Some Information on superconductivity

Before describing the Josephson effect, let's consider some basic properties of superconductors.

(1) <u>Infinite conductivity</u>. The major property of supercond. is the flow of dissipationless current. This assumes that its resistance equates to zero when the temperature is below T_c critical temperature. Typical values for critical current density is $\sim 10^6\ A/cm^2$. Niobium and Aluminum have $T_c \sim 1.2\,K$ and $9.2\,K$ respectively.

(2) <u>Macroscopic wave function of the condensate.</u>

The conventional superconductivity is explained with the Bardeen-Cooper-Schrieffer theory by coupling electrons, which form the so-called "Cooper pairs" with zero spin. The size of a Cooper-pair is characterized by it "coherence length" ξ_0. The aggregate of such bosons forms the Bose-Einstein condensate: all those paired electrons are situated on the same quantum level and they can be described by a single wavefunction

$\psi(\vec{r}) = |\psi(\vec{r})| \, e^{i\varphi(\vec{r})}$. This global wavefunction is normalized by the density of the Cooper pairs $|\psi(\vec{r})| = \sqrt{n_s}$, where n_s represents the density.

For the current density of Cooper pairs with charge $2e$ and mass $2m$, we have

$$\vec{J}_s = (2e)\left(\psi^* \frac{\vec{p} - 2e\vec{A}/c}{2(2m)}\psi + c.c.\right)$$

$$= -i\frac{e\hbar}{2m}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right) - \frac{2e^2}{mc}|\psi|^2\vec{A} =$$

$$= 2e\,|\psi|^2\,\frac{\hbar\nabla\varphi - 2e\vec{A}/c}{2m} \equiv 2e\,n_s\,\vec{V}_s$$

This means that the phase gradient $\nabla\varphi$ defines the current in the system via the superfluid velocity $\vec{V}_s$. Here it was taken into account that the presence of a magnetic field corresponding to $\vec{p} \longrightarrow \vec{p} - (2e)\vec{A}/c$ and $\vec{p} = -i\hbar\nabla$.

So the supercurrent density is related to the phase gradient as follows:

$$\vec{J}_s = 2e\,n_s\,\frac{\hbar\nabla\varphi - 2e\vec{A}/c}{2m}$$

(3) Perfect diamagnetism.

A magnetic field does not penetrate into bulk of superconductors, unlike normal metals, but it is rather expelled from them. This is related to the appearance of the surface current, of which the field shields the external magnetic field, known as "Meissner effect".

In order to understand this, let's consider a homogeneous superconductor in a weak magnetic field and in equilibrium.

From the Eq. of the superc. density, we get

$$\mathrm{rot}\,\vec{J} = -\frac{2e^2 n_s}{mc} B \qquad \text{(London equation)}$$

By adding the Maxwell equations $\mathrm{rot}\,\vec{B} = \frac{4\pi}{c}\vec{J}$; $\mathrm{div}\,\vec{B} = 0$ and using that $\mathrm{rot}\,\mathrm{rot}\,\vec{B} = \nabla(\mathrm{div}\vec{B}) - \Delta\vec{B}$, we get

$$\mathrm{rot}\,\vec{J} = \frac{c}{4\pi}\mathrm{rot}\,\mathrm{rot}\,\vec{B} = -\frac{c}{4\pi}\Delta\vec{B} \hat{=} -\frac{2e^2 n_s}{mc}\vec{B} \Rightarrow \Delta\vec{B} = \frac{1}{\lambda^2}\vec{B}$$

with $\lambda^2 = \frac{mc^2}{8\pi e^2 n_s}$ is the so called "field penetration depth".

The magnetic field penetrates only to the depth $\sim\lambda$ ($\sim 0.1\,\mu m$).

(4) Magnetic flux quantization.

Consider a doubly connected supercond. (a ring).
Let's consider a circuit C inside the ring and integrate along it the eq. for the superc. density, we will get:

$$0 = \int_C d\vec{l} \left(\hbar \nabla \varphi - 2e\vec{A}/c \right) .$$ For Stokes theorem

$$\int_C d\vec{l}\, \vec{A} = \int_{S_C} d\vec{S} \cdot \nabla \times \vec{A} = \vec{B} \int_S d\vec{S} = BS = \Phi$$ $\Big\{$ magnetic flux inside the ring

We obtain $\Phi = \dfrac{\hbar c}{2e} \displaystyle\int_C d\vec{l}\, \nabla \varphi$. On the other hand, the requirement of the wavefunction to be single-valued gives that over the full path-tracing $\displaystyle\int_C d\vec{l}\, \nabla\varphi = \delta\varphi = -2\pi n$, $n \in \mathbb{N}$

We obtain $\Phi = n \Phi_0$; $\Phi_0 = \dfrac{hc}{2|e|}$

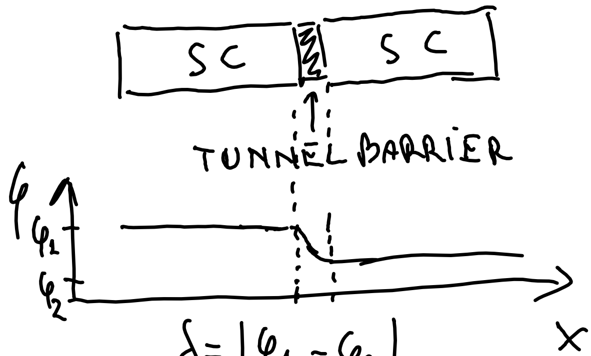
This means that the magnetic flux trapped by the superc. ring, $\Phi = BS$, can take only values multiple of the "flux quantum" $\Phi_0 \simeq 2 \cdot 10^{-15}$ Wb. Note that this is also a way to change the phase of the superconducting wavefunction.

(5) Quasiparticle excitations.

In the non-stationary regime, the response of a superconductor is defined both by the superconductive condensate and the quasiparticle excitations over the ground state. The excitations are separated by the gap $2\Delta_s$ from the ground state, and therefore, they can be neglected at sufficiently low temp., $k_B T \ll \Delta_s$.
The value $2\Delta_s$ corresponds to the energy, which is necessary for splitting one Cooper pair.
Another condition on the absence of quasiparticle excitations is the absence of high-freq. fields with $\hbar\omega \sim \Delta_s$, which means that it is necessary also to have

$$\omega \ll \Delta_s / \hbar.$$

The Josephson Junction as a NON-LINEAR INDUCTOR



$\delta = |\varphi_1 - \varphi_2|$

INDUCTION LAW: $V = -L\dot{I}$ ⑥

JOSEPHSON RELATIONS → ① : $I = I_0 \sin\delta$

(supercurrent across the junction)

② : $V = \dfrac{\Phi_0}{2\pi}\dot{\delta}$

(time-dependent phase if voltage is applied)

We can find (from ①)

$$\dot{I} = I_0 \cos\delta\,\dot{\delta} \Rightarrow \dot{\delta} = \dot{I}/I_0\cos\delta$$

and placing it into the ② Josephson relation:

$$V = \frac{\Phi_0}{2\pi}\frac{1}{I_0}\frac{1}{\cos\delta}\dot{I} \equiv L_J\,\dot{I} \quad \text{with} \quad L_J = \frac{\Phi_0}{2\pi I_0}\frac{1}{\cos\delta}$$

This relation allows us to interpret the quantity L_J (proportionality constant between V across the junction and the $\frac{d}{dt}(I)$ through the junction) as an inductor L_J.

$$L_J = L_{J_0}\frac{1}{\cos\delta} \quad \text{with } L_{J_0} = \frac{\Phi_0}{2\pi I_0} \quad \text{specific Josephson inductance}$$

$\hookrightarrow$ NON-LINEARITY

N.B. I_0 is the "critical current" of the Josephson junction
Φ_0 is the reduced flux quantum $= \hbar/2e$

The two Josephson relations relate the current and voltage across a Josephson junction to the superconducting phase-difference δ between the two superconducting islands.

Defining an effective flux variable $\tilde{\Phi} = \Phi_0\delta$ and the Josephson inductance $L_J = \Phi_0/I_0$ we obtain

$$I = \frac{\Phi_0}{L_J}\sin\left(\tilde{\Phi}/\Phi_0\right) = \frac{\tilde{\Phi}}{L_J} - \frac{1}{6}\frac{\tilde{\Phi}^3}{L_J\Phi_0^2} + \dots \quad \text{for } \delta \ll 1; \text{ i.e. } I \ll I_0$$

$$V = \dot{\tilde{\Phi}}$$

The phase difference δ across the junction can be regarded as a normalized magnetic flux.

- In this form the Josephson equations are analogous to the relations between V, I and the magnetic flux in a linear inductor.
- However, the current depends nonlinearly on $\tilde{\Phi}$.

For our purposes, the Josephson junction thus acts like a nonlinear inductor.

- Furthermore, current flows without any dissipation, a fundemental feature for their use in quantum coherent devices.

- Combining the Josephson junctions, which we can schematically represent by the following symbol , with inductors  and capacitors  allows us to build nonlinear electrical circuits, which can be operated in a quantum mechanical regime.

Josephson Inductance and Josephson Energy

- Josephson energy

$$E_J = \int V\, I\, dt =$$

$$= \int \frac{\Phi_0}{2\pi}\dot{\delta}\, I_0 \sin\delta\, dt =$$

$$= \frac{\Phi_0 I_0}{2\pi} \cos\delta$$

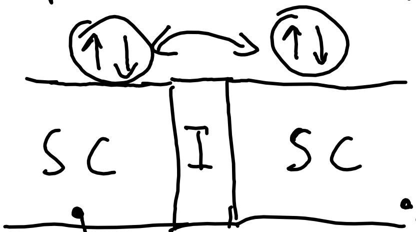
$$= E_{J0} \cos\delta \quad \text{with } E_{J0} = \frac{\Phi_0 I_0}{2\pi}$$

- Typical parameters: $I_0 \cong 100\ \text{nA}$

$$\Rightarrow L_{J0} = \frac{\Phi_0}{2\pi I_0} \simeq 3\ \text{nH} \quad \left\{ \begin{array}{l} \text{equivalent} \\ \text{of a wire} \\ \text{of} \sim 3\,\text{mm} \end{array} \right.$$

$$\Rightarrow E_{J0} = \frac{\Phi_0 I_0}{2\pi} \simeq 50\ \text{GHz}$$

JOSEPHSON EFFECT (1962)



GAP OXIDE LAYER

$$\psi_1 = \sqrt{\frac{n_{s1}}{2}}\, e^{i\Phi_1}$$

$$\psi_2 = \sqrt{\frac{n_{s2}}{2}}\, e^{i\Phi_2}$$

HYBRIDIZATION

$$\hat{\psi} = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \; ; \; H = \begin{bmatrix} 0 & K \\ K & 0 \end{bmatrix}$$

$$i\hbar \partial_t \hat{\psi} = H \hat{\psi}$$

$$\Rightarrow \frac{dn_{s1}}{dt} = -\frac{dn_{s2}}{dt} \propto K \sin(\Phi_2 - \Phi_1)$$

$$I = I_c \sin \Delta\Phi$$

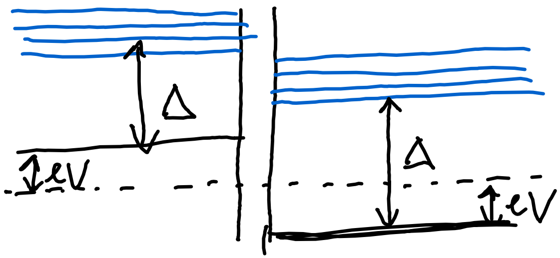
CURRENT-PHASE RELATION FOR DC JOSEPHSON JUNCTION

IF VOLTAGE IS APPLIED:

$$H = \begin{bmatrix} eV & B \\ B & -eV \end{bmatrix}$$

$$(\Delta\dot{\Phi}) = \frac{2e}{\hbar} V \quad \text{(AC JOSEPHSON EFFECT)}$$

Assume now that the electric potential difference V is applied to the junction.



The two states of the junctions banks form a two-level system with the energy levels defined by the shift of the chemical potential and equal to $\pm eV$.

Let us define the vector-states for the junction so that they form the basis of a two-level system

$|\psi_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$; $|\psi_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Let's expand the wave function in the basis $|\psi\rangle = a_1|\psi_1\rangle + a_2|\psi_2\rangle = \begin{pmatrix} \sqrt{n_{s1}}\, e^{i\varphi_1} \\ \sqrt{n_{s2}}\, e^{i\varphi_2} \end{pmatrix}$

The system's Hamiltonian has the diagonal elements which are equal to the system's energy in the respective state: $H_{11} = \langle\psi_1|H|\psi_1\rangle = eV$; $H_{22} = -eV$

The off-diagonal elements describe transitions between the levels, which are related to the tunneling through the barrier, which we characterize by the value B: $H_{12} = H_{21} = B$

$$H = \begin{pmatrix} eV & B \\ B & -eV \end{pmatrix} = eV\sigma_z + B\sigma_x$$

And from the Schrödinger equation $i\hbar \frac{\partial}{\partial t}|\psi\rangle = H|\psi\rangle$

we obtain

$$\begin{cases} \dot{n}_{s_1} = -\dot{n}_{s_2} = \frac{2Bn_{s0}}{\hbar}\sin\varphi & (1) \\ \dot{\varphi}_1 = -\frac{B}{\hbar}\cos\varphi - \frac{eV}{\hbar} & (2) \\ \dot{\varphi}_2 = -\frac{B}{\hbar}\cos\varphi + \frac{eV}{\hbar} & (3) \end{cases}$$

n_{s0} = density of Cooper pairs in the banks without the junction.

Since the current through the tunneling junction is $\propto$ to the rate of change of electron density, $I_s \propto dn_s/dt$

we get $I_J = I_c \sin\varphi$ [stationary Josephson effect]

with I_c the junction critical current.

Subtracting (2) - (3) we get $\dot{\varphi} = \frac{2eV}{\hbar}$ [non-stationary Josephson effect]

So $\varphi(t) = \varphi_0 + \frac{2e}{\hbar}Vt \Rightarrow I_S = I_c \sin\left(\varphi_0 + \frac{2e}{\hbar}Vt\right)$

By applying the voltage results in flowing of alternating current with the angular freq. $\omega = \frac{2eV}{\hbar}$

Note that the average power, consumed from the external source for the supercurrent drive, is zero, $\overline{I_J V} = 0$

This means that also this alternating supercurrent does not dissipate energy!

If we invert $V = \frac{\hbar}{2e} \dot{\varphi}$ it becomes clear that the non-stationary Josephson effect consists in the appearance of the dc-voltage on the junction, if the phase difference linearly depends on time.

SIMPLEST SUPERCONDUCTING DEVICE

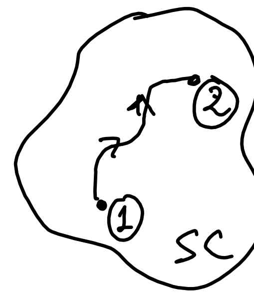
SQUID (Superconducting Quantum Interference Device)

- $J_s = -\frac{e^2 n_s}{m}\left(\vec{A} - \frac{\hbar}{2e}\nabla\Phi\right)$

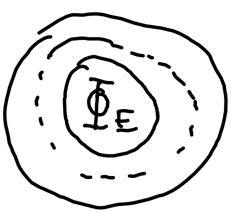
  ↳ It is zero inside a superconductor

  So $\vec{A} = \frac{\hbar}{2e}\nabla\Phi \Rightarrow \int_1^2 A\,dl = \frac{\Phi_0}{2\pi}\int_1^2 \nabla\Phi =$

  $= \frac{\Phi_0}{2\pi}\left[\Phi_2 - \Phi_1\right]$



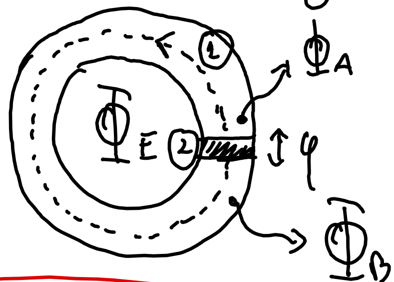
- Superconducting loop $\Rightarrow$ QUANTIZED FLUX



$$\oint A\,dl = \frac{\Phi_0}{2\pi}\oint \nabla\Phi = \frac{\Phi_0}{2\pi}2\pi n \qquad n \in \mathbb{Z}$$

$$\boxed{\Phi_E = \Phi_0\, n}$$

- Superconducting Loop with one Junction (RF-SQUID)

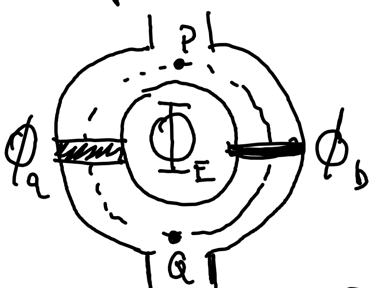


$$\oint \nabla \Phi = 2\pi n$$

$$\oint \nabla \Phi = \underset{(1)}{\int_A^B} \nabla \Phi + \underset{(2)}{\int_B^A} \nabla \Phi = \frac{2\pi}{\Phi_0} \underbrace{\int_A^B A\,dl}_{\Phi_E} + \varphi$$

$$\boxed{2\pi n = \frac{2\pi}{\Phi_0}\Phi_E + \varphi}$$

- Superconducting Loop with Two Junctions (DC-SQUID)



$$\Phi_Q - \Phi_P = \Phi_a + \frac{2\pi}{\Phi_0}\int_P^Q \vec{A}\cdot d\vec{l} = \Phi_b + \frac{2\pi}{\Phi_0}\int_P^Q \vec{A}\, d\vec{l}$$

$$\Phi_a - \Phi_b = \frac{2\pi}{\Phi_0}\oint \vec{A}\cdot d\vec{l} = \frac{2\pi}{\Phi_0}\Phi_E$$

$$I_{TOT} = I_c\left[\sin\Phi_a + \sin\Phi_b\right]$$

$$\Phi = \Phi_a - \pi\frac{\Phi_E}{\Phi_0} \quad ; \quad \Phi = \Phi_b - \pi\frac{\Phi_E}{\Phi_0}$$

$$I = 2I_c \cos\left[\pi\,\Phi_E/\Phi_0\right] \sin\varphi$$

THE CRITICAL CURRENT IS MODULATED BY THE EXTERNAL FIELD