Particle Physics 1: Exercise 7

Exercise 1

Verify the statement that the Einstein energy-momentum relationship is recovered if any of the four orthogonal Dirac spinors derived in the lecture are substituted into the Dirac equation written in terms of momentum, $(\gamma^\mu p_\mu - m)u = 0$.

Exercise 2

For a particle with four-momentum $p^\mu = (e, \vec{p})$, the general solution to the free-particle Dirac equation can be written

$$\psi(p) = \left[a u_1(p) + b u_2(p)\right] e^{i(\vec{p}\cdot\vec{x} - Et)}$$

Using the explicit form for u_1 and u_2, show that the four-vector current $j^\mu = (\rho, \vec{j})$ is given by

$$j^\mu = 2p^\mu.$$

Furthermore, show that the resulting probability density and probability current are consistent with a particle moving with velocity $\beta = p/E$.

Exercise 3

Show that up to an overall phase factor

$$\hat{P} u_\uparrow(\theta, \phi) = u_\downarrow(\pi - \theta, \pi + \phi)$$

where $u_\uparrow$ and $u_\downarrow$ are, respectively, the right-handed and left-handed helicity particle spinors. Comment on the result.

Exercise 4

Under the combined operation of parity and charge conjugation ($\hat{C}\hat{P}$) spinors transform as

$$\psi \to \psi^C = \hat{C}\hat{P}\psi = i\gamma^2\gamma^0\psi^*.$$

Show that up to an overall phase factor

$$\hat{C}\hat{P} u_\uparrow(\theta, \phi) = v_\downarrow(\pi - \theta, \pi + \phi).$$

Exercise 1 $p_\nu = (E, -p_x, -p_y, -p_z)$

Verify the statement that the Einstein energy-momentum relationship is recovered if any of the four orthogonal Dirac spinors derived in the lecture are substituted into the Dirac equation written in terms of momentum, $(\gamma^\mu p_\mu - m)u = 0$.

$$\gamma^\nu p_\nu - m = E \underbrace{\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}}_{\gamma^0} - p_x \underbrace{\begin{bmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{bmatrix}}_{\gamma^1} - p_y \begin{bmatrix} & & & -i \\ & & i & \\ & i & & \\ -i & & & \end{bmatrix}$$

$$-p_z \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} - mI \rightarrow \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$(\gamma^\nu p_\nu - m)u = \begin{bmatrix} E-m & 0 & -p_z & -p_x+ip_y \\ 0 & E-m & -p_x-ip_y & +p_z \\ p_z & p_x-ip_y & -(E+m) & 0 \\ p_x+ip_y & -p_z & 0 & -(E+m) \end{bmatrix} \overbrace{\begin{bmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{bmatrix}}^{u_1} = 0$$

$$E^2 = p^2 + m^2$$

$$\begin{bmatrix} (E-m) + 0 - \frac{p_z^2}{E+m} + \frac{-p_x^2 - p_y^2}{E+m} \\ 0 + 0 - \frac{p_z(p_x+ip_y)}{E+m} + \frac{p_z(p_x+ip_y)}{E+m} \\ p_z + 0 - p_z + 0 \\ (p_x+ip_y) + 0 + 0 - (p_x+ip_y) \end{bmatrix} = \frac{1}{E+m} \begin{bmatrix} E^2 - m^2 \underbrace{- p_x^2 - p_y^2 - p_z^2}_{p^2} \\ 0 \\ 0 \\ 0 \end{bmatrix} = 0$$

Exercise 2

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Using the explicit form for u_1 and u_2, show that the four-vector current $j^\mu = (\rho, \vec{j})$ is given by

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For arbitrary spinors ψ and ϕ

$$\underset{\swarrow}{\bar{\psi}} \gamma^0 \phi = (\psi_1^*, \psi_2^*, -\psi_3^*, -\psi_4^*) \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix}$$

$\psi^\dagger \gamma^0$

$$(\psi_1^*, \psi_2^*, \psi_3^*, \psi_3^*) \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

$$= (\psi_1^*, \psi_2^*, -\psi_3^*, -\psi_4^*) \begin{pmatrix} \phi_1 \\ \phi_2 \\ -\phi_3 \\ -\phi_4 \end{pmatrix}$$

$$\bar{\psi} \gamma^0 \phi = \psi_1^* \phi_1 - \psi_2^* \phi_2 + \psi_3^* \phi_3 + \psi_4^* \phi_4$$

Repeat this procedure for $\mu = 1, 2, 3$

$$\bar{\psi} \gamma^1 \phi = \psi_1^* \phi_4 + \psi_2^* \phi_3 + \psi_3^* \phi_2 + \psi_4^* \phi_1$$

$$\bar{\psi} \gamma^2 \phi = -i\left(\psi_1^* \phi_4 - \psi_2^* \phi_3 + \psi_3^* \phi_2 - \psi_4^* \phi_1\right)$$

$$\bar{\psi} \gamma^3 \phi = \psi_1^* \phi_3 - \psi_2^* \phi_4 + \psi_3^* \phi_1 - \psi_4^* \phi_2$$

For the free particle spinor $u_1(p) = \sqrt{E+m}\begin{pmatrix} 1 \\ 0 \\ p_z/E+m \\ (p_x+ip_y)/E+m \end{pmatrix}$

$$\bar{u}_1 \gamma^0 u_1 = (E+m)\left[1 + \frac{p_z^2}{(E+m)^2} + \frac{(p_x^2+p_y^2)}{(E+m)^2}\right]$$

$$= (E+m)\left[1 + \frac{p^2}{(E+m)^2}\right]$$

$$= \frac{(E+m)^2+p^2}{E+m} = \frac{E^2+2Em+\overbrace{m^2+p^2}^{E^2}}{E+m} = \frac{2E^2+2Em}{E+m}$$

$$= \frac{2E\,\cancel{(E+m)}}{\cancel{(E+m)}} = 2E$$

Repeating this $\Rightarrow$ $\bar{u}_1 \gamma^\mu u_1 = (2E, 2p_x, 2p_y, 2p_z) = 2p^\mu$

$$\bar{u}_1 \gamma^\mu u_1 = \bar{u}_2 \gamma^\mu u_2 = \bar{v}_1 \gamma^\mu v_1 = \bar{v}_2 \gamma^\mu v_2 = 2p^\mu$$

All corresponding cross terms give zero

$$\bar{u}_1 \gamma^\mu u_2 = \bar{u}_2 \gamma^\mu u_1 = \ldots\ldots$$

For a particle $\psi = u(p)e^{ip.x}$

$$\bar{\psi} = \psi^\dagger \gamma^0 = u(p)^\dagger \gamma^0 e^{-ipx} = \bar{u}(p)\, e^{-ipx}$$

$$j^\mu = \bar{\psi}\gamma^\mu \psi = \bar{u}\gamma^\mu u = \bar{v}\gamma^\mu v$$

for anti-particle spin

Given $u = \alpha_1 u_1 + \alpha_2 u_2$ with $|\alpha_1|^2 + |\alpha_2|^2 = 1$

?

$$\bar{u}\gamma^\mu u = |\alpha_1|^2 \underbrace{\bar{u}_1\gamma^\mu u_1}_{2p^\mu} + |\alpha_2|^2 \underbrace{\bar{u}_2\gamma^\mu u_2}_{2p^\mu} = 2p^\mu$$

$$j^\mu = \bar{u}\gamma^\mu u = 2p^\mu$$

$$j^\mu = (\rho, \vec{j})$$

$$\rho = 2E \qquad \vec{j} = 2\vec{p}$$

$*\ E = \gamma mc = \gamma m \qquad \vec{p} = \gamma m\vec{v} = E\vec{v} \quad \rightsquigarrow \quad \vec{j} = \underbrace{2E}_{\rho}\vec{v}$

$$\vec{j} = \rho\vec{v}$$

Exercise 3

Show that up to an overall phase factor

$$\hat{P}u_\uparrow(\theta,\phi) = u_\downarrow(\pi-\theta,\pi+\phi)$$

where $u_\uparrow$ and $u_\downarrow$ are, respectively, the right-handed and left-handed helicity particle spinors. Comment on the result.

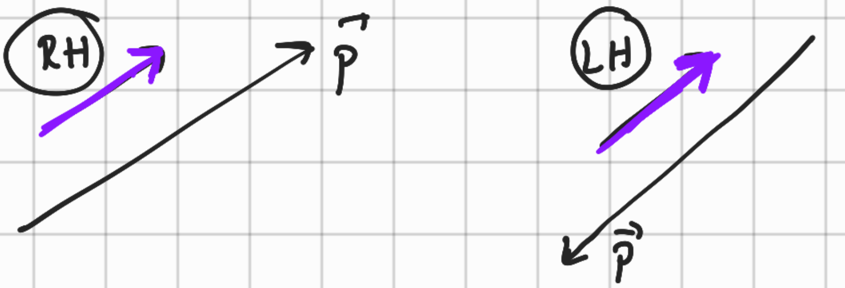
$$\hat{P} = \gamma^0 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \qquad u_\uparrow(\theta,\phi) = \sqrt{E+m}\begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right)e^{i\phi} \\ \frac{p}{E+m}\cos\left(\frac{\theta}{2}\right) \\ \frac{p}{E+m}e^{i\phi}\sin\left(\frac{\theta}{2}\right) \end{pmatrix}$$

$$\hat{P}u_\uparrow(\theta,\phi) = \sqrt{E+m}\begin{pmatrix} c \\ se^{i\phi} \\ -\frac{p}{E+m}c \\ -\frac{p}{E+m}se^{i\phi} \end{pmatrix}$$

$$v_\downarrow(\pi-\theta,\ \pi+\phi) = \sqrt{E+m}\begin{pmatrix} -\overbrace{\sin\left(\frac{\pi-\theta}{2}\right)}^{\to \cos\theta/2} \\ \cos\left(\frac{\pi-\theta}{2}\right) e^{i(\cancel{\pi}+\phi)}\ \ {\scriptstyle -1} \\ \left(\frac{p}{E+m}\right)\overbrace{\sin\left(\frac{\pi-\theta}{2}\right)}^{\to\cos\frac{\theta}{2}} \\ -\frac{p}{E+m}\underbrace{\cos\left(\frac{\pi-\theta}{2}\right)}_{\sin(\theta/2)} e^{i(\phi+\cancel{\pi})}\ \ {\scriptstyle -1} \end{pmatrix}$$

$$= \sqrt{E+m}\begin{pmatrix} -c \\ -se^{i\phi} \\ \frac{p}{E+m}c \\ \frac{p}{E+m}se^{i\phi} \end{pmatrix}$$

$\vec{p} \to -\vec{p}$ → The parity operator reverses $\vec{p}$ but leaves the spin orientation UNCHANGED



$RH \to LH$

Exercise 4

Under the combined operation of parity and charge conjugation ($\hat{C}\hat{P}$) spinors transform as

$$\psi \to \psi^C = \hat{C}\hat{P}\psi = i\gamma^2\gamma^0\psi^*.$$

Show that up to an overall phase factor

$$\hat{C}\hat{P}u_\uparrow(\theta,\phi) = v_\downarrow(\pi-\theta, \pi+\phi).$$

$$\gamma^0 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \qquad \gamma^2 = \begin{pmatrix} & & & -i \\ & & i & \\ & i & & \\ -i & & & \end{pmatrix} \qquad u_\uparrow(\theta,\phi) = \sqrt{E+m}\begin{pmatrix} c \\ se^{i\phi} \\ \frac{p}{E+m}c \\ \frac{p}{E+m}se^{i\phi} \end{pmatrix}$$

$$\hat{C}\hat{P}\, v_\uparrow(\theta,\phi) = i\gamma^2\gamma^0\, v_\uparrow(\theta,\phi)^* = i \underbrace{\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}} \underbrace{\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}} v_\uparrow^*$$

$\hat{C} \to i\gamma^2\psi^*$ $\quad$ $\hat{P} \to \gamma^0\psi$

$$= \sqrt{E+m} \begin{pmatrix} & & & 1 \\ & & -1 & \\ & -1 & & \\ 1 & & & \end{pmatrix} \begin{pmatrix} c \\ s e^{-i\phi} \\ -p/E+m\; c \\ -p/E+m\; s e^{-i\phi} \end{pmatrix}$$

$$v_\downarrow(\pi,\theta,\pi+\phi) = \sqrt{E+m} \underbrace{\begin{pmatrix} \frac{-p}{E+m} s e^{-i\phi} \\ \frac{p}{E+m} c \\ -s e^{i\phi} \\ c \end{pmatrix}}$$

$$v_\downarrow(\pi-\theta,\phi+\pi) = \sqrt{E+m} \begin{pmatrix} \frac{p}{E+m} s \\ -\frac{p}{E+m} c e^{i\phi} \\ s \\ -c e^{i\phi} \end{pmatrix} = -e^{i\phi}\sqrt{E+m} \underbrace{\begin{pmatrix} \frac{-p}{E+m} s e^{-i\phi} \\ \frac{p}{E+m} c \\ -s e^{-i\phi} \\ c \end{pmatrix}}$$

$$\hat{C}\hat{P}\, v_\uparrow(\theta,\phi) = \underbrace{-e^{i\phi}}_{\text{overall phase factor}} v_\downarrow(\pi-\theta,\pi+\phi)$$