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Conservation rules for entanglement transfer between qubits

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Abstract
We consider an entangled but non-interacting qubit pair a_1 and b_1 that is independently coupled to a set of local qubit systems, a_I and b_J, of 0 bit value, respectively. We derive rules for the transfer of entanglement from the pair a_1–b_1 to an arbitrary pair a_I–b_J, for the case of qubit-number conserving local interactions. It is shown that the transfer rule depends strongly on the initial entangled state. If the initial entanglement is in the form of the Bell state corresponding to anti-correlated qubits, the sum of the square of the nonlocal pairwise concurrences is conserved. If the initial state is the Bell state with correlated qubits, this sum can be reduced, even to zero in some cases, to reveal a complete and abrupt loss of all nonlocal pairwise entanglement. We also identify that for the nonlocal bipartitions A–b_J involving all qubits at one location, with one qubit b_J at the other location, the concurrences satisfy a simple addition rule for both cases of the Bell states that the sum of the square of the nonlocal concurrences is conserved.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

Entanglement is not only crucial to the transition between classical and quantum behaviour [1], but is essential to many key applications in quantum information [2]. An understanding of how entanglement is transferred between systems and the existence of associated conservation rules is of fundamental and practical importance.

The seminal works of Bell focused on entanglement that is shared between two distant and non-interacting 'qubit' (two-level) systems [3]. This 'nonlocal' entanglement can be maintained over long distances with exciting implications for tests of quantum mechanics and applications such as quantum cryptography [4]. However, a fundamental issue is the degradation of entanglement brought about because each party inevitably interacts 'locally' with other systems. This local coupling can lead to an abrupt depletion of the original entanglement [5–9], an effect which has been recently experimentally confirmed [10]. Under some circumstances, the already lost entanglement can revive after a finite time [11–14]. The dynamical behaviour of entanglement under the action of the environment is regarded as a central issue in quantum information [15, 16].

Intuition tells us that the two-qubit entanglement is not truly 'lost', but simply redistributed among the interacting parties [16–20]. While it is the case that entanglement can be created between local systems, due to the local couplings, it is logical to investigate under what circumstances a global nonlocal entanglement is conserved, to reflect that no further 'nonlocal' interaction has taken place, and to ask whether a rule exists to describe the entanglement transfer and to express a conserved global entanglement in terms of the constituent entanglement.

Indeed a requirement of a measure of entanglement between remotely separated parties is that the entanglement does not increase under certain local operations assisted by classical communication [21]. Our interest here is more specific. We construct subsets of local systems, and consider only qubit-conserving interactions between them, so a transfer

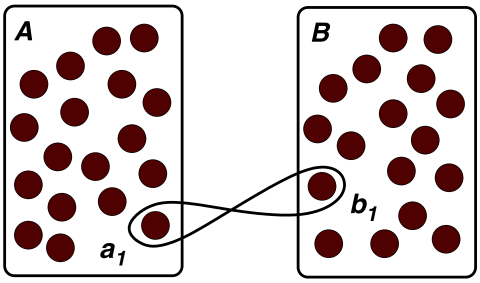

Figure 1. We derive rules to describe how the entanglement is transferred among qubits, when each member of an entangled non-interacting qubit pair a_1, b_1 is locally and independently coupled to a set of qubits, a_I and b_I, respectively.

of qubits is described. We do not allow further interaction between the two remotely separated systems themselves.

The question of whether a universal conservation rule exists to describe the entanglement transfer is a difficult one in full, requiring knowledge of necessary and sufficient entanglement measures where entanglement can be shared among more than two qubits. For example, the recent work of Hiesmayr *et al* [22] focuses on the development of computable measures of entanglement in the multipartite scenario. Nonetheless, in this paper we take a first step by deriving some simple conservation rules that apply for the fundamental case of an initial two-qubit 'Bell state' entanglement. The rules allow a quantitative knowledge of the degree of the 'nonlocal' entanglement, in terms of the concurrence measure [23], shared between remaining parties, given a knowledge of the entanglement between one spatially separated pair.

Specifically, we examine entanglement transfer from one qubit pair a_1, b_1 into a set of qubits pairs a_I, b_J, where members of all 'nonlocal' pairs a_I, b_J are non-interacting, as illustrated in figure 1. The initial entanglement is in the form of a Bell state, with all other qubits having a 0 bit value, and the local interaction that can exist within the qubit sets, $A \equiv \{a_I\}$ and $B \equiv \{b_J\}$. The interaction is constrained only by the requirement that there is a transfer of the qubit value, so the total local qubit number, or Hamming weight, is conserved. Such interactions are relevant to quantum networks, in creating qubit superposition states, and are fundamental in describing system–environment interactions [5, 24].

Yonac *et al* [17, 18] have presented a conservation rule for the case of Bell states with anti-correlated qubits

$$|\Psi\rangle = \cos\alpha|10\rangle + \sin\alpha\,\mathrm{e}^{\mathrm{i}\beta}|01\rangle, \tag{1}$$

where β is an arbitrary phase and $|10\rangle$ represents qubit values of 1 and 0 for the qubit pair a_1 and b_1 respectively. Yonac *et al* observed that $C_{11} + C_{22}$ is constant, where C_{IJ} is the concurrence measure [23] of entanglement between qubits a_I and b_J. Their result however was specific to one local Hamiltonian, that of the Jaynes–Cummings model, and is not valid generally.

Lopez *et al* [16] presented a very different result for Bell states with correlated qubits

$$|\Phi\rangle = \cos\alpha|11\rangle + \sin\alpha\,\mathrm{e}^{\mathrm{i}\beta}|00\rangle. \tag{2}$$

They showed that when such a qubit pair is coupled to independent reservoirs, the death of qubit entanglement precedes the birth of reservoir entanglement, so there is a temporary loss of all nonlocal pairwise entanglement. Again, their results applied to one form of local interaction Hamiltonian only.

Our conservation rules apply to all cases of qubit-preserving 'local' interactions, and allow new insight. Which conservation rule is valid depends only on the type of global entanglement that is imprinted onto the system. When the initial 'global' entanglement is that of the Bell state $|\Psi\rangle$, a remarkably simple conservation result exists:

$$\sum_{I=1}^{N}\sum_{J=1}^{M} C_{IJ}^2(t) = C_{AB}^2, \tag{3}$$

which shows that the total nonlocal pairwise entanglement is conserved. However, this conservation rule manifests in the *square* of the concurrence (the 'tangle'), not in the concurrence itself. The conserved quantity corresponds to the initial entanglement, but only when evaluated as C_{AB}^2, the square of the concurrence C_{AB} of the Bell state. For this case, we will prove an additivity of constituent entanglement that the entanglement shared between any two nonlocal partitions is the sum of the nonlocal pairwise entanglement of the constituents of the partitions. This rule applies to closed systems, and is investigated for open systems.

When the initial 'global' entanglement is that of the Bell state $|\Phi\rangle$, the following inequality holds:

$$0 \leqslant \sum_{I,J} C_{IJ}^2(t) \leqslant C_{AB}^2. \tag{4}$$

We will show that there can be a vanishing of all nonlocal pairwise constituent entanglement C_{IJ}, despite conservation of the global nonlocal entanglement C_{AB}, to give consistency with the result of Lopez *et al* [16].

The two different scenarios may be thought of in the following way. Suppose entanglement exists between A and B, where A is made up of subsystems measured by Alice, Ann, Agatha respectively, and B is composed of subsystems measured by Bob, Bill and Brian. In the Ψ scenario, the global nonlocal entanglement can always be evaluated, through measurements shared only by pairs: Alice–Bob, Alice–Bill, Ann–Bob, Ann–Bill, etc. In the Φ scenario, this is not the case. Examples exist where all pairwise entanglement would be zero, despite there being a global entanglement, between A and B. This has potential implications for quantum cryptography, where measurement of shared entanglement between two parties, A and B, at different locations is used to determine security [25].

We will show however that in both cases, the global (original) entanglement could be inferred with communication between all parties at A and one at B: Alice–Ann–Agatha–Bob; Alice–Ann–Agatha–Bill, and so on. This follows from the simple addition rule that can be proved in both cases:

$$C_{AB}^2 = C_{AB_1}^2 + C_{AB_2}^2 + \cdots + C_{AB_N}^2. \tag{5}$$

This represents saturation of the CKW inequality [26], which constrains the concurrences for any three parties A, B_1, B_2, according to

$$C^2_{AB_1} + C^2_{AB_2} + \cdots + C^2_{AB_N} \leqslant C^2_{AB}. \tag{6}$$

Equality (5) is useful to cryptography scenarios, where B and A share an entangled Bell state, but a third party Eve (who we call B_2) may eavesdrop through an interaction that conserves qubit number. In this case, regardless of the type of Bell state used to transport the entanglement, Bob (who we call B_1) and Alice (who we call A) can deduce the degree of entanglement that Eve can possess $\left(C^2_{AB_2} = C^2_{AB} - C^2_{AB_1}\right)$.

Studies of entanglement distribution among m parties have revealed that for some states (GHZ-type states) entanglement can exist among m parties, but not exist when measurements are performed on less than m parties. On the other hand, for other states, the W states, distributed entanglement satisfies a simple rule based on saturation of the CKW inequality. The rules describing entanglement transfer are largely determined by what states can be formed under the local entanglement transfer interactions.

2. Qubit-conserving local interaction Hamiltonian model

We begin by considering the two global non-interacting systems, the qubit sets $A \equiv \{a_I\}$ and $B \equiv \{b_J\}$. For the initial states we consider, the total qubit number (or Hamming weight) $Q_{A/B}$ of A/B has a value 0 or 1. We thus define eigenstates $|Q_A\rangle|Q_B\rangle$, where $Q_A, Q_B = 0, 1$ are the outcomes for the total qubit number at A and B, respectively. We assume that each system A, B has constituent subsystems, and there are internal interactions between them, described by Hamiltonians H_A and H_B respectively, so that the total Hamiltonian is

$$H = H_A + H_B. \tag{7}$$

Importantly, it is assumed in this model that each of H_A and H_B conserves the qubit number (Hamming weight) of the system A and B, respectively.

A measure of bipartite entanglement between two qubit systems, such as A and B, is the concurrence, defined as $C_{AB} = \max(0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4})$, where $\lambda_1, \ldots, \lambda_4$ are the eigenvalues, in decreasing order, of the density matrix $\rho' = \rho(\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$. Here ρ is the two-qubit system density operator, ρ^* is the complex conjugation of ρ in the standard basis and σ_y is the Pauli matrix in the y direction expressed in the same basis. The maximum possible entanglement is given by $C_{AB} = 1$, while $C_{AB} = 0$ implies no entanglement. The state is entangled when $C_{AB} > 0$.

The global bipartite entanglement defined for systems A and B is invariant throughout the evolution. This follows from the fact that the Hamiltonian (7) does not allow interaction between the two systems, or with external systems.

It is useful to first examine the full qubit description of the system, which does evolve. Since the total qubit value at each of A and B is assumed conserved under the action of the Hamiltonian (7) and we consider the case where initially only a_1 and b_1 can have a non-zero qubit value, we then see that at any time there can be at most one qubit system at each of A and B with a bit value 1; all other bits are 0. The possible states of the system can thus be written as

$$|\{1\}_n\rangle_A|\{1\}_m\rangle_B, \tag{8}$$

where $|\{1\}_n\rangle_A = |0\ldots\{1\}_n\ldots0\rangle$ denotes that the qubit a_n is in the state $|1\rangle$, while all other qubits $a_{I\neq n}$ are in $|0\rangle$, and $|\{1\}_m\rangle_B$ similarly defines the qubit states of B. Also possible are the states, written as $|0\rangle_A$ and $|0\rangle_B$, with zero qubit value, for all qubits. Thus, explicitly, if we write the initial state as a superposition of the two Bell states:

$$\begin{aligned}|\psi(0)\rangle &= d_{00}(0)|0\rangle_A|0\rangle_B + d_{01}(0)|0\rangle_A|\{1\}_1\rangle_B\\ &\quad + d_{10}(0)|\{1\}_1\rangle_A|0\rangle_B + d_{11}(0)|\{1\}_1\rangle_A|\{1\}_1\rangle_B,\end{aligned} \tag{9}$$

then the evolution is given by

$$\begin{aligned}\psi(t)\rangle &= \mathrm{e}^{-\mathrm{i}Ht/\hbar}|\psi(0)\rangle\\ &= d_{00}(0)|0\rangle_A|0\rangle_B + d_{01}(0)|0\rangle_A\mathrm{e}^{-\mathrm{i}H_Bt/\hbar}|\{1\}_1\rangle_B\\ &\quad + d_{10}(0)\,\mathrm{e}^{-\mathrm{i}H_At/\hbar}|\{1\}_1\rangle_A|0\rangle_B\\ &\quad + d_{11}(0)\,\mathrm{e}^{-\mathrm{i}H_At/\hbar}|\{1\}_1\rangle_A\,\mathrm{e}^{-\mathrm{i}H_Bt/\hbar}|\{1\}_1\rangle_B\\ &= d_{00}(0)|0\rangle_A|0\rangle_B + \sum_{m=1}^{M} d_{0m}(t)|0\rangle_A|\{1\}_m\rangle_B\\ &\quad + \sum_{n=1}^{N} d_{n0}(t)|\{1\}_n\rangle_A|0\rangle_B + \sum_{n=1}^{N}\sum_{m=1}^{M} d_{nm}(t)|\{1\}_n\rangle|\{1\}_m\rangle,\end{aligned} \tag{10}$$

where

$$\begin{aligned}d_{00}(t) &= d_{00}(0), &\qquad |d_{10}(0)|^2 &= \sum_{n=1}^{N}|d_{n0}(t)|^2,\\ |d_{01}(0)|^2 &= \sum_{m=1}^{M}|d_{0m}(t)|^2, &\qquad |d_{11}(0)|^2 &= \sum_{n=1}^{N}\sum_{m=1}^{M}|d_{nm}(t)|^2.\end{aligned} \tag{11}$$

In the derivation of (10) we have used the Hamiltonian (7), that $H = H_A + H_B$, and that the unitary operators conserve probability, so for example

$$\mathrm{e}^{-\mathrm{i}H_Bt/\hbar}|\{1\}_1\rangle_B = \sum_{n=1}^{N} c_n(t)|\{1\}_n\rangle_B, \tag{12}$$

where $\sum_{n=1}^{N}|c_n(t)|^2 = 1$. Note that the coefficient d_{00} is time independent, because a system in the ground state remains in the ground state under the action of the Hamiltonian. Moreover, the global bipartite entanglement between systems A and B remains constant. This is because for evaluation of the bipartite concurrence C_{AB}, the expansion of $|\psi(t)\rangle$ in terms of the eigenstates $|Q_A\rangle|Q_B\rangle$ of the *total* qubit values Q_A and Q_B at A and B is all that is required. The factorization $\exp(-\mathrm{i}Ht/\hbar) = \exp(-\mathrm{i}H_At/\hbar)\exp(-\mathrm{i}H_Bt/\hbar)$ of the unitary operator leads to the invariance of C_{AB}, since H_A, H_B each conserve the total bit values of A and B, respectively. Hence, the concurrence C_{AB} is invariant.

3. Entanglement evolution and conservation rules for Bell states

We now examine the two types of pure state entanglement that can characterize the two-qubit system A–B [5]. These are given by the two Bell states, one with anticorrelated 'spins'

$$|\Psi\rangle = \cos\alpha|1\rangle|0\rangle + \mathrm{e}^{\mathrm{i}\beta}\sin\alpha|0\rangle|1\rangle \tag{13}$$

and the other with correlated spins

$$|\Phi\rangle = \cos\alpha|1\rangle|1\rangle + \mathrm{e}^{\mathrm{i}\beta}\sin\alpha|0\rangle|0\rangle. \tag{14}$$

In both cases the concurrence is $C_{AB} = 2\sin\alpha\cos\alpha$. It indicates that the maximal entanglement occurs at $\alpha = \pi/4$.

Next we examine what happens when each of the systems A and B is composed of a collection of *interacting* qubits, which we denote $\{a_I\}$ and $\{b_J\}$, respectively. Since the overall qubit value at each of A and B must be conserved under the Hamiltonian (7), for the initial states $|\Psi\rangle$ and $|\Phi\rangle$, there can be at most one qubit at each of A or B with a bit 1 and all other bits with 0. The possible states for $t > 0$ are described by (10).

3.1. Conservation rule for the Bell state $|\Psi\rangle$

We first assume that the initial state is the Bell state $|\Psi\rangle$. Using the result (10) with $d_{00}(0) = d_{11}(0) = 0$, we find that the evolution for this form of global entanglement $|\Psi\rangle$ can be written as

$$|\Psi(t)\rangle = \sum_{I=1}^{N} d_{AI}(t)|\{1\}_I\rangle|0\rangle + \sum_{J=1}^{M} d_{BJ}(t)|0\rangle|\{1\}_J\rangle. \tag{15}$$

Here, $d_{AI}(t)$ and $d_{BJ}(t)$ satisfy, after using (11):

$$\sum_{I=1}^{N}|d_{AI}(t)|^2 = \cos^2\alpha, \qquad \sum_{J=1}^{M}|d_{BJ}(t)|^2 = \sin^2\alpha. \tag{16}$$

For $N = M = 2$ this state can be written as

$$|\Psi(t)\rangle = d_{A1}|1000\rangle + d_{A2}|0100\rangle + d_{B1}|0010\rangle + d_{B2}|0001\rangle, \tag{17}$$

where $|1000\rangle$ denotes that the qubit a_1 has value 1, while others are 0..., etc. This conservation of probability holds because there is no external coupling, nor coupling between the systems A and B, that allows a transfer of qubit excitation [19].

We now relate the global bipartite entanglement C_{AB}, which is conserved, to the nonlocal pairwise concurrences C_{IJ} of subsystems a_I and b_J. This pairwise concurrence is calculated by tracing over all other systems. Calculation shows that the reduced density matrix for the a_I, b_J system, written in the basis states $|11\rangle$, $|10\rangle$, $|01\rangle$ and $|00\rangle$, is of 'X-state' form [17]:

$$\rho_{IJ} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & |d_{AI}|^2 & d_{AI}d_{BJ}^* & 0 \\ 0 & d_{AI}^*d_{BJ} & |d_{BJ}|^2 & 0 \\ 0 & 0 & 0 & \sum_{i\neq I}|d_{Ai}|^2 + \sum_{j\neq J}|d_{Bj}|^2 \end{pmatrix}. \tag{18}$$

In this case, the pairwise concurrence is simply given by

$$C_{IJ} = 2|d_{AI}||d_{BJ}|. \tag{19}$$

Theorem 1. *For the case of a global Bell entanglement C_{AB} of type $|\Psi\rangle$ between two non-interacting systems A and B, the sum of the square of the pairwise constituent 'nonlocal' concurrences (SSPC) is conserved: SSPC $= \sum_{I=1}^{N}\sum_{J=1}^{M} C_{IJ}^2 = C_{AB}^2$. The entanglement shared by any two nonlocal partitions $\{a_i, a_j, \ldots\}$ and $\{b_m, b_n, \ldots\}$ satisfies a simple Pythagorean addition of constituent entanglement*

$$C_{AB}^2 = C_{\{i,j,\ldots\}\{m,n,\ldots\}}^2 = \sum_{k=i,j,\ldots}\ \sum_{l=m,n,\ldots} C_{kl}^2. \tag{20}$$

In addition, we can write the sum rule for the entanglement shared between system A and each of the subsystems of B:

$$C_{AB}^2 = C_{A\{m,n,\ldots\}}^2 = \sum_{l=m,n,\ldots} C_{Al}^2. \tag{21}$$

Proof. For any Hamiltonian of the form $H = H_A + H_B$, the probability sums (16) are constant [19]. Hence, given that the pairwise concurrence is derived from (18), the conjecture must hold:

$$\begin{aligned} \mathrm{SSPC} &= \sum_I\sum_J |C_{IJ}|^2 \\ &= 4\left(\sum_{I=1}^{N}|d_{AI}(t)|^2\right)\times\left(\sum_{J=1}^{M}|d_{BJ}(t)|^2\right) \\ &= 4\cos^2\alpha\sin^2\alpha = C_{AB}^2. \end{aligned} \tag{22}$$

The result (21) follows in the same manner from direct evaluation of concurrences after tracing. We note other sum rules follow for this system. There is an additivity of constituent entanglement that the entanglement shared between any two nonlocal partitions is the sum of the nonlocal pairwise entanglement of the constituents of the partitions.

We note that the state (15) reduces to the $2N$-partite W state $(|100\ldots\rangle + |010\ldots\rangle + |001\ldots\rangle + \ldots)/\sqrt{2N}$ when the probability amplitudes are equal. We note that if we consider only the entanglement between system A, which is written as $A \equiv \{i, j, \ldots\}$, and the subsystems of B, which we write as $B_1, B_2, \ldots$, then (21) reduces to

$$C_{AB}^2 = C_{AB_1}^2 + C_{AB_2}^2 + \cdots + C_{AB_N}^2, \tag{23}$$

which is the well-known monogamy relation for W states, for which the CKW inequality is saturated [26, 27]. The W states [28] are known to be robust with respect to particle losses, and this is reflected in the conservation rule, which states that entanglement is preserved, with concurrence $C_{IJ} = 1/N$, after tracing over all but two parties I, J, that is, if all but two parties lose the qubit information. That the W state has the greatest amount of pairwise bipartite entanglement possible after tracing over all other parties was conjectured by Dur *et al* [28]. If we consider the three sub-systems A, B_1, B_2, then the statement (23) reduces to the result that the 3-tangle defined as $C_{A(B_1B_2)}^2 - C_{AB_1}^2 - C_{AB_2}^2$ is 0, which is a known result for the four-qubit W state [29]. □

3.2. Concurrence inequality rule for the Bell state $|\Phi\rangle$

Where the global entanglement of the qubits a_1, b_1 is that of the Bell state $|\Phi\rangle$, the Pythagorean sum of the pairwise concurrences is no longer conserved, but satisfies inequality (4). In this case, the wavefunction evolves as

$$|\Phi(t)\rangle = \sum_{i=1}^{N}\sum_{j=1}^{M} c_{ij}(t)|\{1\}_i\rangle|\{1\}_j\rangle + c_0|0\rangle|0\rangle. \tag{24}$$

Here

$$|c_0(t)|^2 = \sin^2\alpha, \qquad \sum_{i,j}|c_{ij}(t)|^2 = \cos^2\alpha \tag{25}$$

as follows from (11). For $N = 2$, the state can be expressed as

$$|\Phi(t)\rangle = c_{11}|1010\rangle + c_{12}|1001\rangle + c_{21}|0110\rangle + c_{22}|0101\rangle + c_0|0000\rangle. \tag{26}$$

In this case, the reduced density matrix ρ_{IJ} for the qubits a_I and b_J written in terms of basis states $|11\rangle$, $|10\rangle$, $|01\rangle$ and $|00\rangle$ is

$$\rho_{IJ} = \begin{pmatrix} |c_{IJ}|^2 & 0 & 0 & c_{IJ}c_0^* \\ 0 & \sum_{n\neq J}|c_{In}|^2 & 0 & 0 \\ 0 & 0 & \sum_{m\neq I}|c_{mJ}|^2 & 0 \\ c_{IJ}^*c_0 & 0 & 0 & \sum_{n,m\neq J,I}|c_{m,n}|^2 + |c_0|^2 \end{pmatrix}. \tag{27}$$

For example, for the case of $N = M = 2$, the reduced density matrix for qubits a_1 and b_1 is

$$\rho_{\rm red} = (c_0|00\rangle + c_{11}|11\rangle)\left(c_0^*\langle 00| + c_1^*\langle 11|\right) + |c_{12}|^2|10\rangle\langle 10| + |c_{21}|^2|01\rangle\langle 01| + |c_{22}|^2|00\rangle\langle 00|. \tag{28}$$

The concurrence of (27) is

$$C_{IJ} = \max\left\{0, 2\left(|c_{IJ}||c_0| - \sqrt{\left(\sum_{n\neq J}|c_{In}|^2\right)\left(\sum_{m\neq I}|c_{mJ}|^2\right)}\right)\right\}, \tag{29}$$

from which we note immediately that $C_{IJ}^2 \leqslant 4|c_0|^2|c_{IJ}|^2$ and thus, using (25), inequality (4) must hold. Hence, we write the following theorem.

Theorem 2. *For the case of a global Bell entanglement C_{AB} of type $|\Phi\rangle$ between two non-interacting systems A and B, SSPC is constrained by an upper bound. The entanglement shared by any two nonlocal partitions $\{a_i, a_j, \ldots\}$, $\{b_m, b_n, \ldots\}$ satisfies*

$$C_{AB}^2 = C_{\{i,j,\ldots\}\{m,n,\ldots\}}^2 \leqslant \sum_{k=i,j,\ldots}\sum_{l=m,n,\ldots} C_{kl}^2. \tag{30}$$

It is possible for all pairwise entanglement to vanish. However, we find that the following sum rule does hold to give a saturation of the monogamy relation [26] for the total system at A, with the components of B:

$$C_{AB}^2 = C_{A\{m,n,\ldots\}}^2 = \sum_{l=m,n,\ldots} C_{Al}^2. \tag{31}$$

Proof. The proof of the inequality follows from above. In fact, inequality (30) can be proved for any state via the CKW inequality (6). One merely applies the CKW inequality a second time to each of the terms on the left-hand side of (6).

To prove equality (31), we note that the state (24) written in terms of the total qubit system at A is $\sum_{j=1}^{M}\tilde{c}_{1j}|1\rangle|\{1\}_j\rangle + c_0|0\rangle|0\rangle$, which can be written explicitly as

$$|\Phi(t)\rangle = \tilde{c}_1|1; 100\ldots\rangle + \tilde{c}_2|1; 010\ldots\rangle + \tilde{c}_3|1; 001\ldots\rangle + \cdots + c_0|0; 000\ldots\rangle, \tag{32}$$

where $|c_0|^2 = \sin^2\alpha$ and $\sum_j|\tilde{c}_j|^2 = \cos^2\alpha$. Tracing over all qubits of B except B_1 gives

$$\rho_{AB_1} = \{\tilde{c}_1|1; 1\ldots\rangle + c_0|0; 0\ldots\rangle\} \times \{\tilde{c}_1^*\langle 1; 1\ldots| + c_0^*\langle 0; 0\ldots|\} + \{|\tilde{c}_2|^2 + |\tilde{c}_3|^2 + \ldots\}|1; 0\ldots\rangle\langle 1; 0\ldots|, \tag{33}$$

which is of the X-form (18), from which the concurrence can be calculated. More generally, we find $C_{AB_J} = 2|\tilde{c}_J||c_0|$, to confirm the required result.

We can see directly from (28) that all nonlocal pairwise entanglement C_{IJ} can vanish (SSPC $= 0$), even where the maximal global entanglement C_{AB} exists. This is consistent with behaviour of other multi-partite states, such as the GHZ state $(|000\rangle + |111\rangle)/\sqrt{2}$, which display no bipartite entanglement once one of the qubits is traced out [28]. Calculation reveals that the four-qubit state (32) has zero bipartite concurrence for all nonlocal bi-partitions when there are equal probability amplitudes: $|c_{11}| = |c_{12}| = |c_{21}| = |c_{22}| = |c_0| = 1/\sqrt{5}$.

We note however this loss of entanglement is not the case for all nonlocal partitions. Theorem 2 reveals for $|\Phi\rangle$ states that the entanglement C_{AJ} defined for nonlocal partitions $\{a_1, \ldots, a_N\}\{b_J\}$ satisfies a simple addition rule (31), $\sum_J C_{AJ}^2 = C_{AB}^2$. Thus for the case of four qubits,

$$C_{\{1,2\}\{1,2\}}^2 = C_{\{1,2\}\{1\}}^2 + C_{\{1,2\}\{2\}}^2, \tag{34}$$

which implies a zero 3-tangle between A and qubits b_1 and b_2 in this case, as for the W state considered above. However, there is a non-zero 3-tangle for systems A_1, B_1, B_2, for example, defined as $\tau = C_{A_1B_1B_2}^2 = C_{A_1B}^2 - C_{A_1B_1}^2 - C_{A_1B_2}^2$. For the state, C_{A_1B} can be calculated similarly as C_{AB_1} (see (32)) to yield a non-zero value, which implies a 3-tangle of $\tau_3 = C_{A_1B}^2$ when $C_{A_1B_1} = C_{A_1B_2} = 0$, that is, when the probability amplitudes are equal. This gives a different behaviour to the four-qubit GHZ-type states, which have zero 3-tangle upon tracing out over one of the four parties [29]. Such GHZ states cannot be formed from the two-qubit GHZ-type state $|\Phi\rangle$ under the local transformations considered. □

4. Jaynes–Cummings example

We give an example of the conservation rule by examining the case of $N, M = 2$, where the qubits a_1, b_1 are two-level atoms with transition frequency ω_0, and the qubits a_2, b_2 are cavity modes with resonant frequencies ω_a and ω_b, respectively. We consider the local interaction Hamiltonian of the Jaynes–Cummings form [30]:

$$H_A = \hbar\omega_0 S_A^z + \hbar\omega_a\left(a^\dagger a + \tfrac{1}{2}\right) + \hbar g_a\left(a^\dagger S_A^- + aS_A^\dagger\right), \tag{35}$$

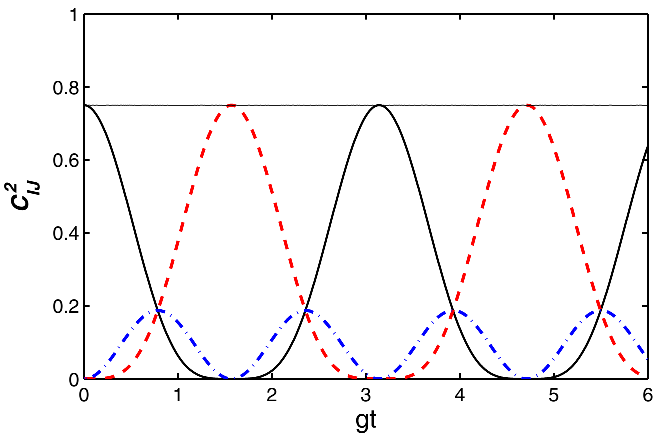

Figure 2. Evolution of the two-qubit concurrences for the state $|\Psi\rangle$ with the Hamiltonian (35) and for different nonlocal partitions: C_{11}^2 (solid line); C_{22}^2 (dashed line); $C_{12}^2 = C_{21}^2$ (dash-dotted line); C_{AB}^2 (thin solid line). The following rule, universal for $|\Psi\rangle$, holds: SSPC $= C_{AB}^2$. Here $\Delta_a = \Delta_b = 0$, $g_a = g_b$ and $\alpha = \pi/6$. In the case of symmetric interactions $g_a = g_b$ illustrated here, the simple conservation rule $C_{AB} = C_{11} + C_{22}$ of Yonac *et al* [17, 18] also holds.

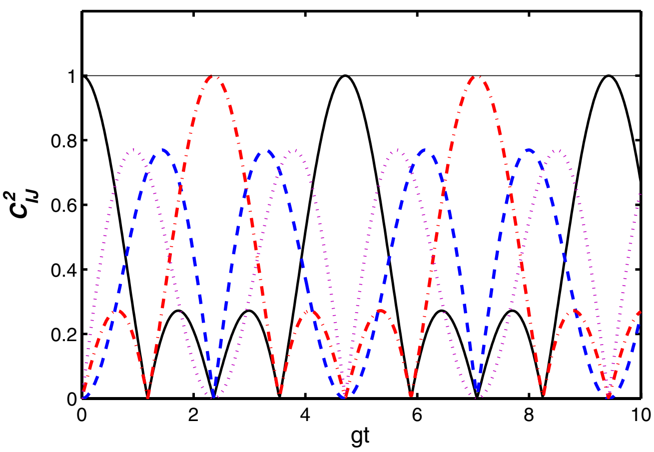

Figure 3. Evolution of two-qubit concurrences for the state $|\Psi\rangle$ with the Hamiltonian (35) and for different nonlocal partitions. The plots confirm the rule, universal for $|\Psi\rangle$, that SSPC $= C_{AB}^2$. C_{11} (solid line), C_{22} (dashed line), C_{12} (dashed-dotted line), C_{21} (dotted line). Here, $\Delta_a = \Delta_b = 0$, $g_a = 2g_b$, $g = (g_a + g_b)/2$ and $\alpha = \pi/4$.

where S_A^+, S_A^- and S_A^z are respectively raising, lowering and spin-z operators for the atom qubit a_1, and $a^\dagger$ (a) are the creation (annihilation) operators for the cavity mode qubit a_2. The parameter g_a is the strength of the coupling between the atom and the cavity mode. The local Hamiltonian for B is defined similarly.

The Jaynes–Cummings interaction is fundamental in describing couplings between field and atoms in cavities [31]. The Jaynes–Cummings interaction Hamiltonian has also been used to model the qubit–cavity coupling in circuit QED experiments that use superconducting qubits, and which have recently realized an entangled two-qubit nonlocal Bell state [32, 33]. Our conservation results enable prediction of the entanglement between cavity–atom pairs, for the two types of Bell state entanglement, and could in principle be tested in these experimental situations as well as in the all-optical entanglement experiment of Almeida *et al* [10].

4.1. Jaynes–Cummings example for $|\Psi\rangle$

Solutions for the concurrences describing pairwise entanglement can be evaluated for the case of an initial Bell state entanglement of type $|\Psi\rangle$, for the full case of arbitrary couplings and detunings, defined as $\Delta_a = (\omega_0 - \omega_a)/2$ and $\Delta_b = (\omega_0 - \omega_b)/2$. Figures 2 and 3 show concurrences for symmetric and asymmetric interactions, to confirm the conservation law (3).

4.2. Jaynes–Cummings example for $|\Phi\rangle$

The loss of all nonlocal pairwise concurrence for evolution of the Bell state $|\Phi\rangle$ is evident in the model (35). This is illustrated in figure 4, where we plot the time evolution of pairwise concurrences for different nonlocal partitions and for the state $|\Phi\rangle$. Where the entanglement C_{AB} is low, we can identify regions where each C_{IJ} is zero (SSPC $= 0$). The SSPC is regained when the transfer of entanglement from qubits a_1, b_1

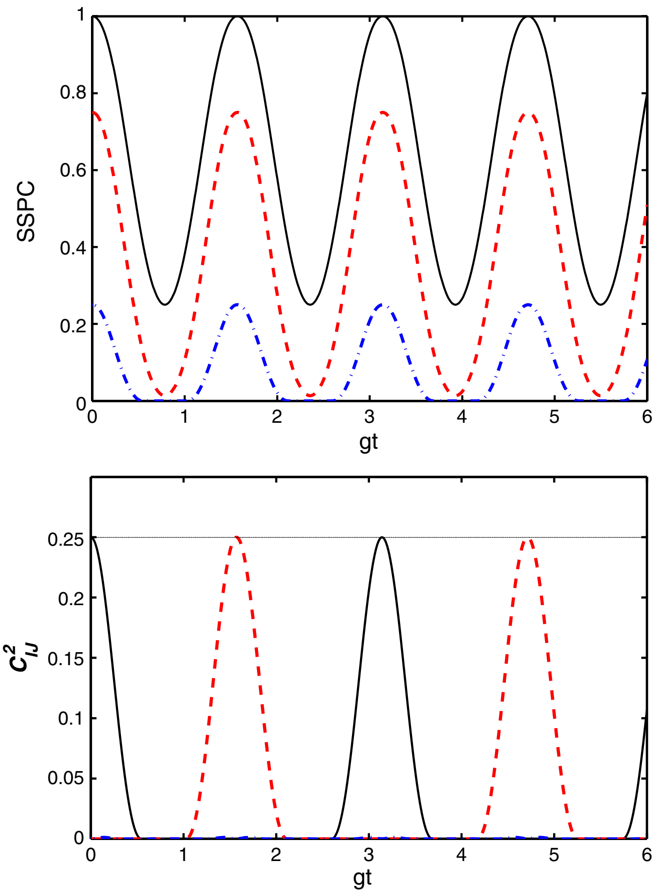

Figure 4. Evolution of pairwise concurrences for different nonlocal partitions, for the state $|\Phi\rangle$ and the Hamiltonian (35). Here, $\Delta_a = \Delta_b = 0$ and $g_a = g_b = g$. The top figure plots SSPC for $\alpha = \pi/4$ (solid line), $\alpha = \pi/6$ (dashed line), $\alpha = \pi/12$ (dashed-dotted line). The bottom figure plots the individual concurrences for $\alpha = \pi/12$: C_{11}^2 (solid line), C_{22}^2 (dashed line), C_{AB}^2 (thin solid line). $C_{12}^2 = C_{21}^2$ (dash-dotted line) is below 0.01 in this case. As the global entanglement C_{AB} weakens, regions of total loss of nonlocal pairwise entanglement are evident (SSPC = 0).

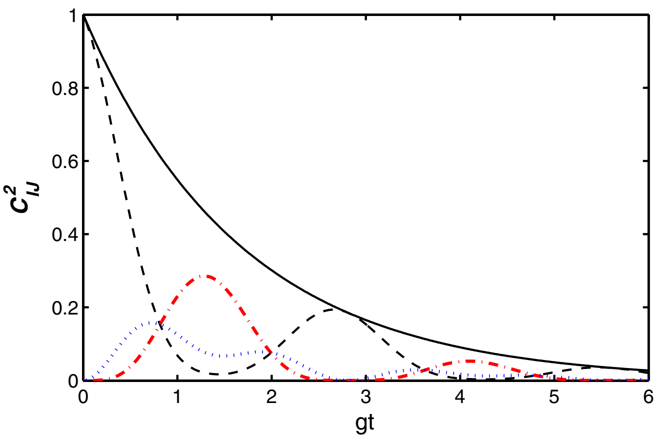

Figure 5. Evolution of pairwise concurrences for the state $|\Psi\rangle$ under the Hamiltonian (35) and with coupling to reservoirs. C_{11}^2 (dashed line); C_{22}^2 (dashed-dotted line); $C_{12}^2 = C_{21}^2$ (dotted line); C_{AB}^2 (solid line). Here, $\Delta_a = \Delta_b = 2g$, $g_a = g_b = g$, $\alpha = \pi/4$, and $\gamma = 0.3g$.

into the qubits a_2, b_2 is complete, so that the inequality of theorem 2 is saturated at regular intervals determined by the Rabi frequency, at which all excitations (qubits with bit value 1) are in the same type of qubit (atoms or field modes). We see that C_{AB} is low and observe the feature predicted by Lopez *et al* [16] for reservoir interactions that the 'birth' of entanglement in cavity modes is delayed a finite time after the 'death' of entanglement in the atoms.

5. Effect of environment on entanglement transfer

We show that the constituent addition rule (3) for the Bell state $|\Psi\rangle$ also applies to describe entanglement transfer for open systems, where energy loss is modelled by zero-temperature reservoir interactions

$$H_R = a_I\Gamma^\dagger + a_I^\dagger\Gamma, \tag{36}$$

if the qubits are cavity modes, or

$$H_R = \sigma_I\Gamma^\dagger + \sigma_I^\dagger\Gamma, \tag{37}$$

if the qubits are two-level atoms. Here, $\Gamma = \sum_R g_R b_R$ and b_R is the boson destruction operator for one of many vacuum modes that comprise the reservoir. Inclusion in the Hamiltonian (35) of such coupling, under the Markovian assumption and assuming equal cavity and atomic damping rates γ, leads to a simple damping envelope for the concurrences $C_{IJ}(t) \to C_{IJ}(t)\exp(-\gamma t)$, as illustrated in figure 5. The global entanglement C_{AB} shows asymptotic decay $C_{AB}(t) = C_{AB}(0)\exp(-\kappa t)$, where κ is the global decay rate, to confirm the rule $\sum_{IJ} C_{IJ}^2(t) = C_{AB}^2$ in this case. Finally, we point out that the conservation rule will apply to other qubit-number conserving interactions, and can be studied in relation to other models of decoherence such as due to dephasing, important to, for example, electron spins in quantum dots [34, 35].

While the one-sided addition rule (31) holds for both types of Bell state entanglement, the different nonlocal pairwise concurrence relations allow us to deduce that the robustness of this addition rule with respect to coupling to the environment will be very different. If the system a_1 initially prepared in a $|\Phi\rangle$ state is coupled to an environment via interactions such as (36), the inequality indicates that there can be a complete loss of pairwise concurrence between a_1 and b_1 at finite times (ESD) [5].

6. Conclusion

How entanglement is lost, or transferred, when a system is coupled to an environment is a fundamental issue. In this paper we have quantified how such transfer takes place in a useful subset of scenarios, where the initial entanglement is in the form of a two-qubit Bell state, and the local interactions preserve total qubit number at each site. When the Bell state has only one excitation, there is a conservation of the sum of the tangle (the concurrence squared) associated with each resulting nonlocal two qubit bipartition. When the Bell state is a superposition of zero and two excitations, the sum of the tangle for the nonlocal bipartite partitions has an upper bound, but can be zero, to indicate a total absence of pairwise two-qubit entanglement. We have shown that a conservation rule for a nonlocal partition can be found in both cases. This is that the sum of the tangle between qubit A and each of the qubits at B is conserved throughout the evolution. In order to further analyse the entanglement distributed among the qubits, new analyses of measures for multipartite entanglement could be useful [22].

The rules derived in this paper apply where qubit number at each location is conserved, and hence are fundamental to a quantitative understanding of the distribution of the entanglement that will be inevitably shared between a system and its environment, because of decoherence mechanisms. Our results should be directly testable in the experimental arrangements such as those of Almeida *et al* [10].

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