Dirac Equation Part II

$$\left.\begin{aligned}\gamma^0 &= \beta \\ \gamma^1 &= \beta\alpha_x \\ \gamma^2 &= \beta\alpha_y \\ \gamma^3 &= \beta\alpha_z\end{aligned}\right\} \quad (i\gamma^\mu \partial_\mu - m)\Psi = 0$$

$$\not{a} = \gamma^\mu a_\mu = \gamma^0 a_0 - \vec{\gamma}\cdot\vec{a}$$

slash notation

$$\boxed{(i\not{\partial} - m)\Psi = 0}$$

$$\alpha_x = \alpha_x^\dagger \;,\quad \alpha_y = \alpha_y^\dagger \;,\quad \alpha_z = \alpha_z^\dagger \;,\quad \beta = \beta^\dagger$$

Properties of γ matrices

$$(\gamma^0)^2 = \beta^2 = 1 \;(I)$$

anti-commutation $\rightarrow$ $\beta\alpha_x + \alpha_x\beta = 0$

$$(\gamma^1)^2 = \beta\alpha_x\beta\alpha_x = -\alpha_x\underbrace{\beta\beta}_{I}\alpha_x = -\alpha_x^2 = -1$$

$$(\gamma^1)^2 = (\gamma^2)^2 = (\gamma^3)^3 = -1$$

$$\gamma^j\gamma^k + \gamma^k\gamma^j = 0 \quad (j \neq k)$$

$\downarrow$ Anti-Commutation Rule

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$$

$$g^{\mu\nu} \rightarrow \begin{array}{c|c} & \nu \\ \mu & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \end{array}$$

☆ Gamma Matrices are also Hermitian

$$\gamma^{0\dagger} = \gamma^0$$

$$\gamma^{1\dagger} = (\beta\alpha_x)^\dagger = \alpha_x^\dagger\beta^\dagger \overset{\text{Hermitian}}{=} \alpha_x\beta \overset{\text{anti-commuting}}{=} -\beta\alpha_x = -\gamma^1$$

$$\gamma^{2\dagger} = -\gamma^2$$

$$\gamma^{3\dagger} = -\gamma^3$$

$$\boxed{(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0} = -\gamma^\mu\underbrace{\gamma^0\gamma^0}_{I}$$

$$\gamma^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}$$

$$\gamma^0 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \qquad \gamma^1 = \left(\begin{array}{cc|cc} & & & 1 \\ & & 1 & \\ \hline & -1 & & \\ -1 & & & \end{array}\right) \qquad \gamma^2 = \left(\begin{array}{cc|cc} & & & -i \\ & & i & \\ \hline & i & & \\ -i & & & \end{array}\right)$$

$$\gamma^3 = \left(\begin{array}{cc|cc} & & 1 & 0 \\ & & 0 & -1 \\ \hline -1 & 0 & & \\ 0 & 1 & & \end{array}\right)$$

Four-vector current

$\rho = \psi^\dagger \psi$ $\quad\quad$ $\vec{j} = \psi^\dagger \vec{\alpha} \psi$

$$j^\mu = (\rho, \vec{j}) = \psi^\dagger \gamma^0 \gamma^\mu \psi$$

$$\boxed{\partial_\mu j^\mu = 0}$$ ⤳ continuity equation

Adjoint Spinor $\Rightarrow \psi^\dagger \gamma^0$

$$\bar{\psi} = \psi^\dagger \gamma^0 = (\psi^*)^T \gamma^0 = (\psi_1^*, \psi_2^*, \psi_3^*, \psi_4^*) \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

$$\bar{\psi} = (\psi_1^*, \psi_2^*, -\psi_3^*, -\psi_4^*)$$

$$j^\mu = \bar{\psi} \gamma^\mu \psi$$

⤳ $$(i \partial_\mu \bar{\psi} \gamma^\mu + m \psi) = 0$$

$$i \partial_\mu \bar{\psi} \gamma^\mu = -m \bar{\psi}$$

$$\bar{\psi} (i \cancel{\partial} + m) = 0$$

$$\boxed{i \bar{\psi} \cancel{\partial} = -m \bar{\psi}}$$

Dirac Equation Solution: free particle at rest

$$\psi_0^{(1)} = N_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} e^{-imt} \qquad \psi_0^{(3)} = N_3 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} e^{-imt}$$

$$\psi_0^{(2)} = N_2 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} e^{+imt} \qquad \psi_0^{(4)} = N_4 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} e^{+imt}$$

Solution for a moving particle

$$p^\mu = (E, \vec{p}) \quad \text{for} \quad (\gamma^\mu p_\mu - m)\psi = 0$$

$$\psi = u(E, \vec{p})\, e^{i(\vec{p}.\vec{r} - Et)}$$

$$\gamma^\mu p_\mu - m = \gamma^0 E - \vec{\gamma}.\vec{p} - m$$

$$= \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} E - \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} .\vec{p} - m \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

$$= \begin{pmatrix} (E-m)I & -\vec{\sigma}.\vec{p} \\ \vec{\sigma}.\vec{p} & -(E+m)I \end{pmatrix}_{4\times 4}$$

$$u(E, \vec{p}) \longrightarrow u = \begin{pmatrix} u_A \\ u_B \end{pmatrix}$$

$$= \begin{pmatrix} (E-m)I & -\vec{\sigma}.\vec{p} \\ \vec{\sigma}.\vec{p} & -(E+m)I \end{pmatrix}_{4\times 4} \begin{pmatrix} u_A \\ u_B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} (E-m)\,I\,u_A - \vec{\sigma}\cdot\vec{p}\,u_B = 0 \\ (\vec{\sigma}\cdot\vec{p})\,u_A - (E+m)\,u_B = 0 \end{cases}$$

$$\begin{cases} (\vec{\sigma}\cdot\vec{p})\,u_B = (E-m)\,u_A \\ (\vec{\sigma}\cdot\vec{p})\,u_A = (E+m)\,u_B \end{cases}$$

$$\vec{\sigma}\cdot\vec{p} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} p_x + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} p_y + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} p_z$$

$$\boxed{\vec{\sigma}\cdot\vec{p} = \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}}$$

$$u_B = \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\,u_A = \frac{1}{E+m}\begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix} u_A$$

Simplest choices for u_A

$$u_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow u_1 = N_1 \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x + ip_y}{E+m} \end{pmatrix}$$

normalization

Positive Energy Solutions

$$u_A = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow u_2 = N_2 \begin{pmatrix} 0 \\ 1 \\ \frac{p_z}{E+m} \\ \frac{p_x + ip_y}{E+m} \end{pmatrix}$$

$$u_A = \frac{\vec{\sigma}\cdot\vec{p}}{E-M}\, u_B$$

$$u_B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \quad u_3 = N_3 \begin{pmatrix} \frac{p_z}{E-M} \\ \frac{p_x+ip_y}{E-M} \\ 1 \\ 0 \end{pmatrix}$$

Negative energy solutions

$$u_B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow \quad u_4 = N_4 \begin{pmatrix} \frac{p_x-ip_y}{E-m} \\ \frac{-p_z}{E-m} \\ 0 \\ 1 \end{pmatrix}$$

Antiparticles : the first approach

→ Define antiparticle wave-function by flipping the sign of E and $\vec{p}$ and with E now being positive $E = \sqrt{|\vec{p}|^2 + M^2}$

$$u_1(E,\vec{p})\, e^{-i(\vec{p}\cdot\vec{r} - Et)} = u_4(-E,-\vec{p})\, e^{i(\vec{p}\cdot\vec{r} - Et)}$$

$$u_2(E,\vec{p})\, e^{-i(\vec{p}\cdot\vec{r} - Et)} = u_3(-E,-\vec{p})\, e^{i(\vec{p}\cdot\vec{r} - Et)}$$

2^{nd} Approach

Find negative energy plane wave solutions to the Dirac equation of the form $\psi = v(E,\vec{p})\, e^{-i(\vec{p}.\vec{r}-Et)}$, where $E = \sqrt{|\vec{p}|^2+m^2}$

Although $E>0$, there're still neg. energy solutions

$$\hat{H} v_1 = i\frac{\partial}{\partial t} v_1 = -E v_1$$

Putting ψ into Dirac equation $\rightarrow$ $\boxed{(i\gamma^\mu \partial_\mu - m)\psi = 0}$

$$\left(-\gamma^0 E + \gamma^1 p_x + \gamma^2 p_y + \gamma^3 p_z - m\right) v = 0$$

$$(\gamma^\mu p_\mu + m)\, v = 0$$

$v_1 = u_3$ & $v_2 = u_4$

Solutions for $\psi_i = u_i(E,\vec{p})\, e^{i(\vec{p}.\vec{r}-Et)}$

$$\underbrace{u_1 = N_1 \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix} \quad u_2 = N_2 \begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}}_{E = +\sqrt{|\vec{p}|^2+m^2}} \quad \underbrace{u_3 = N_3 \begin{pmatrix} \frac{p_z}{E-m} \\ \frac{p_x+ip_y}{E-m} \\ 1 \\ 0 \end{pmatrix} \quad u_4 = N_4 \begin{pmatrix} \frac{p_x-ip_y}{E-m} \\ \frac{-p_z}{E-m} \\ 0 \\ 1 \end{pmatrix}}_{E = -\sqrt{|p|^2+m^2}}$$

Solutions for $\Psi_i = v_i(E,\vec{p})\, e^{-i(\vec{p}\cdot\vec{r} - Et)}$

$$\boxed{v_1 = N_1' \begin{pmatrix} \frac{p_x - ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix} \qquad v_2 = N_2' \begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \\ 1 \\ 0 \end{pmatrix}} \qquad v_3 = N_3' \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E-m} \\ \frac{p_x+ip_y}{E-m} \end{pmatrix} \qquad v_4 = N_4' \begin{pmatrix} 0 \\ 1 \\ \frac{p_x - ip_y}{E-m} \\ \frac{-p_z}{E-m} \end{pmatrix}$$

$$E = +\sqrt{|\vec{p}|^2 + m^2} \qquad\qquad E = -\sqrt{|\vec{p}|^2 + m^2}$$

One can choose to work with $\{u_1, u_2, u_3, u_4\}$ or $\{v_1, v_2, v_3, v_4\}$

But it's natural to use positive energy solutions $\{u_1, u_2, v_1, v_2\}$

Normalization and orthogonality $\rightarrow$ Normalize wave functions to $2|E|$ particles per unit volume

$$\psi = u_1\, e^{i(\vec{p}\vec{r} - Et)} \qquad\qquad \rho = \psi^{+}\psi = u_1^{+} u_1$$

$$u_1^{+} u_1 = |N|^2 \left(1 + \frac{p_z^2}{(E+m)^2} + \frac{p_x^2 + p_y^2}{(E+m)^2} \right)$$

$$= |N|^2 \left(\frac{(E+m)^2 + |\vec{p}|^2}{(E+m)^2} \right) = |N|^2 \frac{(E+m)^2 + E^2 - m^2}{(E+m)^2}$$

$|\vec{p}|^2 \rightarrow E^2 - m^2$

$$= |N|^2 \frac{2E^2 + 2Em}{(E+m)^2} = |N|^2 \frac{2E\,\cancel{(E+m)}}{(E+m)^{\cancel{2}}} = |N|^2 \frac{2E}{E+m}$$

$$N = \sqrt{E+m}$$

$$u_j^\dagger u_k = 0 \qquad \text{for} \quad j \neq k$$

$$u_j^\dagger u_k = 2|E|\,\delta_{jk} \qquad j,k = 1,2,3,4$$

Spin of a Dirac Particle

$$\vec{L} = \vec{x} \times \vec{p} = -i\vec{x} \times \vec{\nabla} \quad \text{and} \quad \text{check if}$$

angular momentum operator

it commutes with $\mathcal{H}_{Dirac}$:

$$[\vec{L}, \mathcal{H}_{Dirac}] = [\vec{x} \times \vec{p}, (\vec{\alpha}\cdot\vec{p} + \beta m)] = [\vec{x} \times \vec{p}, \vec{\alpha}\cdot\vec{p}]$$

$$= (\vec{x} \times \vec{p})\,\vec{\alpha}\cdot\vec{p} - \vec{\alpha}\cdot\vec{p}\,(\vec{x} \times \vec{p})$$

$$= \vec{x}(\vec{\alpha}\cdot\vec{p}) \times \vec{p} - (\vec{\alpha}\cdot\vec{p})\,\vec{x} \times \vec{p} = [\vec{x}, \vec{\alpha}\cdot\vec{p}] \times \vec{p}$$

$$[\vec{x}, \vec{\alpha}\cdot\vec{p}] = \left[(x_1, x_2, x_3), \sum_j \alpha_j p_j\right] = i(\alpha_1, \alpha_2, \alpha_3)$$

$$[x_i, p_j] = i\,\delta_{ij}$$

$$[\vec{L}, \mathcal{H}_{Dirac}] = i\vec{\alpha} \times \vec{p} \neq 0$$

! Conclusion: The orbital angular momentum is not a conserved quantity of the quantum system.

Define $\vec{\Sigma}$ operator $(\Sigma_1, \Sigma_2, \Sigma_3) \equiv \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$

$$\Sigma_1 = \begin{pmatrix} \overset{\sigma_x}{\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}} & \\ & \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \end{pmatrix}; \quad \Sigma_2 = \begin{pmatrix} \begin{matrix} 0 & -i \\ i & 0 \end{matrix} & \\ & \begin{matrix} 0 & -i \\ i & 0 \end{matrix} \end{pmatrix}; \quad \Sigma_3 = \begin{pmatrix} \begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix} & \\ & \begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix} \end{pmatrix}$$

$$[\vec{\Sigma}, \mathcal{H}_{Dirac}] = [\vec{\Sigma}, (\vec{\alpha}\cdot\vec{p} + \beta m)] = \left[\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}, \begin{pmatrix} 0 & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & 0 \end{pmatrix} + \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} m \right]$$

$$\left[\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}, \begin{pmatrix} 0 & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & 0 \end{pmatrix} \right] = \begin{pmatrix} 0 & [\vec{\sigma}, \vec{\sigma}\cdot\vec{p}] \\ [\vec{\sigma}, \vec{\sigma}\cdot\vec{p}] & 0 \end{pmatrix} = -2i \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \times \vec{p}$$

$$[\vec{\sigma}, \vec{\sigma}\cdot\vec{p}] = [(\sigma_1, \sigma_2, \sigma_3), \sigma_j]\, p_j$$

$$= ([\sigma_1, \sigma_j] p_j, [\sigma_2, \sigma_j] p_j, [\sigma_3, \sigma_j] p_j)$$

$$= (2i\epsilon_{1jk} \sigma_k p_j, 2i\epsilon_{2jk} \sigma_k p_j, 2i\epsilon_{3jk} \sigma_k p_j)$$

$$= -2i (\epsilon_{1jk}, \epsilon_{2jk}, \epsilon_{3jk})\, \sigma_j p_k$$

$$= -2i (\vec{\sigma} \times \vec{p})$$

$$\left[\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}, \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \right] = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} - \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$$

$= 0$?

$$\downarrow$$

Thus, $\left[\vec{\Sigma}, \mathcal{H}_{Dirac}\right] = -2i \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \times \vec{p} = -2i\left(\vec{\alpha} \times \vec{p}\right)$

$$\left[\vec{L}, \mathcal{H}_{Dirac}\right] = i\, \vec{\alpha} \times \vec{p}$$

Total Angular Momentum Operator $\vec{J}$

Define $\vec{J} \equiv \vec{L} + \frac{1}{2}\vec{\Sigma} = \vec{L} + \vec{S}$

$$\left[\vec{J}, \mathcal{H}_{Dirac}\right] = \left[\vec{L} + \vec{S}, \mathcal{H}_{Dirac}\right] = 0$$

$$\Downarrow$$

J is conserved

★ Dirac equation describes a relativistic particle with spin-1/2

$$\vec{S} = (S_1, S_2, S_3) = \frac{1}{2}\vec{\Sigma} = \frac{1}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}_{4\times 4}$$

$$\left[S_i, S_j\right] = 2i\, \epsilon_{ijk}\, S_k$$

$s = \frac{1}{2}$ (?) $\frac{1}{2} \cdot \frac{3}{2} = \frac{3}{4}$

Spin Magnitude $\rightarrow$ $S^2 \psi = s(s+1)\psi$

$$\vec{S}^2 = \frac{1}{4}\left(\Sigma_1^2 + \Sigma_2^2 + \Sigma_3^2\right) = \frac{3}{4}\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

Spin of a Dirac Particle: Particle at rest

Spinors for the $\psi_0^{(i)}$ $(i = 1, ..., 4)$

$$\psi_0^{(1)} = N\begin{pmatrix}1\\0\\0\\0\end{pmatrix}e^{-imt} \quad ; \quad \psi_0^{(2)} = N\begin{pmatrix}0\\1\\0\\0\end{pmatrix}e^{-imt} \quad \text{with } E>0$$

$$\psi_0^{(3)} = N\begin{pmatrix}0\\0\\1\\0\end{pmatrix}e^{+imt} \quad ; \quad \psi_0^{(4)} = N\begin{pmatrix}0\\0\\0\\1\end{pmatrix}e^{+imt} \quad \text{with } E<0$$

✶ They're eigenstates of the diagonal operator S_3

$$S_3 = \frac{1}{2}\Sigma_3 = \frac{1}{2}\begin{pmatrix}\sigma_3 & 0\\ 0 & \sigma_3\end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$$

✶ Represent spin-up $|\uparrow\rangle$ & spin down $|\downarrow\rangle$ eigenstates

Spin of a Dirac Particle

Particle traveling in z-direction $\vec{p} = (0,0,\pm p)$

$$\psi^{(1,2)} = u_z^{(1,2)} e^{(-ipx)} \qquad E>0$$

$$\psi^{(3,4)} = u_z^{(3,4)} e^{(+ipx)} \qquad E<0$$

$$u_z^{(1)} = N \begin{pmatrix} 1 \\ 0 \\ \frac{\pm p}{E+m} \\ 0 \end{pmatrix} \qquad u_z^{(2)} = N \begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{\pm p}{E+m} \end{pmatrix} \qquad u_z^{(3)} = N \begin{pmatrix} \frac{\pm p}{E-m} \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$u_z^{(4)} = N \begin{pmatrix} 0 \\ \frac{\pm p}{E-m} \\ 0 \\ 1 \end{pmatrix}$$

for $\vec{p} = (0,0,\pm p)$

$$S_3 \psi^{(1)} = +\frac{1}{2} \psi^{(1)}$$

$$S_3 \psi^{(2)} = -\frac{1}{2} \psi^{(2)}$$

$$S_3 \psi^{(3)} = +\frac{1}{2} \psi^{(3)}$$

$$S_3 \psi^{(4)} = -\frac{1}{2} \psi^{(4)}$$

$\psi^{(1)}$ & $\psi^{(3)}$ represent spin-up

represent spin-down

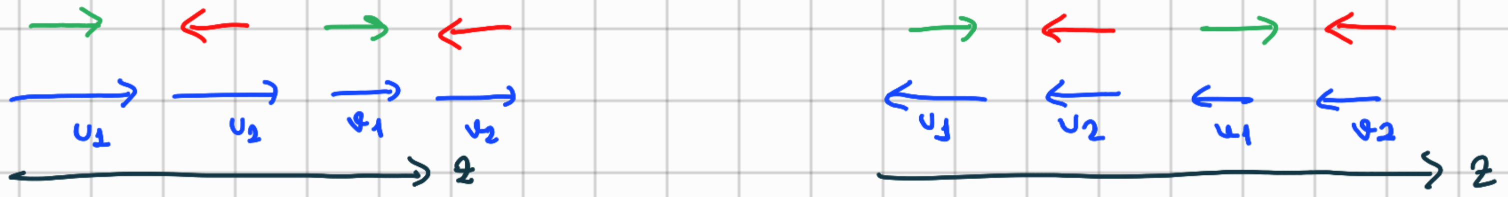
Spin States

Here $\underbrace{u_1, u_2, v_1, v_2}$ are eigenstates of S_3

$\downarrow$

In general they're not eigenstates of S_3.

This is only the case for particles & anti-particles traveling in the $\boxed{z\text{-direction}}$



$(0,0,p)$ $(0,0,-p)$

"Good Quantum Numbers" $\longrightarrow$ "A set of commuting observables"

z component of spin $\rightarrow$ S_3 is not a good quantum number.

Since $[\mathcal{H}, S_3] \neq 0$

$\downarrow$

THEREFORE INTRODUCE HELICITY

$$\vec{S} = \frac{1}{2}\Sigma = \frac{1}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$$
$$|\vec{S}| = \frac{1}{2}$$

Helicity Operator

Normalized projection of the spin along the flight direction of the particle

$$h = \frac{\vec{S}\cdot\vec{p}}{\underbrace{|\vec{S}|}_{1/2}|\vec{p}|} = \frac{\overbrace{2\vec{S}}^{\Sigma}\cdot\vec{p}}{|\vec{p}|} = \frac{\vec{\Sigma}\cdot\vec{p}}{|\vec{p}|}$$

* Helicity operator commutes with $\mathcal{H}_{Dirac}$ for a free particle.

$$[\vec{\Sigma}\cdot\vec{p}, \mathcal{H}_{Dirac}] = [\Sigma\cdot\vec{p}, \vec{\alpha}\cdot\vec{p} + \beta m] = [\Sigma\cdot\vec{p}, \vec{\alpha}\cdot\vec{p}] = 0$$

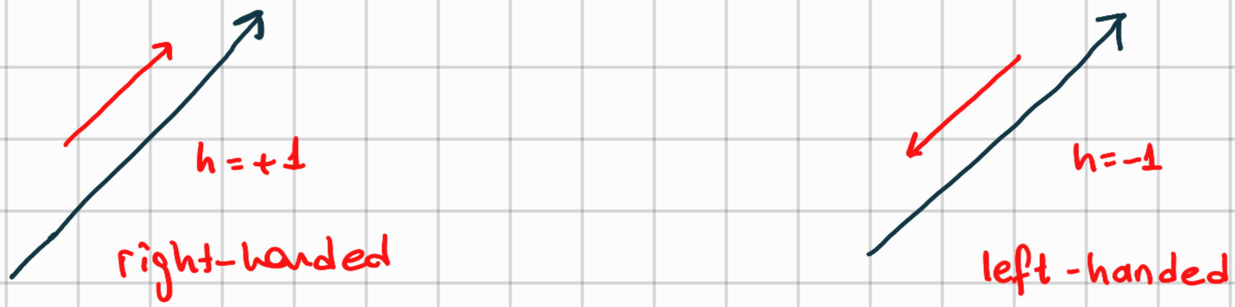
since $[\vec{\Sigma}, \vec{\alpha}] = 0$

Also $h^2 = I$

$$h^2 = \frac{1}{p^2}\begin{pmatrix} (\vec{\sigma}\cdot\vec{p})^2 & 0 \\ 0 & (\vec{\sigma}\cdot\vec{p})^2 \end{pmatrix} = \frac{1}{p^2}\begin{pmatrix} (\vec{p})^2 I & 0 \\ 0 & (\vec{p})^2 I \end{pmatrix} = I$$

$h = \pm 1$ for a spin 1/2 particle, the spin is quantized to be either "up" or "down"

* Helicity is a good quantum number with eigenvalues respectively $+1$ or -1

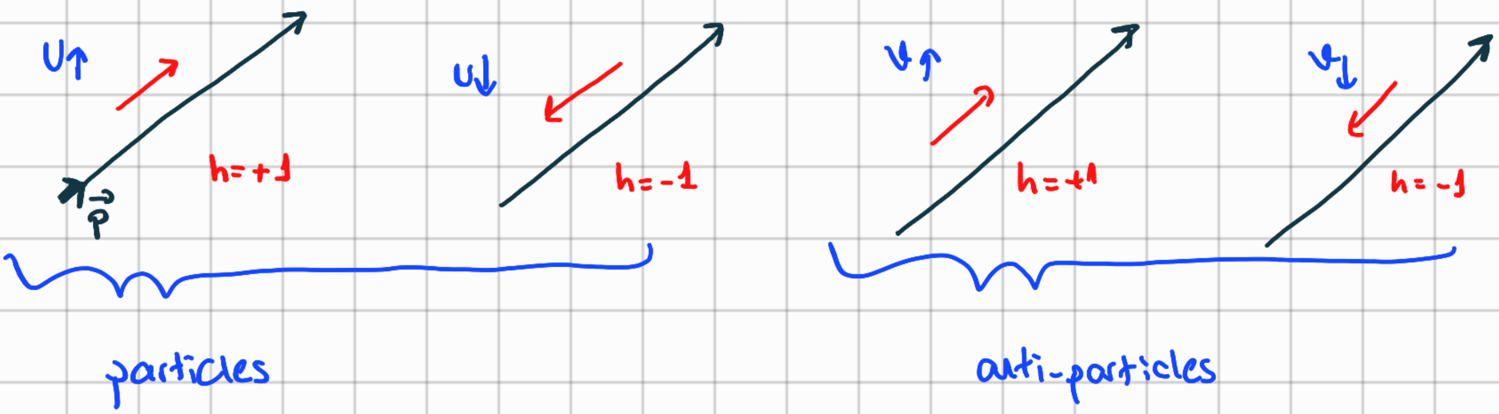


Helicity is conserved for free Dirac particle, Helicity is not a Lorentz Invariant Quantity

It is only invariant for massless particles)

like photon, neutrino.

Helicity Eigenstates



$$(\vec{\Sigma}\cdot\vec{p})\,u_\uparrow = +u_\uparrow$$

$$(\vec{\Sigma}\cdot\vec{p})\,u_\downarrow = -u_\downarrow$$

$$\begin{pmatrix} \vec{\sigma}\cdot\vec{p} & 0 \\ 0 & \vec{\sigma}\cdot\vec{p} \end{pmatrix}\begin{pmatrix} u_A \\ u_B \end{pmatrix} = \pm\begin{pmatrix} u_A \\ u_B \end{pmatrix}$$

$$(\vec{\sigma}\cdot\vec{p})\, u_A = \pm\, u_A$$

$$(\vec{\sigma}\cdot\vec{p})\, u_B = \pm\, u_B$$

$$\vec{p} = (\sin\theta\cos\phi \,,\, \sin\theta\sin\phi \,,\, \cos\theta)$$

$$\vec{\sigma}\,\vec{p} = \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta\cos\phi - i\sin\theta\sin\phi \\ \sin\theta\cos\phi + i\sin\theta\sin\phi & -\cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta \end{pmatrix}$$

Let's call $u_A = \begin{pmatrix} a \\ b \end{pmatrix}$ or $u_B = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \pm\begin{pmatrix} a \\ b \end{pmatrix}$$

$$\Downarrow$$

$$a\cos\theta + b\sin\theta e^{-i\phi} = \pm a$$

$$a(\mp 1 + \cos\theta) = -b\sin\theta\, e^{-i\phi}$$

$$\frac{b}{a} = \frac{\pm 1 - \cos\theta}{\sin\theta} e^{i\phi}$$

For the right-handed helicity +1 $\left(\vec{\Sigma}\cdot\vec{p}\right) v_\uparrow = -v_\uparrow$

$$\frac{2\sin^2\left(\frac{\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)} e^{i\phi} = e^{i\phi}\frac{\sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)} = \frac{b}{a}$$

$$U_{A\uparrow} \simeq \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ e^{i\phi}\sin\left(\frac{\theta}{2}\right) \end{pmatrix}$$

$$U_{B\uparrow} \simeq \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ e^{i\phi}\sin\left(\frac{\theta}{2}\right) \end{pmatrix}$$

$$U_\uparrow = \begin{pmatrix} U_A \\ U_B \end{pmatrix} = \begin{pmatrix} k_1\cos\left(\frac{\theta}{2}\right) \\ k_1 e^{i\phi}\sin\left(\frac{\theta}{2}\right) \\ k_2\cos\left(\frac{\theta}{2}\right) \\ k_2 e^{i\phi}\sin\left(\frac{\theta}{2}\right) \end{pmatrix}$$

Also from the Dirac Equation: to find Normalization

$$\boxed{(\vec{\sigma}\cdot\vec{p})\, u_A = (E+m)\, u_B}$$

$$u_B = \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\, u_A = \frac{|\vec{p}|}{E+m}\underbrace{\left(\frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}\right) u_A}_{\pm u_A} = \pm\frac{|\vec{p}|}{E+m}\, u_A$$

$$u_\uparrow = N\begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ e^{i\phi}\sin\left(\frac{\theta}{2}\right) \\ \frac{|p|}{E+m}\cos\left(\frac{\theta}{2}\right) \\ \frac{|\vec{p}|}{E+m}e^{i\phi}\sin\left(\frac{\theta}{2}\right) \end{pmatrix}$$

$$u_\uparrow = N\begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ e^{i\phi}\sin\left(\frac{\theta}{2}\right) \\ \frac{|\vec{p}|}{E+m}\cos\left(\frac{\theta}{2}\right) \\ \frac{|\vec{p}|}{E+m}e^{i\phi}\sin\left(\frac{\theta}{2}\right) \end{pmatrix} \qquad u_\downarrow = N\begin{pmatrix} -\sin\left(\frac{\theta}{2}\right) \\ e^{i\phi}\cos\left(\frac{\theta}{2}\right) \\ \frac{|\vec{p}|}{E+m}\sin\left(\frac{\theta}{2}\right) \\ \frac{-|\vec{p}|}{E+m}e^{i\phi}\cos\left(\frac{\theta}{2}\right) \end{pmatrix}$$

$$v_\uparrow = N\begin{pmatrix} \frac{|\vec{p}|}{E+m}\sin\left(\frac{\theta}{2}\right) \\ \frac{-|\vec{p}|}{E+m}e^{i\phi}\cos\left(\frac{\theta}{2}\right) \\ -\sin\left(\frac{\theta}{2}\right) \\ e^{i\phi}\cos\left(\frac{\theta}{2}\right) \end{pmatrix} \qquad v_\downarrow = N\begin{pmatrix} \frac{|\vec{p}|}{E+m}\cos\left(\frac{\theta}{2}\right) \\ \frac{|\vec{p}|}{E+m}e^{i\phi}\sin\left(\frac{\theta}{2}\right) \\ \cos\left(\frac{\theta}{2}\right) \\ e^{i\phi}\sin\left(\frac{\theta}{2}\right) \end{pmatrix}$$

where $N = \sqrt{E+m}$

Parity & Time Reversal

Parity P : $x^0 \longrightarrow x^0$; $x^i \longrightarrow -x^i$

Time Reversal T: $x^0 \rightarrow -x^0$; $x^i \longrightarrow x^i$ $(i=1,2,3)$

$S(P)$ and $S(T)$ of them on a Dirac spinor :

$$\Psi(t,\vec{x}) \rightarrow S(P)\,\Psi(t,-\vec{x})$$

$$\Psi(t,\vec{x}) \rightarrow S(T)\,\Psi(-t,\vec{x})$$

Parity Reversal

$P^{-1} = P$ & $P^2 = I$ should satitsy

Operator P reverses the momentum $\vec{p}$ but retains its spin.

consider a particle moving in z- direction with $\vec{p}=(0,0,p)$

$$S(P)\, u_z^{(1)}(E,-p) = u_z^{(1)}(E,p)$$

$$N\begin{pmatrix} 1 & & & \\ & ? & & \\ & & -1 & \\ & & & ? \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ \frac{-p}{E+m} \\ 0 \end{pmatrix} = N\begin{pmatrix} 1 \\ 0 \\ +\frac{p}{E+m} \\ 0 \end{pmatrix}$$

$$S(P)\, u_2^{(2)}(E,-p) = u_2^{(2)}(E,p)$$

$$N\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{-p}{E+m} \end{pmatrix} = N\begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{+p}{E+m} \end{pmatrix}$$

γ^0

$$\boxed{u_2^{(i)}(E,\vec{p}) \longleftrightarrow u_2^{(i)}(E,-\vec{p})}$$

$$P: \quad \psi \to S(P)\,\psi = \eta_p \gamma^0 \psi$$

η_p → overall unobservable phase

For a particle/antiparticle at rest the solutions to the Dirac equations:

$\psi = u_1 e^{-imt}$ $\qquad u_1 = N\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $\qquad u_2 = N\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

$\psi = u_2 e^{-imt}$

$\psi = v_1 e^{+imt}$ $\qquad v_1 = N\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ $\qquad v_2 = N\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

$\psi = v_2 e^{+imt}$

$$S(P)\,u_1 = \pm \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \pm u_1$$

$$S(P)\,u_1 = \pm u_1 \qquad S(P)\,v_1 = \mp v_1$$

$$S(P)\,u_2 = \pm u_2 \qquad S(P)\,v_2 = \mp v_2$$

* Hence an anti-particle at rest has opposite intrinsic parity to a particle at rest.

Convention $\rightarrow$ $S(P) = +\gamma^0$

$P(s) = s$ $\rightarrow$ scalar $\rightarrow$ spin-0 particle with pos. parity 0^+ f_0 mesons

$P(p) = -p$ $\rightarrow$ pseudoscalar $\rightarrow$ spin 0 " " neg. " 0^- ; pions $\pi^{\pm}, \pi^0$

$P(\vec{v}) = -\vec{v}$ $\rightarrow$ vector $\rightarrow$ spin-1 particle " " " 1^-, ρ, ω, γ, gluon

$P(\vec{a}) = \vec{a}$ $\rightarrow$ pseudovector $\rightarrow$ Spin 1 " " pos " 1^+, f_1 mesons

Charge Conjugation

keeping the spin unchanged, operation that replaces each particle

by its antiparticle $\Rightarrow$ charge conjugation

$$\psi \to \psi_c = C\psi^* = C\psi^{+T}$$

rac Eqn for a particle of charge e in an external EM field

$$\left(i\gamma^\mu(\partial_\mu - eA_\mu) - m\right)\psi = 0 \qquad (\text{Eq } 1)$$

$$\left(i\gamma^\mu(\partial_\mu + eA_\mu) - m\right)\psi_c = 0 \qquad (\text{Eq } 2)$$

$\downarrow$

charge $-e$

Taking complex conjugate and multiplying with C

$$C\left(-i(\gamma^\mu)^*(\partial_\mu - eA_\mu) - m\right)\overbrace{C^{-1}C}^{I}\underbrace{\psi^*}_{} =$$

$$-i\underbrace{C(\gamma^\mu)^* C^{-1}}_{-\gamma^\mu}(\partial_\mu - eA_\mu) - m)\,\psi_c = 0$$

$$C = -i\gamma^2$$

$$\psi \to \psi_c = -i\eta_c\gamma^2\psi^*$$