The recipe to create a qubit consists of taking any system that behaves quantum mechanically and (1)
to make sure to limit the dynamics of the system to two eigenstates. This is also translated in limits in the control pulses, so that only the transition to a single excited level is possible.

The general time-dependent state of a quantum bit is denoted by a Dirac Ket $|\psi(t)\rangle$. A generic superposition of two basic states $|0\rangle$ and $|1\rangle$ is $|\psi(t)\rangle = c_0(t)|0\rangle + c_1(t)|1\rangle$.
The qubit state obeys the Schrödinger equation:

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}(t)|\psi(t)\rangle$$

where $\hat{H}(t)$ it the time-dependent Hamiltonian operator.

The Schrödinger Equation in matrix notation reads: $i\hbar\begin{pmatrix} c_0(t) \\ c_1(t) \end{pmatrix} = \begin{pmatrix} h_{00}(t) & h_{01}(t) \\ h_{10}(t) & h_{11}(t) \end{pmatrix}\begin{pmatrix} c_0(t) \\ c_1(t) \end{pmatrix}$

Note that any two-states Hamiltonian (a 2 x 2 matrix) can be decomposed into a weighted sum of Pauli matrices an the unit matrix. (2)

This means that the dynamics of a two-state system can be always mapped onto the problem of a spin-$\frac{1}{2}$ particle in a (time dependent) magnetic field.

As you may recall, a constant magnetic field makes the spin precess along an axis pointing along the direction of the field with an angular velocity proportional to the field strength.

Let's see how it works in details, by calculating the dynamics of a spin $\frac{1}{2}$ particle during some pulses.

$$H = -\frac{\hbar\gamma}{2}\left[B_x(t)\,\sigma_x + B_y(t)\,\sigma_y + B_z(t)\,\sigma_z\right]$$

A pulse in the z-direction:

Let's pulse $B_z(t)$ $\begin{cases}\text{time-dependent but direction of} \\ \text{the field is constant.}\end{cases}$

The Schrödinger eq. reads:

$$i\hbar \begin{bmatrix} \dot{c}_0(t) \\ \dot{c}_1(t) \end{bmatrix} = -\frac{\hbar\gamma}{2} \begin{bmatrix} B_z(t) & 0 \\ 0 & B_z(t) \end{bmatrix} \begin{bmatrix} c_0(t) \\ c_1(t) \end{bmatrix}.$$

Having two uncoupled differential equations, we can immediately write down the solution

$$\begin{bmatrix} c_0(t) \\ c_1(t) \end{bmatrix} = \begin{bmatrix} e^{i\delta/2} & 0 \\ 0 & e^{-i\delta/2} \end{bmatrix} \begin{bmatrix} c_0(0) \\ c_1(0) \end{bmatrix}, \quad \delta = \gamma \int_0^t dt' B_z(t')$$

Write the time-evolution operator $R_z(-\delta) = \begin{pmatrix} e^{i\delta/2} & \\ & e^{-i\delta/2} \end{pmatrix}$

we see that is has the following effect on a general qubit state $R_z(-\delta)|\psi(\theta,\phi)\rangle = |\psi(\theta,\phi-\delta)\rangle$.

So, a pulse of magnetic field:

$$B_z(t) \Rightarrow \hat{H}(t) = -\frac{\hbar}{2}\gamma B_z(t)\,\sigma_z$$

which represents a rotation with an angle

$$\phi = \gamma \int_0^t dt'\, B_z(t') \quad \text{around the } z\text{-axis.}$$

We see that it is only the integral of the pulse and not its exact shape which matters.

Since the z-axis is an arbitrary chosen direction in space, we realize that any pulse along a fixed direction give rise to the same physics, i.e. a rotation around that axis.

Since we can reach any point on the Bloch-sphere with two consecutive rotations along two not-parallel axes, there is not need to use other pulses than simple rotations.

Performing single qubit gates

A natural method to implement unitary single qubit gates consists in using small amplitude harmonic perturbation of some qubit parameters. Those high frequency pulses are usually on resonance with the qubit energy splitting.

This method was first discussed by Rabi in the context of nuclear magnetic resonance (NMR).

Let us assume that the artificial atom can be described as a simple two-level system with the ground state $|g\rangle$ and excited $|e\rangle$ having energy difference $\Delta E = E_e - E_g = \hbar\omega_0$. When the electromagnetic radiation is shined onto an atom, a photon can be absorbed by the atom in the ground state; the atom will so transit to its excited state. This process only occurs if the radiation is quasi resonant $\omega \simeq \omega_0$.

The Hamiltonian of the field-atom interaction in the dipole approximation is $\hat{H}_{int} = -\vec{d}\cdot\vec{E}(t)$, where $\vec{d} = -e\,\vec{r}$ is the dipole moment operator.

We now consider two atomic levels with different parity, called ground state $|g\rangle$ and excited state $|e\rangle$ [with transition frequency $\omega_0 = (E_e - E_g)/\hbar$].

We apply harmonic radiation $\vec{E}(t) = \vec{E}_0 \cos(\omega t + \phi)$ with frequency close to that of the atom resonance $|\omega_0 - \omega| \ll \omega_0$.

The truncated two-level Hamiltonian then reads:

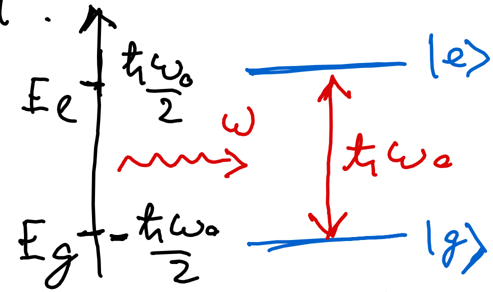
$$\boxed{\hat{H} = -\frac{\hbar\omega_0}{2}\sigma_z - A\cos(\omega t + \phi)\sigma_x} \; ; \quad A = \langle e|\vec{d}\cdot\vec{E}|g\rangle \equiv \hbar\Omega_1$$

Writing the ansatz for the state

$$|\psi(t)\rangle = C_g(t)\, e^{-iE_g t/\hbar}|g\rangle + C_e(t)\, e^{-iE_e t/\hbar}|e\rangle$$

with $E_g = -\hbar\omega_0/2$ and $E_e = +\hbar\omega_0/2$

From the Schrödinger Eq., we get:



$$\dot{C}_g = \frac{i}{\hbar} A\cos(\omega t+\phi)\, e^{-i\omega_0 t} C_e$$

$$\dot{C}_e = \frac{i}{\hbar} A\cos(\omega t+\phi)\, e^{i\omega_0 t} C_g$$

Expanding $\cos(\omega t+\phi) = \left[e^{i(\omega+\phi)t} + e^{-i(\omega+\phi)t}\right]/2$

we will find slowly rotating terms $e^{\pm i(\omega-\omega_0)t}$ and fast oscillating terms $e^{\pm i(\omega+\omega_0)t}$.

Since the time evolution induced by the applied field is much slower than ω_0, we can neglect the quickly rotating terms [Rotating Wave Approximation - RWA].

Leaving:

$$\dot{C}_g = \frac{i}{2\hbar} A\, e^{i\phi} e^{i(\omega-\omega_0)t} C_e$$

$$\dot{C}_e = \frac{i}{2\hbar} A\, e^{-i\phi} e^{-i(\omega-\omega_0)t} C_g$$

By eliminating C_g: $\ddot{C}_e + i(\omega-\omega_0)\dot{C}_e + \frac{1}{4}\frac{A^2}{\hbar^2} C_e = 0$

From the trial solution $C_e(t) = e^{i\lambda t}$ we find:

$$\lambda_\pm = \frac{1}{2}\left(\Delta \pm \sqrt{\Delta^2 + A^2/\hbar^2}\right) = \frac{1}{2}\left(\Delta \pm \Omega_R\right)$$

where $\Delta = \omega_0 - \omega$ is the frequency detuning and $\Omega_R^2 = \Delta^2 + \Omega_1^2$ is the generalised Rabi frequency.

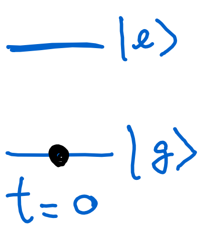
The general solution can be written as:

$$C_e(t) = C_+ e^{i\lambda_+ t} + C_- e^{i\lambda_- t}$$

$$C_g(t) = \frac{2\hbar}{iA} e^{i\phi} e^{-i\Delta t}\left[i\lambda_+ C_+ e^{i\lambda_+ t} + i\lambda_- C_- e^{i\lambda_- t}\right].$$

Let us look at an example by considering the atom being at time $t=0$ in $|g\rangle$, so we have $C_g(0) = 1$; $C_e(0) = 0$, with $\phi = 0$

We find $C_e(t) = i\frac{A}{\Omega\hbar} e^{i\Delta t/2} \sin(\Omega_R t/2)$



$$C_g(t) = e^{i\phi} e^{-i\Delta t/2}\left[\cos(\Omega_R t/2) + i\frac{\Delta}{\Omega_R}\sin(\Omega_R t/2)\right].$$

The probability to find the atom in state $|e\rangle$ is given

$$P_e(t) = |C_e(t)|^2 = \frac{A^2}{\Omega_R^2 \hbar^2} \sin(\Omega_R t/2)^2 = \left(\frac{\Omega_1}{\Omega_R}\right)^2 \sin(\Omega_R t/2)^2 \quad (9)$$

a) [Try to plot this relation for $P_e(t)$ for 3 different detuning values Δ: $\Delta = 0$; $|\Delta| > 0$ and $|\Delta| \gg 0$]

b) [Try to plot $P_{g\to e}$ vs. frequency detuning Δ]

Those $P_e(t)$ / $P_g(t)$ population oscillations are called

RABI OSCILLATIONS

We can now show how the Rabi Hamiltonian allows manipulations of a qubit!

In order to better understand what this light-spin interaction does to the qubit on the Bloch sphere, it's convenient to move to the Rotating frame at frequency ω_0.

Move to a rotating frame

(rotating around z-axis with the free precession freq. ω_0)

$$\hat{R}_z(+\delta) = e^{-i\frac{\delta}{2}\hat{\sigma}_z} = e^{-i\frac{\omega_0 t}{2}\hat{\sigma}_z}$$

Using relation: $\hat{H}'_{Rot} = \hat{R}\,\hat{H}\,\hat{R}^{\dagger} + i\hbar\frac{\partial \hat{R}}{\partial t}\hat{R}^{\dagger}$

$$H'_{ROT} = e^{-i\frac{\omega_0}{2}t\hat{\sigma}_z}\left(-\frac{\hbar\omega_0}{2}\hat{\sigma}_z - A\cos(\omega t+\phi)\hat{\sigma}_x\right)e^{i\frac{\omega_0 t}{2}\hat{\sigma}_z} + i\hbar\left(-i\frac{\omega_0}{2}\hat{\sigma}_z\right)$$

$$= \dots$$

$$= -\frac{\hbar\omega_0}{2}\hat{\sigma}_z \underbrace{-A\cos(\omega t+\phi)}_{= -\frac{A}{2}\left[e^{i(\omega t+\phi)} + e^{-i(\omega t+\phi)}\right]}\begin{pmatrix} 0 & e^{-i\omega_0 t} \\ e^{i\omega_0 t} & 0 \end{pmatrix} + \frac{\hbar\omega_0}{2}\hat{\sigma}_z$$

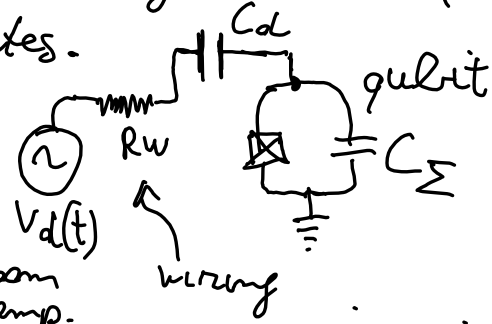
RWA

$$= -\frac{A}{2}\left[\begin{pmatrix} 0 & e^{i\phi}e^{+i(\omega-\omega_0)t} \\ e^{i\phi}e^{+i(\omega+\omega_0)t} & 0 \end{pmatrix} + \begin{pmatrix} 0 & e^{-i\phi}e^{-i(\omega+\omega_0)t} \\ e^{-i\phi}e^{-i(\omega-\omega_0)t} & 0 \end{pmatrix}\right]$$

for $\Delta = 0$ $\approx -\frac{A}{2}\begin{pmatrix} 0 & e^{i\phi} \\ e^{-i\phi} & 0 \end{pmatrix} = -\frac{A}{2}\left[\cos\phi\,\hat{\sigma}_x - \sin\phi\,\hat{\sigma}_y\right]$

Single-qubit gate:

Here we will review the steps necessary to demonstrate that capacitive coupling of MW to a superconducting circuit can be used to drive single-qubit gates.



Capacitive coupling: X, Y control

$$H = -\frac{\omega_q}{2}\sigma_z + \Omega V_d(t)\,\sigma_y = H_0 + H_{int}$$

$$\Omega = (C_d/C_\Sigma)\,Q_{zpf} \qquad \omega_q = (E_1 - E_0)/\hbar$$

C_d = coupling capacitance
C_Σ = qubit capacitance

To elucidate the role of the driving, we move into a frame rotating with the qubit at freq. ω_q ("rotating frame" or "interaction frame"). To see the usefulness of this frame, consider a state $|\psi_0\rangle = (1\ \ 1)^T/\sqrt{2}$ (on the equatorial frame). Considering the propagator U_{H_0} corresponding to H_0, by the time dependent Schröd. eq. we have

$$|\psi_0(t)\rangle = U_{H_0}|\psi_0\rangle = 1/\sqrt{2}\begin{pmatrix} e^{i\omega_q t/2} \\ e^{-i\omega_q t/2} \end{pmatrix}.$$

By calculating e.g. $\langle \psi_0 | \sigma_x | \psi_0 \rangle = \cos(\omega_q t)$, it is evident that the phase is winding with a frequency ω_q due to the σ_z term. By going into a frame rotating with the qubit at frequency ω_q, the action of the driving can be more clearly studied.

We can do this by $U_{rf} = e^{iH_0 t} = U_{H_0}^{\dagger}$. The new state in the rotating frame is $|\psi_{rf}(t)\rangle = U_{rf} |\psi_0\rangle$

and the Schröd. eq. is now:

$$i \partial_t |\psi_{rf}(t)\rangle = i (\partial_t U_{rf}) |\psi_0\rangle + i U_{rf} (\partial_t |\psi_0\rangle), =$$

$$= i \dot{U}_{rf} U_{rf}^{\dagger} |\psi_{rf}\rangle + U_{rf} H_0 |\psi_0\rangle =$$

$$= \underbrace{\left(i \dot{U}_{rf} U_{rf}^{\dagger} + U_{rf} H_0 U_{rf}^{\dagger} \right)}_{\tilde{H}_0} |\psi_{rf}\rangle$$

$\tilde{H}_0$ is the transformed H_0 in the new rotating frame.

We can see that the new $\tilde{H}_0 = 0$, as expected because the rotating frame takes care of the time dependence.

This transformation allows also to obtain the new form of the interaction term in the rotating frame.

$$\tilde{H}_d = \Omega V_d(t)\left[\cos(\omega_q t)\sigma_y - \sin(\omega_q t)\sigma_x\right]$$

We can assume that the time dependent part of the voltage $V_d(t) = V_0 v(t)$, has the generic form

$$v(t) = S(t)\sin(\omega_d t + \phi) =$$

$$= s(t)\left[\cos(\phi)\sin(\omega_d t) + \sin(\phi)\cos(\omega_d t)\right]$$

[dimentionless envelope function] that sets the shape and amplitude $\underline{V_0 S(t)}$ of the drive

$I = \cos(\phi)$ ["in-phase" component]

$Q = \sin(\phi)$ ["out-phase" component]

We can rewrite the driving Hamiltonian as

$$\tilde{H}_d = \Omega V_0 S(t) \left[I \sin(\omega_d t) - Q \cos(\omega_d t) \right] \times$$
$$\times \left[\cos(\omega_q t) \sigma_y - \sin(\omega_q t) \sigma_x \right].$$

RWA: we can dropp the fast rotating terms $(\omega_q + \omega_d)$ that will average to zero.

$$\tilde{H}_d = \frac{1}{2} \Omega V_0 s(t) \left[\left(-I \cos(\delta\omega t) + Q \sin(\delta\omega t) \right) \sigma_x + \right.$$
$$\boxed{\delta\omega = \omega_q - \omega_d} \qquad \left. + \left(I \sin(\delta\omega t) - Q \cos(\delta\omega t) \right) \sigma_y \right]$$

$$\tilde{H}_d = -\frac{\Omega}{2} V_0 S(t) \begin{bmatrix} 0 & e^{i(\delta\omega t + \phi)} \\ e^{-i(\delta\omega t + \phi)} & 0 \end{bmatrix}$$

This last expression for the driving Hamiltonian in the rotating frame is a powerful tool for understanding single qubit gates in superconducting qubits.

- Let's apply a pulse at the qubit frequency ω_q

$\delta\omega = 0$:
$$\tilde{H}_d = -\frac{\Omega}{2} V_0 S(t) \left(\underbrace{I}_{\cos(\phi)} \sigma_x + \underbrace{Q}_{\sin(\phi)} \sigma_y \right)$$

This shows that an in-phase pulse ($\phi = 0$, or the I-quadrature) gives a rotation around the x-axis; while an out-of-phase pulse ($\phi = \frac{\pi}{2}$, or the Q-quadrature) corresponds to a rotation around the y-axis.

- As a concrete example of an in-phase pulse, we can define the unitary operation
$$U_d^{\phi=0}(t) = \exp\left[\left(\frac{i}{2} \Omega V_0 \int_0^t S(t')\, dt' \right) \sigma_x \right]$$
which depends on the envelope of the pulse $S(t)$ and amplitude V_0 that are controlled using the AWG. This $U_d^{\phi=0}$ is known as Rabi driving.

(AWG = Arbitrary Waveform Generator)

We can define $\Theta(t) = -\Omega V_0 \int_0^t S(t')\,dt'$ representing the angle by which a state is rotated given the circuit parameters, the magnitude V_0 and the shape of the pulse envelope $S(t)$.

This means that to implement a π-pulse around the x-axis, one would solve the eq. $\Theta(t) = \pi$ and output the signal in-phase on the qubit driving line.

In this framework, a sequence of pulses $\Theta_k, \Theta_{k-1}, \dots, \Theta_0$ is converted to a sequence of gates operating on a qubit

$$U_k \cdots U_1 U_0 = \prod_{n=0}^{k} \exp\left[-\frac{i}{2}\Theta_n(t)\left(I_n \sigma_x + Q_n \sigma_y\right)\right]$$

Microwave pulses (I-Q) $\Longrightarrow$ rotations around X, Y, Z $\Longrightarrow$ Set of unitary gates

Multiple qubit states

Register of $n=2$ classical bits

BIT A	BIT B
0	0
0	1
1	0
1	1

2^n different states

NOTE: only 1 state is realized at any given time

Register of $n=2$ quantum bits

QUBIT A	QUBIT B
$\|0\rangle$	$\|0\rangle$
$\|0\rangle$	$\|0\rangle$
$\|1\rangle$	$\|1\rangle$
$\|1\rangle$	$\|1\rangle$

2^n basis states

BUT: quantum registers can be in any SUPERPOSITION of basis states

Formal description of general state of $n=2$ quantum register

$$|\psi\rangle = |A\rangle \otimes |B\rangle = |AB\rangle$$

e.g. $|A\rangle = \alpha_A |0\rangle + \beta_A |1\rangle \; ; \; |B\rangle = \alpha_B |0\rangle + \beta_B |1\rangle$

$$|\psi\rangle = \alpha_A \alpha_B |00\rangle + \alpha_A \beta_B |01\rangle + \beta_A \alpha_B |10\rangle + \beta_A \beta_B |11\rangle$$

with $\sum_{i,j} |\alpha_i \beta_j|^2 = 1$

Generalize to "n" qubits $|\psi\rangle = \sum_{i_1, \dots, i_N = 0}^{1} \alpha_{i_1, \dots, i_N} | i_1, \dots, i_N \rangle$ 2

$\hookrightarrow 2^n$ independent amplitudes

Register of N qubits:

- 2^N basis states
- general superposition state is described by 2^N complex coefficients

- For comparison: $2^{500} \gtrsim$ # of atoms in the universe
- Impossible to store this information classically!
- This is why it is difficult to simulate Q.M. on a classical computer.
- A quantum system, in principle, processes all these amplitudes in parallel $\Longrightarrow$ exploit for computation!

Two-qubit gates:

- product-gate : U, V single qubit gates

$U \otimes V$ simple single gates acting independently on each qubit.

Boring! This does not generate entanglement.

- Controlled gates: "if-then-else" type of operation

- CNOT GATE [CX] (addition MOD 2: $|A, A \oplus B\rangle$)

if qubit A (control) is in $|0\rangle \longrightarrow$ Do NOTHING on the TARGET

" " " " " " $|1\rangle \longrightarrow$ Do $\hat{X}$ gate on the TARGET

Action on the basis states: $|\overset{C}{0}\overset{T}{0}\rangle \longrightarrow |00\rangle$

$$CNOT = \begin{bmatrix} 1 & 0 & & \\ 0 & 1 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{bmatrix}$$

$|01\rangle \longrightarrow |01\rangle$

$|10\rangle \longrightarrow |11\rangle$

$|11\rangle \longrightarrow |10\rangle$

- CPHASE GATE [CZ]
- If qubit A (control) is in $|0\rangle$: Do NOTHING on target
  " " " " " " $|1\rangle$: Do $\hat{Z}$ GATE on "

Action on the basis states:

$$\begin{array}{c} \quad\ \ C\ T \\ |0\,0\rangle \longrightarrow |0\,0\rangle \end{array}$$

$$CZ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$|0\,1\rangle \longrightarrow |0\,1\rangle$$

$$|1\,0\rangle \longrightarrow |1\,0\rangle$$

$$|1\,1\rangle \longrightarrow -|1\,1\rangle$$

- Other 2-qubit gates
- SWAP: EXCHANGE excitations

$$SWAP = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$|0\,0\rangle \longleftrightarrow |0\,0\rangle$; $|1\,1\rangle \longleftrightarrow |1\,1\rangle$

$|0\,1\rangle \longleftrightarrow |1\,0\rangle$

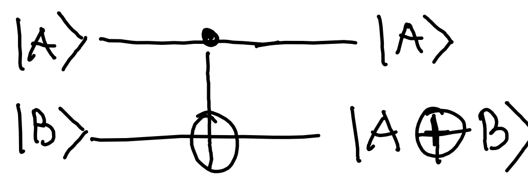
Does not create entanglement.

- iSWAP $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$; $\sqrt{SWAP} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2}(1+i) & \frac{1}{2}(1-i) & 0 \\ 0 & \frac{1}{2}(1-i) & \frac{1}{2}(1+i) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

<u>They create entanglement.</u>

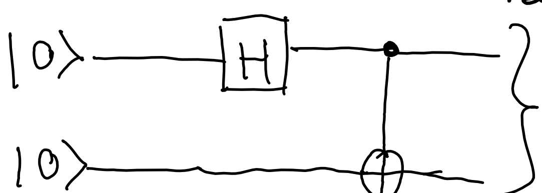
N.B. CNOT is a powerful entangling gate:

1 single qubit gate + CNOT create a fully entangled



- CNOT is reversible (unitary)
- " " universal
- " can be realized using any 2-qubit interaction combined with single qubit manipulation

BELL STATE



$$\frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

Universality of CNOT and single qubit rotations (6)

It can be shown that a $2^n \times 2^n$ unitary matrices, , representing the most general quantum computation gate on a n-qubits register, can be obtained by concatenating only single qubit rotations and two-qubit gates (like CNOT or CZ).

- In other words, any unitary matrix can be expressed exactly as a product of CNOT gates and single qubit rotations in between.
- The gain is that now the problem has been reduced to finding a dense subset of 2 qubit gates and 1 qubit gates, which are much simpler to implement.
- Example of universal set: $\left\{ T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}; H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}; C_Z = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix} \right.$

These gates have a very intuitive meaning:
T is chosen specifically to generate non-trivial phases;
the Hadamard gate creates superpositions; C_Z creates entanglement.

Comparison of Classical and Quantum logic:

- Overall, quantum circuits are realized by concatenating a given number of these gates; $\{Toff, H\}$ or $\{T, C_Z, H\}$, , the exact sequence being specified by the algorithm. We can see already from this that a classical computing is a subset of a quantum computer.
- When expressed in terms of the other boolean universal gate sets (AND, XOR, NOT) classical gates can combine a number of input bits into a single output bit. This classical computation so constructed is irreversible, , the input cannot be reconstructed from the output. Quantum gates instead are unitary! They conserve the number of qubits. Q.C. is reversible.

Measurements instead are irreversible! but they may always be moved at the end of the calculation.

OBSERVATIONS: UNIVERSAL SET of Quantum GATES (8)

Like in digital electronics you would like to operate with a minimal set of gates because it's easier to engineer them.

The most popular set of universal gate $\{H, T, S, CNOT\}$. Any unitary operator U on n qubits can be approximated to accuracy ε by finite sequence of universal gates.

This approximation is unfortunately not efficient. For an arbitrary U it takes this order $O[2^n \log(1/\varepsilon)/\log(n)]$ of resources to approximate it to an accuracy ε. It is an exponential amount of resources.

- The goal of Q.C. is to device specific quantum algorithms that are executed on circuits with at most a polynomial $O(poly(n))$ order in the number of qubits to carry out useful computational tasks.

This is the goal of quantum software engineers, to think of algorithms that execute useful computational tasks without requiring an exp. number of gates.

- Fun facts: $T^2 = S$, so why we have both T and S in the universal set? This is due to reasons that are related to quantum error correction. The gate T presents some difficulties with q. error correction. Whenever one can apply S as single operation, better to do it.

- The gates $\{H, S, CNOT\}$ are the generators of the Clifford group. This set is NOT UNIVERSAL. If you make circuits just out of those elementary gates you are unable to cover the all Hilbert Space of the system. You cannot generate any arbitrary quantum gate. In particular, there is the Gottesman-Knill theorem that tells you that any quantum circuit made of only Cliffor gates can be simulate efficiently on a classical computer.

This is really surprising because with those 3 gates you can generate high entangled states. In other words we are saying that entanglement does not necessarely mean heavy to simulate. This theorem shows that entanglement is responsible of the power of Q.C., but there must be something else.

Properties of classical computation:

1) bit can be copied (FANOUT)
2) additional coworking bits are allowed (ANCILLAS)
3) values of bits can be interchanged (crossOVER)
4) number of output bits may be smaller then input bits

- Which of these properties are preserved in the Q. context?

1) Copying of qubits is not possible! [NO CLONING THEOREM]
2) ancillas qubits are allowed
3) crossover ok $\Longleftrightarrow$ SWAP operation
4) number of OUTPUT qubits = number of INPUT qubits

Physical implementation of CNOT gate:

Let's consider 2 capacitively coupled superconducting charge qubits. Writing the Hamiltonian of the two qubit with $\hat{z}$ along the charge axis:

$$H_{tot} = H_{qubit\,1} + H_{qubit\,2} + H_{INT}$$

$$H_{qubit_1} = -\frac{E_{ch}^{1}(t)}{2}\left(\sigma_1^z \otimes \mathbb{I}_2\right) - \frac{E_J^1(t)}{2}\left(\sigma_1^x \otimes \mathbb{I}_2\right)$$

$$H_{qubit_2} = -\frac{E_{ch}^{2}(t)}{2}\left(\mathbb{I}_1 \otimes \sigma_2^z\right) - \frac{E_J^2(t)}{2}\left(\mathbb{I}_1 \otimes \sigma_2^x\right)$$

$$H_{INT} = \frac{E_{INT}(t)}{2}\left(\sigma_1^z \otimes \sigma_2^z\right)$$ [induced by the capacitive interaction]

It is possible to show that we can switch on this interaction term for a time giving the following transformation in the computational basis:

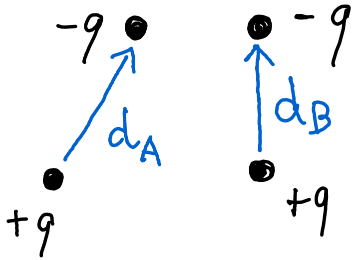
$$\begin{bmatrix} c^f_{00} \\ c^f_{01} \\ c^f_{10} \\ c^f_{11} \end{bmatrix} = \begin{bmatrix} e^{-i\delta/2} & & & \\ & e^{+i\delta/2} & & \\ & & e^{+i\delta/2} & \\ & & & e^{-i\delta/2} \end{bmatrix} \begin{bmatrix} c^i_{00} \\ c^i_{01} \\ c^i_{10} \\ c^i_{11} \end{bmatrix} =$$

$$= e^{-i\delta/2} \begin{bmatrix} 1 & & & \\ & e^{+i\delta} & & \\ & & e^{+i\delta} & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \psi \\ \\ \\ \\ \end{bmatrix} \quad \text{with } \delta = \frac{1}{\hbar}\int dt' E_{int}(t')$$

By accumulating $\delta = \frac{3}{2}\pi$ we get $e^{-i 3/4 \pi} \begin{bmatrix} 1 & & & \\ & -i & & \\ & & -i & \\ & & & 1 \end{bmatrix}$

and by applying some 1-qubit gates we can arrive at the CNOT gate.

TWO QUBIT GATE via DIPOLAR INTERACTION



ATOM A : $\{E_{0A}, E_{1A}\}$; $\{|0\rangle_A, |1\rangle_A\}$,

ATOM B : $\{E_{0B}, E_{1B}\}$; $\{|0\rangle_B, |1\rangle_B\}$

$$H_A = E_{0A}|0\rangle_A\langle 0|_A + E_{1A}|1\rangle_A\langle 1|_A =$$

$$= \begin{bmatrix} E_{0A} & 0 \\ 0 & E_{1A} \end{bmatrix} = \begin{bmatrix} \frac{E_{0A}-E_{1A}}{2} & 0 \\ 0 & \frac{E_{0A}-E_{10}}{2} \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{E_{0A}+E_{1A}}{2} & 0 \\ 0 & \frac{E_{0A}+E_{10}}{2} \end{bmatrix}}_{\text{Const. } \mathbb{I}}$$

$$H_A = -\frac{1}{2}\underbrace{(E_{1A}-E_{10})}_{\hbar\omega_A}\sigma_{zA}$$

Similarly $H_B = -\frac{\hbar}{2}\omega_B\sigma_{zB}$

(11)

$H_{INT} = -J\, \vec{d}_A \cdot \vec{d}_B$ dipole-dipole interaction

$$H_{INT} = -J \left[\sum_{ij} |i\rangle_A {}_A\langle i | \vec{d}_A | j \rangle_A {}_A\langle j| \right] \otimes \left[\sum_{ij} |i\rangle_B {}_B\langle i | \vec{d}_B | j\rangle_B {}_B\langle j| \right]$$

$\langle 0 | \vec{d} | 0 \rangle = 0 = \langle 1 | \vec{d} | 1 \rangle$ due to symmetry

in fact, $\vec{d} = \vec{r}\, q \Rightarrow q \langle 0 | \vec{r} | 0 \rangle = \int d^3 r\, \underbrace{e^{-r^2/a^2}}_{\text{even}}\, \underset{\text{odd}}{\vec{r}}\, e^{-r^2/a^2} = 0$

so $J\, {}_A\langle 0 | \vec{d}_A | 1 \rangle_A \, {}_B\langle 0 | \vec{d}_B | 1 \rangle_B \overset{\text{def}}{\equiv} \hbar g$

$$H_{INT} = -\hbar g \left[|0\rangle_A {}_A\langle 1| + |1\rangle_A {}_A\langle 0| \right] \otimes \left[|0\rangle_B {}_B\langle 1| + |1\rangle_B {}_B\langle 0| \right] =$$

$$= -\hbar g\, \sigma_{xA}\, \sigma_{xB}$$

so $H_{TOT} = -\frac{1}{2}\hbar \omega_A \sigma_{zA} - \frac{1}{2}\hbar\omega_B \sigma_{zB} - \hbar g\, \sigma_{xA}\sigma_{xB}$

Going in the rotating frame of the qubits + RWA

$$U = \exp[-i\omega_A t\, \sigma_{zA}/2] \otimes \exp[-i\omega_B t\, \sigma_{zB}/2]$$

we will get ...

$$\tilde{H} = -\frac{1}{4}\hbar g\left[e^{-i(\omega_A-\omega_B)}\sigma_{+A}\sigma_{-B} + e^{i(\omega_A-\omega_B)}\sigma_{-A}\sigma_{+B}\right]$$

$\sigma_+ = \sigma_x + i\sigma_y = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ and $\sigma_- = \sigma_x - i\sigma_y = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$

for $\omega_A = \omega_B$ (resonance) $\Rightarrow \tilde{H} = -\frac{1}{2}\hbar g\left[\sigma_{xA}\sigma_{xB} + \sigma_{yA}\sigma_{yB}\right]$

<u>XY coupling</u> very common in superconducting qubits.

- What kind of 2-qubit gate this H_{INT} generates?

$$U = \exp\left[-i\frac{\tilde{H}_{INT}}{\hbar}t\right] = \exp\{igt\sigma_{xA}/2\} \otimes \exp\{igt\sigma_{xB}/2\} \cdot \exp\{igt\sigma_{yA}/2\} \otimes \exp\{igt\sigma_{yB}/2\}$$

By using $\exp\{-igt\sigma_{xA}/2\} = \mathbb{I}\cos\left(\frac{gt}{2}\right) - i\sin\left(\frac{gt}{2}\right)\sigma_{xA}$

we can get $U = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(gt) & i\sin(gt) & 0 \\ 0 & i\sin(gt) & \cos(gt) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xRightarrow[t=\frac{\pi}{2g}]{\text{if}} U = \begin{bmatrix} 1 & & & \\ & 0 & i & \\ & i & 0 & \\ & & & 1 \end{bmatrix}$

$\boxed{i\,\text{SWAP gate}}$