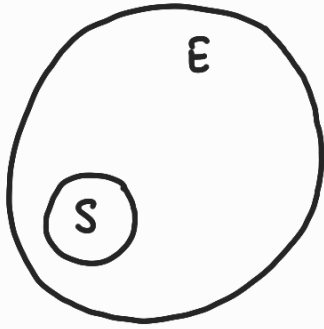


THE MASTER EQUATION



$$\mathcal{E}_S(\rho_S) = \text{Tr}_E \left(U_{SE} \rho_S(0) \otimes \rho_E(0) U_{SE}^\dagger \right)$$

↓
completely positive map

Assumption: At t_0 , $\rho_{SE}(0) = \rho_S(0) \otimes \rho_E(0)$

- E.g. Either
- System has been insulated from E design at $t=t_0$
 - System and environment are weakly coupled.

Hilbert Space : $\mathcal{H}_S \otimes \mathcal{H}_E$

↓

often finite dimensional

often infinite # of harmonic oscillators

Dynamics: $H_{\text{tot}} = \underbrace{H_S + H_E}_{H_0 \text{ "free part"}} + V$

↓
interaction btw the system and the env.

Assume, "reservoir" or environment is in thermal equilibrium at temperature T :

$$\rho_E(0) = \frac{e^{-\frac{H_E}{k_B T}}}{\text{Tr}(e^{-H_E/k_B T})} = \frac{e^{-\beta H_E}}{\text{Tr}(e^{-\beta H_E})}$$

Total state of S+E @ time t :

$$\rho_{\text{tot}}(t) = e^{-iH_{\text{tot}}t} (\rho_S(0) \otimes \rho_E(0)) e^{iH_{\text{tot}}t}$$

$$\frac{d}{dt} \rho_{\text{tot}}(t) = -i [H_0 + V, \rho_{\text{tot}}(t)]$$

$$\rho_S(t) = \text{Tr}_E(\rho_{\text{tot}}(t))$$

Assume $H_0 = H_S + H_E$ is solvable

↙
finite
dim.

↘
quasi-free ?

Move to rotating frame: (interaction picture)

$$V_{\text{int}}(t) = e^{iH_0t} V e^{-iH_0t}$$

$$\rho_{\text{int,tot}}(t) = e^{-iH_0t} \rho_{\text{tot}}(t) e^{iH_0t}$$

$$\frac{d}{dt} \rho_{\text{int,tot}}(t) = -i [V_{\text{int}}(t), \rho_{\text{int,tot}}(t)] \quad (*)$$

$$\begin{aligned} \frac{d}{dt} \rho_{\text{int,tot}}(t) = & +i \left(e^{-iH_0t} [H_0, \rho_{\text{int,tot}}(t)] e^{iH_0t} \right) \quad ? \text{ How?} \\ & -i \left(e^{-iH_0t} [H_0 + V, \rho_{\text{int,tot}}(t)] e^{iH_0t} \right) \end{aligned}$$

We drop the "int" subscript. Now (*)

$$\rho_{\text{tot}}(t) = \int_0^t \frac{d}{ds} \rho_{\text{tot}}(s) ds + \rho_{\text{tot}}(0)$$

$$\rho_{\text{tot}}(t) = -i \int_0^t [V(s), \rho_{\text{tot}}(s)] ds + \rho_{\text{tot}}(0) \quad (**)$$

Substituting (**) \rightarrow (*)

$$\frac{d}{dt} \rho_{\text{tot}}(t) = -i [V(t), \rho_{\text{tot}}(0)] - \int_0^t dt_1 [V(t), [V(t_1), \rho_{\text{tot}}(t_1)]]$$

$$\frac{d}{dt} \rho_S(t) = -i \text{Tr}_E \left([V(t), \rho_{\text{tot}}(0)] \right) - \int_0^t dt_1 \text{Tr}_E [V(t), [V(t_1), \rho_{\text{tot}}(t_1)]] \quad (***)$$

Assumption 1: $\rho_{\text{tot}}(0) = \rho_S(0) \otimes \rho_E(0)$

Split $V(t)$ into $V(t) = V_S(t) + V_{SE}(t)$ where $V_S = V_S \otimes I_E$

$$\text{and } \text{Tr}_E (V_{SE}(t) \rho_{\text{tot}}(0)) = 0$$

Assumption 2: $V = \lambda H_I$ with λ small

Assumption 3 (Born Approximation)

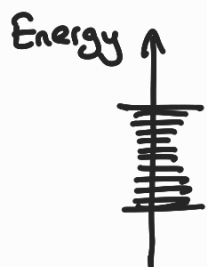
We can replace $\rho_{\text{tot}}(t_1)$ in (***) by

$$\rho_{\text{tot}}(t_1) = \rho_S(t_1) \otimes \rho_E(t_1) \cong \rho_S(t_1) \otimes \rho_E(0)$$

$$\frac{d}{dt} \rho_S(t) = -i [V_S(t), \rho_S(0)] - \int_0^t \text{Tr}_E \left([V_{SE}(t), [V_{SE}(t_1), \rho_S(t_1) \otimes \rho_E(0)]] \right) dt_1$$

Nonlocal in time

Assumption 4: The physics of bath is such that system couples roughly equally to many closely spaced energy levels of bath.



$$\rho_S(t_1) \simeq \rho_S(t)$$

$$\frac{d}{dt} \rho_S(t) = -i [V_S(t), \rho_S(0)] - \int_0^t dt_1 \text{Tr}_E \left([V_{SE}(t), [V_{SE}(t_1), \rho_S(t) \otimes \rho_E(0)]] \right)$$

"Red-Field Equation"

Assumption 5: (Markov Approximation - Memoryless)

Assume we have memoryless bath and replace lower limit of integral by $-\infty$:

$$\frac{d}{dt} \rho_S(t) = -i [V_S(t), \rho_S(0)] - \int_{-\infty}^t dt_1 \text{Tr}_E \left([V_{SE}(t), [V_{SE}(t_1), \rho_S(t) \otimes \rho_E(0)]] \right)$$

"Born-Markov Master Equation"

(Here, energy levels are continuous)

Requires H_E to have continuous spectra in relevant energy

The - radiative - damping master equation:

Following Bohr & Einstein, Wigner & Weisskopf explained how radiative decay of atom from excited state could be explained in Q.M.

Environment : Free-Field

$$H_E = \sum_k \omega_k \hat{b}_k^\dagger \hat{b}_k \quad \text{with} \quad [\hat{b}_k, \hat{b}_l^\dagger] = \delta_{k,l}$$

System: $H_S = \omega_a \frac{\sigma^z}{2}$

Interaction : "dipole coupling"

$$V = \sum_k (g_k \hat{b}_k + g_k \hat{b}_k^\dagger) (\sigma^+ + \sigma^-) \quad \text{where}$$

$$\sigma^+ = |0\rangle\langle 1|, \quad \sigma^- = |1\rangle\langle 0| \quad \text{with } g_k \text{ proportional to}$$

dipole matrix element for transition: they depend on

structure of mode k :

$$g_k = \frac{1}{\sqrt{V_k}}$$

V_k = volume of mode k

(?)

$$V_{int}(t) = \sum_k \left(g_k \hat{b}_k e^{-i\omega_k t} + g_k \hat{b}_k^\dagger e^{i\omega_k t} \right) (\sigma^+ e^{i\omega_a t} + \sigma^- e^{-i\omega_a t})$$

$\omega_a \sim 10^{15} \text{ s}^{-1}$. we approximate V_{int} by:

$$V_{int}(t) \approx \underbrace{\sum_k g_k \hat{b}_k \sigma^+ e^{-i(\omega_k - \omega_a)t}}_{V_+} + \underbrace{\sum_k g_k \hat{b}_k^\dagger \sigma^- e^{i(\omega_k - \omega_a)t}}_{V_-}$$

Rotating Wave Approximation (RWA)

Now choose $P_E(0) = |\Omega \times \Omega\rangle \rightarrow \text{vacuum, no photon}$

In Born Approximation:

$$\frac{d}{dt} P_S(t) = - \int_0^t dt_1 \left\{ \overset{\text{gamma}}{\Gamma(t-t_1)} (\sigma^+ \sigma^- P_S(t_1) - \sigma^- P_S(t_1) \sigma^+) + \text{h.c.} \right\}$$

hermitian conjugate \uparrow