WEEK 12 Part 1 Von Neumann Entropy

Mixed state as density matrix ρ → $D \times D$ matrix

Recall $\rho = \rho^\dagger$, ρ is positive semi-definite matrix

All eigenvalues are in $\mathbb{R}_+$

$\mathrm{Tr}\,\rho = 1$

$$S(\rho) = -\mathrm{Tr}\,(\rho \log \rho) = -\sum_{\alpha=1}^{D} \lambda_\alpha \log \lambda_\alpha$$

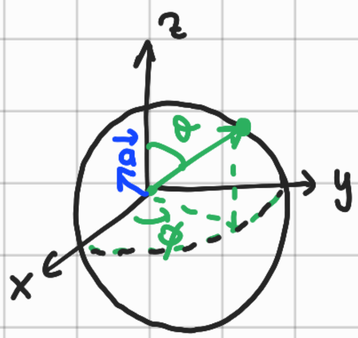
where λ_α are eigenvalues of ρ, $\underbrace{\alpha = 1 \dots D}_{\dim \mathcal{H}}$

Note: $0 \leq \lambda_\alpha \leq 1$, $\sum_{\alpha=1}^{D} \lambda_\alpha = 1$ so $S(\rho)$ = Shannon Entropy of $\{\lambda_\alpha, \alpha = 1 \dots D\}$

Entropy of one quantum bit (or one two level system)

$\mathcal{H} \in \mathbb{C}^2$ pure states $|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)|1\rangle$

$\rho_{|\psi\rangle} = |\psi\rangle\langle\psi|$



For a general mixed state:

$$\rho = \frac{1}{2}\left(I + \vec{a}\cdot\vec{\sigma}\right)$$

$\vec{a} = (a_x, a_y, a_z)$ with $\|\vec{a}\| \leq 1$

$\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z) = (X, Y, Z)$

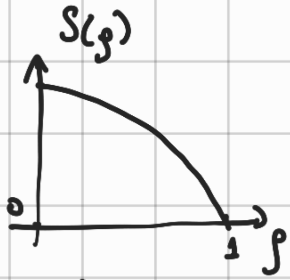
$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\vec{a} = 0$ $\qquad \rho = \frac{1}{2} I = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$

maximally mixed state

$\|\vec{a}\| = 1$ $\qquad \rho$ is rank-one matrix $\Rightarrow \rho^2 = \rho$

Eigenvalues: $\qquad \frac{1}{2}\left(1 \pm \|\vec{a}\|\right) = \lambda_\pm$



$$S(\rho) = -\left(\frac{1+\|\vec{a}\|}{2}\right)\log\left(\frac{1+\|\vec{a}\|}{2}\right) - \left(\frac{1-\|\vec{a}\|}{2}\right)\log\left(\frac{1-\|\vec{a}\|}{2}\right)$$

Remark: $\qquad p = \dfrac{1+\|\vec{a}\|}{2}$

$S(p) = -p\log(p) - (1-p)\log(1-p)$ $\rightarrow$ Classical Formula (Shannon Entropy)

Examples of Entangled Systems and "entanglement entropy"

1. $|B_{00}\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B\right)$

$$\rho_{B_{00}} = |B_{00}\rangle\langle B_{00}| = \frac{1}{2}\left(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|\right)$$

$$\rho_A = \mathrm{Tr}_B\, \rho = \frac{1}{2}|0\rangle_A\langle 0|_A \underbrace{\mathrm{Tr}_B |0\rangle_B\langle 0|_B}_{1} + \frac{1}{2}|0\rangle_A\langle 1|_A \underbrace{\cancel{\mathrm{Tr}_B |0\rangle_B\langle 1|_B}}_{0}$$

$$+ \frac{1}{2}|1\rangle_A\langle 0|_A \underbrace{\cancel{\mathrm{Tr}_B |1\rangle_B\langle 0|_B}}_{0} + \frac{1}{2}|1\rangle_A\langle 1|_A \underbrace{\mathrm{Tr}_B |1\rangle_B\langle 1|_B}_{1}$$

$\vec{a}=0$ $\quad\rho_A = \frac{1}{2}\left(|0\rangle\langle 0|_A + |1\rangle_A\langle 1|_A\right) = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}_A$

$\vec{a}=0$ $\quad\rho_B = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_B$

total entropy

$S(AB) = 0$ Since $|B_{00}\rangle$ is pure

$$S(\rho_A) = S(\rho_B) = -\frac{1}{2}\log\frac{1}{2} - \frac{1}{2}\log\frac{1}{2} = \left(+\frac{1}{2}\log 2\right)\cdot 2$$
$$= \log 2$$

! Main difference in classical and quantum entropy

$$S(AB) - S(A) < 0$$

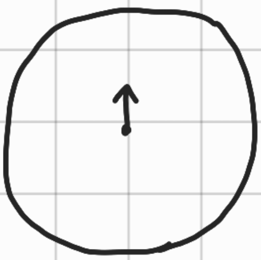
can be

Example with W state $(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C)$

$$|W\rangle = \frac{1}{\sqrt{3}}\left(|001\rangle + |010\rangle + |100\rangle\right)$$

$\rho_W = |W\rangle\langle W|$ $\qquad S(ABC) = 0$

$$\rho_A = \mathrm{Tr}_{\mathcal{H}_B\otimes\mathcal{H}_C}\left(|W\rangle\langle W|\right) = \frac{1}{3}|0\rangle\langle 0|_A + \frac{1}{3}|0\rangle\langle 0|_A + \frac{1}{3}|1\rangle\langle 1|_A$$



$$\rho_A = \frac{2}{3}|0\rangle\langle 0|_A + \frac{1}{3}|1\rangle\langle 1|_A = \begin{pmatrix} 2/3 & 0 \\ 0 & 1/3 \end{pmatrix}_A$$

$$S(A) = \frac{2}{3}\log\frac{3}{2} + \frac{1}{3}\log 3 \quad < \log 2$$

$$\rho_{BC} = \mathrm{Tr}_A\, \rho_W = \frac{1}{3}|01\rangle_{BC}\langle 01|_{BC} + \frac{1}{3}|10\rangle\langle 10|_{BC} + \boxed{\frac{1}{3}|00\rangle\langle 00|_{BC}}$$

$$+\frac{1}{3}|01\rangle\langle 10|_{BC} + \frac{1}{3}|10\rangle\langle 01|_{BC}$$

$$\rho_{BC} = \frac{1}{3}|00\rangle\langle 00| + \frac{2}{3}\,|B_{01}\rangle\langle B_{01}|$$

$$(|01\rangle + |10\rangle)(\langle 01| + \langle 10|)$$

$$\lambda_{\pm}^{BC} = \frac{1}{3}, \frac{2}{3}, 0, 0$$

$$S(BC) = S(\rho_A)$$