QUANTUM CHANNELS

$$\hat{\rho} \longrightarrow \boxed{} \longrightarrow \hat{\rho}'$$

1) Amplitude Damping Channel

Model for spontaneous emission

System S (qubit)

Environment E

$$\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_E$$

Unitary evolution:

$$|0\rangle \otimes |0\rangle \xrightarrow{\text{Nothing happens}} |0\rangle \otimes |0\rangle$$

$$|1\rangle \otimes |0\rangle \longrightarrow \underbrace{\sqrt{p}}_{\text{emission prob}}\, |0\rangle \otimes |1\rangle + \underbrace{\sqrt{1-p}}_{\text{nothing happens}}\, |1\rangle \otimes |0\rangle$$

$$\hat{S} : \hat{\rho}_S \longrightarrow \hat{\rho}_S' = \mathrm{Tr}_E\left[\hat{U}\rho\hat{U}^+\right]$$

Initialize : $\hat{\rho} = \hat{\rho}_S \otimes |0\rangle\langle 0|_E$

$$\hat{\rho}' = \langle 0_E|\,\hat{U}\hat{\rho}\hat{U}^+|0_E\rangle + \langle 1_E|\hat{U}\hat{\rho}\hat{U}^+|1_E\rangle$$

$$\hat{\rho}' = \hat{M}_0\,\hat{\rho}_S\,\hat{M}_0^+ + \hat{M}_1\,\hat{\rho}_S\,M_1^+$$

$\hat{\rho}_S, \hat{M}_0, \hat{M}_1$ operators on $\mathcal{H}_S$

- $\hat{M}_0 = ?$

$$\langle 0_E|\,\hat{U}\,(\hat{\rho}_S \otimes |0\rangle\langle 0|_E)\,\hat{U}^+\,|0_E\rangle = \hat{M}_0\,\hat{\rho}_S\,\hat{M}_0^+$$

$$\hat{M}_0 = \langle 0_E|\,\hat{U}\,|0_E\rangle$$

$$
= \begin{array}{cc}
|0\rangle|0\rangle \longrightarrow |0\rangle|0\rangle\ (1) & |0\rangle|0\rangle \longrightarrow |1\rangle|0\rangle\ (0) \\
\left[\begin{array}{cc}
\underbrace{\langle 0_S| \otimes \langle 0_E|\hat{U}|0_S\rangle \otimes |0_E\rangle}_{M_{0,00}} & \underbrace{\langle 0_S| \otimes \langle 0_E|\hat{U}|1_S\rangle \otimes |0_E\rangle}_{M_{0,01}} \\
\underbrace{\langle 1_S| \otimes \langle 0_E|\hat{U}|0_S\rangle \otimes |0_E\rangle}_{M_{0,10}} & \underbrace{\langle 1_S| \otimes \langle 0_E|\hat{U}|1_S\rangle \otimes |0_E\rangle}_{M_{0,11}}
\end{array}\right]_{2\times 2} \\
|1\rangle|0\rangle \longrightarrow |0\rangle|0\rangle\ (0) & |1\rangle|0\rangle \longrightarrow |1\rangle|0\rangle\ \sqrt{1-p}
\end{array}
$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{bmatrix}$$

$$\langle 1_E | \hat{U} \left(\hat{\rho}_s \otimes |0_E\rangle\langle 0| \right) \hat{U}^{\dagger} | 1_E \rangle$$

$$\hat{M}_1 = \langle 1_E | \hat{U} | 0_E \rangle$$

$|0\rangle|0\rangle \longrightarrow |0\rangle|1\rangle$ 0 $\qquad$ $|10\rangle \longrightarrow |01\rangle$ $\sqrt{p}$

$$= \begin{bmatrix} \underset{M_{1,00}}{\langle 0|_s \otimes \langle 1_E | \hat{U} | 0_s \rangle \otimes |0_E\rangle} & \underset{M_{1,01}}{\langle 0_s| \otimes \langle 1_E | \hat{U} | 1_s \rangle \otimes |0_E\rangle} \\ \langle 1_s| \otimes \langle 1_E | \hat{U} | 0_s \rangle \otimes |0_E\rangle & \langle 1_s| \otimes \langle 1_E | \hat{U} | 1_s \rangle \otimes |0_E\rangle \end{bmatrix}$$

$M_{1,10}$ $\quad$ $|00\rangle \longrightarrow |11\rangle$ 0 $\qquad$ $M_{1,11}$ $\quad$ $|10\rangle \longrightarrow |11\rangle$ 0

$$= \begin{bmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{bmatrix}$$

Check: $\hat{M}_0^{\dagger} \hat{M}_0 + \hat{M}_1^{\dagger} \hat{M}_1 = \mathbb{I}$

$$\hat{\rho}_s = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} \quad \text{then} \quad \hat{\rho}_s' = \begin{pmatrix} \rho_{00} + p\,\rho_{11} & \sqrt{1-p}\,\rho_{01} \\ \sqrt{1-p}\,\rho_{10} & \rho_{11}\,(1-p) \end{pmatrix}$$

- Probability of transition p between 0 and 1
- $\sqrt{1-p}$ reduction in the coherence!

On Bloch Sphere:

$$\hat{\rho}_s = \frac{1}{2}\left(\hat{\mathbb{I}} + \vec{a}\cdot\vec{\hat{\sigma}}\right)$$

$$\begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} \longrightarrow \quad \vec{a}' = \begin{pmatrix} \sqrt{1-p}\; a_x \\ \sqrt{1-p}\; a_y \\ (1-p)\,a_z + p \end{pmatrix}$$

$$Tr\left(\rho_s'^2\right) \leq Tr\left(\rho_s^2\right) \qquad \forall p \in [0,1]$$

Interpretation in terms of measurements

$$\hat{E}_0 = \hat{M}_0^\dagger M_0 \qquad \hat{E}_1 = \hat{M}_1^\dagger M_1$$

- $\hat{M}_1 : \ |\psi\rangle \longrightarrow \dfrac{\hat{M}_1|\psi\rangle}{\sqrt{p}}$ $\qquad \hat{M}_1 = \sqrt{p}\,|0\rangle\langle 1|$

"1 click" $\qquad \hat{E}_1 = p\,|1\rangle\langle 1|$

$$|\psi\rangle \longrightarrow |0\rangle \quad (\text{Heralded preparation of } |0\rangle)$$

- $\hat{M}_0 : \ |\psi\rangle \longrightarrow \dfrac{\hat{M}_0|\psi\rangle}{\sqrt{p(0)}}$ $\qquad \hat{M}_0 = |0\rangle\langle 0| + \sqrt{1-p}\,|1\rangle\langle 1|$

"0 click" $\qquad \hat{E}_0 = |0\rangle\langle 0| + (1-p)|1\rangle\langle 1|$

$$\left.\begin{aligned} |\psi\rangle &= a|0\rangle + b|1\rangle \\ \hat{M}_0|\psi\rangle &\propto a|0\rangle + b\sqrt{1-p}\,|1\rangle \end{aligned}\right\} \quad |\psi\rangle \neq |\psi'\rangle$$

No click $\neq$ No information

Remark: Repeat n consecutive amp. damping ch:

$$\rho_{11} \longrightarrow \rho_{11}(1-p)^n$$

Decay Rate Γ: $\qquad p = \frac{\Gamma t}{n}$

$$\rho_{11} \longrightarrow \rho_{11} e^{-\Gamma t} \qquad (n \to \infty)$$

2) Phase Damping Channel

Unitary operator onto $\mathcal{H}_S \otimes \mathcal{H}_E$

$$|0\rangle \otimes |0\rangle \longrightarrow \sqrt{p}\, |0\rangle \otimes |0\rangle + \sqrt{1-p}\, |0\rangle \otimes |1\rangle$$

$$|1\rangle \otimes |0\rangle \longrightarrow \sqrt{p}\, |1\rangle \otimes |0\rangle - \sqrt{1-p}\, |1\rangle \otimes |1\rangle$$

$$|0\rangle \otimes |0\rangle \rightarrow |0\rangle \otimes \left(\sqrt{p}\, |0\rangle + \sqrt{1-p}\, |1\rangle \right)$$

$$|1\rangle \otimes |0\rangle \rightarrow |1\rangle \otimes \left(\sqrt{p}\, |0\rangle - \sqrt{1-p}\, |1\rangle \right)$$

Remark: It is an entangling operation:

$$\hat{\rho}_S' = \mathrm{Tr}_E \left[\hat{U} \overbrace{\left(\hat{\rho}_S \otimes |0\rangle\langle 0|_E \right)}^{\hat{\rho}} \hat{U}^+ \right]$$

$$= \langle 0_E | \hat{U} \hat{\rho} \hat{U}^+ | 0_E \rangle + \langle 1_E | \hat{U} \hat{\rho} \hat{U}^+ | 1_E \rangle$$

$$= \hat{M}_0 \hat{\rho}_S \hat{M}_0^\dagger \quad + \quad \hat{M}_1 \hat{\rho}_S \hat{M}_1^\dagger$$

$$\hat{M}_0 = \sqrt{p} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \hat{M}_1 = \sqrt{1-p} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{\sigma_z}$$

Check that $\hat{M}_0^\dagger M_0 + \hat{M}_1^\dagger M_1 = \hat{I}$

Bloch vector:

$$\hat{\rho} = \frac{1}{2} \left(\hat{I} + \vec{a} \cdot \vec{\sigma} \right) \qquad \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} \rightarrow a' = \begin{pmatrix} a_x(2p-1) \\ a_y(2p-1) \\ a_z \end{pmatrix}$$

Unless $p = 0$ or 1 we have $\|\vec{a}'\|^2 \leq \|\vec{a}\|^2$

Interpretation: $\frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes |0\rangle = \sqrt{p} \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}} \right) \otimes |0\rangle + \sqrt{1-p} \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right) \otimes |1\rangle$

If $p = 1/2 \rightarrow \hat{\rho}' = \frac{1}{2} \left(|0\rangle\langle 0| + |1\rangle\langle 1| \right)$

Remark: ⋆ Irreversible

Not possible to "inflate the Bloch Sphere"

* generalize to harmonic oscillator

* Physical motivation: Random Rotations about z

$$R_z(\hat{\theta}) = e^{i\theta\hat{\sigma}_z}$$

Average over possible values of θ:

$$\hat{\rho} = \frac{1}{\sqrt{4\pi\lambda}} \int d\theta \, \hat{R}_z(\theta) \hat{\rho} \hat{R}_z(\theta)^\dagger e^{-\theta^2/4\lambda}$$

$$\hat{\rho}' = \begin{pmatrix} |a|^2 & ab^* e^{-\lambda} \\ a^* b e^{-\lambda} & |b|^2 \end{pmatrix}$$

3) Depolarizing Channel:

Phase damping picks a preferred direction: z

$$\hat{M}_0 = \sqrt{1-p}\,\hat{\mathbb{I}} + \sqrt{p}\,\hat{\sigma}_z$$

Now, define a channel through Kraus Operators

$$\hat{M}_0 = \sqrt{1-p}\,\hat{\mathbb{I}}$$

$$\hat{M}_1 = \sqrt{\frac{p}{3}}\,\hat{\sigma}_x \qquad \hat{M}_2 = \sqrt{\frac{p}{3}}\,\hat{\sigma}_y \qquad \hat{M}_3 = \sqrt{\frac{p}{3}}\,\hat{\sigma}_z$$

$$\sum_i \hat{M}_i^\dagger M_i = \hat{\mathbb{I}}$$

Action on the Bloch vector:

$$\hat{\rho}' = \sum_{\nu} \hat{M}_\nu \hat{\rho} \hat{M}_\nu^+ = \frac{1}{2}\left(\hat{I} + \vec{a}\cdot\begin{pmatrix} \sum_\nu \hat{M}_\nu \hat{\sigma}_x \hat{M}_\nu^+ \\ \vdots \end{pmatrix}\right)$$

$$\sum_\nu \hat{M}_\nu \hat{\sigma}_x \hat{M}_\nu^+ = \dots = \left(1 - \frac{4p}{3}\right)$$, same for $\hat{\sigma}_y, \hat{\sigma}_z$

So $$\vec{a}' = \vec{a}\cdot\left(1 - \frac{4p}{3}\right)$$

Uniform "contraction" of the Bloch Sphere