



## Supporting Online Material for

### **Sudden Death of Entanglement**

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#### **This PDF file includes:**

SOM Text

Fig. S1

References

# Supporting Online Material – Sudden Death of Entanglement

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# 1 Entanglement dynamics of simple quantum systems

A particular pathway of two-party decoherence called entanglement sudden death (ESD) has been discussed extensively recently (1) as part of recent interest in the dynamics of quantum correlations. Attractive proposals to use quantum entanglement in practice to enhance computational power and information processing capacity (2) have drawn international attention and have provided motivation. Preservation of useful entanglement is vital in this context, but unexpected developments have recently revealed the surprising fact that the life of entanglement under the influence of environmental noise can be unpredictably brief.

To bring the creative ideas proposed for quantum information processing into fruition, one must learn to control and manipulate quantum systems called qubits (quantum bits of information) when they are employed as information carriers. An especially daunting task is to combat loss of entanglement, which can be caused by background noise or manipulation errors (3–5).

Since early in this decade researchers have been gradually gaining a better understanding of the evolution of small-system entanglement in the presence of noise by examining calculable dynamical systems of two parties (6–9). Consideration has also been given to issues that are still not resolved, such as reliable measures of entanglement applicable to more than two parties (10–12).

By focusing on two-party quantum systems for which entanglement is quantifiable and its evolution in a noisy environment is calculable, entanglement dynamics has been extensively investigated recently. Research has touched on issues ranging from quantum foundations (13, 14), to coherence control (15, 16), error correction (17) and entanglement generation (18), to name few. Besides, ESD has been examined in several distinctive model situations involving pairs of atomic, photonic and spin qubits (19–21), continuous Gaussian states (22, 23), multiple qubits (24) and spin chains (25–27). ESD is also examined for different environments including

random matrix environments (28, 29) and thermal noise (30–32). ESD occurs whether noise is local to the two parties separately or the same noise affects both. The effect of global noise on entanglement decay may depend on whether the initial two-party state occupies a decoherence-free subspace (DFS) or not (33).

## 2 Entangled qubit descriptions

The entangled state of two cats encoded by the bracket notation introduced in the review (1)

$$[(Ws) \Leftrightarrow (Sw)] \tag{S1}$$

doesn't have the property of separability, in the sense of factorization, and is thus an entangled state. The reader can check this by comparing it with the state represented by this slightly different two-cat bracket:

$$[(W) \Leftrightarrow (S)][(w) \Leftrightarrow (s)]. \tag{S2}$$

The two forms (S1) and (S2) provide similar information in that both convey no definite information about either cat. Both (S1) and (S2) indicate that each cat is awake and asleep at the same time. However, (S2) provides the information in factored form, and this has an important effect because learning the state of the big cat does not provide more information about the little cat. For example in (S2), upon learning the big cat is found to be awake, the  $(S)$  possibility must be discarded, and then the overall two-cat state is reduced to the bracket  $(W)[(w) \Leftrightarrow (s)]$ . This still says the little cat is both awake and sleeping, unchanged from its situation before we learned about the big cat. The non-factorable first bracket (S1) contains more joint information because, as the review text indicates, it reduces to  $(Ws)$ , which provides definite information about the small cat.

These simple examples explain why a non-entangled state is called “separable” and an entangled state is called “non-separable”. The second state (S2) above can be separated (its bracket

can be factored, just as written) into a big-cat part times a little-cat part, whereas the first example is represented by a bracket more intimately mixing the big-cat and little-cat possibilities, making an overall (big cat)-(little cat) factoring impossible.

Two states of a qubit, designated  $(+)$  and  $(-)$  in the text, provide four elementary joint possibilities  $++$ ,  $+-$ ,  $-+$ ,  $--$ , and these supply the row and column labels for a two-qubit density matrix, a  $4 \times 4$  matrix with 16 elements. Most of the analyses carried out on two-qubit systems have been specialized to so-called  $X$  matrices, with only diagonal or anti-diagonal elements. All two-particle  $X$  matrices have purely diagonal reduced density matrices for each qubit separately, so all coherence applies jointly to the two parties, and not to either separately. A standard  $X$  matrix for two parties  $A$  and  $B$  can be written (in the basis described above)

$$\rho^{AB} = \begin{pmatrix} a & 0 & 0 & w \\ 0 & b & z & 0 \\ 0 & z^* & c & 0 \\ w^* & 0 & 0 & d \end{pmatrix}, \quad (\text{S3})$$

where  $a + b + c + d = 1$  to preserve correct normalization. If we use letters  $i$  and  $j$  to stand for either  $+$  or  $-$  as the case may be, we can say that the diagonal element in the  $(ij)(ij)$  position represents the probability that the first party is in state  $i$  and the second in state  $j$ . The off-diagonal positions such as  $(-+)(+-)$  indicate a cat-type situation, in which each party is both  $+$  and  $-$  at the same time, but in such a way that if the first party is  $+$ , then the second is definitely  $-$ , and vice versa. Generally the elements of such a density matrix are such that  $\rho^2 \neq \rho$ , so it usually represents a mixed rather than pure state. Such a simple two-party density matrix is actually not unusual. Experience shows that it arises naturally in a wide variety of physical situations and it includes pure Bell states and the Werner mixed state as special cases (34).

The most common measure of two-party entanglement is the concurrence used in (I), and

for this  $X$  matrix it is given by (35)

$$C(\rho) = 2 \max[0, Q(t)] = 2 \max[0, |z| - \sqrt{ad}, |w| - \sqrt{bc}], \quad (\text{S4})$$

where the second form is found by evaluating  $Q(t) = \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}$ , and the quantities  $\lambda_i$  are the eigenvalues in decreasing order of the auxilliary matrix  $\zeta(\rho)$ :

$$\zeta(\rho) \equiv \rho(\sigma_y^A \otimes \sigma_y^B) \rho^*(\sigma_y^A \otimes \sigma_y^B)$$

where  $\rho^*$  denotes the complex conjugation of  $\rho$  in the standard ( $\pm$ ) basis and  $\sigma_y$  is the Pauli matrix. The definition ensures that  $C$  is real and is limited to the range  $1 \geq C \geq 0$ , from maximal entanglement to complete disentanglement. All of the elements, and therefore  $C$ , may be time dependent if the qubits are exposed to external forces such as provided by environmental noise, and examples are given below.

### 3 Entanglement vs. decoherence

The word “decoherence” is commonly used as shorthand for loss of, or lack of, correlation in many contexts – quantum (36) and classical. Entanglement is a purely quantum form of correlation, and is applicable by definition only to multi-part quantities (collections of two or more objects being treated quantum mechanically such as atoms, spins, photons, quantum dots, etc.). It is important to keep in mind that loss of entanglement doesn’t require complete loss of coherence. A single electron exhibits a form of coherence, for example, when its spin precesses in a regular predictable way. Weak environmental influences will slowly disturb the regularity of the precession and thus reduce the coherence, i.e., lead to decoherence, whether the electron is entangled with another or not.

It is easy to see that ESD is a form of decoherence, and clearly an unwanted prospect for any program intending to utilize the entanglement of multi-qubit systems. In principle, entanglement can be restored from any low degree no matter how small (37), but restoration is possible

only before entanglement reaches zero. ESD means a finite disentanglement time, so recovery after that time is not possible by any method. This emphasizes the value of efforts to find a way to estimate ESD times.

If qubits are employed for practical goals such as quantum computing, or communication or information processing in general, they must be considered open systems, because they must be made accessible to external agents programmed to execute desired operations on them to make them functional. Thus useful qubits will inevitably be open to sources of background noise, implying interactions that are uncontrolled and often unidentified. Thus the available information about the qubits becomes degraded, and so a consequence of openness and the accompanying decoherence is that practical systems will almost always be in mixed rather than pure quantum states.

## 4 Kraus operator-sum representation

Solutions of the appropriate equations for noisy evolution of two-qubit density matrices can be expressed by a set of Kraus operators (2) if the noise is weak and has self-correlation times so short that they can be considered zero (the so-called Born-Markov case). Given an initial state  $\rho$  (pure or mixed), its evolution can be written compactly as

$$\rho(t) = \sum_{\mu} K_{\mu}(t) \rho(0) K_{\mu}^{\dagger}(t),$$

where the so-called Kraus operators  $K_{\mu}$  satisfy  $\sum_{\mu} K_{\mu}^{\dagger} K_{\mu} = 1$  for all  $t$ .

For longitudinal (amplitude) noise, such as provided by vacuum fluctuations, the Kraus

operators are given by

$$\begin{aligned}
K_1 &= \begin{pmatrix} \gamma_A & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} \gamma_B & 0 \\ 0 & 1 \end{pmatrix}, \\
K_2 &= \begin{pmatrix} \gamma_A & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ \omega_B & 0 \end{pmatrix}, \\
K_3 &= \begin{pmatrix} 0 & 0 \\ \omega_A & 0 \end{pmatrix} \otimes \begin{pmatrix} \gamma_B & 0 \\ 0 & 1 \end{pmatrix}, \\
K_4 &= \begin{pmatrix} 0 & 0 \\ \omega_A & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ \omega_B & 0 \end{pmatrix},
\end{aligned}$$

and the time-dependent Kraus matrix elements for amplitude noise are

$$\begin{aligned}
\gamma_A(t) &= \exp[-\Gamma_{am}^A t/2], \quad \gamma_B(t) = \exp[-\Gamma_{am}^B t/2], \\
\omega_A(t) &= \sqrt{1 - \gamma_A^2(t)}, \quad \omega_B(t) = \sqrt{1 - \gamma_B^2(t)}.
\end{aligned}$$

For transverse (phase) noise, we have the following compact Kraus operators for qubits  $A$  and  $B$ :

$$\begin{aligned}
K_1 &= \begin{pmatrix} 1 & 0 \\ 0 & \gamma_A \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & \gamma_B \end{pmatrix}, \\
K_2 &= \begin{pmatrix} 1 & 0 \\ 0 & \gamma_A \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & \omega_B \end{pmatrix}, \\
K_3 &= \begin{pmatrix} 0 & 0 \\ 0 & \omega_A \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & \gamma_B \end{pmatrix}, \\
K_4 &= \begin{pmatrix} 0 & 0 \\ 0 & \omega_A \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & \omega_B \end{pmatrix},
\end{aligned}$$

where

$$\gamma_A(t) = \exp[-\Gamma_{ph}^A t/2] \quad \text{and} \quad \omega_A(t) = \sqrt{1 - \gamma_A^2(t)},$$

and similar expressions for  $\gamma_B(t)$  and  $\omega_B(t)$ .



## 5 Evolution of concurrence for $X$ in the review (I)

The time dependence of the matrix elements of  $X$  state (S3) under amplitude noise can be easily evaluated by using Kraus operators:

$$\begin{aligned}
z(t) &= z\gamma_A\gamma_B, \\
w(t) &= w\gamma_A\gamma_B, \\
a(t) &= a\gamma_A^2\gamma_B^2, \\
b(t) &= b\gamma_A^2 + a\gamma_A^2\omega_B^2, \\
c(t) &= c\gamma_B^2 + a\gamma_B^2\omega_A^2, \\
d(t) &= 1 - a + b\omega_A^2 + c\omega_B^2 + \omega_A^2\omega_B^2a.
\end{aligned}$$

As in the text, we have assumed equivalent noise strengths for the two parties, so  $\Gamma_{am}^A = \Gamma_{am}^B = \Gamma_{am}$ , leading to  $\gamma_A = \gamma_B = \exp[-\Gamma_{am}t/2]$  and  $\omega_A = \omega_B = \sqrt{1 - \exp[-\Gamma_{am}t]}$ .

## 6 Special examples

Fig. 2 in the review (I) is the plot of degradation of concurrence appropriate to vacuum fluctuation noise, for an initial state that is a special example of (S3), namely:

$$\rho^{AB} = \frac{1}{3} \begin{pmatrix} \kappa & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 - \kappa \end{pmatrix}. \quad (\text{S5})$$

This initial density matrix  $\rho^{AB}$  has a concurrence value that ranges from  $2/3$  to  $1/3$  to  $2/3$ , corresponding to the range of values of  $\kappa$  from  $0$  to  $1/2$  to  $1$ . Note that the notational connection between the general  $X$  matrix and this example is provided by  $a = \kappa/3, b = c = 1/3, d = (1 - \kappa)/3, z = 1/3$  and  $w = 0$ . For any initial setting of  $\kappa$ , the concurrence can be evaluated as a function of time. As seen from Fig. 2 in the review (I), the onset of ESD is not uniquely

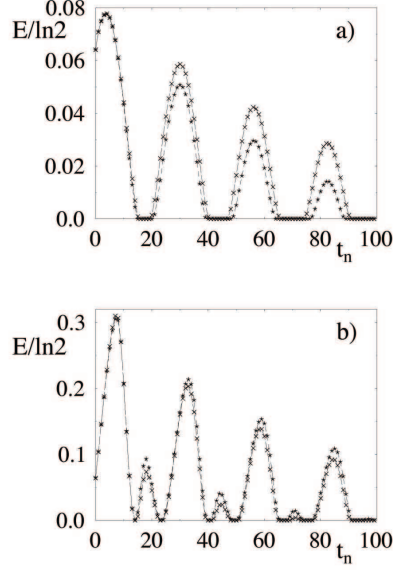


Figure 1: Normalized entanglement of formation  $E$  of two entangled qubits as a function of time. Note repeated birth and death of entanglement (adapted from (6)).

determined by the initial degree of entanglement. In the case of (S5), the non-zero matrix elements at time  $t$  simplify as

$$\begin{aligned}
 z(t) &= \exp[-\Gamma t], \\
 a(t) &= \frac{\kappa}{3} \exp[-2\Gamma t], \\
 b(t) = c(t) &= \exp[-\Gamma t] \left(1 + \frac{\kappa}{3} - \frac{\kappa}{3} \exp[-\Gamma t]\right), \\
 d(t) &= 1 - \frac{\kappa}{3} + 2(1 - \exp[-\Gamma t]) + \frac{\kappa}{3}(1 - \exp[-\Gamma t])^2.
 \end{aligned}$$

In contrast to the monotonic decay of entanglement under vacuum noise, a model for two-party decoherence driven by classical jump noise (6) can show degradation that is underdamped, as in Fig. (1), where entanglement of formation rather than concurrence is plotted.

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