CORRECTION OF ENSEMBLE AVERAGED DENSITY-OPERATOR

$$H(t) = -\frac{1}{2}\mu\left[B(t)(\sigma_z^A + \sigma_z^B) + b_A(t)\,\sigma_z^A + b_B(t)\,\sigma_z^B\right]$$

$$\langle B(t)\rangle = 0 \qquad \langle b_i(t)\rangle = 0$$

$$\langle B(t)B(t')\rangle = \frac{\Gamma}{N^2}\,\delta(t-t') \qquad \langle b_i(t)\,b_i'(t)\rangle = \frac{\Gamma_i}{N^2}\,\delta(t-t')$$

$$U(t) = \exp\left[-i\int_0^t dt'\,H(t')\right]$$

$$U(t) = \underbrace{e^{-i\left(\frac{\mu}{2}\right)\int_0^t dt'\left[B(t)+b_A(t)\right]\sigma_z^A\otimes I^B}}_{e^{-iK\sigma_z^A\otimes I^B}}\;\underbrace{e^{-i\left(\frac{\mu}{2}\right)\int_0^t dt'\left[B(t)+b_B(t)\right] I^A\otimes\sigma_z^B}}_{e^{-iL\,I^A\otimes\sigma_z^B}}$$

$$e^{-iL\,I^A\otimes\sigma_z^B} \downarrow \quad I^A\otimes e^{-iL\sigma_z^B}$$

$$L = -i\left(-\frac{\mu}{2}\right)\int_0^t dt'\left[B(t)+b_B(t)\right]$$

$$e^{-iK\sigma_z^A\otimes I^B} = \sum_{n=0}^{\infty}\frac{(-ik)^n}{n!}\underbrace{(\sigma_z^A\otimes I^B)^n}_{(\sigma_z^A)^n\otimes\underbrace{(I^B)^n}_{I}}$$

$$= \underbrace{\left(\sum_{n=0}^{\infty}\frac{(-ik)^n}{n!}(\sigma_z^A)^n\right)}_{e^{-iK\sigma_z^A}}\otimes I$$

$$e^{-iK\sigma_z^A}\otimes I$$

$$K = -i\left(-\frac{\mu}{2}\right)\int_0^t dt'\left(B(t)+b_A(t)\right)$$

$$e^{-ik\sigma_z^A} \otimes I = \left[\cos(k) I^A + i\sin(k)\sigma_z^A \right] \otimes I^B$$

$$\begin{bmatrix} \overbrace{\cos(k) + i\sin k}^{e^{ik}} & 0 \\ 0 & \underbrace{\cos k - i \sin k}_{e^{-ik}} \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} e^{ik} & & & 0 \\ & e^{ik} & & \\ & & e^{-ik} & \\ 0 & & & e^{-ik} \end{bmatrix}_{4\times4}$$

$$I^A \otimes e^{-iL\sigma_z^B} = I^A \otimes \left[\cos(L) I^B + i\sin(L)\sigma_z^B \right]$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} e^{iL} & 0 \\ 0 & e^{-iL} \end{bmatrix}$$

$$= \begin{bmatrix} e^{iL} & & & 0 \\ & e^{-iL} & & \\ & & e^{iL} & \\ 0 & & & e^{-iL} \end{bmatrix}_{4\times4}$$

$$U(t) = \begin{bmatrix} e^{i(k+L)} & & & \\ & e^{i(k-L)} & & \\ & & e^{-i(k-L)} & \\ & & & e^{-i(k+L)} \end{bmatrix}_{4\times4}$$

$$U^{\dagger}(t) = \begin{bmatrix} e^{-i(k+L)} & & & \\ & e^{-i(k-L)} & & \\ & & e^{+i(k-L)} & \\ & & & e^{+i(k+L)} \end{bmatrix}$$

$$\rho_{st}(t) = U(t)\,\rho(0)\,U^{\dagger}(t)$$

$$\rho_{st}(t) = \begin{bmatrix} \rho_{11} & \rho_{12}e^{i2L} & \rho_{13}e^{i2k} & \rho_{14}e^{i2(k+L)} \\ \rho_{21}e^{-i2L} & \rho_{22} & \rho_{23}e^{i2(k-L)} & \rho_{24}e^{i2k} \\ \rho_{31}e^{-i2k} & \rho_{32}e^{-i(k-L)} & \rho_{33} & \rho_{34}e^{i2L} \\ \rho_{41}e^{-i2(k+L)} & \rho_{42}e^{-i2k} & \rho_{43}e^{-i2L} & \rho_{44} \end{bmatrix}$$

Now compute $\langle\langle\langle \rho_{st}(t) \rangle\rangle\rangle$

Characteristic Function of Gaussian Stochastic Process

$$E\left(e^{i\int \varphi(s)\,X(s)\,ds}\right) = e^{+\frac{1}{2}\iint \varphi(s)\,C(s-s')\,\varphi(s')\,ds\,ds'}$$

where $E(X(s)) = 0$

$E(X(s)\,X(s')) = C(s-s')$ (Autocorrelation)

$$L = -i\left(-\frac{\mu}{2}\right)\int_0^t dt'\left[B(t') + b_B(t')\right]$$

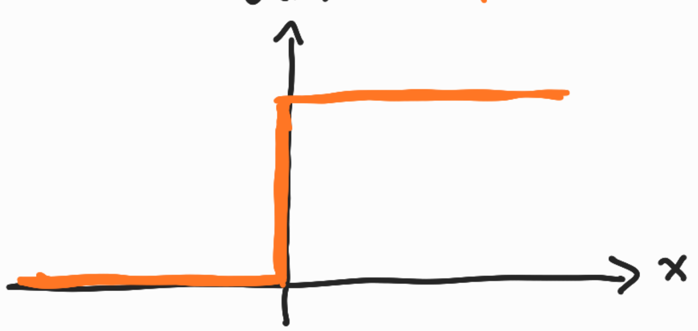
How

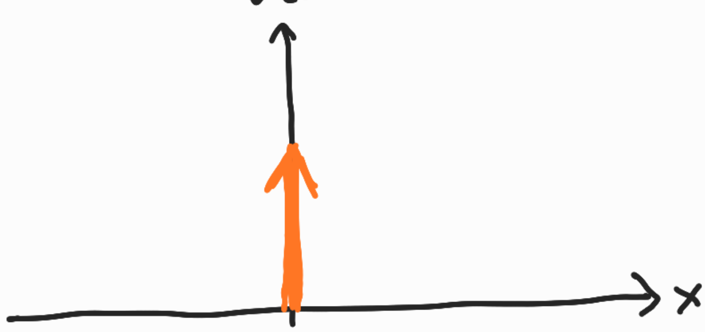
$$e^{i2L} = e^{\,i\int_0^t \underbrace{(i\mu)}_{p(s)}\underbrace{B(t')}_{X(s)}\,dt'} \cdot e^{\,i\int_0^t (i\mu)\, b_B(t')\,dt'}$$

$$e^{+\frac{1}{2}\iint (i\mu)\langle B(t'), B(t'')\rangle (i\mu)\, dt'dt''} \times e^{+\frac{1}{2}\iint (i\mu)\langle b_B(t'), b_B(t'')\rangle (i\mu)\, dt'dt''}$$

$$e^{-\frac{1}{2}\iint (i\mu)\underbrace{\langle B(t'), B(t'')\rangle}_{\frac{\Gamma}{\mu^2}\delta(t'-t'')} (i\mu)\, dt'dt''} = e^{-\frac{1}{2}\mu^2\frac{\Gamma}{\mu^2}\underbrace{\int_0^t\int_0^t \delta(t'-t'')\,dt'dt''}_{t}} = e^{-\frac{1}{2}\Gamma t}$$

Step Function





$$\int_0^t dt' \int_0^t dt'' \underbrace{\delta(t''-t')}_{\frac{d}{dt''}\theta(t''-t')}$$

$$= \int_0^t dt' \left[\underbrace{\theta(t-t')}_{1} - \underbrace{\theta(0-t')}_{0} \right] = \int_0^t dt' = t$$

$$e^{i2Z} = e^{i\int_0^t (i\mu) B(t')\, dt'} \cdot e^{i\int_0^t (i\mu) b_B(t')\, dt'}$$

$$e^{i2Z} = e^{-\frac{1}{2}\Gamma t} \cdot e^{-\frac{1}{2}\Gamma_B t} = e^{-\frac{1}{2}(\Gamma+\Gamma_B)t}$$

Using Characteristic Function, the density operator can be obtained as below:

$$
\begin{bmatrix}
\rho_{11} & \rho_{12}\, e^{-\frac{t(\Gamma+\Gamma_B)}{2}} & \rho_{13}\, e^{-\frac{t(\Gamma+\Gamma_A)}{2}} & \rho_{14}\, e^{-\frac{t(\Gamma_A+\Gamma_B)}{2}-2t\Gamma} \\
\rho_{21}\, e^{-\frac{t(\Gamma+\Gamma_B)}{2}} & \rho_{22} & \rho_{23}\, e^{-\frac{t}{2}(\Gamma_A+\Gamma_B)} & \rho_{24}\, e^{-\frac{t}{2}(\Gamma+\Gamma_A)} \\
\rho_{31}\, e^{-\frac{t(\Gamma+\Gamma_A)}{2}} & \rho_{32}\, e^{-\frac{t(\Gamma_A+\Gamma_B)}{2}} & \rho_{33} & \rho_{34}\, e^{-\frac{t}{2}(\Gamma+\Gamma_B)} \\
\rho_{41}\, e^{-\frac{t(\Gamma+\Gamma_A)}{2}-2t\Gamma} & \rho_{42}\, e^{-\frac{t}{2}(\Gamma+\Gamma_A)} & \rho_{43}\, e^{-\frac{t}{2}(\Gamma+\Gamma_B)} & \rho_{44}
\end{bmatrix}
$$

APPENDIX

where does it come from?

$$E\left(e^{i\int \varphi(s)\, X(s)\, ds}\right) = e^{-\frac{1}{2}\iint \varphi(s)\, C(s-s')\, \varphi(s')\, ds\, ds'}$$

Gaussian random process
$\neq$
Characteristic function of the Gaussian probability density:

probability density

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

characteristic function

$$\tilde{p}(k) = \langle e^{ikx} \rangle = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} dx\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, e^{ikx}$$

$$\tilde{p}(k) = e^{-\frac{1}{2}k^2\sigma^2 + ik\mu}$$

Gaussian probability density: Probability distribution of a random variable

Gaussian random process: Any finite number of random variables has a Joint Gaussian distribution. It's fully described by its mean function and covariance function.

Characteristic Function of Gaussian Random Process:

Let $X(t)$ be a Gaussian random process with mean function $\mu(t)$ and covariance function $K(s,t)$ where s and t are in time domain

$$\Phi(t) = E\left[e^{i\int m(s)ds + i\iint k(s,t)\,dt\,ds} \right]$$

or $$\Phi(t) = e^{i\mu(t)t - \frac{1}{2}\iint \overbrace{K(s,t)}^{\text{covariance}}\,ds\,dt}$$

Note: $\underbrace{C_{xx}(t_1,t_2)}_{\text{covariance}} = \underbrace{R_{xx}(t_1,t_2)}_{\text{autocorrelation}} - \mu_x(t_1)\mu_x^*(t_2)$