QUANTUM TRAJECTORIES

I. Stochastic Schrödinger Equation

1) Introduction

Duration Δt $\quad \{\hat{M}_\mu\} \rightarrow$ measurement operators

Update rule:

$$\boxed{|\psi_c(t+\Delta t)\rangle = \frac{\hat{M}_\mu |\psi_c(\mu)\rangle}{\sqrt{p_\mu}}}$$

, conditioned on outcome μ

- $\hat{M}_\mu = \sqrt{\Delta t}\,\hat{L}_\mu \qquad \mu \geq 1$
- $\hat{M}_0 = \hat{\mathbb{I}} - i\hat{H}\Delta t - \sum_{\mu\geq 1} \frac{\hat{L}_\mu^+ L_\mu}{2}\Delta t = \mathbb{I} - i\hat{H}_{eff}\Delta t$

$$p_\mu = \langle \psi_{c(t)} | \hat{M}_\mu^+ \hat{M}_\mu | \psi_{c(t)} \rangle = \begin{cases} \langle \psi_{c(t)} | \hat{L}_\mu^+ \hat{L}_\mu | \psi_{c(t)} \rangle . \Delta t & \text{for } \mu \geq 1 \\ 1 - \sum_{\mu\geq 1} \langle \psi_{c(t)} | \hat{L}_\mu^+ \hat{L}_\mu | \psi_{c(t)} \rangle & \text{for } \mu = 0 \end{cases}$$

$\downarrow$
probability

2) Unraveling

From the point of view of the "environment": $N_\mu(t)$

# of clicks of detector μ.
in the interval $[0,t]$

$N_\mu(t)$ is a classical random variable

Let $\Delta t \to dt$:

$$dN_\mu(t) = \begin{cases} 1 & \text{if the detector } \mu \text{ clicked} \\ & \text{btw } t \text{ and } t+\Delta t \\ 0 & \text{otherwise} \end{cases}$$

$$dN_0(t) = 1 - \sum_{\mu \geq 1} dN_\mu(t)$$

dN_μ is a classical random process $\Rightarrow \begin{cases} dN_\mu = 1 & \text{w prob. } p_\mu = \langle L_\mu^\dagger L_\mu \rangle dt \\ dN_0 = 1 & \text{w prob. } 1 - p_\mu \end{cases}$

ensemble average

$$\overline{dN_\mu} = p_\mu$$

"point process" : $dN_\mu dN_\nu = \delta_{\mu\nu} dN_\mu$, $dN_\mu dt \ll dt$

3) Derivation

$$|\Psi_c(t+dt)\rangle = \sum_\mu dN_\mu(t) \frac{\hat{M}_\mu |\Psi_c(t)\rangle}{\sqrt{p_\mu}}$$

Separate $\mu=0$ and $\mu \geq 1$

$$|\Psi_c(t+dt)\rangle = \left(1 - \sum_{\mu\geq 1} dN_\mu(t)\right) \frac{\hat{M}_0 |\Psi_c\rangle}{\sqrt{1 - \underbrace{\sum_{\mu \geq 1} p_\mu}_{O(dt)}}} + \underbrace{\sum_{\mu\geq 1} dN_\mu(t) \frac{\hat{M}_\mu |\Psi_c(t)\rangle}{\sqrt{p_\mu}}}_{\sum_{\mu\geq 1} dN_\mu(t) \frac{\hat{L}_\mu |\Psi_c(t)\rangle}{\sqrt{\langle \hat{L}_\mu^\dagger \hat{L}_\mu\rangle}}}$$

$$|\Psi_c(t+dt)\rangle = \left(1 - \sum_{\mu\geq 1} dN_\mu(t)\right)\left(1 + \frac{dt}{2}\sum_\mu \langle \hat{L}_\mu^\dagger \hat{L}_\mu\rangle\right)\left(\hat{\mathbb{I}} - i\hat{H}dt - \frac{dt}{2}\sum_\mu \hat{L}_\mu^\dagger \hat{L}_\mu\right)|\Psi_c(t)\rangle$$

$$+ \sum_{\mu\geq 1} dN_\mu(t) \frac{\hat{L}_\mu |\Psi_c(t)\rangle}{\sqrt{\langle \hat{L}_\mu^\dagger \hat{L}_\mu\rangle}}$$

$$|\Psi_c(t+dt)\rangle - |\Psi_c(t)\rangle = \Big[-i\hat{H}dt\,|\Psi_c\rangle - \frac{dt}{2}\sum_\mu \left(\hat{L}_\mu^\dagger \hat{L}_\mu - \langle \hat{L}_\mu^\dagger \hat{L}_\mu\rangle\right)|\Psi_c\rangle$$

$$+ \sum_\mu dN_\mu(t)\left(\frac{\hat{L}_\mu |\Psi_c(t)\rangle}{\sqrt{\langle \hat{L}_\mu^\dagger L_\mu\rangle}} - \mathbb{I}\right)|\Psi_c\rangle\Big]$$

$\wr$

$$\Downarrow$$

$$d|\psi_c\rangle = \Big(\underbrace{-i\hat{H} - \frac{1}{2}\sum_\mu \hat{L}_\mu^\dagger \hat{L}_\mu}_{H_{eff}} - \langle\psi_c|\hat{L}_\mu^\dagger\hat{L}_\mu|\psi_c\rangle\Big)dt\,|\psi_c\rangle + \sum_\mu dN_\mu \left(\frac{\hat{L}_\mu}{\sqrt{\langle\hat{L}_\mu^\dagger\hat{L}_\mu\rangle}} - 1\right)|\psi_c\rangle$$

"Stochastic Schrödinger Equation"

Remarks: * Solution to SSE $\{N_\mu(t), |\psi_c(t)\rangle\}$ is called quantum trajectory.

* Nonlinear

II. Interpretation

1) Lindblad Equation

Consider $\overline{|\psi_c(t)\rangle\langle\psi_c(t)|}$ average over the random process dN_μ

$|\psi_c(t)\rangle + d|\psi_c(t)\rangle$

$$\overline{|\psi_c(t+dt)\rangle\langle\psi_c(t+dt)|} = \overline{|\psi_c(t)\rangle\langle\psi_c(t)|} + \overline{d|\psi_c(t)\rangle\langle\psi_c(t)|} + \overline{|\psi_c(t)\rangle d\langle\psi_c(t)|} + \overline{d|\psi_c\rangle d\langle\psi_c|}$$

SSE ① ② ③

it's not 2nd order (due to point process $dN_\mu dN_\nu = \delta_{\mu\nu} dN_\mu$)

$$|\psi_c(t+dt)\rangle\langle\psi_c(t+dt)| - |\psi_c(t)\rangle\langle\psi_c(t)| = d\hat{\rho}$$

$$d\hat{\rho} = \left[\left(-i\hat{H}_{eff} + \overbrace{\frac{1}{2}\sum_N \langle \hat{L}_N^{\dagger}\hat{L}_N\rangle}^{\square}\right)dt\,|\psi_c\rangle\langle\psi_c|\right]$$

$$+ \sum_N \overline{dN_N}\left(\frac{\overbrace{\hat{L}_N|\psi_c\rangle\langle\psi_c|}^{(*)}}{\sqrt{\langle L_N^{\dagger}L_N\rangle}} - \underbrace{|\psi_c\rangle\langle\psi_c|}_{\square}\right) \quad \Bigg\} \; ①$$

$$+ |\psi_c(t)\rangle\langle\psi_c(t)|\left(+i\hat{H}_{eff}^{\dagger} + \overbrace{\frac{1}{2}\sum_N \langle \hat{L}_N^{\dagger}\hat{L}_N\rangle}^{\square}\right)dt$$

$$+ \sum_N \overline{dN_N}\left(\frac{\overbrace{|\psi_c\rangle\langle\psi_c|\hat{L}_N^{\dagger}}^{(**)}}{\sqrt{\langle \hat{L}_N^{\dagger}\hat{L}_N\rangle}} - \underbrace{|\psi_c\rangle\langle\psi_c|}_{\square}\right) \quad \Bigg\} \; ②$$

$dt\,dt \} \to 0$
$dt\,dN_N$

$$+ \sum_{NN'} \overline{dN_N\,dN_{N'}}\left(\frac{\hat{L}_N}{\sqrt{\langle \hat{L}_N^{\dagger}\hat{L}_N\rangle}} - \hat{\mathbb{I}}\right)|\psi_c\rangle\langle\psi_c|\left(\frac{\hat{L}_N^{\dagger}}{\sqrt{\langle \hat{L}_N^{\dagger}\hat{L}_N\rangle}} - \hat{\mathbb{I}}\right) \quad \Big\} \; ③$$

↳ last term can be written

$$\left.\begin{aligned} &\to \overline{dN_N\,dN_{N'}} = \delta_{NN'}\,\overline{dN_N} \\ &\to \overline{dN_N} = \langle \hat{L}_N^{\dagger}\hat{L}_N\rangle\,dt \end{aligned}\right\} \sum_N \langle \hat{L}_N^{\dagger}L_N\rangle dt\left(\frac{-\overbrace{\hat{L}_N|\psi\rangle\langle\psi|}^{(*)} - \overbrace{|\psi\rangle\langle\psi|\hat{L}_N^{\dagger}}^{(**)}}{\sqrt{\langle L_N^{\dagger}L_N\rangle}} + \frac{\hat{L}_N|\psi\rangle\langle\psi|\hat{L}_N^{\dagger}}{\sqrt{\langle L_N^{\dagger}L_N\rangle}} + \underbrace{|\psi\rangle\langle\psi|}_{\square}\right)$$

$$\boxed{d\hat{\rho} = -i\left[\hat{H}_{eff}\,\hat{\rho} - \hat{\rho}\,\hat{H}_{eff}^{+}\right] + \sum_{\mu} \hat{L}_{\mu}\,\hat{\rho}\,\hat{L}_{\mu}^{+}}$$

"Lindblad Equation"

2) Monte-Carlo wave function Algorithm

$\dim \mathcal{H} = N$

$\hat{\rho} \sim O(N^2)$ entries

propagating $\hat{\rho} \sim O(N^4)$

- Pure state $|\Psi\rangle \sim O(N)$ entries

  Propagation of $|\Psi\rangle \sim O(N^2)$

M different trajectories $\longrightarrow$ recovering $\hat{\rho} = \overline{|\Psi_c\rangle\langle\Psi_c|}$

$(M \ll N^2)$ $\sim O(MN^2)$

⊕ parallel task

- Naive algorithm: . Define δt time step.

  . Each δt: draw random number $R \in [0,1]$

  . Jump prob.: $p_\mu = \langle\Psi_c|\hat{L}_\mu^{+}\hat{L}_\mu|\Psi_c\rangle\,\delta t$

  . Total jump probability : $p_N = \sum_\mu p_\mu$

If $R < p$: jump happens, determine randomly which one $\{\mu\}_{\mu \geq 1}$

Apply corresponding $\hat{L}_\mu |\Psi_c\rangle$ + normalize to obtain $|\Psi_c(t+\delta t)\rangle$

Record the jump event on "channel" μ.

If $R > p$:

$$\delta|\Psi_c\rangle = \left[-i\hat{H} - \frac{1}{2}\sum_\mu \left(\hat{L}_\mu^\dagger \hat{L}_\mu - \overset{\text{related to normalization}}{\langle \hat{L}_\mu^\dagger \hat{L}_\mu \rangle}\right)\right]\delta t |\Psi_c\rangle$$

Determine $\delta|\Psi_c\rangle$ as above.

$$|\Psi_c(t+\delta t)\rangle = |\Psi_c(t)\rangle + \delta|\Psi_c\rangle$$

* Clever Algorithm

Use non-normalized wavefunction for the no-jump evolution:

$$\frac{d|\tilde{\Psi}\rangle}{dt} = -\left(i\hat{H} + \frac{1}{2}\underbrace{\sum_\mu \hat{L}_\mu^\dagger \hat{L}_\mu}_{\text{decrease of } \langle\tilde{\Psi}|\tilde{\Psi}\rangle}\right)|\tilde{\Psi}\rangle$$

Draw random R, evolve until $\langle\tilde{\Psi}|\tilde{\Psi}\rangle = R$

$\downarrow$

then jump (like previously)

3) Different types of unravelings

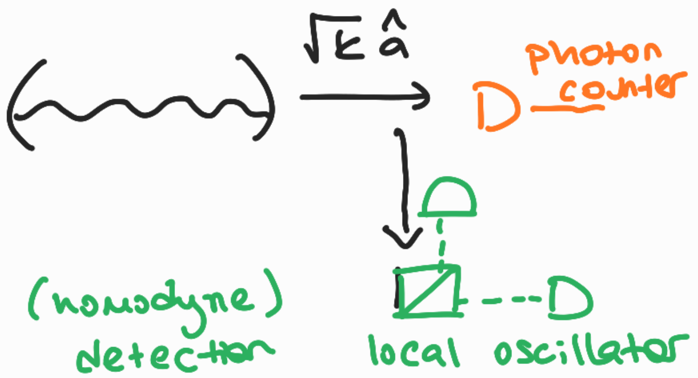
Remark: Unitary transformations on krauss operators:

$$\hat{K}_\vartheta = \sum_{\mu,\vartheta} U_{\mu\vartheta} \hat{M}_\mu$$

$$\sum_\vartheta \hat{K}_\vartheta \hat{\rho} \hat{K}_\vartheta^+ = \sum_\mu \hat{M}_\mu \hat{\rho} \hat{M}_\mu^+ \; ; \; \sum_\vartheta \hat{K}_\vartheta^+ \hat{K}_\vartheta = \mathbb{I} = \sum_\mu \hat{M}_\mu^+ \hat{M}_\mu$$

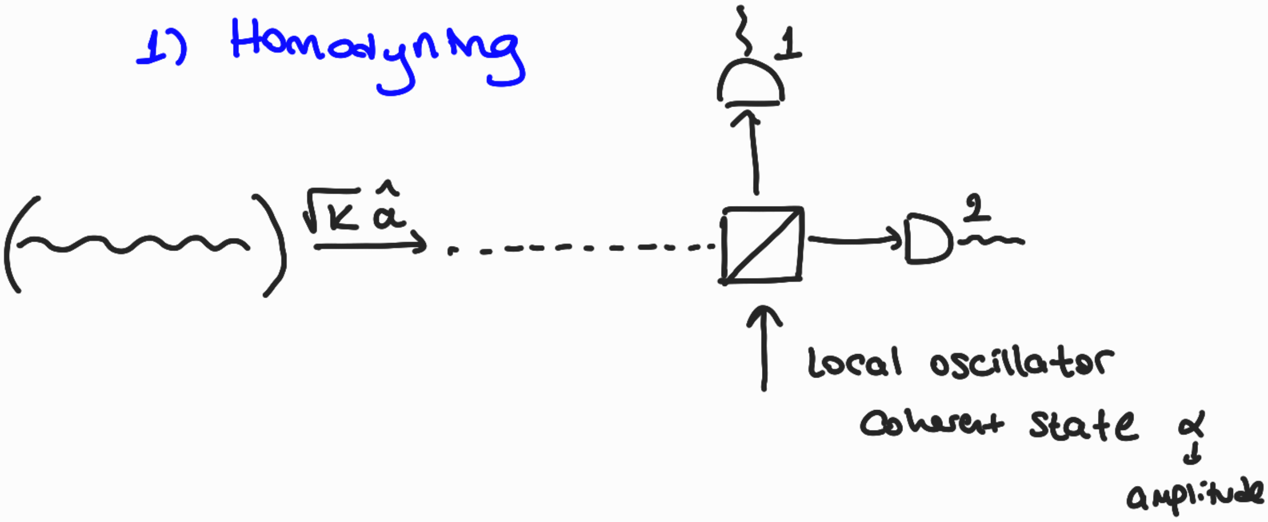
Lindblad Equation: Similar

Example: Photon detection



III. Weak Continous Measurement

1) Homodyning



$$\hat{L} = \sqrt{\kappa}\,\hat{a} \xrightarrow{\text{(mixing with coherent state)}} \begin{cases} \sqrt{\kappa}\,\dfrac{\hat{a} - \alpha \hat{\mathbb{I}}}{\sqrt{2}} = \hat{c}_1 \\[2ex] \sqrt{\kappa}\, i\, \dfrac{\hat{a} + \alpha \hat{\mathbb{I}}}{\sqrt{2}} = \hat{c}_2 \end{cases}$$

Transformations $\quad \hat{M}_0^\dagger \hat{M}_0 + \hat{M}_1^\dagger \hat{M}_1 = \hat{\mathbb{I}}$

$$(\hat{\mathbb{I}} - i\hat{H}_{eff}\delta t)^\dagger (\hat{\mathbb{I}} - i H_{eff}\,\delta t) + \kappa \hat{a}^\dagger \hat{a}\, \delta t = \hat{\mathbb{I}}$$

δt terms $\rightarrow$ $i(\hat{H}_{eff}^\dagger - \hat{H}_{eff}) + \kappa \hat{a}^\dagger \hat{a} = 0$

$$\hat{H}_{eff} = \hat{H} - i\,\frac{\kappa \hat{a}^\dagger \hat{a}}{2}$$

Now: $\hat{M}_0^\dagger \hat{M}_0 + \hat{M}_1^\dagger \hat{M}_1 + \hat{M}_2^\dagger \hat{M}_2 = \mathbb{I}$

$$(\hat{\mathbb{I}} - i\hat{H}_{eff}\delta t)^\dagger(\hat{\mathbb{I}} - i\hat{H}_{eff}\delta t) + \frac{\kappa}{2}\delta t(\hat{a}^\dagger\hat{a} + |\alpha|^2 - \cancel{\hat{a}\alpha^*} - \cancel{\hat{a}^\dagger\alpha})$$

$$+ \frac{\kappa}{2}\delta t(\hat{a}^\dagger\hat{a} + |\alpha|^2 + \cancel{\hat{a}\alpha^*} + \cancel{\hat{a}^\dagger\alpha})$$

$$H_{eff}^\dagger = \hat{H} - \frac{i\kappa}{2}(\hat{a}^\dagger\hat{a} - |\alpha|^2)$$

↳ constant is irrelevant (normalized out)

For detector 1:

$$p_1 = \delta t\langle\hat{c}_1^\dagger\hat{c}_1\rangle = \frac{\kappa\delta t}{2}\left(|\alpha|^2 + \langle\hat{a}^\dagger\hat{a}\rangle - \langle\alpha\hat{a}^\dagger + \alpha^*\hat{a}\rangle\right)$$

let $\alpha = |\alpha|e^{i\phi}$ and $\hat{x}_\phi = e^{i\phi}\hat{a}^\dagger + \hat{a}e^{-i\phi}$

$$p_1 = \frac{\kappa\delta t}{2}\left(|\alpha|^2 + \langle\hat{a}^\dagger\hat{a}\rangle - |\alpha|\langle\hat{x}_\phi\rangle\right)$$

$$p_2 = \frac{\kappa\delta t}{2}\left(|\alpha|^2 + \langle\hat{a}^\dagger\hat{a}\rangle + |\alpha|\langle\hat{x}_\phi\rangle\right)$$

★ Let $|\alpha|$ is very large : $|\alpha|^2 \gg \langle\hat{a}^\dagger\hat{a}\rangle$

Integrate $\int_0^{\delta\tau} dN_1$: choose $\delta\tau$ such that

$\langle x_\phi\rangle(t+\varepsilon) \sim \langle x_\phi\rangle(t)$

for $\varepsilon \ll \delta\tau$

$|\psi_c(t+\varepsilon)\rangle \sim |\psi_c(t)\rangle$

Let $\overset{\substack{\text{current}\\ \uparrow}}{I}_{1,2} = \frac{\delta N_1}{\delta \tau}$: photocurrent

Large # of counts, uncorrelated (Poisson) distributed

↳ Gaussian

$$\delta N_1 = \overline{\delta N_1} + \delta W_1$$

→ continuously fluctuating random variable. (noise)

$$\overline{\delta N_1} = \int_0^{\delta\tau} dN_1 = \frac{\kappa}{2}\delta\tau \left(|\alpha|^2 + \langle \hat{x}_\phi \rangle(0) \cdot |\alpha| \right)$$

can be ignored

$$\overline{\delta W_1} = 0 \quad \text{and} \quad \overline{\delta W_1^2} = \overline{\delta N_1} \sim \frac{K}{2}|\alpha|^2 \delta\tau$$

So: $$\boxed{dN_1(t) = \frac{K|\alpha|^2}{2}\left(1 + \frac{\langle x_\phi \rangle (t)}{|\alpha|}\right) dt + |\alpha|\sqrt{\frac{K}{2}}\, dW}$$

dW: Wiener process

$$I_{1,2}(t) = \frac{K|\alpha|^2}{2}\left(1 \mp \frac{\langle x_\phi \rangle (t)}{|\alpha|}\right) + |\alpha|\sqrt{\frac{K}{2}}\, \overset{\substack{\text{Langevin}\\ \uparrow}}{\xi_{1,2}(t)}$$

$\xi(t)\,dt = dW$

Balanced Homodyne Detection: Record $I_2 - I_1$

$$I_{bh}(t) = K|\alpha|\langle x_\phi\rangle(t) + |\alpha|\sqrt{\gamma_c}\,\xi(t)$$

$$\xi(t) = \frac{\xi_2 - \xi_1}{\sqrt{2}}$$

Remark: Unbalanced also works:

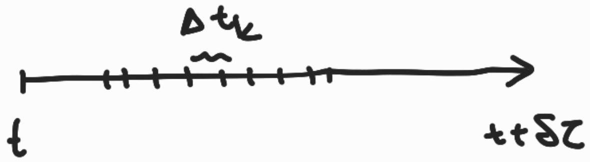
One can also use heterodyne detection:

$$LO \rightarrow \alpha e^{i\Omega t}$$

2) Quantum state diffusion

During $\delta\tau$

$$|\psi_c(t+\delta\tau)\rangle = \prod_i \hat{M}_{\nu_i}(\Delta t_i)\,|\psi_c(t)\rangle$$



$$|\tilde{\psi}_c(t+\delta\tau)\rangle \sim \hat{M}_0(\delta\tau)\,\hat{c}_1^{m_1}\,\hat{c}_2^{m_2}\,|\tilde{\psi}_c(t)\rangle$$

conditionized on m_1 clicks on 1

m_2 clicks on 2

$$\hat{c}_1^{\,m_1} = \left(\sqrt{\frac{\kappa}{2}}\right)^{m_1} (\hat{a}-\alpha)^{m_1} = \left(\sqrt{\frac{\kappa}{2}}\,(-\alpha)\right)^{m_1} \left(\mathbb{I} - \frac{\hat{a}}{\alpha}\right)^{m_1}$$

$$\left[\sqrt{\frac{\kappa}{2}}\,|\alpha|\right]^{m_1} \rightarrow \text{normalized out}$$

$$(-e^{i\phi})^{m_1} \rightarrow \text{the only object relevant } |\psi_c\rangle\langle\psi_c|$$

$$|\tilde{\psi}_c(t+\delta\tau)\rangle = \left(\hat{\mathbb{I}} - i\hat{H}_{eff}\,\delta\tau\right)\left(\hat{\mathbb{I}} - \frac{m_1\hat{a}}{\alpha}\right)\left(\hat{\mathbb{I}} + \frac{m_2\hat{a}}{\alpha}\right)|\tilde{\psi}_c(t)\rangle$$

$$= \left(\hat{\mathbb{I}} - i\hat{H}_{eff}\,\delta\tau + \frac{\hat{a}}{\alpha}(m_2 - m_1)\right)|\tilde{\psi}_c(t)\rangle$$

$$= \left(\hat{\mathbb{I}} - i\hat{H}_{eff}\,\delta\tau + \frac{\hat{a}}{\alpha}\left[\kappa\frac{|\alpha|^2}{2}\left(1 + \frac{\langle x_\phi\rangle}{|\alpha|}\right)\delta\tau + |\alpha|\sqrt{\frac{\kappa}{2}}\,\delta W_2\right.\right.$$

$$\left.\left. - \kappa\frac{|\alpha|^2}{2}\left(1 - \frac{\langle x_\phi\rangle}{|\alpha|}\right)\delta\tau - |\alpha|\sqrt{\frac{\kappa}{2}}\,\delta W_1\right]\right)|\tilde{\psi}_c\rangle$$

$$\boxed{d|\tilde{\psi}_c\rangle = \left(-i\hat{H}_{eff}\,dt + \hat{a}e^{-i\phi}\left(\kappa\langle\hat{x}_\phi(t)\rangle dt + \sqrt{\kappa}\,dW\right)\right)|\tilde{\psi}_c\rangle}$$

"Quantum state diffusion"

also: $$d|\tilde{\psi}_c\rangle = \left(-i\hat{H}_{eff} + \frac{I_{hom}}{\alpha}\,\hat{a}e^{-i\phi}\right)dt\,|\tilde{\psi}_c\rangle$$

Normalization:

$$\langle \tilde{\Psi}_c(t+dt) | \tilde{\Psi}_c(t+dt) \rangle = 1 + \langle d\Psi | \Psi \rangle + \langle \Psi | d\Psi \rangle + \langle d\Psi | d\Psi \rangle$$

⋮

$$d|\Psi_c\rangle = \Big[-i \hat{H}_{eff}\, dt + \kappa \langle \hat{x}_\phi(t) \rangle dt \left(\hat{a} e^{-i\phi} - \frac{\langle \hat{x}_\phi \rangle(t)}{2} \right)$$

$$+ \kappa\, dW \left(a e^{-i\phi} - \frac{\langle x_\phi \rangle}{2} \right) \Big] |\Psi_c\rangle$$

Remarks

. Read as drift-diffusion equation in the Hilbert space.

. fully general (equivalent to click-based SSE)