Solid state systems for quantum information, Session 2

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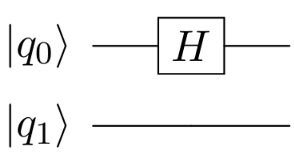
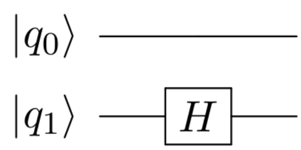
Exercise 1 : Quantum computation with 2 qubits

1. Matrix representation for 2 qubit gates
   A single qubit lies in a Hilbert space $\mathcal{H}_1$ and is represented by a state vector $|\psi_1\rangle = \alpha|0\rangle + \beta|1\rangle$. When we have two qubits we represent their Hilbert space as the tensor product of the constituent Hilbert spaces: $\mathcal{H}_1 \otimes \mathcal{H}_2$. Operators $\mathcal{O}$ acting on a single qubit can be represented by a 2×2 matrix. Operators acting on n qubits are given by a $2^n \times 2^n$ matrix. If we have two simultaneous single qubit operations we can represent their combined oeprator as $\mathcal{O}_1 \otimes \mathcal{O}_2$.

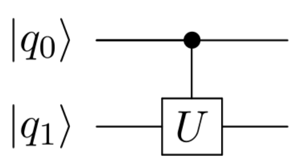
   (a) What are the 4×4 unitary matrices for the following circuits in the computational basis?



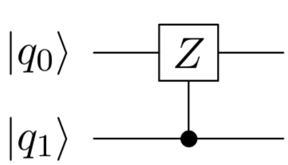
$$H \otimes I = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}$$

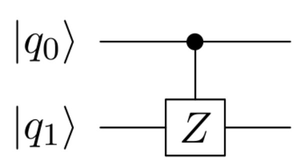


2. Control and target in two qubit gates
   Control gates are among the most common two qubit gates: in general they can be written as in the following picture, where the unitary operator acts on the state of the qubit $U|q_1\rangle$ only when the state of qubit 0 is $|1\rangle$.

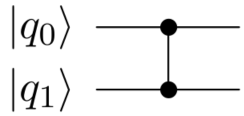


   (a) Show explicitly (e.g. by matrix multiplication) that the following two quantum circuits are equivalent





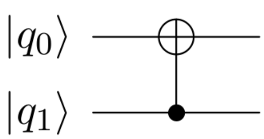
Note that to stress this symmetry the controlled-Z gate if normally written as

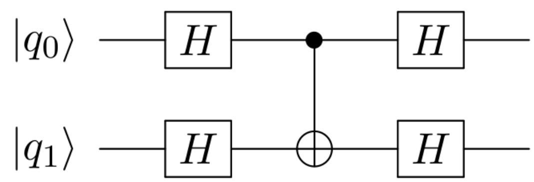


(b) Construct a CNOT gate from one controlled-Z gate and two Hadamard gates. Specify the control and target gates.

3. Change of basis
What will happen to the role of the control and the target qubit if we change basis? Note that in this question we use $\oplus$ to represent the X gate when controlled by another qubit.

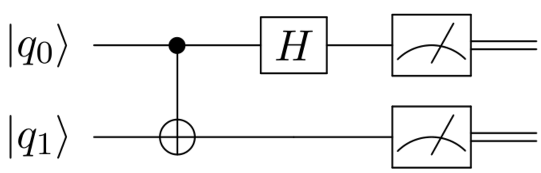
(a) Show that the following two circuits are equivalent (this time without using matrices!). Then use this fact to answer the following: in the basis $|\pm\rangle$ which qubit controls which in the CNOT gate ?





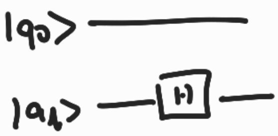
It is also possible to perform projective measurements in the other bases than just tje computational one.

4 (a) Show that this circuit performs in the basis of Bell states



5 Equivalent gates
Many times a single- and a two-qubit gate can be expressed the product of other gates. In this problem, we investigate a few examples for this. (Using the predefined gate in QuTip is encouraged)

1)



$I \otimes H$ or

$$\frac{1}{\sqrt{2}}\begin{array}{c} \\ 00 \\ 01 \\ 10 \\ 11 \end{array}\begin{array}{c} \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \end{array}$$

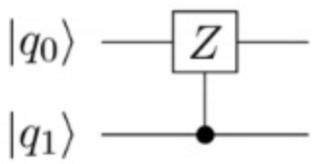
$|0\rangle|0\rangle \longrightarrow |0\rangle \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

$|0\rangle|1\rangle \longrightarrow |0\rangle \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

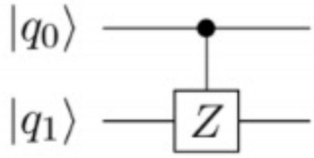
$|1\rangle|0\rangle \longrightarrow |1\rangle \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

$|1\rangle|1\rangle \longrightarrow |1\rangle \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

2)



Show that they're equal $\longrightarrow$



$|00\rangle \longrightarrow |00\rangle$

$|01\rangle \longrightarrow |01\rangle$

$|10\rangle \longrightarrow |10\rangle$

$|11\rangle \longrightarrow -|11\rangle$

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$$

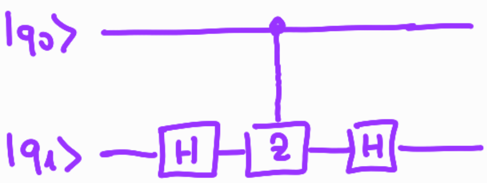
$|00\rangle \longrightarrow |00\rangle$

$|01\rangle \longrightarrow |01\rangle$

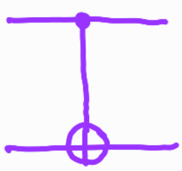
$|10\rangle \longrightarrow |10\rangle$

$|11\rangle \longrightarrow -|11\rangle$

(b) Construct a CNOT gate from one controlled-Z gate and two Hadamard gates. Specify the control and target gates.



$\equiv$



$|00\rangle \xrightarrow{H} |0\rangle \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \xrightarrow{Z} |0\rangle \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \xrightarrow{H} |00\rangle$

$|01\rangle \xrightarrow{H} |0\rangle \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \xrightarrow{Z} |0\rangle \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \xrightarrow{H} |01\rangle$

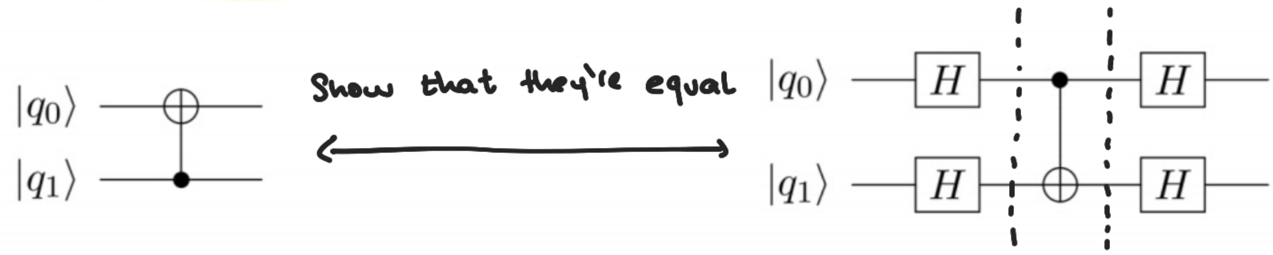
$|10\rangle \xrightarrow{H} |1\rangle \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \xrightarrow{Z} |1\rangle \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \xrightarrow{H} |11\rangle$

$|11\rangle \xrightarrow{H} |1\rangle \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \xrightarrow{Z} |1\rangle \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \xrightarrow{H} |10\rangle$

3. Change of basis

What will happen to the role of the control and the target qubit if we change basis? Note that in this question we use $\bigoplus$ to represent the X gate when controlled by another qubit.

(a) Show that the following two circuits are equivalent (this time without using matrices!). Then use this fact to answer the following: in the basis $|\pm\rangle$ which qubit controls which in the CNOT gate ?



$$|{++}\rangle = \frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right) \xrightarrow{\text{CNOT}} \frac{1}{2}\left(|00\rangle + |11\rangle + |10\rangle + |01\rangle\right) = |{++}\rangle$$

$$|{+-}\rangle = \frac{1}{2}\left(|00\rangle - |01\rangle + |10\rangle - |11\rangle\right) \xrightarrow{\text{CNOT}} \frac{1}{2}\left(|00\rangle - |01\rangle + |11\rangle - |10\rangle\right) = |{--}\rangle$$

$$|{-+}\rangle = \frac{1}{2}\left(|00\rangle + |01\rangle - |10\rangle - |11\rangle\right) \rightarrow \frac{1}{2}\left(|00\rangle + |01\rangle - |11\rangle - |10\rangle\right) = |{-+}\rangle$$

$$|{--}\rangle = \frac{1}{2}\left(|00\rangle - |01\rangle - |10\rangle + |11\rangle\right) \rightarrow \frac{1}{2}\left(|00\rangle - |01\rangle - |11\rangle + |10\rangle\right) = |{+-}\rangle$$

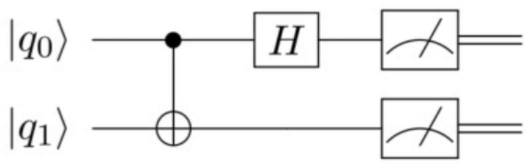
$$|00\rangle \xrightarrow{H_1 \otimes H_2} |{++}\rangle \xrightarrow{\text{CNOT}} |{++}\rangle \xrightarrow{H_1 \otimes H_2} |00\rangle$$

$$|01\rangle \longrightarrow |{+-}\rangle \longrightarrow |{--}\rangle \longrightarrow |11\rangle$$

$$|10\rangle \longrightarrow |{-+}\rangle \longrightarrow |{-+}\rangle \longrightarrow |10\rangle$$

$$|11\rangle \longrightarrow |{--}\rangle \longrightarrow |{+-}\rangle \longrightarrow |01\rangle$$

4 (a) Show that this circuit performs in the basis of Bell states



$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\overbrace{(|00\rangle + |10\rangle)}^{|+0\rangle} \xrightarrow{H\otimes I} |00\rangle$$

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\overbrace{(|00\rangle - |10\rangle)}^{|-0\rangle} \xrightarrow{H\otimes I} |10\rangle$$

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\overbrace{(|01\rangle + |11\rangle)}^{|+1\rangle} \xrightarrow{H\otimes I} |01\rangle$$

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\overbrace{(|01\rangle - |11\rangle)}^{|-1\rangle} \xrightarrow{H\otimes I} |11\rangle$$

5. (a) Show that for the single-qubit gates $X = HZH$.

(b) Show that the two-qubit entangling CNOT and CZ gates are related through the following relationship $\text{CNOT} = [\mathbb{1} \otimes R_y(\pi/2)] \cdot CZ \cdot [\mathbb{1} \otimes R_y(-\pi/2)]$

a) $$Z = HXH = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} = Z$$

$$HZH = \underbrace{HH}_{I} \times \underbrace{HH}_{I} \Rightarrow X = HZH$$

b) $$R_y(\overset{\pi/2}{\theta}) = \begin{pmatrix} \cos\overset{\pi/4}{\theta/2} & -\sin\overset{\pi/4}{\theta/2} \\ \sin\theta/2 & \cos\theta/2 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$\text{CNOT} = [I \otimes R_y(\tfrac{\pi}{2})] \cdot CZ \cdot [I \otimes R_y(-\tfrac{\pi}{2})]$$

$$\frac{1}{2}\begin{bmatrix} 1 & -1 & & \\ 1 & 1 & & \\ & & 1 & -1 \\ & & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 & & \\ -1 & 1 & & \\ & & 1 & 1 \\ & & -1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix} = \text{CNOT}$$

(a) Show that for the single-qubit gates $X = HZH$.

(b) Show that the two-qubit entangling CNOT and CZ gates are related through the following relationship CNOT $= [\mathbb{1} \otimes R_y(\pi/2)] \cdot CZ \cdot [\mathbb{1} \otimes R_y(-\pi/2)]$

Exercise 2 : Qubit in the presence of an Oscillatory field: The Rabi Hamiltonian

A qubit in the presence of an oscillatory driving field follows the following Hamiltonian.

$$\hat{H} = -\frac{\hbar\omega_0}{2}\hat{\sigma}_z + \hbar\Omega_1 \cos{(\omega t + \phi)}\hat{\sigma}_x. \tag{1}$$

The first term represents the qubit Hamiltonian, where $\pm\frac{\hbar\omega_0}{2}$ are the ground/excited state of the qubit and $\hat{\sigma}_z$ is the Pauli-Z operator. The second term represents the interaction of the qubit with the driving field, of amplitude $\hbar\Omega_1$, frequency ω and phase ϕ, and $\hat{\sigma}_x$ is the Pauli-X operator. In this exercise, we are going to investigate the solutions of this Hamiltonian.

✓ 1. Write the eigenvector of the system, $|\Psi(t)\rangle$, in the interaction picture as a function of ω_0, $c_g(t)$ the probability of being in the ground state and $c_e(t)$ the probability of being in the excited state.

✓ 2. Solve the time-dependent Schrödinger equation for the Hamiltonian of eq 1 using the eigenvector you found in the previous question. Apply the rotating-wave approximation. Write your results as a function of $c_g(t)$, $c_e(t)$, Ω_1, ϕ and the detuning $\Delta = \omega_0 - \omega$.
Reminder: In the rotating-wave approximation we get rid of the fast-oscillating terms and keep only the slowly-oscillating terms. This approximation is valid as the fast oscillating terms give a negligible average contribution.

✓ 3. Derive the general solution for the eigenvector of the system, $\Psi(t, \Delta)$, by solving the set of coupled differential equations.
Hint: Assume that you have oscillating solutions for the system of the type $a(t) = a(t_0)e^{i\lambda t}$

✓ 4. We are now interested in the problem where the qubit is initialized in the ground state, for $t_0 = 0$, we have: $c_g(0) = 1$, $c_e(0) = 0$ and $\phi = 0$.

✓ 5. Find the probability $P_{g\to e}(t, \Delta) = |\langle e|\,\Psi(t, \Delta)|^2$ of the system being in excited state with the system being initially in the ground state.

✓ 6. Interpretation of the previous question:

   (a) Plot $P_{g\to e}(t, \Delta)$ for $\Delta = 0$ and $\Delta \neq 0$, what is different between these two cases? For what values of t and Δ, $P_{g\to e}(t, \Delta)$ is maximum?

   (b) Plot $P_{g\to e}(t, \Delta)$ vs Δ for the case in which t was making $P_{g\to e}(t, \Delta)$ maximum in the previous question.

   (c) Bonus: Plot numerically $P_{g\to e}(t, \Delta)$ as a function of t in the interval $[0; 6\pi]$ and Δ in the interval $[-2; 2]$ for $\Omega_1 = 1$ and $\hbar = 1$.

In this part, we investigate the effects of the Rabi Hamiltonian on the Bloch sphere. In the laboratory frame, the qubit will always precess around the z axis at a frequency ω_0. In order to visualize better the effect of the Rabi Hamiltonian we need to work in the rotating frame of the qubit, using the rotation matrix, $\hat{R}_z(\delta) = e^{-i\frac{\delta}{2}t\hat{\sigma}_z}$.

✔ 7. Rewrite the rotation matrix, $\hat{R}_z(\delta) = e^{-i\frac{\delta}{2}t\hat{\sigma}_z}$, in a full matrix form.
Hint: $\hat{\sigma}_z^{2n} = \mathbb{1}$ and $\hat{\sigma}_z^{2n+1} = \hat{\sigma}_z$.

✔ 8. Rewrite the Hamiltonian of equation 1 in the rotating frame as:

$$\hat{H}' = \hat{R}_z(-\omega_0)\hat{H}\hat{R}_z^\dagger(-\omega_0) + i\hbar\frac{d\hat{R}_z(-\omega_0)}{dt}\hat{R}_z^\dagger(-\omega_0). \tag{2}$$

Apply the rotating-wave approximation and assume that the driving field is in resonance with the qubit, i.e. $\omega_0 = \omega$.

9. Using the results of exercise 1 of last week, what would you do to perform experimentally rotation operations on the x, y and z axes of the Bloch sphere?

10. Draw on the Bloch sphere the process of question 6.(a) for $\Delta = 0$. What do you think is happening on the Bloch sphere for $\Delta \neq 0$?

?

1. Write the eigenvector of the system, $|\Psi(t)\rangle$, in the interaction picture as a function of ω_0, $c_g(t)$ the probability of being in the ground state and $c_e(t)$ the probability of being in the excited state.

excited state?

interaction btw qubit and the driving field

$$\hat{H} = -\frac{\hbar\omega_0}{2}\hat{\sigma}_z + \hbar\Omega_1\cos(\omega t+\phi)\hat{\sigma}_x$$

$$\hat{H} = \hat{H}_0 + \hat{H}_S$$

Since we're in the interaction picture, only diagonalize H_0

$$\lambda = \pm\frac{\hbar\omega_0}{2}$$

$$\lambda = \frac{\hbar\omega_0}{2} \longrightarrow |g\rangle = \begin{pmatrix}1\\0\end{pmatrix}$$

$$\lambda = -\frac{\hbar\omega_0}{2} \longrightarrow |e\rangle = \begin{pmatrix}0\\1\end{pmatrix}$$

$$|\psi(t)\rangle = \sum_k c_k(t)\, e^{-i\frac{E_k t}{\hbar}} |k\rangle = \begin{pmatrix} c_g(t)\, e^{i\frac{\omega_0 t}{2}} \\ c_e(t)\, e^{-\frac{i\omega_0 t}{2}} \end{pmatrix}$$

2. Solve the time-dependent Schrödinger equation for the Hamiltonian of eq 1 using the eigenvector you found in the previous question. Apply the rotating-wave approximation. Write your results as a function of $c_g(t)$, $c_e(t)$, Ω_1, ϕ and the detuning $\Delta = \omega_0 - \omega$.
Reminder: In the rotating-wave approximation we get rid of the fast-oscillating terms and keep only the slowly-oscillating terms. This approximation is valid as the fast oscillating terms give a negligible average contribution.

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle$$

$$i\cancel{\hbar}\begin{pmatrix} \dot{c}_g(t)\, e^{\frac{i\omega_0 t}{2}} + i\frac{\omega_0}{2} c_g(t)\, e^{\frac{i\omega_0 t}{2}} \\ \dot{c}_e(t)\, e^{-\frac{i\omega_0 t}{2}} - i\frac{\omega_0}{2} c_e(t)\, e^{-\frac{i\omega_0 t}{2}} \end{pmatrix} = \cancel{\hbar}\begin{pmatrix} -\frac{\omega_0}{2} & \Omega_1\cos(\omega t+\phi) \\ \Omega_1\cos(\omega t+\phi) & \frac{\omega_0}{2} \end{pmatrix}\begin{pmatrix} c_g(t)\, e^{\frac{i\omega_0 t}{2}} \\ c_e(t)\, e^{-\frac{i\omega_0 t}{2}} \end{pmatrix}$$

$$i\dot{c}_g(t)\, e^{\frac{i\omega_0 t}{2}} - \cancel{\frac{\omega_0}{2} c_g(t)\, e^{\frac{i\omega_0 t}{2}}} = \cancel{-\frac{\omega_0}{2} c_g(t)\, e^{\frac{i\omega_0 t}{2}}} + \Omega_1 c_e(t)\cos(\omega t+\phi)\, e^{-\frac{i\omega_0 t}{2}}$$

$$\dot{c}_g(t) = -i\,\Omega_1 \cos(\omega t+\phi)\, c_e(t)\, e^{-i\omega_0 t} \qquad \cos(\omega t+\phi) \to \frac{\left(e^{i(\omega t+\phi)} + e^{-i(\omega t+\phi)}\right)}{2}$$

$$\dot{c}_e(t) = -i\,\Omega_1 \cos(\omega t+\phi)\, c_g(t)\, e^{i\omega_0 t}$$

Why did we find H_{rot} before? Why? In Quantum Optics

$\Delta = \omega - \omega_0$

$$\dot{c}_g(t) = -i\frac{\Omega_1}{2} c_e(t)\left(e^{i\phi}\cdot e^{i\underbrace{(\omega-\omega_0)}_{-\Delta}t} + e^{-i\phi}\cdot \cancel{e^{-i(\omega+\omega_0)t}}\right) \quad \text{RWA}$$

$$\dot{c}_e(t) = -i\frac{\Omega_1}{2} c_g(t)\left(e^{i\phi}\, \cancel{e^{i(\omega+\omega_0 t)}} + e^{-i\phi}\, e^{-i\overbrace{(\omega-\omega_0)}^{-\Delta}t}\right) \quad \text{RWA}$$

$$\dot{c}_g(t) \simeq -i\frac{\Omega_1}{2} c_e(t)\, e^{i\phi}\, e^{-i\Delta t} \quad (1)$$

$$\dot{c}_e(t) \simeq -i\frac{\Omega_1}{2} c_g(t)\, e^{-i\phi}\, e^{i\Delta t} \quad (2)$$

3. Derive the general solution for the eigenvector of the system, $\Psi(t,\Delta)$, by solving the set of coupled differential equations.
Hint: Assume that you have oscillating solutions for the system of the type $a(t) = a(t_0)e^{i\lambda t}$

Take derivative of Eq (2) and insert it to the Eq (1)

$$\ddot{c}_e(t) = -i\frac{\Omega_1}{2}\left(\dot{c}_g(t)\, e^{-i\phi} e^{i\Delta t} + c_g(t)\, e^{-i\phi}(i\Delta)\, e^{i\Delta t}\right)$$

$$\left[\frac{2\ddot{c}_e(t)}{-i\Omega_1} - c_g(t)(i\Delta)e^{-i\phi}\cdot e^{i\Delta t}\right] e^{i\phi} e^{-i\Delta t} = \dot{c}_g(t)$$

$$\frac{2\dot{c}_e}{-i\Omega_1} e^{+i\phi}\cdot e^{-i\Delta t}$$

$$\frac{2\ddot{c}_e(t)}{-i\Omega_1} e^{i\phi} e^{-i\Delta t} - (i\Delta)\, c_g(t) = \dot{c}_g(t)$$

$$\dot{c}_g(t) = \frac{2}{-i\Omega_1}\cancel{e^{i\phi}}\,\cancel{e^{-i\Delta t}}\left[\ddot{c}_e(t) - (i\Delta)\dot{c}_e(t)\right] = -\frac{i\Omega_1}{2} c_e(t)\,\cancel{e^{+i\phi}}\,\cancel{e^{-i\Delta t}}$$

$$\left[\ddot{c}_e(t) - (i\Delta)\dot{c}_e(t)\right] = \left(\frac{-i\Omega_1}{2}\right)^2 \cdot c_e(t)$$

$$\boxed{\ddot{c}_e(t) - (i\Delta)\dot{c}_e(t) + \frac{\Omega_1^2}{4} c_e(t) = 0}$$

Use the ansatz: $c_e(t) = c_e(t_0)\, e^{i\lambda t}$

$$\dot{c}_e(t) = c_e(t_0)(i\lambda)\, e^{i\lambda t} = (i\lambda)\, c_e(t)$$

$$\ddot{c}_e(t) = c_e(t_0)(i\lambda)(i\lambda)\, e^{i\lambda t} = (i\lambda)^2 c_e(t) = -\lambda^2 c_e(t)$$

$$-\lambda^2 \cancel{c_e(t)} - (i\Delta)(i\lambda)\cancel{c_e(t)} + \frac{\Omega_1^2}{4}\cancel{c_e(t)} = 0$$

$$-\lambda^2 + \Delta\lambda + \frac{\Omega_1^2}{4} = 0$$

$$\lambda^2 - \Delta\lambda - \frac{\Omega_1^2}{4} = 0 \qquad \lambda_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a}$$

$$\lambda_\pm = \frac{1}{2}\left(\Delta \pm \sqrt{\Delta^2 + \cancel{4}\cdot 1\left(\frac{-\Omega_1^2}{\cancel{4}}\right)}\right)$$

$$\lambda_\pm = \frac{1}{2}\left(\Delta \pm \Omega\right) \quad \text{where} \quad \Omega = \sqrt{\Delta^2 + \Omega_1^2}$$

HOW ?

generalized Rabi frequency

$$\boxed{c_e(t) = c_+(t_0)\, e^{i\lambda_+ t} + c_-(t_0)\, e^{i\lambda_- t}} \quad (3)$$

From Eq 2, $c_g(t) = -\frac{2}{i\Omega_1} e^{i\phi} e^{-i\Delta t}\left[i\lambda_+ c_+(t_0)\, e^{i\lambda_+ t} + i\lambda_- c_-(t_0)\, e^{i\lambda_- t}\right]$ (4)

4. We are now interested in the problem where the qubit is initialized in the ground state, for $t_0 = 0$, we have: $c_g(0) = 1$, $c_e(0) = 0$ and $\phi = 0$.

$$c_g(0) = 1 = -\frac{2}{\cancel{i}\Omega_1} e^0 \cdot e^0 \left[\cancel{i}\lambda_+ C_+(t_0) e^0 + \cancel{i}\lambda_- C_-(t_0) e^0 \right]$$

$$\left.\begin{aligned} 1 &= \frac{-2}{\Omega_1}\left(\lambda_+ C_+ + \lambda_- C_- \right) \\ c_e(0) = 0 &= C_+ + C_- \end{aligned}\right\}$$

where

$$\lambda_\pm = \frac{1}{2}(\Delta \pm \Omega)$$

$$\lambda_+ \underbrace{C_+}_{-C_-} + \lambda_- C_- = -\frac{\Omega_1}{2}$$

$$(\Delta - \Omega) - (\Delta + \Omega)$$

$$\cancel{\Delta} - \Omega - \cancel{\Delta} - \Omega$$

$$C_-(\lambda_- - \lambda_+) = -\frac{\Omega_1}{2}$$

$$\begin{cases} C_- = -\frac{\Omega_1}{2} \cdot \frac{1}{(\lambda_- - \lambda_+)} = \cancel{-}\frac{\Omega_1}{\cancel{2}} \cdot \frac{\cancel{2}}{\cancel{-}2\Omega} = +\frac{\Omega_1}{2\Omega} \\ C_+ = -C_- = -\frac{\Omega_1}{2\Omega} \end{cases}$$

Plug these into Eq (3), (4)

$$C_e(t) = -\frac{\Omega_1}{2\Omega} e^{i\frac{\Delta}{2}t}\left(e^{i\frac{\Omega}{2}t} - e^{-i\frac{\Omega}{2}t} \right) = -i\frac{\Omega_1}{\Omega} e^{i\frac{\Delta}{2}t} \sin\left(\frac{\Omega}{2}t\right)$$

$$c_g(t) = e^{i\phi} e^{i\frac{\Delta}{2}t}\left(i\frac{\Delta}{\Omega}\sin\left(\frac{\Omega}{2}t\right) + \cos\left(\frac{\Omega}{2}t\right) \right)$$

5. Find the probability $P_{g\to e}(t,\Delta) = |\langle e|\,\Psi(t,\Delta)|^2$ of the system being in excited state with the system being initially in the ground state.

$$|\Psi(t)\rangle = \begin{pmatrix} c_g(t)\, e^{i\frac{\omega_0 t}{2}} \\ c_e(t)\, e^{-\frac{i\omega_0 t}{2}} \end{pmatrix}$$

$$P = \left| \overset{\langle e|}{(0 \;\; 1)} \begin{pmatrix} c_g(t)\, e^{i\omega_0 t/2} \\ c_e(t)\, e^{-\frac{i\omega_0 t}{2}} \end{pmatrix} \right|^2 = \left| c_e(t)\, e^{-i\omega_0 t/2} \right|^2$$

$$P_{g\to e}(t,\Delta) = |c_e(t)|^2 = \left| \cancel{-i}\,\frac{\Omega_1}{\Omega}\, \cancel{e^{\frac{i\Delta t}{2}}} \sin\left(\frac{\Omega}{2}t\right)\right|^2$$

$$\boxed{P_{g\to e}(t,\Delta) = \left(\frac{\Omega_1}{\Omega}\right)^2 \sin^2\left(\frac{\Omega}{2}t\right)}$$

6. Interpretation of the previous question:

(a) Plot $P_{g\to e}(t,\Delta)$ for $\Delta = 0$ and $\Delta \neq 0$, what is different between these two cases? For what values of t and Δ, $P_{g\to e}(t,\Delta)$ is maximum?

(b) Plot $P_{g\to e}(t,\Delta)$ vs Δ for the case in which t was making $P_{g\to e}(t,\Delta)$ maximum in the previous question.

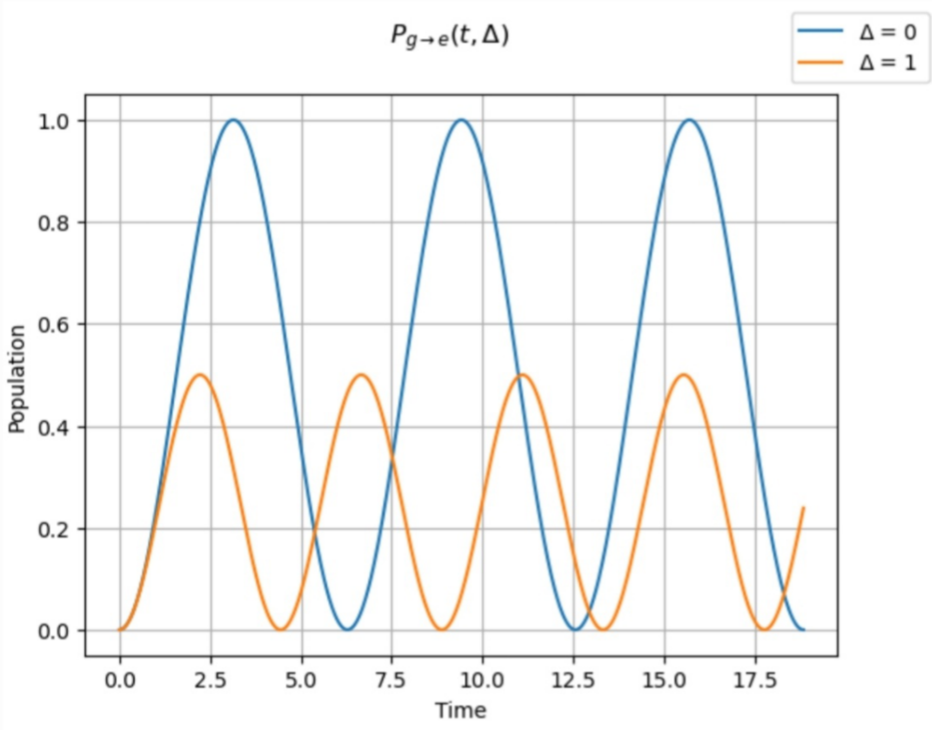
(c) Bonus: Plot numerically $P_{g\to e}(t,\Delta)$ as a function of t in the interval $[0;6\pi]$ and Δ in the interval $[-2;2]$ for $\Omega_1 = 1$ and $\hbar = 1$.

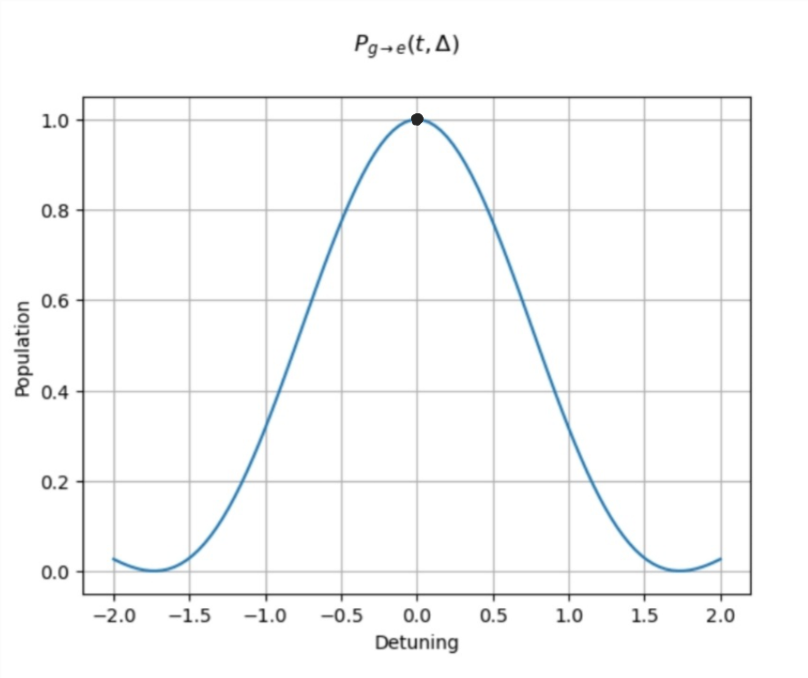
$$\Omega = \sqrt{\Delta^2 + \Omega_1^2}$$

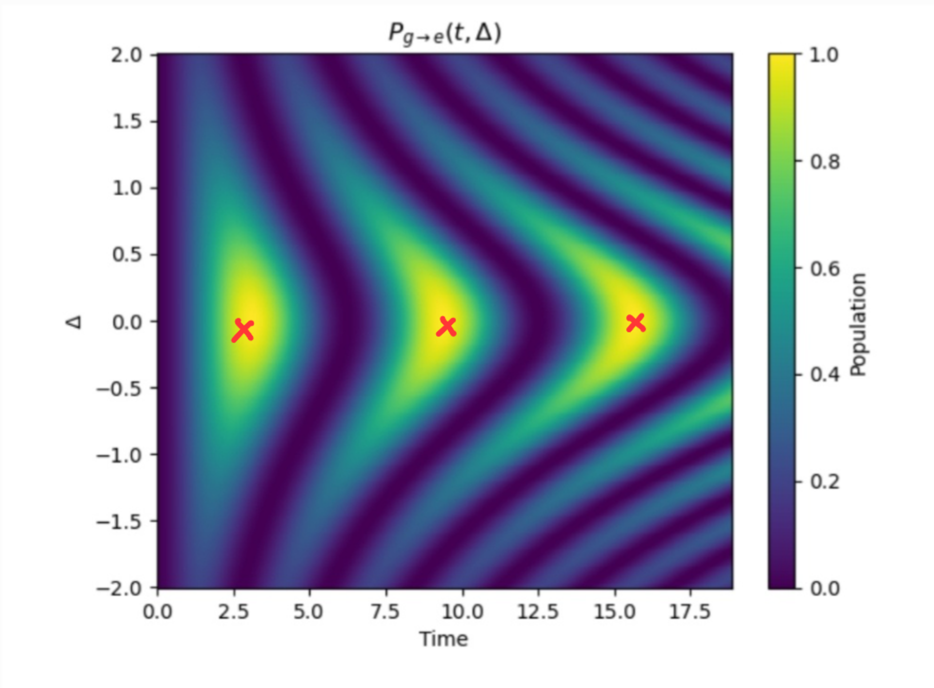
for $\Delta = 0 \longrightarrow \quad \Omega = \Omega_1$

$$P_{g\to e}(t,0) = \left(\cancel{\frac{\Omega_1}{\Omega_1}}\right)^{\cancel{2}} \cdot \sin^2\left(\frac{\Omega_1 t}{2}\right)$$

$$P_{g\to e}(t,\Delta\neq 0) = \frac{\Omega_1^2}{\Delta^2 + \Omega_1^2} \sin^2\left(\frac{\Omega_1 + \Delta^2}{2}t\right)$$


$P_{g\to e}(t,\Delta)$
$\Delta = 0$
$\Delta = 1$
Population
Time
1.0
0.8
0.6
0.4
0.2
0.0
0.0
2.5
5.0
7.5
10.0
12.5
15.0
17.5


$P_{g\to e}(t,\Delta)$
Population
Detuning
1.0
0.8
0.6
0.4
0.2
0.0
−2.0
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
2.0


$P_{g\to e}(t,\Delta)$
Δ
Time
Population
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
0.0
2.5
5.0
7.5
10.0
12.5
15.0
17.5
1.0
0.8
0.6
0.4
0.2
0.0

In this part, we investigate the effects of the Rabi Hamiltonian on the Bloch sphere. In the laboratory frame, the qubit will always precess around the z axis at a frequency ω_0. In order to visualize better the effect of the Rabi Hamiltonian we need to work in the rotating frame of the qubit, using the rotation matrix, $\hat{R}_z(\delta) = e^{-i\frac{\delta}{2}t\hat{\sigma}_z}$.

7. Rewrite the rotation matrix, $\hat{R}_z(\delta) = e^{-i\frac{\delta}{2}t\hat{\sigma}_z}$, in a full matrix form.
   Hint: $\hat{\sigma}_z^{2n} = \mathbb{1}$ and $\hat{\sigma}_z^{2n+1} = \hat{\sigma}_z$.

$$\hat{R}_z(\delta) = e^{-i\frac{\delta}{2}t\sigma_z} = \sum_n \frac{\left(-i\frac{\delta}{2}t\sigma_z\right)^n}{n!}$$

$$= \mathbb{I} + \left(-i\frac{\delta}{2}t\right)\sigma_z + \frac{\left(-i\frac{\delta}{2}t\right)^2}{2}\underbrace{\sigma_z^2}_{\mathbb{I}} + \frac{\left(-i\frac{\delta}{2}t\right)^3}{3!}\sigma_z$$

$$\hat{R}_z(\delta) = \cos\left(\frac{\delta}{2}t\right)\mathbb{I} - i\sin\left(\frac{\delta}{2}t\right)\sigma_z = \begin{pmatrix} \overbrace{\cos\left(\frac{\delta}{2}t\right) - i\sin\left(\frac{\delta}{2}t\right)}^{e^{-i\frac{\delta}{2}t}} & 0 \\ 0 & \underbrace{\cos\left(\frac{\delta}{2}t\right) + i\sin\left(\frac{\delta}{2}t\right)}_{e^{i\frac{\delta}{2}t}} \end{pmatrix}$$

$$\hat{R}_z(\delta) = \begin{pmatrix} e^{-i\frac{\delta}{2}t} & 0 \\ 0 & e^{i\frac{\delta}{2}t} \end{pmatrix}$$

8. Rewrite the Hamiltonian of equation 1 in the rotating frame as:

$$\hat{H}' = \hat{R}_z(-\omega_0)\hat{H}\hat{R}_z^\dagger(-\omega_0) + i\hbar\frac{d\hat{R}_z(-\omega_0)}{dt}\hat{R}_z^\dagger(-\omega_0). \tag{2}$$

Apply the rotating-wave approximation and assume that the driving field is in resonance with the qubit, i.e. $\omega_0 = \omega$.

$H_{rot} = UHU^\dagger + i\hbar\dot{U}U^\dagger$ where $\hat{U} = \hat{R}_z(-\omega_0) \overset{?}{=} \hat{R}_z(\omega)$

$$\hat{H} = -\frac{\hbar\omega_0}{2}\hat{\sigma}_z + \hbar\Omega_1\cos(\omega t + \phi)\hat{\sigma}_x$$

$$\hat{H} = \begin{pmatrix} -\frac{\hbar\omega_0}{2} & +\hbar\Omega_1\cos(\omega t + \phi) \\ \hbar\Omega_1\cos(\omega t + \phi) & +\frac{\hbar\omega_0}{2} \end{pmatrix}$$

$$\hat{R}_z(\omega) = \begin{pmatrix} e^{-i\omega t/2} & 0 \\ 0 & e^{+i\omega t/2} \end{pmatrix} \qquad \frac{d}{dt}\hat{R}_z(\omega_0) = \begin{pmatrix} -\frac{i\omega}{2} e^{-i\omega t/2} & 0 \\ 0 & +\frac{i\omega_0}{2} e^{+i\omega t/2} \end{pmatrix}$$

$$\hat{R}_z^+(\omega) = \begin{pmatrix} e^{+i\omega t/2} & 0 \\ 0 & e^{-i\omega t/2} \end{pmatrix}$$

$$\hat{R}_z(-\omega_0)\,\hat{H}\,\hat{R}_z^+(\omega_0) = \begin{pmatrix} e^{-i\omega t/2} & 0 \\ 0 & e^{+i\omega t/2} \end{pmatrix} \begin{pmatrix} -\frac{\hbar\omega_0}{2} & +\hbar\Omega_1\cos(\omega t+\phi) \\ \hbar\Omega_1\cos(\omega t+\phi) & +\frac{\hbar\omega_0}{2} \end{pmatrix} \hat{R}_z^+(\omega_0)$$

$$= \begin{pmatrix} e^{-i\omega t/2}\left(-\frac{\hbar\omega_0}{2}\right) & e^{-i\omega t/2}\left(\hbar\Omega_1\cos(\omega t+\phi)\right) \\ e^{+i\omega t/2}\left(\hbar\Omega_1\cos(\omega t+\phi)\right) & e^{+i\omega t/2}\left(\frac{\hbar\omega_0}{2}\right) \end{pmatrix} \begin{pmatrix} e^{+i\omega t/2} & 0 \\ 0 & e^{-i\omega t/2} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{\hbar\omega_0}{2} & e^{-i\omega t}\left(\hbar\Omega_1\cos(\omega t+\phi)\right) \\ e^{+i\omega t}\left(\hbar\Omega_1\cos(\omega t+\phi)\right) & \frac{\hbar\omega_0}{2} \end{pmatrix}$$

$$i\hbar\,\dot{\hat{R}}_z(\omega_0)\,\hat{R}_z^+(\omega_0) = i\hbar \begin{pmatrix} \frac{i\omega}{2} e^{i\omega t/2} & 0 \\ 0 & -\frac{i\omega}{2} e^{-i\omega t/2} \end{pmatrix} \begin{pmatrix} e^{-i\omega t/2} & 0 \\ 0 & e^{+i\omega t/2} \end{pmatrix}$$

$$= i\hbar \begin{pmatrix} \frac{i\omega}{2} & 0 \\ 0 & -\frac{i\omega}{2} \end{pmatrix} = \begin{pmatrix} +\frac{\hbar\omega}{2} & 0 \\ 0 & -\frac{\hbar\omega}{2} \end{pmatrix}$$

$$\hat{H}' = \begin{pmatrix} \cancel{-\frac{\hbar\omega_0}{2}} & e^{-i\omega t}\left(\hbar\Omega_1\cos(\omega t+\phi)\right) \\ e^{+i\omega t}\left(\hbar\Omega_1\cos(\omega t+\phi)\right) & \cancel{\frac{\hbar\omega_0}{2}} \end{pmatrix} + \begin{pmatrix} \cancel{+\frac{\hbar\omega}{2}} & 0 \\ 0 & \cancel{-\frac{\hbar\omega}{2}} \end{pmatrix}$$

$$\downarrow$$

$$e^{+i\omega t}\left(\hbar\Omega_1\,\frac{e^{i(\omega t+\phi)}+e^{-i(\omega t+\phi)}}{2}\right) \qquad e^{-i\omega t}\cdot e^{-i\phi}$$

$$\hat{H}_{rot} = \frac{\hbar\Omega_1}{2}\begin{pmatrix} 0 & e^{i\phi}\underbrace{e^{0}}_{1} + e^{-i\phi}\underbrace{\cancel{e^{-2i\omega t}}}_{RWA} \\ e^{i\phi}\underbrace{\cancel{e^{2i\omega t}}}_{RWA} + e^{-i\phi}\cdot\underbrace{e^{0}}_{1} & 0 \end{pmatrix}$$

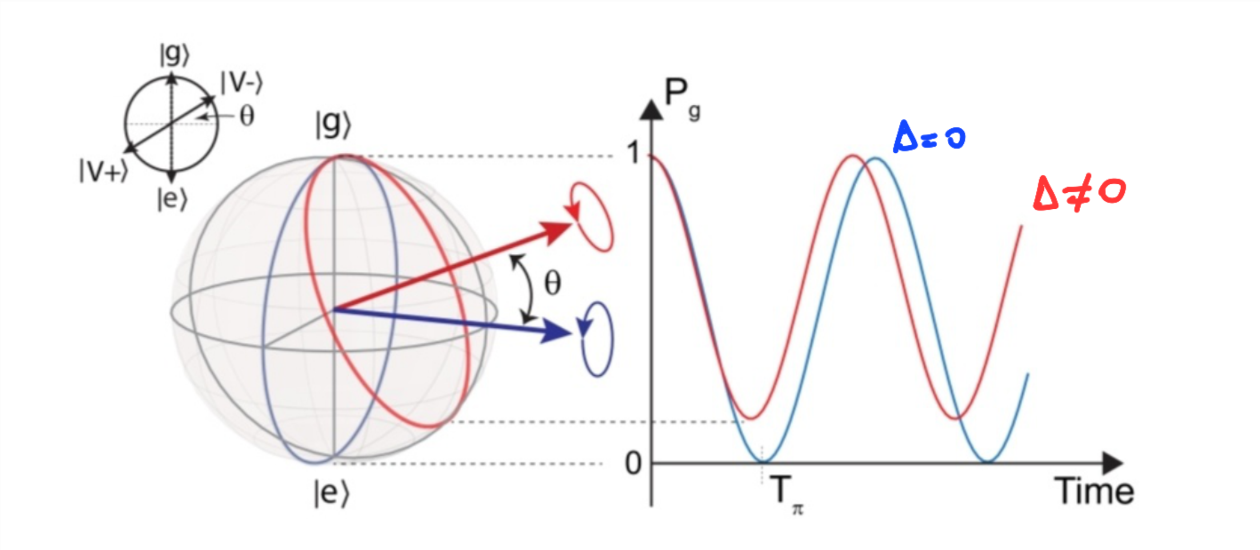
$$\hat{H}_{rot} = \frac{\hbar\Omega_1}{2}\begin{pmatrix} 0 & e^{i\phi} \\ e^{-i\phi} & 0 \end{pmatrix} = \boxed{\frac{\hbar\Omega_1}{2}\left[\cos(\phi)\hat{\sigma}_x - \sin(\phi)\hat{\sigma}_y\right]}$$

9. Using the results of exercise 1 of last week, what would you do to perform experimentally rotation operations on the x, y and z axes of the Bloch sphere?

10. Draw on the Bloch sphere the process of question 6.(a) for $\Delta = 0$. What do you think is happening on the Bloch sphere for $\Delta \neq 0$?

9. To perform rotations around the x and y axis, one can control the phase of the signal ϕ.

10.



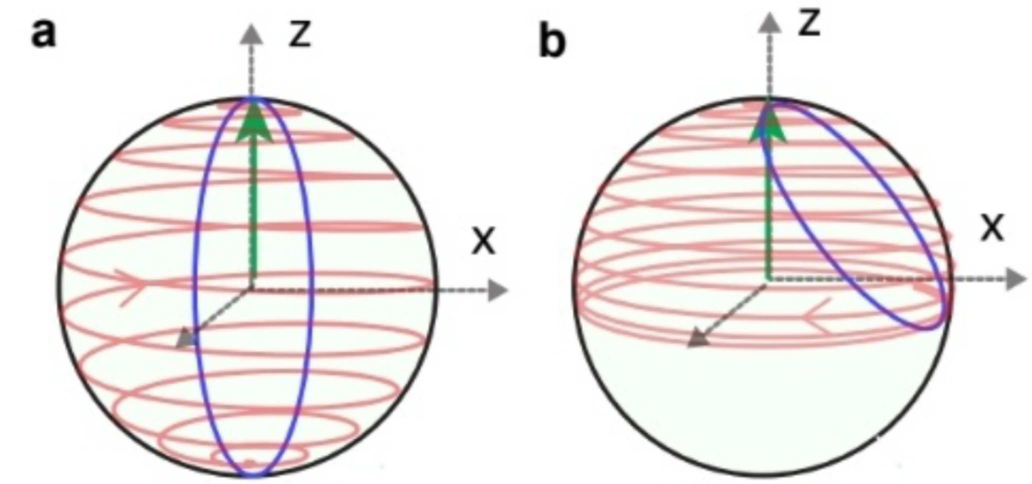


Figure 2.15: **Driven qubit state evolution in the Bloch sphere:** The red (blue) line shows the evolution of a driven qubit in the lab frame (rotating frame of the drive) **a**, for an on-resonant drive and **b**, for a detuned drive.

* Note for the first solution method which was solving the Schrödinger Equation for this time-dependent Hamiltonian

where $A = Ed$ quantifies how strong the interaction is. Now we want to know how the qubit evolves under this Hamiltonian. There are couple of ways we may solve this Hamiltonian. The first way is to solve the Schrödinger equation for this time-dependent Hamiltonian. We start with an *ansatz* instead of starting from scratch. The idea is that if we have no electric field or turn off the interaction, then we know the solution for Hamiltonian (2.4.2) would be $|\psi\rangle = C_g|g\rangle + C_e|e\rangle$ and its time evolution would be $|\psi(t)\rangle = C_g e^{+i\frac{\omega_q}{2}}|g\rangle + C_e e^{-i\frac{\omega_q}{2}}|e\rangle$. Now, we hope to find the solutions for (2.4.2) in the form of,

$$|\psi(t)\rangle = C_g(t)e^{+i\frac{\omega_q}{2}}|g\rangle + C_e(t)e^{-i\frac{\omega_q}{2}}|e\rangle, \tag{2.4.3}$$

Reference: Introduction to Experimental Quantum Measurement with Superconducting Qubits