MAT 380 REAL ANALYSIS

• A sequence (a_n) is **increasing** if and only if $a_n \leqslant a_{n+1}$ for all $n \in \mathbb{N}$;
decreasing if and only if $a_n \geqslant a_{n+1}$ for all $n \in \mathbb{N}$;
monotone if it is either increasing of decreasing.
• Set $A \subset \mathbb{R}$ is **bounded** if and only if there exists a number C such that $|a| \leqslant C$ for all $a \in A$.
• A sequence of numbers $(a_n)_{n=1}^{\inf}$ **converges** to the point L, if for every $\epsilon > 0$, there exists a natural number $N > 0$ such that $|an - L| < \epsilon$ for all $n \geqslant \mathbb{N}$.
• **(Ex13)** Assume $s \in R$ is an upper bound for a set $M \subset \mathbb{R}$. Then $s = sup(M)$ if and only if for every $\epsilon > 0$ there exists an element $x \in M$ such that $s - \epsilon < M$ ($x < i + \epsilon$ for infimum as i is a lowerbound).

Exercise 32. (*Resubmit,*)
Prove that if a sequence is monotone and bounded, then it converges.

Proof. Suppose a sequence $A = \{a_n | n \in \mathbb{N}\}$ that is monotone and bounded. Since A has an upper-bound, by axiom of completeness, there exists $s = sup(A)$. By Ex13, for some $N \in \mathbb{N}$,

$$a_N > s - \epsilon$$

Assuming that (a_n) is increasing, for all $n \geqslant N$,

$$a_n \geqslant a_N$$

for all $a_n \in A$, since s is an upperbound of a_n,

$$s \geqslant a_n$$

$$s + \epsilon > s \geqslant a_n \geqslant a_N > s - \epsilon$$

Thus, $|a_n - s| < \epsilon$ for all $n \leqslant N$, by def of convergence, $A = \{a_n | n \in \mathbb{N}\}$ converges when A is increasing. Similarly, when A is decreasing, consider $i = inf(A)$, by property of infimum, definition of decreasing, and Ex13, for all $n \geqslant N$,

$$i - \epsilon < s \leqslant a_n \leqslant a_N < i + \epsilon$$

Thus, $|a_n - i| < \epsilon$ for all $n \leqslant N$, by def of convergence, $A = \{a_n | n \in \mathbb{N}\}$ converges when A is decreasing. Therefore, combining both cases, $A = \{a_n | n \in \mathbb{N}\}$ converges. □

• Let (a_n) be a sequence of real numbers and let $n_1 < n_2 < n_3 < n_4 < ...$ be an increasing sequences of natural numbers. Then the sequence $a_{n_1}, a_{n_2}, a_{n_3}, ...$ is called a **subsequence** of (a_n) and is denoted by (a_{n_j}), where $j \in \mathbb{N}$ indexes the subsequence.
• **Nested Interval Property**. For each $n \in \mathbb{N}$, assume we are given a closed interval $I_n = [a_n, b_n] = x \in \mathbb{R} : a_n \leqslant x \leqslant b_n$. Assume also that I_n contains I_{n+1}. Then, the resulting nested sequence of closed intervals $I_1 \supseteq I2 \supseteq I3 \supseteq I4 \supseteq ...$ has a nonempty intersection; that is $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

Exercise 34. (*Resubmit,*)
Prove that every bounded sequence has a convergent subsequence.

Proof. [Constructed a nested interval with a subsequence] Let $(a_n)_{n=1}^{\infty}$ be a bounded sequence such that for all $n \in \mathbb{N}, \exists c \in \mathbb{R}^+$, $|a_n| \leqslant c$, so interval $[-c, c]$ of length $2c$ contains infinite many terms of (a_n). At least one half of the interval denoted as I_1, is $[-c, 0]$ or $[0, c]$ with a length of c, contains infinite many terms of (a_n), choose one term $n_1 \in (a_n)$ denoted as (a_{n_1}). Then, divide I_1 by half, pick the interval that contains infinite many terms of (a_{n_1}) with a length of $c/2$ as I_2, and pick one of the terms as such that $n_2 > n_1$, then $a_{n_2} \in I_2$. Continuing this fashion, a sequence of nested intervals can be constructed

$$I_1 \supseteq I_2 \supseteq I_3 ... \supseteq I_n$$

where I_n has a length of $c/2^n$. Since each of the interval contains infinite many terms of (a_n), a subsequence (a_{n_j}) can be construct ed by picking one term out of each interval from I_1 to I_j.

[Prove convergence] By Nested Interval Property, there is at least one $x \in (a_n)$ contained in $\bigcap_{j=1}^{\infty} I_j$. Let $\epsilon > 0$. Since the length of $I_n = c/2^n$ converges to 0, let (by Archimedian there exists N) $N \in \mathbb{N}$ such that for all $j \geqslant N$, the length of $I_j < \epsilon$. Then, for all $j \geqslant N$,

since a_{n_j} and x are both in the closed interval I_j,

$$|a_{n_j} - x| < \epsilon$$

Thus, by definition of convergence, the subsequence (a_{n_j}) converges. □

• A sequence (a_n) is called a **Cauchy sequence** if, for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that whenever $n, m \geqslant N$ we have $|a_n a_m| < \epsilon$.

• 1. A function f is **continuous** at the point $x = c$ if for every sequence $(a_n)_{n=1}^{\infty}$ that converges to c, the sequence $(f(a_n))_{n=1}^{\infty}$ converges to $f(c)$. If a function is continuous at every point of some set S, we say f is continuous on S.
• 2. A function f is **continuous** at a point x if for every $\epsilon > 0$ there exists $\delta > 0$ such that $|f(y) - f(x)| < \epsilon$ for all y such that $|x - y| < \delta$.

Exercise 43. (*Resubmit,*)
Prove that $f(x) = -5x + 3$ is continuous for all real numbers.

Proof by definition. Let $\epsilon > 0$, for $x, x_0 \in \mathbb{R}$, choose $\cancel{\delta > 0}$ ~~and~~ $\delta = min\{\epsilon/5, 1\}$ ~~such that~~ suppose $|x - x_0| < \delta$, then,

$$|f(x) - f(x_0)| = |-5x + 3 + 5x_0 - 3| = 5|x - x_0| < 5\epsilon/5 = \epsilon$$

Since $f(x)$ converges to $f(x_0)$ for $x, x_0 \in \mathbb{R}$, $f(x)$ f(x) is continuous for all real numbers. □

• **(Ex26) Convergence Property** Suppose $(a_n)_{n=1}^{\infty}$ converges to L and $(b_n)_{n=1}^{\infty}$ converges to K, then $(a_n b_n)_{n=1}^{\infty}$ converges to LK.

Exercise 44. (*Resubmit,*)
Prove that $f(x) = x^2$ is continuous for all real numbers.

Proof by decomposing function. Let $(a_n) \in \mathbb{R}$ be a sequence that converges to c, by Ex26, $(a_n^2)_{n=1}^{\infty}$ converges to $c^2 = f(c)$. Since the sequence $(f(a_n))_{n=1}^{\infty}$ converges to $f(c)$, $f(x)$ is continuous for all real numbers. □

Exercise 45. (*Resubmit,*)
Prove that the function $f(x)$ where $f(x) = 1$ for all $x \neq c$ and $f(c) = 2$ is not continuous at $x = c$.

Prove by a counter-example. define (x_n): let (x_n) be a sequence. Since $|f(x_n) - f(c)| = 1 > \epsilon$ for any arbitrary $\epsilon > 1$, Thus, $f(x_n)$ does not converge to $f(c)$, and by definition, $f(x)$ is not continuous. □

Exercise 46. (*Resubmit,*)
Prove that our two definitions of continuity are equivalent.

Proof. $\Rightarrow$ Since for (a_n) that converges to c, the sequence $(f(a_n))$ converges to $f(c)$. By definition of convergence, for every $\delta > 0$ and $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geqslant N$, $|a_n - c| < \delta$ and $|f(a_n) - f(c)| < \epsilon$.

$\Leftarrow$ Let (a_n) be a sequence that converges to c, then by definition of convergence, there exists $\delta > 0$ such that $|a_n - c| < \delta$. □

• 1. A function f is **continuous** at the point $x = c$ if for every sequence $(a_n)_{n=1}^{\infty}$ that converges to c, the sequence $(f(a_n))_{n=1}^{\infty}$ converges to $f(c)$. If a function is continuous at every point of some set S, we say f is continuous on S.
• 2. A function f is **continuous** at a point x if for every $\epsilon > 0$ there exists $\delta > 0$ such that $|f(y) - f(x)| < \epsilon$ for all y such that $|x - y| < \delta$.
• A sequence (a_n) is called a **Cauchy sequence** if, for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that whenever $n, m \geqslant N$ we have $|a_n - a_m| < \epsilon$.

!Exercise 51. (*Resubmit, 2021/02/08*)
Let f be a function defined on all of $\mathbb{R}$ and assume there is a constant c such that $0 < c < 1$ and $|f(x) - f(y)| \leqslant c|x - y|$ for all $x, y \in \mathbb{R}$.

(1) Show that f is continuous on $\mathbb{R}$.

Proof. Let $\epsilon > 0$, for $x, y \in \mathbb{R}$, choose $\delta > 0$ and $\delta = \dfrac{\epsilon}{c}$ such that $|x - y| < \delta$. Then

$$|f(x) - f(y)| \leqslant c|x - y| < c\delta = c\frac{\epsilon}{c} = \epsilon$$

Since $f(x)$ converges to $f(y)$ for $x, y \in \mathbb{R}$, f is continuous on $\mathbb{R}$. □

(2) Pick some point $y_1 \in \mathbb{R}$ and construct the sequence $y_1, f(y_1), f(f(y_1)), \ldots$ In general, $y_{n+1} = f(y_n)$. Show that the sequence (y_n) is a Cauchy sequence. (Hence we may let $y = \lim\limits_{n\to\infty} y_n$.)

Proof. Let $y_1 \in \mathbb{R}$ and construct the sequence $y_1, f(y_1), f(f(y_1)), \ldots$ □

(3) Prove that $f(y) = y$ and that it is unique in this regard.

prove. ??? □

(4) Prove that if x is any arbitrary point in $\mathbb{R}$ the the sequence $x, f(x), f(f(x)), ...$ converges to the y defined in (b).

prove. ??? □

• A function $f : \mathbb{R} \to \mathbb{R}$ is **uniformly continuous** on a set A if for every $\epsilon > 0$ there exists $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$.

Exercise 52. (*Homework 12, 2021/02/08*)
Prove that $f(x) = 4x + 2$ is uniformly continuous on $\mathbb{R}$.

Proof. Let $\epsilon > 0$. There exists $\delta = \epsilon/4 > 0$ such that for every $x, y \in \mathbb{R}$, $|x - y| < \delta = \epsilon/4$.

$$|f(x) - f(y)| = |4x + 2 - 4y - 2| = 4|x - y| < 4\epsilon/4 = \epsilon$$

Thus, by definition, since for every $x, y \in \mathbb{R}$, $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$, the function is uniformly continuous on $\mathbb{R}$. □

Exercise 53. (*Homework 12, 2021/02/08*)
Prove that $f(x) = x^2 + 2$ is uniformly continuous on $[a, b]$. (not uniformly continuous on $\mathbb{R}$)

Proof. Let $\epsilon > 0$. There exists $\delta > 0$ such that for every $x, y \in \mathbb{R}$, $|x - y| < \delta$. Since $x, y \in [a, b]$, without loss of generality, assume that $a < b$. Then $a \leqslant x \leqslant b$ and $a \leqslant y \leqslant b$ implies $2a \leqslant x + y \leqslant 2b$. Let $c = max\{|2a|, |2b|\}$, then $|x + y| \leqslant c$. Choose $\delta = \frac{\epsilon}{c}$, then

$$|f(x) - f(y)| = |x^2 + 2 - y^2 - 2| = |x^2 - y^2| = |x + y||x - y| \leqslant c|x - y| < c\delta = c\frac{\epsilon}{c} = \epsilon$$

Thus, by definition, since for every $x, y \in [a, b]$, $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$, the function is uniformly continuous on $[a, b]$. □

Exercise 54. (*Homework 12, 2021/02/08*)
Prove that $f(x) = 1/x^2$ is uniformly continuous on $[1, \infty)$.

Proof. Let $\epsilon > 0$. There exists $\delta > 0$ such that for every $x, y \in \mathbb{R}$, $|x - y| < \delta$. Since $x, y \in [1, \infty)$, $1 \leqslant x$ and $1 \leqslant y$ implies $\frac{1}{(xy)^2} \leqslant 1$ and $|x + y| \geqslant 2$. Choose $\delta = \frac{\epsilon}{2}$, then

$$|f(x) - f(y)| = |1/x^2 - 1/y^2| = \frac{|x^2 - y^2|}{x^2y^2} \leqslant \frac{|x+y||x-y|}{(xy)^2} \leqslant 2|x - y| < 2\delta = 2\frac{\epsilon}{2} = \epsilon$$

Thus, by definition, since for every $x, y \in [1, \infty)$, $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$, the function is uniformly continuous on $[1, \infty)$. □

- A function $A \to \mathbb{R}$ is called **Lipschitz** if there exists a bound $M > 0$ such that $|\frac{f(x) - f(y)}{x - y}| \leqslant M$ for all $x, y \in A$ (and $x \neq y$).

Exercise 55. (*Homework 12, 2021/02/08*)
Show that if $f : A \to \mathbb{R}$ is Lipschitz, then it is uniformly continuous on A.

Proof. Let $\epsilon > 0$. There exists $\delta > 0$ such that for every $x, y \in A$, $|x - y| < \delta$. Since f is Lipschitz, $|\frac{f(x) - f(y)}{x - y}| \leqslant M$ implies $|f(x) - f(y)| \leqslant M|x - y|$. Choose $\delta = \frac{\epsilon}{M}$, then

$$|f(x) - f(y)| \leqslant M|x - y| < M\delta = M\frac{\epsilon}{M} = \epsilon$$

Thus, by definition, since for every $x, y \in A$ and $x \neq y$, $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$, the function is uniformly continuous on A. □

- Let $f : A \to R$, for $A \subset R$, $x, c \in A$. f is **continuous** at c if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|x - c| < \delta$ and $x \in A$ implies that $|f(x) - f(c)| < \epsilon$.

Exercise 56. (*Homework 12,*)
Suppose a function f is continuous on $[a, b]$ and $c \in [a, b]$ that satisfies $f(c) > 0$. Prove that there exists an open interval A such that $f(x) > 0$ for all $x \in A$.

Proof by contradiction. Since f is continuous on $[a, b]$ and let $c \in [a, b]$, then for every $\epsilon > 0$, there exists $\delta > 0$ such that, $|x - c| < \delta$ implies $|f(x) - f(c)| < \epsilon$ for all $x \in [a, b]$. Suppose that there exists $z \in [a, b]$ such that $f(z) \leqslant 0$. Then, $|x - c| < \delta$ implies $|f(x) - f(c)| < \epsilon$ for every $\epsilon > 0$. Since $f(x) > 0, f(c) \leq 0$, $|f(x) - f(c)| > 0$ contradicting with the assumption that $|f(x) - f(c)| < \epsilon$. Thus, $f(x) > 0$ for all $x \in A$. □

Exercise 57. (*Homework 12,*)
Suppose that f is a continuous function defined on the interval $[a, b]$ and assume that $f(a) < 0 < f(b)$. Prove that the set $A = \{x \in [a, b] : f(x) < 0\}$ has a supremum c such that $a \leq c \leq b$.

Proof. If $f(a) < 0 < f(b)$, then there exists at least $a \in A$ such that $f(a) < 0$, so set A is not empty. Since $f(a) < 0 < f(b)$, A is upper-bounded by b, by axiom of completeness, a supremum exists. Let $c = sup(A)$, since c is the supremum, b is an upperbound of A, and a is an element of A, by definition of supremum, $a \leq c \leq b$. □

Exercise 58. (*Homework 12,*)
Suppose that f is a continuous function defined on the interval $[a, b]$ and assume that $f(a) < 0 < f(b)$. Prove that the set $A = \{x \in [a, b] : f(x) < 0\}$ has a supremum c such that $a \leq c \leq b$.

Proof. If $f(a) < 0 < f(b)$, then there exists at least $a \in A$ such that $f(a) < 0$, so set A is not empty. Since $f(a) < 0 < f(b)$, A is upper-bounded by b, by axiom of completeness, a supremum exists. Let $c = sup(A)$, since c is the supremum, b is an upperbound of A, and a is an element of A, by definition of supremum, $a \leq c \leq b$. □

Exercise 59. (*Homework 12,*)
Suppose f is a continuous function defined on the interval $[a, b]$ and assume that $f(a) < 0 < f(b)$. Prove that there exists a number $c \in (a, b)$ such that $f(c) = 0$.

Proof by contradiction. (copied from HW 13, added in the contradiction marked as highlighted)

Let $c \in [a, b]$ and $f(c) = 0$. Since $f(a), f(b) \neq 0$, $c \neq a, b$ so that $c \in (a, b)$. Since f is continuous, for every $\epsilon > 0$, there exists $\delta > 0$ such that $|x - c| < \delta$ implies $|f(x) - f(c)| < \epsilon$ for every $x \in [a, b]$. Suppose that $f(c) \neq 0$, and choose $\epsilon = 0.1|f(c)|$, then $|x - c| < \delta$ implies $|f(x) - f(c)| < 0.1|f(c)|$. If $f(c) < 0$,

$$0.1f(c) + f(c) < f(x) < -0.1f(c) + f(c) < 0 \rightarrow f(x) < 0.9f(c) < 0$$

Then, since $x \in (c - \delta, c + \delta)$ and $(c - \delta, c + \delta) \subset [a, b]$, so for the set $A = \{x \in [a, b] : f(x) < 0\}$, $c + \delta > c$ would be the upperbound of A contradicting with Ex56 where c is the upperbound.

Else If $f(c) > 0$, then similarly, $|f(x) - f(c)| < 0.1f(c)$ gives

$$0 < -0.1f(c) + f(c) < f(x) < 0.1f(c) + f(c) \to 0 < 0.9f(c) < f(x)$$

Then, since $x \in (c-\delta, c+\delta)$ and $(c-\delta, c+\delta) \subset [a,b]$, for $f(x) < 0$, the interval of x is $[a, c-\delta) \cup (c+\delta, b]$ in either way, contradicting that c is the least upper bound proven in Ex56. Thus, by contradiction, there must exist a number $c \in (a,b)$ such that $f(c) = 0$.

□

Exercise 60. (*Homework 12,*)
Suppose that f is a continuous function defined on the interval $[a,b]$ and assume that $f(a) < L < f(b)$. Prove that these exists a number $c \in (a,b)$ such that $f(c) = L$.

Proof. Define function $g(x) = f(x) - L$ for $x \in [a,b]$. Then,

$$f(a) - L < L - L < f(b) - L \to g(a) < 0 < g(b)$$

By Ex56, there exists a number $c \in (a,b)$ such that $g(c) = 0$. Since by definition of g, $g(c) = f(c) - L = 0$, $f(c) = L$. □

Suppose that f is a continuous function defined on the interval $[a,b]$ and assume that $f(b) < L < f(a)$. Prove that there exists a number $c \in (a,b)$ such that $f(c) = L$.

Proof. Define function $g(x) = -f(x)$ for $x \in [a,b]$, then,

$$f(b) < L < f(a) \to -f(b) > -L > -f(a) \to g(a) < -L < g(b)$$

By Ex57(a), since $g(a) < -L < g(b)$, there exists a number $c \in (a,b)$ such that $g(c) = -L$. Then, $g(c) = -f(c) = -L$ implies $f(c) = L$. □

• Let $f : D \to \mathbb{R}$ be a function, and suppose $c \in D$. Then f has a **derivative** $f'(c)$ at c if and only if for every sequence $(a_n)_{n=1}^{\infty}$ converging to c where $a_n \neq c$ and $a_n \in D$ for all $n \in \mathbb{N}$, the sequence $(\dfrac{f(a_n) - f(c)}{a_n - c})_{n=1}^{\infty}$ converges to $f'(c)$.

Exercise 61. (*Homework 12,*)
Let $f(x) = x^2 + 2$. Use the definition to prove that $f'(-3) = -6$.

Proof. Let $(a_n) \in D$ be a sequence and $-3 \in D$ such that $(a_n)_{n=1}^{\infty} \to -3$. Then,

$$f'(c) = (\frac{f(a_n) - f(c)}{a_n - c})_{n=1}^{\infty} = (\frac{((a_n)^2 + 2) - (3^2 + 2)}{a_n + 3})_{n=1}^{\infty} = (\frac{((a_n)^2 - 9}{a_n + 3})_{n=1}^{\infty} = (a_n - 3)_{n=1}^{\infty}$$

Since $(a_n)_{n=1}^{\infty} \to -3$, $(a_n)_{n=1}^{\infty} - 3 \to -3 - 3 = -6$. Thus, $f'(c) \to -6$. □

Exercise 62. (*Homework 12,*)
Let $f(x) = x^2$. Use the definition to prove that $f'(x) = 2x$, for all $x \in \mathbb{R}$.

Proof. Let $(a_n) \in \mathbb{R}$ be a sequence and $c \in \mathbb{R}$ such that $(a_n)_{n=1}^{\infty} \to c$. Then,

$$f'(c) = (\frac{f(a_n) - f(c)}{a_n - c})_{n=1}^{\infty} = (\frac{((a_n)^2) - (c^2)}{a_n - c})_{n=1}^{\infty} = (a_n + c)_{n=1}^{\infty}$$

Since $(a_n)_{n=1}^{\infty} \to c$, $(a_n)_{n=1}^{\infty} + c \to c + c = 2c$ for all $c \in \mathbb{R}$. Thus, $f'(c) \to 2c$ and $f'(x) = 2x$ for all $x \in \mathbb{R}$. □