Introduction to Quantum Information Theory

The Bloch Sphere

1. a) Any pure quantum state can be written as the pure state vector,

$$|\psi\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)e^{i\phi}|1\rangle$$

or as the density matrix,

$$|\psi\rangle\langle\psi| = \frac{1}{2}(\mathbb{1} + s_x X + s_y Y + s_z Z)\,.$$

Find an expression for $\boldsymbol{s}$ in terms of θ and ϕ. What does the vector $\boldsymbol{s}$ denote? What is $|\boldsymbol{s}|^2$?

b) Sketch on the Bloch sphere the states: $|0\rangle$, $|+\rangle := \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, $|-\rangle := \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$, and $\frac{1}{\sqrt{3}}(|0\rangle - i\sqrt{2}|1\rangle)$.

2. a) Show that any mixed state of a single qubit can be written as a point in the Bloch sphere.

b) Sketch the points $\frac{1}{2}\mathbb{1}$ and $\frac{1}{3}|+\rangle\langle+| + \frac{2}{3}|-\rangle\langle-|$.

c) Give a geometric argument to show that $\frac{1}{2}(|+\rangle\langle+| + |-\rangle\langle-|) = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$. (Is this surprising?)

3. a) Let σ_i denote the i_{th} Pauli matrix. Show that $\sigma_i^2 = \mathbb{1}$ for all i.

b) Hence, use a series expansion to show that $e^{-i\theta\sigma_i/2} = \cos(\theta/2)\mathbb{1} - i\sin(\theta/2)\sigma_i$.

c) What is the effect of the state $|+\rangle$ evolving under $e^{-i\theta\sigma_z/2}$ for $\theta = \pi/2$? State the final state and sketch this evolution on the Bloch sphere.

4. The Hadamard gate takes the form

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\,. \tag{1}$$

Show that $H|0\rangle = |+\rangle$, $H|1\rangle = |-\rangle$ and $X = HZH$.

Two-Qubit Gates and Entangled States

1. a) Write out the truth tables and explicit matrices for the NOT and CNOT gates.

b) Show that C-NOT can be written as

$$\text{C-NOT} = |0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes X$$

.

c) The C-Phase gate implements a Z gate on the second qubit conditional on the first qubit being in state $|1\rangle$. Write out the truth table and matrix for the C-Phase gate. The C-Phase gate is sometimes said to be a symmetric gate- why do you think this is?

d) Show that C-Phase $= (\mathbb{1} \otimes H)$(C-NOT)$(\mathbb{1} \otimes H)$. (Bonus points if you do it without matrix multiplication).

2. a) The SWAP gate swaps the state of system 1 and system 2, i.e. SWAP$|\psi\rangle \otimes |\phi\rangle = |\phi\rangle \otimes |\psi\rangle$. Write out the explicit matrix form for SWAP.

   b) Show that 3 CNOTs can be used to implement the SWAP gate. (Do try and figure this out at first without google!)

3. a) Write out the four Bell states. Show that it is possible to transform between them using a single qubit gate.

   b) Sketch the circuits you would use to prepare each of the Bell states from the all zero state $|00\rangle$.

   c) What is the effect of applying the SWAP gate to a Bell state?

   d) Suppose you measure the first qubit of the $|\Psi^-\rangle$ Bell state in the computational basis. What is the probability of finding the first qubit to be in the state $|1\rangle$? Suppose *after* this measurement you now measure the second qubit- what's the probability of finding it to be in the state $|1\rangle$?

   e) Show that the Bell states form an orthonormal basis. Draw a circuit to perform the $|\Psi^-\rangle\langle\Psi^-|$ projective measurement. What is the probability of measuring the state $|+-\rangle$ to be in the state $|\Psi^-\rangle$?

Extension

Harder! Give it a shot and see how far you get.

1. a) Write the SWAP gate i. as a sum of Pauli matrices and ii. as a sum of Bell states.

   b) Show that $\mathrm{Tr}[(A \otimes B)\,\mathrm{SWAP}] = \mathrm{Tr}[AB]$

   c) Hence draw a circuit and describe a post processing strategy to determine $|\langle\psi|\phi\rangle|^2$ using a quantum computer. (Hint use b) with $A = |\psi\rangle\langle\psi|$ and $B = |\phi\rangle\langle\phi|$).

Bloch Sphere

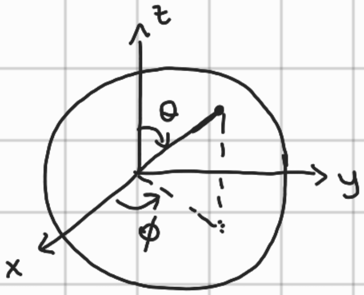
1. a) Any pure quantum state can be written as the pure state vector,

$$|\psi\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)e^{i\phi}|1\rangle$$

or as the $\boxed{\text{density matrix,}}$

$$|\psi\rangle\langle\psi| = \frac{1}{2}(\mathbb{1} + s_x X + s_y Y + s_z Z)\,.$$

Find an expression for $\boldsymbol{s}$ in terms of θ and ϕ. What does the vector $\boldsymbol{s}$ denote? What is $|\boldsymbol{s}|^2$?



S is a vector of state on Bloch sphere

$$S = \begin{pmatrix} S_x \\ S_y \\ S_z \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}$$

$$|S|^2 = S^* S = (S_x \; S_y \; S_z)\begin{pmatrix} S_x \\ S_y \\ S_z \end{pmatrix}$$

$$|S|^2 = S_x^2 + S_y^2 + S_z^2 = \underbrace{\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi}_{\sin^2\theta} + \cos^2\theta = 1$$

$$|\psi\rangle = \cos\left(\tfrac{\theta}{2}\right)|0\rangle + \sin\left(\tfrac{\theta}{2}\right)e^{i\phi}|1\rangle$$

$$\langle\psi| = \cos\left(\tfrac{\theta}{2}\right)\langle 0| + \sin\left(\tfrac{\theta}{2}\right)e^{-i\phi}\langle 1|$$

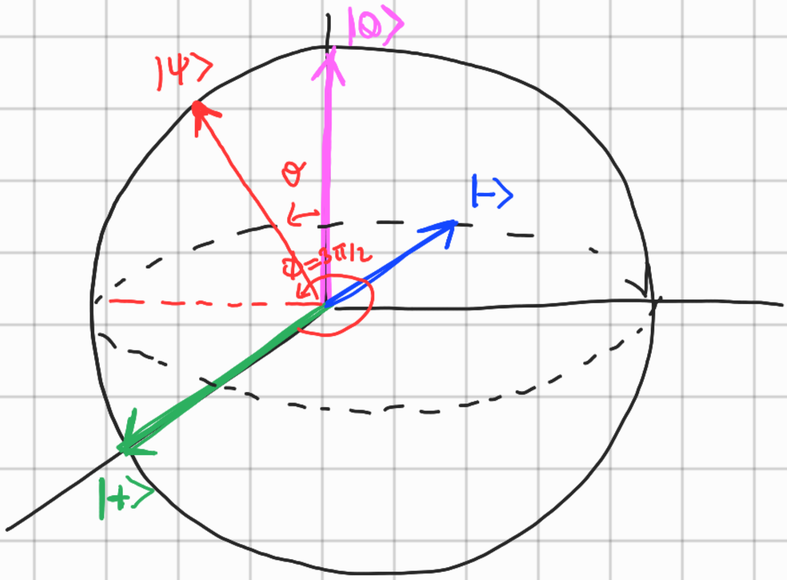
$$|\psi\rangle\langle\psi| = \begin{pmatrix} \overbrace{\cos^2\frac{\theta}{2}}^{\frac{1+\cos\theta}{2}} & \overbrace{\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)e^{-i\phi}}^{\frac{\sin\theta}{2}(\cos\phi - i\sin\phi)} \\ \underbrace{\sin\frac{\theta}{2}\cos\frac{\theta}{2}e^{i\phi}}_{\frac{\sin(\theta)}{2}(\cos\phi + i\sin\phi)} & \underbrace{\sin^2\frac{\theta}{2}}_{\frac{1-\cos\theta}{2}} \end{pmatrix}$$

$$|\psi\rangle\langle\psi| = \cos^2\left(\tfrac{\theta}{2}\right)|0\rangle\langle 0| + \cos\left(\tfrac{\theta}{2}\right)\sin\left(\tfrac{\theta}{2}\right)e^{-i\phi}|0\rangle\langle 1| + \sin\tfrac{\theta}{2}\cos\left(\tfrac{\theta}{2}\right)e^{i\phi}|1\rangle\langle 0| + \sin^2\tfrac{\theta}{2}|1\rangle\langle 1|$$

$$|\psi\rangle\langle\psi| = \frac{1}{2}\Big(\underset{\begin{pmatrix}1&0\\0&1\end{pmatrix}}{\mathbb{I}} + \cos\theta\,\underset{\begin{pmatrix}1&0\\0&-1\end{pmatrix}}{\sigma_z} + \sin\theta\cos\phi\,\sigma_x + \sin\theta\sin\phi\,\sigma_y\Big)$$

$$|\psi\rangle\langle\psi| = \frac{1}{2}\Big(\mathbb{I} + \underset{\sin\theta\cos\phi}{S_x}\sigma_x + \underset{\sin\theta\sin\phi}{S_y}\sigma_y + \underset{\cos\theta}{S_z}\sigma_z\Big)$$

b) Sketch on the Bloch sphere the states: $|0\rangle$, $|+\rangle := \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, $|-\rangle := \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$, and $\frac{1}{\sqrt{3}}(|0\rangle - i\sqrt{2}|1\rangle)$.

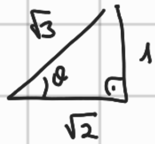


$$|\psi\rangle = \cos\frac{\theta}{2}\,|0\rangle + e^{i\phi}\sin\frac{\theta}{2}\,|1\rangle$$

$$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

$$\theta = \frac{\pi}{2} \qquad \phi = \pi \rightarrow e^{i\pi} = -1$$

$$\cos\frac{\theta}{2} = \frac{1}{\sqrt{3}}, \quad \sin\frac{\theta}{2} = \frac{\sqrt{2}}{\sqrt{3}}$$



$$e^{i\phi} = -i \qquad \phi = \frac{3\pi}{2}$$

state	coordinates
$\lvert 0\rangle$	$(0, 0, 1)$
$\lvert +\rangle$	$(1, 0, 0)$
$\lvert -\rangle$	$(-, 0, 0)$

2. a) Show that any $\boxed{\text{mixed state}}$ of a single qubit can be written as a point in the Bloch sphere.

b) Sketch the points $\frac{1}{2}\mathbb{1}$ and $\frac{1}{3}|+\rangle\langle+| + \frac{2}{3}|-\rangle\langle-|$.

c) Give a geometric argument to show that $\frac{1}{2}(|+\rangle\langle+| + |-\rangle\langle-|) = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$. (Is this surprising?)

$\{I, X, Y, Z\}$ forms a basis for 2x2 matrices (*)

Density matrices' Properties

(1) Hermitian $\rightarrow$ $\rho = \rho^{+}$

(2) positive semi-definiteness: $\langle\psi|\rho|\psi\rangle \geq 0$, $\forall |\psi\rangle$

(3) Unit trace $Tr(\rho) = 1$

$$\rho = \frac{1}{2}\left(\mathbb{I} + s_x X + s_y Y + s_z Z \right)$$

s is a real vector such that $|s|^2 < 1$ for Mixed states → inside the Bloch sphere

$|s|^2 = 1$ for pure states → on the Bloch sphere

$$\mathrm{Tr}[\rho^2] = \frac{1}{4}\mathrm{Tr}\left[\mathbb{1} + \sum_i |s_i|^2 \mathbb{I} \right] \overset{?}{=} \frac{1}{2}(1 + \|s\|^2)$$

$$\|S\|^2 = 2\mathrm{Tr}[\rho^2] - 1$$

Hermitian

Also ρ is (1) and (2) → Thus, One can always write it in its eigenbasis

↘ $\sum_i \lambda_i |\lambda_i\rangle\langle\lambda_i|$ ⟶ $\mathrm{Tr}[\rho^2] = \sum_i |\lambda_i|^2$

since eigen states are orthogonal and eigenvalues real and positive

$\|S\|^2 = 2\sum_i |\lambda_i|^2 - 1$ → $= 1$ ⟶ if state is pure $\lambda_0 = 1$ and $\lambda_j = 0$ for $j \neq 0$

↘ ≤ 1 ⟶ if the state is mixed $\sum_i |\lambda_i|^2 \leq 1$ $(0 \leq \lambda_i \leq 1)$

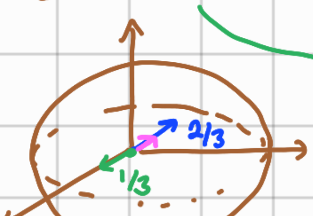
↘ $\boxed{\|S\|^2 \leq 1}$

b) $$\rho = \frac{1}{2}\left(\mathbb{I} + s_x X + s_y Y + s_z Z \right)$$

$\frac{1}{2}\mathbb{I} \Rightarrow \rho = \frac{1}{2}\mathbb{I} + \frac{1}{2}(0.X + 0.Y + 0.Z) \rightarrow (0,0,0)$

$= \frac{1}{3}|+\rangle\langle+| + \frac{2}{3}|-\rangle\langle-|$ → $+\frac{2}{3}$ along $-X$ (i.e $|-\rangle$) $= \frac{1}{2}\left(\mathbb{I} - \frac{1}{3}X\right)$

→ $+\frac{1}{3}$ along X (i.e $|+\rangle$)

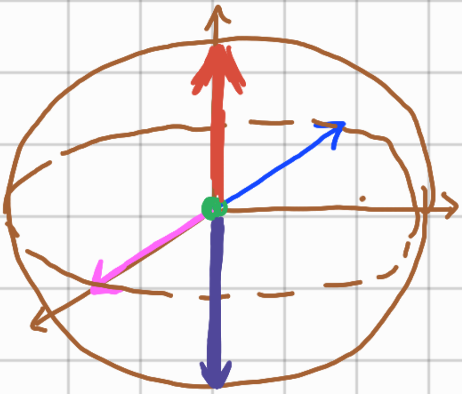


$+1/3$ along $-X$

or

$-1/3$ along X

c) $\frac{1}{2}\big(\overbrace{|+\rangle\langle+| + |-\rangle\langle-|}^{\text{gives origin}}\big) = \frac{1}{2}\big(\overbrace{|0\rangle\langle 0| + |1\rangle\langle 1|}^{\text{gives origin}}\big)$



⚠ $-\frac{1}{2}$ along $-X$ = $-\frac{1}{2}$ along X

But $-|-\rangle\langle-| \neq |+\rangle\langle+|$

↳ Since λ_i's are not negative

3. a) Let σ_i denote the i_{th} Pauli matrix. Show that $\sigma_i^2 = \mathbb{1}$ for all i. → Matrix Multiplication

b) Hence, use a series expansion to show that $e^{-i\theta\sigma_i/2} = \cos(\theta/2)\mathbb{1} - i\sin(\theta/2)\sigma_i$.

c) What is the effect of the state $|+\rangle$ evolving under $e^{-i\theta\sigma_z/2}$ for $\theta = \pi/2$? State the final state and sketch this evolution on the Bloch sphere.

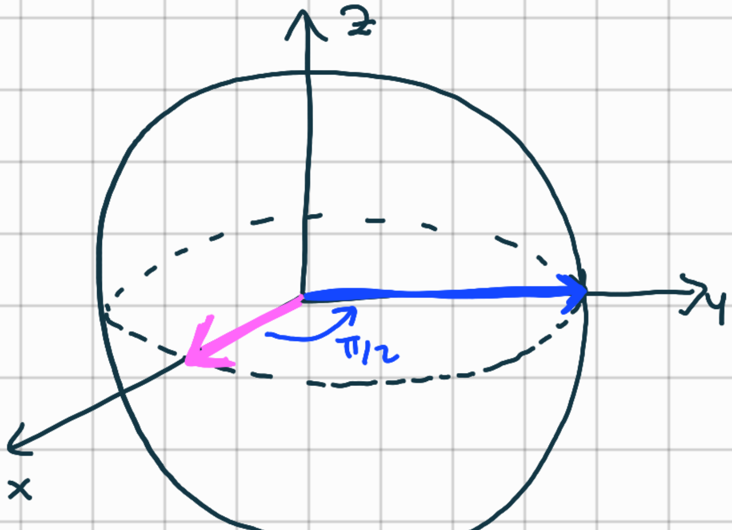
b) $$e^{-i\theta\sigma_i/2} = \sum_{k=0}^{\infty} \frac{(-i\theta/2)^k \sigma_i^k}{k!} = \mathbb{I}\sum_{n=0}^{\infty}\frac{(-i\theta/2)^{2n}}{(2n)!} - \sigma_i \sum_{k=0}^{\infty}\frac{(-i\theta/2)^{2n+1}}{(2n+1)!}$$

$\sigma_i^{2k} = \mathbb{I}$ $\quad \sigma_i^{2k+1} = \sigma_i$; $(-i)^{2n} = (-1)^n$;

$(-i)^{2n+1} = (-1)^n(-i)$

$$e^{-i\theta\sigma_i/2} = \mathbb{I}\underbrace{\sum_{n=0}^{\infty}\frac{(-1)^n(\theta/2)^{2n}}{2n!}}_{\cos(\theta/2)} - i\sigma_i \underbrace{\sum_{k=0}^{\infty}\frac{(-1)^n(\theta/2)^{2n+1}}{(2n+1)!}}_{\sin(\theta/2)}$$

c)



Rotation around z axis with $\theta = \frac{\pi}{2}$

$$e^{-i\theta\sigma_z/2} = \cos\left(\frac{\theta}{2}\right) - i\sin\left(\frac{\theta}{2}\right)\sigma_z$$

where $\theta = \frac{\pi}{2}$

$$e^{-i\frac{\pi}{2}\frac{\sigma_z}{2}}|+\rangle = \left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\sigma_z\right)|+\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle - i|-\rangle\right)$$

$$\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\-1\end{pmatrix} = |-\rangle$$

$$|\psi\rangle = \frac{1-i}{2}|0\rangle + \frac{1+i}{2}|1\rangle = \frac{1}{2}\begin{pmatrix}1-i\\1+i\end{pmatrix}$$

$$\frac{1}{4}\begin{pmatrix}1-i\\1+i\end{pmatrix}\begin{pmatrix}1+i & 1-i\end{pmatrix}$$

$$|\psi\rangle\langle\psi| = \frac{1}{2}\begin{pmatrix}1 & -i\\ i & 1\end{pmatrix} = \frac{1}{2}\left(I + Y\right)$$

Or $|\psi\rangle = \frac{1-i}{2}\left(|0\rangle + i|1\rangle\right) = e^{-i\pi/4}\frac{\left(|0\rangle + i|1\rangle\right)}{\sqrt{2}}$

$\theta = \pi/2$, $\phi = \pi$

$\frac{1-i}{2} \to \frac{e^{-i\pi/4}}{\sqrt{2}}$

$= e^{-i\pi/4}|\psi_+\rangle \approx |y_+\rangle$

$e^{-i\pi/4}$: Global phase

(it is not physical!)

✱ In general, with $\vec{n} = \begin{pmatrix}n_x\\n_y\\n_z\end{pmatrix}$ unit norm vector in $\mathbb{R}^3$

and $\vec{\sigma} = \begin{pmatrix}\sigma_x\\\sigma_y\\\sigma_z\end{pmatrix} \rightsquigarrow e^{-i\theta/2\,\vec{n}\cdot\vec{\sigma}}$ is a rotation of angle θ around n.

4. The Hadamard gate takes the form

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{1}$$

Show that $H|0\rangle = |+\rangle$, $H|1\rangle = |-\rangle$ and $X = HZH$.

$$H|0\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = |+\rangle$$

$$H|1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = |-\rangle$$

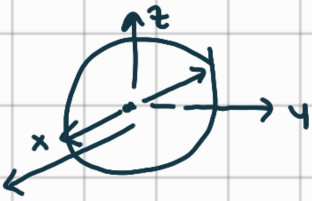
$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$Z = |0\rangle\langle 0| - |1\rangle\langle 1|$$

$H|0\rangle = |+\rangle$, $\quad H|1\rangle = |-\rangle$

$$H|0\rangle\langle 0|H = |+\rangle\langle +| \qquad H|1\rangle\langle 1|H = |-\rangle\langle -|$$

$$HZH = H\left(|0\rangle\langle 0| - |1\rangle\langle 1|\right)H = |+\rangle\langle +| - |-\rangle\langle -| = \hat{X}$$



Two-Qubit Gates and Entangled States

1. a) Write out the truth tables and explicit matrices for the NOT and CNOT gates.

b) Show that C-NOT can be written as

$$\text{C-NOT} = |0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes X$$

.

c) The C-Phase gate implements a Z gate on the second qubit conditional on the first qubit being in state $|1\rangle$. Write out the truth table and matrix for the C-Phase gate. The C-Phase gate is sometimes said to be a symmetric gate- why do you think this is?

d) Show that C-Phase $= (\mathbb{1} \otimes H)(\text{C-NOT})(\mathbb{1} \otimes H)$. (Bonus points if you do it without matrix multiplication).

a,b)

NOT

0	1
1	0

$$NOT = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_x = \hat{X}$$

CNOT

00	00
01	01
10	11
11	10

$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes \hat{X}$$

$$\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

c) C- Phase Gate = C-Z Gate

	In	Out
	00	00
	01	01
apply Z	10	10
	11	-11

It is symmetric in the sense that the control and target qubits can be interchanged

$$|0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes Z = 1 \otimes |0\rangle\langle 0| + Z \otimes |1\rangle\langle 1|$$

For the CNOT

$$|0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X \neq 1 \otimes |0\rangle\langle 0| + X \otimes |1\rangle\langle 1| \quad !$$

d) $(I \otimes H)(C\text{-}NOT)(I \otimes H) = C\text{-}Z$ gate

For the control qubit ⇒ $H^2 = I$ ✓

" " $=1$ ⇒ $HXH = Z$ ✓

$$X = \underbrace{H^\dagger H}_{I} Z \underbrace{H H^\dagger}_{I} \rightarrow \underbrace{H^\dagger}_{H} X \underbrace{H^\dagger}_{H} = HXH = Z$$

$$(\mathbb{1}\otimes H)\,\text{CNOT}\,(\mathbb{1}\otimes H) = (I\otimes H)\left(|0\rangle\langle 0|\otimes I + |1\rangle\langle 1|\otimes X\right)(\mathbb{1}\otimes H)$$

$$= \left(|0\rangle\langle 0| \otimes HI + |1\rangle\langle 1| \otimes HX\right)(I\otimes H)$$

$$\text{C-Z Gate} = \left(|0\rangle\langle 0| \otimes \underbrace{H^2}_{I} + |1\rangle\langle 1| \otimes \underbrace{HXH}_{Z}\right)$$

2. a) The SWAP gate swaps the state of system 1 and system 2, i.e. SWAP$|\psi\rangle \otimes |\phi\rangle = |\phi\rangle \otimes |\psi\rangle$. Write out the explicit matrix form for SWAP.

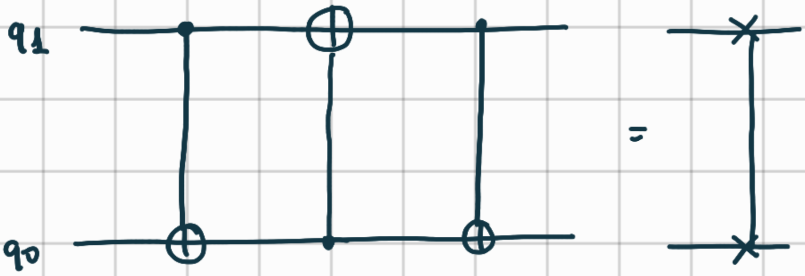
b) Show that 3 CNOTs can be used to implement the SWAP gate. (Do try and figure this out at first without google!)

SWAP

00	00
01	10
10	01
11	11

$$\text{SWAP} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$q_1 q_0$	after CNOT_1	after CNOT_2	after CNOT_3
00	00	00	00
01	01	11	10
10	11	01	01
11	10	10	11



* $\text{SWAP}_{i,j} = \text{SWAP}_{j,i} = \text{CNOT}_{i,j}\ \text{CNOT}_{j,i}\ \text{CNOT}_{i,j}$

SWAP is a symmetric gate.

$$\boxed{\underbrace{HXH}_{Z} \overset{?}{=} \underbrace{HH}_{I}X \rightarrow \text{not commute} \qquad Z \neq X}$$

3. a) Write out the four Bell states. Show that it is possible to transform between them using a single qubit gate.

b) Sketch the circuits you would use to prepare each of the Bell states from the all zero state $|00\rangle$.

c) What is the effect of applying the SWAP gate to a Bell state?

d) Suppose you measure the first qubit of the $|\Psi^-\rangle$ Bell state in the computational basis. What is the probability of finding the first qubit to be in the state $|1\rangle$? Suppose *after* this measurement you now measure the second qubit- what's the probability of finding it to be in the state $|1\rangle$?

e) Show that the Bell states form an orthonormal basis. Draw a circuit to perform the $|\Psi^-\rangle\langle\Psi^-|$ projective measurement. What is the probability of measuring the state $|+-\rangle$ to be in the state $|\Psi^-\rangle$?

a) Bell States

$$|\Phi^\pm\rangle = \frac{|00\rangle \pm |11\rangle}{\sqrt{2}}$$

$$|\Psi^\pm\rangle = \frac{|01\rangle \pm |10\rangle}{\sqrt{2}}$$

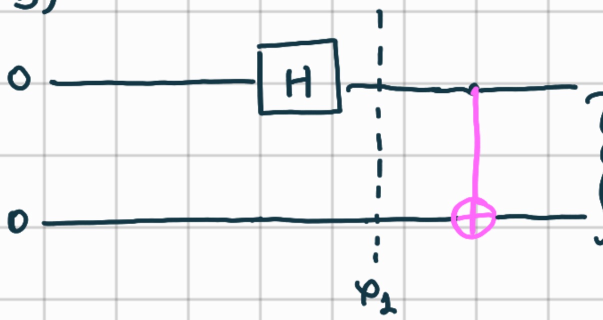
$$(\mathbb{1}\otimes Y)\,|\Phi_\pm\rangle = i\,|\Psi_\mp\rangle$$

$$(\mathbb{1}\otimes X)\,|\Phi_\pm\rangle = |\Psi_\pm\rangle$$

$$(I\otimes Z)\,|\Phi_+\rangle = |\Phi_-\rangle = (I\otimes Z)\,|\Phi_+\rangle$$

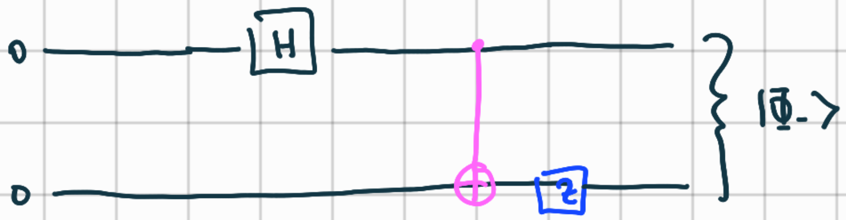
$$(Z\otimes I)\,|\Psi_+\rangle = |\Psi_-\rangle$$

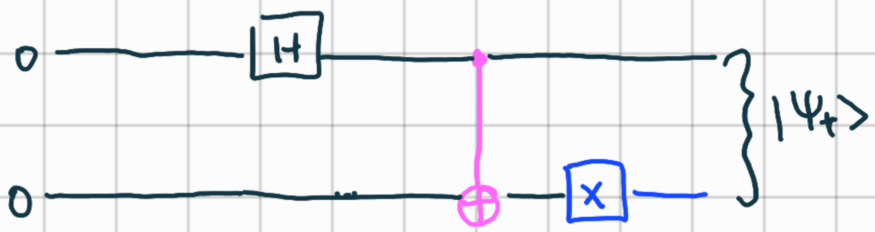
b)

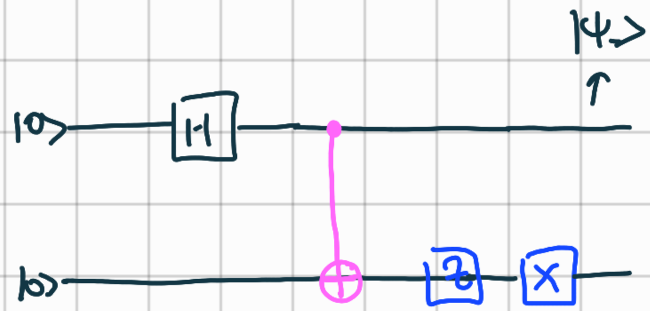


CNOT $|11\rangle$

$$\varphi_1 = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\otimes|0\rangle = \left(\frac{|00\rangle + |10\rangle}{\sqrt{2}}\right)$$







c) SWAP to Bell States

$$|\Phi^\pm\rangle = \frac{|00\rangle \pm |11\rangle}{\sqrt{2}} \rightarrow \text{SWAP}|\Phi^\pm\rangle = |\Phi^\pm\rangle$$

$$|\Psi^\pm\rangle = \frac{|01\rangle \pm |10\rangle}{\sqrt{2}} \rightarrow \text{SWAP}|\Psi^+\rangle = |\Psi^+\rangle$$

$$\text{SWAP}|\Psi^-\rangle = \underset{\text{global phase}}{-}|\Psi^-\rangle$$

Bell states are invariant under SWAP operation

d) $|\Psi_-\rangle = \frac{1}{\sqrt{2}}\left(|\underset{1st}{0}1\rangle - |\underset{1st}{1}0\rangle\right)$

$P(1^{st}\text{ qubit} = 1) = \frac{1}{2} \quad (|10\rangle)$

$P(2^{nd}\text{ qubit} = 1 \mid 1^{st}\text{ qubit} = 1) = 0$

Since the state collapsed to $|10\rangle$

From Solution:

$$\frac{|1\rangle\langle 1| \otimes \mathbb{I}\,|\Psi_-\rangle}{\left\| |1\rangle\langle 1| \otimes \mathbb{I}\,|\Psi_-\rangle \right\|} = -|10\rangle \sim |10\rangle$$

measurement

inner product

$$\text{Prob}(|1\rangle_2) = \langle 10|\,\mathbb{I}\otimes|1\rangle\langle 1|\,|10\rangle = \langle 0|1\rangle\langle 1|0\rangle = 0$$

$$\langle 10|1\rangle\langle 1|10\rangle$$

$$\langle 1|\langle 0|1\rangle|0\rangle = \langle 0|1\rangle\langle 1|0\rangle$$

e) Bell states form an orthonormal basis

$$|\Phi^{\pm}\rangle = \frac{|00\rangle \pm |11\rangle}{\sqrt{2}}$$

$$\langle \Phi_+ | \Phi_- \rangle = 0$$

$$|\Psi^{\pm}\rangle = \frac{|01\rangle \pm |10\rangle}{\sqrt{2}}$$

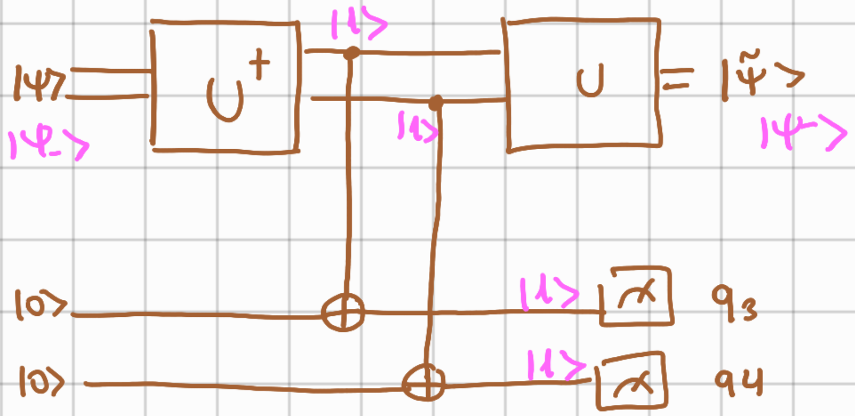
$$\frac{\left(\langle 00| + \langle 11|\right)\left(|00\rangle - |11\rangle\right)}{2} = \frac{1 - 0 + 0 - 1}{2} = 0$$

$$\frac{\left(\langle 00| + \langle 11|\right)\left(|01\rangle + |10\rangle\right)}{2} = \frac{0+0+0+0}{2} = 0$$

$$\langle \Phi_{\pm} | \Psi_{\pm} \rangle = 0$$

$$\langle 0\,\underbrace{0|0}\,1\rangle = \underbrace{\langle 0|0\rangle}_{1}\,\underbrace{\langle 0|1\rangle}_{0}$$

Circuits perform $|\Psi_-\rangle\langle\Psi_-|$ measurement



Given state $|\psi\rangle$, we can apply $U^{\dagger} = (H \otimes I)\,\mathrm{CNOT}$ and perform the measurement in the comp. basis, then outcomes $|00\rangle$ for $|\Phi_+\rangle$, $|10\rangle$ for $|\Phi_-\rangle$...

$\Downarrow$

$$\boxed{\begin{array}{ll} U|00\rangle = |\Phi_+\rangle & U|10\rangle = |\Phi_-\rangle \\ U|01\rangle = |\Psi_+\rangle & U|11\rangle = |\Psi_-\rangle \end{array}}$$

$\mathrm{CNOT}(H \otimes I)$

Measuring

$$|+-\rangle = \frac{1}{2}\left(|00\rangle + |10\rangle - |01\rangle - |11\rangle\right) = \frac{1}{\sqrt{2}}\left(|\Phi_-\rangle - |\Psi_-\rangle\right)$$

$(|0\rangle + |1\rangle)_1 \otimes (|0\rangle - |1\rangle)_2$

$$P_{|+-\rangle}(|\Psi_-\rangle) = \left|-\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$$

Harder! Give it a shot and see how far you get.

1. a) Write the SWAP gate i. as a sum of Pauli matrices and ii. as a sum of Bell states.

   b) Show that $\mathrm{Tr}[(A \otimes B)\,\mathrm{SWAP}] = \mathrm{Tr}[AB]$

   c) Hence draw a circuit and describe a post processing strategy to determine $|\langle\psi|\phi\rangle|^2$ using a quantum computer. (Hint use b) with $A = |\psi\rangle\langle\psi|$ and $B = |\phi\rangle\langle\phi|$).

a)

i.) $\mathrm{SWAP} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$X \otimes X = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ $\quad Y \otimes Y = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$ $\quad Z \otimes Z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$$\frac{1}{2}\left(\mathbb{I} \otimes \mathbb{I} + X \otimes X + Y \otimes Y + Z \otimes Z \right)$$

ii) $\mathrm{SWAP} = \sum_{|\psi\rangle \in \text{Bell}} |\psi\rangle\langle\psi|$

$|\Phi^\pm\rangle\langle\Phi^\pm| = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & \pm 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \pm 1 & 0 & 0 & 1 \end{pmatrix}$ $\quad |\psi^\pm\rangle\langle\psi^\pm| = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \pm 1 & +1 & 0 \\ 0 & +1 & \pm 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\frac{1}{2}$

?

b) $\mathrm{Tr}[(A\otimes B)\,\mathrm{SWAP}] = \mathrm{Tr}[AB]$

$$\left.\begin{aligned} A &= a_{ij}\,|i\rangle\langle j| \\ B &= b_{ij}\,|i\rangle\langle j| \end{aligned}\right\} \quad A\otimes B = a_{i_1 j_1} b_{i_2 j_2} |i_1, i_2\rangle\langle j_1, j_2|$$

$$\mathrm{Tr}(\mathrm{SWAP}(A\otimes B)) = a_{i_1 j_1} b_{i_2 j_2} \underbrace{\mathrm{Tr}(|i_2, i_1\rangle\langle j_1, j_2|)}_{\langle j_1, j_2 | i_2, i_1\rangle = \delta_{j_1 i_2}\,\delta_{j_2 i_1}}$$

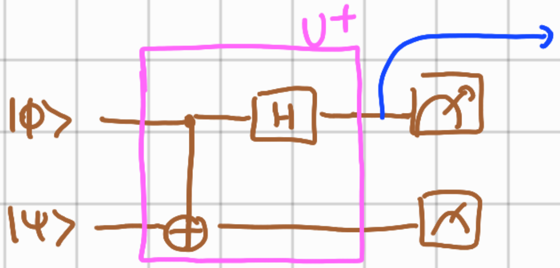
$j_1 \to i_2$, $j_2 \to i_1$

$$\mathrm{Tr}(\mathrm{SWAP}(A\otimes B)) = a_{i_1 i_2} b_{i_2 i_1} = \mathrm{Tr}(AB)$$

$$a_{ij} b_{jk} |i\rangle\langle k| = a_{ij} b_{jk}\, \delta_{ik} = a_{ij} b_{ji}$$

$$a_{ij} |i\rangle\langle j|\, b_{jk} |j\rangle\langle k|$$

c) Suppose that we are able to prepare states $|\phi\rangle$ and $|\psi\rangle$ then we initialize 1 qubit in $|\phi\rangle$ and the other in $|\psi\rangle$.



$$|\psi_{out}\rangle = \frac{|00\rangle}{\sqrt{2}}\left(\langle 0|\psi\rangle\langle 0|\Phi\rangle + \langle 1|\phi\rangle\langle 1|\psi\rangle\right)$$
$$+ \frac{|10\rangle}{\sqrt{2}}\left(\langle 0|\psi\rangle\langle 0|\Phi\rangle - \langle 1|\Phi\rangle\langle 1|\psi\rangle\right)$$
$$+ \frac{|01\rangle}{\sqrt{2}}\left(\langle 0|\phi\rangle\langle 1|\psi\rangle + \langle 1|\phi\rangle\langle 0|\psi\rangle\right)$$
$$+ \frac{|11\rangle}{\sqrt{2}}\left(\langle 0|\phi\rangle\langle 1|\psi\rangle - \langle 1|\phi\rangle\langle 0|\psi\rangle\right)$$

$$|\langle\psi|\phi\rangle|^2 = |\langle\psi|\mathbb{I}|\phi\rangle|^2$$
$$= |\langle\psi|0\rangle\langle 0|\phi\rangle + \langle\psi|1\rangle\langle 1|\phi\rangle|^2$$

$$= \underbrace{|\langle\psi|0\rangle\langle 0|\phi\rangle|^2 + |\langle\psi|1\rangle\langle 1|\psi\rangle|^2}_{P_{00}+P_{10}} + \langle\psi|0\rangle\langle 1|\psi\rangle\langle\phi|1\rangle\langle 0|\Phi\rangle$$

$$\underbrace{+\langle\psi|1\rangle\langle 0|\psi\rangle\langle\phi|0\rangle\langle 1|\phi\rangle}_{P_{01}-P_{11}\ (?)}$$

Outcome probabilities

$$|\langle\psi|\phi\rangle|^2 = |\langle 00|\Psi_{out}\rangle|^2 + |\langle 10|\Psi_{out}\rangle|^2 + |\langle 01|\Psi_{out}\rangle|^2 - |\langle 11|\Psi_{out}\rangle|^2$$

$$\left.\begin{aligned} A &= |\psi\rangle\langle\psi| \\ B &= |\phi\rangle\langle\phi| \\ Tr[AB] &= |\langle\psi|\phi\rangle|^2 \end{aligned}\right\}$$

$U(C\text{-}Z)U^{+} = SWAP$

$\downarrow$

$U = (C\text{-}NOT)\, H \otimes I$

SWAP is diagonal in Bell state basis

Thus 1.5) $Tr[(A\otimes B)\, SWAP] = Tr[(A\otimes B)\, U\,(C\text{-}Z)\, U^{+}] =$

$$= \langle\psi|\otimes\langle\phi|(U(C\text{-}Z)U^{+})|\psi\rangle\otimes|\phi\rangle$$

$|\langle\psi|\phi\rangle|^2$ is the expectation value of $C\text{-}Z$ in state $U^{+}|\psi\rangle\otimes|\phi\rangle$

Questions

** Qiskit Tutorial **

- Pure States
- Mixed States
- Density Matrix

* Mixed States consist of an ensemble of n pure states can be expressed in the form of a list of outcome elements.

$$\{|\psi_j\rangle\}_{j=1}^{n} = \{|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_n\rangle\}$$

$$\{p_j\}_{j=1} = \{p_1, p_2, \ldots, p_n\}$$

Density Matrix

$$\rho = \sum_j p_j |\psi_j\rangle\langle\psi_j| \rightsquigarrow \text{for mixed states}$$

$$\rho = \sum_j |\psi_j\rangle\langle\psi_j| \rightarrow \text{for pure states}$$