_{n=1}^{\infty}$ that converges to c, the sequence $(f(a_n))_{n=1}^{\infty}$ converges to $f(c)$. If a function is continuous at every point of some set S, we say f is continuous on S.

A function f is continuous at a point x if for every $\epsilon > 0$ there exists $\delta > 0$ such that $|f(y) - f(x)| < \epsilon$ for all y such that $|x - y| < \delta$.

A function $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous on a set A if for every $\epsilon > 0$ there exists $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$.

Continuity of a function on an interval refers to that it is continuous at each point of the interval. However, uniform continuity refers a global property of a function.

When proving, for continuous at a point, we will first choose a point c, an $\epsilon > 0$, and let $\delta = \delta(\epsilon, c)$ so that we assume for any x, $|x - c| < \delta$. But for uniform continuity, we choose $\epsilon > 0$ and $\delta = \delta(\epsilon)$. Then suppose that for any x, x_0, $|x - x_0| < \delta$.

10. Determine the following statement is true or false and give a brief explanation on your choices.

    (a) If a function is continuous on $[a, b]$, then it is differentiable on $[a, b]$.
    False. Consider the function $f = |x|$ at $x = 0$.

    (b) If a function is differentiable on $[a, b]$, then it must be continuous on $[a, b]$.
    True. $\lim_{x \to c} f(x) - f(c) = \lim_{x \to c} \dfrac{f(x) - f(c)}{x - c}(x - c) \to f'(c) * 0 = 0.$

    (c) If a function f is differentiable on $[a, b]$, then $f'(x)$ must be continuous.
    False. Consider the categories of C^0 functions to C^{∞} functions.

11. prove that $\lim\limits_{x\to 1} x+2=3$.

Proof. Let $\epsilon > 0$. $(a_n) \to 1$ and $(a_n) \neq 1$. By definition of convergence, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - 1| < \epsilon$, Then

$$|a_n + 2 - 3| = |a_n - 1| < \epsilon$$

Thus, $f(x)$ converges to 3 and $\lim\limits_{x\to 1} x+2=3$. $\square$

12. Let $f(x) = x^2 + 2$. Use the definition to prove that $f'(-3) = -6$.

Proof. Let $(a_n) \in D$ be a sequence and $-3 \in D$ such that $(a_n)_{n=1}^{\infty} \to -3$ and $(a_n)_{n=1}^{\infty} \neq -3$. Then,

$$f'(c) = (\frac{f(a_n) - f(c)}{a_n - c})_{n=1}^{\infty} = (\frac{((a_n)^2 + 2) - (3^2 + 2)}{a_n + 3})_{n=1}^{\infty} = (\frac{((a_n)^2 - 9}{a_n + 3})_{n=1}^{\infty} = (a_n - 3)_{n=1}^{\infty}$$

Since $(a_n)_{n=1}^{\infty} \to -3$, $(a_n)_{n=1}^{\infty} - 3 \to -3 - 3 = -6$. Thus, $f'(c) \to -6$. $\square$

13. State any one of the **derivative rules** and prove it. (Possible choices: Product Rule, Quotient Rule, Chain Rule, etc).

Proof. Want to Prove: Let α be a constant, and let $f : [a, b] \to \mathbb{R}$ be a differentiable function on the set (a, b). Define $h(x) = \alpha f(x)$ for all $x \in [a, b]$. Then $h(x)$ is differentiable on (a, b) and that $h'(x) = \alpha f'(x)$ for all $x \in (a, b)$.

Since f is differentiable on (a, b), let (a_n) be a sequence in $[a, b]$ such that $(a_n) \to c$ with $a_n \neq c$ for all $n \in \mathbb{N}$, then,

$$\alpha f'(x) = \alpha(\frac{f(a_n) - f(c)}{a_n - c})_{n=1}^{\infty} = (\frac{\alpha f(a_n) - \alpha f(c)}{a_n - c})_{n=1}^{\infty} = (\frac{h(a_n) - h(c)}{a_n - c})_{n=1}^{\infty} = h'(x)$$

Since f is differentiable over (a, b), $\alpha f(x)$ is differentiable over (a, b). Since, $\alpha f'(x) = h'(x)$, h is differentiable over (a, b). $\square$

14. State what the following theorems mean.

   (a) Intermediate Value Theorem: If f is a continuous function on [a, b], let s be a number with $f(a) < s < f(b)$. Then,
   there exists some x between a and b such that $f(x) = s$.

   (b) Extreme Value Theorem: If f is continuous on $[a, b]$, then there exist numbers c and d in $[a, b]$ such that for any $x \in [a, b]$
   $f(c) \leq f(x) \leq f(d)$

   (c) Mean Value Theorem: If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b). Then there exists a point $c \in (a, b)$ such that
   $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

15. Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, prove there exists a point $c \in (a, b)$ such that $f'(c) = 0$.

Proof. By extreme value theorem, since f is continuous, f has both max and min on $[a, b]$. If either max or min is at a or b, then since $f(a) = f(b)$, the function is a constant function with $f'(c) = 0$ for any $c \in (a, b)$. Else, if max and min are not at a or b, by Ex65, the derivative at the max or min is 0. Thus, there exists $c \in (a, b)$ which is max or min such that $f'(c) = 0$. $\square$

16. State the definitions.

(a) We say that a number s is a Riemann sum for a bounded function f on $[a, b]$ over the partition $P = t_0, t_1, ..., t_n$ of $[a, b]$ if there exists a finite sequence of numbers $x_1, x_2, ..., x_n$ such that $x_i \in [t_{t-1}, t_i]$ for $i = 1, 2, ..., n$ and $s = \sum_{i=1}^{n} f(x_i)(t_i - t_{i1})$

(b) We say that $U_P f$ is an upper Riemann sum for a bounded function f on [a, b] over the partition $P = t_0, t_1, ..., t_n$ of $[a, b]$ if there is a finite sequence of numbers $y_1, y_2, ..., y_n$ such that for each positive integer i :
$U_P f = \sum_{i=1}^{n} sup\{f(x) : x \in [t_{i1}, t_i]\}(t_i - t_{i-1})$

(c) The upper Riemann integral of a bounded function f on the interval $[a, b]$ is defined to be $U \int_a^b f = inf\{U_P f :$ P is a partition of $[a, b]\}$

17. Assume that P and Q are partitions of the interval $[a, b]$ and that f is a bounded function on $[a, b]$. Further assume that Q is a refinement of P. Determine whether $L_Q f \leq L_P f$, or $L_Q f \geq L_P f$. Then prove your assertion.

$L_Q f \geq L_P f$. Let P be a partition of $[a, b]$ such that $P = \{p_1, p_2, ..., p_n\}$ for $i = 1, 2, 3, ..., n$, and f be a bounded function on $[a, b]$. Let Q be a partition of $[a, b]$ such that $Q = P \cap [p_{i-1}, p_i] = q_0 = p_{i-1}, q_1, q_2, ..., q_{m-1}, q_m = p_i$ for $j = 1, 2, 3, ..., m$. Then, by infinmum properties, $inf\{f(x) : x \in [q_{j-1}, q_j]\} \geq inf\{f(x) : x \in [p_{i-1}, p_i]\}$. Also, since $\sum_{q=1}^{n}(q_i - q_{i-1}) = |a - b| = \sum_{p=1}^{n}(p_i - p_{i-1})$,

$$L_Q f = \sum_{j=1}^{m} inf\{f(x) : x \in [q_{j-1}, q_j]\}(q_j - q_{j-1}) \geq \sum_{q=1}^{m} inf\{f(x) : x \in [p_{i-1}, p_i]\}(q_j - q_{j-1})$$

$$= inf\{f(x) : x \in [p_{i-1}, p_i]\} \sum_{j=1}^{m}(q_j - q_{j-1}) = inf\{f(x) : x \in [p_{i-1}, p_i]\}(p_i - p_{i-1})$$

$$\geq \sum_{i=1}^{n} inf\{f(x) : x \in [p_{i-1}, p_i]\}(p_i - p_{i-1}) = L_P f$$

Thus, $L_Q f \geq L_P f$ if Q is a refinement of P. □

18. Prove the following statement: If f is Riemann integrable on $[a, b]$ then for every $\epsilon > 0$ there exists a partition P such that $U_P f - L_P f < \epsilon$

Proof. Let $\epsilon > 0$. For some partition R, Q on $[a, b]$, by Question 4's result and other properties related to question 4's result

$$U \int_a^b f + \frac{\epsilon}{2} > U_R f$$

$$L \int_a^b f - \frac{\epsilon}{2} < L_Q f \text{ implies } - L \int_a^b f + \frac{\epsilon}{2} > L_R f$$

Since f is Riemann integrable, $U \int_a^b f = L \int_a^b f$. Adding together, we get

$$U \int_a^b f + \frac{\epsilon}{2} - L \int_a^b f + \frac{\epsilon}{2} > U_R f - L_Q f \text{ implies } U_R f - L_Q f < \epsilon$$

Let P be a partition such that $P = R \cup Q$, then, P is a refinement of R and Q. Thus, by question 17 and other properties related to question 17,

$$U_P f - L_P f \leq U_R f - L_Q f < \epsilon$$

Thus, such partition exists. □

19. Prove that a constant function $f(x) = 1$ on [0,1] is Riemann integrable such that $\int_0^1 1dx = 1$.

Proof. Let P be a partition of $[a, b]$ such that $P = \{0 = p_1, p_2, ..., p_n = 1\}$ for $i = 1, 2, 3, ..., n$ and $\sum_{p=1}^{n}(p_i - p_{i-1}) = (b - a) = 1$. Since f is constant, $sup\{f(x) : x \in [p_{i-1}, p_i]\} = inf\{f(x) : x \in [p_{i-1}, p_i]\} = 1$, then

$$U_P f = L_P f = 1 * \sum_{i=1}^{n}(p_i - p_{i-1}) = 1 - 0 = 1$$

Thus, by definition, f is Riemann integrable such that $\int_0^1 1dx = 1$. □