Problem 1. The total number of pages is n; the size of the cache is k pages; and the total number of requests is T.

(1) Find the expected competitive ratio of LRU assuming that $T \gg n$ and $n \gg k$.

Since the total number of distinct pages is much greater than k, $\frac{k}{n}$ would be really small. Then the probability that there will be a page miss would be $1 - \frac{k}{n}$which is quite close to 1 and the expected page miss would be close to T. For this case, all the algorithms would behave similarly leading to $E[\frac{\text{\# LRU misses}}{\text{\# OPT misses}}] \approx 1$.

(2) Find the expected competitive ratio of LRU assuming that $T \gg n$ and $n = k + 1$.

Define a_i to be blocks of requests. Assume that LRU has at most k misses in each block. Since once a page is in the cache, it will not be evicted until k other distinct pages being requested. Then, at each block, there would be at most one miss per distinct page. Let p denotes the probability that a miss happened for a page. Let b be number of blocks. Since in the first block request, there must be k misses, and in the rest of blocks, there must be at least one page that must be missed. Then, there will be $1 * k + (b-1)(1 + (k-1)p)$ expected page misses. For the optimal algorithm, since it also has a cache of k. Then for every block after the first block which contains a set of brand-new k requests, it will have at least 1 miss. Thus, the expected miss will be no less than $(b-1)$. Since there are $k+1$ distinct items with each request being chosen randomly, $p = \frac{1}{k+1}$. Then, the expected ratio is $\frac{k + (\frac{T}{k} - 1)(1 + \frac{k-1}{k+1})}{\frac{T}{k} - 1} \approx k$.

Problem 2. Design FPT algorithms for the following problems and prove that they are correct.

(1) Finding a cycle of length exactly k in a graph G.

Algorithm:
Randomly assign a coloring $c : V \to \{1, 2, ..., k-1\}$ to G with each vertex receives a random color independent from others.
For a vertex $s \in V$, look for the longest colorful path p in G
If $|p| = k - 1$ and there exists an edge between the end point of p and s, return p
Else repeat for e^{k-1} times

Proof:
Since finding a cycle of length exactly k can also be regarded as finding a path in graph G of length $k-1$ with the beginning vertex and ending vertex sharing the same edge, we can apply the color coding algorithm. Then,
$Pr\{$if there is a path of $k-1$ with the correct shared edge, then path is colorful$\}$
$= \frac{(k-1)!}{(k-1)^{k-1}} \geq \frac{(\frac{k-1}{e})^{k-1}}{(k-1)^{k-1}} = e^{-(k-1)}$ according to Stirling's approximation.

Then, the probability of not finding such a cycle is at most $1 - e^{-(k-1)}$. By repeating e^{k-1} times, the probability of error is bounded by $\frac{1}{e}$.

(2) Finding a cycle of length from k to $2k$ in a graph G.

Algorithm:
Execute algorithm (1) and getting the result p_1
If p_1 does not exist, return
Else, for each vertex v_i in p_1, run algorithm (1) again with v_i as the starting vertex and a new terminating condition that if $|p_2| \leq k-1$ and the ending point of $p_2 \in p_1$, then return $p2 \cup p1$.
Repeat for e^{2k-1} times.

Proof:
$Pr\{$get a good result p_1 from algorithm (1)$\} \leq e^{-(k-1)}$
$Pr\{$get a good result p_2 from modified algorithm (1)$\} \leq e^{-(k-1)}$
Then, the probability of finding a cycle satisfying the condition is at least $(e^{-(k-1)})^2$ which is a lower bound, so when we repeat the process for e^{2k-1} times with high probability the algorithm is correct.

(3) Finding a cycle of length at least k in a graph G.

Algorithm:
Randomly assign a coloring $c : V \rightarrow \{1, 2, ..., 2k-1\}$ to G with each vertex receives a random color independent from others.
For a vertex $s \in V$, look for the longest colorful path p in G
If $|p| \geq k-1$ and there exists an edge between the end point of p and s, return p
Else contract an edge between u, v and repeat the algorithm with $G/E(uv)$ for up to $|V|^2$ times

(4) Prove the following claim: If a graph G contains a cycle of length at least 2k, then after contracting an arbitrary edge, it still contains a cycle of length at least k.

Let u be any vertex in the cycle and v be a neighbor of u. Let $G' = G/e$ denote the graph after contracting the edge between u and v in G. Then, $|V(G')| = |V(G)| - 1$
If we contract an edge on the cycle that has length at least 2k, since there's no cut vertex on the cycle, then contraction leads to a cycle of length $2k-1 \geq k$ for $k \geq 1$. For an edge not on the cycle, if u, v has no neighbor vertices $\in C$, then the contraction would resulting G' still contains a cycle of length at least 2k. Otherwise, denote the obtained vertex from contraction as t. Since G' is still connected, for the worst case, t would be a cut vertex leading to a cycle of length k.

Problem 3. Consider the following linear program (LP).

(1) Write the dual LP for the LP above.

Maximize $6y_1 + 5y_2 + y_3 + y_4$
subject to
$y_1 + y_2 + y_3 + y_4 \leq 5$
$y_1 + y_2 - y_4 \leq 3$
$y_1 - y_3 \leq 2$
$y_1, y_2, y_3, y_4 \geq 0$

(2) Solve the primal and dual LPs: Find the optimal solutions $x*$ and $y*$ and then compare the LP values for $x*$ and $y*$.

$x_1^* = 3, x_2^* = 2, x_3^* = 1$ with LP value $= 5 * 3 + 3 * 2 + 2 * 1 = 23$
$y_1^* = 2, y_2^* = 2, y_3^* = 0, y_4^* = 1$ with LP value $= 6 * 2 + 5 * 2 + 1 * 1 = 23$
Since the values are equal, this is a strong duality.

(3) Verify that the complementary slackness conditions hold for $x*$ and $y*$.

Since $x_1^* = 3, x_2^* = 2, x_3^* = 1$ are non-zero, the constraints of y has no slack. Then,
$y_1 + y_2 + y_3 + y_4 = 5$
$y_1 + y_2 - y_4 = 3$
$y_1 - y_3 = 2$
y_j^* is a solution to the system of equations $A^T y_j$ such that $A^T y_j = c_j$. Similarly, for primal, since $y_1, y_2, y_4 > 0$, the first, second and fourth constraint of primal cannot be slack, and the results of Ax_i^* are consistent with the conditions such that $Ax_1^* = b_1, Ax_2^* = b_2, Ax_4^* = b_4$ but $Ax_3^* = 3 - 1 \geq b_3 = 2$. Therefore, the complementary slackness conditions hold.