

Week 4 Particle Decay Rates

Cross Sections & Decay Rates

Fermi's Golden Rule

$$\Gamma_{fi} = 2\pi |\mathcal{T}_{fi}|^2 \rho(E_f)$$

\downarrow
of transitions per unit time from $|i\rangle$ to final $|f\rangle$

\mathcal{T}_{fi} : transition matrix element with perturbing Hamiltonian \hat{H}

$$\mathcal{T}_{fi} = \langle f | \hat{H} | i \rangle + \sum_{j \neq i} \frac{\langle f | \hat{H} | j \rangle \langle j | \hat{H} | i \rangle}{E_i - E_j} + \dots$$

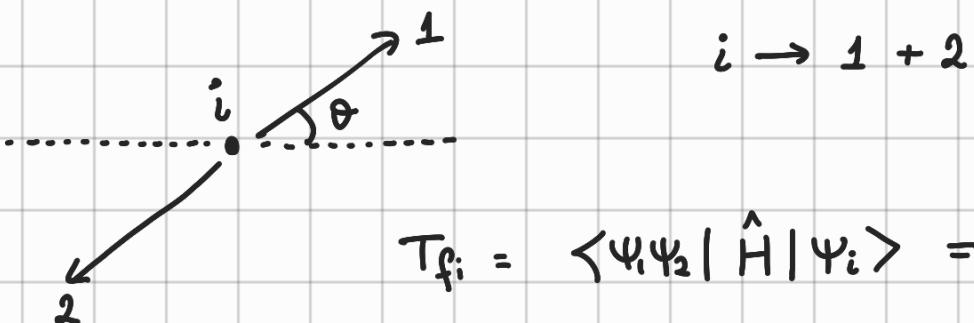
↳ intermediate particle j

Phase Space and wavefunction normalisation

Non-relativistic treatment

$\rho(E_f)$: density of final states

$$\left| \frac{dn}{dE} \right|_{E_f}$$



$$\mathcal{T}_{fi} = \langle \Psi_1 \Psi_2 | \hat{H} | \Psi_i \rangle = \int_V \Psi_1^* \Psi_2^* \hat{H} \Psi_i d^3x$$

Let's calculate the decay rate in first order perturbation theory using plane-wave description of the particles
 → Born approxim.

$$\Psi_1 = N e^{i(\vec{p} \cdot \vec{r} - Et)}$$

$$= N e^{-i(p_x x - Et)}$$

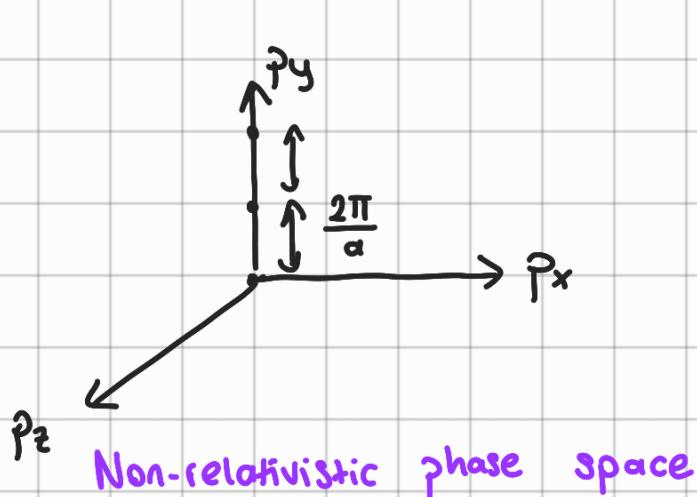
→ Normalization



$$\int \int \int \Psi^* \Psi dx dy dz = 1$$

Volume a^3

Normalization $\rightarrow \int \Psi \Psi^* dV = N^2 a^3 = 1 \Rightarrow N^2 = 1/a^3 = 1/\sqrt{a^3}$



$$p_x = \frac{2\pi n_x}{a}$$

$$p_y = \frac{2\pi n_y}{a}$$

$$p_z = \frac{2\pi n_z}{a}$$

$$\Psi(x+a, y, z)$$

↓
Periodic wavefunction

$$\Psi(x, y, z)$$

Volume of single state in momentum space:

$$\left(\frac{2\pi}{a}\right)^3 = \frac{(2\pi)^3}{V}$$

$$dn = \frac{d^3 \vec{p}}{(2\pi)^3 / V} \times \frac{1}{V} = \frac{d^3 \vec{p}}{(2\pi)^3} = \frac{dp_x dp_y dp_z}{(2\pi)^3}$$

of states
momentum

Magnitude of momentum

$$dn = 4\pi p^2 dp \frac{V}{(2\pi)^3}$$

$$\frac{dn}{dp} = \frac{4\pi p^2}{(2\pi)^3} V$$

$$\rho(E) = \left| \frac{dn}{dE} \right| = \left| \frac{dn}{dp} \right| \left| \frac{dp}{dE} \right|$$

$$dn_i = \frac{d^3 p_i}{(2\pi)^3}$$

$$dn = \prod_{i=1}^{N-1} dn_i = \prod_{i=1}^{N-1} \frac{d^3 p_i}{(2\pi)^3}$$

$$dn = \prod_{i=1}^{N-1} \frac{d^3 p_i}{(2\pi)^3} \delta^3 \left(\vec{p}_a - \sum_{i=1}^N \vec{p}_i \right) d^3 p_N$$

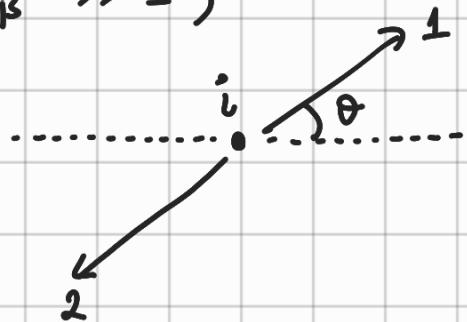
$$dn = (2\pi)^3 \prod_{i=1}^N \frac{d^3 p_i}{(2\pi)^3} \delta^3 \left(\vec{p}_a - \sum_{i=1}^N \vec{p}_i \right)$$

General non-relativistic expr. for N-body phase space

$$E^2 = p^2 + m^2$$

$$\hookrightarrow 2E dE = 2p dp$$

$$\frac{dp}{dE} = \frac{\bar{E}}{p} \approx \frac{1}{\beta} \quad (\beta \gg 1)$$



$$\rho(E_f) = \frac{4\pi p^2}{(2\pi)^3} \times \frac{1}{\beta}$$

energy conversion ↑

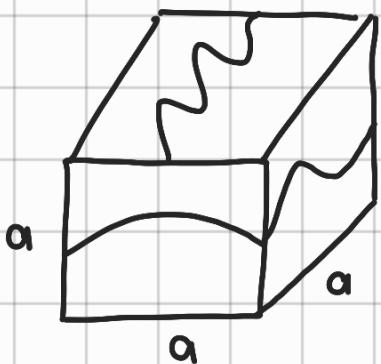
$$\rho(E_f) = \left| \frac{dn}{dE} \right|_{E_f} = \int \frac{dn}{dE} \delta(E - E_i) dE \quad \text{since } E_f = E_i$$

$$\Gamma_{fi} = 2\pi \int |\tau_{fi}|^2 \delta(E_i - E) dn$$

$$\Gamma_{fi} = (2\pi)^4 \int |\tau_{fi}|^2 \delta(E_i - E_1 - E_2) \delta^3(\vec{p}_i - \vec{p}_1 - \vec{p}_2) \frac{d^3 p_1}{(2\pi)^3} \frac{d^3 p_2}{(2\pi)^3}$$

↑ for x, y, z
E conservation → conservation density of states

Lorentz Invariant Phase Space



Particle density therefore increases by $\gamma = E/m$

Usual convention: Normalize to $2E$ particles/unit volume $\int \Psi^* \Psi dV = 2E$

$$\Psi' = (2E)^{1/2} \Psi$$

→ Lorentz invariant matrix element, M_{fi}

$$M_{fi} = \langle \Psi_1' \Psi_2' \dots | \hat{H} | \dots \Psi_{n-1}' \Psi_n' \rangle$$

$$= (2E_1 2E_2 \dots 2E_{n-1} 2E_n)^{1/2} T_{fi}$$

$$M_{fi} = \langle \Psi_1' \Psi_2' | \hat{H} | \Psi_i' \rangle = (2E_1 2E_2 2E_3)^{1/2} \langle \Psi_1 \Psi_2 | \hat{H} | \Psi_i \rangle$$

$$= (2E_1 2E_2 2E_3)^{1/2} T_{fi}$$

|

$$\Gamma_{fi} = \frac{(2\pi)^4}{2E_i} \int |M_{fi}|^2 \delta(E_i - E_1 - E_2) \delta^3(\vec{p}_i - \vec{p}_1 - \vec{p}_2) \frac{d^3 p_1}{(2\pi)^3 2E_1} \frac{d^3 p_2}{(2\pi)^3 2E_2}$$

$\frac{d^3 \vec{p}}{(2\pi)^3 2E}$ \rightarrow Lorentz Invariant Phase Space

Two Body Decay

Since this integral Lorentz invariant

It can be evaluated in any frame

In Center of Mass (COM) frame: $E^2 = M^2 + \vec{p}^2$ ($c = 1$)

$$\epsilon_i = M_i, p_i = 0$$

$$\Gamma_{fi} = \frac{1}{8\pi^2 E_i} \int |M_{fi}|^2 \delta(M_i - E_1 - E_2) \delta^3(\vec{p}_1 + \vec{p}_2) \frac{d^3 \vec{p}_1}{2E_1} \frac{d^3 \vec{p}_2}{2E_2}$$

Integrate over p_2 using δ function $\rightarrow p_2 = -p_1$

$$\Gamma_{fi} = \frac{1}{8\pi^2 E_i} \int |M_{fi}|^2 \delta(M_i - E_1 - E_2) \frac{d^3 p_1}{4E_1 E_2}$$

$$E_1^2 = M_1^2 + |\vec{p}_1|^2$$

$$E_2^2 = M_2^2 + |\vec{p}_2|^2$$

$$|\vec{p}_2|^2 d\vec{p}_1 \underbrace{\sin\theta d\theta d\phi}_{d\Omega}$$

$$(p_1 = -p_2)$$

$$\Gamma_{fi} = \frac{1}{8\pi^2 \bar{\epsilon}_i} \int |M_{fi}|^2 \delta(m_i - \sqrt{m_1^2 + p_1^2} - \sqrt{m_2^2 + p_1^2}) \frac{p_1^2 dp_1 d\Omega}{\bar{\epsilon}_1 \bar{\epsilon}_2}$$

↓ Write in this form

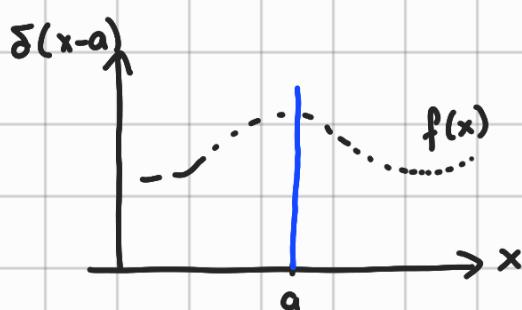
$$\Gamma_{fi} = \frac{1}{32\pi^2 \bar{\epsilon}_i} \int |M_{fi}|^2 g(p_1) \delta(f(p_1)) dp_1 d\Omega$$

$$g(p_1) = \frac{p_1^2}{\bar{\epsilon}_1 \bar{\epsilon}_2} = p_1^2 \frac{1}{\sqrt{(m_1^2 + p_1^2)} \cdot \sqrt{(m_2^2 + p_1^2)}}$$

$$f(p_1) = m_i - (m_1^2 + p_1^2)^{1/2} - (m_2^2 + p_1^2)^{1/2}$$

$\therefore f(p_1) = 0$ for $p_1 = -p_2 = p^*$ Q.M.

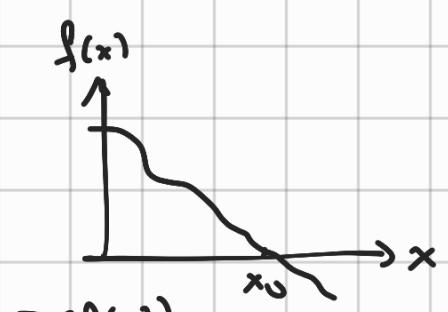
Dirac δ -function Review



$$\int_{-\infty}^{\infty} \delta(x-a) dx = 1$$

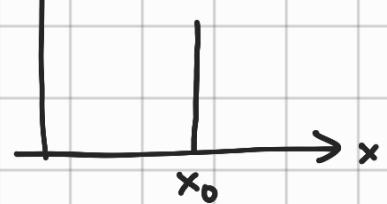
$$\int_{-\infty}^{\infty} f(x) \delta(x-a) dx = f(a)$$

$$\int_{y_1}^{y_2} \delta(y) dy = \begin{cases} 1 & \text{if } y_1 < 0 < y_2 \\ 0 & \text{otherwise} \end{cases}$$



$$y = f(x)$$

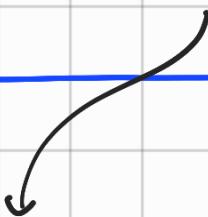
$$\int_{x_1}^{x_2} \delta(f(x)) \frac{df}{dx} dx = \begin{cases} 1 & \text{if } x_1 < x_0 < x_2 \\ 0 & \text{otherwise} \end{cases}$$



$$\left| \frac{df}{dx} \right| \int_{x_1}^{x_2} \delta(f(x)) dx = \begin{cases} 1 & \text{if } x_1 < x_0 < x_2 \\ 0 & \text{otherwise} \end{cases}$$

$$\int_{x_1}^{x_2} \delta(f(x)) dx = -\frac{1}{\left| \frac{df}{dx} \right|_{x_0}} \int_{x_1}^{x_2} \delta(x-x_0) dx$$

$$\delta(f(x)) = \left| \frac{df}{dx} \right|_{x_0}^{-1} \delta(x-x_0)$$



Using this equation :

$$\begin{aligned} \int g(p_1) \delta(f(p_1)) dp_1 &= \frac{1}{\left| \frac{df}{dp_1} \right|_{p^*}} \int g(p_1) \delta(p-p^*) dp_1 \\ &= \frac{g(p^*)}{\left| \frac{df}{dp_1} \right|_{p^*}} \quad \text{where } f(p^*)=0 \end{aligned}$$

Let's evaluate $\frac{df}{dp_1}$

$$\frac{df}{dp_1} = -\frac{p_1}{(N_1^2 + p_1^2)^{1/2}} - \frac{p_1}{(N_2^2 + p_1^2)^{1/2}} = -\frac{p_1}{E_1} - \frac{p_1}{E_2} = -p_1 \frac{E_1 + E_2}{E_1 E_2}$$

$$\Gamma_{fi} = \frac{1}{32\pi^2 E_i} \int |M_{fi}|^2 \left| \frac{\left| \frac{df}{dp_1} \right|_{p^*}^{-1}}{\frac{E_1 E_2}{p_1(E_1+E_2)}} \frac{p_1^2}{\frac{E_1 E_2}{p_1=E_2}} \right| d\Omega$$

$$f_{fi} = \frac{1}{32\pi^2 E_i} \int |N_{fi}|^2 \left| \frac{p_1}{\epsilon_1 + \epsilon_2} \right| d\Omega$$

\downarrow

$\epsilon_1 = \epsilon^*$
 m_i

$$f_{fi} = \frac{|\vec{p}^*|}{32\pi^2 \epsilon_i m_i} \int |N_{fi}|^2 d\Omega$$

ϵ_i

$\frac{1}{T} = f_{fi} = \frac{|\vec{p}^*|}{32\pi^2 m_i^2} \int |N_{fi}|^2 d\Omega$

Notes on " $\frac{d^3 \vec{p}}{(2\pi)^3 2E}$ is Lorentz Invariant"

Consider a Lorentz transformation along the z-axis

$$d^3 p' = dp'_x dp'_y dp'_z = dp_x dp_y \frac{dp'_z}{dp_z} dp_z = \frac{dp'_z}{dp_z} d\vec{p}$$

From Einstein energy-momentum relation:

$$E^2 = p_x^2 + p_y^2 + p_z^2 + m^2$$

$$p_z' = \gamma(p_z - \beta E) \rightsquigarrow \frac{dp'_z}{dp_z} = \gamma \left(1 - \beta \frac{\partial E}{\partial p_z} \right) = \gamma \left(1 - \beta \frac{p_z}{E} \right)$$

$$E' = \gamma(E - \beta p_z)$$

$$E = \gamma(E' + \beta p_z')$$

$$\frac{\partial E}{\partial p_z} = \frac{\partial E}{\partial p_z'} \frac{\partial p_z'}{\partial p_z} = \gamma \beta \cdot \gamma = \gamma^2 \beta \frac{p_z}{E}$$

$\frac{dp'_z}{dp_z} = \frac{E'}{E} \rightsquigarrow \frac{d^3 p'}{E'} = \frac{d^3 p}{E}$