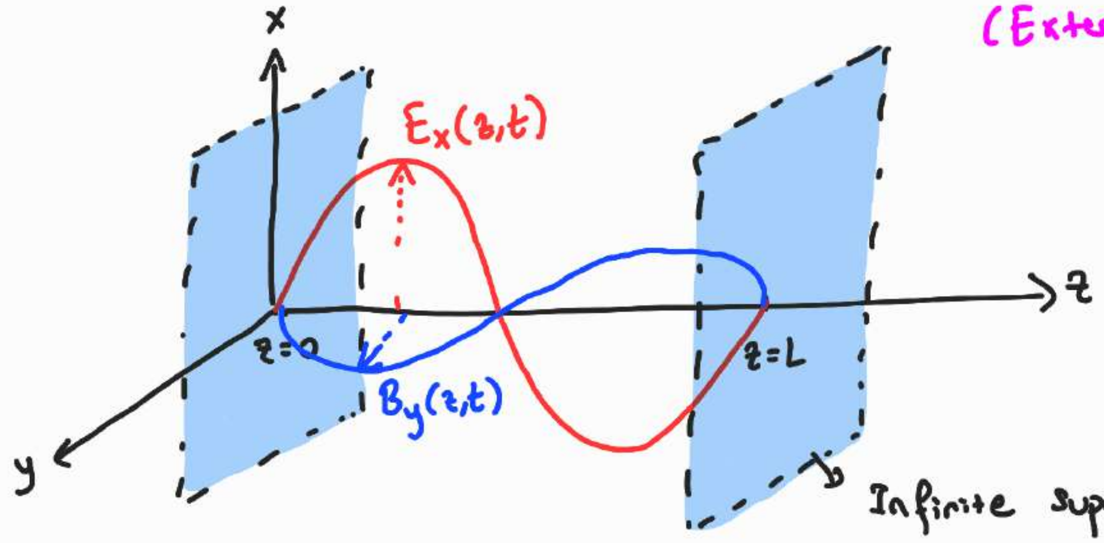
QUBIT-CAVITY INTERACTION

Reference: Introduction to Experimental Quantum Measurement with Superconducting Qubits / Naghiloo

1. One-dimensional cavity mode

How $\vec{E}$ is generated?
(External potential difference)



Infinite superconducting walls

Due to translational symmetry in x and y directions, the EM fields are only functions of z.

time evolution for modes, position x(t)

From
$$E_x(z,t) = \mathcal{E}\, q(t) \sin(kz)$$

$$B_y(z,t) = \mathcal{E} \frac{\mu_0 \epsilon_0}{k} \dot{q}(t) \cos(kz)$$

Normalization constant $\mathcal{E} = \sqrt{\dfrac{2\omega_c^2}{V\epsilon_0}}$

V: effective volume of the cavity

$k = \dfrac{m\pi}{L}$ $\quad m = 1, 2, \ldots$ wave number

$\omega_c : k / \sqrt{\mu_0 \epsilon_0}$

$$H = \frac{1}{V} \int dV \left(\frac{\varepsilon_0}{2} |E_x(z,t)|^2 + \frac{1}{2\mu_0} |B_y(z,t)|^2 \right)$$

Total Energy

$$H = \frac{1}{2} \left[p^2(t) + \omega_c^2 q^2(t) \right] \quad \text{where} \quad p(t) = \dot{q}(t)$$

$$p, q \longrightarrow \hat{p}, \hat{q}$$

quantum Hamiltonian $\leftarrow$

$$\hat{H} = \frac{1}{2} \left[\hat{p}^2(t) + \omega_c^2 \hat{q}^2(t) \right]$$

$$\hat{a} = \frac{1}{\sqrt{2\omega_c}} (\omega_c \hat{q} + i\hat{p})$$

$$\hat{a}^+ = \frac{1}{\sqrt{2\omega_c}} (\omega_c \hat{q} - i\hat{p})$$

$$\hat{E}_x(z,t) = \mathcal{E}_0 (\hat{a} + \hat{a}^+) \sin(kz)$$

$$\hat{B}_y(z,t) = i B_0 (\hat{a} - \hat{a}^+) \cos(kz)$$

$$\hat{H} = \omega_c \left(\hat{a}^+ \hat{a} + \frac{1}{2} \right) = \omega_c \left(\hat{n} + \frac{1}{2} \right)$$

$$\hat{H} |n\rangle = E_n |n\rangle \quad n = 0, 1, 2, \ldots$$

↳ Fock States (photon number states)

$$W(q,p) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \langle q + \tfrac{x}{2} | \psi \rangle \langle \psi | q - \tfrac{x}{2} \rangle e^{+ipx} dx$$

Wigner function probability distribution of photons in phase space.

↳ intuitive for classical light (coherent light, thermal light)

nonintuitive for non-classical light (single photon state) vacuum state

* Fock states are orthogonal to each other. Thus $\langle E \rangle$ and $\langle B \rangle = 0$ for a Fock states. However for a vacuum states they're nonzero. (from vacuum fluctuations)

$$|0\rangle \longrightarrow W_0(q,p) = \frac{1}{2\pi} e^{-(q^2+p^2)}$$

$$|1\rangle \longrightarrow W_1(q,p) = \frac{1}{2\pi} (2q^2 + 2p^2 - 1) e^{-(q^2+p^2)}$$

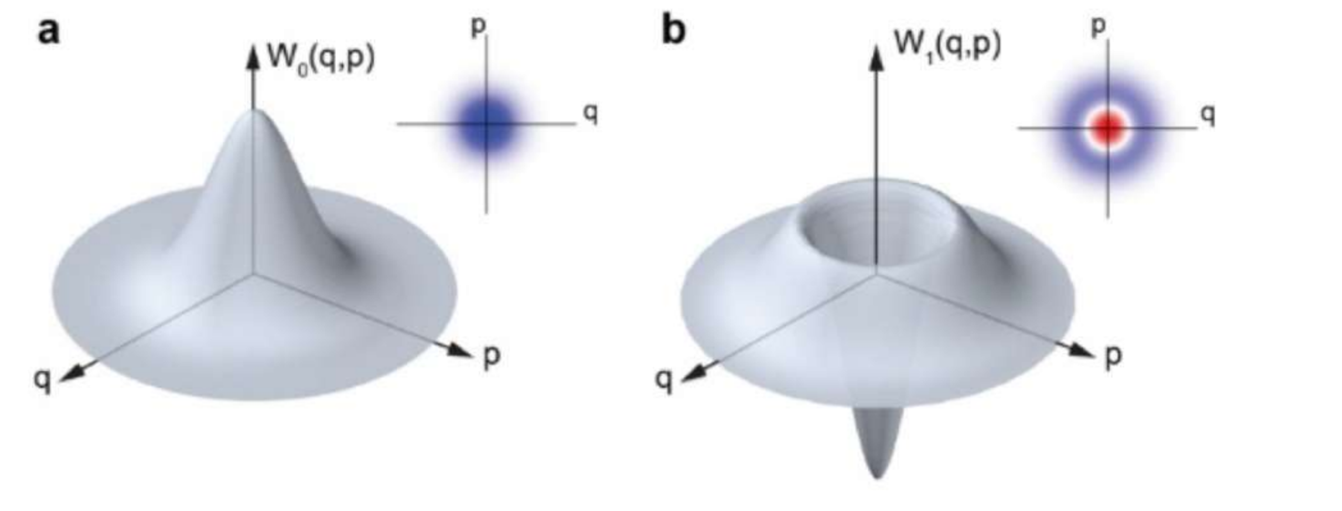


Figure 2.2: **Wigner distribution for photon-number states:** **a**, The vacuum state $|0\rangle$ has a Gaussian distribution centered at the origin of the phase space. **b**, The single photon state $|1\rangle$ exhibits negative probabilities around the origin.

Fock states are eigenstates of the harmonic oscillator. Thus a Fock state Wigner functions are stationary, do not evolve in time.

Coherent State

$$|\psi\rangle = |\alpha\rangle = \sum_n c_n |n\rangle \quad , \quad c_n = e^{-|\alpha|^2/2} \frac{\alpha^n}{\sqrt{n!}}$$

is constant $\langle \hat{n} \rangle = |\alpha|^2$

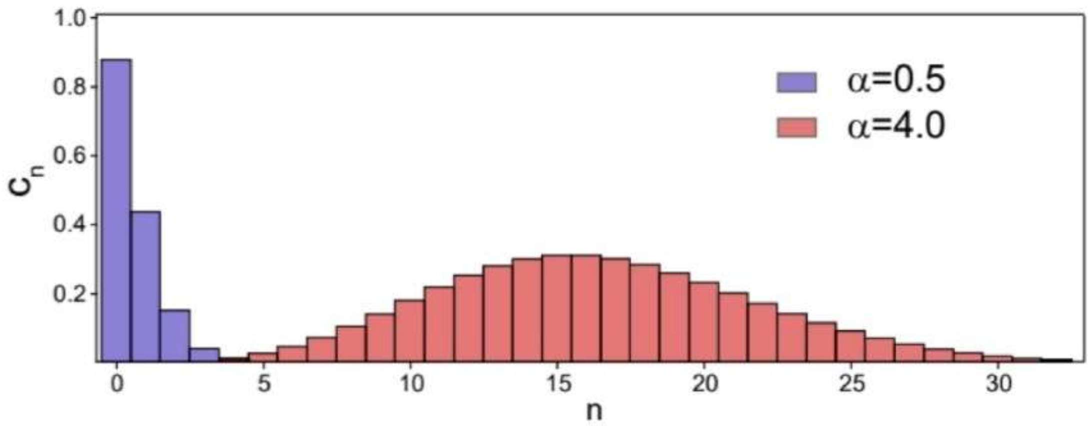


Figure 2.3: **Photon number distributions for coherent states:** The blue (red) distribution shows the photon number distribution for a coherent state which has an average photon number $\bar{n} = 1/4$ ($\bar{n} = 16$). The photon number distribution for the higher average number of photons is more like a Gaussian distribution.

follows "Central Limit Theorem" for a Poisson distribution

Unlike the photon-number state, the coherent state is not an eigenstate of the Hamiltonian, therefore it has time evolution.

$$|\alpha(t)\rangle = e^{-i\omega_c t}\alpha \quad \text{and} \quad |\alpha\rangle = |\alpha| e^{i\phi}$$

Coherent light ≈ Classical Oscillatory Motion

$$\langle \alpha_t | E | \alpha_t \rangle = 2\,\mathrm{Re}(\alpha_t)\, \mathcal{E}_0 \sin(kz) \cos(\omega_c t)$$

$$= 2|\alpha|\, \mathcal{E}_0 \sin(kz) \cos(\omega_c t + \phi)$$

$$\langle \alpha_t | B | \alpha_t \rangle = 2\,\mathrm{Im}(\alpha_t)\, B_0 \cos(kz) \sin(\omega_c t)$$

$$= 2|\alpha|\, B_0 \cos(kz) \sin(\omega_c t - \phi)$$

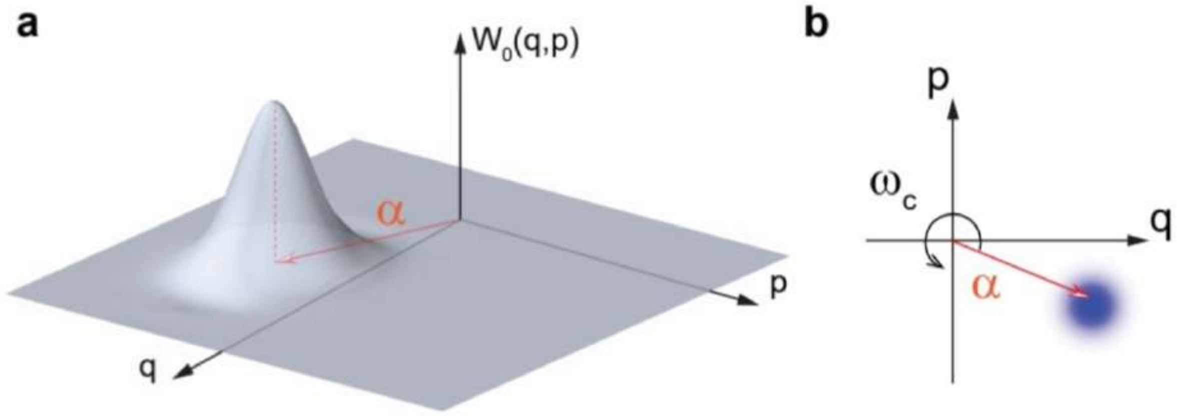


Figure 2.4: **Winger function for a coherent state:** **a**, The Wigner function for a coherent state is a Gaussian distribution displaced from the origin by amount of α. The coherent state has minimum uncertainty in each quadrature like a vacuum state. **b**, The evolution of coherent state under harmonic oscillator Hamiltonian is simply a rotation around the origin.

* In rotating frame, the coherent state does not rotate anymore in phase space.
* The state stands along q axis, which means all the energy is potential

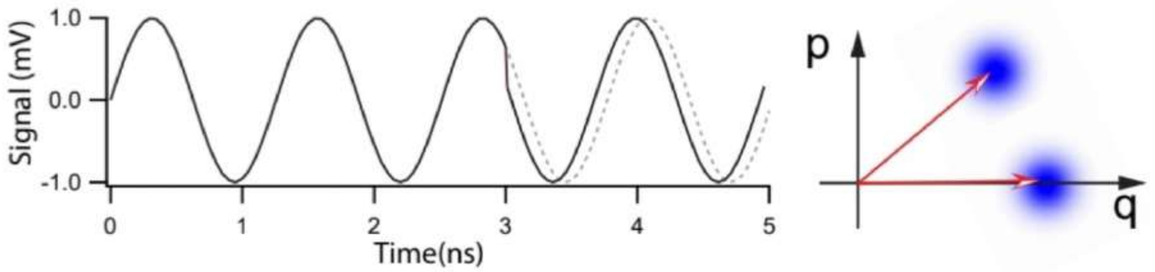
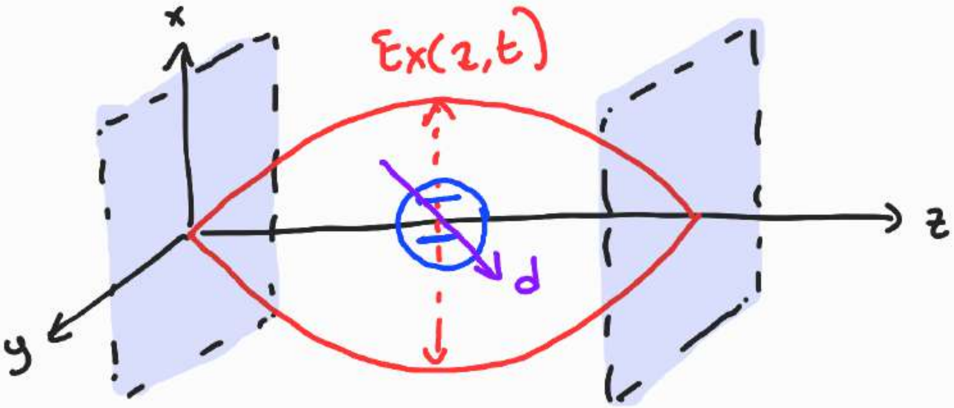


Figure 2.5: **Phase shifts for coherent state in the rotating frame:** The phase shift of a coherent signal is easily detectable in the rotating frame.

2. Qubit Cavity Interaction



$n=1$ First mode cavity only min frequency exists.

. Qubit only interacts with $E_x\left(z=\frac{L}{2}, t\right)$

. The qubit interacts via its electric dipole moment $\vec{d}$ to the electric field of the cavity via the interaction Hamiltonian

why minus?

$$H_{int} = - \hat{d} \cdot \hat{E}_x\left(\frac{L}{2}, t\right)$$

$$\hat{d} = \begin{pmatrix} 0 & d \\ d^* & 0 \end{pmatrix}$$

aligned with the $\vec{E}$ field

$$\hat{d} = d_x \sigma_x = d_x(\sigma_+ + \sigma_-)$$

Assume d_x is real

$$\hat{E}_x\left(\frac{L}{2}, t\right) = \mathcal{E}_0(\hat{a} + \hat{a}^+) \underbrace{\sin(kz)}_{1 \text{ for } z=\frac{L}{2}}$$

$$H_{int} = d_x(\sigma_+ + \sigma_-)(\hat{a} + \hat{a}^+)\mathcal{E}_0 = -\underbrace{g}_{d\,\mathcal{E}_0}(\sigma_+ + \sigma_-)(\hat{a} + \hat{a}^+)$$

2.1 Jaynes-Cummings Model

$\hbar = 1$

$$H_{Rabi} = \omega_c\left(\hat{a}^\dagger \hat{a} + \frac{1}{2}\right) - \frac{1}{2}\omega_q \sigma_z - g(\hat{a} + \hat{a}^\dagger)(\sigma_- + \sigma_+)$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$H_{Cavity} \qquad H_{qubit} \qquad H_{int}$$

If $g = 0$ (no interaction)

- Eigenstates are tensor product of the cavity and qubit eigenstates $\{ |g\rangle|n\rangle, |e\rangle|n\rangle \}$

If $g \neq 0$

- Using RWA (where $g \ll \omega_q, \omega_c$ and $|\omega_c - \omega_q| \ll |\omega_c + \omega_q|$

$$H_{int} = g(\hat{a}^\dagger \sigma_- + \hat{a}\sigma_+ + \hat{a}^\dagger \sigma_+ + \hat{a}\sigma_-)$$

much less occur

$\Delta E \Delta t \geq \frac{\hbar}{2}$

$\Delta E = \pm(\omega_c + \omega_q)$ requires substantial energy

creation of photon and decay of qubit

annihilation of a photon and excitation of qubit

$\Delta E = \pm(\omega_c - \omega_q)$ somehow conserves energy

$\Delta E \ll E_{tot} \sim (\omega_c + \omega_q)$

$$\boxed{H_{JC} = \omega_c\left(\hat{a}^\dagger \hat{a} + \frac{1}{2}\right) - \frac{1}{2}\omega_q \sigma_z - g(\hat{a}^\dagger \sigma_- + \hat{a}\sigma_+)}$$

$$
H_{JC} = \begin{pmatrix}
\frac{1}{2}\omega_c - \frac{\omega_q}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{3}{2}\omega_c - \frac{\omega_q}{2} & g & 0 & 0 & 0 \\
0 & g & \frac{3}{2}\omega_c + \frac{\omega_q}{2} & 0 & 0 & 0 \\
 & & & \cdots & & \\
0 & 0 & 0 & 0 & (n+\frac{1}{2})\omega_c - \frac{\omega_q}{2} & \sqrt{n+1}g \\
0 & 0 & 0 & 0 & \sqrt{n+1}g & (n+\frac{1}{2})\omega_c + \frac{\omega_q}{2}
\end{pmatrix},
$$

$$
M_n = \begin{pmatrix}
(n+\frac{1}{2})\omega_c - \frac{\omega_q}{2} & \sqrt{n+1}\,g \\
\sqrt{n+1}\,g & (n+\frac{1}{2})\omega_c + \frac{\omega_q}{2}
\end{pmatrix}
$$

$$
E_{\pm} = (n+1)\omega_c \pm \frac{1}{2}\sqrt{4g^2(n+1)+\Delta^2}\,, \qquad \Delta = \omega_q - \omega_c
$$

Dressed States:

$$
|n,-\rangle = \cos(\theta_n)\,|g\rangle|n+1\rangle - \sin(\theta_n)\,|e\rangle|n\rangle
$$

$$
|n,+\rangle = \sin(\theta_n)\,|g\rangle|n+1\rangle - \cos(\theta_n)\,|e\rangle|n\rangle
$$

where $\theta_n = \frac{1}{2}\tan^{-1}\left(\frac{2g\sqrt{n+1}}{\Delta}\right)$

For $\Delta \to 0$ $\theta_n = \frac{\pi}{4}$ $\longrightarrow$ $|n,\pm\rangle = \frac{1}{\sqrt{2}}\left(|g\rangle|n+1\rangle \pm |e\rangle|n\rangle\right)$ $\rightarrow$ Polariton States

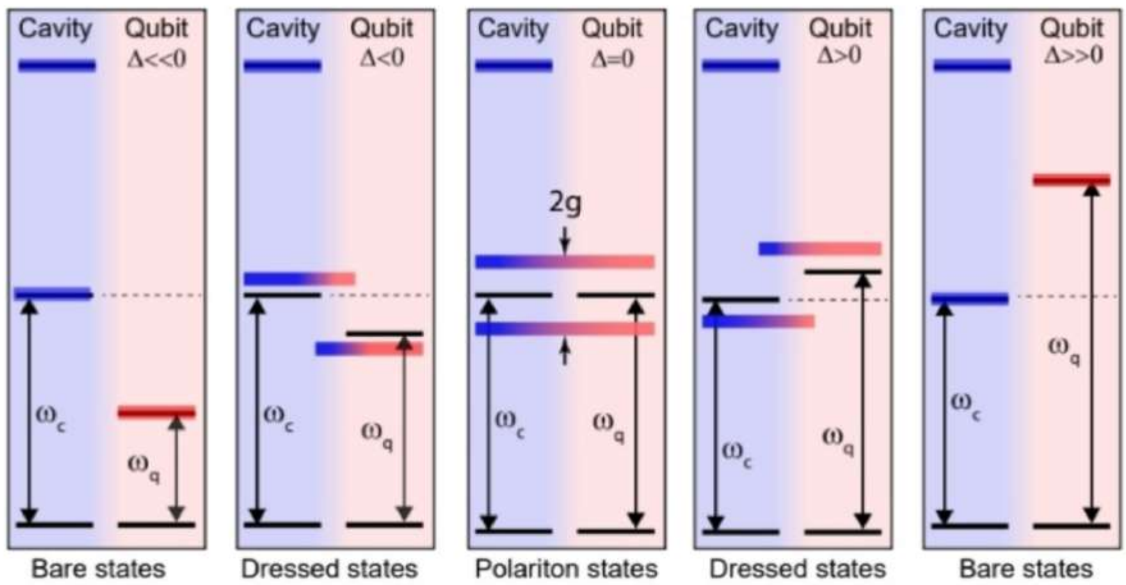


Figure 2.11: **Dressed states vs bare states:** The panels illustrate the dressed states of the qubit-cavity system for different qubit-cavity detunings in comparison with the bare states (refer to the main text for a more detailed description). Note that this illustration is not accurate and lacks some details but we rather to avoid them here.

Every time qubit level crosses one of the cavity levels, we may expect an avoided crossing and hybridization.

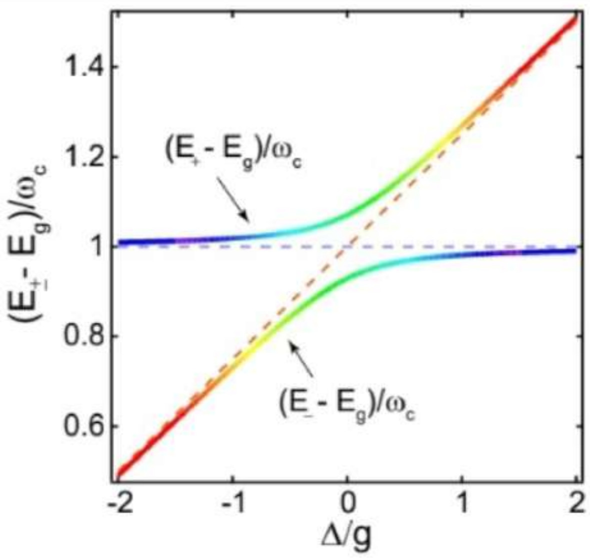


Figure 2.12: **Avoided crossing:** The transition energy from higher and lower dressed states to the ground state versus the detuning Δ. The transition energy is scaled by the energy of the cavity ω_c and the detuning is scaled by the coupling rate g. The dashed lines indicate the bare states' transition. Note that you can somehow see a similar avoided-crossing curve in Figure 2.11 by connecting the upper (lower) dressed states in different detunings together.