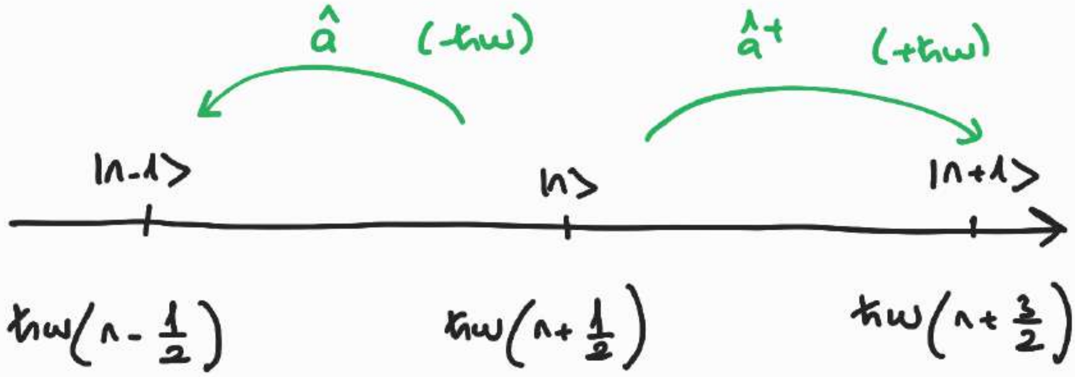
COHERENT STATES

REVIEW

Quantum Harmonic Oscillator:

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m \omega^2 \hat{x}^2 = \hbar\omega\left(\hat{a}^+\hat{a} + \frac{1}{2}\right) = \hbar\omega\left(\hat{N} + \frac{1}{2}\right)$$

$$\hat{H}|n\rangle = E_n|n\rangle \qquad E_n = \hbar\omega\left(n + \frac{1}{2}\right), \quad n = 0, 1, 2, \ldots$$



Coherent States (canonical coherent states)

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$$

It's not Hermitian since $\alpha \in \mathbb{C}$

$$\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle$$

Energy eigenstates

Coherent states in the energy basis

$$\hat{H}|n\rangle = E_n|n\rangle$$

$$\boxed{|\alpha\rangle = \sum_{n=0}^{\infty} c_n(\alpha)|n\rangle} \qquad c_n(\alpha) = \langle\alpha|n\rangle$$

$$\hat{a}|\alpha\rangle = \hat{a}\sum_{n=0}^{\infty} c_n(\alpha)|n\rangle = \sum_{n=0}^{\infty} c_n(\alpha)\,\hat{a}|n\rangle$$

$n \geq 1$: $\hat{a}|n\rangle \rightarrow \sqrt{n}|n-1\rangle$, $n=0$: $\hat{a}|n\rangle = 0$

$$\hat{a}|\alpha\rangle = \sum_{n=1}^{\infty} c_n(\alpha)\sqrt{n}\,|n-1\rangle$$

$n \longrightarrow n+1$: $\quad \hat{a}|\alpha\rangle = \sum_{n=0}^{\infty} c_{n+1}(\alpha)\sqrt{n+1}\,|n\rangle$

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle \Rightarrow \sum_{n=0}^{\infty} c_{n+1}(\alpha)\sqrt{n+1}\,|n\rangle = \alpha\sum_{n=0}^{\infty} c_n(\alpha)|n\rangle$$

$$\Rightarrow \sum_{n=0}^{\infty}\Big[\underbrace{c_{n+1}(\alpha)\sqrt{n+1} - \alpha c_n(\alpha)}_{=0}\Big]|n\rangle = 0$$

$$c_{n+1}(\alpha) = \frac{\alpha}{\sqrt{n+1}}c_n(\alpha) = \frac{\alpha}{\sqrt{n+1}}\frac{\alpha}{\sqrt{n}}c_{n-1}(\alpha) = \dots$$

$$c_n(\alpha) = \frac{\alpha}{\sqrt{n}} \frac{\alpha}{\sqrt{n-1}} \cdots\cdots \frac{\alpha}{\sqrt{2}} \frac{\alpha}{\sqrt{1}} c_0(\alpha)$$

$$\boxed{c_n(\alpha) = \frac{\alpha^n}{\sqrt{n!}} c_0(\alpha)}$$

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n(\alpha)|n\rangle = \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} c_0(\alpha)|n\rangle$$

Normalization:

not depend on

$$1 = \sum_{n=0}^{\infty} |c_n(\alpha)|^2 = \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} |c_0(\alpha)|^2$$

$$= |c_0(\alpha)|^2 \underbrace{\sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!}}_{e^{|\alpha|^2}}$$

$$1 = |c_0(\alpha)|^2 e^{|\alpha|^2} \rightarrow |c_0(\alpha)|^2 = e^{-|\alpha|^2}$$

$$c_0(\alpha) = e^{-|\alpha|^2/2}$$

$$|\alpha\rangle = c_0(\alpha) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

Poisson distribution: (IMPORTANT OBSERVATION for QUANTUM OPTICS)

$$\underbrace{P}_{\text{prob.}}(\underbrace{E_n}_{\text{energy level } n}) = |c_n(\alpha)|^2 = \left| e^{-|\alpha|^2/2} \frac{\alpha^n}{\sqrt{n!}} \right|^2 = \frac{|\alpha|^{2n}}{n!} e^{-|\alpha|^2}$$

Time Evolution

$\hat{H}$ is time independent

$$|\Psi(0)\rangle = |\alpha_0\rangle = e^{-|\alpha_0|^2/2} \sum_{n=0}^{\infty} \frac{\alpha_0^n}{\sqrt{n!}} |n\rangle$$

$$|\Psi(t)\rangle = e^{-|\alpha_0|^2/2} \sum_{n=0}^{\infty} \frac{\alpha_0^n}{\sqrt{n!}} e^{-iE_n t/\hbar} |n\rangle$$

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right)$$

$$|\Psi(t)\rangle = e^{-|\alpha_0|^2/2} \sum_{n=0}^{\infty} \frac{\alpha_0^n}{\sqrt{n!}} e^{-i\hbar\omega\left(n+\frac{1}{2}\right)t/\hbar} |n\rangle$$

$$= e^{-i\omega t/2} e^{-|\alpha_0|^2/2} \sum_{n=0}^{\infty} \frac{(\alpha_0 e^{-i\omega t})^n}{n!} |n\rangle$$

$$|\Psi(0)\rangle = |\alpha_0\rangle \longrightarrow |\Psi(t)\rangle = e^{-i\omega t/2}\, e^{-\underbrace{|\alpha_0|^2}_{|\alpha|^2}/2} \sum_{n=0}^{\infty} \frac{(\underbrace{\alpha_0 e^{-i\omega t}}_{\alpha})^n}{n!} |n\rangle$$

$$|\Psi(t)\rangle = e^{-i\omega t/2} |\alpha\rangle \quad \text{for} \quad \alpha = \alpha_0 e^{-i\omega t}$$

$$|\Psi(0)\rangle = |\alpha_n\rangle \longrightarrow |\Psi(t)\rangle = e^{-i\omega t/2} |\alpha = e^{-i\omega t}\alpha_0\rangle$$

global phase factor
does not change physics

Displacement Operator

$$|n\rangle = \frac{(\hat{a}^+)^n}{\sqrt{n!}} |0\rangle$$

$$\hat{a}|n\rangle = \sqrt{n}\, |n-1\rangle$$

$$\hat{a}^+|n\rangle = \sqrt{n+1}\, |n+1\rangle$$

$$\hat{D}(\alpha) = e^{\alpha \hat{a}^+ - \alpha^* \hat{a}} \qquad \alpha \in \mathbb{C}$$

$$\left.\begin{array}{l}[\hat{A},[\hat{A},\hat{B}]]=0\\ [\hat{B},[\hat{A},\hat{B}]]=0\end{array}\right\} e^{\hat{A}+\hat{B}}=e^{\hat{A}}e^{\hat{B}}e^{-\frac{1}{2}[\hat{A},\hat{B}]}$$

(BCH Formula)

$$[\hat{a},\hat{a}^+]=1 \Rightarrow [\hat{a},[\hat{a},\hat{a}^+]]=0, \quad [\hat{a}^+,[\hat{a},\hat{a}^+]]=0$$

$$\hat{D}(\alpha)=e^{\alpha\hat{a}^+-\alpha^*\hat{a}}=e^{\alpha\hat{a}^+}e^{-\alpha^*\hat{a}}e^{-\frac{1}{2}[\alpha\hat{a}^+,-\alpha^*\hat{a}]}$$

$$[\alpha\hat{a}^+,-\alpha^*\hat{a}]=-\alpha\alpha^*[\hat{a}^+,\hat{a}]=-|\alpha|^2.(-1)=|\alpha|^2$$

$$\boxed{\hat{D}(\alpha)=e^{\alpha\hat{a}^+}e^{-\alpha^*\hat{a}}.e^{-|\alpha|^2/2}}$$

$$|\alpha\rangle=\hat{D}(\alpha)|0\rangle$$

$$\hat{D}(\beta)|\alpha\rangle=\hat{D}(\beta)\hat{D}(\alpha)|0\rangle=e^{(\beta\alpha^*-\beta^*\alpha)/2}\underbrace{\hat{D}(\alpha+\beta)|0\rangle}_{|\alpha+\beta\rangle}$$

$$\hat{D}(\beta)|\alpha\rangle=e^{(\beta\alpha^*-\beta^*\alpha)/2}|\alpha+\beta\rangle$$

"Displacement Operator"

A. Partially Entangled Bell State $|\Phi_{AB}\rangle$

$$|\Phi(0)\rangle = |\Phi_{AB}\rangle \otimes |\alpha_A, \alpha_B\rangle$$

$$= \left(\cos\alpha \, |e_A, e_B\rangle + \sin\alpha \, |g_A, g_B\rangle \right) \otimes |\alpha_A, \alpha_B\rangle$$

$$= \cos\alpha \, |e_A, \alpha_A\rangle \otimes |e_B, \alpha_B\rangle + \sin\alpha \, |g_A, \alpha_A\rangle \otimes |g_B, \alpha_B\rangle$$

CAN IT BE DONE

$$|e_A, \alpha_A\rangle = |e_A, \overbrace{\hat{D}(\alpha_A)|0\rangle}^{|\alpha_A\rangle}\rangle \overset{?}{=} \hat{D}(\alpha_A) \, |e_A, 0\rangle$$

displacement operator

$$\hat{D}(\alpha_A) = e^{-|\alpha|^2/2} \, e^{(\alpha \hat{a}^{+} - \alpha^{*} \hat{a})}$$

$$|\Phi(0)\rangle = \cos\alpha \left(\hat{D}(\alpha_A) |e_A, 0_a\rangle \otimes \hat{D}(\alpha_B) |e_B, 0_b\rangle \right)$$
$$+ \sin\alpha \left(\hat{D}(\alpha_A) |g_A, 0_a\rangle \otimes \hat{D}(\alpha_B) |g_B, 0_b\rangle \right)$$

$$|\Phi(0)\rangle = \cos\alpha \left(\hat{D}(\alpha_A) \left(c_0 |\Psi_1^{+}\rangle_A - s |\Psi_1^{-}\rangle_A \right) \otimes \right.$$
$$\left. \left(\hat{D}(\alpha_B) \left(c_0 |\Psi_1^{+}\rangle_B - s |\Psi_1^{-}\rangle_B \right) \right) + \right.$$
$$\sin\alpha \left(\hat{D}(\alpha_A) |\Psi^0\rangle_A \otimes \hat{D}(\alpha_B) |\Psi^0_B\rangle \right)$$

$$H_{JC}\,|\psi_n^{\pm}\rangle = \lambda_n^{\pm}\,|\psi_n^{\pm}\rangle$$

$$|\psi^{\pm}(t)\rangle = e^{-i\lambda^{\pm}t}\,|\psi^{\pm}(0)\rangle$$

$$|\Phi(t)\rangle = \cos\alpha\left(\hat{D}(\alpha_A)\left(c_0 e^{-i\lambda^+ t}|\psi_1^+\rangle_A - s\, e^{-i\lambda^- t}|\psi_1^-\rangle_A\right)\otimes\right.$$
$$\left.\left(\hat{D}(\alpha_B)\left(c_0 e^{-i\lambda^+ t}|\psi_1^+\rangle_B - s\, e^{-i\lambda^- t}|\psi_1^-\rangle_B\right)\right) +$$
$$\sin\alpha\left(\hat{D}(\alpha_A)|\psi^0\rangle_A \otimes \hat{D}(\alpha_B)|\psi^0_B\rangle\right)$$

$$|\Phi(t)\rangle = \cos\alpha\left(\hat{D}(\alpha_A)\left(c_0 e^{-i\lambda^+ t}\left(c_0|e_A,0_a\rangle + s_0|g_A,1_a\rangle\right) - \right.\right.$$
$$\left. s_0 e^{-i\lambda^- t}\left(-s_0|e_A,0_a\rangle + c_0|g_A,1_a\rangle\right)\right)\otimes$$
$$\left(\hat{D}(\alpha_B)\left(c_0 e^{-i\lambda^+ t}\left(c_0|e_B,0_b\rangle + s_0|g_B,1_b\rangle\right) - \right.\right.$$
$$\left. s_0 e^{-i\lambda^- t}\left(-s_0|e_B,0_b\rangle + c_0|g_B,1_b\rangle\right)\right) +$$

$$+\sin\alpha\,\hat{D}(\alpha_A)|g_A,0_a\rangle \otimes \hat{D}(\alpha_B)|g_B,0_b\rangle$$

$$\left.\begin{aligned}\hat{D}(\alpha_A) &= e^{\alpha_A\hat{a}^+}\,e^{-\alpha_A^*\hat{a}}\cdot e^{-|\alpha_A|^2/2}\\ \hat{D}(\alpha_B) &= e^{\alpha_B\hat{a}^+}\,e^{-\alpha_B^*\hat{a}}\,e^{-|\alpha_B|^2/2}\end{aligned}\right\}\quad \hat{D}(\alpha_A)\hat{D}(\alpha_B) = e^{(\alpha_A+\alpha_B)\hat{a}^+}\cdot e^{-(\alpha_A^*+\alpha_B^*)\hat{a}}\cdot e^{-\frac{(|\alpha_A|^2+|\alpha_B|^2)}{2}}$$

$$\Phi(t) = x_1 \,|\uparrow\uparrow 00\rangle + x_2 \,|\downarrow\downarrow 11\rangle + x_3 \,|\uparrow\downarrow 01\rangle$$

$$+ x_4 \,|\downarrow\uparrow 10\rangle + x_5 \,|\downarrow\downarrow 00\rangle$$

$$\rightarrow x_1 = \cos\alpha \left(\hat{D}(\alpha_A)\hat{D}(\alpha_B)\, c_0^2 e^{-i\lambda^+ t} + \hat{D}(\alpha_A)\hat{D}(\alpha_B)\, s_0^2 e^{-i\lambda^- t} \right)^2$$

$$= \cos\alpha\, \underbrace{\hat{D}(\alpha_A)\hat{D}(\alpha_B)} \left(\underbrace{c_0^2}_{L} e^{-i\lambda^+ t} + \underbrace{s_0^2}_{M} e^{-i\lambda^- t} \right)$$

$$\rightarrow x_2 = \cos\alpha \left(\hat{D}(\alpha_A)\hat{D}(\alpha_B)\, c_0^2 s_0^2 e^{-i\lambda^+ t} - \hat{D}(\alpha_A)\hat{D}(\alpha_B)\, c_0^2 s_0^2 e^{-i\lambda^- t} \right)^2$$

$$= \cos\alpha\, \hat{D}(\alpha_A)\hat{D}(\alpha_B) \left(\underbrace{c_0^2 s_0^2}_{LM} e^{-i\lambda^+ t} - \underbrace{c_0^2 s_0^2}_{LM} e^{-i\lambda^- t} \right)^2$$

$$\rightarrow x_3 \,|e_A, 0_a\rangle \otimes |g_B, 1\rangle$$

$$\cos\alpha\, \hat{D}(\alpha_A) \left(c_0^2 e^{-i\lambda^+ t} + s_0^2 e^{-i\lambda^- t} \right) |e_A, 0_a\rangle \otimes$$

$$\hat{D}(\alpha_B) \left(c_0 s_0 e^{-i\lambda^+ t} - c_0 s_0 e^{-i\lambda^- t} \right) |g_B, 1_b\rangle$$

$$x_3 = \cos\alpha\, \hat{D}(\alpha_A)\hat{D}(\alpha_B) \left(\underbrace{c_0^2}_{L} e^{-i\lambda^+ t} + \underbrace{s_0^2}_{M} e^{-i\lambda^- t} \right) \left(\underbrace{c_0 s_0}_{\sqrt{LM} = N} e^{-i\lambda^+ t} - \underbrace{c_0 s_0}_{\sqrt{LM}} e^{-i\lambda^- t} \right)$$

$$x_3 = \cos\alpha\, N \left(e^{-i\lambda^+ t} - e^{-i\lambda^- t} \right) \left(L e^{-i\lambda^+ t} + M e^{-i\lambda^- t} \right) \hat{D}(\alpha_A)\hat{D}(\alpha_B)$$

$$\rightarrow x_4 = x_3$$

$$\rightarrow x_5 = \sin\alpha$$

$\Phi(t)$ in terms of coherent states

$$\Phi(t) = x_1 |\uparrow\uparrow 00\rangle + x_2 |\downarrow\downarrow 11\rangle + x_3 |\uparrow\downarrow 01\rangle + x_4 |\downarrow\uparrow 10\rangle + x_5 |\downarrow\downarrow 00\rangle$$

x_3 $\quad$ $\sin\alpha$

$\hat{D}(\alpha_A)\hat{D}(\alpha_B) x_1'$ $\qquad$ $\hat{D}(\alpha_A)\hat{D}(\alpha_B) x_2'$ $\qquad$ $\hat{D}(\alpha_A)\hat{D}(\alpha_B) x_3'$

$$\hat{D}(\alpha_A)|0\rangle = \alpha_A$$

$$\hat{D}(\alpha_A)|1\rangle = \hat{D}(\alpha_A)\underbrace{\hat{D}(1)|0\rangle}_{|1\rangle} = e^{(\alpha_A 1^* - \alpha_A^* \cdot 1)/2}\underbrace{\hat{D}(\alpha_A+1)}_{|\alpha_A+1\rangle}$$

$$\hat{D}(\alpha_B)|0\rangle = \alpha_B$$

$$\hat{D}(\alpha_B)|1\rangle = e^{(\alpha_B - \alpha_B^*)/2}|\alpha_B+1\rangle$$

$$\boxed{\begin{aligned}\Phi(t) = {} & x_1' |\uparrow\uparrow \alpha_A\alpha_B\rangle + x_2' e^{\frac{(\alpha_A-\alpha_A^*)+(\alpha_B-\alpha_B^*)}{2}} |\downarrow\downarrow(\alpha_A+1)(\alpha_B+1)\rangle \\ & + x_3' e^{(\alpha_B-\alpha_B^*)/2} |\uparrow\downarrow \alpha_A (\alpha_B+1)\rangle + x_4' e^{(\alpha_A-\alpha_A^*)/2} |\downarrow\uparrow(\alpha_A+1)\alpha_B\rangle \\ & + x_5 |\downarrow\downarrow 00\rangle\end{aligned}}$$

A.1 $C_{AB}(t)$

$$C_{AB}(t) = 2|x_1||x_5| - 2|x_3||x_4|$$

$$|x_3| = \left|x_3' \, e^{(\alpha_B - \alpha_B^*)/2}\right| = |x_3'| \left|e^{(\alpha_B-\alpha_B^*)/2}\right| = |x_3'| \, |\underbrace{e^{ib_B}}_{1}|$$

Assume $\alpha_B = a_B + i b_B$ $\quad \alpha_B - \alpha_B^* = (a_B + i b_B) - (a_B - i b_B)$

$$= +2 i b_B$$

Then $C_{AB}(t)$ is same as with the 1 photon case:

$$C_{AB}(t) = \cos^2\left(\frac{Gt}{2}\right)\left[\,|\sin(2\alpha)| - 2\sin^2\left(\frac{Gt}{2}\right)\cos^2\alpha\right]$$

Then, other concurrence results will be same as with single mode case investigated before.

WHAT IF IT CAN'T BE DONE?

$$|e_A, \alpha_A\rangle = |e_A, \overbrace{\hat{D}(\alpha_A)|0\rangle}^{|\alpha_A\rangle}\rangle = \hat{D}(\alpha_A)\,|e_A, 0\rangle$$