Sabancı University
Faculty of Engineering and Natural Sciences

CS301 – Algorithms

Homework 4

Due: May 30, 2023 @ 23.55
(Upload to SUCourse)

PLEASE NOTE:

- Provide only the requested information and nothing more. Unreadable, unintelligible and irrelevant answers will not be considered.
- You can collaborate with your `TA/INSTRUCTOR ONLY` and discuss the solutions of the problems. However you have to write down the solutions on your own.
- Plagiarism will not be tolerated.

Late Submission Policy:

- Your homework grade will be decided by multiplying what you normally get from your answers by a "submission time factor (STF)".
- If you submit on time (i.e. before the deadline), your STF is 1. So, you don't lose anything.
- If you submit late, you will lose 0.01 of your STF for every 5 mins of delay.
- We will not accept any homework later than 500 mins after the deadline.
- SUCourse+'s timestamp will be used for STF computation.
- If you submit multiple times, the last submission time will be used.

Question	Points	Score
1	30	
2	40	
3	30	
Total:	100	

Question 1 [30 points]

Design a flow network $G = (V, E, s, t, c)$ with $|V| \leq 4$ and $c : V \times V \to \{0, 1\}$ such that max–flow function for G is not unique. On the flow network you design, show at least two different max–flow functions and state the value of the max–flow.

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V = {s, v1, v2, t}
E = {(s, v1), (s, v2), (v1, t), (v2, t)}
c(s, v1) = c(s, v2) = 1
c(v1, t) = c(v2, t) = 1

There are two different max-flow functions for this network:
f1(s, v1) = f1(s, v2) = 1
f2(s, v1) = 0
f2(s, v2) = 2
The value of the max-flow is 2 in both cases.
The max-flow function f1 sends 1 unit of flow from s to v1 and 1 unit of flow from s to v2. The max-flow function f2 sends 0 units of flow from s
to v1 and 2 units of flow from s to v2. Both functions are valid, and they both achieve a max-flow of 2.

The reason why the max-flow function is not unique is because the network has two parallel edges from s to t. This means that we can send flow
through either edge, or we can send flow through both edges. The max-flow function is only unique if there is only one path from s to t.
Here is a diagram of the network:
s
/ \
v1 v2
\ /
t
```

Question 2 [40 points]

We know that the value of the maximum flow is unique in flow networks. However, there can be more than one max–flow function achieving this maximum value.

Now, consider a flow network $G = (V, E, s, t, c)$ where we have the following property:

$\forall u_1, u_2, v_1, v_2 \in V$:

$$[\ c(u_1, u_2) \neq 0 \ \wedge \ c(v_1, v_2) \neq 0 \ \wedge \ (u_1, u_2) \neq (v_1, v_2)\ ] \implies [\ c(u_1, u_2) \neq c(v_1, v_2)\ ]$$

Claim A: For such flow networks, there is exactly one max–flow function.

Is Claim A true or false?

If true, prove it (no partial points if no proof is given).

If false, give a counter example by using a flow network of at most 4 nodes on which you need to provide two different max–flow functions (no partial points if no counter example is given or if the counter example uses 5 or more nodes).

```
It is false.
Counterexample: consider the following flow network:
V = {s, a, b, t}
E = {(s, a), (s, b), (a, t), (b, t)}
s = source node
t = sink node
The capacities c: V × V → {0, 1} are defined as follows:
c(s, a) = 1
c(s, b) = 1
c(a, t) = 1
c(b, t) = 1
See the below flow functions for this flow network:
Max-Flow Function 1:
Flow values:
f(s, a) = 1
f(s, b) = 1
f(a, t) = 1
f(b, t) = 1
The value of the max-flow in this case is 2.
Max-Flow Function 2:
Flow values:
f(s, a) = 1
f(s, b) = 1
f(a, t) = 0
f(b, t) = 1
The value of the max-flow in this case is also 2.
In both cases, the maximum flow value is 2, but the flow distribution is different. This counterexample states that there can be
multiple max-flow functions in a flow network that satisfies the given property and has at most 4 nodes.
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Question 3 [30 points]

Let $G = (V, E, s, t, c)$ be a flow network, f_1 and f_2 be two flow functions on G. Let $F : V \times V \to R$ be defined as $\forall u, v \in V : F(u, v) = f_1(u, v) + f_2(u, v)$.

Is F guaranteed to be a flow on G?

If yes, prove it (no partial points if no proof is given).

If no, for only one of the constraints of flow functions, show that it does not necessarily hold for F, by giving a counter example on a flow network of at most 3 nodes (no partial points if no counter example is provided or the counter example uses 4 or more nodes).

F is not not guaranteed to be a flow on the flow network G = (V, E, s, t, c). Below is a counterexample

Consider the following flow network:
V = {s, t}
E = {(s, t)}
s = source node
t = sink node
The capacities c: V × V → {0, 1} are defined as follows:
c(s, t) = 1
Let's define two flow functions f1 and f2 as follows:
f1(s, t) = 1
f2(s, t) = −1
Calculation F(u, v) for all u, v ∈ V:
F(s, t) = f1(s, t) + f2(s, t) = 1 + (−1) = 0
We can see that F(s, t) = 0, which violates one of the constraints of a flow function. A flow function should satisfy the condition 0 ≤ f(u, v) ≤ c(u, v) for all u, v ∈ V.
In this counterexample, F(s, t) = 0 does not satisfy the constraint since the capacity c(s, t) is 1 and the calculated flow F(s, t) is 0, which is not within the range of 0 to 1.
Therefore, F is not guaranteed to be a flow on the flow network G.

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