Lindblad Equation

idea: $\hat{\rho}(0) \longrightarrow \hat{\rho}(t) = \underset{\substack{\downarrow \\ \text{Super operator}}}{S}[\hat{\rho}(0)]$

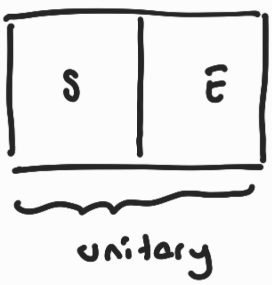
For unitary evolution we know that $i\dot{\hat{\rho}} = [\hat{H}, \hat{\rho}]$

What about more general evolution?

I. Markov Approximation

. There is no 1 to 1 correspondence btw t and $t' > t$

Actually:



There's a 1-1 correspondence btw t and t' for S+E.

When measurements are done on E → some information is lost.

E can also evolve in time → if E is traced out
No history of E!

In particular $\hat{\rho}_S(t)$ may depend on $\hat{\rho}_E(t')$, for all $t' < t$

to progress: hypothesis

- E is very large: reservoir in the thermodynamic sense.
  Coupling S to E does not significantly change E.

- Markov approximation:

Memory time τ for E (property of E)

after which any perturbation introduced by S relaxes

$\rightarrow$ E is stationary over time scales $\gg \tau$

"Coarse grained" description over $t \gg \tau \leftrightarrow$ E is stationary

<u>Examples:</u>

- E: vacuum of electro-magnetic field in free space.

- Large number of spins

II. Derivation of the Lindblad Equation

1) Notations

- $\hat{\rho}(t+\Delta t) = S[\hat{\rho}(t)] = \sum_{\nu} \hat{M}_{\nu} \rho(t) \hat{M}_{\nu}^{+}$

- Markov approximation: $\hat{\rho}(t+\Delta t) = \hat{\rho}(t) + \hat{O}(\Delta t)$

$\left.\begin{array}{l} \cdot\ \Delta t \text{ small} \\ \cdot\ \Delta t \gg \tau \end{array}\right\}$ Coarse grained evolution

Start from Krauss:

- $$\hat{M}_0 = \underbrace{\hat{\mathbb{I}}}_{0^{th}\text{ orde}} - i\underbrace{\hat{K}}_{1^{st}\text{ order}}\Delta t + \hat{O}(\Delta t^2)$$

- $\mu \geq 1$ $\quad \hat{M}_\mu \hat{\rho} \hat{M}_\mu^{\dagger}$ has to be order Δt

  So $\hat{M}_\mu = \hat{O}(\sqrt{\Delta t}) = \sqrt{\Delta t}\,\hat{L}_\mu$

- $$\hat{H} = \frac{\hat{K} + \hat{K}^{\dagger}}{2} \qquad \hat{J} = i\,\frac{\hat{K} - \hat{K}^{\dagger}}{2} \quad \text{st} \quad \hat{K} = \hat{H} - i\hat{J}$$

  $$\hat{H}^{\dagger} = \hat{H} \qquad\qquad \hat{J}^{\dagger} = -\hat{J}$$

2) Expression

$$\hat{M}_0 \hat{\rho} \hat{M}_0^{\dagger} = (\hat{\mathbb{I}} - i\hat{K}\Delta t)\hat{\rho}(\hat{\mathbb{I}} + i\hat{K}^{\dagger}\Delta t) + \dots$$

$$= \hat{\rho} - i\Delta t(\hat{K}\hat{\rho} - \hat{\rho}\hat{K}^{\dagger}) + \dots$$

$$= \hat{\rho} - i\Delta t[\hat{H}, \hat{\rho}] - \Delta t(\hat{J}\hat{\rho} + \hat{\rho}\hat{J}) + \dots$$

Remark: for unitary evolution: only $\hat{M}_0$ in the sum, and

$$\hat{M}_0^\dagger M_0 = \hat{\mathbb{I}}$$

$$(\hat{\mathbb{I}} + i\hat{K}^\dagger \Delta t)(\hat{\mathbb{I}} - i\hat{K}\Delta t) = \hat{\mathbb{I}} + i\Delta t(\cancelto{0}{K^\dagger - K}) + \dots$$

$$\Rightarrow \hat{J} = 0$$

$$\underbrace{\hat{M}_0 \hat{\rho} \hat{M}_0^\dagger}_{} = \hat{\rho} - i[\hat{H}, \hat{\rho}]$$ ⇝ Where's $\Delta t = ?$

$$\hat{\rho}(t+\Delta t) = \hat{\rho}(t) - i[\hat{H}, \hat{\rho}] \Rightarrow i\dot{\hat{\rho}} = [\hat{H}, \hat{\rho}]$$

$\hat{H}$ has to be interpreted as the Hamiltonian

But it can be $\neq$ from the Hamiltonian of S alone

↓

the difference is the Lamb-shift

For $\mu \geq 1$: $$\sum_{\mu=0} \hat{M}_\mu^\dagger \hat{M}_\mu = \hat{\mathbb{I}}$$

$$\hat{M}_0^\dagger \hat{M}_0 + \sum_{\mu \geq 1} \hat{M}_\mu^\dagger \hat{M}_\mu = \mathbb{I}$$

$$\hat{M}_0^\dagger \hat{M}_0 = \cancel{\hat{\mathbb{I}}} - 2\Delta t \hat{J} + \Delta t \sum_{\mu \geq 1} \hat{L}_\mu^\dagger \hat{L}_\mu = \cancel{\hat{\mathbb{I}}}$$

$$\hat{J} = \frac{1}{2} \sum_{\mu \geq 1} \hat{L}_\mu^\dagger \hat{L}_\mu$$

$$\frac{\partial \hat{\rho}}{\partial t} = \frac{1}{\Delta t} \cdot \left(\hat{\rho}(t+\Delta t) - \hat{\rho}(t) \right)$$

$$= \frac{1}{\Delta t} \left(\sum_{\mu} \hat{M}_\mu \rho(t) \hat{M}_\mu^\dagger - \hat{\rho}(t) \right)$$

"Lindblad Equation"

$$\boxed{\frac{\partial \hat{\rho}}{\partial t} = -i[\hat{H}, \hat{\rho}] + \sum_{\mu \geq 1} \hat{L}_\mu \hat{\rho} \hat{L}_\mu^\dagger - \frac{1}{2} \hat{L}_\mu^\dagger \hat{L}_\mu \hat{\rho} - \frac{1}{2} \hat{\rho} \hat{L}_\mu^\dagger \hat{L}_\mu}$$

unitary part

krauss operators (POVM)

$\hat{J} \rightarrow$ non-unitary part of $\hat{K}$

$\hookrightarrow$ normalization of krauss representation

· $\hat{L}_\mu$: Jump operators $\propto \sqrt{Hz}$

III. Interpretation

· As POVM: $\{ \hat{M}_\mu^\dagger \hat{M}_\mu , \mu \}$

$$\hat{E}_0 = \hat{M}_0^\dagger M_0 = \hat{I} - 2\Delta t \hat{J}$$

$$\hat{E}_\mu = \hat{M}_\mu^\dagger \hat{M}_\mu = \Delta t \hat{L}_\mu^\dagger \hat{L}_\mu$$

P_0 = probability to obtain $\overset{\text{zero}}{0}$: $\mathrm{Tr}(\hat{\rho}E_0) = 1 - 2\Delta t\,\mathrm{Tr}(\hat{\rho}\hat{J})$

$$\mathrm{Tr}(\hat{\rho}E_0) = 1 - \Gamma_i\,\Delta t$$

$$\langle \hat{J} \rangle = \mathrm{Tr}(\hat{\rho}\hat{J}) = \frac{\Gamma}{2}$$

Spontaneous Emission

P_μ = probability to obtain μ: $\mathrm{Tr}(\hat{E}_\mu \hat{\rho}) = \Delta t\,\mathrm{Tr}(\hat{\rho}\hat{L}_\mu^{+}\hat{L}_\mu)$
$= \Delta t\,\Gamma_\mu$

Normalization: $\hat{J} = \frac{1}{2}\sum_{\mu \geq 1} \hat{L}_\mu^{+}\hat{L}_\mu$

Unitary evolution over an extended Hilbert Space
$\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_R$

$|\psi_S\rangle \in \mathcal{H}_S$

$$\hat{U}\left(|\psi_S\rangle \otimes |0_R\rangle\right) = \left(\hat{M}_0 |\psi_S\rangle\right) \otimes |0_R\rangle + \sum_{\mu \geq 1} \hat{M}_\mu |\psi_S\rangle \otimes |\mu\rangle_R$$

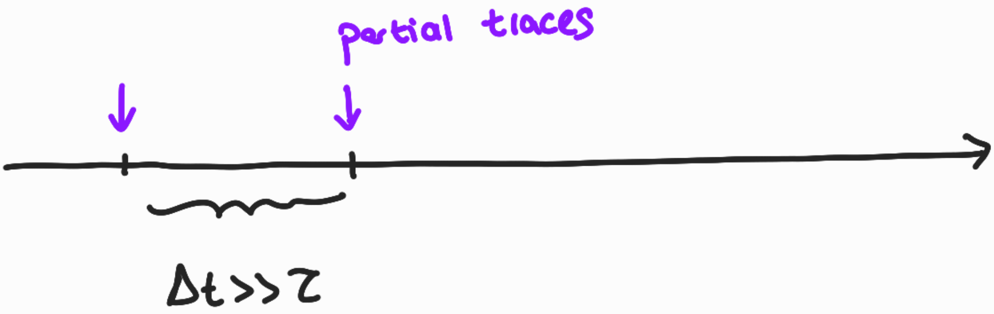
$$= \left[\left(\hat{\mathbb{I}} - i\hat{H}\Delta t - \hat{J}\Delta t\right)|\psi_S\rangle\right] \otimes |0_R\rangle + \sqrt{\Delta t}\sum_{\mu} \hat{L}_\mu |\psi_S\rangle \otimes |\mu\rangle$$

Projective measurement on the reservoir:

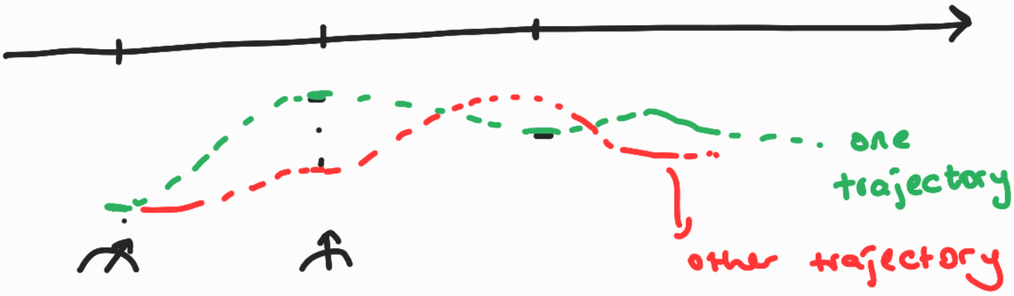
- with prob. $P_0 \sim O(1) \rightarrow$ reservoir stays in state $|0_R\rangle$
- with prob P_μ $|\mu_R\rangle$

Quantum Trajectories:

. Lindblad Equation describes at each point in time the "tracing out" of the environment



. One realization followed in time:



averaging over trajectories provide $\hat{\rho}(t)$

→ Monte-Carlo Wavefunction Method