Particle Physics 1: Exercise 4

Exercise 1

For the decay $a \to 1 + 2$, show that the modulus of the momentum of both daughter particles in the centre-of-mass frame is:

$$|\vec{p}\,| = \frac{\sqrt{(m_a^2 - (m_1 + m_2)^2)(m_a^2 - (m_1 - m_2)^2)}}{2m_a}$$

Exercise 2

Assuming that $f(\vec{p})$ is an arbitrary function of the momentum $\vec{p}$, verify the following relation:

$$\int \mathrm{d}^4 p\, f(\vec{p})\, \delta(p^2 - m^2)\, \theta(p^0) = \int \frac{\mathrm{d}^3\vec{p}}{2E} f(\vec{p})$$

with $E = \sqrt{\vec{p}^{\,2} + m^2}$. Using this relation, show the invariance of $\frac{\mathrm{d}^3\vec{p}}{2E}$, i.e. that $\frac{\mathrm{d}^3\vec{p}}{2E}$ is a Lorentz scalar.

Here the step function $\theta(p^0)$ is used to select only positive values of p^0 and is defined as

$$\theta(p^0) = \begin{cases} 1 & \text{if } p^0 \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Hint: Integrate the left side of the equation over $\mathrm{d}p^0$ and use the following property of the δ-function: if x_i $(i = 1,\ldots,n)$ are zeroes of the function $g(x)$, i.e. $g(x_i) = 0$, then

$$\delta\Big(g(x)\Big) = \sum_{i=1}^{n} \frac{\delta(x - x_i)}{|g'(x_i)|} \quad \text{with} \quad g'(x) = \frac{\mathrm{d}g(x)}{\mathrm{d}x}$$

Exercise 3

Calculate the branching ratio for the decay of $\mathrm{K}^+ \to \pi^+\pi^0$ given the partial decay width $\Gamma(\mathrm{K}^+ \to \pi^+\pi^0) = 1.2 \times 10^{-8}$ eV and the mean kaon lifetime $\tau(\mathrm{K}^+) = 1.2 \times 10^{-8}$ s.

Exercise 1

For the decay $a \to 1 + 2$, show that the modulus of the momentum of both daughter particles in the centre-of-mass frame is:

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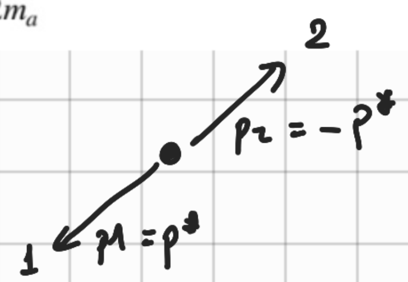
In the CoM frame:



Conservation of Energy

$$E_a = E_1 + E_2$$

$$E_a^2 = p_a^2 + M_a^2 \xrightarrow[p_a=0]{} E_a^2 = M_a^2 \rightarrow E_a = m_a$$

$$\boxed{m_a = E_1 + E_2}$$

$$(m_a - E_2)^2 = (E_1)^2$$

$$m_a^2 - 2M_aE_2 + E_2^2 = E_1^2$$

$$M_a^2 - 2M_aE_2 + m_2^2 + \overset{p^{*2}}{p_2^2} = M_1^2 + \overset{p^{*2}}{p_1^2}$$

$$\boxed{\left(m_a^2 + (m_2^2 - M_1^2)\right)^2 = \left(2M_aE_2\right)^2}$$ to eliminate E_2

$$m_a^4 + 2M_a^2(M_2^2 - M_1^2) + (M_2^2 - M_1^2)^2 = 4m_a^2(M_2^2 + p^{*2})$$

$$M_a^4 + 2M_a^2(M_2^2 - M_1^2) + (M_2^2 - M_1^2)^2 - 4M_a^2M_2^2 = 4M_a^2p^{*2}$$

$$M_a^4 - 2M_a^2(M_1^2 + M_2^2) + (M_2 - M_1)^2(M_1 + M_2)^2 = 4M_a^2p^{*2}$$

$$(M_1+M_2)^2 + (M_1-M_2)^2$$

$$Ma^4 - 2Ma^2 \, \frac{1}{2}\left[(M_1+M_2)^2 + (M_1-M_2)^2 \right] + (M_1-M_2)^2 (M_1+M_2)^2 = 4Ma^2 p^{*2}$$

$$a^4 - a^2 [b^2 + c^2] + b^2 c^2 = (a^2-b^2)(a^2-c^2)$$

$$\begin{array}{ll} a^2 & -b^2 \\ a^2 & -c^2 \end{array}$$

$a = Ma^2$ $\quad b = M_1 - M_2$

$c = M_1 + M_2$

$$\left[Ma^2 - (M_1-M_2)^2 \right] \left[Ma^2 - (M_1+M_2)^2 \right] = 4Ma^2 p^{*2}$$

$$p^* = \frac{\sqrt{\left[Ma^2 - (M_1-M_2)^2 \right] \left[Ma^2 - (M_1+M_2)^2 \right]}}{2ma}$$

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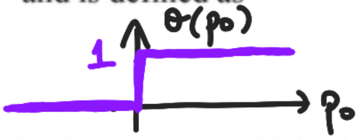

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$$I \equiv \int d^4 p \, f(\vec{p}) \, \delta(p^2 - m^2) \, \theta(p^0) = \int d^3\vec{p} \, f(\vec{p}) \int dp^0 \, \delta\left((p^0)^2 - \vec{p}^2 - m^2 \right) \theta(p^0)$$

4 vector

$$= \int d^3\vec{p} \, f(\vec{p}) \int dp^0 \, \delta\left(g(p^0) \right) \theta(p^0)$$

with $g(p^0) = (p^0)^2 - \vec{p}^2 - m^2$

Thus $\quad g(p^0) = 0 \Rightarrow p^0 = \pm \sqrt{\vec{p}^2 + m^2}$ and $\frac{dg(p^0)}{dp^0} = 2p^0$

$$I = \int d^3\vec{p}\, f(\vec{p}) \int dp^0 \left[\frac{\delta\left(p^0 - \sqrt{\vec{p}^{\,2} + M^2}\right)}{2p_0} + \cancel{\frac{\delta\left(p^0 + \sqrt{\vec{p}^{\,2} + M^2}\right)}{2p_0}} \right] \theta(p^0)$$

$$= \int d^3\vec{p}\, f(\vec{p}) \int dp^0 \, \frac{\delta\left(p^0 - \sqrt{\vec{p}^{\,2} + M^2}\right)}{2p^0} \cdot 1$$

$\theta(p^0) = 1$

$$I = \int d^3\vec{p}\, f(\vec{p})\, \frac{1}{2\sqrt{p^2 + m^2}} = \int \frac{d^3\vec{p}}{2E}\, f(\vec{p})$$

with $E = +\sqrt{\vec{p}^{\,2} + M^2}$

$$\int d^4p \, \delta(p^2 - m^2)\, \theta(p^0) = \int \frac{d^3\vec{p}}{2E}$$

LI $\quad \delta = \delta' \,, \theta = \theta'$

Show this is LI.

$$p'^\mu = \Lambda^\mu{}_\nu p^\nu$$

$$d^4p' = \underbrace{\left| \frac{\partial(p'^0, p'^1, p'^2, p'^3)}{\partial(p^0, p^1, p^2, p^3)} \right|}_{\text{Jacobian}} d^4p$$

$\det(\Lambda) = 1$

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In Natural Units

SI

$$\tau = \frac{1}{\Gamma} \rightarrow \Gamma = \frac{1}{\tau} \longrightarrow \Gamma = \frac{\hbar}{\tau}$$

$$Br = \frac{\Gamma(K^+ \to \pi^+\pi^0)}{\Gamma} = \frac{1.2 \times 10^{-8}\ eV}{\frac{\hbar}{1.2\times10^{-8}}}$$

$$= \frac{1.2\times10^{-8} \times 1.6\times10^{-19}}{1.06\times10^{-34}\ \frac{1}{1.2\times10^{-8}}} = 21\%$$

Natural to SI Units

Quantity	[kg, m, s]	[$\hbar$, c, GeV]	$\hbar = c = 1$
Energy	kg m^2 s^{-2}	GeV	GeV
Momentum	kg m s^{-1}	GeV/c	GeV
Mass	kg	GeV/c^2	GeV
Time	s	(GeV/$\hbar$)$^{-1}$	GeV^{-1}
Length	m	(GeV/$\hbar c$)$^{-1}$	GeV^{-1}
Area	m^2	(GeV/$\hbar c$)$^{-2}$	GeV^{-2}

Table 2.1 Relationship between S.I. and natural units.

$$\hbar c = 0.197\ \text{GeV fm}$$

$$1\ \text{fm} = 10^{-15}\ \text{m}$$