GROUP THEORY

Definition of a group: A group is a set of elements G together with a binary operation * that satisfies the following axioms:

1. **Closure:** For all elements a, b in G, $a*b$ is also in G.

2. **Associativity:** For all elements a, b, c in G,

$$(ab)c = a(bc)$$

3. **Identity:** There exists an element e in G, $ea = ae = a$

4. **Inverse:** For each element e in G, there exists an element a^{-1} in G such that

$$aa^{-1} = a^{-1}a = e$$

Ex $(\mathbb{Z}, +)$ Group of integers under addition

1. It is closed $(a+b) \in \mathbb{Z}$
2. It is associative $(a+b)+c = a+(b+c)$
3. 0 is the identity $a+0 = 0+a = a$
4. Every integer has an additive inverse $a \to -a$
   $3 \to -3$

Ex $GL(2)$ 2x2 matrices with nonzero determinant under matrix multiplication

1. Closed
2. Associative $A \times (B \times C) = (A \times B) \times C$
3. I is the identity matrix
4. Every matrix has inverse since $\Delta \neq 0$

Symmetry → reparametrization under which the form of the eqs. of motion is unchanged.

Ex

$$m\ddot{q} = 0$$

$$q \to q' = q + c + vt$$

constant parameters

Galileian transformation

$$\ddot{q}' = \ddot{q}$$

or another parametrization:

$$q'(t') = q(t)$$

$$t' = t + a$$

$$\boxed{q'(t) = q(t-a)}$$

$$\frac{d^2}{dt^2} q = \frac{d'^2}{dt'^2} q'$$

→ Two classes of symmetry groups ↗ discrete ↘ continuous

→ Lie Group : its elements g depend continously on a set of real parameters $\alpha_1 \ldots\ldots \alpha_N$

A Lie group G is both a group and a differentiable Manifold

Group operations are differentiable

I. $g(\alpha) \circ g(\beta) = g(p(\alpha, \beta))$

II. $g^{-1}(\alpha) = g(r(\alpha))$

$p(\alpha, \beta)$ & $r(\alpha)$ are differentiable functions

Usual convention is to choose coordinate such that ?

$e = \text{identity} = g(0)$ $\qquad$ $\alpha = 0 \Leftrightarrow \text{identity}$

Realization of a group

Realization: writing concerетly group elements as transformations $x \longrightarrow g(x)$ over same space X

Ex Euclidean group in 2 dimensions: ISO(2) ?

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \rightarrow \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}}_{\text{rotations}} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}}_{\text{translations}}$$

Group Representations

Given a group G and a linear vector space V a representation D is an operation

$$g \longrightarrow D(g) \qquad D \in GL(V)$$

satisfying:

1. $D(e) = \mathbb{I}$
2. $D(g_1)\,D(g_2) = \overbrace{D(g_1 \cdot g_2)}^{?}$

D is a homomorphism ?

$$G \longrightarrow GL(V)$$

→ V = vector space of dimension N

. V ≡ basis of the representation

. N ≡ dimension of representation ($N \times N$ matrices)

→ $v \in V$ $\qquad v = (v^1, \ldots\ldots, v^N) \rightarrow v^i \quad i = 1, \ldots N$

→ $g \longrightarrow D(g) \equiv (D(g))^i$

$$v \rightarrow D(g)v \; : \; v^i \longrightarrow D(g)^i$$

Group Representation: Basic Notations & Results

Reducibility ⇝ ?

D is reducible if $\exists\, V' \subset V$, $V' \neq \{0\}$

such that $D(g)V' \subseteq V'$ $\quad \forall g$

V' ≡ invariant subspace

→ if no invariant subspace exists, D is said to be irreducible.

Reducible

$$\left(\begin{array}{c|c} M\times M & M\times N \\ \hline 0 & N\times N \end{array}\right)\left(\begin{array}{c} \\ \\ \end{array}\right)\begin{array}{l}\big\}\, M \\ \big\}\, N-M\end{array}$$

$D(g)$

Complete Reducibility

$\exists$ a choice of basis in V such that

$$D(g) = \begin{pmatrix} D_1(g) & 0 & 0 & \\ 0 & \boxed{D_2(g)} & 0 & \cdots \\ 0 & 0 & \ddots & \\ 0 & \cdots & 0 & \ddots \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ \vdots \end{pmatrix}$$

and $D_i(g)$ are all irreducible.

direct sum

$$V = \bigoplus_i V_i$$

$$D = \bigoplus_i D_i$$

Equivalence

$$G \longrightarrow D_1(g)$$

$$G \longrightarrow D_2(g)$$

and S exists, $D_1(g) = S^{-1} D_2(g)\, S$

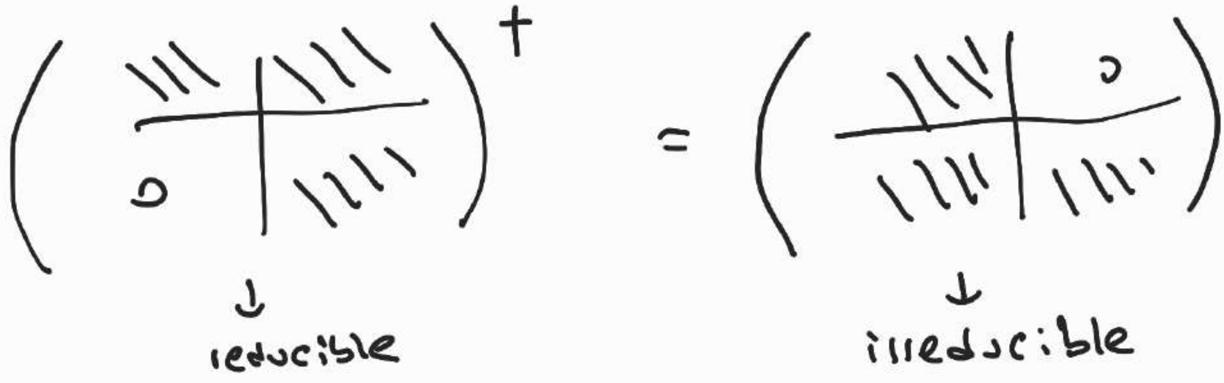
D_1 and D_2 are equivalent

D_1 and D_2 are related by change of basis

Unitary

D is unitary if $\forall g$ $D(g)$ is unitary

$$D^{-1}(g) = D(g^{-1}) = D(g)^{\dagger}$$



Schur's Lemma (1)

$$\begin{tikzcd}
 & G \arrow[dl, bend right, "D_1(g)"'] \arrow[dr, bend left, "D_2(g)"] & \\
V_1 \arrow[rr, "A"] & & V_2
\end{tikzcd}$$

Hypothesis: A = linear operator : $AV_1 \subseteq V_2$

$$\boxed{A D_1(g) = D_2(g) A} \quad \forall g$$

D_1 and D_2 irreducible

Thesis: either $A = 0$

or A = bijective ?

Proof: Consider K = kernel of A

$$AK = 0$$

$$0 = D_2(g) AK = A \underbrace{D_1(g) K}_{D_1(g) K \subseteq K}$$

D_1 irreducible $\Rightarrow$ either $K = \{0\}$ or $K = V_1$

1) $K = V_1 \implies A = 0$ obvious

2) $K = \{0\} \implies A$ is bijective (needs proof)

$$V_1' \equiv A V_1 \subseteq V_2$$

$$D_2(g) V_1' = D_2(g) A V_1 = A \underbrace{D_1(g) V_1}_{\subseteq V_1} \subseteq V_1'$$

$V_1' \equiv$ invariant subspace of D_2

$$D_2 \text{ is irreducible} \implies V_1' \begin{cases} \{0\} \\ V_2 \end{cases}$$

$$V_1' = V_2$$

$$\begin{cases} A V_1 = V_2 \\ A K = 0 \end{cases}$$

A is bijective

Schur Lemma (2)

Thesis:

Hypo: $V_1 = V_2 \equiv V$ and $AD = DA \Rightarrow A = \lambda \mathbb{I}$

$D_1 = D_2 \equiv D$ = irreducible

A will have at least one eigenvector v

$$Av = \lambda v$$

$$A' \equiv A - \lambda \mathbb{I}$$

$\rightarrow A' D(g) = D(g) A'$ (commutes)

$\rightarrow$ A has null eigenvalue : $A'v = Av - \lambda v = 0$

A' not irrevertible $\quad$ A' not bijective

by Schur (1) $\quad$ $A' = 0$

$$\Rightarrow A' = A - \lambda \mathbb{I} = 0$$

$$\Rightarrow A = \lambda \mathbb{I}$$