# MAT 380 REAL ANALYSIS

**Exercise 33.** (*Resubmission,* )
Prove that subsequences of a convergent sequence converge to the same limit as the original sequence.

*Proof.* Let $(a_n)$ be a convergence sequence with $L$ as the limit. Then suppose that $(a_{n_j})$ is a subsequence of $(a_n)$. Let $\epsilon > 0$. Since $a_n \to L$ when $n \to \infty$, there exists a $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - L| < \epsilon$. Then, by definition of subsequence, $n_J \geq N$ implies that for all $j \geq J$, $|a_{n_j} - L| < \epsilon$. Thus, by definition of convergence, the subsequence convergences to the same limit. □

**Exercise 36.** (*Resubmission,* )
Prove that Cauchy sequences are bounded.

*Proof.* Let $(a_n)$ be a cauchy sequence. Let $\epsilon = 1$, by definition of cauchy, there exists $N$ such that for every $m, n > N$, $|a_n - a_m| < \epsilon = 1$. Let $m$ be some $M$th term in the sequence. Then by triangle inequality,

$$|a_n| = |a_n - a_m + a_m| \leq |a_n - a_m| + |a_m| < 1 + |a_M|$$

Then, $|a_n| \leq max\{|a_1|, |a_2|, ...., |a_{M-1}|, 1 + |a_M|\}$ for all $n$. Thus, we can find a constant $c$ in $max\{|a_1|, |a_2|, ...., |a_{M-1}|, 1 + |a_M|\}$ such that $|a_n| \leq c$, and therefore, the cauchy sequences are bounded. □

• 1. A function $f$ is **continuous** at the point $x = c$ if for every sequence $(a_n)_{n=1}^{\infty}$ that converges to $c$, the sequence $(f(a_n))_{n=1}^{\infty}$ converges to $f(c)$. If a function is continuous at every point of some set $S$, we say $f$ is continuous on $S$.
• 2. A function $f$ is **continuous** at a point $x$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that $|f(y) - f(x)| < \epsilon$ for all $y$ such that $|x - y| < \delta$.

**Exercise 47.** (*Resubmission,* )
Use the second definition of continuity to prove that $f(x) = x^2 + 2$ is continuous at $x = 2$.

*Proof.* Let $\epsilon > 0$. Choose $\delta = min\epsilon/5, 1$. Suppose $|x-2| < \delta$, then, by triangle inequality, $|x| - |2| \leq |x-2| < 1$ implies $|x| + 2 < 5$. Then,

$$|f(x) - f(2)| = |x^2 + 2 - (2^2 + 2)| = |x^2 - 4| = |x+2||x-2| \leq (|x|+2)|x-2| < 5\frac{\epsilon}{5} = \epsilon$$

Since $f(x)$ converges to $f(2)$ for $x \in \mathbb{R}$, $f(x)$ is continuous at $x = 2$. □

**Exercise 50.** (*Resubmission,* )
Given $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$, assume that the range $f(A) = f(x) : x \in A$ is contained in the domain $B$ so that the composition $(g \circ f)(x) = g(f(x))$ is well defined on $A$. Prove the following: If $f$ is continuous at $c \in A$ and if $g$ is continuous at $f(c) \in B$, then $(g \circ f)$ is continuous at $c$.

*Proof.* Let $\epsilon > 0$. By definition of continuity, there exists $\delta_1 > 0$ such that $|f(x) - f(c)| < \delta_1$ implies $|g(f(x)) - g(f(c))| < \epsilon$. Similarly, there exists $\delta_2 > 0$ such that $|x - c| < \delta_2$ implies $|f(x) - f(c)| < \delta_1$. Thus, for all $\epsilon > 0$, $|x - c| < \delta_2$ implies $|g(f(x)) - g(f(c))|$, and therefore, $g(f(x)) = (g \circ f)(x)$ is continuous at $c$. □

**Exercise 51(d).** (*Resubmission,* )
Let $f$ be a function defined on all of $\mathbb{R}$ and assume there is a constant $c$ such that $0 < c < 1$ and $|f(x) - f(y)| \leqslant c|x - y|$ for all $x, y \in \mathbb{R}$. Prove that if $x$ is any arbitrary point in $\mathbb{R}$ the the sequence $x, f(x), f(f(x)), \ldots$ converges to the $y$ defined in $(b)$.

*Proof.* By (b), $y_{n+1} = f^n(y)$, then

$$|f^n(y) - x| = |f(f^{n-1}(y)) - f(x)| \leq C|f^{n-1}(y) - x| \leq \ldots \leq C^n|y - x|$$

Since $n \to 0$ implies that $|C^n| \to 0$, $f^n(y) \to x$; therefore, the sequence converges. □

- A function $f : \mathbb{R} \to \mathbb{R}$ is **uniformly continuous** on a set $A$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$.

**Exercise 93.** (*Homework 18, 2021/02/08*)
Prove that $f(x) = 4x + 2$ is uniformly continuous on $\mathbb{R}$.

*Proof.* Let $\epsilon > 0$. There exists $\delta = \epsilon/4 > 0$ such that for every $x, y \in \mathbb{R}$, $|x - y| < \delta = \epsilon/4$.

$$|f(x) - f(y)| = |4x + 2 - 4y - 2| = 4|x - y| < 4\epsilon/4 = \epsilon$$

Thus, by definition, since for every $x, y \in \mathbb{R}$, $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$, the function is uniformly continuous on $\mathbb{R}$. □

**Exercise 94.** (*Homework 18, 2021/02/08*)
Prove that $f(x) = x^2 + 2$ is uniformly continuous on $[a, b]$. (not uniformly continuous on $\mathbb{R}$)

*Proof.* Let $\epsilon > 0$. There exists $\delta > 0$ such that for every $x, y \in \mathbb{R}$, $|x - y| < \delta$. Since $x, y \in [a, b]$, without loss of generality, assume that $a < b$. Then $a \leqslant x \leqslant b$ and $a \leqslant y \leqslant b$ implies $2a \leqslant x + y \leqslant 2b$. Let $c = max\{|2a|, |2b|\}$, then $|x + y| \leqslant c$. Choose $\delta = \frac{\epsilon}{c}$, then

$$|f(x) - f(y)| = |x^2 + 2 - y^2 - 2| = |x^2 - y^2| = |x + y||x - y| \leqslant c|x - y| < c\delta = c\frac{\epsilon}{c} = \epsilon$$

Thus, by definition, since for every $x, y \in [a, b]$, $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$, the function is uniformly continuous on $[a, b]$. □

- A function $f : \mathbb{R} \to \mathbb{R}$ is **uniformly continuous** on a set $A$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$.

**Exercise 95.** (*Homework 18, 2021/02/08*)
Prove that $f(x) = 4x + 2$ is uniformly continuous on $\mathbb{R}$.

*Proof.* Let $\epsilon > 0$. There exists $\delta = \epsilon/4 > 0$ such that for every $x, y \in \mathbb{R}$, $|x - y| < \delta = \epsilon/4$.

$$|f(x) - f(y)| = |4x + 2 - 4y - 2| = 4|x - y| < 4\epsilon/4 = \epsilon$$

Thus, by definition, since for every $x, y \in \mathbb{R}$, $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$, the function is uniformly continuous on $\mathbb{R}$. □

**Exercise 96.** (*Homework 18, 2021/02/08*)
Prove that $f(x) = 1/x^2$ is uniformly continuous on $[1, \infty)$.

*Proof.* Let $\epsilon > 0$. There exists $\delta > 0$ such that for every $x, y \in \mathbb{R}$, $|x - y| < \delta$. Since $x, y \in [1, \infty)$, $1 \leqslant x$ and $1 \leqslant y$ implies $\frac{1}{(xy)^2} \leqslant 1$ and $|x + y| \geqslant 2$. Choose $\delta = \frac{\epsilon}{2}$, then

$$|f(x) - f(y)| = |1/x^2 - 1/y^2| = \frac{|x^2 - y^2|}{x^2 y^2} \leqslant \frac{|x + y||x - y|}{(xy)^2} \leqslant 2|x - y| < 2\delta = 2\frac{\epsilon}{2} = \epsilon$$

Thus, by definition, since for every $x, y \in [1, \infty)$, $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$, the function is uniformly continuous on $[1, \infty)$. □

- A function $A \to \mathbb{R}$ is called **Lipschitz** if there exists a bound $M > 0$ such that $|\frac{f(x) - f(y)}{x - y}| \leqslant M$ for all $x, y \in A$ (and $x \neq y$).

**Exercise 97.** (*Homework 18, 2021/02/08*)
Show that if $f : A \to \mathbb{R}$ is Lipschitz, then it is uniformly continuous on $A$.

*Proof.* Let $\epsilon > 0$. There exists $\delta > 0$ such that for every $x, y \in A$, $|x - y| < \delta$. Since $f$ is Lipschitz, $|\dfrac{f(x) - f(y)}{x - y}| \leqslant M$ implies $|f(x) - f(y)| \leqslant M|x - y|$ . Choose $\delta = \dfrac{\epsilon}{M}$, then

$$|f(x) - f(y)| \leqslant M|x - y| < M\delta = M\frac{\epsilon}{M} = \epsilon$$

Thus, by definition, since for every $x, y \in A$ and $x \neq y$, $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$, the function is uniformly continuous on $A$. □

**Exercise 60.** (*Resubmission,* )
Let $f(x) = x^2 + 2$. Use the definition to prove that $f'(-3) = -6$.

*Proof.* Let $(a_n) \in D$ be a sequence and $-3 \in D$ such that $(a_n)_{n=1}^{\infty} \to -3$ and $(a_n)_{n=1}^{\infty} \neq -3$. Then,

$$f'(c) = (\frac{f(a_n) - f(c)}{a_n - c})_{n=1}^{\infty} = (\frac{((a_n)^2 + 2) - (3^2 + 2)}{a_n + 3})_{n=1}^{\infty} = (\frac{((a_n)^2 - 9}{a_n + 3})_{n=1}^{\infty} = (a_n - 3)_{n=1}^{\infty}$$

Since $(a_n)_{n=1}^{\infty} \to -3$, $(a_n)_{n=1}^{\infty} - 3 \to -3 - 3 = -6$. Thus, $f'(c) \to -6$. □

**Exercise 61.** (*Resubmission,* )
Let $f(x) = x^2$. Use the definition to prove that $f'(x) = 2x$, for all $x \in \mathbb{R}$.

*Proof.* Let $(a_n) \in \mathbb{R}$ be a sequence and $c \in \mathbb{R}$ such that $(a_n)_{n=1}^{\infty} \to c$ and $(a_n)_{n=1}^{\infty} \neq c$ . Then,

$$f'(c) = (\frac{f(a_n) - f(c)}{a_n - c})_{n=1}^{\infty} = (\frac{((a_n)^2) - (c^2)}{a_n - c})_{n=1}^{\infty} = (a_n + c)_{n=1}^{\infty}$$

Since $(a_n)_{n=1}^{\infty} \to c$, $(a_n)_{n=1}^{\infty} + c \to c + c = 2c$ for all $c \in \mathbb{R}$. Thus, $f'(c) \to 2c$ and $f'(x) = 2x$ for all $x \in \mathbb{R}$. □

**Exercise 69.** (*Resubmission,* )
Determine appropriate hypotheses and then state and prove the Chain Rule for derivatives. *$f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ satisfy $f(A) \subseteq B$ so that the composition $g \circ f(x) = g(f(x))$ is well defined. If $f$ is differentiable at $c \in A$, and $g$ is differentiable at $f(c) \in B$, then, $g \circ f$ is differentiable with $(g \circ f)'(c) = g'((f(c)))f'(c)$.*

*Proof.* Let $(a_n) toc$ with $a_n \neq c$. Since $g(c)$ is differentiable at the point $f(c)$, as long as $f(a_n) \neq f(c)$,

$$(\frac{g(f(a_n)) - g(f(c))}{f(a_n) - f(c)})_{n=1}^{\infty} \to g'(f(c))$$

Let $(b_n)$ be a sequence such that $(b_n) = 0$ if $f(a_n) = f(c)$, otherwise, let

$$b_n = (\frac{g(f(a_n)) - g(f(c))}{f(a_n) - f(c)})_{n=1}^{\infty} - g'(f(c))$$

then, $(d_n) \to 0$.

$$(g(f(a_n)) - g(f(c))_{n=1}^{\infty} = (g'(f(c) + d_n))_{n=1}^{\infty}(f(a_n) - f(c))_{n=1}^{\infty}$$

Consequently,

$$(\frac{g(f(a_n)) - g(f(c))}{a_n - c})_{n=1}^{\infty} = ((g'(f(c)) + d_n)_{n=1}^{\infty}\frac{f(a_n) - f(c)}{a_n - c})_{n=1}^{\infty} \to g'(f(c))f'(c)$$

Thus, $g(f(x))$ is differentiable and converges to $g'(f(c))f'(c)$. □

**Exercise 70.** (*Resubmission,* )
Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on $(a, b)$. If $f(a) = f(b)$, prove there exists a point $c \in (a, b)$ such that $f'(c) = 0$.

*Proof.* By extreme value theorem, since $f$ is continuous, f has both max and min on $[a, b]$. If either max or min is at a or b, then since $f(a) = f(b)$, the function is a constant function with $f'(c) = 0$ for any $c \in (a, b)$. Else, if max and min are not at $a$ or $b$, by Ex65, the derivative at the max or min is 0. Thus, there exists $c \in (a, b)$ which is max or min such that $f'(c) = 0$. □

**Exercise 71.** (*Resubmission,* )
*Mean Value Theorem.* Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on $(a, b)$. Prove there exists a point $c \in (a, b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

*Proof.* Define the secant line function $d(x) = f(x) - (\frac{f(b) - f(a)}{b - a}(x - a) + f(a))$. Since f is continuous and differentiable, by continuous and differentiable function properties, their sum and linear functions are still continuous and differentiable. Thus, $d(x)$ is continuous on $[a, b]$ and differentiable on $(a, b)$. Since $d(a) = d(b) = 0$, by Ex70, there exists a point $c$ such that $d'(c) = 0$. □

**Exercise 72.** (*Resubmission,* )
Let $f(x) = 0$ for each number in $[0, 1]$ except $x = 0$, and let $f(0) = 1$. Show that if $P$ is any partition of $[0, 1]$, then $0 < U_P f$, and if $\epsilon > 0$, then there exists a partition $P$ of $[0, 1]$ such that $U_P f < \epsilon$.

*Proof.* Let $P$ be a partition over $[0,1]$ such that $P = t_0, t_1, t_2, ..., t_n$ for $i = 1, 2, 3, ..., n$. Then, by how the function $f$ is defined, $Y_1 = sup\{f(x) : x \in [t_0, t_1]\} = sup 0, 1 = 1$, and if $i \neq 1$, then $Y_i = sup\{f(x) : x \in [t_i, t_{i-1}]\} = sup(0) = 0$. Thus,

$$U_P f = \sum_{i=1}^{n} Y_i(t_i - t_{i-1}) = 1 * (t_1 - t_0) = t_1 - t_0 = t_1 > 0$$

Thus, $0 < U_P f$.
Let $\epsilon > 0$. If $t_1 \geq \epsilon$. Let $P_1 = P \cup \{\frac{\epsilon}{2}\}$, otherwise, let $P_1 = P$ and use the work above. Then,

$$U_P f = t_1(\text{which is the first partition point beyond } t_0 = 0) = \frac{\epsilon}{2} < \epsilon$$

Thus, $U_P f < \epsilon$. □

**Exercise 73.** (*Resubmission,* )
Using $f$ as in the previous problem, show that $L_P f = 0$ for all partitions $P$.

*Proof.* Let $P$ be a partition over $[0,1]$ such that $P = t_0, t_1, t_2, ..., t_n$ for $i = 1, 2, 3, ..., n$. By Ex8, $y_i = inf\{f(x) : x \in [t_{i-1}, t_i]\} = 0$. Then, $L_P f = \sum_{i=1}^{n} y_i(t_i - t_{i-1}) = \sum_{i=1}^{n} 0(t_i - t_{i-1}) = 1 * (t_1 - t_0) = 0$. Thus, $L_P f = 0$ for all partitions $P$. □

**Exercise 74.** (*Resubmission,* )
Prove that if $f$ is a bounded function with domain that includes $[a, b]$, and $P$ is a partition of $[a, b]$ then $LPf \leq UPf$

*Proof.* Since $y_i = inf\{f(x) : x \in [t_{i-1}, t_i]\} \leq sup\{f(x) : x \in [t_{i-1}, t_i]\} = Y_i$,

$$L_P f = \sum_{i=1}^{n} y_i(t_i - t_{i-1}) \leq \sum_{i=1}^{n} Y_i(t_i - t_{i-1}) = UPf$$

Thus, $LPf \leq UPf$. □

**Exercise 75.** (*Resubmission,* )
Assume that $P$ and $Q$ are partitions of the interval $[a, b]$ and that $f$ is a bounded function on $[a, b]$. Further assume that $Q$ is a refinement of $P$. Determine whether $U_Q f \leq U_P f$, or $U_Q f \geq U_P f$. Then prove your assertion. $[U_Q f \leq U_P f]$

*Proof.* Let $P$ be a partition over $[0,1]$ such that $P = t_0, t_1, t_2, ..., t_n$ for $i = 1, 2, 3, ..., n$. Let $Q = P \cup t^*$. Suppose that $t_{i-1} < t^* < t_i$ for some $i \leq n$. Let $Y_1 = sup\{f(x) : x \in [t_{i-1}, t^*]\}$,

$Y_2 = sup\{f(x) : x \in [t^*, t_i]\}$, and $Y_0 = sup\{f(x) : x \in [t_{i-1}, t_i]\}$. Then, by property of supremum, $Y_0 \geq Y_1$ and $Y_0 \geq Y_2$. Then,

$$U_Q f - U_P f = Y_1(t^* - t_{i-1}) + Y_2(t_i - t^*) - Y_0(t_i - t_{i-1}) \leq 0$$

Thus, $U_Q f \leq U_P f$. □

**Exercise 76.** (*Resubmission,* )
Assume that $P$ and $Q$ are partitions of the interval $[a, b]$ and that $f$ is a bounded function on $[a, b]$. Further assume that $Q$ is a refinement of $P$. Determine whether $L_Q f \leq L_P f$, or $L_Q f \geq L_P f$. Then prove your assertion. $[L_Q f \geq L_P f]$

*Proof.* Let $P$ be a partition of $[a, b]$ such that $P = \{p_1, p_2, ..., p_n\}$ for $i = 1, 2, 3, ..., n$, and f be a bounded function on $[a, b]$. Let $Q$ be a partition of $[a, b]$ such that $Q = P \cap [p_{i-1}, p_i] = q_0 = p_{i-1}, q_1, q_2, ...., q_{m-1}, q_m = p_i$ for $j = 1, 2, 3, ..., m$. Then, by infinmum properties, $inf\{f(x) : x \in [q_{j-1}, q_j]\} \geq inf\{f(x) : x \in [p_{i-1}, p_i]\}$. Also, since $\sum_{q=1}^{n}(q_i - q_{i-1}) = |a - b| = \sum_{p=1}^{n}(p_i - p_{i-1})$,

$$L_Q f = \sum_{j=1}^{m} inf\{f(x) : x \in [q_{j-1}, q_j]\}(q_j - q_{j-1}) \geq \sum_{q=1}^{m} inf\{f(x) : x \in [p_{i-1}, p_i]\}(q_j - q_{j-1})$$

$$= inf\{f(x) : x \in [p_{i-1}, p_i]\} \sum_{j=1}^{m}(q_j - q_{j-1}) = inf\{f(x) : x \in [p_{i-1}, p_i]\}(p_i - p_{i-1})$$

$$\geq \sum_{i=1}^{n} inf\{f(x) : x \in [p_{i-1}, p_i]\}(p_i - p_{i-1}) = L_P f$$

Thus, $L_Q f \geq L_P f$ if $Q$ is a refinement of $P$. □

**Exercise 78.** (*Resubmission,* )
Prove the following statement: If $f$ is a bounded function with domain $[a, b]$ and for every $\epsilon > 0$, there exists a partition $P$ of $[a, b]$ such that $U_P f - L_P f < \epsilon$, then $f$ is Riemann Integrable on $[a, b]$. (Note: This condition is known as Riemann's condition.).

*Proof.* Let $P$ be a partition of $[a, b]$. By Ex 1(b), Since $U_P f - L_P f < \epsilon$ for every $\epsilon > 0$, $U_P f - L_P f = 0$. Then, $U \int_a^b f \leq L \int_a^b f$. If $U \int_a^b f - L \int_a^b f \geq 0$, Let $P$ be a partition. Then, for any partition $Q$, since $L_Q f \leq U_P f$, $L_P f$ is a lowerbound for the set of upper Riemann sums. By definition of infinmum, $L_P f$ has to be less than the greatest lowerbound. Thus, $L_P f \leq U \int_a^b f$. Since $L \int_a^b f$ is the least upperbound, $L \int_a^b f \leq U \int_a^b f$. Thus, since

$U\int_a^b f \leq L\int_a^b f$ and $L\int_a^b f \leq U\int_a^b f$, $L\int_a^b f = U\int_a^b f$ and by definition, $f$ is Riemann Integrable on $[a,b]$. $\square$

**Exercise 79.** (*Resubmission,* )
Prove the following statement: If $f$ is Riemann integrable on $[a,b]$ then for every $\epsilon > 0$ there exists a partition $P$ such that $U_P f - L_P f < \epsilon$

*Proof.* Let $\epsilon > 0$. For some partition $R, Q$ on $[a,b]$, by Ex13 (sup/inf property with epsilon)

$$U\int_a^b f + \frac{\epsilon}{2} > U_R f$$

$$L\int_a^b f - \frac{\epsilon}{2} < L_Q f \text{ implies } -L\int_a^b f + \frac{\epsilon}{2} > L_R f$$

Since f is Riemann integrable, $U\int_a^b f = L\int_a^b f$. Adding together, we get

$$U\int_a^b f + \frac{\epsilon}{2} - L\int_a^b f + \frac{\epsilon}{2} > U_R f - L_Q f \text{ implies } U_R f - L_Q f < \epsilon$$

Let $P$ be a partition such that $P = R \cup Q$, then, $P$ is a refinement of $R$ and $Q$. Thus, by Ex75 and 76,

$$U_P f - L_P f \leq U_R f - L_Q f < \epsilon$$

Thus, such partition exists. $\square$

**Exercise 81.** (*Resubmission,* )
Prove the following: If $f$ and $g$ are differentiable functions on the interval $(a,b)$ and satisfy $f'(x) = g'(x)$ for all $x \in (a,b)$, then $f(x) = g(x) + k$ for some constant $k \in \mathbb{R}$.

*Proof.* Let $h(x) = f(x) - g(x)$. Since $f'(x) = g'(x)$, $h'(x) = 0$. By Mean Value Theorem (Quiz2 last question), $h(x)$ has to be a constant function. Thus, $f(x) - g(x) = k$ which implies $f(x) = g(x) + k$ for some constant $k \in \mathbb{R}$. $\square$

**Exercise 83.** (*Resubmission,* )
Prove that a bounded function $f$ is integrable on $[a,b]$ if and only if there exists a sequence of partitions $(P_n)_{n=1}^{\infty}$ satisfying $\lim\limits_{n\to\infty} [U_{P_n} f - L_{P_n} f] = 0$.

*Proof.* $\Rightarrow$ Since $f$ is Riemann integrable, by Ex78, for any $\epsilon$ there exists a partition $P$ such that $U_P f - L_P f < \epsilon$. Choose $\epsilon = \dfrac{1}{n}$, then $U_P f - L_P f < \dfrac{1}{n}$. By squeeze theorem proved in the homework, $\lim\limits_{n\to\infty} [U_{P_n} f - L_{P_n} f] = 0$.

$\Leftarrow$ Let $\epsilon > 0$. Since $\lim\limits_{n\to\infty} [U_{P_n} f - L_{P_n} f] = 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$|U_{P_n}f - L_{P_n}f - 0| < \epsilon$. Since $U_{P_n}f - L_{P_n}f > 0$, $U_{P_n}f - L_{P_n}f < \epsilon$. Thus, by Ex78, $f$ is integrable. □

**Exercise 87.** (*Resubmission,* )
Suppose $f$ is a Riemann integrable function on the interval $[a, b]$ and $m \leq f(x) \leq M$. Prove that $m(b-a) \leq \int_b^a f(x)dx \leq M(b-a)$.

*Proof.* Let $P = \{a, b\}$, then, by integral properties,

$$m(b-a) \leq L_P f < \int_a^b f(x)dx \leq U_P f \leq M(b-a)$$

Thus, $m(b-a) \leq \int_b^a f(x)dx \leq M(b-a)$.

□

**Exercise 88.** (*Resubmission,* )
Prove if $f$ and $g$ are Riemann integrable on $[a, b]$, and $f(x) \leq g(x)$ for all $x \in [a, b]$. Then $\int_a^b f(x)dx \leq \int_a^b g(x)dx$.

*Proof.* Let $P = \{t_0, t_1, ...., t_n\}$ be a partition over $[a, b]$. Then, because $\int_a^b f(x)dx \leq \int_a^b g(x)dx$, $inf\{f(x) : x \in [t_{i-1}, t_i]\} \leq inf\{g(x) : x \in [t_{i-1}, t_i]\}$. Then,

$$L_P f = \sum_{i=1}^{n} sup\{f(x) : x \in [t_{i-1}, t_i]\}(t_i - t_{i-1}) \leq \sum_{i=1}^{n} inf\{g(x) : x \in [t_{i-1}, t_i]\}(t_i - t_{i-1})$$

Thus, since $f, g$ are integrable on $[a, b]$, $L_P f \leq L_P g$. □