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Prof. Dr. Jean-Philippe Brantut
Dr. Rohit Prasad Bhatt
Francesca Orsi

Quantum Optics 2 – Spring semester 2023 – 02/03/2023
Problem Set 2 : Decoherence and Tomography

For questions contact : Francesca Orsi (francesca.orsi@epfl.ch)

I. QUANTUM STATE TOMOGRAPHY

Quantum state tomography is the attempt to discover the quantum-mechanical state of a physical system, or more precisely, of a finite set of systems prepared by the same process. The experimenter acquires a set of measurements of different non-commuting observables and tries to estimate what the density matrix of the systems must have been before the measurements were made, with the goal of being able to predict the statistics of future measurements generated by the same process. In this sense, quantum state tomography characterizes a state preparation process that is assumed to be stable over time.
The goal of this exercise is to understand, develop and apply two different methods of quantum state tomography (direct inversion tomography and Bayesian mean estimate) to the simple example of a qubit.
The state of a two-level system can be expressed as

$$\hat{\rho} = \frac{1}{2}(\mathbb{1} + x\hat{\sigma}_x + y\hat{\sigma}_y + z\hat{\sigma}_z) = \frac{1}{2}(\mathbb{1} + \boldsymbol{r}\cdot\hat{\boldsymbol{\sigma}})$$

$\frac{1}{2}\mathrm{Tr}(\mathbb{1} + \hat{r}\cdot\hat{\sigma}) = 1$

$\mathrm{Tr}(\rho) = I$

where the $\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z$ are the Pauli matrices and $\boldsymbol{r} = (x, y, z)$ and $\hat{\boldsymbol{\sigma}} = (\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)$.

1. Show that $\hat{\rho}$ is indeed a density matrix for a specific set of $\boldsymbol{r} \in \mathrm{BS}$.
2. For projective measurements along the axis $\boldsymbol{n}$ (with $\|\boldsymbol{n}\| = 1$), defined by the operator $\hat{\sigma}_{\boldsymbol{n}} = \boldsymbol{n}\cdot\hat{\boldsymbol{\sigma}}$, calculate the expectation value $\langle\hat{\sigma}_n\rangle$ and the probabilities, $p_\uparrow(\boldsymbol{n})$ and $p_\downarrow(\boldsymbol{n})$, for detecting the qubit in the "up" ($\hat{\sigma}_{\boldsymbol{n}}|\boldsymbol{n}\uparrow\rangle = +|\boldsymbol{n}\uparrow\rangle$) or "down" ($\hat{\sigma}_{\boldsymbol{n}}|\boldsymbol{n}\downarrow\rangle = -|\boldsymbol{n}\downarrow\rangle$) state.

In order to infer the density matrix many projective measurement along different axes $\boldsymbol{n}_i$ have to be performed. For simplicity we will here focus on the scenario of N_x, N_y, N_z measurements taken along the three basis axis x, y, z. The measurement outcomes are $N_{x\uparrow}, N_{x\downarrow}, N_{y\uparrow}, N_{y\downarrow}, N_{z\uparrow}, N_{z\downarrow}$, where $N_{i\uparrow}$ stands for the number of "up" states measured for direction i.

3. The simplest method of tomography is called "direct inversion". It assumes that the sample mean of the measurement outcomes $\langle\langle\hat{\sigma}_n\rangle\rangle$ equals the quantum mechanical mean $\langle\hat{\sigma}_n\rangle$. Derive an expression for the reconstructed Bloch vector $\boldsymbol{r}_d$ by calculating $\langle\hat{\sigma}_n\rangle$.

An alternative approach, known as the "Bayesian mean estimate", is to construct a likelihood $\mathcal{L}(\boldsymbol{r}) = \mathcal{L}(\hat{\rho}(\boldsymbol{r}))$ and interpret it as a density in state space (here BS). Calculating the weighted mean gives an estimate of a Boch vector as

$$\boldsymbol{r}_{\mathrm{BME}} = \frac{\int_{\mathrm{BS}} \boldsymbol{r}\mathcal{L}(\boldsymbol{r})d^3\boldsymbol{r}}{\int_{\mathrm{BS}} \mathcal{L}(\boldsymbol{r})d^3\boldsymbol{r}}.$$

4. Calculate the probability $P(N_{x\uparrow}, N_{x\downarrow}, N_{y\uparrow}, N_{y\downarrow}, N_{z\uparrow}, N_{z\downarrow}|\hat{\rho})$ of measuring $N_{x\uparrow}, N_{x\downarrow}, N_{y\uparrow}, N_{y\downarrow}, N_{z\uparrow}, N_{z\downarrow}$ given a density matrix $\hat{\rho}$.

Given a prior probability on density matrices $\mathcal{C}(\hat{\rho})$ the likelihood can then be defined as

$$\mathcal{L}(\hat{\rho}) = \mathcal{C}(\hat{\rho})P(N_{x\uparrow}, N_{x\downarrow}, N_{y\uparrow}, N_{y\downarrow}, N_{z\uparrow}, N_{z\downarrow}|\hat{\rho}).$$

5. As a final part of the exercise we want to compare the different methods by testing them on some "real" data. Open the jupyter notebook `tomography.ipynb` and reconstruct the density matrix of the state that was used to generate the data (`data_x_1,data_y_1,data_z_1` and `data_x_2,data_y_2,data_z_2`) with the direct inversion and the Bayesian mean estimate. What is the problem with `data_x_2,data_y_2,data_z_2`? Why does the direct inversion give nonphysical results and the Bayesian mean doesn't?

To read more on the problem, check Schmied, Roman. *"Quantum state tomography of a single qubit: comparison of methods."* Journal of Modern Optics 63.18 (2016): 1744-1758. `https://arxiv.org/abs/1407.4759`

II. DECOHERENCE OF SCHRÖDINGER'S CAT, A TOY MODEL

To read more on the problem, check pages 407-409, 141-143, 130-134 of Haroche, Serge, and J-M. Raimond. Exploring the quantum: atoms, cavities, and photons. Oxford university press, 2006.

In this exercise, we want to develop a toy model to understand the fast decoherence of quantum systems as they approach larger sizes (toward classical regime). The problem we want to study here are so called Schrödinger's cat states (SC) in an optical cavity coupled to a bath of ground-state harmonic oscillators. Schrödinger's cat states are superposition states made of two coherent states of same amplitude α but different phase ϕ

$$|\Psi_{\text{cat}}\rangle = \frac{e^{i\psi_1}}{\sqrt{2}}|e^{i\phi}\alpha\rangle + \frac{e^{i\psi_2}}{\sqrt{2}}|e^{-i\phi}\alpha\rangle\,.$$

In order to understand this problem in detail we first have to understand the decay of a coherent state inside a cavity. The cavity can be modeled as a harmonic oscillator of frequency ω_c coupled to a bath of harmonic oscillators of frequency ω_j. The corresponding annihilation operators are a and b_j. The cavity-bath coupling can be modeled with a beam-splinter Hamiltonian

$$H_{\text{int}} = -\sum_j \frac{\hbar g_j}{2}\left(a^\dagger b_j + b_j^\dagger a\right). \tag{1}$$

1. Describe the action of the unitary operator $U = \exp(-iH_{\text{BS}}\delta\tau/\hbar)$ on a coherent state $|\alpha, 0\rangle$ where

$$H_{\text{BS}} = \frac{\hbar g}{2}(e^{i\varphi}a^\dagger b + e^{-i\varphi}b^\dagger a)$$

describes the beam-splitter Hamiltonian of two modes a and b.

Hint: Write the coherent state in terms of the displacement operator $D(\alpha) = \exp[i\ (\alpha a^\dagger + \alpha^* a)]$ and use the identity $U^\dagger U = \mathbb{1}$. Use Baker-Hausdorff formula.
Result: $U|\alpha, 0\rangle = |\alpha\cos(\theta/2), ie^{-i\phi}\alpha\sin(\theta/2)\rangle$ where $\theta = g\ \delta\tau$.

2. Show that the evolution of a coherent state in the cavity can be described by an exponential depletion of the coherent state amplitude in time like

$$|\alpha e^{-\kappa t/2}\rangle \prod_j |\beta_j\rangle\,,$$

where the partial amplitudes β_i are such that:

$$\sum_j |\beta_j|^2 = \bar{n}(1 - e^{-t\kappa}). \tag{2}$$

Here $\kappa = \sum_j \frac{g_j^2\delta\tau}{4}$ and $\bar{n}$ is the initial intracavity photon number $|\alpha|^2 = \bar{n}$.

Hints:

- Consider the full hamiltonian of the system in case of many bath modes (Hamiltonian 1) and rewrite it in the interaction picture.
- Use the result of part 1 to calculate the state after a time evolution of time $\delta\tau$, $U(\delta\tau)(|\alpha\rangle \prod_j |0\rangle_j)$

- Look at the dynamics for a small time interval $\delta\tau$ for which you can assume the Hamiltonian to be static ($\theta_j \ll 1$)
- Assume that the amplitude of the environment states is sufficiently low such that the scattering process back into the cavity can be neglected and the evolution of the cavity field can be written as a consecutive application of the BS Hamiltonian like in part 1, $U(t)\left|\alpha\right\rangle \prod_j \left|0\right\rangle_j \sim (U(\delta\tau)(\left|\alpha\right\rangle \prod_j \left|0\right\rangle_j))^{\frac{t}{\delta\tau}}$.
- For small $\kappa\delta\tau \ll 1$ we find $(1 - \frac{\kappa\delta\tau}{2})^{t/\delta\tau} \approx e^{-\kappa t/2}$
- To show equation 2 use energy conservation.

3. Show that the coherence of Schrödinger's cat state in the cavity decays rapidly for large $\bar{n}$ as $\exp(-\bar{n}\kappa t)$.

   Hint: Explain why the result of part 2 of the exercise can be generalized to Schrödinger's cat states by taking a superposition of the result of the last part of the exercise. Write down the density matrix of the whole system (cavity + bath) and trace out the bath degrees of freedom. The coherence is the remaining term proportional to $\left|\alpha(t)e^{i\phi}\right\rangle \left\langle\alpha(t)e^{-i\phi}\right|$. Use the formula of the scalar product of two coherent states $\langle\alpha|\beta\rangle = e^{|\alpha|^2/2}e^{|\beta|^2/2}e^{\alpha^*\beta}$.

1. Show that $\hat{\rho}$ is indeed a density matrix for a specific set of $\boldsymbol{r} \in$ BS.

$$\mathrm{Tr}(\hat{\rho}) = 1 \quad , \hat{\rho} = \frac{1}{2}(\mathbb{I} + \vec{n}.\vec{\sigma}).$$

Operators can be represented by $\hat{O} = \frac{1}{2}\{\mathbb{I}\,\mathrm{Tr}(\hat{O}) + \vec{n}.\vec{\sigma}\}$

2. For projective measurements along the axis $\boldsymbol{n}$ (with $\|\boldsymbol{n}\| = 1$), defined by the operator $\hat{\sigma}_{\boldsymbol{n}} = \boldsymbol{n} \cdot \hat{\boldsymbol{\sigma}}$, calculate the expectation value $\langle \hat{\sigma}_{\boldsymbol{n}} \rangle$ and the probabilities, $p_{\uparrow}(\boldsymbol{n})$ and $p_{\downarrow}(\boldsymbol{n})$, for detecting the qubit in the "up" ($\hat{\sigma}_{\boldsymbol{n}} |\boldsymbol{n} \uparrow\rangle = +|\boldsymbol{n} \uparrow\rangle$) or "down" ($\hat{\sigma}_{\boldsymbol{n}} |\boldsymbol{n} \downarrow\rangle = -|\boldsymbol{n} \downarrow\rangle$) state.

$$\langle \hat{\sigma}_n \rangle = \mathrm{Tr}(\hat{\sigma}_n \rho) = \frac{1}{2}\mathrm{Tr}\left((\vec{n}.\vec{\sigma})(\mathbb{I} + \vec{r}\,\vec{\sigma})\right)$$

$$= \frac{1}{2}\left[\underbrace{\mathrm{Tr}(\vec{n}.\vec{\sigma})}_{0} + \mathrm{Tr}\left((\vec{n}.\vec{\sigma}).(\vec{r}.\vec{\sigma})\right)\right]$$

$$\sigma_i \sigma_j = \delta_{ij}\hat{\mathbb{I}} + i\varepsilon_{ijk}\sigma_k$$

$$= \frac{1}{2}\mathrm{Tr}\left[\delta_{nr}\mathbb{I} + i\varepsilon_{nrk}\sigma_k\right] \qquad \sigma_n \sigma_r = \delta_{nr} + i\varepsilon_{nrk}\sigma_k$$

$$= \frac{1}{2}\left[(\vec{n}.\vec{r})\underbrace{\mathrm{Tr}(\mathbb{I})}_{2} + i\,\underbrace{\mathrm{Tr}[i\varepsilon_{nrk}\sigma_k]}_{0}\right] = \vec{n}.\vec{r}$$

$|\langle \psi | n\uparrow \rangle|^2$

Born Distribution

$$P_{\uparrow n} = \mathrm{Tr}(\hat{\rho}\,|n\uparrow\rangle\langle n\uparrow|)$$

prob

$$= \frac{1}{2}\mathrm{Tr}(\hat{\rho}\,(\mathbb{I} + n\sigma))$$

$$\frac{1}{2}(\hat{\mathbb{I}} + \vec{n}.\vec{\sigma}) = \frac{1}{2}\left(|n\uparrow\rangle\langle n\uparrow| + \cancel{|n\downarrow\rangle\langle n\downarrow|} + |n\uparrow\rangle\langle n\uparrow| - \cancel{|n\downarrow\rangle\langle n\downarrow|}\right)$$

$|n\uparrow\rangle\langle n\uparrow| - |n\downarrow\rangle\langle n\downarrow|$

$|n\uparrow\rangle\langle n\uparrow| + |n\downarrow\rangle\langle n\downarrow|$

$$= \frac{1}{2}\mathrm{Tr}(\hat{\rho}(\hat{\mathbb{I}} + \underbrace{\vec{n}.\vec{\sigma}}_{\sigma_n})) = \frac{1}{2}\mathrm{Tr}(\hat{\rho}) + \frac{1}{2}\underbrace{\mathrm{Tr}(\rho\hat{\sigma}_n)}_{\langle \sigma_n \rangle}$$

$$= \frac{1}{2}\mathrm{Tr}(\hat{\rho}) + \frac{1}{2}\underbrace{\langle \sigma_n \rangle}_{\vec{n}.\vec{r}}$$

$$P_{n\uparrow} = \frac{1}{2}(1 + \vec{n}.\vec{r})$$

$$P_{n\downarrow} = \mathrm{Tr}\left(\hat{\rho}\, |n\downarrow\rangle\langle n\downarrow|\right) = \frac{1}{2}\mathrm{Tr}\left(\rho\,(\mathbb{I} - \vec{n}\vec{\sigma})\right)$$

$|\langle\psi|n\downarrow\rangle|^2$

$$|n\downarrow\rangle\langle n\downarrow| = \frac{1}{2}\left(\mathbb{I} - \vec{n}\cdot\vec{\sigma}\right) \qquad = \frac{1}{2}\left(1 - \vec{n}\cdot\vec{r}\right)$$

$$= \frac{1}{2}\left(\cancel{|n\uparrow\rangle\langle n\uparrow|} + |n\downarrow\rangle\langle n\downarrow| - \cancel{|n\uparrow\rangle\langle n\uparrow|} + |n\downarrow\rangle\langle n\downarrow|\right)$$

In order to infer the density matrix many projective measurement along different axes $\boldsymbol{n}_i$ have to be performed. For simplicity we will here focus on the scenario of N_x, N_y, N_z measurements taken along the three basis axis x, y, z. The measurement outcomes are $N_{x\uparrow}, N_{x\downarrow}, N_{y\uparrow}, N_{y\downarrow}, N_{z\uparrow}, N_{z\downarrow}$, where $N_{i\uparrow}$ stands for the number of "up" states measured for direction i.

3. The simplest method of tomography is called "direct inversion". It assumes that the sample mean of the measurement outcomes $\langle\langle\hat{\sigma}_n\rangle\rangle$ equals the quantum mechanical mean $\langle\hat{\sigma}_n\rangle$. Derive an expression for the reconstructed Bloch vector $\boldsymbol{r}_d$ by calculating $\langle\hat{\sigma}_n\rangle$.

$$P(N_{x\uparrow}, N_{x\downarrow}, N_{y\downarrow}, N_{y\downarrow}, N_{z\uparrow}, N_{z\downarrow})$$

$$= P(N_{x\uparrow}, N_{x\downarrow} | \hat{\rho}) \cdot P(N_{y\uparrow}, N_{y\downarrow} | \hat{\rho}) \cdot P(N_{z\uparrow}, N_{z\downarrow} | \hat{\rho})$$

$$P(N_{x\uparrow}, N_{x\downarrow} | \hat{\rho}) = P(N_{x\uparrow} | \rho) \cdot P(N_{x\downarrow} | \rho)$$

$$\binom{N_x}{N_x'} \quad \binom{N_{x\uparrow} + N_{x\downarrow}}{N_{x\uparrow}} \underbrace{P(n_{x\uparrow} | \rho)}_{\frac{1}{2}(1 + \vec{n}\cdot\vec{r}_x)}{}^{N_{x\uparrow}} \underbrace{P(n_{x\downarrow} | \rho)}_{\frac{1}{2}(1 - \vec{n}\cdot\vec{r}_x)}{}^{N_{x\downarrow}}$$

↳ # of combinations to get $N_{x\uparrow}, N_{x\downarrow}$

Thus,

$$P(N_{x\uparrow}, N_{x\downarrow}, \ldots | \hat{\rho}) = \binom{N_x}{N_{x\uparrow}} \left[\frac{1}{2}(1+x)\right]^{N_{x\uparrow}} \left[\frac{1}{2}(1-x)\right]^{N_{x\downarrow}}$$

$$\times \binom{N_y}{N_{y\uparrow}} \left[\frac{1}{2}(1+y)\right]^{N_{y\uparrow}} \left[\frac{1}{2}(1-y)\right]^{N_{y\downarrow}}$$

$$\times \binom{N_z}{N_{z\uparrow}} \left[\frac{1}{2}(1+z)\right]^{N_{z\uparrow}} \left[\frac{1}{2}(1-z)\right]^{N_{z\downarrow}}$$

Sample mean $\langle\langle\sigma_x\rangle\rangle$

$$\boldsymbol{r}_\mathrm{d} = \left(\frac{N_{x\uparrow} - N_{x\downarrow}}{N_{x\uparrow} + N_{x\downarrow}}, \frac{N_{y\uparrow} - N_{y\downarrow}}{N_{y\uparrow} + N_{y\downarrow}}, \frac{N_{z\uparrow} - N_{z\downarrow}}{N_{z\uparrow} + N_{z\downarrow}}\right).$$

An alternative approach, known as the "Bayesian mean estimate", is to construct a likelihood $\mathcal{L}(\boldsymbol{r}) = \mathcal{L}(\hat{\rho}(\boldsymbol{r}))$ and interpret it as a density in state space (here BS). Calculating the weighted mean gives an estimate of a Boch vector as

$$\boldsymbol{r}_{\text{BME}} = \frac{\int_{\text{BS}} \boldsymbol{r}\mathcal{L}(\boldsymbol{r})d^3\boldsymbol{r}}{\int_{\text{BS}} \mathcal{L}(\boldsymbol{r})d^3\boldsymbol{r}}.$$

4. Calculate the probability $P(N_{x\uparrow}, N_{x\downarrow}, N_{y\uparrow}, N_{y\downarrow}, N_{z\uparrow}, N_{z\downarrow}|\hat{\rho})$ of measuring $N_{x\uparrow}, N_{x\downarrow}, N_{y\uparrow}, N_{y\downarrow}, N_{z\uparrow}, N_{z\downarrow}$ given a density matrix $\hat{\rho}$. (3)

Given a prior probability on density matrices $\mathcal{C}(\hat{\rho})$ the likelihood can then be defined as

$$\mathcal{L}(\hat{\rho}) = \mathcal{C}(\hat{\rho})P(N_{x\uparrow}, N_{x\downarrow}, N_{y\uparrow}, N_{y\downarrow}, N_{z\uparrow}, N_{z\downarrow}|\hat{\rho}).$$

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$$H_{\text{int}} = -\sum_j \frac{\hbar g_j}{2}\left(a^\dagger b_j + b_j^\dagger a\right). \qquad (1)$$

mode a affects mode b

1. Describe the action of the unitary operator $U = \exp(-iH_{\text{BS}}\delta\tau/\hbar)$ on a coherent state $|\alpha, 0\rangle$ where

$$H_{\text{BS}} = \frac{\hbar g}{2}(e^{i\varphi}a^\dagger b + e^{-i\varphi}b^\dagger a)$$

describes the beam-splitter Hamiltonian of two modes a and b.

Hint: Write the coherent state in terms of the displacement operator $D(\alpha) = \exp[i\ (\alpha a^\dagger + \alpha^* a)]$ and use the identity $U^\dagger U = \mathbb{1}$. Use Baker-Hausdorff formula.
Result: $U|\alpha, 0\rangle = |\alpha\cos(\theta/2), ie^{-i\phi}\alpha\sin(\theta/2)\rangle$ where $\theta = g\ \delta\tau$.

$$|\alpha,0\rangle = D_a(\alpha)|0,0\rangle$$

$$U|\alpha,0\rangle = U D_a(\alpha) \underbrace{U^\dagger U}_{I} \underbrace{|0,0\rangle}_{} \quad (U|0,0\rangle \to |0,0\rangle)$$

$$U|0,0\rangle = \underbrace{\exp\left(-i\frac{g}{2}\delta\tau\left(\overbrace{e^{i\varphi}a^\dagger b + e^{-i\varphi}b^\dagger a}^{0}\right)\right)}_{1}|0,0\rangle = |0,0\rangle$$

$$a^\dagger b|0,0\rangle = 0\cdot|0,0\rangle \quad ; \quad b^\dagger a|0,0\rangle = 0$$

$U D_a(\alpha) U^\dagger$ $\quad\left(\theta = g\delta_z \;;\; G = -(e^{i\varphi}\hat{a}^\dagger b + e^{-i\varphi}\hat{a}\hat{b}^\dagger)\right)$

$$D_a(\alpha) = \exp\left[i\left(\alpha\hat{a}^\dagger + \alpha^*\hat{a}\right)\right]$$

How did we do this?

$$U D_a(\alpha) U^\dagger = \exp\left[i\left(\alpha\, U\hat{a}^\dagger U^\dagger + \alpha^*\, U a U^\dagger\right)\right]$$

$$U a U^\dagger = e^{iG\frac{\theta}{2}}\, a\, e^{-iG\frac{\theta}{2}} = a + \frac{i\theta}{2}[G,a] + \frac{i^2\theta^2}{2!\cdot 2^2}[G,[G,a]] + \dots \frac{i^n\theta^n}{n!\,2^n}[G,[\dots[G,a]]..]$$

e^a

Baker - Hausdorf Formula

$$[G,\hat{a}] = e^{i\varphi} b\,\underbrace{[\hat{a}^\dagger,\hat{a}]}_{1} = e^{i\varphi}\hat{b}$$

$$[G,[G,\hat{a}]] = [G, e^{i\varphi}\hat{b}] = [\cancel{e^{-i\varphi}} a\hat{b}^\dagger, \cancel{e^{i\varphi}}\hat{b}] = a[\hat{b}^\dagger,\hat{b}]$$

$$U a U^\dagger = a\left(1 + \frac{i^2\theta^2}{2!\cdot 2^2} + \text{even terms}\right) - e^{i\varphi} b\left(\frac{i\theta}{2} + \text{odd terms}\right)$$

$$= \hat{a}\cos\left(\frac{\theta}{2}\right) - i e^{i\varphi}\hat{b}\sin\left(\frac{\theta}{2}\right)$$

Thus, $$U D_a(\alpha) U^\dagger = \exp\left(i\left[\alpha\left(\cos\left(\frac{\theta}{2}\right)\hat{a} + i e^{i\varphi}\hat{b}\sin\left(\frac{\theta}{2}\right)\right) + \alpha^*\left(\cos\left(\frac{\theta}{2}\right)\hat{a}^\dagger - i e^{-i\varphi}\sin\left(\frac{\theta}{2}\right)\hat{b}^\dagger\right)\right]\right.$$

$$= D_a\left(\alpha\cos\frac{\theta}{2}\right) D_b\left(i\alpha e^{i\varphi}\sin\left(\frac{\theta}{2}\right)\right)$$

Therefore $\quad U D_a(\alpha) U^{+} \, |0,0\rangle = D_a\left(\alpha \cos\frac{\theta}{2}\right) D_b\left(i\alpha e^{i\varphi} \sin\frac{\theta}{2}\right)$

$$|\psi'\rangle = \left|\alpha \cos\frac{\theta}{2}, +i\alpha e^{i\varphi} \sin\left(\frac{\theta}{2}\right)\right\rangle$$

2. Show that the evolution of a coherent state in the cavity can be described by an exponential depletion of the coherent state amplitude in time like

$$|\alpha e^{-\kappa t/2}\rangle \prod_j |\beta_j\rangle ,$$

where the partial amplitudes β_i are such that:

$$\sum_j |\beta_j|^2 = \bar{n}(1 - e^{-t\kappa}). \tag{2}$$

Here $\kappa = \sum_j \frac{g_j^2 \delta\tau}{4}$ and $\bar{n}$ is the initial intracavity photon number $|\alpha|^2 = \bar{n}$.

Hints:

- Consider the full hamiltonian of the system in case of many bath modes (Hamiltonian 1) and rewrite it in the interaction picture.
- Use the result of part 1 to calculate the state after a time evolution of time $\delta\tau$, $U(\delta\tau)(|\alpha\rangle \prod_j |0\rangle_j)$
- Look at the dynamics for a small time interval $\delta\tau$ for which you can assume the Hamiltonian to be static ($\theta_j \ll 1$)
- Assume that the amplitude of the environment states is sufficiently low such that the scattering process back into the cavity can be neglected and the evolution of the cavity field can be written as a consecutive application of the BS Hamiltonian like in part 1, $U(t) |\alpha\rangle \prod_j |0\rangle_j \sim (U(\delta\tau)(|\alpha\rangle \prod_j |0\rangle_j))^{\frac{t}{\delta\tau}}$.
- For small $\kappa\delta\tau \ll 1$ we find $(1 - \frac{\kappa\delta\tau}{2})^{t/\delta\tau} \approx e^{-\kappa t/2}$
- To show equation 2 use energy conservation.

$$(1) \quad H_{int} = -\sum_j \frac{\hbar g_j}{2} \left(\hat{a}^{+} b_j + b_j^{+} \hat{a}\right)$$

$$\left.\begin{aligned} \hat{a}^{+} &\longrightarrow e^{i\omega_c t} \hat{a}^{+} \\ \hat{b}_j^{+} &\longrightarrow e^{i\omega_j t} \hat{b}_j^{+} \end{aligned}\right\} \text{ In the interaction frame}$$

$$\hat{b}_j \longrightarrow e^{-i\omega_j t} \hat{b}_j$$

rotating frame

$\delta_j = \omega_c - \omega_j$

$$\tilde{H}_{int} = -\sum_j \frac{\hbar g_j}{2} \left(\hat{a}^{+} b_j e^{i\omega_c t} e^{-i\omega_j t} + \hat{a} \hat{b}_j^{+} e^{i\omega_j t} e^{-i\omega_c t}\right)$$

$$\tilde{H}_{int} = -\sum_j \underbrace{\frac{\hbar g_j}{2} \left(\hat{a}^{+} b_j e^{i\delta_j t} + \hat{a} \hat{b}_j^{+} e^{-i\delta_j t}\right)}_{\tilde{H}_{int\,j}}$$

* For small $\delta\tau$ the Hamiltonian can be assumed to be static.

Initial state of the system $|\alpha\rangle \prod_j |0\rangle_j$

After $\delta\tau$, system will be:

$$U(\delta\tau)|\alpha\rangle \prod_j |0\rangle_j = \prod \exp\left[-i \sum_j \tilde{H}_{int_j} \delta\tau/\hbar\right] |\alpha\rangle \prod_j |0\rangle_j$$

commutator from Becker-Ham

$$= \left[\prod_j \exp\left[-i \tilde{H}_{int_j} \delta\tau/\hbar\right] + \cancel{O(\delta\tau^2)}\right] |\alpha\rangle \prod_j |0\rangle_j$$

Small angle approximation

How?

$$= |\alpha \textstyle\prod_j \cos(\frac{\theta_j}{2})\rangle \prod_j |i\alpha \prod_{k<j} \cos(\frac{\theta_k}{2}) e^{i\varphi_j} \sin\frac{\theta_j}{2}\rangle$$

$$= |\alpha\left(1 - \sum_j \left(\frac{\theta_j}{2}\right)^2\right)\rangle \prod_j |i\alpha e^{i\varphi_j} \frac{\theta_j}{2}\rangle$$

Approximation

$\sin(\theta) \approx \theta$

$$\theta_j = g_j \delta\tau$$

small amplitudes compared to α

$$= |\alpha\left(1 - \underbrace{\sum_j \frac{g_j^2 \delta\tau}{2}}_{K} \frac{\delta\tau}{2}\right)\rangle \prod_j |\ldots\rangle_j$$

$$= |\alpha\left(1 - \frac{k}{2}\delta\tau\right)\rangle \prod_j |\ldots\rangle_j$$

At state at time $t = n\delta t$

$$|\psi(t)\rangle = |\alpha\left(1 - \frac{k}{2}\delta\tau\right)^n\rangle \prod_j |\beta_j\rangle \quad \text{where } |\beta_j| \ll |\alpha|$$

$$\boxed{|\psi(t)\rangle = |\alpha \overbrace{\left(1 - \frac{k}{2}\frac{t}{n}\right)^n}^{e^{-\frac{k}{2}t}}\rangle \prod_j |\beta_j\rangle = |\alpha e^{-\frac{k}{2}t}\rangle \prod_j |\beta_j\rangle}$$

Photon number $\bar{n} = |\alpha|^2$ → Initially

Initial ph # $=$ Final ph #

$$\bar{n} = |\alpha|^2 e^{-\kappa t} + \sum_j |\beta_j|^2$$

$$\underset{|\alpha|^2}{\bar{n}}\,(1 - e^{-\kappa t}) = \sum_j |\beta_j|^2$$

3. Show that the coherence of Schrödinger's cat state in the cavity decays rapidly for large $\bar{n}$ as $\exp(-\bar{n}\kappa t)$.

Hint: Explain why the result of part 2 of the exercise can be generalized to Schrödinger's cat states by taking a superposition of the result of the last part of the exercise. Write down the density matrix of the whole system (cavity + bath) and trace out the bath degrees of freedom. The coherence is the remaining term proportional to $|\alpha(t)e^{i\phi}\rangle\langle\alpha(t)e^{-i\phi}|$. Use the formula of the scalar product of two coherent states $\langle\alpha|\beta\rangle = e^{|\alpha|^2/2}e^{|\beta|^2/2}e^{\alpha^*\beta}$.

↑ $|C+\rangle\langle C-|$

Initial State $\quad |\Psi_{cat}\rangle \prod_j |0\rangle_j = \frac{1}{\sqrt{2}}\left(|\alpha e^{i\phi}\rangle \prod_j |0\rangle + |\alpha e^{-i\phi}\rangle \prod_j |0\rangle\right)$

Using part2)

$$|\Psi(t)\rangle = \frac{1}{\sqrt{2}}\Big(\overbrace{|\alpha e^{i\phi} e^{-\frac{\kappa}{2}t}\rangle}^{|+\rangle} \overbrace{\prod_j |\beta_j e^{i\phi}\rangle}^{|B+\rangle} + \overbrace{|\alpha e^{-i\phi} e^{-\frac{\kappa}{2}t}\rangle}^{|C-\rangle} \overbrace{\prod_j |\beta_j e^{-i\phi}\rangle}^{|B-\rangle}\Big)$$

$$= \frac{1}{\sqrt{2}}\left(|C+\rangle|B+\rangle + |C-\rangle|B-\rangle\right)$$

$$\rho(t) = |\Psi(t)\rangle\langle\Psi(t)| = \frac{1}{2}\Big(|C+\rangle\langle C+| \, |B+\rangle\langle B+| + |C+\rangle\langle C-| \, |B+\rangle\langle B-| +$$

$$|C-\rangle\langle C-| \, |B-\rangle\langle B-| + |C-\rangle\langle C-| \, |B-\rangle\langle B-| \Big)$$

$$\rho_A(t) = Tr_B(\rho(t)) = \frac{1}{2}|C+\rangle\langle C+| \overbrace{Tr(|B+\rangle\langle B+|)}^{1} + \frac{1}{2}|C+\rangle\langle C-| \, Tr(|B+\rangle\langle B-|)$$

$$+ \frac{1}{2}|C-\rangle\langle C-| \, Tr(|B-\rangle\langle B-|) + \frac{1}{2}|C-\rangle\langle C-| \, Tr(|B+\rangle\langle B-|)$$

Then coherence is : $Tr(|B+\rangle\langle B-|)$

$$\sum_{\Psi \in Basis} \langle\Psi|B+\rangle\langle B-|\Psi\rangle$$

or from cyclicity

$$= \langle B+|B+\rangle\langle B-|B+\rangle \qquad Tr(|B+\rangle\langle B-|) = Tr(\langle B-|B+\rangle)$$

$$\prod_j \langle \beta_j e^{-i\phi} | \beta_j e^{+i\phi} \rangle$$

$\downarrow$ using formula $\underbrace{\langle\alpha|\beta\rangle}_{\text{coherent state}} = e^{|\alpha|^2/2} e^{|\beta|^2/2} e^{\alpha^*\beta}$

$$\prod_j e^{\frac{|\beta_j|^2}{2}} e^{\frac{|\beta_j|^2}{2}} e^{\beta_j^2 e^{2i\phi}} = \prod_j e^{\beta_j^2} e^{\beta_j^2 e^{2i\phi}}$$

$$\exp\Big[\underbrace{\sum_j \beta_j^2}_{\bar{n}(1-e^{-kt})} (1 + \underbrace{e^{2i\phi}}_{-kt}) \Big] \propto \exp(-\bar{n}kt)$$