Proof G.4 Distance in Graphs. Let G be a connected graph such that $rad(G) \leq k \leq diam(G)$. Prove that there exists a vertex $v \in V(G)$ such that $e(v) = k$.

By contradiction. Let $A = \{ecc(v) : \forall v \in V(G)\}$ which is the set of eccentricity value of each vertex in G. Assume that the graph is non-trivial so $V \neq \emptyset$ and $A \neq \emptyset$. Suppose that there does not exist a vertex $v \in V(G)$ such that $ecc(v) = k$. Then, all the vertices in G would have eccentricity to be either less than $rad(G)$ or greater than $diam(G)$. By definition, $diam(G)$ is the value of greatest eccentricity, and $rad(G)$ is the value of smallest eccentricity of G, so that $diam(G)$ is an upperbound of A, and $rad(G)$ is a lowerbound of A. Since for every $a \in A$, $a < rad(G)$ or $a > diam(G)$, it contradicts with $diam(G)$ being the upperbound and $rad(G)$ being the lowerbound of A. Thus, by contradiction, there must exists a vertex $v \in V(G)$ such that $e(v) = k$.

□

Since I'm allowed to use my Book, I firstly read about the definition of $rad(G)$ and $diam(G)$ in page 18 chapter 1.2.1. Then, by definition, I think that there must exists at least one vertex $v \in V(G)$ such that $rad(G) \leq e(v) \leq diam(G)$ otherwise it violates the definition. Since I'm less confident in saying things clearly in a direct proof, I used contradiction and relate the definition of "greatest" and "smallest" to bounds as I'm really familiar with them in the real analysis class.

Proof G.7 Line Graphs. Let G be Eulerian. Prove $L(G)$ is also Eulerian. *(You can also use this question to earn G.6 if you did not earn this on Quiz 1.)*

Proof. Let G be a Eulerian graph, by theorem 1.20, every vertex of G has even degree, which means that there are correspondingly $2k_n$ edges share each vertex of G for $(k_n) \in \mathbb{N}$. Let e be any edges in G. By definition of edge, e must connect to two end vertices denoted as v_1 and v_2. Assume that there are $2k_1$ edges share the same vertex v_1, and $2k_2$ edges share the same vertex v_2. Since v_1 and v_2 are connected by e, there are in total $2k_1 - 1 + 2k_2 - 1$ number of edges in G sharing a vertex of either v_1 or v_2 with e. By definition of line graph, edge e in G is considered to be a vertex denoted as e_1 in $L(G)$ and the edges that vertex e_1 in $L(G)$ connects are the corresponding edges in G that share a vertex. Since the vertex e_1 in $L(G)$ have in total $2k_1 - 1 + 2k_2 - 1 = 2(k_1 + k_2 - 1)$ edges which is an even number and e_1 is chosen arbitrarily, it implies that every vertex in $L(G)$ has even degree. Thus, by theorem 1.20, $L(G)$ is Eulerian. □

Since I cannot remember any of the definition and I can use the book, I first looked up the definition of L(G) found in Page 16 and the property of Eulerian in Page 56 referred as theorem 1.20. Now, the question I interpreted becomes if every vertex in G has even degree, then every vertex in L(G) has even degree. Further, by reading the definition of edges in line graph, the question becomes assuming that every vertex in G has even degree, WLOG find an arbitrary edge in G, find all vertices it connects and all the rest edges that share the same vertex, then try to prove that the total number of edges in common vertex are even. Then, I know that number corresponding to the number of edges connect to the arbitrary vertex in L(G).

Proof G.8 Planarity. Use induction to prove that all trees are planar.

Proof. Let T be a tree with at least one vertex. Let n be the order of T, q be the size of T, and r be the number of regions of T. Assume T is planar. Then, by Euler's formula, $n - q + r = 2$.
Base Case: Let T has only one vertex, then n = 1, q = 0, r = 1 implies that n - q + r = 1 - 0 + 1 = 2.
Inductive Step: Let T has n vertices, q edges. Consider T' to be a tree after removing in a vertex in T. Since T' is a tree and remain connected, the vertex being removed can only be leaves in T because only leaves in T are not bridges. Then, for T', there will be $n-1$ vertices. Since every leaf would have a degree of 1, there will be $q-1$ edges in T'. Because tree is acyclic, removing an edge does not result in an decrease in regions. Then, $(n-1) - (q-1) + r = n - 1 + q + 1 + r = n - q + r = 2$. Thus, by induction, for any tree, $n - q + r = 2$. Therefore, by Euler's formula, all trees are planar.

□

I first think about using tree's property to prove this, it's natural to see that since it is a tree, it has n vertices with n-1 edges, and since it's acyclic, the region is 1 so $n - (n-1) + 1 = 2$. however, because I need to use induction, I then think about assuming that the Euler's formula holds, then either adding or removing a vertex of the tree of n vertices would still lead to the same result.

Proof G.9 Chromatic Number. Let G be a graph of order n. Prove that:

$$\frac{n}{\alpha(G)} \le \chi(G) \le n + 1 - \alpha(G).$$

You are not allowed to use your homework for this exercise...Only your brain and the book, and the coloring worksheet you worked through last week.

Proof. Let G be a graph of order n. Let $\alpha(G) = a$ and $\chi(G) = k$.
RHS: Since at minimum using k color can color all vertices in G, if we remove all the vertices that are in the largest independent set, then either we still need k proper coloring, or we need $k - 1$ proper coloring since by definition of independence set, all vertices in them can be colored in the same coloring because they are not pairwise adjacent.
Case 1: If we still need k proper coloring after removing the vertices in the largest independent set. Since the worst case is that all the rest vertices need a unique coloring, $\chi(G) \le n - \alpha(G)$ that implies $\chi(G) \le n + 1 - \alpha(G)$.
Case 2: If we need $k-1$ proper coloring, then the worst case $\chi(G)$ equals the total number of remaining vertices, so $\chi(G) - 1 \le n - \alpha(G)$ that implies $\chi(G) \le n + 1 - \alpha(G)$.

LHS: Since we can label all the vertices G by at least k colors, if we put all the vertices into k sets based on their coloring, we would get k independent sets, because two vertices will have the same color if they are not pairwise adjacent according to the greedy algorithm. Let $v_1, v_2, ..., v_k$ denote the size of each independent vertex sets. By definition of independence number, $\alpha(G) = max(v_1, v_2, ..., v_k) \neq 0$ for non-trivial graphs. Then, the total number of vertices in G is the sum of number of all vertices in each set. Thus, we get $n = \sum_{i=1}^{k} v_i \le k * max(v_1, v_2, ..., v_k) = k\alpha(G)$ which implies $\frac{n}{\alpha(G)} \le \chi(G)$.

Therefore, considering both sides, $\frac{n}{\alpha(G)} \le \chi(G) \le n + 1 - \alpha(G)$.

□

For right hand side, I first visualize how the coloring works for each vertices and what is special to the coloring of independence set in a graph. Also, by observing the inequality, in n - $\alpha(G)$, the subtraction implies that we need to somehow remove the largest independent set and see how the rest can be colored. Based on the fact that if a coloring can work for the whole graph, then it must work for the subtracted graph, I split the scenario into two cases depending on the worst case (every vertex needs a new coloring) to derive the inequality. For the left hand side, since I'm not sure how the division may reflect the coloring process, I rearrange it into $n \le \alpha(G)\chi(G)$. I split the vertices in a graph based on their coloring and glued those pieces back to set up the inequalities. And inspired by the infinity norm derivation in the numerical linear algebra class, I transformed the inequality into the desired form.