TWO LEVEL SYSTEMS

1- State and Operators

- Hilbert space $\mathcal{H} = \mathrm{Sp}\{|\uparrow\rangle, |\downarrow\rangle\}$, $|g\rangle, |e\rangle$ or $|0\rangle, |1\rangle$

  State $|\psi\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle$ with $|\alpha|^2 + |\beta|^2 = 1$

one possible parametrization: $|\psi\rangle = \cos\theta/_2\, |\uparrow\rangle + e^{i\phi} \sin\frac{\theta}{2} |\downarrow\rangle$

- Hermitian operators on $\mathcal{H}$: spanned by $\hat{\mathbb{I}}, \hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z$

$$\hat{\mathbb{I}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\hat{\sigma}_+ = |\uparrow\rangle\langle\downarrow| = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \hat{\sigma}_- = |\downarrow\rangle\langle\uparrow| = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\hat{\sigma}_+ = \frac{\hat{\sigma}_x + i\hat{\sigma}_y}{2} \qquad \hat{\sigma}_- = \frac{\hat{\sigma}_x - i\hat{\sigma}_y}{2}$$

Properties of Pauli Matrices: $\mathrm{Tr}\,\hat{\sigma}_i = 0 \qquad i = x, y, z$

commutation

$$\{\hat{\sigma}_i, \hat{\sigma}_j\} = 2\delta_{ij}\hat{\mathbb{I}} \implies \hat{\sigma}_i^2 = \hat{\mathbb{I}}$$

$$[\hat{\sigma}_x, \hat{\sigma}_y] = 2i\,\hat{\sigma}_z + \text{circ permutations}$$

Notation: $\vec{\hat{\sigma}} = \begin{pmatrix} \hat{\sigma}_x \\ \hat{\sigma}_y \\ \hat{\sigma}_z \end{pmatrix}$

Representation of operators : $\hat{O} = \frac{1}{2}\left\{ \hat{\mathbb{I}}\, Tr(\hat{O}) + \sum_i Tr(\hat{O}\hat{\sigma}_i).\hat{\sigma}_i \right\}$

Think about $A = \sum_i \langle A, u_i \rangle u_i$

$$\boxed{\hat{O} = \frac{1}{2}\left\{ \mathbb{I}\, Tr(\hat{O}) + \vec{n}_0\, \vec{\sigma} \right\}}$$ where $\vec{n}_0 = \begin{pmatrix} n_{0x} \\ n_{0y} \\ n_{0z} \end{pmatrix}$

2. Bloch Sphere

1) Density Matrix

$\hat{\rho}$: $Tr(\hat{\rho}) = 1$ and $\hat{\rho} = \frac{1}{2}\left\{ \mathbb{I} + \vec{n}.\vec{\sigma} \right\}$

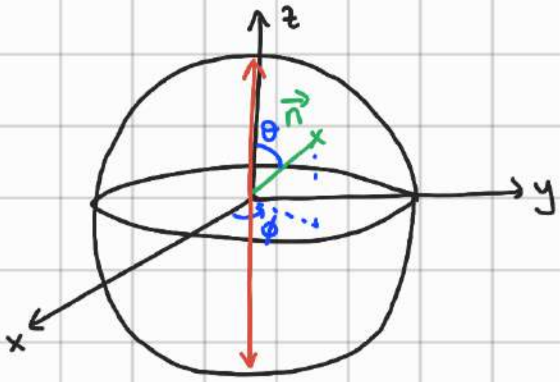
Bloch vector

Pure State

$Tr(\hat{\rho}^2) \leq 1$ with $Tr(\hat{\rho}^2) = 1 \Rightarrow \hat{\rho} = |\psi\rangle\langle\psi|$

$$Tr(\hat{\rho}^2) = Tr\left\{ \frac{1}{4} (\mathbb{I} + \vec{n}.\vec{\sigma})(\mathbb{I} + \vec{n}.\vec{\sigma}) \right\} = \frac{1 + \|\vec{n}\|^2}{2}$$

Pure States are characterized by $\|\vec{n}\|^2 = 1$



$$\vec{n} = \|\vec{n}\| \begin{pmatrix} \sin\theta \cos\phi \\ \sin\phi \sin\phi \\ \cos\theta \end{pmatrix}$$

for pure states $\|\vec{n}\| = 1$

$|\psi\rangle = \cos\frac{\vartheta}{2}\,|\uparrow\rangle + e^{i\phi}\sin\frac{\vartheta}{2}\,|\downarrow\rangle$

$\hat{\rho} = |\psi\rangle\langle\psi|$ $\quad \mathrm{Tr}(\hat{\rho}\hat{\sigma}_x) = \ldots$ → Hilbert-Schmidt Product

Example: • $|\uparrow\rangle$: north pole for $\theta = 0$

• $|\downarrow\rangle$: South pole for $\theta = 2\pi$

• Equal weight superposition of $|\uparrow\rangle$ and $|\downarrow\rangle$: $|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle + e^{i\phi}|\downarrow\rangle\right)$

"equator of the BS"

• Equal weight incoherent mixture: $\hat{\rho} = \frac{1}{2}|\uparrow\rangle\langle\uparrow| + \frac{1}{2}|\downarrow\rangle\langle\downarrow|$ → center of the sphere

• General incoherent mixture : $\hat{\rho} = p_e|\uparrow\rangle\langle\uparrow| + (1-p_e)|\downarrow\rangle\langle\downarrow|$

$\hat{\rho} = \hat{I} + p_e\hat{\sigma}_z$

2) Hamiltonian

$$\hat{H} = \hat{\mathbb{I}} - \vec{h}\cdot\vec{\hat{\sigma}}$$

Ex: spin 1/2 in a $\vec{B}$ field, $\hat{H} = -\vec{\hat{\mu}}\cdot\vec{B}$ $\quad \vec{\hat{\mu}} = \mu\vec{\hat{\sigma}}$

Dynamics: $$i\hbar\frac{\partial\hat{\rho}}{\partial t} = [\hat{H}, \hat{\rho}]$$

{ Remark, don't mix this with Heisenberg picture $i\hbar\partial_t\hat{O} = [\hat{O}, \hat{H}]$ } → density matrix is fixed state

Look similar but different physical meaning

E_0 does not generate any dynamics

$\hat{\rho}(t) = \frac{1}{2}\left(\mathbb{I} + \vec{a}(t)\vec{\hat{\sigma}}\right)$

(

Bloch vector

$$\hat{\rho}(t) = \frac{1}{2}\left(\mathbb{I} + \vec{a}(t)\cdot\hat{\vec{\sigma}}\right)$$

$$i\hbar\dot{\hat{\rho}} = [\hat{H}, \hat{\rho}] = \left[\mathbb{I} - \vec{h}\cdot\hat{\vec{\sigma}}, \frac{1}{2}\left(\mathbb{I} + \vec{a}(t)\hat{\vec{\sigma}}\right)\right]$$

$$\hat{H}\hat{\rho} - \hat{\rho}\hat{H}$$

$$\Bigg(\cancel{\frac{1}{2}\mathbb{I}} + \cancel{\frac{1}{2}\vec{a}(t)\vec{\sigma}} - \cancel{\frac{1}{2}\vec{h}\vec{\sigma}} + \frac{1}{2}\vec{h}\vec{\sigma}\,\vec{a}(t)\vec{\sigma}$$

$$-\frac{1}{2}\left(\mathbb{I} + \vec{a}(t)\hat{\vec{\sigma}}\right)\left(\mathbb{I} - \vec{h}\cdot\hat{\vec{\sigma}}\right) \qquad ?$$

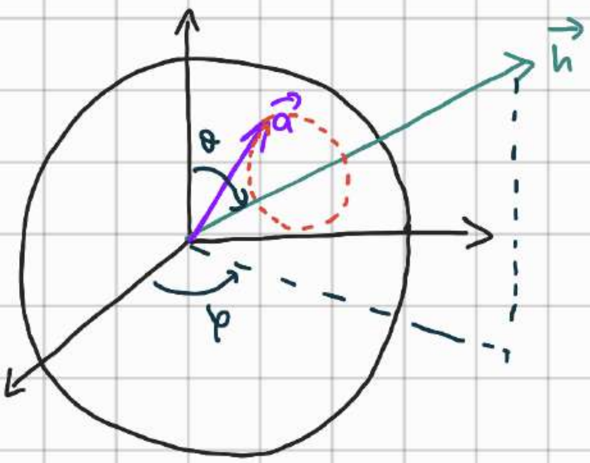
$$-\cancel{\frac{1}{2}\mathbb{I}} + \cancel{\frac{1}{2}\vec{h}\vec{\sigma}} - \cancel{\frac{1}{2}\vec{a}(t)\vec{\sigma}} + \frac{1}{2}\vec{a}(t)\vec{\sigma}\,\vec{h}\vec{\sigma} \Bigg)$$

$$i\hbar\dot{\hat{\rho}} = \dots$$

$$\boxed{\dot{\vec{a}} = \frac{2\vec{h}}{\hbar} \times \vec{a}}$$

Precession of $\vec{a}$ about the vector $\vec{h}$

Frequency: $\frac{2|\vec{h}|}{\hbar}$



Eigenstates of $\hat{H}$:

$$|\psi_+\rangle = \cos\frac{\theta}{2}|\uparrow\rangle + e^{i\varphi}\sin\frac{\theta}{2}|\downarrow\rangle$$

$|\psi_+\rangle$

$|\psi_-\rangle$

$$|\psi_-\rangle = \sin\frac{\theta}{2}|\uparrow\rangle - e^{i\varphi}\cos\frac{\theta}{2}|\downarrow\rangle$$

Rotations on the Bloch Sphere

$e^{-i\frac{\hat{H}t}{\hbar}}$ is a rotation by angle $\frac{2\|\vec{h}\|}{\hbar}.t$

Time Evolution Operator

around the axis $\vec{h}$

$$\hat{R}_{\vec{n}}(\theta) = e^{-i\frac{\theta}{2}\vec{n}.\vec{\sigma}}$$

Example: Rabi Oscillations $\vec{h}$ along x axis $\longrightarrow$

any $\vec{h}$ in the xy plane creates Rabi Oscillations

HARMONIC OSCILLATORS

I. Definitions and Notations

2 conjugate variable $\hat{x}, \hat{p}$ $\quad [\hat{x}, \hat{p}] = i\hbar$

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2$$

Natural energy scale $\hbar\omega$

Natural length scale $\sqrt{\frac{\hbar}{m\omega}} = x_0$

$$\frac{\hat{H}}{\hbar\omega} = \frac{1}{2}\left(\frac{\hat{p}}{\hbar}\right)^2 x_0^2 + \frac{1}{2}\left(\frac{\hat{x}}{x_0}\right)^2$$

1) Ladder Operators

$$\frac{\hat{H}}{\hbar\omega} = \underbrace{\frac{1}{\sqrt{2}}\left(\frac{\hat{x}}{x_0} + i\frac{\hat{p}}{\hbar}x_0\right)^{\dagger}}_{\hat{a}^{+}} \underbrace{\frac{1}{\sqrt{2}}\left(\frac{\hat{x}}{x_0} - \frac{i\hat{p}x_0}{\hbar}\right)}_{\hat{a}} + \frac{1}{2}\mathbb{I}$$

$$\frac{\hat{H}}{\hbar\omega} = \hat{a}^{+}\hat{a} + \frac{1}{2}\mathbb{I}$$

$$\boxed{\hat{x} = \frac{x_0}{\sqrt{2}}\left(\hat{a} + \hat{a}^{+}\right)}$$

Define $\hat{n} = \hat{a}^{+}\hat{a}$

$$\boxed{\frac{\hat{p}}{\hbar} = \frac{1}{i\sqrt{2}x_0}\left(\hat{a} - \hat{a}^{+}\right)}$$

Commutation Relation

$$[\hat{a}, \hat{a}^+] = \mathbb{I}$$

$$[\hat{n}, \hat{a}^+] = \hat{a}^+$$

$$[\hat{n}, \hat{a}] = -\hat{a}$$

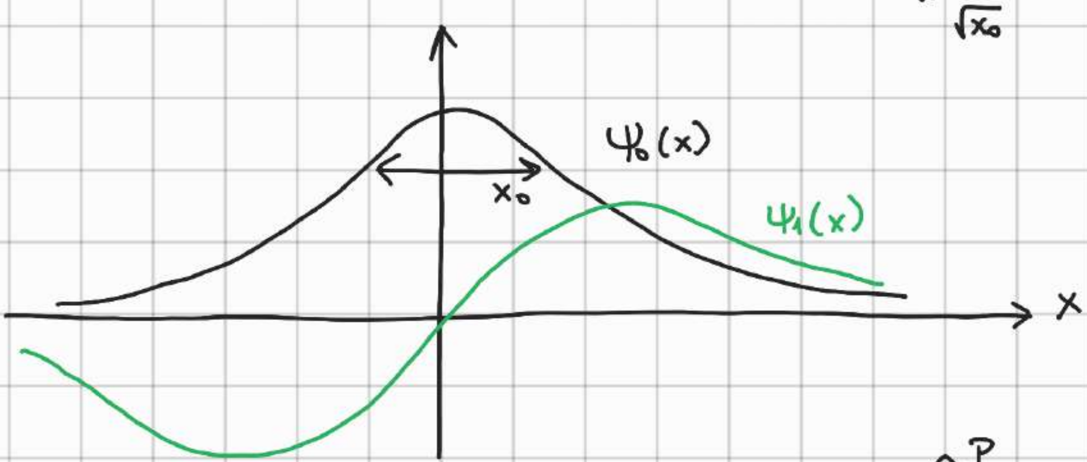
Eigen State:

$|n\rangle \quad , \quad n \in N \qquad \hat{n}|n\rangle = n|n\rangle$

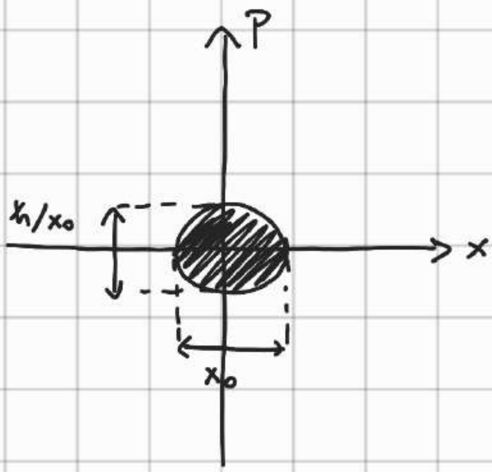
$$\begin{cases} \hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle \\ \hat{a}^+|n\rangle = \sqrt{n+1}\,|n+1\rangle \end{cases} \qquad \hat{a}|0\rangle = 0$$

Wavefunctions:

$$\langle x|0\rangle = \psi_0(x) = \Bigg(\underbrace{\frac{m\omega}{\hbar\pi}}_{\sim \frac{1}{\sqrt{x_0}}}\Bigg)^{1/4} e^{-x^2/2x_0^2}$$



Phase Space Representation:



II. Dynamics

1) Free Harmonic Oscillator

Heisenberg Picture : $i\hbar \dot{\hat{a}} = [\hat{a}, \hat{H}] = \hbar\omega \hat{a}$

$$\hat{a}(t) = \hat{a}(0)\, e^{-i\omega t}$$

2) Driven Oscillator

$$\hat{H} = \hbar\omega \hat{a}^{+}\hat{a} + \underbrace{F_0}_{\text{some classical force}} \hat{x} \cos(\omega_e t + \varphi)$$

$$\hat{H} = \hbar\omega \hat{a}^{+}\hat{a} + \frac{F_0 x_0}{\sqrt{2}} (\hat{a} + \hat{a}^{+}) \frac{1}{2} \left(e^{i(\omega_e t + \varphi)} + e^{-i(\omega_e t + \varphi)} \right)$$

Moving to a "rotating frame"

$$|\tilde{\Psi}(t)\rangle = \hat{U}(t)\, |\Psi(t)\rangle \qquad U^{+}|\tilde{\Psi}(t)\rangle = |\Psi(t)\rangle$$

$$i\hbar \partial_t |\tilde{\Psi}(t)\rangle = (i\hbar \partial_t \hat{U}) . |\Psi(t)\rangle + \hat{U}(t) \underbrace{i\hbar \partial_t |\Psi(t)\rangle}_{\hat{H}|\Psi(t)\rangle}$$

$$i\hbar \partial_t |\tilde{\Psi}\rangle = [\underbrace{(i\hbar \partial_t \hat{U}) . \hat{U}^{+} + \hat{U}(t) \hat{H} U^{+}}_{\tilde{H}}] |\tilde{\Psi}(t)\rangle$$

Here : $\hat{U}(t) = e^{i\omega_e \hat{n} t} = \sum_n e^{i\omega_e n t} |n\rangle\langle n|$

$$\tilde{H} = (i\hbar\partial_t\hat{U}).\hat{U}^+ + \hat{U}\hat{H}U^+$$

$$\partial_t\hat{U} = i\omega_e n \underbrace{\sum_n e^{i\omega_e n t} |n\rangle\langle n|}_{U} = i\omega_e n\,\hat{U}$$

$$\hat{H} = \hbar\omega\hat{n} + \dots$$

$$\tilde{H} = -\hbar\omega_e\hat{n} + \hat{U}\hat{H}U^+$$

$$\hat{U}\hat{a}\hat{U}^+ = e^{-i\omega_e t}\hat{a}$$

So that $\tilde{H} = \hbar(\omega-\omega_e)\hat{n} + \frac{F_0x_0}{2\sqrt{2}}\left(\hat{a}^+e^{i\varphi} + \hat{a}e^{-i\varphi} + \cancel{\hat{a}^+e^{i(2\omega_e t+\varphi)}} + \cancel{\hat{a}e^{-i(2\omega_e t+\varphi)}}\right)$

RWA

$\omega-\omega_e$, F_0x_0 ... $\ll 2\omega_e$

Rotating Wave Approximation

(provides Time Independent Hamiltonian)

$$\tilde{H} = \hbar\Delta\hat{a}^+\hat{a} + \frac{F_0x_0}{2\sqrt{2}}\left(\hat{a}^+e^{i\varphi} + \hat{a}e^{-i\varphi}\right)$$

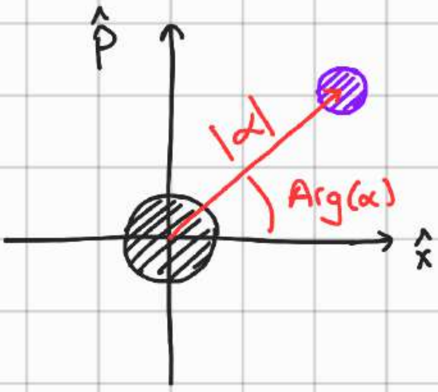
$$\tilde{H} = \hbar\Delta\left(\hat{a} + \frac{F_0x_0}{2\sqrt{2}\,\hbar\Delta}e^{i\varphi}\right)^+.\left(\hat{a} + \frac{F_0x_0}{2\sqrt{2}\,\hbar\Delta}e^{i\varphi}\right) + \text{constant}$$

3) Coherent State

$|\alpha\rangle$ such that $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$

Displacement operator

$\hat{D}(\alpha)|0\rangle = |\alpha\rangle \quad , \quad \hat{D}(\alpha) = e^{\alpha\hat{a}^+ - \alpha^*\hat{a}}$



Example: $\alpha \in \mathbb{R}$

$$\hat{D}(\alpha) = e^{\alpha(\hat{a}^+ - \hat{a})} = \underbrace{e^{-\alpha i x_0 \sqrt{2}\frac{\hat{p}}{\hbar}}}_{\text{physical displacement } \alpha\sqrt{2}.x_0}$$

Ex $\alpha = i\beta$, $\beta \in \mathbb{R}$

$$\hat{D}(\alpha) = e^{i\beta(\hat{a}^+ + \hat{a})} = \underbrace{e^{i\beta\frac{\hat{x}\sqrt{2}}{x_0}}}_{\text{momentum kick by } \alpha\frac{\hbar}{x_0}}$$

Back to driven oscillator:

$$\tilde{H} = \hbar\Delta\,\hat{b}^+\hat{b} + \text{constant}$$

$$\hat{b} = \hat{D}\left(\frac{-F_0 x_0}{2\sqrt{2}\hbar\Delta}e^{i\varphi}\right)\hat{a}\,\hat{D}\left(-\frac{F_0 x_0}{2\sqrt{2}\hbar\Delta}e^{i\varphi}\right)^+$$

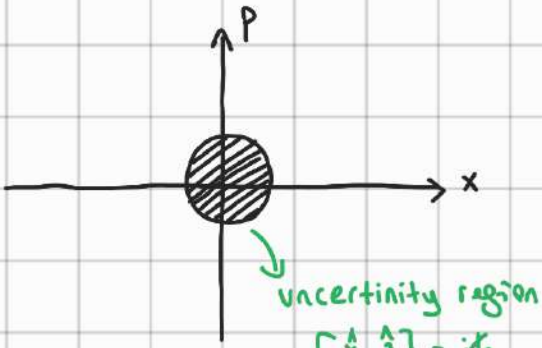
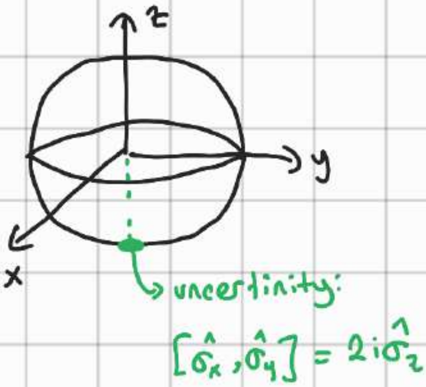
$$\hat{D}(\alpha)\,\hat{a}\,\hat{D}(\alpha)^+ = \hat{a} - \alpha\,\mathbb{I}$$

when $\Delta \to 0$: displacement goes to ∞ $\Rightarrow$ RESONANCE

Dynamics : $\hat{a}(t) = e^{-i\omega t}\hat{a}(0) \longrightarrow$ Schrödinger Picture

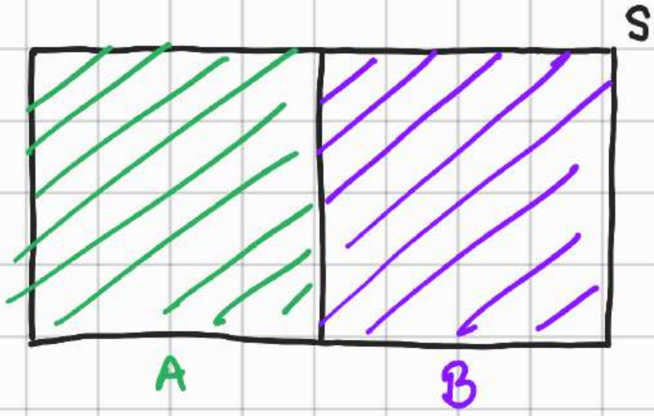
$$|\alpha(t)\rangle = |\alpha(0)e^{-i\omega t}\rangle$$

III. Comparison with two-level systems

	Harmonic Oscillator	Two Level System
Spectrum		
Ladder Operator	$[\hat{a}, \hat{a}^+] = \hat{\mathbb{I}}$	$[\hat{\sigma}_-, \hat{\sigma}_+] = \hat{\sigma}_z$
Phase Space		
Displacement	$\hat{D}(\alpha)$	$\hat{R}_{\vec{n}}(\theta)$

Bipartite Systems - Entanglement

I. Density Matrix



1) Bipartite System

$$\mathcal{H}_S = \mathcal{H}_A \otimes \mathcal{H}_B$$

$$|\Psi_{AB}\rangle \in \mathcal{H}_S$$

$\hat{M}_A$: observable on system A

$\hat{M} = \hat{M}_A \otimes \hat{\mathbb{I}}$: observeble on S
(acts on A only)

$$\langle \hat{M} \rangle = \langle \Psi_{AB} | \hat{M} | \Psi_{AB} \rangle$$

now with $|\psi_{AB}\rangle = \sum_{i\mu} \alpha_{i\mu} |i_A\rangle \otimes |\mu_B\rangle$

$$\langle \hat{M} \rangle = \sum_{\substack{i\mu \\ j\nu}} \langle i_A | \hat{M}_A | j_A \rangle \langle \mu_B | \hat{\mathbb{I}} | \nu_B \rangle \alpha^*_{i\mu} \alpha_{j\nu}$$

$$\langle \hat{M} \rangle = \sum_{ij\mu} \alpha_{i\mu}^* \alpha_{j\mu} \langle i | \hat{M}_A | j \rangle$$

$$\hat{\rho}_A = \sum_{\mu, j, i} \alpha_{i\mu}^* \alpha_{j\mu} |j\rangle\langle i|$$ density matrix A

$$\langle \hat{M} \rangle = \mathrm{Tr}\left(\hat{\rho}_A \hat{M}_A \right)$$

$\rightarrow \hat{\rho}_A$ is obtained from $|\psi_{AB}\rangle$:

Partial trace $\hat{\rho}_A = \mathrm{Tr}_B \left[|\psi\rangle_{AB} \langle \psi|_{AB} \right]$

2. States

- Pure state on A: $|\psi_A\rangle \in \mathcal{H}_A$ such that $\hat{\rho}_A = |\psi_A\rangle\langle\psi_A|$

- Product state of S: $|\psi_{AB}\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$ for some
(Seperate)
$|\psi_A\rangle$ and $|\psi_B\rangle$

- A state which is not a product state is <u>entangled.</u>

$\Leftrightarrow$ A is entangled with B when S is in state $|\psi_{AB}\rangle$

3. Discussion

- Here we defined entanglement for bipartite systems

- $\hat{\rho}_A$ can be diagonalized $\hat{\rho}_A = \sum_i p_i |i\rangle\langle i|$
prob.

$\hat{\rho}_A$ represents an ensemble of states

$\langle \hat{M}_A \rangle = Tr(\hat{\rho}_A \hat{M}_A) \longrightarrow$ ① quantum mech. expectation value for each state of the ensemble

$\langle i | \hat{M}_A | i \rangle$

$\downarrow$

(2) Classical average over all possible states $|i\rangle$

- Purity:

$$\hat{\rho}_A^2 = \sum_i p_i^2 |i\rangle\langle i|$$

So $\mathrm{Tr}(\hat{\rho}_A^2) = \sum_i p_i^2 \leq \sum_i p_i = 1$

II. ENTANGLEMENT

* Traces
* Stat. mech.
* Canonical exp.
* Schmidt

Bipartite system $\hat{\rho}_A = |\Psi_A\rangle\langle\Psi_A|$

$$|\Psi_{AB}\rangle = |\Psi_A\rangle \otimes |\Psi_B\rangle$$

$$Tr_B\left[|\Psi_{AB}\rangle\langle\Psi_{AB}|\right] = \sum_{\nu} |\Psi_A\rangle\langle\nu_B|\Psi_B\rangle\langle\Psi_A|\langle\Psi_B|\nu_B\rangle$$

$$= |\Psi_A\rangle\langle\Psi_A| \underbrace{\sum_{\nu} |\langle\nu_B|\Psi_B\rangle|^2}_{1}$$

If $\hat{\rho}_A$ is mixed : A and B were entangled

1) Entropy

Given $\hat{\rho}_A$: → Renyi entropy $S_n(\hat{\rho}_A) = -\text{Log}\left[Tr(\hat{\rho}_A^n)\right]\frac{1}{n-1}$ More practical

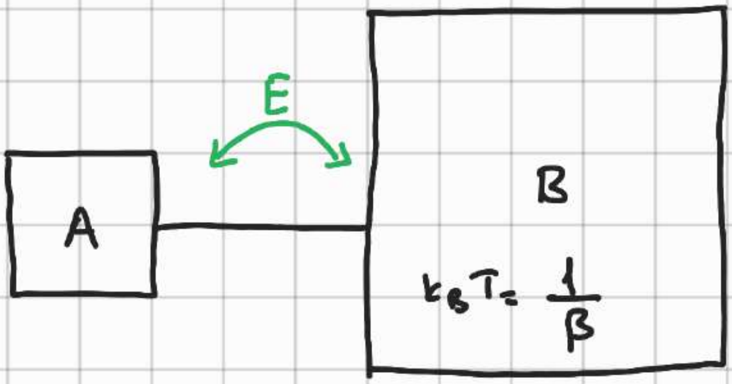
→ Entanglement entropy $S(\hat{\rho}_A) = -Tr\left[\hat{\rho}_A \ln \hat{\rho}_A\right]$

$$= \lim_{n\to 1} S_n(\hat{\rho}_A)$$

$$S(|\Psi\rangle\langle\Psi|) = 0$$

Ex Statistical Mechanics, canonical example

$$\hat{\rho}_A = \frac{1}{Z} e^{-\beta \hat{H}_A}$$



$$S(\hat{\rho}_A) = \; .... \; = F \; \beta \langle \hat{H}_A \rangle$$

Free energy

$$(e^{-\beta F} = Z)$$

Remarks → For any bipartition of S into A/B

$$S(\hat{\rho}_A) = S(\hat{\rho}_B)$$

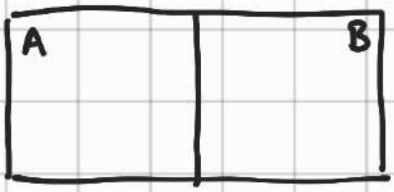
→ Maximum entropy: $\hat{\rho}_A = \frac{1}{N} \sum_i |i\rangle_A \langle i|_A$ $\quad N = \dim(HA)$

$$S(\hat{\rho}_A) = \log N$$

$\hat{\rho}_A = \frac{1}{N} \sum_i |i\rangle_A \langle i_A|$ infinite temperature

→ Thermalization: Building up entanglement

S



$S \propto$ area of the boundary A/B

$S \propto$ volume of A

→ Connection between S and quantum gravity (?)

2) Schmidt Decomposition

let $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ and $|\psi\rangle \in \mathcal{H}_{AB}$

Here exists an orthonormal set in $\mathcal{H}_A$: $\{|i_A\rangle\}$

" " " $\mathcal{H}_B$: $\{|i_B\rangle\}$

Such that $|\psi\rangle = \sum_i \lambda_i |i_A\rangle \otimes |i_B\rangle$

$\{\lambda_i\}$: Schmidt coef.

$\#\ \lambda_i$'s : Schmidt Rank $1 \rightarrow$ pure state

Density matrix: $\hat{\rho}_A = \mathrm{Tr}_B \left[|\psi\rangle\langle\psi| \right]$

let $\{i_B\}$ the 'Schmidt basis'

$$= \sum_i \langle i_B | \Psi \rangle \langle \Psi | i_B \rangle$$

$$\hat{\rho}_A = \sum_i |\lambda_i|^2 \, |i_A\rangle\langle i_A|$$

$$S(\hat{\rho}_A) = - \sum_i |\lambda_i|^2 \operatorname{Log} \lambda_i^2$$

Remark: $\lambda_i = e^{-\xi_i/2}$ $\quad |\Psi\rangle = \sum_i e^{-\xi_i/2} |i_A\rangle \otimes |i_B\rangle$

$\{\xi_i\}$: entanglement spectrum

Given $\hat{\rho}_A = \sum_i p_i |i_A\rangle\langle i_A|$

there exists system R, with Hilbert space $\mathcal{H}_R$

such that $\exists |\Psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_R \quad \hat{\rho}_A = \mathrm{Tr}_R [\,|\Psi\rangle\langle\Psi|\,]$

purification of $\hat{\rho}_A$

3) Example

2 qubit system : A, B

$$\mathcal{H}_A = S_p \{ |0\rangle, |1\rangle \}$$

$$\mathcal{H}_B = S_p \{ |0\rangle, |1\rangle \}$$

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \qquad \mathcal{H} = \mathrm{Sp}\{ |00\rangle, |01\rangle, |10\rangle, |11\rangle \}$$

$$\left(|ij\rangle = |i\rangle_A \otimes |j\rangle_B \right)$$

- $\frac{1}{\sqrt{2}} \left(|00\rangle + |01\rangle \right) \longrightarrow$ Not entangled $\left(\frac{1}{\sqrt{2}} |0\rangle \otimes (|0\rangle + |1\rangle) \right)$

$|\Phi^+\rangle = \cdot \frac{1}{\sqrt{2}} \left(|00\rangle + |11\rangle \right) \longrightarrow$ Entangled

$$\mathrm{Tr}_B \left[|\Phi^+\rangle\langle\Phi^+| \right] = \frac{1}{2} \left(|0\rangle\langle 0| + |1\rangle\langle 1| \right)$$

$\downarrow$

Maximally mixed state

$$|\phi_\varepsilon\rangle = \frac{1}{\sqrt{\varepsilon^2 + (1-\varepsilon)^2}} \left(\varepsilon |00\rangle + (1-\varepsilon) |11\rangle \right) \rightarrow \text{Yes}$$

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}} \left(|00\rangle + |11\rangle \right) \qquad |\Psi^+\rangle = \frac{1}{\sqrt{2}} \left(|01\rangle + |10\rangle \right)$$

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}} \left(|00\rangle - |11\rangle \right) \qquad |\Psi^-\rangle = \frac{1}{\sqrt{2}} \left(|01\rangle - |10\rangle \right)$$

QUANTUM MEASUREMENT

I. PROJECTIVE MEASUREMENT

Let $\hat{M}$ be an observable $\hat{M} = \sum_m m \hat{P}_m$

where $\hat{P}_m$ projector onto the eigenspace of m

· Measurement: Upon observing outcome m, the state of the system evolves:

$$|\psi\rangle \longrightarrow \frac{\hat{P}_m |\psi\rangle}{\sqrt{p(m)}} = |\psi_m\rangle$$

The probability to observe m is $p(m) = \langle\psi|\hat{P}_m|\psi\rangle$

For a mixed state $\hat{\rho}$: $p(m) = \mathrm{Tr}(\hat{\rho}\hat{P}_m) = E[m] = \langle\hat{M}\rangle$

2 averages : - Quantum average $\langle\psi|\hat{P}_m|\psi\rangle$

- Classical prob. average

If m has been observed (as outcome of the measurement)

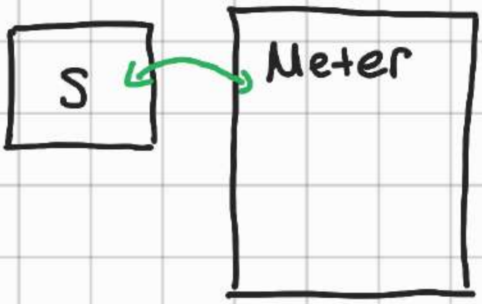
$$\hat{\rho}_m = \frac{\hat{P}_m \hat{\rho} \hat{P}_m^{+}}{p(m)}$$

If the measurement is performed but the outcome is not recorded

$$\hat{\rho} \longrightarrow \sum_m \hat{\rho}_m \, p(m) = \sum_m \hat{P}_m \hat{\rho} \hat{P}_m^{+}$$

if $|\psi\rangle\langle\psi| \longrightarrow \sum_m p(m) \hat{P}_m |\psi\rangle\langle\psi| \hat{P}_m^{+} = \sum_m p(m) |\psi_m\rangle\langle\psi_m|$

II. System - meter formulation



$\downarrow$

projective measurement

① Coupling between S and M

$$|\psi_{SM}\rangle \in \mathcal{H}_S \otimes \mathcal{H}_{meter}$$

$$|\Psi_{SM}\rangle = |\theta\rangle \otimes |\varphi\rangle \quad , |\theta\rangle \in \mathcal{H}_M$$

$$|\varphi\rangle \in \mathcal{H}_S$$

$$|\Psi_{SM}\rangle \longrightarrow |\Psi'\rangle = \hat{U}\,|\Psi_{SM}\rangle \neq |\theta'\rangle \otimes |\varphi'\rangle$$

② Projective Measurement of Meter:

Let r the outcome

$$\hat{P}_r = \hat{\Pi}_r \otimes \hat{\mathbb{I}}_S$$

$$|\Psi'_r\rangle = \frac{\hat{P}_r\,|\Psi'\rangle}{\sqrt{p(r)}} = \frac{(\hat{\Pi}_r \otimes \hat{\mathbb{I}}_S)\,|\Psi'\rangle}{\sqrt{p(r)}}$$

$$|\Psi'_r\rangle = \frac{(\hat{\Pi}_p \otimes \hat{\mathbb{I}})\,\hat{U}\,|\theta\rangle \otimes |\varphi\rangle}{\sqrt{p(r)}}$$

Suppose $\hat{\Pi}_r = |r\rangle\langle r|$

$$|\Psi'_r\rangle = \frac{|r\rangle \otimes \langle r|\hat{U}|\theta\rangle \cdot |\varphi\rangle}{\sqrt{p(r)}}$$

operator onto $\mathcal{H}_S$

**

let $\hat{M}_r = \langle r|\hat{U}|\theta\rangle$ operator on $\mathcal{H}_S$

So after the 2 steps:

$$|\Psi_r'\rangle = \frac{|r\rangle \otimes \hat{M}_r|\varphi\rangle}{\sqrt{p(r)}}$$

Consequence:

→ Starting from seperable state $|\theta\rangle \otimes |\varphi\rangle$ end with a seperable state $|\Psi_r'\rangle$

$$|\varphi\rangle \longrightarrow \frac{\hat{M}_r|\varphi\rangle}{\sqrt{p(r)}}$$

→ $\hat{M}_r$ is not a projector : "Measurement Operator"

→ $p(r) = \langle\Psi'|\hat{P}_r|\Psi'\rangle = \langle\varphi|M_r^{\dagger}M_r|\varphi\rangle$

III. Generalized Measurements

1. Definition: let $\hat{M}_r$ measurement operators,

and $\hat{E}_r = \hat{M}_r^+ \hat{M}_r$ such that

$\rightarrow \sum_r \hat{E}_r = \hat{I}$ $\quad \hat{E}_r$ are probability operators

Hermitian, positive

$\rightarrow$ Measurement outcomes are labelled by r

$\{\hat{E}_r, r\}$ = Positive operator-valued measurement (POVM)

Then $p(r)$ = probability of outcome r

$$p(r) = \mathrm{Tr}(\hat{\rho}\hat{E}_r)$$

After the measurement:

$\rightarrow$ If r has been observed $\hat{\rho} \rightarrow \hat{\rho}_r = \dfrac{\hat{M}_r \hat{\rho} \hat{M}_r^+}{p(r)}$

→ If the outcome is not recorded : $\hat{\rho} \to \sum_r \hat{\rho}_r \cdot p(r)$

$$= \sum_r \hat{M}_r \hat{\rho} \hat{M}_r^\dagger$$

Remarks: • It is not the most general formali.

• Given $\hat{E}_r$, positive Hermitian operator

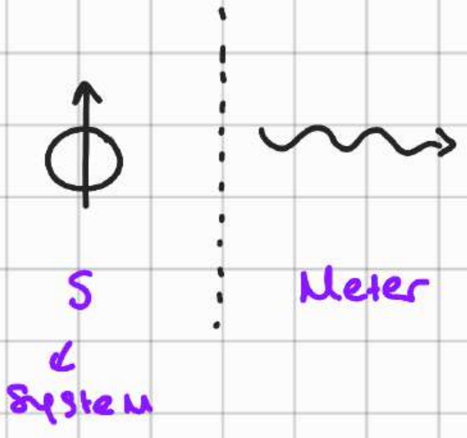
$$\hat{M}_r = \sqrt{\hat{E}_r}$$

(Cannot be repeated, photon killed)

NOT PROJECTIVE !

Example:

Photon counting + spontaneous emission



$|\Psi_{SM}\rangle$: pure state $\in \mathcal{H}_S \otimes \mathcal{H}_M$

Measurement of the # of photon :

j photon : $\hat{M}_j = \hat{I} \otimes |0\rangle\langle j|$

$\hat{I} \otimes \langle 0 | j \rangle \, |0\rangle\langle j|$

? How

$\hat{E}_j = \hat{M}_j^\dagger M_j = \hat{\mathbb{I}} \otimes |j\rangle\langle j|$, positive operator

$$\sum_j E_j = \hat{\mathbb{I}} \qquad \{\hat{E}_j, j\} \text{ POVM !}$$

After the measurement: $|\psi_{SM,j}\rangle = \frac{|\psi_S\rangle \otimes |0\rangle}{pr(j)}$

(pure) product State

If the result is not recorded:

$$|\psi_{SM}\rangle\langle\psi_{SM}| \longrightarrow \sum_j \hat{M}_j^\dagger |\psi_{SM}\rangle\langle\psi_{SM}| \hat{M}_j^\dagger$$

$$|\psi_{SM}\rangle = \sum_{kl} \alpha_{kl} |k\rangle \otimes |l\rangle$$

$$|\psi_{SM}\rangle\langle\psi_{SM}| \to \sum_j (\mathbb{I} \otimes |0\rangle\langle j|) \sum_{kl} \alpha_{kl} (|k\rangle \otimes |l\rangle)$$

$$\ldots (\langle k'| \otimes \langle l'|)\, \alpha^A_{k,l} \sum_j (\mathbb{I} \otimes |j\rangle\langle 0|)$$

$$\sum_{kk'} \left(\sum_j \alpha^*_{k'j} \hat{\alpha}_j \, \alpha_{kj} \right) |k\rangle\langle k'| \otimes |0\rangle\langle 0|$$

$$\rightarrow \mathrm{Tr}_M \left(|\psi_{SM}\rangle\langle\psi_{SM}| \right) \otimes |0\rangle\langle 0|$$

EVOLUTION OF DENSITY MATRICES

I. Super-operators

Preskill Notes

1) Evolution Paths for $\hat{\rho}$

→ Unitary evolutions (closed system)

$$i\hbar \partial_t \hat{\rho} = [\hat{H}, \hat{\rho}]$$

$$\hat{\rho}(t) = \hat{U}(t) \hat{\rho} \hat{U}^{+}(t)$$

→ Measurements

- with results recorded:

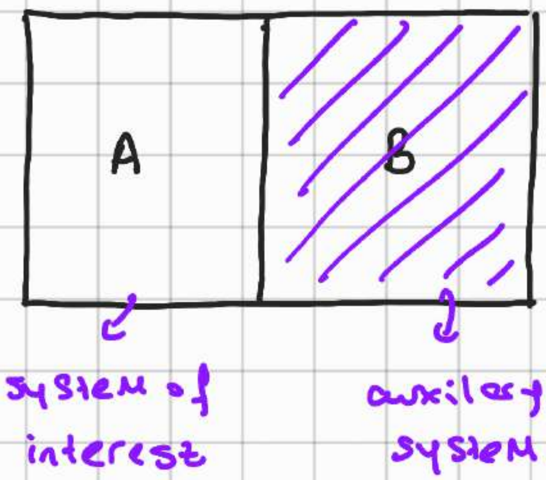
$$\hat{\rho} \longrightarrow \hat{\rho}' = \frac{1}{\sqrt{p(r)}} \hat{M}_r \hat{\rho} \hat{M}_r^{+}$$

with $\hat{M}_r$ measurement operators $\sum_r \hat{M}_r^{+} \hat{M}_r = \mathbb{I}$

- without recording the result:

$$\hat{\rho}' = \sum_r \hat{M}_r \hat{\rho} \hat{M}_r^{+}$$

→ Unitary evolution on extended space



$$\hat{\rho} = \hat{\rho}_A \otimes \underbrace{|0\rangle\langle 0|}_{B}$$

unitary evolution:

$$\hat{\rho}' : \hat{U}(\hat{\rho}_A \otimes |0\rangle\langle 0|)\hat{U}^\dagger$$

operator on $\mathcal{H}_A \otimes \mathcal{H}_B \neq \hat{U}_A \otimes \hat{U}_B$

Partial-trace over B:

$$\hat{\rho}_A' = \mathrm{Tr}_B\, \hat{\rho}'$$

$$= \sum_\mu \langle \mu_B | \hat{U}(\hat{\rho}_A \otimes |0\rangle\langle 0|)\hat{U}^\dagger | \mu_B \rangle$$

$$= \sum_\mu \underbrace{\langle \mu_B | \hat{U} | 0 \rangle}_{\substack{\text{operator on } \mathcal{H}_A \\ \hat{M}_\mu}} \cdot \hat{\rho}_A \cdot \underbrace{\langle 0 | \hat{U}^\dagger | \mu_B \rangle}_{\hat{M}_\mu^\dagger}$$

Common feature: $\hat{\rho}' = \sum_\mu \hat{M}_\mu \hat{\rho} \hat{M}_\mu^\dagger$ → operator sum representation

2) Completely Positive Maps

$$\mathcal{S}: \hat{\rho} \longrightarrow \hat{\rho}'$$

$\hat{\rho}$: operator on $\mathcal{H}_A$

Requirements:
- preserves Hermicity
- trace-preserving
- positive $\hat{\rho} \geq 0 \rightarrow \hat{\rho}' \geq 0$
- $\forall \mathcal{H}_B$ Hilbert space of system B

  $\mathcal{S} \otimes \mathbb{I}$ is a positive operator

complete positivity

Completely positive map: Super-Operator $\hat{S} = \hat{\rho} \rightarrow \hat{\rho}'$

Krauss Theorem: Any completely positive map has an operator sum representation.

$$\hat{S}(\hat{\rho}) = \sum_{\mu} \hat{M}_{\mu} \hat{\rho} \hat{M}_{\mu}^{+} \quad \text{for a set of } \{\hat{M}_{\mu}\}$$

such that $\sum_{\mu} \hat{M}_{\mu}^{+} \hat{M}_{\mu} = \mathbb{I}$

Remarks :

- if $\dim(\mathcal{H}_A) = N$ then there are at most N^2 operators
- the representation is not unique

unitary matrix

$$\hat{N}_\nu = \sum_\mu U_{\nu\mu} \hat{M}_\mu$$

- Any super-operator can be interpreted as a POVM

Decoherence

- In general, a super operator IS NOT invertible

the only case where S.O IS invertible is unitary evolution.

Quantum Channels

$$\hat{\rho} \longrightarrow \boxed{} \longrightarrow \hat{\rho}'$$

1) Amplitude Damping Channel

Model for spontaneous emission

System S (qubit)

Environment E

$$\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_E$$

Unitary evolution:

$$|0\rangle \otimes |0\rangle \xrightarrow{\text{Nothing happens}} |0\rangle \otimes |0\rangle$$

$$|1\rangle \otimes |0\rangle \longrightarrow \underbrace{\sqrt{p}}_{\text{emission prob}}\, |0\rangle \otimes |1\rangle + \underbrace{\sqrt{1-p}}_{\text{nothing happens}}\, |1\rangle \otimes |0\rangle$$

$$\hat{S}: \hat{\rho}_S \longrightarrow \hat{\rho}_S' = Tr_E[\hat{U}\rho\hat{U}^+]$$

Initialize: $\hat{\rho} = \hat{\rho}_S \otimes |0\rangle\langle 0|_E$

$$\hat{\rho}' = \langle 0_E|\hat{U}\hat{\rho}\hat{U}^+|0_E\rangle + \langle 1_E|\hat{U}\hat{\rho}\hat{U}^+|1_E\rangle$$

$$\hat{\rho}' = \hat{M}_0\hat{\rho}_S\hat{M}_0^+ + \hat{M}_1\hat{\rho}_S M_1^+$$

$\hat{\rho}_S, \hat{M}_0, \hat{M}_1$ operators on $\hat{\mathcal{H}}_S$

- $\hat{M}_0 = ?$

$$\langle 0_E|\hat{U}(\hat{\rho}_S \otimes |0\rangle\langle 0|_E)\hat{U}^+|0_E\rangle = \hat{M}_0\hat{\rho}_S\hat{M}_0^+$$

$$\hat{M}_0 = \langle 0_E|\hat{U}|0_E\rangle$$

$$= \begin{array}{cc} |0\rangle|0\rangle \rightarrow |0\rangle|0\rangle\ (1) & |0\rangle|0\rangle \rightarrow |1\rangle|0\rangle\ (0) \\ \left[\begin{array}{cc} \underbrace{\langle 0_S|\otimes\langle 0_E|\hat{U}|0_S\rangle\otimes|0_E\rangle}_{M_{0,00}} & \underbrace{\langle 0_S|\otimes\langle 0_E|\hat{U}|1_S\rangle\otimes|0_E\rangle}_{M_{0,01}} \\ \underbrace{\langle 1_S|\otimes\langle 0_E|\hat{U}|0_S\rangle\otimes|0_E\rangle}_{M_{0,10}} & \underbrace{\langle 1_S|\otimes\langle 0_E|\hat{U}|1_S\rangle\otimes|0_E\rangle}_{M_{0,11}} \end{array}\right]_{2\times 2} & \\ |1\rangle|0\rangle \rightarrow |0\rangle|0\rangle\ (0) & |1\rangle|0\rangle \rightarrow |1\rangle|0\rangle\ \sqrt{2p} \end{array}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{bmatrix}$$

$$\langle 1_E | \hat{U} (\hat{\rho}_S \otimes |0_E\rangle\langle 0|) \hat{U}^{+} | 1_E \rangle$$

$$\hat{M}_1 = \langle 1_E | \hat{U} | 0_E \rangle$$

$|0\rangle|0\rangle \longrightarrow |0\rangle|1\rangle$ 0 $\qquad$ $|10\rangle \longrightarrow |01\rangle$ $\sqrt{p}$

$$= \begin{bmatrix} \underbrace{\langle 0|_S \otimes \langle 1_E | \hat{U} | 0_S \rangle \otimes |0_E\rangle}_{M_{1,00}} & \underbrace{\langle 0_S| \otimes \langle 1_E | \hat{U} | 1_S \rangle \otimes |0_E\rangle}_{M_{1,01}} \\ \langle 1_S| \otimes \langle 1_E | \hat{U} | 0_S \rangle \otimes |0_E\rangle & \langle 1_S| \otimes \langle 1_E | \hat{U} | 1_S \rangle \otimes |0_E\rangle \end{bmatrix}$$

$M_{1,10}$ $\quad |00\rangle \longrightarrow |11\rangle$ 0 $\qquad$ $M_{1,11}$

$|10\rangle \longrightarrow |11\rangle$ 0

$$= \begin{bmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{bmatrix}$$

Check: $\hat{M}_0^{+} \hat{M}_0 + \hat{M}_1^{+} \hat{M}_1 = \mathbb{I}$

$$\hat{\rho}_s = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} \text{ then } \hat{\rho}_s' = \begin{pmatrix} \rho_{00} + p\,\rho_{11} & \sqrt{1-p}\,\rho_{01} \\ \sqrt{1-p}\,\rho_{10} & \rho_{11}(1-p) \end{pmatrix}$$

- Probability of transition p between 0 and 1
- $\sqrt{1-p}$ reduction in the coherence!

On Bloch Sphere:

$$\hat{\rho}_s = \frac{1}{2}\left(\hat{\mathbb{I}} + \vec{a} \cdot \vec{\hat{\sigma}} \right)$$

$$\begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} \longrightarrow \vec{a}' = \begin{pmatrix} \sqrt{1-p}\, a_x \\ \sqrt{1-p}\, a_y \\ (1-p)\, a_z + p \end{pmatrix}$$

$$\mathrm{Tr}\left(\rho_s'^2\right) \leq \mathrm{Tr}\left(\rho_s^2\right) \qquad \forall p \in [0,1]$$

Interpretation in terms of measurements

$$\hat{E}_0 = \hat{M}_0^\dagger M_0 \qquad \hat{E}_1 = \hat{M}_1^\dagger M_1$$

. $\hat{M}_1 : |\psi\rangle \longrightarrow \dfrac{\hat{M}_1|\psi\rangle}{\sqrt{p}}$ $\qquad \hat{M}_1 = \sqrt{p}\, |0\rangle\langle 1|$

"1 click" $\qquad \hat{E}_1 = p\, |1\rangle\langle 1|$

$|\psi\rangle \longrightarrow |0\rangle$ (Heralded preparation of $|0\rangle$)

. $\hat{M}_0 : |\psi\rangle \longrightarrow \dfrac{\hat{M}_0|\psi\rangle}{\sqrt{p(0)}}$ $\qquad \hat{M}_0 = |0\rangle\langle 0| + \sqrt{1-p}\, |1\rangle\langle 1|$

"0 click" $\qquad \hat{E}_0 = |0\rangle\langle 0| + (1-p)|1\rangle\langle 1|$

$$\left.\begin{array}{l} |\psi\rangle = a|0\rangle + b|1\rangle \\ \hat{M}_0|\psi\rangle \propto a|0\rangle + b\sqrt{1-p}\,|1\rangle \end{array}\right\} \quad |\psi\rangle \neq |\psi'\rangle$$

No click $\neq$ No information

Remark: Repeat n consecutive amp. damping ch:

$$\rho_{11} \longrightarrow \rho_{11}(1-p)^n$$

Decay Rate Γ : $\quad p = \frac{\Gamma t}{n}$

$$\rho_{11} \longrightarrow \rho_{11} e^{-\Gamma t} \quad (n \to \infty)$$

2) Phase Damping Channel

Unitary operator onto $\mathcal{H}_S \otimes \mathcal{H}_E$

$$|0\rangle \otimes |0\rangle \longrightarrow \sqrt{p}\, |0\rangle \otimes |0\rangle + \sqrt{1-p}\, |0\rangle \otimes |1\rangle$$

$$|1\rangle \otimes |0\rangle \longrightarrow \sqrt{p}\, |1\rangle \otimes |0\rangle - \sqrt{1-p}\, |1\rangle \otimes |1\rangle$$

$$|0\rangle \otimes |0\rangle \rightarrow |0\rangle \otimes \left(\sqrt{p}\, |0\rangle + \sqrt{1-p}\, |1\rangle \right)$$

$$|1\rangle \otimes |0\rangle \rightarrow |1\rangle \otimes \left(\sqrt{p}\, |0\rangle - \sqrt{1-p}\, |1\rangle \right)$$

Remark: It is an entangling operation:

$$\hat{\rho}_S' = \mathrm{Tr}_E \Big[\hat{U} \big(\overbrace{\hat{\rho}_S \otimes |0\rangle\langle 0|_E}^{\hat{\rho}} \big) \hat{U}^\dagger \Big]$$

$$= \langle 0_E | \hat{U} \hat{\rho} \hat{U}^\dagger | 0_E \rangle + \langle 1_E | \hat{U} \hat{\rho} \hat{U}^\dagger | 1_E \rangle$$

$$= \hat{M}_0 \hat{\rho}_S \hat{M}_0^\dagger + \hat{M}_1 \hat{\rho}_S \hat{M}_1^\dagger$$

$$\hat{M}_0 = \sqrt{p} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \hat{M}_1 = \sqrt{1-p} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{\sigma_z}$$

Check that $\hat{M}_0^\dagger M_0 + \hat{M}_1^\dagger M_1 = \hat{\mathbb{I}}$

Bloch vector:

$$\hat{\rho} = \frac{1}{2} (\hat{\mathbb{I}} + \vec{a}.\vec{\sigma}) \qquad \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} \longrightarrow a' = \begin{pmatrix} a_x(2p-1) \\ a_y(2p-1) \\ a_z \end{pmatrix}$$

Unless $p=0$ or 1 we have $\|\vec{a}'\|^2 \leq \|\vec{a}\|^2$

Interpretation: $\frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes |0\rangle = \sqrt{p} \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}} \right) \otimes |0\rangle + \sqrt{1-p} \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right) \otimes |1\rangle$

If $p = 1/2 \rightarrow \hat{\rho}' = \frac{1}{2} \left(|0\rangle\langle 0| + |1\rangle\langle 1| \right)$

Remark: ⋆ Irreversible

Not possible to "inflate the Bloch Sphere"

* generalize to harmonic oscillator

* Physical motivation: Random Rotations about z

$$R_z(\hat{\theta}) = e^{i\theta\hat{\sigma}_z}$$

Average over possible values of θ:

$$\hat{\rho} = \frac{1}{\sqrt{4\pi\lambda}} \int d\theta \, \hat{R}_z(\theta) \hat{\rho} \hat{R}_z(\theta)^{+} e^{-\theta^2/4\lambda}$$

$$\hat{\rho}' = \begin{pmatrix} |a|^2 & ab^* e^{-\lambda} \\ a^* b e^{-\lambda} & |b|^2 \end{pmatrix}$$

3) Depolarizing Channel:

Phase damping picks a preferred direction: z

$$\hat{M}_0 = \sqrt{1-p}\,\hat{\mathbb{I}} + \sqrt{p}\,\hat{\sigma}_z$$

Now, define a channel through Kraus Operators

$$\hat{M}_0 = \sqrt{1-p}\,\hat{\mathbb{I}}$$

$$\hat{M}_1 = \sqrt{\frac{p}{3}}\,\hat{\sigma}_x \qquad \hat{M}_2 = \sqrt{\frac{p}{3}}\,\hat{\sigma}_y \qquad \hat{M}_3 = \sqrt{\frac{p}{3}}\,\hat{\sigma}_z$$

$$\sum_i \hat{M}_i^{+} M_i = \hat{\mathbb{I}}$$

Action on the Bloch vector:

$$\hat{\rho}' = \sum_{\nu} \hat{M}_{\nu}\, \hat{\rho}\, \hat{M}_{\nu}^{+} = \frac{1}{2}\left(\hat{I} + \vec{a} \cdot \begin{pmatrix} \sum_{\nu} \hat{M}_{\nu}\, \hat{\sigma}_x\, \hat{M}_{\nu}^{+} \\ \vdots \end{pmatrix} \right)$$

$$\sum_{\nu} \hat{M}_{\nu}\, \hat{\sigma}_x\, \hat{M}_{\nu}^{+} = \dots = \left(1 - \frac{4p}{3}\right)$$, same for $\hat{\sigma}_y, \hat{\sigma}_z$

So $\vec{a}' = \vec{a} \cdot \left(1 - \frac{4p}{3}\right)$

Uniform "contraction" of the Bloch Sphere

Lindblad Equation

idea: $\hat{\rho}(0) \longrightarrow \hat{\rho}(t) = \underset{\substack{\downarrow \\ \text{Superoperator}}}{S}[\hat{\rho}(0)]$

For unitary evolution we know that $i\dot{\hat{\rho}} = [\hat{H}, \hat{\rho}]$

What about more general evolution?

I. Markov Approximation

. There is no 1 to 1 correspondence btw t and $t' > t$

Actually: $\underbrace{\boxed{S \mid E}}_{\text{unitary}}$ There's a 1-1 correspondence btw t and t' for $S+E$.

When measurements are done on E $\rightarrow$ some information is lost.

E can also evolve in time $\rightarrow$ if E is traced out No history of E!

In particular $\hat{\rho}_S(t)$ may depend on $\hat{\rho}_E(t')$, for all $t' < t$

to progress: hypothesis

- E is very large: reservoir in the thermodynamic sense. coupling S to E does not significantly change E.

• Markov approximation:

Memory time τ for E (property of E)

after which any perturbation introduced by S relaxes

$\rightarrow$ E is stationary over time scales $\gg \tau$

"Coarse grained" description over $t \gg \tau \leftrightarrow$ E is stationary

Examples:

- E: vacuum of electro-magnetic field in free space.
- Large number of spins

II. Derivation of the Lindblad Equation

1) Notations

- $\hat{\rho}(t+\Delta t) = S[\hat{\rho}(t)] = \sum_{\nu} \hat{M}_\nu \rho(t) \hat{M}_\nu^+$
- Markov approximation: $\hat{\rho}(t+\Delta t) = \hat{\rho}(t) + \hat{O}(\Delta t)$

$$\left.\begin{array}{l} \cdot\ \Delta t \text{ small} \\ \cdot\ \Delta t \gg \tau \end{array}\right\} \text{Coarse grained evolution}$$

Start from Krauss:

- $$\hat{M}_0 = \underbrace{\hat{\mathbb{I}}}_{0^{th}\text{ order}} - i\underbrace{\hat{K}}_{1^{st}\text{ order}}\Delta t + \hat{O}(\Delta t^2)$$

- $\nu \geq 1$ $\quad \hat{M}_\nu \hat{\rho} \hat{M}_\nu^+$ has to be order Δt

  So $\hat{M}_\nu = \hat{O}(\sqrt{\Delta t}) = \sqrt{\Delta t}\,\hat{L}_\nu$

- $$\hat{H} = \frac{\hat{K} + \hat{K}^+}{2} \qquad \hat{J} = i\,\frac{\hat{K} - \hat{K}^+}{2} \quad \text{st} \quad \hat{K} = \hat{H} - i\hat{J}$$

  $$\hat{H}^+ = \hat{H} \qquad\qquad \hat{J}^+ = -\hat{J}$$

2) Expression

$$\hat{M}_0 \hat{\rho} \hat{M}_0^+ = (\hat{\mathbb{I}} - i\hat{K}\Delta t)\hat{\rho}(\hat{\mathbb{I}} + i\hat{K}^+\Delta t) + \dots$$

$$= \hat{\rho} - i\Delta t(\hat{K}\hat{\rho} - \hat{\rho}\hat{K}^+) + \dots$$

$$= \hat{\rho} - i\Delta t[\hat{H}, \hat{\rho}] - \Delta t(\hat{J}\hat{\rho} + \hat{\rho}\hat{J}) + \dots$$

Remark: for unitary evolution: only $\hat{M}_0$ in the sum, and

$$\hat{M}_0^\dagger M_0 = \hat{\mathbb{I}}$$

$$(\hat{\mathbb{I}} + i\hat{K}^\dagger \Delta t)(\hat{\mathbb{I}} - i\hat{K}\Delta t) = \hat{\mathbb{I}} + i\,\Delta t(\underbrace{K^\dagger - K}_{0}) + \dots$$

$$\Rightarrow \quad \hat{J} = 0$$

$$\underbrace{\hat{M}_0 \hat{\rho} \hat{M}_0^\dagger}_{} = \hat{\rho} - i[\hat{H}, \hat{\rho}]$$ ⇝ Where's $\Delta t = ?$

$$\hat{\rho}(t+\Delta t) = \hat{\rho}(t) - i[\hat{H}, \hat{\rho}] \quad \Rightarrow i\dot{\hat{\rho}} = [\hat{H}, \hat{\rho}]$$

$\hat{H}$ has to be interpreted as the Hamiltonian

But it can be $\neq$ from the Hamiltonian of S alone

↓

the difference is the Lamb-shift

For $\nu \geq 1$: $$\sum_{\nu=0} \hat{M}_\nu^\dagger \hat{M}_\nu = \hat{\mathbb{I}}$$

$$\hat{M}_0^\dagger \hat{M}_0 + \sum_{\nu \geq 1} \hat{M}_\nu^\dagger \hat{M}_\nu = \mathbb{I}$$

$$\hat{M}_0^\dagger \hat{M}_0 = \cancel{\hat{\mathbb{I}}} - 2\Delta t \hat{J} + \Delta t \sum_{\nu \geq 1} \hat{L}_\nu^\dagger \hat{L}_\nu = \cancel{\hat{\mathbb{I}}}$$

$$\hat{J} = \frac{1}{2} \sum_{\nu \geq 1} \hat{L}_\nu^\dagger \hat{L}_\nu$$

$$\frac{\partial \hat{\rho}}{\partial t} = \frac{1}{\Delta t} \cdot \left(\hat{\rho}(t+\Delta t) - \hat{\rho}(t) \right)$$

$$= \frac{1}{\Delta t} \left(\sum_{\mu} \hat{M}_{\mu} \rho(t) \hat{M}_{\mu}^{+} - \hat{\rho}(t) \right)$$

"Lindblad Equation"

$$\boxed{\frac{\partial \hat{\rho}}{\partial t} = \underbrace{-i[\hat{H}, \hat{\rho}]}_{\text{unitary part}} + \sum_{\mu \geq 1} \underbrace{\hat{L}_{\mu} \hat{\rho} \hat{L}_{\mu}^{+}}_{\substack{\text{krauss operators} \\ \text{(POVM)}}} - \underbrace{\frac{1}{2} \hat{L}_{\mu}^{+} \hat{L}_{\mu} \hat{\rho} - \frac{1}{2} \hat{\rho} \hat{L}_{\mu}^{+} \hat{L}_{\mu}}_{\hat{J}}}$$

$\hat{J} \rightarrow$ non-unitary part of $\hat{K}$

$\hookrightarrow$ normalization of krauss representation

- $\hat{L}_{\mu}$: Jump operators $\propto \sqrt{Hz}$

III. Interpretation

- As POVM: $\{ \hat{M}_{\mu}^{+} \hat{M}_{\mu}, \mu \}$

$$\hat{E}_0 = \hat{M}_0^{+} M_0 = \hat{I} - 2\Delta t \hat{J}$$

$$\hat{E}_{\mu} = \hat{M}_{\mu}^{+} \hat{M}_{\mu} = \Delta t \hat{L}_{\mu}^{+} \hat{L}_{\mu}$$

P_0 = probability to obtain $\overset{\text{zero}}{0}$: $Tr(\hat{\rho} E_0) = 1 - 2\Delta t \, Tr(\hat{\rho}\hat{J})$

$$Tr(\hat{\rho} E_0) = 1 - \Gamma_i \Delta t$$

Spontaneous Emission

$$\langle \hat{J} \rangle = Tr(\hat{\rho}\hat{J}) = \frac{\Gamma}{2}$$

P_ν = probability to obtain ν: $Tr(\hat{E}_\nu \hat{\rho}) = \Delta t \, Tr(\hat{\rho} \hat{L}_\nu^\dagger \hat{L}_\nu)$

$$= \Delta t \, \Gamma_\nu$$

Normalization: $\hat{J} = \frac{1}{2} \sum_{\nu \geq 1} \hat{L}_\nu^\dagger \hat{L}_\nu$

Unitary evolution over an extended Hilbert Space

$$\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_R$$

$|\psi_s\rangle \in \mathcal{H}_S$

$$\hat{U}(|\psi_s\rangle \otimes |0_R\rangle) = (\hat{M}_0 |\psi_s\rangle) \otimes |0_R\rangle + \sum_{\nu \geq 1} \hat{M}_\nu |\psi_s\rangle \otimes |\nu\rangle_R$$

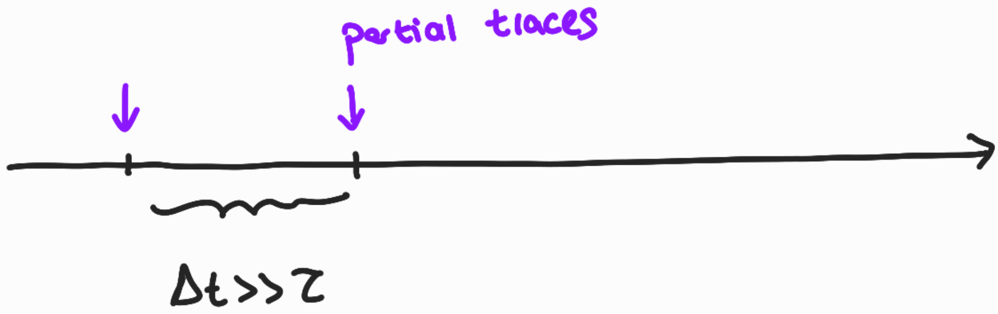
$$= \left[(\hat{\mathbb{I}} - i\hat{H}\Delta t - \hat{J}\Delta t)|\psi_s\rangle\right] \otimes |0_R\rangle + \sqrt{\Delta t} \sum_\nu \hat{L}_\nu |\psi_s\rangle \otimes |\nu\rangle$$

Projective measurement on the reservoir:

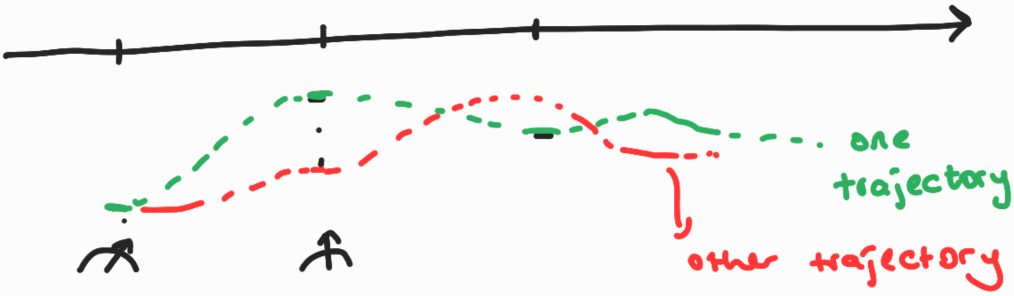
- with prob. $P_0 \sim O(1)$ → reservoir stays in state $|0_R\rangle$
- with prob P_ν $|\mu_R\rangle$

Quantum Trajectories:

. Lindblad Equation describes at each point in time the "tracing out" of the environment



. One realization followed in time:

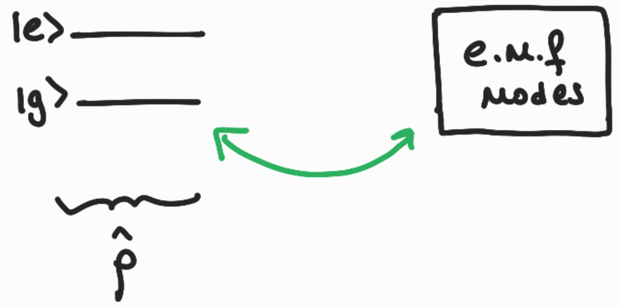


averaging over trajectories provide $\hat{\rho}(t)$

→ Monte-Carlo Wavefunction Method

OPTICAL BLOCH EQUATIONS

Two level system + environment



I. Derivation

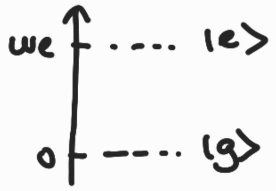
1) Model

$$\boxed{\hat{H} = \Delta |e\rangle\langle e| + \frac{\Omega}{2}\overbrace{\left(|e\rangle\langle g| + |g\rangle\langle e| \right)}^{\hat{\sigma}_x}}$$

→ can be anything

$$\hat{H} = \frac{\Delta}{2}\hat{\mathbb{I}} + \vec{h}\cdot\vec{\sigma}$$

$\Delta = \omega_e - \omega_{drive}$



- $\hat{H}$ in the frame rotating at the drive frequency + RWA

- We have chosen the phase of the external drive

$$\vec{h} \in xz \text{ plane}$$

In general, $\frac{\Omega}{2}\left(|e\rangle\langle g| e^{i\phi} + |g\rangle\langle e| e^{-i\phi} \right)$, ϕ arbitrary

Spontaneous emission

$$\hat{L} = \sqrt{\Gamma}\, |g\rangle\langle e|$$

jump operator

- jumps produce transitions $|e\rangle \rightarrow |g\rangle$

- Note: $\mathrm{Tr}(\hat{\rho}\hat{L}^{+}\hat{L}) = \Gamma .\ \mathrm{Tr}(\hat{\rho}\, |e\rangle\langle e|) = \Gamma\, p_e$

prob. being in $|e\rangle$ state

2. Justification

Extended Hilbert Space : $\mathcal{H} = \mathcal{H}_{\text{2-lev sys}} \otimes \mathcal{H}_{\text{emf}}$ (fock space)

Light-Matter interaction : $\hat{H}_{int} = \hat{d} \otimes \hat{\vec{E}}$

(dipole) moment operator

(electro magnetic field operator)

$$\hat{\vec{d}} = \vec{e}_x\, d_0 \left(|e\rangle\langle g| + |g\rangle\langle e| \right)$$ (Dipolar approximation)

polarization vector

reduced dipole moment

$$\hat{\vec{E}} = -i \sum_{\vec{k},\lambda} \sqrt{\frac{\hbar \omega_{\vec{k}}}{2\varepsilon_0 V}}\, \vec{\varepsilon}_\lambda \left(e^{-i(\vec{k}.\vec{r} - \omega_{\vec{k}} t)}\, \hat{a}_{\vec{k},\lambda} - \text{hermit conj} \right)$$

$$\omega_{\vec{k}} = c|\vec{k}|$$

V quantization volume

Fermi Golden Rule $\Gamma_{\vec{k},\lambda} = \frac{2\pi}{\hbar} \left| \langle \{0\}_{\vec{k},\lambda}, e | \hat{\vec{d}} \otimes \hat{\vec{E}} | \{1\}_{\vec{k}}, g \rangle \right|^2$

$$= \delta(\omega_e - \omega_{\vec{k}})$$

Lindblad Equation

$$\dot{\hat{\rho}} = -i[\hat{H}, \hat{\rho}] + \sum_{\vec{k},\lambda} \left(\hat{L}_{\vec{k},\lambda} \hat{\rho} \hat{L}^{\dagger}_{k,\lambda} - \frac{1}{2} \hat{L}^{\dagger}_{k\lambda} \hat{L}_{k,\lambda} \hat{\rho} - \frac{1}{2} \hat{\rho} \hat{L}^{\dagger}_{k,\lambda} \hat{L}_{k,\lambda} \right)$$

Here $\hat{L}_{\vec{k},\lambda} = \sqrt{\Gamma_{k,\lambda}}\, |g\rangle\langle e|$

$$\dot{\hat{\rho}} = -i[\hat{H}, \hat{\rho}] + \left(\sum_{k,\lambda} \Gamma_{k,\lambda} \right) |g\rangle\langle e| \hat{\rho} |e\rangle\langle g| - \frac{1}{2} |e\rangle\langle e| \hat{\rho} - \frac{1}{2} \hat{\rho} |e\rangle\langle e|$$

Replace all $L_{\vec{k},\lambda}$ by $\hat{L} = \sqrt{\Gamma}\, |g\rangle\langle e|$, $\Gamma = \sum_{\vec{k},\lambda} \Gamma_{\vec{k},\lambda}$

Remark Momentum is not conserved ($\hbar\vec{k}$ is taken by the field)

3. Bloch vector dynamics

$$\dot{\hat{\rho}} = -i[\hat{H}, \hat{\rho}] + \Gamma \left(|g\rangle\langle e| \hat{\rho} |e\rangle\langle g| - \frac{1}{2} \hat{\rho} |e\rangle\langle e| - \frac{1}{2} |e\rangle\langle e| \hat{\rho} \right)$$

Matrix element $\dot{\rho}_{ee} = \langle e | \dot{\hat{\rho}} | e \rangle$

$$= -i \langle e | \hat{H}\hat{\rho} - \hat{\rho}\hat{H} | e \rangle - \Gamma \rho_{ee}$$

$$= -i \frac{\Omega}{2} (\rho_{ge} - \rho_{eg}) - \Gamma \rho_{ee}$$

$$\dot{\rho}_{eg} = -i\langle e|\hat{H}\hat{\rho} - \hat{\rho}\hat{H}|g\rangle - \frac{\Gamma}{2}\rho_{eg}$$

$$= -i\left(\Delta \rho_{eg} + \frac{\Omega}{2}(\rho_{gg} - \rho_{ee})\right) - \frac{\Gamma}{2}\rho_{eg}$$

$$\rho_{ee} + \rho_{gg} = 1 \quad \Rightarrow \quad \dot{\rho}_{ee} = -\dot{\rho}_{gg}$$

$$\rho_{ge} = \rho_{eg}^*$$

? $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$

$|e\rangle\langle e|$ $|g\rangle\langle g|$ $\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

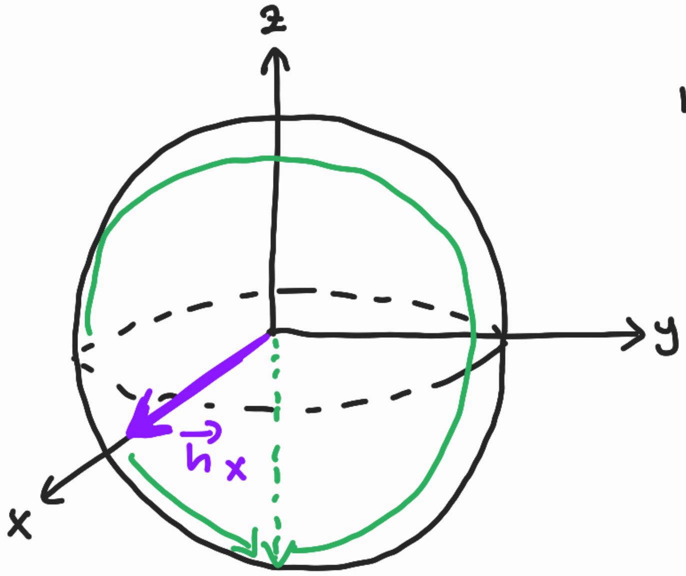
$$\boxed{\hat{\rho} = \frac{1}{2}(\mathbb{I} + \vec{a}\cdot\vec{\sigma}) \qquad \begin{aligned} a_z &= \rho_{ee} - \rho_{gg} = 2\rho_{ee} - 1 \\ a_x &= \rho_{eg} - \rho_{ge} \\ a_y &= i(\rho_{eg} - \rho_{ge}) \end{aligned}}$$

$$\boxed{\begin{aligned} \dot{a}_z &= \Omega a_y - \Gamma(a_z + 1) \\ \dot{a}_x &= -\frac{\Gamma}{2}a_x - \Delta a_y \\ \dot{a}_y &= -\frac{\Gamma}{2}a_y + \Delta a_x - \underbrace{\Omega a_z}_{\text{external drive}} \end{aligned}}$$

detuning

"Optical Bloch" Equation

$\star$ choice of phase appears in the a_y equation



$\vec{h}_x$ causes rotation in yz plane

II. Solutions

Stationary solutions $(\dot{\vec{a}} = 0)$

Let $s = \dfrac{2\Omega^2}{\Gamma^2 + 4\Delta^2}$ = saturation parameter (dimensionless)

$$a_x = -\frac{2\Delta}{\Omega}\frac{s}{1+s}$$

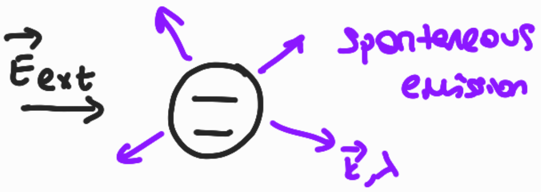
$$a_y = \frac{\Gamma}{\Omega}\frac{s}{(1+s)}$$

$$a_z = \frac{-1}{(1+s)}$$

Interpretation: $p_e = \langle e|\hat{\rho}|e\rangle = \rho_{ee} = \dfrac{1+a_z}{2} = \dfrac{1}{2}\dfrac{s}{(1+s)}$

prob. being in $|e\rangle$

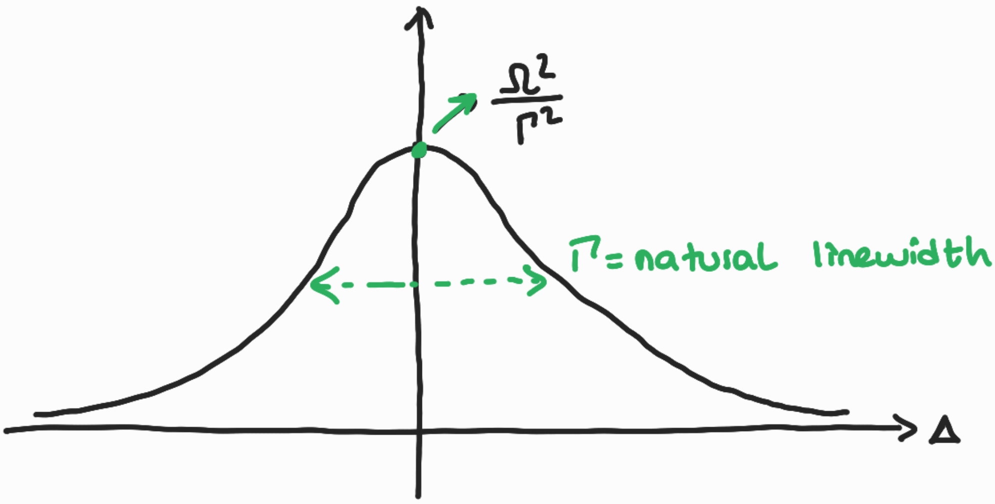
$\Leftrightarrow$ Rate of photon scattering



$\Gamma \rho_{ee}$ = rate of spontaneous emission = photon scattering

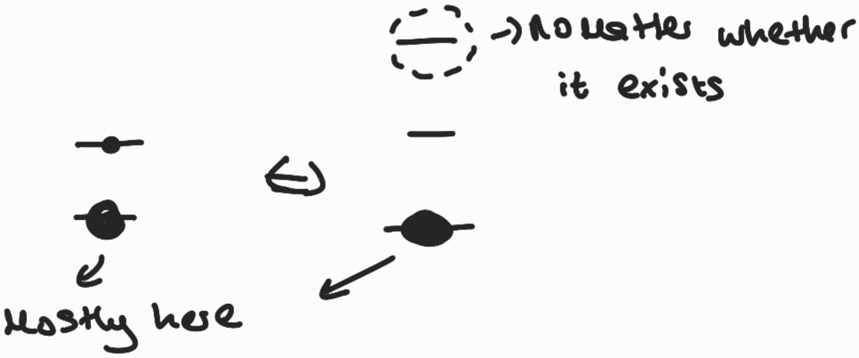
. $s \ll 1$: low saturation $\nearrow$ either $\Omega \ll \Gamma, \Delta$ ...
$\searrow$ large Δ

Then $\rho_{ee} = \frac{s}{2} = \frac{\Omega^2}{\Gamma^2 + 4\Delta^2} \ll 1$ "Lorentzian Profile"



Remark: Classical harmonic oscillator with damping rate Γ

Low s $\Rightarrow$ low ρ_{ee}



• $s \gg 1$ Strong saturation $\rho_{ee} = \frac{1}{2}$, $\Omega \gg \Gamma, \Delta$

forget Γ : at the scale of $T_{rabi} = \frac{2\pi}{\Omega}$,

there is no spontaneous emission.

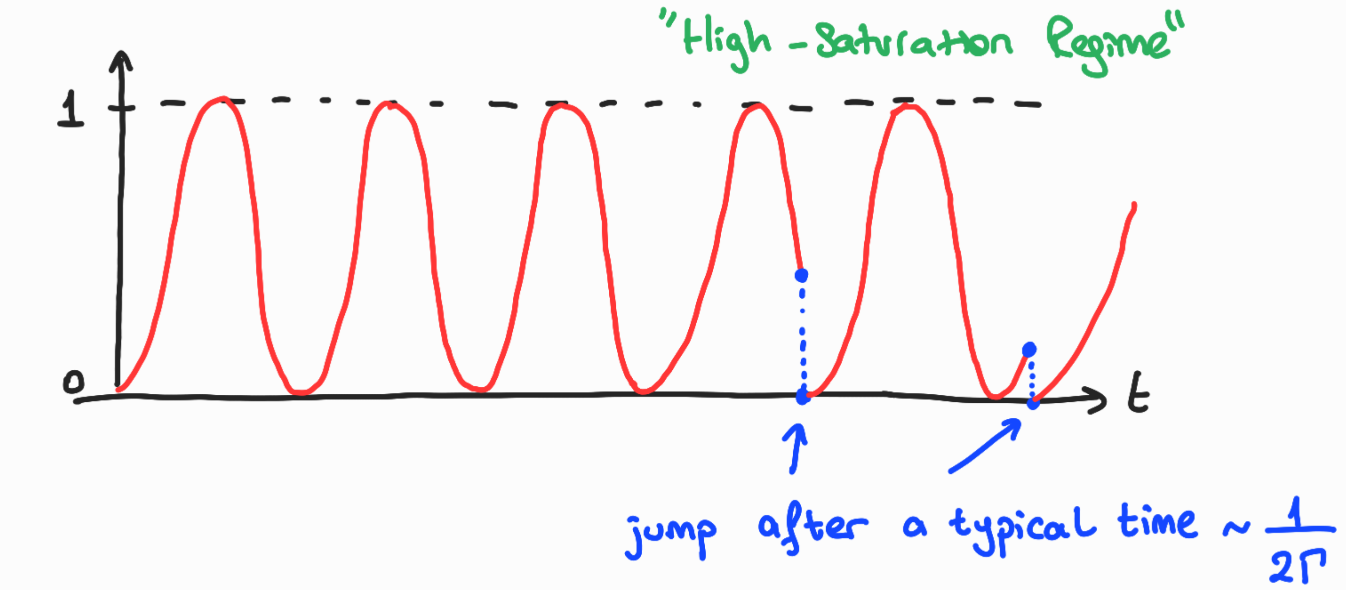
$\frac{1}{2}$ = time average of the Rabi oscillation

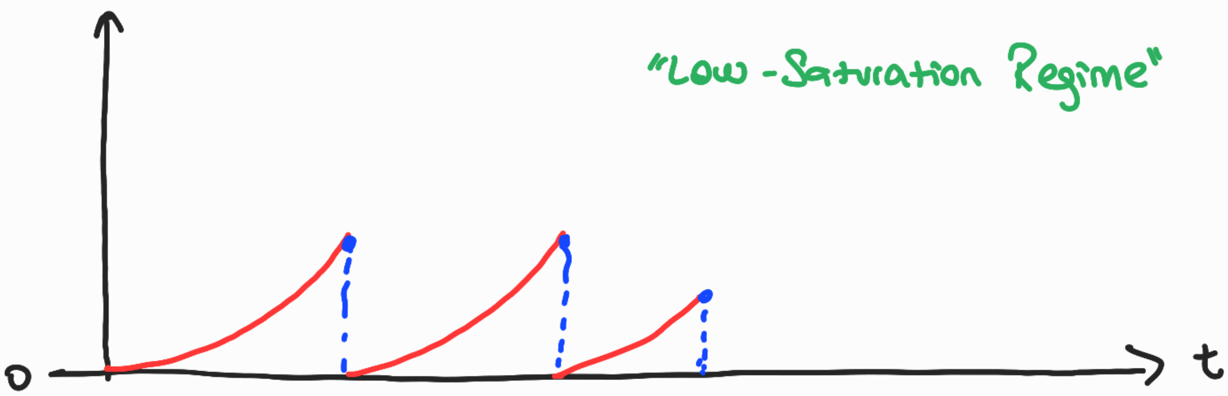
• Intermediate region: $\rho_{ee} = \frac{\overset{?}{4}\Omega^2}{4\Delta^2 + \Gamma'^2}$, $\Gamma' = \sqrt{\Gamma^2 + 2\Omega^2}$

power brodening

Quantum Trajectory Interpretation

$|\langle e|\psi(t)\rangle|^2$





Remarks: We can include other processes than sp. emission

$\hat{L}_1 = \sqrt{\Gamma_1}\, |g\rangle\langle e|$

$\hat{L}_2 = \sqrt{\Gamma_2}\, |e\rangle\langle e|$ Dephasing

then: $\dot{\rho}_{ee} = -i\langle e|\hat{H}\hat{\rho} - \hat{\rho}\hat{H}|e\rangle - \Gamma_1 \rho_{ee}$

$\dot{\rho}_{eg} = -i\langle e|\hat{H}\hat{\rho} - \hat{\rho}\hat{H}|g\rangle - \frac{\Gamma_1}{2}\rho_{eg} - \frac{\Gamma_2}{2}\rho_{eg}$

In general, T_1 time $\longrightarrow$ decay time of ρ_{ee}

T_2 time $\longrightarrow$ " " " ρ_{eg}

Without dephasing : $T_2 = 2T_1$ $\rightarrow$ performance criteria

DAMPED HARMONIC OSCILLATOR

I. Introduction

Single mode of the em field 

★ Hamiltonian $\hat{H} = \omega_c \hat{a}^\dagger \hat{a} + \sqrt{\kappa}\,\mathcal{E}\left(\hat{a}\, e^{i\omega_e t} + \text{hermit conj.}\right)$

in a rotating frame at ω_e $\quad U(t) = e^{i\omega_e t (\hat{a}^\dagger \hat{a})}$

$$\boxed{\hat{H} = \Delta \hat{a}^\dagger \hat{a} + \sqrt{\kappa}\,\mathcal{E}(\hat{a} + \hat{a}^\dagger)} \qquad \Delta = \omega_c - \omega_e$$

★ jump operator $\hat{L} = \sqrt{\kappa}\,\hat{a}$

Lindblad Equation $\dot{\hat{\rho}} = -i[\hat{H}, \hat{\rho}] + \kappa\left(\hat{a}\hat{\rho}\hat{a}^\dagger - \frac{1}{2}\hat{\rho}\hat{a}\hat{a}^\dagger - \frac{1}{2}\hat{a}^\dagger\hat{a}\hat{\rho}\right)$

II. Field Amplitude

$$\langle \hat{a} \rangle = \mathrm{Tr}(\hat{\rho}\hat{a})$$

Cyclicity

$\mathrm{Tr}(ABC) = \mathrm{Tr}(CAB)$

$$\frac{d}{dt}\langle \hat{a} \rangle = \frac{d}{dt}\mathrm{Tr}[\hat{\rho}\hat{a}] = \mathrm{Tr}(\dot{\hat{\rho}}\hat{a})$$

$$\mathrm{Tr}\left[\hat{a}\left(-i[\hat{H},\hat{\rho}] + \kappa\left(\hat{a}\hat{\rho}\hat{a}^\dagger - \frac{1}{2}\hat{\rho}\hat{a}^\dagger\hat{a} - \frac{1}{2}\hat{a}^\dagger\hat{a}\hat{\rho}\right)\right)\right]$$

$$-i\,\mathrm{Tr}(\hat{a}[\hat{H},\hat{\rho}]) = -i\,\mathrm{Tr}(\hat{a}\hat{H}\hat{\rho} - \hat{a}\hat{\rho}\hat{H}) = -i\,\mathrm{Tr}((\hat{a}\hat{H} - \hat{H}\hat{a})\hat{\rho}) = -i\langle[\hat{a},\hat{H}]\rangle$$

$$= -i\langle[\hat{a}, \Delta\hat{a}^\dagger\hat{a} + \sqrt{\kappa}\,\mathcal{E}(\hat{a} + \hat{a}^\dagger)]\rangle$$

$$= -i\Delta\langle\hat{a}\rangle - i\sqrt{\kappa}\,\mathcal{E}$$

$$\boxed{\kappa \, \mathrm{Tr}\left[\hat{a}\, \hat{a} \hat{\rho} \hat{a}^{+} - \frac{1}{2} \hat{a} \hat{\rho} \hat{a}^{+} \hat{a} - \frac{1}{2} \hat{a} \hat{a}^{+} \hat{a} \hat{\rho} \right]}$$

$$= \kappa \, \mathrm{Tr}\left(\cancel{\hat{a}^{+}\hat{a}}\,\hat{a}\hat{\rho}\right) - \frac{\kappa}{2} \mathrm{Tr}\left(\cancel{\hat{a}^{+}\hat{a}}\,\hat{a}\hat{\rho}\right) - \frac{\kappa}{2} \mathrm{Tr}\left(\overbrace{\hat{a}\hat{a}^{+}}^{\cancel{\hat{a}^{+}\hat{a}}+1}\hat{a}\hat{\rho}\right)$$

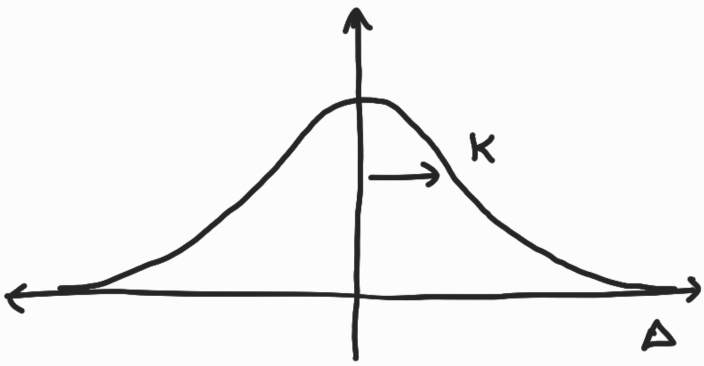
$$= \frac{\kappa}{2} \langle \hat{a} \rangle$$

Thus $\boxed{\langle \dot{\hat{a}} \rangle = -i \Delta \langle \hat{a} \rangle - \frac{\kappa}{2} \langle \hat{a} \rangle - i \sqrt{\kappa}\, \mathcal{E}}$

Steady state $\langle \dot{a} \rangle = 0 \quad \Rightarrow \quad \langle a \rangle = \dfrac{\sqrt{\kappa}\, \mathcal{E}}{-\Delta + i\kappa/2}$

Intensity

$$|\langle a \rangle|^2 = \frac{\kappa\, |\mathcal{E}|^2}{\Delta^2 + \frac{\kappa^2}{4}}$$



$$|\langle \hat{a} \rangle|^2 \sim \# \text{ of photons} = \frac{2|\mathcal{E}|^2}{\kappa}$$

$|\mathcal{E}|^2$: photon flux $\Leftrightarrow$ laser intensity

$\mathcal{E}$ is the laser amplitude

Remark $\quad \hat{H} \longrightarrow \hat{H}_{eff} = \left(\Delta - \frac{i\kappa}{2}\right) \hat{a}^{+} \hat{a}$ not Hermitian

III. Coherent State Decay

Consider $|\alpha\rangle$: $\hat{L}|\alpha\rangle = \sqrt{\kappa}\,\alpha|\alpha\rangle$

Extended Hilbert Space: $\mathcal{H} = \mathcal{H}_s \otimes \mathcal{H}_{env}$

$$\hat{U}(|\alpha\rangle \otimes |0\rangle) = \Big(\underbrace{\hat{M}_0|\alpha\rangle}_{\text{no jump}}\Big)\otimes|0\rangle + \Big(\underbrace{\hat{M}_1|\alpha\rangle}_{\text{jump}}\Big)\otimes|1\rangle$$

$$= \left[\left(\mathbb{I} - i\Delta t\hat{H} - i\Delta t\hat{J}\right)|\alpha\rangle\right]\otimes|0\rangle + \left(\hat{M}_1|\alpha\rangle\right)\otimes|1\rangle$$

$$= \left[\left(\mathbb{I} - i\Delta t\left(\omega_c - \frac{i\kappa}{2}\right)\hat{a}^{\dagger}\hat{a}\right)|\alpha\rangle\right]\otimes|0\rangle + \left(\sqrt{\Delta t\kappa}\,\alpha|\alpha\rangle\right)\otimes|1\rangle$$

Measure the environment

→ 1 is obtained $|\alpha\rangle \longrightarrow |\alpha\rangle$

→ 0 is obtained $\underbrace{\left(\hat{\mathbb{I}} - i\Delta t\left(\omega_c - \frac{i\kappa}{2}\right)\hat{a}^{\dagger}\hat{a}\right)}|\alpha\rangle$

(Δt is sufficiently small) $e^{-i\Delta t\left(\omega_c - \frac{i\kappa}{2}\right)\hat{a}^{\dagger}\hat{a}}|\alpha\rangle$

if $k=0$: $e^{-i\Delta t \omega_c \hat{a}^+\hat{a}} |\alpha\rangle \underset{\downarrow}{=} |\alpha e^{-i\Delta t\omega_c}\rangle$

free evolution

with K finite: $e^{-\Delta t \frac{k}{2} \hat{a}^+\hat{a}} |\alpha(t)\rangle$

IV Alternative descriptions

1) Photon Number

$|n\rangle$ fock state $\mathcal{H}_S = Sp\{|n\rangle, n \in \mathbb{N}\}$

$$\langle n|\hat{\rho}|m\rangle = \rho_{nm}$$

No external drive: $\hat{H} = \omega_c \hat{a}^+\hat{a}$

$$\dot{\rho}_{nm} = \frac{d}{dt}\langle n|\hat{\rho}|m\rangle = \langle n|\dot{\hat{\rho}}|m\rangle$$

$$\dot{\rho}_{nm} = i\omega_c(n-m)\rho_{nm} - \frac{k}{2}(n+m)\rho_{nm} + k\sqrt{n+1}\sqrt{m+1}\,\rho_{n+1\,m+1}$$

Population $p(n) = \rho_{nn}$

$$\Rightarrow \dot{p}(n) = -\underline{kn\,p(n)} + \underline{k(n+1)\,p(n+1)}$$

Remark: Rate of decay of $p(n)$ is nk

Fock states with large n are highly unstable

$\#$ of photons $\langle \dot{\hat{n}} \rangle = \frac{d}{dt}\langle n \rangle = \mathrm{Tr}(\dot{\hat{\rho}} n) = \sum_n \dot{\rho}(n) n = -k\langle n \rangle$

Notice $\left(\langle \dot{\hat{a}} \rangle = -i\left(\omega_c - \frac{k}{2}\right)\langle a \rangle \right)$?

2- Phase space description

Husimi Q function: $Q(\alpha) = \frac{1}{\pi}\langle \alpha | \hat{\rho} | \alpha \rangle$

Lindblad Equation $\longrightarrow$ $\frac{\partial Q}{\partial t} = -i\omega_c\left(\alpha^* \frac{\partial Q}{\partial \alpha^*} \cdot \alpha \frac{\partial Q}{\partial \alpha}\right) + \frac{k}{2}\left(\frac{\partial(\alpha Q)}{\partial \alpha} + \frac{\partial(\alpha^* Q)}{\partial \alpha^*}\right.$

$+ k \frac{\partial^2 Q}{\partial \alpha \partial \alpha^*}$

(Statistical Physics IV) Fokker-Planck equation

V. Extensions

1. Finite temperature

unitary evolution on $\mathcal{H}_S \otimes \mathcal{H}_E$

$\hat{U}(|\psi\rangle \otimes |N\rangle) = (\hat{M}_0|\psi\rangle) \otimes |N\rangle +$

$(\hat{M}_-|\psi\rangle) \otimes |N+1\rangle + (\hat{M}_+|\psi\rangle) \otimes |N+1\rangle$

$$\lambda(\hat{a}\hat{b}^{+} + \underset{\substack{\uparrow \\ \hat{a}^{+}\hat{b}}}{hc}) = \hat{H}_{SE}$$

(sys-env coupling)

$$\hat{M}_{-} \sim \sqrt{\kappa\Delta t}\,\sqrt{N+1}\,\hat{a} \quad \to \quad \hat{L}_{-} = \sqrt{\kappa}\,\sqrt{N+1}\,\hat{a}$$

$$\hat{M}_{+} \sim \sqrt{\kappa\Delta t}\,\sqrt{N}\,\hat{a}^{+} \quad \to \quad \hat{L}_{+} = \sqrt{\kappa}\,\sqrt{N}\,\hat{a}^{+}$$

At temperature $T = \frac{1}{\beta}$ $\quad N = \frac{1}{e^{\beta\omega_c} - 1}$ so that @ $T=0$: $\hat{L}_{+} = 0$

2) Phase Damping

$\hat{L} = \gamma\,\hat{a}^{+}\hat{a}$ (preserves photon number, get rid of coherences)

Environment "measuring the # of photons"

Fock basis: $\quad \dot{\rho}_{nm}(t) = \left[-i\,\omega_c\,(n-m) - \frac{\gamma}{2}(n+m)^2\right]\rho_{nm}$

ρ_{nm} for $n \neq m \longrightarrow 0$ exponentially fast

QUANTUM TRAJECTORIES

I. Stochastic Schrödinger Equation

1) Introduction

Duration Δt $\quad \{\hat{M}_\mu\} \rightarrow$ Measurement operators

Update rule:

$$\boxed{|\psi_c(t+\Delta t)\rangle = \frac{\hat{M}_\mu |\psi_c(t)\rangle}{\sqrt{p_\mu}}}$$

, Conditioned on outcome μ

- $\hat{M}_\mu = \sqrt{\Delta t}\, \hat{L}_\mu \qquad \mu \geq 1$
- $\hat{M}_0 = \hat{\mathbb{I}} - i\hat{H}\Delta t - \sum_{\mu\geq 1} \frac{\hat{L}_\mu^\dagger \hat{L}_\mu}{2}\Delta t = \mathbb{I} - i\hat{H}_{eff}\Delta t$

$$p_\mu = \langle \psi_{c(t)} | \hat{M}_\mu^\dagger \hat{M}_\mu | \psi_{c(t)} \rangle = \begin{cases} \langle \psi_{c(t)} | \hat{L}_\mu^\dagger \hat{L}_\mu | \psi_{c(t)} \rangle . \Delta t & \text{for } \mu \geq 1 \\ 1 - \sum_{\mu\geq 1} \langle \psi_{c(t)} | \hat{L}_\mu^\dagger \hat{L}_\mu | \psi_{c(t)} \rangle & \text{for } \mu = 0 \end{cases}$$

↓ probability

2) Unraveling

From the point of view of the "environment": $N_\mu(t)$

# of clicks of detector μ.
in the interval $[0,t]$

$N_\mu(t)$ is a classical random variable

Let $\Delta t \to dt$:

$$dN_\mu(t) = \begin{cases} 1 & \text{if the detector } \mu \text{ clicked btw } t \text{ and } t+\Delta t \\ 0 & \text{otherwise} \end{cases}$$

$$dN_0(t) = 1 - \sum_{\mu \geq 1} dN_\mu(t)$$

dN_μ is a classical random process $\Rightarrow$ $\begin{cases} dN_\mu = 1 & \text{w prob. } p_\mu = \langle L_\mu^\dagger L_\mu \rangle dt \\ dN_0 = 1 & \text{w prob. } 1 - p_\mu \end{cases}$

ensemble average

$$\overline{dN_\mu} = p_\mu$$

"point process" : $dN_\mu dN_\nu = \delta_{\mu\nu} dN_\mu$, $dN_\mu dt \ll dt$

3) Derivation

$$|\psi_c(t+dt)\rangle = \sum_\mu dN_\mu(t) \frac{\hat{M}_\mu|\psi_c(t)\rangle}{\sqrt{p_\mu}}$$

Separate $\mu=0$ and $\mu\geq 1$

$$|\psi_c(t+dt)\rangle = \left(1-\sum_{\mu\geq1} dN_\mu(t)\right)\frac{\hat{M}_0|\psi_c\rangle}{\sqrt{1-\underbrace{\sum_{\mu\geq1} p_\mu}_{O(dt)}}} + \underbrace{\sum_{\mu\geq1} dN_\mu(t)\frac{\hat{M}_\mu|\psi_c(t)\rangle}{\sqrt{p_\mu}}}_{\sum_{\mu\geq1} dN_\mu(t)\frac{\hat{L}_\mu|\psi_c(t)\rangle}{\sqrt{\langle\hat{L}_\mu^\dagger\hat{L}_\mu\rangle}}}$$

$$|\psi_c(t+dt)\rangle = \left(1-\sum_{\mu\geq1} dN_\mu(t)\right)\left(1+\frac{dt}{2}\sum_\mu\langle\hat{L}_\mu^\dagger\hat{L}_\mu\rangle\right)\left(\hat{\mathbb{I}}-i\hat{H}dt-\frac{dt}{2}\sum_\mu\hat{L}_\mu^\dagger\hat{L}_\mu\right)|\psi_c(t)\rangle + \sum_{\mu\geq1} dN_\mu(t)\frac{\hat{L}_\mu|\psi_c(t)\rangle}{\sqrt{\langle\hat{L}_\mu^\dagger\hat{L}_\mu\rangle}}$$

$$|\psi_c(t+dt)\rangle - |\psi_c(t)\rangle = \left[-i\hat{H}dt|\psi_c\rangle - \frac{dt}{2}\sum_\mu\left(\hat{L}_\mu^\dagger\hat{L}_\mu - \langle\hat{L}_\mu^\dagger\hat{L}_\mu\rangle\right)|\psi_c\rangle + \sum_\mu dN_\mu(t)\left(\frac{\hat{L}_\mu|\psi_c(t)\rangle}{\sqrt{\langle\hat{L}_\mu^\dagger\hat{L}_\mu\rangle}}-\mathbb{I}\right)|\psi_c\rangle\right]$$

$\{$

$$\Downarrow$$

$$d|\psi_c\rangle = \Big(\underbrace{-i\hat{H} - \frac{1}{2}\sum_\mu \hat{L}_\mu^\dagger \hat{L}_\mu}_{H_{eff}} - \langle\psi_c|\hat{L}_\mu^\dagger \hat{L}_\mu|\psi_c\rangle\Big)dt\,|\psi_c\rangle + \sum_\mu dN_\mu \left(\frac{\hat{L}_\mu}{\sqrt{\langle \hat{L}_\mu^\dagger \hat{L}_\mu\rangle}} - 1\right)|\psi_c\rangle$$

"Stochastic Schrödinger Equation"

Remarks: * Solution to SSE $\{N_\mu(t), |\psi_c(t)\rangle\}$ is called quantum trajectory.

* Nonlinear

II. Interpretation

1) Lindblad Equation

Consider $\overline{|\psi_c(t)\rangle\langle\psi_c(t)|}$ average over the random process dN_μ

$$\overline{|\underbrace{\psi_c(t+dt)}_{|\psi_c(t)\rangle + d|\psi_c(t)\rangle}\rangle\langle\psi_c(t+dt)|} = \overline{|\psi_c(t)\rangle\langle\psi_c(t)|} + \overset{①}{\overline{\underbrace{d|\psi_c(t)\rangle}_{SSE}\langle\psi_c(t)|}} + \underbrace{\overline{|\psi_c(t)\rangle d\langle\psi_c(t)|}}_{②} + \underbrace{\overline{d|\psi_c\rangle d\langle\psi_c|}}_{③}$$

it's not 2^{nd} order

(due to point process $dN_\mu dN_\nu = \delta_{\mu\nu} dN_\mu$)

$$|\psi_c(t+dt)\rangle\langle\psi_c(t+dt)| - |\psi_c(t)\rangle\langle\psi_c(t)| = d\hat{\rho}$$

$$d\hat{\rho} = \Big[\Big(-i\hat{H}_{eff} + \frac{1}{2}\sum_\mu \langle \hat{L}_\mu^+ \hat{L}_\mu\rangle\Big)dt\,|\psi_c\rangle\Big]\langle\psi_c| + \sum_\mu \overline{dN_\mu}\left(\frac{\hat{L}_\mu|\psi_c\rangle\langle\psi_c|}{\sqrt{\langle L_\mu^+ L_\mu\rangle}} - |\psi_c\rangle\langle\psi_c|\right) \quad \text{①}$$

$$+ |\psi_c(t)\rangle\langle\psi_c(t)|\Big(+i\hat{H}_{eff}^+ + \frac{1}{2}\sum_\mu \langle \hat{L}_\mu^+ \hat{L}_\mu\rangle\Big)dt + \sum_\mu \overline{dN_\mu}\left(\frac{|\psi_c\rangle\langle\psi_c|\hat{L}_\mu^+}{\sqrt{\langle \hat{L}_\mu^+ \hat{L}_\mu\rangle}} - |\psi_c\rangle\langle\psi_c|\right) \quad \text{②}$$

$dt\,dt \to 0$
$dt\,dN_\mu \to 0$

$$+\sum_{\mu\mu'} \overline{dN_\mu dN_{\mu'}}\left(\frac{\hat{L}_\mu}{\sqrt{\langle \hat{L}_\mu^+ \hat{L}_\mu\rangle}} - \hat{\mathbb{I}}\right)|\psi_c\rangle\langle\psi_c|\left(\frac{\hat{L}_\mu^+}{\sqrt{\langle \hat{L}_\mu^+ \hat{L}_\mu\rangle}} - \hat{\mathbb{I}}\right) \quad \text{③}$$

↳ last term can be written

$\rightarrow \overline{dN_\mu dN_{\mu'}} = \delta_{\mu\mu'}\overline{dN_\mu}$

$\rightarrow \overline{dN_\mu} = \langle \hat{L}_\mu^+ \hat{L}_\mu\rangle\, dt$

$$\sum_\mu \langle \hat{L}_\mu^+ L_\mu\rangle dt\left(\frac{-\hat{L}_\mu|\psi\rangle\langle\psi| - |\psi\rangle\langle\psi|\hat{L}_\mu^+}{\sqrt{\langle L_\mu^+ L_\mu\rangle}} + \frac{\hat{L}_\mu|\psi\rangle\langle\psi|\hat{L}_\mu^+}{\sqrt{\langle L_\mu^+ L_\mu\rangle}} + |\psi\rangle\langle\psi|\right)$$

(*) (**) □

$$\boxed{d\hat{\rho} = -i\left[\hat{H}_{eff}\hat{\rho} - \hat{\rho}\hat{H}_{eff}^{+}\right] + \sum_{\mu}\hat{L}_{\mu}\hat{\rho}\hat{L}_{\mu}^{+}}$$

"Lindblad Equation"

2) Monte-Carlo wave function Algorithm

$\dim \mathcal{H} = N$

$\hat{\rho} \sim O(N^2)$ entries

propagating $\hat{\rho} \sim O(N^4)$

* Pure state $|\psi\rangle \sim O(N)$ entries

Propagation of $|\psi\rangle \sim O(N^2)$

M different trajectories $\longrightarrow$ recovering $\hat{\rho} = \overline{|\psi_c\rangle\langle\psi_c|}$

$(M \ll N^2)$ $\sim O(MN^2)$

$\oplus$ parallel task

* Naive algorithm: . Define δt time step.

. Each δt: draw random number $R \in [0,1]$

. Jump prob.: $p_\mu = \langle\psi_c|\hat{L}_\mu^{+}\hat{L}_\mu|\psi_c\rangle\delta t$

. Total jump probability: $p_N = \sum_\mu p_\mu$

If $R < p$: jump happens, determine randomly which one $\{\mu\}_{\mu \geq 1}$

Apply corresponding $\hat{L}_\mu |\psi_c\rangle$ + normalize to obtain $|\psi_c(t+\delta t)\rangle$

Record the jump event on "channel" μ.

If $R > p$:

related to normalization

$$\text{Determine } \delta|\psi_c\rangle = \left[-i\hat{H} - \frac{1}{2}\sum_\mu \left(\hat{L}_\mu^+ \hat{L}_\mu - \langle \hat{L}_\mu^+ \hat{L}_\mu \rangle\right)\right] \delta t |\psi_c\rangle$$

$$|\psi_c(t+\delta t)\rangle = |\psi_c(t)\rangle + \delta|\psi_c\rangle$$

★ Clever Algorithm

Use non-normalized wavefunction for the no-jump evolution:

$$\frac{d|\tilde{\psi}\rangle}{dt} = -\left(i\hat{H} + \frac{1}{2}\underbrace{\sum_\mu \hat{L}_\mu^+ \hat{L}_\mu}_{\text{decrease of } \langle\tilde{\psi}|\psi\rangle}\right)|\tilde{\psi}\rangle$$

Draw random R, evolve until $\langle\tilde{\psi}|\psi\rangle = R$

↓

then jump (like previously)

3) Different types of unravelings

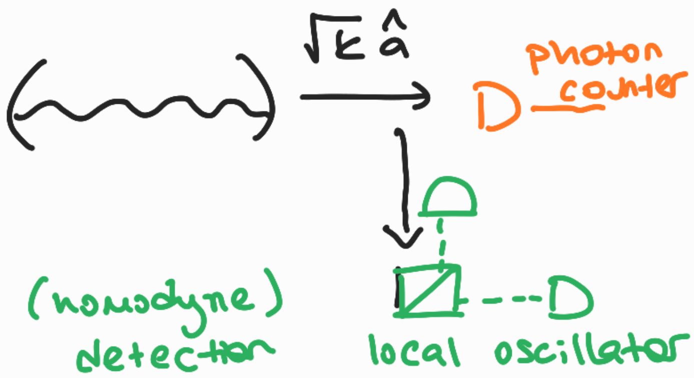
Remark: Unitary transformations on krauss operators:

$$\hat{K}_\nu = \sum_{\mu,\nu} U_{\mu\nu} \hat{M}_\mu$$

$$\sum_\nu \hat{K}_\nu \hat{\rho} \hat{K}_\nu^\dagger = \sum_\mu \hat{M}_\mu \hat{\rho} \hat{M}_\mu^\dagger \; ; \quad \sum_\nu \hat{K}_\nu^\dagger \hat{K}_\nu = \mathbb{I} = \sum_\mu \hat{M}_\mu^\dagger \hat{M}_\mu$$

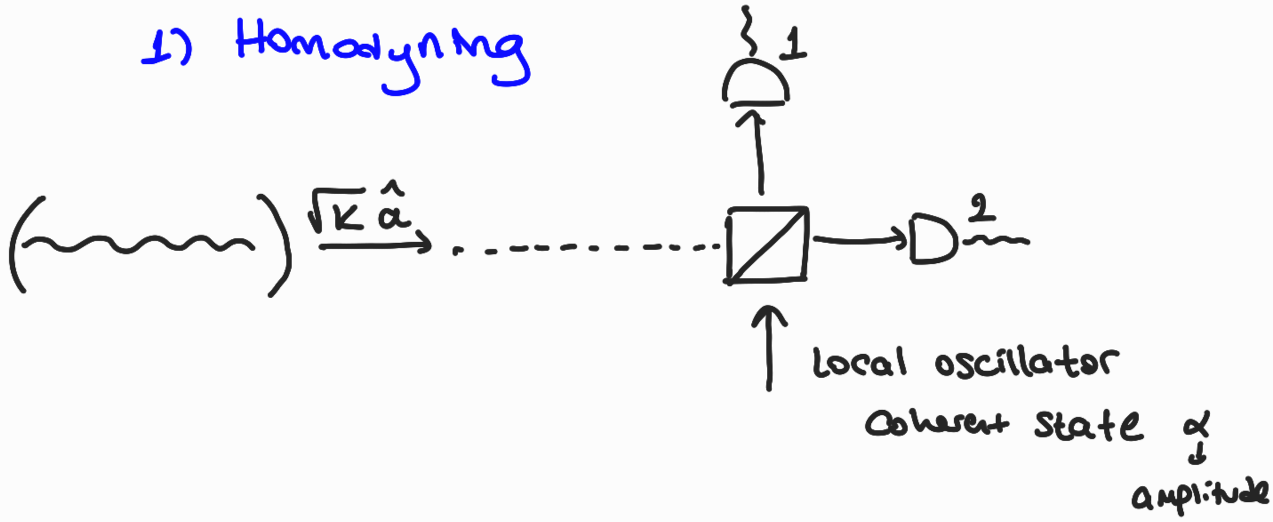
Lindblad Equation: Similar

Example: Photon detection



III. Weak Continous Measurement

1) Homodyning



$$\hat{L} = \sqrt{\kappa}\,\hat{a} \xrightarrow{\text{(Mixing with) coherent state}} \begin{array}{l} \sqrt{\kappa}\,\dfrac{\hat{a} - \alpha\hat{\mathbb{I}}}{\sqrt{2}} = \hat{c}_1 \\[2ex] \sqrt{\kappa}\, i\,\dfrac{\hat{a} + \alpha\hat{\mathbb{I}}}{\sqrt{2}} = \hat{c}_2 \end{array}$$

Transformations $\quad \hat{M}_0^\dagger \hat{M}_0 + \hat{M}_1^\dagger \hat{M}_1 = \hat{\mathbb{I}}$

$$(\hat{\mathbb{I}} - i\hat{H}_{eff}\delta t)^\dagger (\hat{\mathbb{I}} - i H_{eff}\,\delta t) + \kappa\,\hat{a}^\dagger\hat{a}\,\delta t = \hat{\mathbb{I}}$$

δt terms $\rightarrow \quad i\left(\hat{H}_{eff}^\dagger - \hat{H}_{eff}\right) + \kappa\,\hat{a}^\dagger\hat{a} = 0$

$$\hat{H}_{eff} = \hat{H} - i\,\frac{\kappa\hat{a}^\dagger\hat{a}}{2}$$

Now: $\hat{M}_0^\dagger \hat{M}_0 + \hat{M}_1^\dagger \hat{M}_1 + \hat{M}_2^\dagger \hat{M}_2 = \mathbb{I}$

$$(\hat{\mathbb{I}} - i\hat{H}_{eff}\delta t)^\dagger(\hat{\mathbb{I}} - i\hat{H}_{eff}\delta t) + \frac{k}{2}\delta t(\hat{a}^\dagger\hat{a} + |\alpha|^2 - \cancel{\hat{a}\alpha^*} - \cancel{\hat{a}^\dagger\alpha})$$

$$+ \frac{k}{2}\delta t(\hat{a}^\dagger\hat{a} + |\alpha|^2 + \cancel{\hat{a}\alpha^*} + \cancel{\hat{a}^\dagger\alpha})$$

$$H_{eff}^\dagger = \hat{H} - \frac{ik}{2}(\hat{a}^\dagger\hat{a} - |\alpha|^2)$$

↳ constant is irrelevant (normalized out)

For detector 1:

$$p_1 = \delta t\langle \hat{c}_1^\dagger \hat{c}_1\rangle = \frac{k\delta t}{2}\left(|\alpha|^2 + \langle\hat{a}^\dagger\hat{a}\rangle - \langle\alpha\hat{a}^\dagger + \alpha^*\hat{a}\rangle\right)$$

let $\alpha = |\alpha|e^{i\phi}$ and $\hat{x}_\phi = e^{i\phi}\hat{a}^\dagger + \hat{a}e^{-i\phi}$

$$p_1 = \frac{k\delta t}{2}\left(|\alpha|^2 + \langle\hat{a}^\dagger\hat{a}\rangle - |\alpha|\langle\hat{x}_\phi\rangle\right)$$

$$p_2 = \frac{k\delta t}{2}\left(|\alpha|^2 + \langle\hat{a}^\dagger\hat{a}\rangle + |\alpha|\langle\hat{x}_\phi\rangle\right)$$

☆ Let $|\alpha|$ is very large : $|\alpha|^2 \gg \langle\hat{a}^\dagger\hat{a}\rangle$

Integrate $\int_0^{\delta\tau} dN_1$: choose $\delta\tau$ such that

$\langle x_\phi\rangle(t+\varepsilon) \sim \langle x_\phi\rangle(t)$

for $\varepsilon \ll \delta\tau$

$|\psi_c(t+\varepsilon)\rangle \sim |\psi_c(t)\rangle$

Let $\overset{\text{current}}{\overset{\uparrow}{I_{1,2}}} = \dfrac{\delta N_1}{\delta \tau}$: photocurrent

Large # of counts, uncorrelated (Poisson) distributed

$\Rightarrow$ Gaussian

$$\delta N_1 = \overline{\delta N_1} + \delta W_1$$ → continuously fluctuating random variable. (noise)

$$\overline{\delta N_1} = \int_0^{\delta\tau} dN_1 = \frac{K}{2}\delta\tau\left(|\alpha|^2 + \langle \hat{x}_\phi \rangle(0) \cdot |\alpha|\right)$$ can be ignored

$$\overline{\delta W_1} = 0 \quad \text{and} \quad \overline{\delta W_1^2} = \overline{\delta N_1} \sim \frac{K}{2}|\alpha|^2 \delta\tau$$

So: $$\boxed{dN_1(t) = \frac{K|\alpha|^2}{2}\left(1 + \frac{\langle x_\phi \rangle(t)}{|\alpha|}\right)dt + |\alpha|\sqrt{\frac{K}{2}}\, dW}$$

dW: Wiener process

$$I_{1,2}(t) = \frac{K|\alpha|^2}{2}\left(1 \mp \frac{\langle x_\phi \rangle(t)}{|\alpha|}\right) + |\alpha|\sqrt{\frac{K}{2}}\, \overset{\text{Langevin}}{\overset{\uparrow}{\xi_{1,2}(t)}}$$

$$\xi(t)\,dt = dW$$

Balanced Homodyne Detection: Record $I_2 - I_1$

$$I_{bh}(t) = K|\alpha|\langle x_\phi\rangle(t) + |\alpha|\sqrt{\gamma}\,\xi(t)$$

$$\xi(t) = \frac{\xi_2 - \xi_1}{\sqrt{2}}$$

Remark: Unbalanced also works:

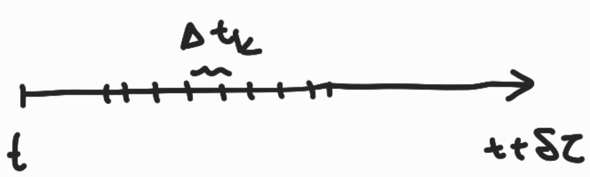
One can also use heterodyne detection:

$$LO \rightarrow \alpha e^{i\Omega t}$$

2) Quantum state diffusion

During $\delta\tau$

$$|\psi_c(t+\delta\tau)\rangle = \prod_i \hat{M}_{N_i}(\Delta t_i)|\psi_c(t)\rangle$$



$$|\tilde{\psi}_c(t+\delta\tau)\rangle \sim \hat{M}_0(\delta\tau)\,\hat{c}_1^{\,m_1}\,\hat{c}_2^{\,m_2}\,|\tilde{\psi}_c(t)\rangle$$

conditionized on m_1 clicks on 1

m_2 clicks on 2

$$\hat{c}_1^{m_1} = \left(\sqrt{\frac{\kappa}{2}}\right)^{m_1} (\hat{a}-\alpha)^{m_1} = \left(\sqrt{\frac{\kappa}{2}}\,-\alpha\right)^{m_1} \left(\mathbb{I} - \frac{\hat{a}}{\alpha}\right)^{m_1}$$

$$\left[\sqrt{\frac{\kappa}{2}}|\alpha|\right]^{m_1} \rightarrow \text{normalized out}$$

$$(-e^{i\phi})^{m_1} \rightarrow \text{the only object relevant } |\Psi_c\rangle\langle\Psi_c|$$

$$|\tilde{\Psi}_c(t+\delta\tau)\rangle = \left(\hat{\mathbb{I}} - i\hat{H}_{eff}\,\delta\tau\right)\left(\hat{\mathbb{I}} - \frac{m_1\hat{a}}{\alpha}\right)\left(\hat{\mathbb{I}} + \frac{m_2\hat{a}}{\alpha}\right)|\tilde{\Psi}_c(t)\rangle$$

$$= \left(\hat{\mathbb{I}} - i\hat{H}_{eff}\,\delta\tau + \frac{\hat{a}}{\alpha}(m_2-m_1)\right)|\tilde{\Psi}_c(t)\rangle$$

$$= \left(\hat{\mathbb{I}} - i\hat{H}_{eff}\,\delta\tau + \frac{\hat{a}}{\alpha}\left[\kappa\frac{|\alpha|^2}{2}\left(1+\frac{\langle x_\phi\rangle}{|\alpha|}\right)\delta\tau + |\alpha|\sqrt{\frac{\kappa}{2}}\,\delta w_2 \right.\right.$$

$$\left.\left. - \kappa\frac{|\alpha|^2}{2}\left(1-\frac{\langle x_\phi\rangle}{|\alpha|}\right)\delta\tau - |\alpha|\sqrt{\frac{\kappa}{2}}\,\delta w_1\right]\right)|\tilde{\Psi}_c\rangle$$

$$\boxed{d|\tilde{\Psi}_c\rangle = \left(-i\hat{H}_{eff}\,dt + \hat{a}e^{-i\phi}\left(\kappa\langle\hat{x}_\phi(t)\rangle dt + \sqrt{\kappa}\,dw\right)\right)|\tilde{\Psi}_c\rangle}$$

"Quantum state diffusion"

also: $$d|\tilde{\Psi}_c\rangle = \left(-i\hat{H}_{eff} + \frac{I_{hh}}{\alpha}\hat{a}e^{-i\phi}\right)dt\,|\tilde{\Psi}_c\rangle$$

Normalization:

$$\langle \tilde{\Psi}_c(t+dt) | \tilde{\Psi}_c(t+dt) \rangle = 1 + \langle d\Psi | \Psi \rangle + \langle \Psi | d\Psi \rangle + \langle d\Psi | d\Psi \rangle$$

$$\vdots$$

$$d|\Psi_c\rangle = \Big[-i \hat{H}_{eff} dt + \kappa \langle \hat{x}_\phi(t) \rangle dt \left(\hat{a} e^{-i\phi} - \frac{\langle \hat{x}_\phi \rangle(t)}{2} \right)$$

$$+ \kappa\, dW \left(a e^{-i\phi} - \frac{\langle x_\phi \rangle}{2} \right) \Big] |\Psi_c\rangle$$

Remarks

. Read as drift-diffusion equation in the Hilbert space.

. fully general (equivalent to click-based SSE)

MECHANICAL EFFECTS OF LIGHT

Idea: . Photon have momentum.

. "Effective" theory

$\longrightarrow E$

motion of atom — motion of e^-

↓ center-of-mass — ↓ internal degrees of freedom

"effective" action ← - - - addressed by laser

I. FORMULATION

1) Hilbert Space

$$\mathcal{H} = \mathcal{H}_{at} \otimes \mathcal{H}_{field} = \mathcal{H}_{c.o.m} \otimes \mathcal{H}_{internal} \otimes \mathcal{H}_{\substack{laser\\mode}} \otimes \mathcal{H}_{vac}$$

$\mathcal{H}_{c.o.m} = Sp\{ \underbrace{|\vec{p}\rangle}_{\text{momentum}} , \vec{p} \in \mathbb{R}^3 \}$ atom in free space

$\mathcal{H}_{internal} = Sp\{ |nLJm_J \dots\rangle \dots \}$ energy levels of the atom

$\mathcal{H}_{internal} = Sp\{ |n\rangle, n \in \mathbb{N} \}$ one particular mode addressed by the laser

$\mathcal{H}_{vac} = \mathcal{F}$ Fock space

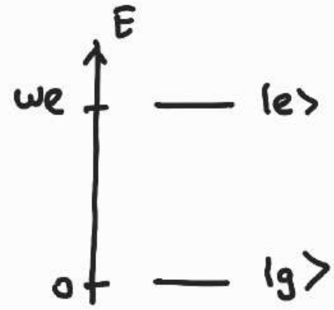
2) Simplification

. Reduce $\mathcal{H}_{int}$ to a 2 dim. Hilbert space $= Sp\{|g\rangle, |e\rangle\}$

↳ cycling transition

- Laser mode is in a coherent state $\to$ classical field
- Vacuum as a reservoir (Markov approximation) $\to$ Lindblad Equation
- C.o.m motion is slow compared with internal dynamics.

3) Hamiltonian



Acts on $\mathcal{H}_{c.o.m} \otimes \mathcal{H}_{int}$

$$\hat{H} = \frac{\vec{P}^2}{2m} \otimes \hat{\mathbb{I}} + \hat{\mathbb{1}} \otimes w_e |e\rangle\langle e| + \frac{\Omega(\hat{\vec{r}})}{2} \left[e^{i(w_0 t - \vec{k}_L \hat{\vec{r}})} \otimes |g\rangle\langle e| + hc \right]$$

key point: Ω and $\phi = w_0 t - \vec{k}_L \hat{\vec{r}}$ are operators on $\mathcal{H}_{com}$

within dipole approximation : "size" of the atom $\ll \lambda$

the atom "feels" Ω and ϕ at the position of its c.o.m.

$$\Omega(\vec{r}) = \langle \Psi_{com} | \Omega(\hat{\vec{r}}) | \Psi_{com} \rangle \text{ for c.o.m in state } |\Psi_{com}\rangle$$

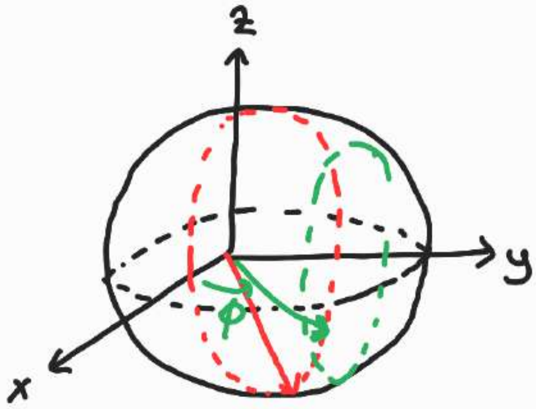
$$\Omega(\hat{\vec{r}}) = -\vec{E}(\hat{\vec{r}})\, d_0 \qquad \phi(\vec{r}) = w_0 t - \vec{k}_L \cdot \hat{\vec{r}}$$

$\to$ how e^- and atom bound together

II. Recoil and Doppler Effects

1) Phase Imprinting

Remainder: $$\hat{H} = \frac{\Omega}{2}\left(e^{-i\phi}|e\rangle\langle g| + hc\right) + \frac{|\Omega|}{2}\vec{n}\vec{\sigma}$$



Here $\phi(\vec{r})$

Consider $|\Psi_{at}\rangle = |\vec{p}\rangle \otimes |g\rangle$

$$\hat{H}_{int}|\Psi_{at}\rangle = \frac{\Omega}{2}\underbrace{\left(e^{-i\phi(\hat{\vec{r}})}|\vec{p}\rangle\right)}\otimes|e\rangle$$

$$\int d^3\vec{R}\, |\vec{R}\rangle\langle\vec{R}|\, e^{-i\phi(\hat{\vec{r}})}|\vec{p}\rangle$$

$$= \int d^3R\, e^{-i\phi(\vec{R})}|\vec{R}\rangle\underbrace{\langle\vec{R}|\vec{p}\rangle}_{e^{i\vec{p}.\vec{R}}}$$

$$= \int d^3R\, e^{-i\phi(\vec{R}) + i\vec{p}.\vec{R}}|\vec{R}\rangle$$

Plane wave: $\phi(\vec{R}) = \omega_0 t - \vec{k}_L \cdot \vec{R}$ then:

$$\hat{H}_{int} |\psi_{at}\rangle = \frac{\Omega}{2} e^{i\omega_0 t} \underbrace{\int d^3R \, e^{i(\vec{k}_L \cdot \vec{R} + \vec{p} \cdot \vec{R})} |\vec{R}\rangle}_{|\vec{p} + \vec{k}_L\rangle}$$

So $\hat{H}_{int} = \frac{\Omega}{2} \sum_{\vec{p}} e^{i\omega_0 t} |\vec{p} g\rangle\langle \vec{p} + \vec{k}_L, e| + hc$

2) Shifts

Move to the rest frame of the atom (from the laboratory frame where momentum $\vec{p}$)

Unitary transformation:

$$\hat{U}(t) = e^{\frac{i\hat{\vec{p}}^2 t}{2m}} \qquad \hat{H} \to (i\partial_t \hat{U})\hat{U}^\dagger + \hat{U}\hat{H}\hat{U}^\dagger$$

In this frame:

$$\hat{H} = \frac{\Omega}{2} \sum_{\vec{p}} e^{i\omega_0 t} \underbrace{\hat{U} |\vec{p} g\rangle\langle \vec{p} + \vec{k}_L, e| \hat{U}^\dagger}_{e^{\frac{i\hat{p}^2 t}{2m}} |\vec{p}, g\rangle\langle \vec{p} + \vec{k}_L, e| e^{-i\frac{(\vec{p} + \vec{k}_L)^2}{2m} t}} + h.c$$

$$= \frac{\Omega}{2} \sum_{\vec{p}} e^{i\left(\omega_0 - \frac{\vec{p}\vec{k_L}}{m} - \frac{\vec{k_L}^2}{2m}\right)t} |\vec{p}, g\rangle\langle \vec{p}+\vec{k_L}, e| + hc$$

$$\omega_0 \longrightarrow \omega_0 - \frac{\vec{p}\vec{k_L}}{m} - \frac{\vec{k_L}^2}{2m}$$

$^6Li \sim 0.1\ ms^{-1}$

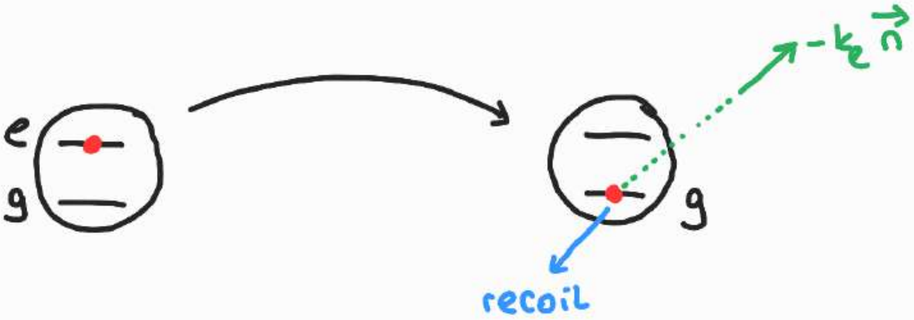
Doppler Shift: $\frac{\vec{p} \cdot \vec{k_L}}{m} = \vec{p}\,\vec{v}_{rec}$, $\vec{v}_{rec} = \frac{\vec{k_L}}{m}$

Ex: ^{6}Li at 300K : 1GHz

Recoil Shift: $E_R = \frac{\hbar^2 k_L^2}{2m} = h \cdot 77kHz$ for ^{6}Li

negligible cmp with doppler shift

3) Jump Operators



$k_e = \frac{\omega_e}{c}$

Family of jump operators: $\hat{L}(\vec{n}) = \sum_{\vec{p}} |\vec{p}+k_e\vec{n}, g\rangle\langle \vec{p}, e|$

$$= e^{ik_e\vec{n}\cdot\vec{r}} \otimes |g\rangle\langle e|$$

Rate $\Gamma_{\vec{n}}$ where $\underbrace{\hat{M}}_{\text{measurement operator}}(\vec{n}) = \sqrt{\Gamma_{\vec{n}}}\, \hat{L}(\vec{n})$

4) Lindblad Equation

$$\dot{\hat{\rho}} = -i[\hat{H}, \hat{\rho}] + \sum_{\vec{n}} \Gamma_{\vec{n}} \left(\hat{L}(n)\hat{\rho}\hat{L}(\vec{n})^{\dagger} - \frac{1}{2}\hat{L}^{\dagger}\hat{L}(\vec{n})\hat{\rho} - \frac{1}{2}\hat{\rho}\hat{L}^{\dagger}\hat{L}(\vec{n}) \right)$$

Note that: $\hat{L}^{\dagger}(\vec{n})\hat{L}(\vec{n}) = |e\rangle\langle e|$

$$\cancel{e^{-i}}\, |e\rangle\langle g|\, |g\rangle\langle e|\, \cancel{e^{+i}}$$

ρ_{ee} ρ_{ee}

Introduce $\Gamma = \sum_{\vec{n}} \Gamma_{\vec{n}}$

$$\dot{\hat{\rho}} = -i[\hat{H}, \hat{\rho}] \underbrace{- \frac{\Gamma}{2}\left(|e\rangle\langle e|\hat{\rho} + \hat{\rho}|e\rangle\langle e| \right)}_{\text{Non-Hermitian "decay" of } |e\rangle} + \underbrace{\sum_{\vec{n}} \hat{L}(\vec{n})\hat{\rho}\hat{L}^{\dagger}(\vec{n})}_{\text{kicks upon spontaneous emission}}$$

Last term $\hat{\rho} = |e, \vec{p}\rangle\langle \vec{p}, e|$

Then $\hat{L}(\vec{n})\hat{\rho}\hat{L}^{\dagger}(\vec{n}) = |g, \vec{p} + k\vec{e}_n\rangle\langle g, \vec{p} + k\vec{e}_n|$

III. Semi-classical Forces

1) Further simplification

. The momentum "spread" of the atom is large $\Delta p \geq k_e, k_L$ ...

$\Leftrightarrow$ position spread is small $\Delta R \ll \lambda$

. k_e is small (momentum exchanges with em field are continous)

. internal dynamics is much faster than motion of c.o.m

Steady state description of the internal state for each position.

$$\Gamma, \Omega \gg \underbrace{E_R}_{\to \text{recoil}}$$

-> Ansatz for the density matrix: $\hat{\rho} = \hat{\rho}_{com} \otimes \hat{\rho}_{int}^{\overset{\text{steady state}}{st}}$

$$\hat{\rho}_{int}^{st}(\vec{R}(t))$$

2) Sponteneous Emission

$$\sum_{\vec{n}} \Gamma_{\vec{n}}\, e^{ik_e \vec{n}.\hat{\vec{r}}} |g\rangle\langle e| \hat{\rho} |e\rangle\langle g| e^{-ik_e \vec{n}.\vec{r}}$$

since k_e is small: $e^{ik_e \vec{n}.\hat{\vec{r}}} \sim 1 + ik_e \vec{n}.\vec{r} - \frac{k_e^2}{2}(\vec{n}.\hat{\vec{r}})^2 + \dots$

So: $\underbrace{\sum_{n} \Gamma_{\vec{n}}}_{\Gamma} |g\rangle\langle e|\hat{\rho}|e\rangle\langle g| + \sum_{n} ik_e \vec{n}.\vec{r}\, |g\rangle\cancel{\langle e|}\hat{\rho}|e\rangle\langle g| + \sum_{n} |g\rangle\cancel{\langle e|}\hat{\rho}|e\rangle\langle g| (-ik_e \vec{n}.\vec{r}) + \cancel{O(k_e^2)}$

$0 \qquad 0$

Up to $O(k_e^2)$: no net effect of spont. emission of the center of motion!

Now, up to $O(k_e^2)$:

$$\dot{\hat{\rho}} = -i[\hat{H}, \hat{\rho}] + \Gamma |g\rangle\langle e|\hat{\rho}|e\rangle\langle g| - \frac{\Gamma}{2}\left(|e\rangle\langle e|\hat{\rho} + \hat{\rho}|e\rangle\langle e|\right)$$

the dissipative part of the Lindblad Eq. is only affecting the internal state!!

3) Forces

$$\vec{F} = \frac{d\langle\vec{p}\rangle}{dt} = \frac{d}{dt}\,\mathrm{Tr}(\hat{\rho}\,\vec{p}) = \mathrm{Tr}(\dot{\hat{\rho}}\,\vec{p})$$

using cyclicity

$$= -i\,\mathrm{Tr}([\hat{H},\hat{\rho}].\hat{\vec{p}})$$

$$= -i\,\mathrm{Tr}(\hat{\rho}.[\vec{p},\hat{H}])$$

$$\vec{F} = -i\,\langle[\hat{p},\hat{H}]\rangle$$

$\hat{H}$ depends on $\hat{\vec{r}}$: move to position representation (via $\Omega(\vec{r})$, $\phi(\vec{r})$ ...)

$$\vec{p} = -i\,\vec{\nabla}_{\vec{r}}$$

then $\langle[\vec{p},\hat{H}]\rangle = -i\,\langle\vec{\nabla}_r\,\hat{H}\rangle$

Proof: for any state $|\alpha\rangle$:

$$\langle\alpha|\,\vec{\nabla}_r\big(\hat{H}|\alpha\rangle\big) = \langle\alpha|\,[\vec{\nabla}_r\,\hat{H}]\,|\alpha\rangle + \langle\alpha|\,\hat{H}\,(\vec{\nabla}_r|\alpha\rangle$$

So, $\boxed{\vec{F} = -\langle\vec{\nabla}_r\,\hat{H}\rangle}$

for an atom at rest:

$$\hat{H} = \omega_e |e\rangle\langle e| + \frac{\Omega(\vec{r})}{2}\left(e^{i(\omega_0 t - \vec{k_L}\cdot\vec{r})} |g\rangle\langle e| + h.c \right)$$

In a frame rotating at ω_0 (for internal degrees of freedom)

$$\hat{H} = \Delta |e\rangle\langle e| + \frac{\Omega(\vec{r})}{2}\left(e^{-ik_L\cdot\vec{r}} |g\rangle\langle e| + h.c \right)$$

$$\langle -\vec{\nabla}_r \hat{H} \rangle = - \frac{\nabla\Omega(\vec{r})}{2}\left(\overbrace{\langle e^{-ik_L\cdot\vec{r}} |g\rangle\langle e| + h.c\rangle}^{\text{internal d.o.f}} \right)$$

internal d.o.f

↓

OBE

optical bloch equation

$$+ik_L \frac{\Omega}{2}\left(\langle e^{-ik_L\cdot\vec{r}} |g\rangle\langle e| + \dots \right)$$

$$\boxed{\vec{F} = \underbrace{\frac{\vec{\nabla}\Omega}{\Omega}\,\Delta.\frac{s(\vec{r})}{1+s(\vec{r})}}_{\text{dipole force}} + \underbrace{k_L \frac{\Gamma}{2}\frac{s(\vec{r})}{1+s(\vec{r})}}_{\text{radiation pressure}}}$$

dipole force

$\vec{F}_{dip}$

↓

$$\frac{\Delta}{2}\frac{\nabla\vec{s}}{1+s}$$

radiation pressure

$$s(\vec{r}) = \frac{\Omega^2(\vec{r})/2}{\Delta^2 + \Gamma^2/4}$$

$\vec{F}_{rad}$

↓

$$\vec{k_L}.\Gamma.\langle \rho_{ee}\rangle_{st}$$

$\rightarrow$ steady state

IV Interpretation

1) Dipole force

$$\vec{F}_{dip} = -\vec{\nabla}\left(-\frac{\Delta}{2}\ln\left(1+s(\vec{r})\right)\right)$$

dipole force is conservative and $\underset{\to \text{potential}}{U_{dip}} = -\frac{\Delta}{2}\ln(1+s)$

Limiting case: $\Delta \gg \Omega, \Gamma$

$$s \ll 1 \Rightarrow \langle \rho_{ee} \rangle \ll 1$$

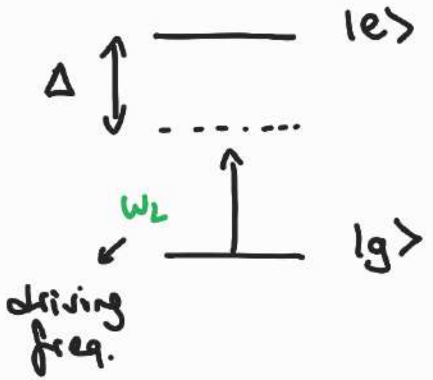
Then $U_{dip} = -\frac{s}{2}\Delta = -\frac{\Omega^2(\vec{r})}{4\Delta}$

- $\Omega \propto \mathcal{E}$ E-field amplitude

  $U_{dip} \propto I$ light intensity

- U_{dip} is also independent of Γ.

Recovering the dipole potential: 2-level atom in external field.

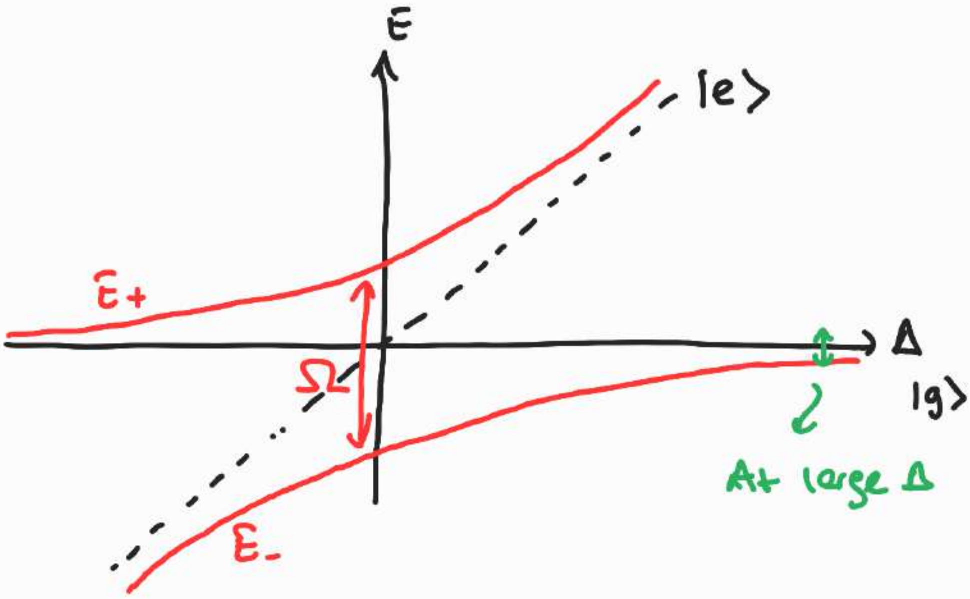


$$\hat{H} = \omega_e |e\rangle\langle e| + \frac{\Omega}{2}\left(e^{-i\omega_L t}|e\rangle\langle g| + hc\right)$$

In a rot. frame:

$$\hat{H} = \Delta |e\rangle\langle e| + \frac{\Omega}{2}\left(|e\rangle\langle g| + hc\right)$$

Energy spectrum:



Large Δ $\qquad E_\pm = \frac{1}{2}\left(\Delta \pm \sqrt{\Delta^2+\Omega^2}\right) \qquad E_- \to \frac{-\Omega^2}{4\Delta}$

and $\quad |-\rangle \to |g\rangle$

Now if $\Omega(\bar{r}) \longrightarrow E_-(\bar{r})$

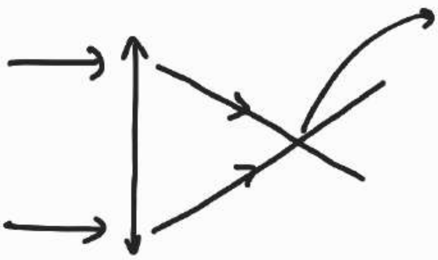
Remarks

* $U_{dip} \Leftrightarrow$ AC- stark shift

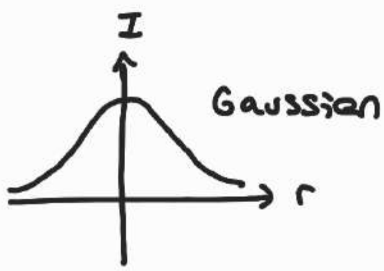
(can also be obtained directly using 2nd order perturbation theory.)

* Also possible for DC fields: $\Delta \to \omega_e$

* Optical dipole trap:

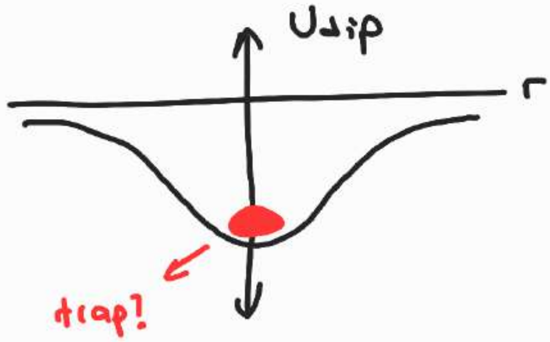


at the focus:

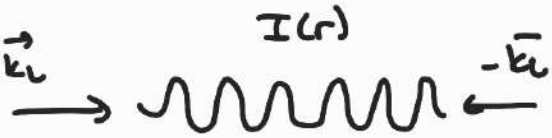


for $\Delta \geqslant 0$: $(\omega_e \geqslant \omega_L)$

"optical tweezer"



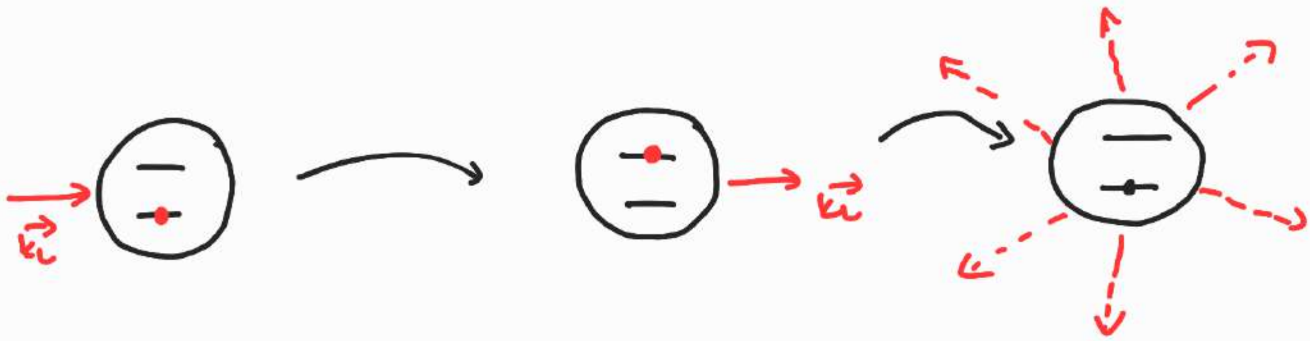
Optical lattices



$$U_{dip} = U_0 \cos^2(k_L r) \Rightarrow \text{band structure}$$

2) Radiation pressure

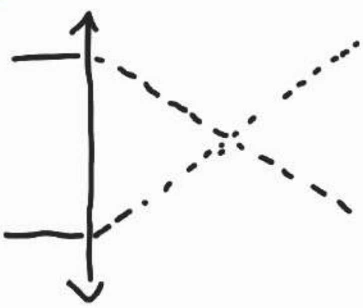
$$\vec{F} = \Gamma \langle \rho_{ee} \rangle^{st} \vec{k_L}$$



Remarks: Intrinsically related to spontaneous emission

$\Leftrightarrow$ incoherent

* Dipole force: stimulated emission



- Radiation pressure is related to photon absorption:

3) Doppler cooling

An atom moving with momentum $\vec{p}$

$$\Delta_0 \longrightarrow \Delta_0 - \vec{p}\cdot\vec{v}_{rec} - \cancel{E_{rec}}$$

negligible

$$\vec{F}_{rad} = \vec{k}_L \frac{\Gamma\Omega^2}{4} \frac{1}{\Delta_0^2 + \Gamma'^2} \quad \Big\} \; \Gamma' = \text{power broadened linewidth}$$

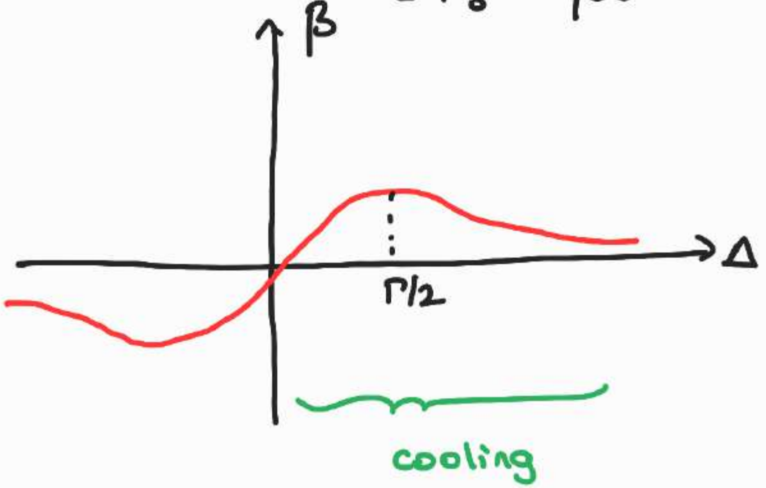
$$= \vec{k}_L \frac{\Gamma\Omega^2}{4} \frac{1}{(\Delta_0 + \vec{p}\cdot\vec{v}_{rec})^2 + \frac{\Gamma'^2}{2}}$$

First order expansion for Doppler shift:

$$\vec{F}_{rad} = \vec{k}_L \frac{\Gamma\Omega^2}{4} \left(\frac{1}{\Delta_0^2 + \Gamma'^2} - \vec{p}\cdot\vec{v}_{rec} \frac{2\Delta_0}{(\Delta_0^2 + \Gamma'^2)^2} + \dots \right)$$

$$= F_0 - \beta\vec{v}$$

β: friction coefficient



Interpretation: Doppler shift bring the atom closer to resonance when it moves against the beam.

Remarks: to cool in all directions, you need 6 laser beams

"optical molasses" 

4) Limitations and Extensions

Limit to Doppler cooling: terms $O(k_e^2)$ in the Lindblad equation.

. Random walk in k (momentum) space $\langle \Delta p^2 \rangle \sim 2m\hbar\Gamma$

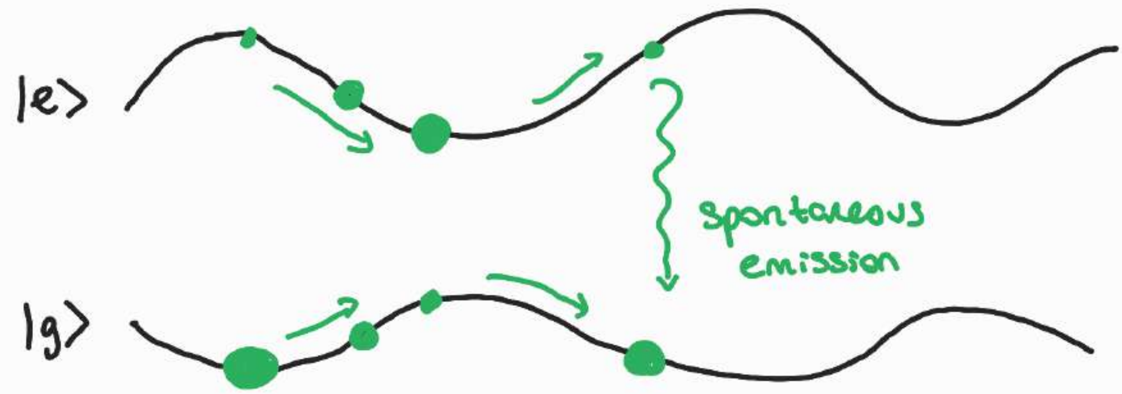
. Limit temperature: $kT \sim \hbar\Gamma$ $\begin{cases} {}^6Li = 280\ \mu K \\ {}^{88}Sr = 400\ nK \end{cases}$

the finite $k_e \Rightarrow$ can not reach $\vec{v} = 0$ in free space.

E_{rec} is the hard limit.

For trapped atoms $\rightarrow$ reaching the ground state is possible.

Other mechanism to cool in free space: Sisyphus cooling

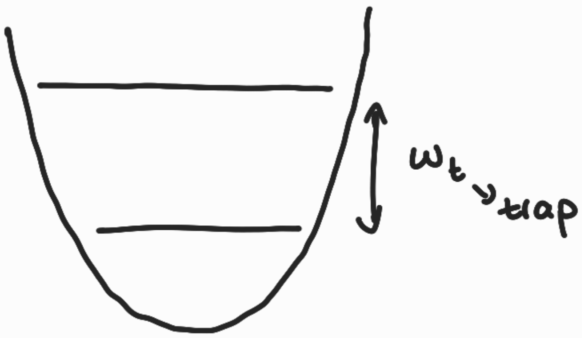

|e>
|g>
spontaneous
emission

Sideband Cooling

So far, free space : $|\vec{k}\rangle$, $\vec{k}$ was continously distributed

Variations of $\vec{k}$ are continous $\Rightarrow$ Force
(continous change of momentum)

Discrete spectrum:



Harmonic trap

Physical realizations
* atoms / ions in traps
* Mechanical oscillators

I- Motional Sidebands

• Hilbert space: $\mathcal{H} = \mathcal{H}_{com} \otimes \mathcal{H}_{internal}$

Field: driven mode $\rightarrow$ coherent state (laser)
vacuum considered as a resavoir.

Two-level atom: $\mathcal{H}_{int} = \mathrm{Sp}\{|g\rangle, |e\rangle\}$

$\mathcal{H}_{com} = \mathrm{Sp}\{|n\rangle, n \in \mathbb{N}\}$

eigenstates of the harmonic trap

- Hamiltonian:

$\hat{z}$ on $\mathcal{H}_{com}$

$$\omega_e |e\rangle\langle e| \otimes \mathbb{I} + \omega_t \hat{\mathbb{I}} \otimes \hat{a}^+ \hat{a} + \frac{\Omega}{2}\left(|e\rangle\langle g| \otimes e^{-i(k\hat{z} - \omega_L t)} + hc\right)$$

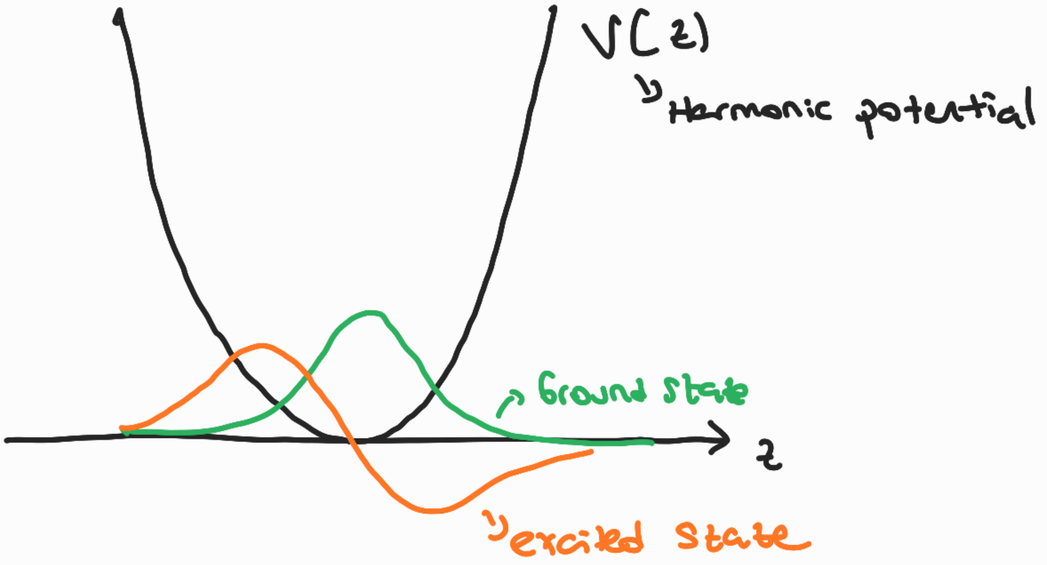
$\hat{a}$: annihilates vibrational quanta ("phonons")

1) Lamb-Dicke parameter

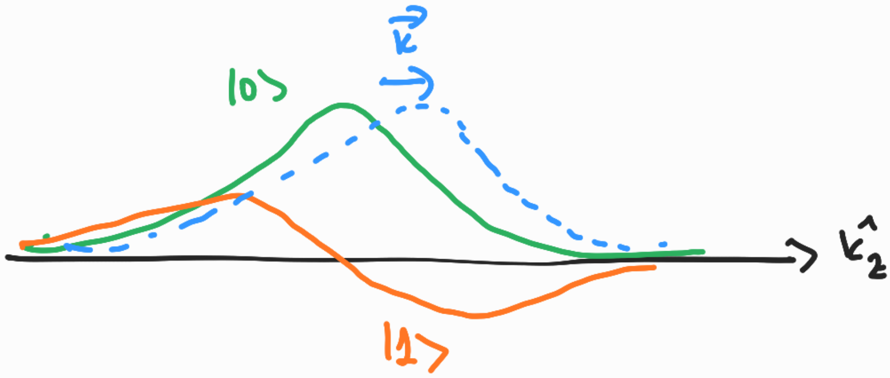
$$\hat{z} = z_0(\hat{a} + \hat{a}^+) \quad \text{with } z_0 = \sqrt{\frac{\hbar}{m\omega_t}} \quad \text{harmonic length}$$

$$e^{ik_L \hat{z}} = e^{ikz_0(\hat{a} + \hat{a}^+)} \qquad \boxed{\eta = kz_0}$$

"Lamb-Dicke parameter



↓ Momentum space



Photon absorption: $|\vec{p}\rangle \longrightarrow |\vec{p}+\vec{k}\rangle$ translation in k space

η measures this displacement w.r to the size of the wavefunction.

2) Sideband Spectrum

because of the commutator

$$e^{ik\underbrace{z_0}_{\eta}(\hat{a}+\hat{a}^+)} = e^{i\eta\hat{a}^+}\, e^{i\eta\hat{a}}\, e^{-\eta^2/2}$$

since $\hat{a}$ & $\hat{a}^+$ are not commuting

$$= \sum_{n,m} \frac{(i\eta)^{n+m}}{n!\,.m!} \hat{a}^{+(m)} \hat{a}^{(n)} e^{-\eta^2/2}$$

Group terms by q, number of quanta exchanges

$$= e^{-\eta^2/2} \Big\{ \sum_m (-1)^m \eta^{2m} \frac{\hat{a}^{+(m)} \hat{a}^{(m)}}{(m!)^2}$$

$$+ \sum_q \underbrace{\boxed{\sum_m (-1)^m \eta^{2m} \frac{\hat{a}^{+(m)} \hat{a}^{(m)}}{m!\,(m+q)!}}}_{f_q(\hat{a}^+\hat{a})} \hat{a}^{(q)} \underbrace{(i\eta)^q}_{q \text{ dependent part}}$$

$$+ \sum_m (-1)^m \eta^{2m} a^{+q} (i\eta)^q \frac{\hat{a}^{+m} \hat{a}^m}{m!\,(m+q)!} \Big\}$$

for $f_q(x) = \sum_m (-1)^m \eta^{2m} \frac{x^{(m)}}{m!\,(m+q)!}$

$$= e^{-\eta^2/2} \Big\{ f_0(\hat{a}^+\hat{a}) + \sum_{q>0} (i\eta)^q \left[f_q(\hat{a}^+\hat{a})\hat{a}^q + \hat{a}^{+q} f_q(\hat{a}^+\hat{a}) \right] \Big\}$$

3) Full Hamiltonian

$$\hat{H} = \omega_e |e\rangle\langle e| + \omega_t \hat{a}^\dagger \hat{a}$$

$$+ \frac{\Omega}{2} e^{-\eta^2/2} \left\{ f_0(\hat{a}^\dagger \hat{a}) |e\rangle\langle g| + \sum_{q>0} \cdots\cdots |e\rangle\langle g| \right\} e^{-i\omega_L t} + hc$$

trap freq.

Interaction picture $\hat{a} \longrightarrow \hat{a} e^{-i\omega_t t}$

$$|e\rangle \longrightarrow e^{-i\omega_e t} |e\rangle$$

$$\hat{H} = \frac{\Omega}{2} e^{-\eta^2/2} \Big\{ f_0(\hat{a}^\dagger \hat{a}) e^{i(\omega_e - \omega_L)t} |e\rangle\langle g| \rightarrow \text{Carrier}$$

$$+ \sum_q f_q(\hat{a}^\dagger \hat{a}) e^{i(\omega_e - \omega_L - q\omega_t)t} (i\eta)^q \hat{a}^q |e\rangle\langle g|$$

Red

$$+ \sum_q e^{i(\omega_e - \omega_L + q\omega_t)t} (i\eta)^q \hat{a}^q |e\rangle\langle g| f_q(\hat{a}^\dagger \hat{a})$$

Blue

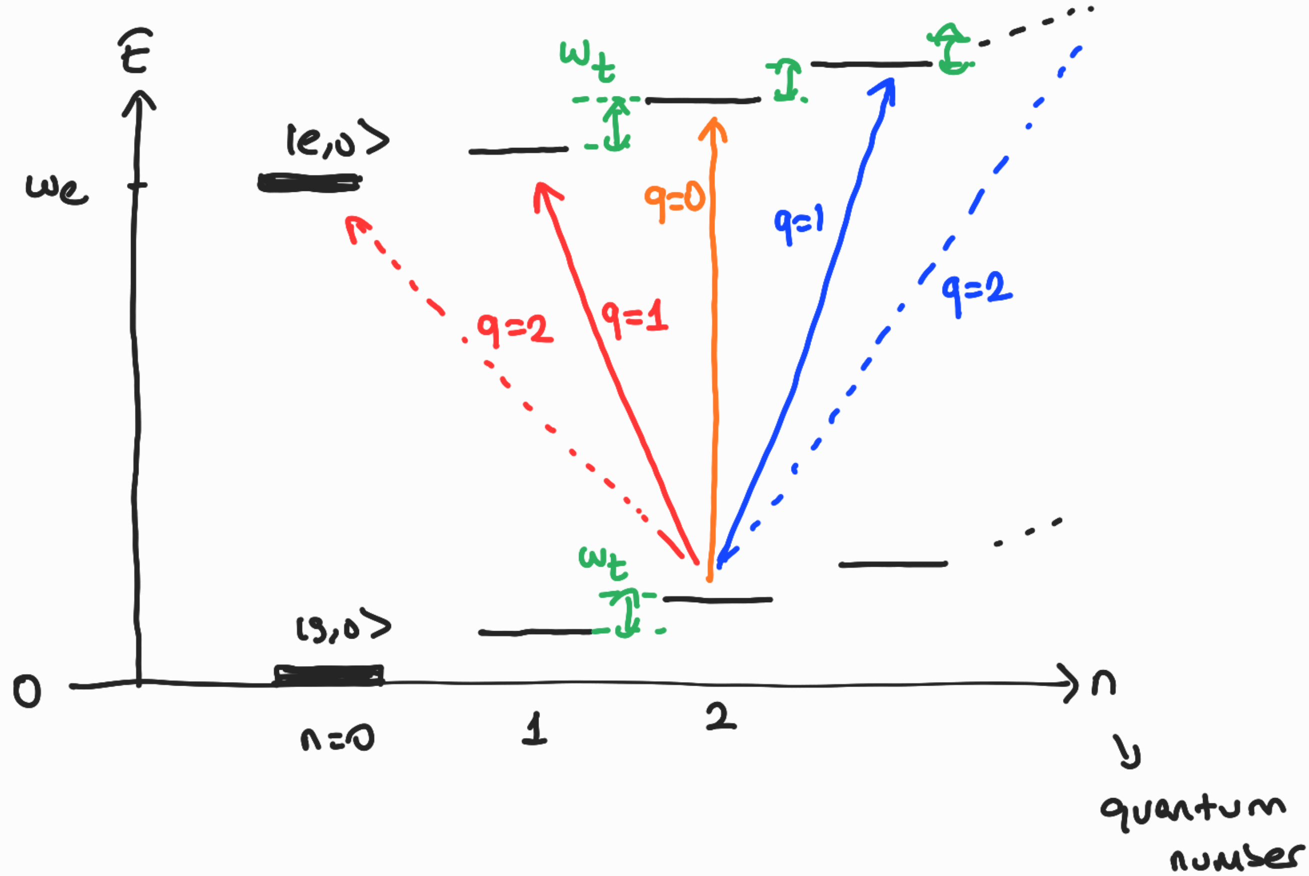
$$+ hc \Big\}$$

Each term is called a motional sideband

Carrier: $\omega_e - \omega_L$ → resonant when $\omega_L = \omega_e$

Red sidebands: $\omega_e - \omega_L - q\omega_t$ → resonant when $\omega_L = \omega_e - q\omega_t$, $q > 0$

Blue sidebands: $\omega_e - \omega_L + q\omega_t$ → " " $\omega_L = \omega_e + q\omega_t$



4) Addressing sidebands

- When $\Omega \ll \omega_t$: rotating wave approximation
  neglect terms oscillating at ω_t or higher

- Choosing $\omega_L = \omega_e \pm \omega_q t$

• Lamb-Dicke regime: $\eta \ll 1$, $f_q(x) = \frac{1}{q!}$, $e^{-\eta^2/2} = 1$

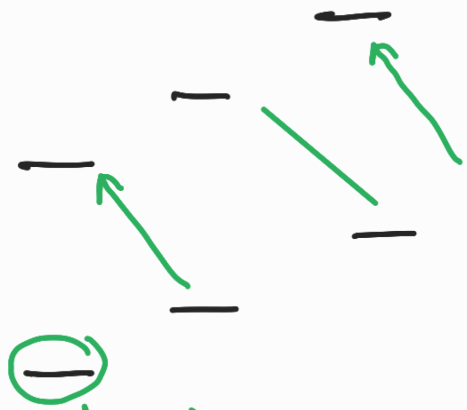
Carrier: $H_c = \frac{\Omega}{2}\left(|g\rangle\langle e| + |e\rangle\langle g|\right)$

Red-Sideband: $H_r = \frac{\Omega}{2}(i\eta)\left(\hat{a}\,|e\rangle\langle g| - \hat{a}^{+}|g\rangle\langle e|\right)$

$\approx$

Jaynes-Cummings Hamiltonian

Remark: There exists a "dark" state



$|n=0\rangle$ not coupled to light

Blue-sideband: $\hat{H}_b$ = "Anti"- Jaynes Cumming Hamiltonian

"No dark states"

II. Cooling

1) Cooling cycle

Lindblad Equation: $\dot{\hat{\rho}} = -i[\hat{H},\hat{\rho}] - \frac{\Gamma}{2}\left(|e\rangle\langle e|\hat{\rho} + \hat{\rho}|e\rangle\langle e|\right) + \sum_{\vec{n}} \Gamma_{\vec{n}} \hat{L}_{\vec{n}} \hat{\rho} \hat{L}_{\vec{n}}^{+}$

with $L_{\vec{n}} = e^{ik\vec{n}.\vec{r}} |g\rangle\langle e|$

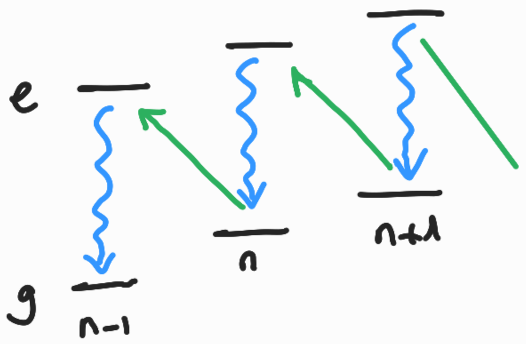
in 1D: $\hat{L}_{\pm} = e^{\pm ik\hat{z}} |g\rangle\langle e| = e^{ikz_0(\hat{a}+\hat{a}^+)} |g\rangle\langle e|$

$= (\hat{\mathbb{I}} \pm i\eta^{\pm(\hat{a}+\hat{a}^+)} + \dots) |g\rangle\langle e|$

Same as free space:

$$\sum_{\vec{n}} \Gamma_{\vec{n}} \hat{L}_{\vec{n}} \hat{\rho} \hat{L}_{\vec{n}}^{+} \sim \sum_{\vec{n}} \Gamma_{\vec{n}} |g\rangle\langle e| \hat{\rho} |e\rangle\langle g| + O(\eta^2)$$

in the Lamb-Dicke regime $\eta \ll 1$: Spontaneous emission does not change n.



Driving the red sideband + sp. emission

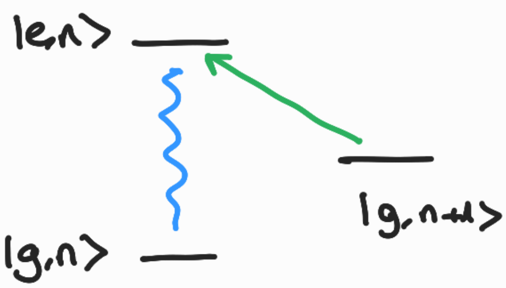
$|n\rangle \longrightarrow |n-1\rangle$ Down to $|n=0\rangle$

Cooling!

(In practise : limited $O(\eta^2)$)

2) Quantative description

Method of adiabatic elimination :



spon. emission rate

3 levels problem for $\Omega_r \ll \Gamma$

red-side band Rabi freq

$\rho_{ee} = 0$

population in $|e\rangle$ state

Effective 2-level theory :

$$\dot{\hat\rho} = -i[\hat H, \hat\rho] - \frac{\Gamma}{2}\left(|e\rangle\langle e|\hat\rho + \hat\rho|e\rangle\langle e|\right) + \Gamma \underbrace{\langle e|\hat\rho|e\rangle}_{\text{operator on } \mathcal{H}_{com}} |g\rangle\langle g| \quad (*)$$

operator on $\mathcal{H}_{com}$

Hilbert space

$$\langle e|\dot{\hat\rho}|e\rangle = \dot{\hat\rho}_{ee} = -i\langle e|\left[i\frac{\Omega_r\eta}{2}\left(\hat a|e\rangle\langle g| - \hat a^{+}|g\rangle\langle e|\right), \hat\rho\right]|e\rangle - \Gamma\hat\rho_{ee}$$

$$= \frac{\Omega_r\eta}{2}\left(\hat a\underbrace{\langle g|\hat\rho|e\rangle}_{\hat\rho_{ge}} + \underbrace{\langle e|\hat\rho|g\rangle}_{\hat\rho_{eg}}\hat a^{+}\right) - \Gamma\hat\rho_{ee}$$

$$\langle g|\dot{\hat{\rho}}|g\rangle = \dot{\hat{\rho}}_{gg} = \frac{\Omega_r \eta}{2}\left(\hat{\rho}_{ge}\hat{a} + \hat{a}^{+}\hat{\rho}_{eg}\right) + \Gamma\hat{\rho}_{ee}$$

$$\langle g|\dot{\hat{\rho}}|e\rangle = \dot{\hat{\rho}}_{eg} = \frac{\Omega_r \eta}{2}\left(\hat{a}\hat{\rho}_{gg} + \hat{\rho}_{ee}\hat{a}^{+}\right) - \frac{\Gamma}{2}\hat{\rho}_{eg}$$

★ Internal degress of freedom are fast!

Steady state is reached : $\hat{\rho}_{eg} = \frac{\Omega_r \eta}{\Gamma}\hat{a}\hat{\rho}_{gg}$, $\hat{\rho}_{ee} \sim 0$

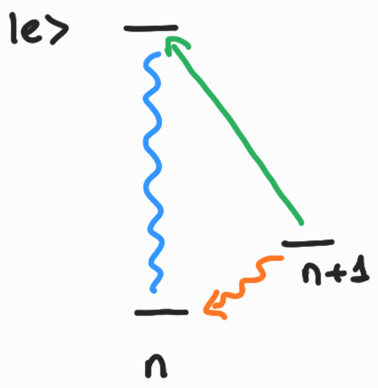
Then $\dot{\hat{\rho}}_{gg} = -\frac{\Omega_r^2\eta^2}{2\Gamma}\left(\hat{a}^{+}\hat{a}\hat{\rho}_{gg} + \hat{\rho}_{gg}\hat{a}^{+}\hat{a}\right) + \frac{\Omega_r^2\eta^2}{\Gamma}\hat{a}\hat{\rho}_{gg}\hat{a}^{+}$

(No $\hat{\rho}_{ee}$ dependence anymore)

Effective Lindblad Eq for $|g\rangle$:

$$\boxed{\dot{\hat{\rho}} = -\frac{\Gamma_1}{2}\left(\hat{a}^{+}\hat{a}\hat{\rho} + \hat{\rho}\hat{a}^{+}\hat{a}\right) + \Gamma_1\hat{a}\hat{\rho}\hat{a}^{+}}$$

$$\hat{L}_{eff} = \sqrt{\Gamma_1}\,\hat{a}$$



$$\Gamma_1 = \frac{\Omega_r^2\eta^2}{\Gamma} \ll \Gamma, \Omega_r$$

RYDBERG ATOMS

Single Atom as quantum information carrier

I. Introduction

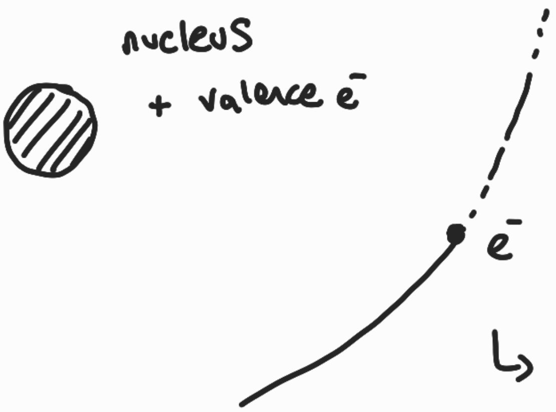
1) Hydrogen-like atoms

Consider 1 e^- all other e^- are "frozen"

(which is ok for highly excited state)

nucleus

+ valence e^-



e^- promoted to highly excited state

↳ bound by Coulomb potential $V(r) = \frac{-e^2}{4\pi\epsilon_0 r}$

It has rotational invariance.

Schrödinger Equation :

$$\left[-\frac{\hbar^2}{2m_e}\Delta + V(r)\right]\Psi_e = E\Psi_e$$

$$[\hat{L}^2, \hat{H}] = 0 \quad , \quad [\hat{L}_z, \hat{H}] = 0$$

So $\psi(r) = \frac{u(r)}{r} Y_\ell^m(\vartheta,\varphi)$

$|\psi\rangle = |n\ell m\rangle$

$\hat{L}^2|\psi\rangle = \hbar^2 \ell(\ell+1)|\psi\rangle$

$\hat{L}_z|\psi\rangle = \hbar m|\psi\rangle$,

$m \in \{-\ell, \dots, \ell\}$

Radial part:

$$-\frac{\hbar^2}{2m_e}\frac{d^2}{dr^2} u_{n\ell}(r) + \underbrace{\left(V(r) + \frac{\hbar^2(\ell+1)\ell}{2mr^2}\right)}_{V_{eff}(r)} u_{n\ell}(r) = E_{n\ell}\, u_{n\ell}(r)$$

Spectrum: $E_n = -\frac{R_y}{n^2}$, independent of ℓ $\quad R_y = \underbrace{\frac{1}{2}\frac{me^4}{\hbar^2}}_{R_{y\infty} = 13{,}6\text{ eV}}\left(\frac{M}{M+m_e}\right)$

Remarks:

- For real atoms $E_{n\ell} = \frac{-R_y}{(n-\mu_\ell)^2}$ $\quad \mu_\ell$: quantum defect

- Length scale $\quad a_0 = \frac{4\pi\epsilon_0\hbar^2}{m_e e^2}$ Bohr radius $a_0 = 5{,}3 \times 10^{-11}$ m

- Rydberg atom: atom with an e^- in state $n \gtrsim 20\text{–}100$

2) Scaling

- "size" of an atom (with n increasing)

$$\left\langle \frac{1}{r} \right\rangle \sim \frac{4\pi\epsilon_0}{-e^2} \langle V(r) \rangle$$

Virial theorem: $\langle V(r) \rangle \propto E_n$

$$\Rightarrow \left\langle \frac{1}{r} \right\rangle \sim \frac{1}{n^2 a_0} \Rightarrow \text{"size"} \sim n^2$$

$\hookrightarrow$ principal quantum number

★ Dipole moment :

$$\langle n\ell m | \hat{\vec{d}} | n'\ell' m \rangle \qquad n' \sim n, \quad \ell' = \ell + 1$$

$$\sim e a_0 n^2$$

$$\left(\Psi_{n\ell m}(r) \sim r^{n-1} e^{-r/a_0} \right)$$

II. Atomic Interactions

1) Van der Walls Potential

Neutral atom, polarizable

Dipole - dipole interaction:

$$U(r) = \frac{1}{4\pi\epsilon_0 r^3} \left(\hat{\vec{d}}_A \cdot \hat{\vec{d}}_B - 3 (\vec{u} \cdot \hat{\vec{d}}_A)(\vec{u} \cdot \hat{\vec{d}}_B) \right)$$

→ from classical electrodynamics

For quantum problem: $\mathcal{H}_{rel} = Sp\{|\vec{r}\rangle, \vec{r} \in \mathbb{R}^3\}$

relative

$\mathcal{H}_A = Sp\{|nlm\rangle, \dots\}$ $\mathcal{H}_B = \dots$

Atom

Born-Oppenheimer Approximation: e^- are fast

Perturbation Theory:

- 1^{st} order $\langle n_A l_A m_A ; n_B l_B m_B | \hat{U} | n_A l_A m_A ; n_B l_B m_B \rangle = 0$

  since $\langle d_A \rangle_{nlm} = 0$

- 2^{nd} order $\Delta E^{(2)} = \sum_{\nu,\nu'} \frac{|\langle n_A l_A m_A ; n_B l_B m_B | \hat{U} | \nu, \nu' \rangle|^2}{E_{n_A l_A n_B l_B} - E_{\nu \nu'}} \neq 0$

Scaling: $\Delta E^{(2)}(r) = -\frac{C_6}{r^6}$ C_6 is a number depending on the "details".

- $\langle n_A | \hat{\vec{d}}_A | \nu \rangle \sim n^2$, same for B

- $U \sim n^4$

- $\langle U \rangle^2 \sim n^8$

$$E_{n_A n_B} - E_{\nu\nu'} = \delta E$$

derivative

Since $E \sim \frac{1}{n^2} \Rightarrow \delta E \sim \frac{1}{n^3}$

$$\Rightarrow C_6 \sim n^{11}$$

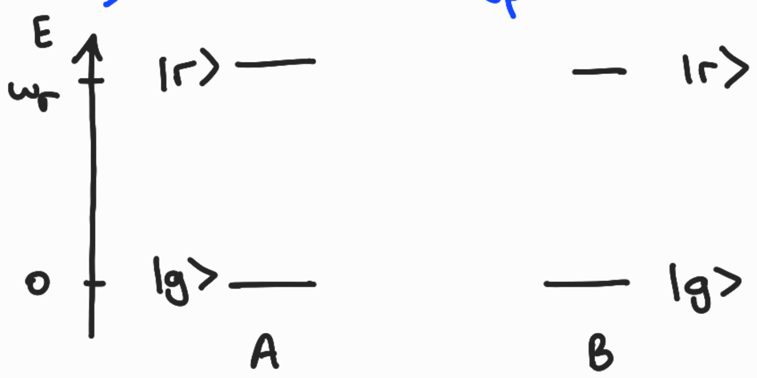
Van der Waals potential: $V_{vdw}(r) = -\frac{C_6}{r^6} \sim n^{11}$

Remark:

Van der Walls force exists between any pair of polarizable objects

There are situation for which first order is non-zero $\frac{C_3}{r^3}$ interaction

2) Two-Atom Spectrum



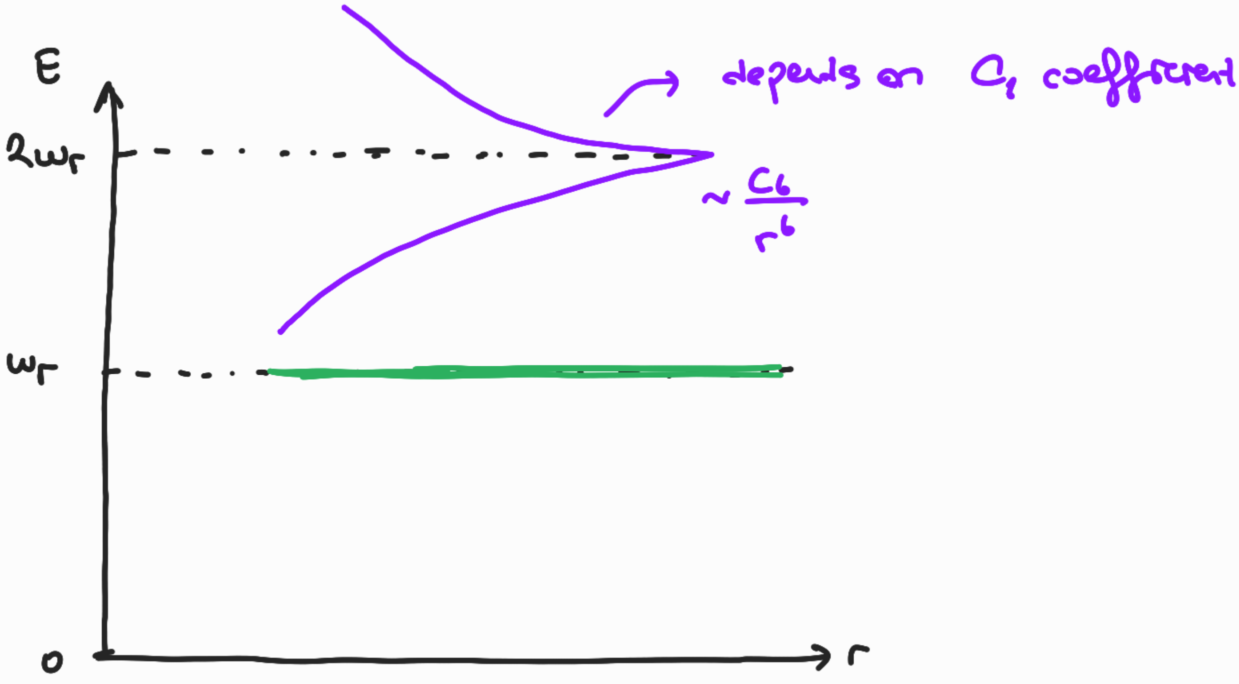
. Very weak interaction between $|g\rangle$ and $|g\rangle$

& btw $|r\rangle$ and $|g\rangle$?

Hilbert space:

$$Sp\{|gg\rangle, |gr\rangle, |rg\rangle, |rr\rangle\}$$

$$\hat{H}_{at} = \omega_r |gr\rangle\langle gr| + \omega_r |rg\rangle\langle rg| + 2\omega_r |rr\rangle\langle rr| - \frac{C_6}{r^6} |rr\rangle\langle rr|$$



3) Rydberg Blockade

We are driving hamiltonian $|g\rangle \longrightarrow |r\rangle$

$$\hat{H} = \hat{H}_{at} + \frac{\Omega}{2}\Big(\underbrace{|r\rangle\langle g| \otimes \hat{\mathbb{I}}}_{\text{atom A}}\, e^{-i\omega_L t} + \hat{\mathbb{I}} \otimes \underbrace{|r\rangle\langle g|}_{\text{Atom B}}\, e^{-i\omega_L t} + h.c.\Big)$$

for $\omega_L = \omega_r$, move to the rotating frame:

$$|r\rangle \longrightarrow |r\rangle e^{i\omega_L t}$$

$$|r,r\rangle \longrightarrow |r,r\rangle e^{+2i\omega_L t}$$

$$\hat{H} = -\frac{C_6}{r^6} |rr\rangle\langle rr| + \frac{\Omega}{2}\big(|r\rangle\langle g| \otimes \hat{\mathbb{I}} + \hat{\mathbb{I}} \otimes |r\rangle\langle g| + h.c\big)$$

• For $C_6 = 0$: Dicke States description

Drive will induce coherent spin states

• For $\frac{C_6}{r^6} \gg \Omega$: $\hat{\mathbb{1}} = |r\rangle\langle r| + |g\rangle\langle g|$

$$\hat{H} = -\frac{C_6}{r^6}|rr\rangle\langle rr| + \frac{\Omega}{2}\Big(|rg\rangle\langle gg| + |rr\rangle\langle gr| + hc$$

$$|gr\rangle\langle gg| + |rr\rangle\langle rg| + hc \Big)$$

Move to rotating frame:

$$\hat{U} = e^{-i\frac{C_6}{r^6}|rr\rangle\langle rr| t}$$

$$|rr\rangle \longrightarrow e^{-i\frac{C_6}{r^6}t}|rr\rangle$$

Rotating wave approximation:

$\frac{C_6}{r^6}$ is a large frequency wr to the other scales.

$$\hat{H}_{RWA} = \frac{\Omega}{2}\Big(|rg\rangle\langle gg| + |gr\rangle\langle gg| + hc \Big)$$

Using RWA: we went from 4d Hilbert space to 3d $\mathcal{H}$ space

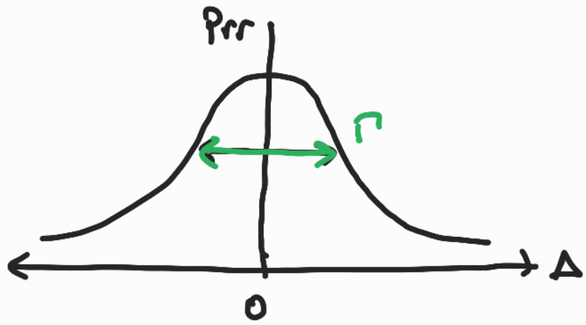
$|rr\rangle$ has dropped.

"Rydberg Blockade"

How far can 2 atoms be for blockade to take place?

$|rg\rangle \longrightarrow |rr\rangle$: 2-level system

$\Rightarrow$ Optical Bloch Equations



$\Delta = (E_{rr} - E_{rg}) - \omega_L$ (ω_L: laser)

$\Delta = \omega_r - \frac{C_6}{r^6} - \omega_r$

$= -\frac{C_6}{r^6}$

If $\frac{C_6}{r^6} >> \Gamma, \Omega$ → Blockade happens

$$r_b = \left(\frac{C_6}{\Gamma}\right)^{1/6}$$

(blockade) radius

In practise $r_b \sim 10\mu m$

III. Applications

$$\hat{H} = \frac{\Omega}{2}\left(|rg\rangle\langle gg| + |gr\rangle\langle gr| + hc\right)$$

$$= \frac{\Omega\sqrt{2}}{2}\left(|\psi^+\rangle\langle gg| + hc \right) \qquad |\psi^+\rangle = \frac{1}{\sqrt{2}}\left(|gr\rangle + |rg\rangle \right)$$

$(g=0,\ r=1)$

Bell State

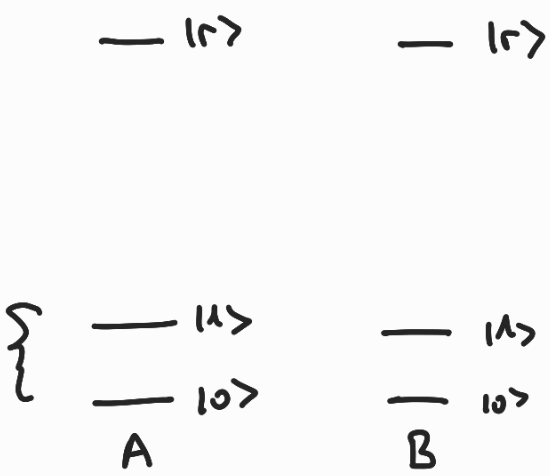
After $1/2$ Rabi cycle: Bell State

Remark
- $\sqrt{2} \leftrightarrow$ superradiance
- For two atoms: Bell / Dicke state $j=1$ are identical. But no way to create Dicke state from $|gg\rangle$ using coherent drive

1) 2 qubit gate

$|r\rangle$ is not a stable state

hyperfine State



· 2 atoms at distance $\ll r_b$

· drive selectively transitions btw $|1\rangle \longrightarrow |r\rangle$ without affecting $|0\rangle$
↓
A or B

Remark: For a 2-level system 2π rotations $\Leftrightarrow \times(-1)$

$$\hat{U}_x(\theta) = e^{-i\frac{\theta}{2}\hat{\sigma}_x} = \cos\frac{\theta}{2}\,\mathbb{I} - i\sin\frac{\theta}{2}\,\hat{\sigma}_x$$

for $\theta = 2\pi$: $\hat{U}_x(2\pi) = -\hat{\mathbb{I}}$

↳ important for CNOT

Pulse sequence:

1. π-rotation of qubit A: $|1\rangle_A \longrightarrow |r\rangle_A$
2. 2π-rotation of qubit B: $|1\rangle_B \longrightarrow -|1\rangle_B$
3. π rotation of A : $|r\rangle_A \rightarrow -|1\rangle_A$

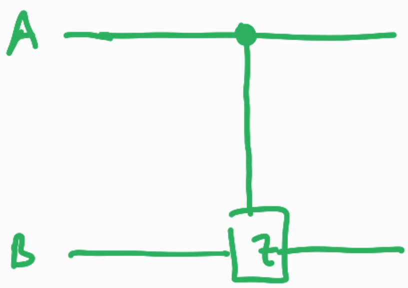
Truth Table:

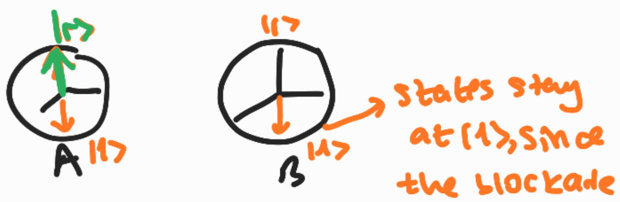
$|00\rangle \longrightarrow |00\rangle$

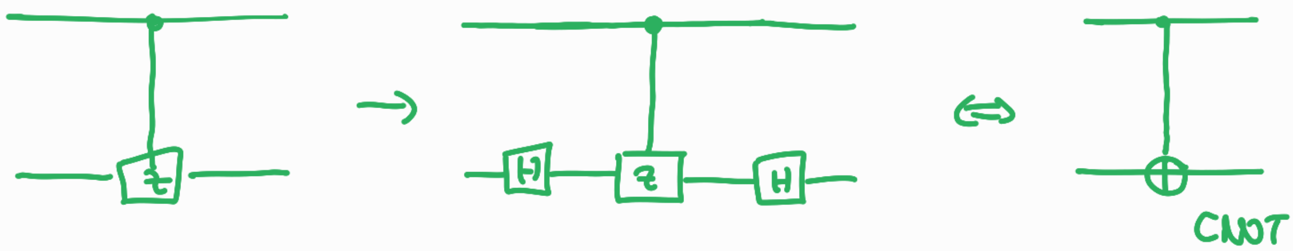
$|10\rangle \longrightarrow -|10\rangle$

$|01\rangle \longrightarrow -|01\rangle$

$|11\rangle \longrightarrow -|11\rangle$







2) Quantum Simulation

$|g\rangle |r\rangle$ describe 2 level system $\Leftrightarrow$ spin 1/2 particle in $\vec{B}$ field

- Rydberg atom, driven by laser $|g\rangle \longrightarrow |r\rangle$

Single atom: $\hat{H} = \omega_r |r\rangle\langle r| + \frac{\Omega}{2}\left(|r\rangle\langle g| e^{-i\omega_L t} + h.c.\right)$

(Rotating frame) $= \delta \hat{\sigma}_z + \frac{\Omega}{2}\hat{\sigma}_x$ + constant

Atomic ensemble: $\hat{H} = \sum_i \delta \hat{\sigma}_z^{\,i} + \frac{\Omega}{2}\hat{\sigma}_x(i)$

$- \sum_{i<j} \frac{C_6}{R_{ij}^6}\hat{n}_i . \hat{n}_j$

$\hat{n}_i = I \otimes I \otimes \cdots \otimes |r\rangle\langle r| \otimes \cdots \otimes I$

$= \frac{1}{2}\left(I + \sigma_z^{(i)}\right)^i$

Rewritten as

$$\hat{H} = \sum_i (\delta + B_i)\, \hat{\sigma}_z^{(i)} + \frac{\Omega}{2} \hat{\sigma}_x^{(i)} + \sum_{i<j} J_{ij}\, \hat{\sigma}_z^{(i)} \hat{\sigma}_z^{(j)}$$

$$B_i = \sum_j -\frac{C_6}{R_{ij}^6}$$

"Ising Model"
with transverse field

Properties:

- Paramagnetic Phase

- Fero/antiferromagnetic phase