HARMONIC OSCILLATORS

I. Definitions and Notations

2 conjugate variable $\hat{x}, \hat{p}$ $\qquad [\hat{x}, \hat{p}] = i\hbar$

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m \omega^2 \hat{x}^2$$

Natural energy scale $\hbar\omega$

Natural length scale $\sqrt{\frac{\hbar}{m\omega}} = x_0$

$$\frac{\hat{H}}{\hbar\omega} = \frac{1}{2}\left(\frac{\hat{p}}{\hbar}\right)^2 x_0^2 + \frac{1}{2}\left(\frac{\hat{x}}{x_0}\right)^2$$

1) Ladder Operators

$$\frac{\hat{H}}{\hbar\omega} = \underbrace{\frac{1}{\sqrt{2}}\left(\frac{\hat{x}}{x_0} + i\frac{\hat{p}}{\hbar}x_0\right)^{\dagger}}_{\hat{a}^{+}} \underbrace{\frac{1}{\sqrt{2}}\left(\frac{\hat{x}}{x_0} - \frac{i\hat{p}x_0}{\hbar}\right)}_{\hat{a}} + \frac{1}{2}\mathbb{I}$$

$$\frac{\hat{H}}{\hbar\omega} = \hat{a}^{+}\hat{a} + \frac{1}{2}\mathbb{I}$$

$$\hat{x} = \frac{x_0}{\sqrt{2}}\left(\hat{a} + \hat{a}^{+}\right)$$

Define $\hat{n} = \hat{a}^{+}\hat{a}$

$$\frac{\hat{p}}{\hbar} = \frac{1}{i\sqrt{2}x_0}\left(\hat{a} - \hat{a}^{+}\right)$$

Commutation Relation

$$[\hat{a}, \hat{a}^+] = \mathbb{I}$$

$$[\hat{n}, \hat{a}^+] = \hat{a}^+$$

$$[\hat{n}, \hat{a}] = -\hat{a}$$

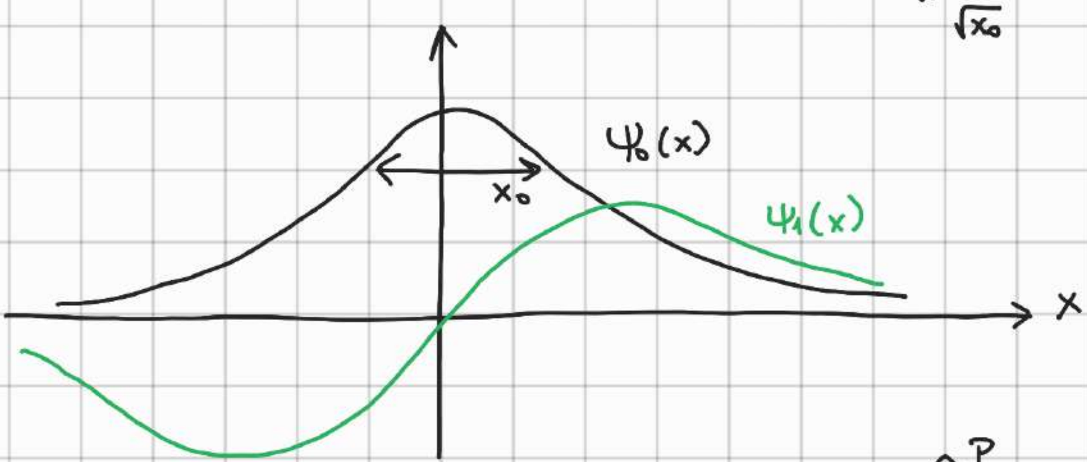
Eigen State:

$|n\rangle \quad , \quad n \in \mathbb{N} \qquad \hat{n}|n\rangle = n|n\rangle$

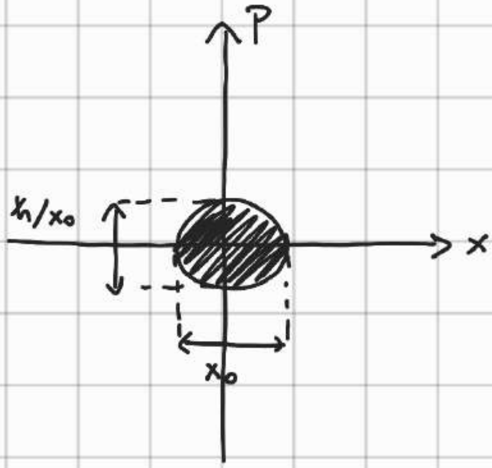
$$\begin{cases} \hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle \\ \hat{a}^+|n\rangle = \sqrt{n+1}\,|n+1\rangle \end{cases} \qquad \hat{a}|0\rangle = 0$$

Wavefunctions:

$$\langle x | 0 \rangle = \psi_0(x) = \Big(\underbrace{\frac{m\omega}{\hbar\pi}}_{\sim \frac{1}{\sqrt{x_0}}}\Big)^{1/4} e^{-x^2/2x_0^2}$$



Phase Space Representation:



II. Dynamics

1) Free Harmonic Oscillator

Heisenberg Picture : $i\hbar\dot{\hat{a}} = [\hat{a}, \hat{H}] = \hbar\omega\hat{a}$

$$\hat{a}(t) = \hat{a}(0)\, e^{-i\omega t}$$

2) Driven Oscillator

$$\hat{H} = \hbar\omega\hat{a}^{+}\hat{a} + \overbrace{F_0}^{\text{some classical force}}\,\hat{x}\cos(\omega_e t + \varphi)$$

$$\hat{H} = \hbar\omega\hat{a}^{+}\hat{a} + \frac{F_0 x_0}{\sqrt{2}}\left(\hat{a} + \hat{a}^{+}\right)\frac{1}{2}\left(e^{i(\omega_e t + \varphi)} + e^{-i(\omega_e t + \varphi)}\right)$$

Moving to a "rotating frame"

$$|\tilde{\Psi}(t)\rangle = \hat{U}(t)\,|\Psi(t)\rangle \qquad U^{+}|\tilde{\Psi}(t)\rangle = |\Psi(t)\rangle$$

$$i\hbar\partial_t|\tilde{\Psi}(t)\rangle = (i\hbar\partial_t\hat{U}).|\Psi(t)\rangle + \hat{U}(t)\underbrace{i\hbar\partial_t|\Psi(t)\rangle}_{\hat{H}|\Psi(t)\rangle}$$

$$i\hbar\partial_t|\tilde{\Psi}\rangle = \Big[\underbrace{(i\hbar\partial_t\hat{U}).\hat{U}^{+} + \hat{U}(t)\hat{H}U^{+}}_{\tilde{H}}\Big]|\tilde{\Psi}(t)\rangle$$

Here : $\hat{U}(t) = e^{i\omega_e\hat{n}t} = \sum_n e^{i\omega_e n t}\,|n\rangle\langle n|$

$$\tilde{H} = (i\hbar\partial_t \hat{U}).\hat{U}^+ + \hat{U}\hat{H}U^+$$

$$\partial_t \hat{U} = i\omega_e n \underbrace{\sum_n e^{i\omega_e n t} |n\rangle\langle n|}_{U} = i\omega_e n \hat{U}$$

$$\hat{H} = \hbar\omega\hat{n} + \dots$$

$$\tilde{H} = -\hbar\omega_e\hat{n} + \hat{U}\hat{H}U^+$$

$$\hat{U}\hat{a}\hat{U}^+ = e^{-i\omega_e t}\hat{a}$$

So that $$\tilde{H} = \hbar(\omega-\omega_e)\hat{n} + \frac{F_0 x_0}{2\sqrt{2}}\left(\hat{a}^+e^{i\varphi} + \hat{a}e^{-i\varphi} + \cancel{\hat{a}^+e^{i(2\omega_e t+\varphi)}} + \cancel{\hat{a}e^{-i(2\omega_e t+\varphi)}}\right)$$

RWA

$\omega-\omega_e , F_0 x_0 \dots \ll 2\omega_e$ Rotating Wave Approximation

(provides Time Independent Hamiltonian)

$$\tilde{H} = \hbar\Delta\hat{a}^+\hat{a} + \frac{F_0 x_0}{2\sqrt{2}}\left(\hat{a}^+e^{i\varphi} + \hat{a}e^{-i\varphi}\right)$$

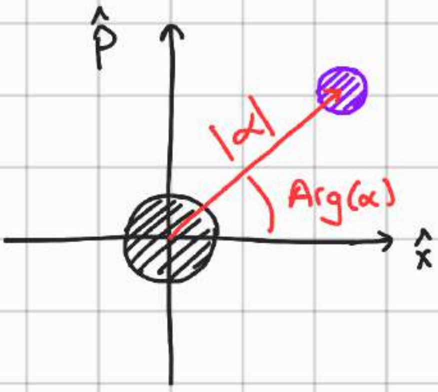
$$\tilde{H} = \hbar\Delta\left(\hat{a} + \frac{F_0 x_0}{2\sqrt{2}\,\hbar\Delta}e^{i\varphi}\right)^+ . \left(\hat{a} + \frac{F_0 x_0}{2\sqrt{2}\,\hbar\Delta}e^{i\varphi}\right) + \text{constant}$$

3) Coherent State

$|\alpha\rangle$ such that $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$

Displacement operator

$$\hat{D}(\alpha)|0\rangle = |\alpha\rangle \quad , \quad \hat{D}(\alpha) = e^{\alpha\hat{a}^{+} - \alpha^{*}\hat{a}}$$



<u>Example:</u> $\alpha \in \mathbb{R}$

$$\hat{D}(\alpha) = e^{\alpha(\hat{a}^{+} - \hat{a})} = \underbrace{e^{-\alpha i x_0 \sqrt{2}\frac{\hat{p}}{\hbar}}}_{\substack{\text{physical displacement} \\ \alpha\sqrt{2}.x_0}}$$

<u>Ex</u> $\alpha = i\beta$, $\beta \in \mathbb{R}$

$$\hat{D}(\alpha) = e^{i\beta(\hat{a}^{+} + \hat{a})} = \underbrace{e^{i\beta\frac{\hat{x}\sqrt{2}}{x_0}}}_{\substack{\text{momentum kick} \\ \text{by } \alpha\frac{\hbar}{x_0}}}$$

Back to driven oscillator:

$$\tilde{H} = \hbar\Delta\hat{b}^{+}\hat{b} + \text{constant}$$

$$\hat{b} = \hat{D}\left(\frac{-F_0 x_0}{2\sqrt{2}\hbar\Delta}e^{i\varphi}\right)\hat{a}\,\hat{D}\left(-\frac{F_0 x_0}{2\sqrt{2}\hbar\Delta}e^{i\varphi}\right)^{+}$$

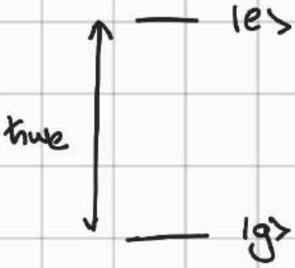
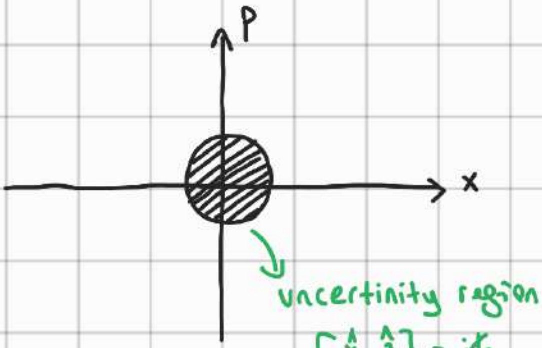
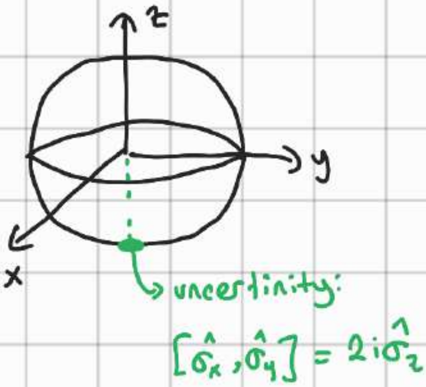
$$\hat{D}(\alpha)\,\hat{a}\,\hat{D}(\alpha)^{+} = \hat{a} - \alpha\,\mathbb{I}$$

When $\Delta \to 0$: displacement goes to ∞ $\Rightarrow$ RESONANCE

Dynamics: $\hat{a}(t) = e^{-i\omega t}\hat{a}(0) \longrightarrow$ Schrödinger Picture

$$|\alpha(t)\rangle = |\alpha(0)e^{-i\omega t}\rangle$$

III. Comparison with two-level systems

	Harmonic Oscillator	Two Level System
Spectrum		
Ladder Operator	$[\hat{a}, \hat{a}^{+}] = \hat{\mathbb{I}}$	$[\hat{\sigma}_-, \hat{\sigma}_+] = \hat{\sigma}_z$
Phase Space		
Displacement	$\hat{D}(\alpha)$	$\hat{R}_{\vec{n}}(\theta)$