Homework 4
Quantum Information Processing

What does it mean to have a vector consisting of matrices?

Exercise 1 *Properties of Pauli matrices*

We collect useful properties of Pauli matrices. Let $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ a vector formed by the 3 Pauli matrices :

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The identity matrix is denoted $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

a) Show that all 2×2 matrices, A, can be written as a linear combination of I and σ_x, σ_y, σ_z :

$$A = a_0 I + a_1 \sigma_x + a_2 \sigma_y + a_3 \sigma_z.$$

This can also be written as $A = a_0 I + \vec{a} \cdot \vec{\sigma}$ where $\vec{a} \cdot \vec{\sigma}$ is an "inner product" between the "vectors" $\vec{a} = (a_1, a_2, a_3)$ et $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$.

Check also that if $A = A^\dagger$ we have a_0, a_1, a_2, $a_3 \in \mathbb{R}$.

b) Check the following algebraic identities :

$$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = I$$
$$\sigma_x \sigma_y = i\sigma_z$$
$$\sigma_y \sigma_z = i\sigma_x$$
$$\sigma_z \sigma_x = i\sigma_y$$

Deduce

$$\sigma_x \sigma_y + \sigma_y \sigma_x = 0$$
$$\sigma_y \sigma_z + \sigma_z \sigma_y = 0$$
$$\sigma_z \sigma_x + \sigma_x \sigma_z = 0$$

c) Let $[A, B] = AB - BA$ be the "commutator". Show (you may use preceding results)

$$[\sigma_x, \sigma_y] = 2i\sigma_z$$
$$[\sigma_y, \sigma_z] = 2i\sigma_x$$
$$[\sigma_z, \sigma_x] = 2i\sigma_y$$

These relations are called "commutation relations for spin".

d) Compute eigenvalues and eigenvectors of σ_x, σ_y, σ_z. Check that the eigenvalues satisfy $\mathrm{Tr}\,\sigma_x = \mathrm{Tr}\,\sigma_y = \mathrm{Tr}\,\sigma_z = 0$ et $\det\sigma_x = \det\sigma_y = \det\sigma_z = -1$.

✓ e) Dirac notation : set

$$|\uparrow\rangle = \begin{pmatrix}1\\0\end{pmatrix} \text{ et } |\downarrow\rangle = \begin{pmatrix}0\\1\end{pmatrix}$$

Check that

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}|\uparrow\rangle\langle\uparrow| + a_{12}|\uparrow\rangle\langle\downarrow| + a_{21}|\downarrow\rangle\langle\uparrow| + a_{22}|\downarrow\rangle\langle\downarrow|$$

$\langle\uparrow|A|\uparrow\rangle = a_{11}$, $\langle\uparrow|A|\downarrow\rangle = a_{12}$, $\langle\downarrow|A|\uparrow\rangle = a_{21}$, $\langle\downarrow|A|\downarrow\rangle = a_{22}$

$\begin{pmatrix}1\\0\end{pmatrix}(1\ 0)$ $\begin{pmatrix}0\\1\end{pmatrix}(0\ 1)$

$$\sigma_z = |\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow| = \begin{pmatrix}1&0\\0&0\end{pmatrix} - \begin{pmatrix}0&0\\0&1\end{pmatrix} = \begin{pmatrix}1&0\\0&-1\end{pmatrix} = \sigma_z$$

$$\sigma_x = |\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow| = \begin{pmatrix}0&1\\0&0\end{pmatrix} + \begin{pmatrix}0&0\\1&0\end{pmatrix} = \begin{pmatrix}0&1\\1&0\end{pmatrix} = \sigma_x$$

$$\sigma_y = i|\downarrow\rangle\langle\uparrow| - i|\uparrow\rangle\langle\downarrow| = i\begin{pmatrix}0&0\\1&0\end{pmatrix} - i\begin{pmatrix}0&1\\0&0\end{pmatrix} = \begin{pmatrix}0&-i\\i&0\end{pmatrix} = \sigma_y$$

Exercise 2 *Exponentials of Pauli matrices*

a) We define the exponential of a matrix A by (for $t \in \mathbb{R}$)

$$e^{tA} = \sum_{n=0}^{\infty} \frac{t^n A^n}{n!} = I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \ldots$$

We want to prove the identity :

$$e^{it\vec{n}\cdot\vec{\sigma}} = I\cos t + i\vec{n}\cdot\vec{\sigma}\sin t$$

where $\vec{n}$ is a unit vector and $t \in \mathbb{R}$. Remark that this is a generalization of Euler's identity :

$$e^{i\theta} = \cos\theta + i\sin\theta$$

To show the identity show first that :

$$(\vec{n}\cdot\vec{\sigma})^2 = I$$

Use Taylor expansions of $\cos t$ and $\sin t$ to deduce the wanted identity above.

b) Explicitly write 2×2 matrices (in component/array form) $\exp(it\sigma_x)$; $\exp(it\sigma_y)$; $\exp(it\sigma_z)$ as well as $\exp(it\vec{n}\cdot\vec{\sigma})$.

Exercise 3 *Rotations on the Bloch sphere*

a) Represent the eigenvectors of σ_x, σ_y et σ_z on the Bloch sphere.

b) Calculate explicitly the matrices $\exp\left(-i\frac{\alpha}{2}\sigma_x\right)$, $\exp\left(-i\frac{\alpha}{2}\sigma_y\right)$, $\exp\left(-i\frac{\alpha}{2}\sigma_z\right)$.

c) Consider the qubit $|\psi\rangle = (\cos\frac{\theta}{2})|\uparrow\rangle + e^{i\frac{\pi}{2}}(\sin\frac{\theta}{2})|\downarrow\rangle$. Calculate the action of the matrices $\exp\left(-i\frac{\gamma}{2}\sigma_z\right)$, $\exp\left(-i\frac{\alpha}{2}\sigma_x\right)$, $\exp\left(-i\frac{\beta}{2}\sigma_y\right)$ on this vector. Represent the "trajectory" as a function of α on the Bloch sphere.

a) Show that all 2×2 matrices, A, can be written as a linear combination of I and σ_x, σ_y, σ_z :

$$A = a_0 I + a_1\sigma_x + a_2\sigma_y + a_3\sigma_z.$$

This can also be written as $A = a_0 I + \vec{a}\cdot\vec{\sigma}$ where $\vec{a}\cdot\vec{\sigma}$ is an "inner product" between the "vectors" $\vec{a} = (a_1, a_2, a_3)$ et $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$.
Check also that if $A = A^\dagger$ we have a_0, a_1, a_2, $a_3 \in \mathbb{R}$.

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$A = \begin{pmatrix} a_0 & 0 \\ 0 & a_0 \end{pmatrix} + \begin{pmatrix} 0 & a_1 \\ a_1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -ia_2 \\ ia_2 & 0 \end{pmatrix} + \begin{pmatrix} a_3 & 0 \\ 0 & -a_3 \end{pmatrix}$$

$$A = \begin{pmatrix} a_0+a_3 & a_1-ia_2 \\ a_1+ia_2 & a_0-a_3 \end{pmatrix} = A^\dagger = \begin{pmatrix} a_0+a_3 & a_1-ia_2 \\ a_1+ia_2 & a_0-a_3 \end{pmatrix}$$

b) Check the following algebraic identities :

$$\checkmark\ \sigma_x^2 = \sigma_y^2 = \sigma_z^2 = I$$
$$\checkmark\ \sigma_x\sigma_y = i\sigma_z$$
$$\checkmark\ \sigma_y\sigma_z = i\sigma_x$$
$$\checkmark\ \sigma_z\sigma_x = i\sigma_y$$

$$\sigma_x^2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$$\sigma_y^2 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$$\sigma_z^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

Deduce

$$\sigma_x\sigma_y + \sigma_y\sigma_x = 0 \rightarrow i\sigma_z - i\sigma_z = 0$$
$$\sigma_y\sigma_z + \sigma_z\sigma_y = 0 \rightarrow i\sigma_x - i\sigma_x = 0$$
$$\sigma_z\sigma_x + \sigma_x\sigma_z = 0 \rightarrow i\sigma_y - i\sigma_y = 0$$

$$\sigma_x\sigma_y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = i\sigma_z$$

$$\sigma_y\sigma_z = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = i\sigma_x$$

$$\sigma_z\sigma_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \times(-i) = \sigma_y \Rightarrow \frac{\sigma_y}{-i}\,(i) = \frac{i\sigma_y}{\underbrace{-i^2}_{-1}}\,+ = i\sigma_y$$

$$\sigma_y\sigma_x = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = -i\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{\sigma_z} = -i\sigma_z$$

c) Let $[A, B] = AB - BA$ be the "commutator". Show (you may use preceding results) } Using these results

$$[\sigma_x, \sigma_y] = 2i\sigma_z = \sigma_x\sigma_y - \sigma_y\sigma_x = 2i\sigma_z$$
$$[\sigma_y, \sigma_z] = 2i\sigma_x = \sigma_y\sigma_z - \sigma_z\sigma_y = 2i\sigma_x$$
$$[\sigma_z, \sigma_x] = 2i\sigma_y = \sigma_z\sigma_x - \sigma_x\sigma_z = 2i\sigma_y$$

These relations are called "commutation relations for spin".

d) Compute eigenvalues and eigenvectors of σ_x, σ_y, σ_z. Check that the eigenvalues satisfy $\mathrm{Tr}\,\sigma_x = \mathrm{Tr}\,\sigma_y = \mathrm{Tr}\,\sigma_z = 0$ et $\det\sigma_x = \det\sigma_y = \det\sigma_z = -1$. $\hookrightarrow$?

$\sigma_x x = \lambda x \hookrightarrow$ x is a eigenvector

$(\sigma_x - \lambda I)\, x = 0$

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} -\lambda & 0 \\ 0 & -\lambda \end{pmatrix} = \begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} \rightarrow \det = \lambda^2 - 1 = 0$

$\lambda = \pm 1 \rightarrow$ eigen values

for $\lambda = 1$ $\quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \rightarrow \left.\begin{matrix} x_2 = x_1 \\ x_1 = x_2 \end{matrix}\right\} \lambda = 1, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ ✓

for $\lambda = -1$ $\quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_1 \\ x_2 \end{pmatrix} = \begin{matrix} x_2 = -x_1 \\ -x_1 = x_2 \end{matrix} \quad \lambda = -1, \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ ✓

eigenvectors

$\sigma_y y = \lambda y$

$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \begin{pmatrix} -\lambda & 0 \\ 0 & -\lambda \end{pmatrix} = \begin{pmatrix} -\lambda & -i \\ i & -\lambda \end{pmatrix} \xrightarrow{\det} \lambda^2 - (\underset{-1}{\overset{+1}{-i^2}}) = 0 \qquad \lambda^2 - 1 = 0$

$\lambda^2 = +1$

$\lambda = \pm 1$

for $\lambda = 1$ $\quad \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \longrightarrow \begin{matrix} -i\overset{i}{y_2} = \overset{1}{y_1} \rightarrow \\ i\underset{1}{y_1} = \underset{i}{y_2} \rightarrow \end{matrix} \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$

for $\lambda = -1$ $\quad \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -y_1 \\ -y_2 \end{pmatrix} \longrightarrow \begin{matrix} \cancel{-}i\overset{1}{y_2} = \cancel{-}\overset{i}{y_1} \\ i\underset{1}{y_1} = -\underset{(-i)}{y_2} \end{matrix} \quad \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \end{pmatrix}$ ✗

Solution gives $\frac{1}{\sqrt{2}}$

$\sigma_z z = \lambda z$

$\begin{pmatrix} 1-\lambda & 0 \\ 0 & -1-\lambda \end{pmatrix} \rightarrow (1-\lambda)(-1-\lambda) = -1 \cancel{-\lambda} + \cancel{\lambda} + \lambda^2 \rightarrow \lambda^2 = 1 \quad \lambda = \pm 1$

for $\lambda = 1$ $\quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \quad \begin{matrix} z_1 = z_1 \\ -z_2 = z_2 \rightarrow z_2 = 0 \end{matrix} \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

for $\lambda = -1$ $\quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} -z_1 \\ -z_2 \end{pmatrix} \quad \begin{matrix} z_1 = -z_1 \\ -z_2 = -z_2 \end{matrix} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Exercise 2 *Exponentials of Pauli matrices*

a) We define the exponential of a matrix A by (for $t \in \mathbb{R}$)

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We want to prove the identity :

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To show the identity show first that :

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Use Taylor expansions of $\cos t$ and $\sin t$ to deduce the wanted identity above.

b) Explicitly write 2×2 matrices (in component/array form) $\exp(it\sigma_x)$; $\exp(it\sigma_y)$; $\exp(it\sigma_z)$ as well as $\exp(it\vec{n}\cdot\vec{\sigma})$.

a) $(\vec{n}.\vec{\sigma})^2 = I$

! $n_x n_y \sigma_x \sigma_y + n_y n_x \sigma_y \sigma_x \to 0$

$n_x n_z \sigma_x \sigma_z + n_z n_x \sigma_z \sigma_x \to 0$

$n_y n_z \sigma_y \sigma_z + n_z n_y \sigma_z \sigma_y \to 0$

$$(n_x\sigma_x + n_y\sigma_y + n_z\sigma_z)^2 = n_x^2 \underbrace{\sigma_x\sigma_x^\dagger}_{I} + n_y^2 \cancel{\underbrace{\sigma_y\sigma_y^\dagger}_{I}} + n_z^2 \underbrace{\sigma_z\sigma_z^\dagger}_{I} = 1$$

$$e^{it\vec{n}.\vec{\sigma}} = \sum_{n=0}^{\infty} \frac{t^n i^n (\vec{n}\vec{\sigma})^n}{n!} = I + t(i\vec{n}\vec{\sigma}) + \frac{t^2}{2!}(i\vec{n}\vec{\sigma})^2 + \frac{t^3}{3!}(i\vec{n}\vec{\sigma})^3 +$$

$$= I + ti(\vec{n}\vec{\sigma}) + \frac{t^2}{2!}(-1)\underbrace{(\vec{n}\vec{\sigma})^2}_{I} + \frac{t^3}{3}(-i)\underbrace{(\vec{n}\vec{\sigma})^3}_{\vec{n}\vec{\sigma}} + \frac{t^4}{4}\overbrace{(i)^4}^{1}\underbrace{(\vec{n}\vec{\sigma})^4}_{I} + \ldots$$

$$= I + ti(\vec{n}\vec{\sigma}) - \frac{t^2}{2!} + \frac{t^3}{3!}(-i)(\vec{n}\vec{\sigma}) + \frac{t^4}{4!} + \frac{t^5}{5!}i\vec{n}\vec{\sigma} - \frac{t^6}{6!} + \frac{t^7}{7!}(-i)(\vec{n}\vec{\sigma})$$

cost

$\sin t\,(i\vec{n}\vec{\sigma})$

Taylor Expansion of

$$\cos(t) = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \ldots$$

$$\sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \ldots$$

$$= I\cos t + i\vec{n}\vec{\sigma}\sin t$$

to make it matrix?

b)

$$\exp(it\sigma_x) = I\cos t + i\sigma_x \sin t = \begin{pmatrix}\cos t & 0\\ 0 & \cos t\end{pmatrix} + \begin{pmatrix}0 & i\sin t\\ i\sin t & 0\end{pmatrix} = \begin{pmatrix}\cos t & i\sin t\\ i\sin t & \cos t\end{pmatrix}$$

$$\exp(it\sigma_y) = I\cos t + i\sigma_y \sin t = \begin{pmatrix}\cos t & 0\\ 0 & \cos t\end{pmatrix} + \begin{pmatrix}0 & \overbrace{-i(i\sin t)}^{+\sin t}\\ (i\sin t)i & 0\end{pmatrix} = \begin{pmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{pmatrix}$$

$$\exp(it\sigma_z) = I\cos t + i\sigma_z \sin t = \begin{pmatrix}\cos t & 0\\ 0 & \cos t\end{pmatrix} + \begin{pmatrix}i\sin t & 0\\ 0 & -i\sin t\end{pmatrix} = \begin{pmatrix}\cos t + i\sin t & 0\\ 0 & \cos t - i\sin t\end{pmatrix}$$

$$\exp(it\vec{n}\cdot\vec{\sigma}) = I\cos t + i\vec{n}\vec{\sigma}\sin t$$

$$= I\cos t + i n_x\sigma_x \sin t + i n_y\sigma_y \sin t + i n_z\sigma_z \sin t$$

$$= \begin{pmatrix}\cos t & 0\\ 0 & \cos t\end{pmatrix} + \begin{pmatrix}0 & i n_x\sin t\\ i n_x\sin t & 0\end{pmatrix} + \begin{pmatrix}0 & n_y\sin t\\ -n_y\sin t & 0\end{pmatrix} + \begin{pmatrix}i n_z\sin t & 0\\ 0 & -i n_z\sin t\end{pmatrix}$$

$$= \begin{bmatrix}\cos t + i n_z\sin t & i n_x\sin t + n_y\sin t\\ i n_x\sin t - n_y\sin t & \cos t - i n_z\sin t\end{bmatrix}$$

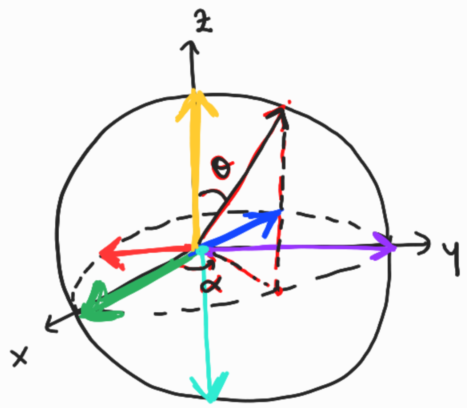
Exercise 3 *Rotations on the Bloch sphere*

a) Represent the eigenvectors of σ_x, σ_y et σ_z on the Bloch sphere.

b) Calculate explicitly the matrices $\exp(-i\frac{\alpha}{2}\sigma_x)$, $\exp(-i\frac{\alpha}{2}\sigma_y)$, $\exp(-i\frac{\alpha}{2}\sigma_z)$.

c) Consider the qubit $|\psi\rangle = (\cos\frac{\theta}{2})|\uparrow\rangle + e^{i\frac{\pi}{2}}(\sin\frac{\theta}{2})|\downarrow\rangle$. Calculate the action of the matrices $\exp(-i\frac{\gamma}{2}\sigma_z)$, $\exp(-i\frac{\alpha}{2}\sigma_x)$, $\exp(-i\frac{\beta}{2}\sigma_y)$ on this vector. Represent the "trajectory" as a function of α on the Bloch sphere.

$\theta = \pi/2 \quad \varphi = 0 \qquad \theta = \pi/2, \alpha = \pi$



$\sigma_x \longrightarrow \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix} \longrightarrow \begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix} \begin{pmatrix}0\\-1\end{pmatrix}$

$\theta = \pi/2, \ \varphi = \pi/2$

$\sigma_y \longrightarrow \begin{pmatrix}1\\i\end{pmatrix}, \begin{pmatrix}i\\1\end{pmatrix} \longrightarrow \begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\i\end{pmatrix},$

$\sigma_z \longrightarrow \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$

$$|\psi\rangle = \cos\frac{\theta}{2}|\uparrow\rangle + e^{i\varphi}\sin\frac{\theta}{2}|\downarrow\rangle$$

$\begin{pmatrix}1\\0\end{pmatrix} \qquad \begin{pmatrix}0\\1\end{pmatrix}$

$$\begin{pmatrix}\cos\theta/2\\ e^{i\alpha}\sin\frac{\theta}{2}\end{pmatrix}$$

b) we found that in 2b

$$\exp(it\sigma_x) = \begin{pmatrix} \cos t & i\sin t \\ i\sin t & \cos t \end{pmatrix}, \quad \exp(it\sigma_y) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}, \quad \exp(it\sigma_z) = \begin{pmatrix} \cos t + i\sin t & 0 \\ 0 & \cos t - i\sin t \end{pmatrix}$$

$$\exp\left(-i\frac{\alpha}{2}\sigma_x\right) = \begin{pmatrix} \cos\frac{\alpha}{2} & -i\sin\alpha/2 \\ -i\sin\alpha/2 & \cos\frac{\alpha}{2} \end{pmatrix} \checkmark$$

$$\exp\left(-i\frac{\alpha}{2}\sigma_y\right) = \begin{pmatrix} \cos\frac{\alpha}{2} & -\sin\frac{\alpha}{2} \\ \sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{pmatrix}$$

$$\exp\left(-i\frac{\alpha}{2}\sigma_z\right) = \begin{pmatrix} \cos\alpha/2 - i\sin\alpha/2 & 0 \\ 0 & \cos\frac{\alpha}{2} + i\sin\frac{\alpha}{2} \end{pmatrix} = \begin{pmatrix} e^{-i\gamma/2} & 0 \\ 0 & e^{i\gamma/2} \end{pmatrix}$$

c)
$$\exp\left(-i\frac{\alpha}{2}\sigma_x\right)|\psi\rangle = \begin{pmatrix} \cos\alpha/2 & -i\sin\alpha/2 \\ -i\sin\alpha/2 & \cos\alpha/2 \end{pmatrix} \begin{pmatrix} \cos\theta/2 \\ \underbrace{e^{i\pi/2}}_{i}\sin\theta/2 \end{pmatrix}$$

Rotation matrix of angle α around the x axis.

$$= \begin{pmatrix} \cos\alpha/2\cos\theta/2 + \sin\alpha/2\sin\theta/2 \\ -i\sin\alpha/2\cos\theta/2 + i\sin\theta/2\cos\alpha/2 \end{pmatrix} = \begin{pmatrix} \cos\left(\frac{\theta}{2}-\frac{\alpha}{2}\right) \\ i\sin\left(\frac{\theta}{2}-\frac{\alpha}{2}\right) \end{pmatrix}$$

$$\exp\left(-i\frac{\alpha}{2}\sigma_y\right)|\psi\rangle = \begin{pmatrix} \cos\alpha/2 & \sin\alpha/2 \\ -\sin\alpha/2 & \cos\alpha/2 \end{pmatrix} \begin{pmatrix} \cos\theta/2 \\ i\sin\theta/2 \end{pmatrix}$$

Rotation matrix

$$= \begin{pmatrix} \cos\alpha/2\cos\theta/2 + i\sin\alpha/2\sin\theta/2 \\ -\sin\alpha/2\cos\theta/2 + i\sin\theta/2\cos\alpha/2 \end{pmatrix} =$$

$$\exp\left(-i\frac{\alpha}{2}\sigma_z\right)|\psi\rangle = \begin{pmatrix} \cos\alpha/2 - i\sin\alpha/2 & 0 \\ 0 & \cos\frac{\alpha}{2} + i\sin\frac{\alpha}{2} \end{pmatrix} \begin{pmatrix} \cos\theta/2 \\ i\sin\theta/2 \end{pmatrix}$$

Rotation around z

$$= \begin{pmatrix} \cos\alpha/2\cos\theta/2 - i\sin\alpha/2\cos\theta/2 \\ i\cos\alpha/2\sin\theta/2 - \sin\alpha/2\sin\theta/2 \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2}\, e^{-i\alpha/2} \\ i\sin\frac{\theta}{2}\, e^{+i\alpha/2} \end{pmatrix}$$

→ global phase

$$e^{-i\alpha/2}\begin{pmatrix} \cos\theta/2 \\ i\sin\theta/2\, e^{+i\alpha} \end{pmatrix}$$