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COM-309 HW10

Exercise 1 Product States and CSHS inequality

a)

$|\varphi_A\rangle$ Alice $\qquad\qquad$ $|\varphi_B\rangle$ Bob

$\{\alpha, \alpha_\perp\}$: Click $a=+1$: $|\langle\alpha|\varphi_A\rangle|^2$; No-click $a=-1$: $|\langle\alpha_\perp|\varphi_A\rangle|^2$

$\{\beta, \beta_\perp\}$: Click $a=+1$: $|\langle\beta|\varphi_B\rangle|^2$; No-click $a=-1$: $|\langle\beta_\perp|\varphi_B\rangle|^2$

Locality: $\quad p(a,b|\alpha,\beta) = p_A(a|\alpha)\cdot p_B(b|\beta)$

From Born Rule

$$p(1,1|\alpha,\beta) = p_A(1|\alpha)\, p_B(1|\beta) = |\langle\alpha|\varphi_A\rangle|^2\, |\langle\beta|\varphi_B\rangle|^2$$

$$p(1,-1|\alpha,\beta) = p_A(1|\alpha)\, p_B(-1|\beta) = |\langle\alpha|\varphi_A\rangle|^2\, |\langle\beta_\perp|\varphi_B\rangle|^2$$

$\Big\}\ |\langle\alpha|\varphi_A\rangle|^2\cdot 1$

$$p(-1,1|\alpha,\beta) = p_A(-1|\alpha)\, p_B(1|\beta) = |\langle\alpha_\perp|\varphi_A\rangle|^2\, |\langle\beta|\varphi_B\rangle|^2$$

$$p(-1,-1|\alpha,\beta) = p_A(-1|\alpha)\, p_B(-1|\beta) = |\langle\alpha_\perp|\varphi_A\rangle|^2\, |\langle\beta_\perp|\varphi_B\rangle|^2$$

$\Big\}\ |\langle\alpha_\perp|\varphi_A\rangle|^2\cdot 1$

+

$$|\langle\alpha|\varphi_A\rangle|^2 + |\langle\alpha_\perp|\varphi_A\rangle|^2 = 1$$

b) $X = \langle\psi|\ A\otimes B + A\otimes B' - A'\otimes B + A'\otimes B'\ |\psi\rangle$

Observables of polarization:

$$A = (+1)\ |\alpha\rangle\langle\alpha| + (-1)|\alpha_\perp\rangle\langle\alpha_\perp|$$

$$A' = (+1)\ |\alpha_\perp\rangle\langle\alpha_\perp| + (-1)|\alpha\rangle\langle\alpha|$$

$$B = (+1)\ |\beta\rangle\langle\beta| + (-1)\ |\beta_\perp\rangle\langle\beta_\perp|$$

$$B' = (+1)\ |\beta_\perp\rangle\langle\beta_\perp| + (-1)\ |\beta\rangle\langle\beta|$$

$$\langle\psi|\ A\otimes B|\ \psi\rangle = \langle\varphi_B|\otimes\langle\varphi_A|\ (A\otimes B)\ |\varphi_A\rangle\otimes|\varphi_B\rangle$$

$$= \langle\varphi_A|A|\varphi_A\rangle\ \langle\varphi_B|B|\varphi_B\rangle$$

$$= \Big[\langle\varphi_A|\ (+1)\ |\alpha\rangle\langle\alpha|\varphi_A\rangle + (-1)\,|\varphi_\perp\rangle\langle\alpha_\perp|\varphi_A\rangle\Big]\Big[\langle\varphi_B|\ +1\,|\beta\rangle\langle\beta|\varphi_B\rangle - 1\cdot|\beta_\perp\rangle\langle\beta_\perp|\varphi_B\rangle\Big]$$

$$= \Big[+|\langle\alpha|\varphi_A\rangle|^2 - |\langle\alpha_\perp|\varphi_A\rangle|^2\Big]\Big[+|\langle\beta|\varphi_B\rangle|^2 - |\langle\beta_\perp|\varphi_B\rangle|^2\Big]$$

$$= |\langle\alpha|\varphi_A\rangle|^2\cdot|\langle\beta|\varphi_B\rangle|^2 - |\langle\alpha|\varphi_A\rangle|^2\cdot|\langle\beta_\perp|\varphi_B\rangle|^2 - |\langle\alpha_\perp|\varphi_A\rangle|^2\,|\langle\beta|\varphi_B\rangle|^2$$
$$+|\langle\alpha_\perp|\varphi_A\rangle|^2\,|\langle\beta_\perp|\varphi_B\rangle|^2$$

$$= p(1,1|\alpha,\beta) - p(1,-1|\alpha,\beta) - p(-1,1|\alpha,\beta) + p(-1,-1|\alpha,\beta)$$

$$\langle\Psi| A\otimes B' |\Psi\rangle = \langle\varphi_B|\otimes\langle\varphi_A| (A\otimes B') |\varphi_A\rangle\otimes|\varphi_B\rangle$$

$$= \langle\varphi_A|A|\varphi_A\rangle \langle\varphi_B|B'|\varphi_B\rangle$$

$$= \left[|\langle\alpha|\varphi_A\rangle|^2 - |\langle\alpha_\perp|\varphi_A\rangle|^2\right] \left[(+1)\, |\langle\beta_\perp|\varphi_B\rangle|^2 - 1.\, |\langle\beta|\varphi_B\rangle|^2\right]$$

$$= (+1)\, p(1,-1|\alpha,\beta) - 1.\, p(1,1|\alpha,\beta) - 1.\, p(-1,-1|\alpha,\beta) + p(-1,1|\alpha,\beta)$$

$$\langle\Psi| A'\otimes B |\Psi\rangle = \langle\varphi_A|A'|\varphi_A\rangle \langle\varphi_B|B|\varphi_B\rangle$$

$$= \left[(+1)\, |\langle\alpha_\perp|\varphi_A\rangle|^2 - 1.\, |\langle\alpha|\varphi_A\rangle|^2\right] \left[+1.\, |\langle\beta|\varphi_B\rangle|^2 - 1.\, |\langle\beta_\perp|\varphi_B\rangle|^2\right]$$

$$= +p(-1,1|\alpha,\beta) - p(-1,-1|\alpha,\beta) - p(1,1|\alpha,\beta) + p(1,-1|\alpha,\beta)$$

$$\langle\Psi| A'\otimes B' |\Psi\rangle = \langle\varphi_A|A'|\varphi_A\rangle \langle\varphi_B|B'|\varphi_B\rangle$$

$$= \left[+|\langle\alpha_\perp|\varphi_A\rangle|^2 - |\langle\alpha|\varphi_A\rangle|^2\right] \left[+|\langle\beta_\perp|\varphi_B\rangle|^2 - |\langle\beta|\varphi_B\rangle|^2\right]$$

$$= p(-1,-1|\alpha,\beta) - p(1,-1|\alpha,\beta) - p(1,-1|\alpha,\beta) + p(1,1|\alpha,\beta)$$

$$X = \langle\Psi| A\otimes B + A\otimes B' - A'\otimes B + A'\otimes B' |\Psi\rangle$$

$$= p(1,1|\alpha,\beta) - p(-1,1|\alpha,\beta) - p(1,-1|\alpha,\beta) + p(-1,-1|\alpha,\beta)$$

$$p(1,-1|\alpha,\beta) - p(1,1|\alpha,\beta) - p(-1,-1|\alpha,\beta) + p(-1,1|\alpha,\beta)$$

$$- \left[p(-1,1|\alpha,\beta) - p(1,1|\alpha,\beta) - p(-1,-1|\alpha,\beta) + p(1,-1|\alpha,\beta)\right]$$

$$p(-1,-1|\alpha,\beta) - p(-1,1|\alpha,\beta) - p(1,-1|\alpha,\beta) + p(1,1|\alpha,\beta)$$

$$X = -2 \cdot p(-1,1|\alpha,\beta) + 2 \cdot p(1,1|\alpha,\beta) + 2p(-1,-1|\alpha,\beta) - 2p(1,1|\alpha,\beta)$$

$$X_{min} = -2\left(\overbrace{p(-1,1|\alpha,\beta) + p(1,-1|\alpha,\beta)}^{1 \ \hookrightarrow \text{ others are "0"}}\right) = -2$$

$$X_{max} = 2 \cdot \left(\underbrace{p(1,1|\alpha,\beta) + p(-1,-1|\alpha,\beta)}_{\hookrightarrow \text{ others are "0"}}\right) = 2$$

$$\boxed{-2 \leq X \leq 2}$$

Exercise 2 The difference between a Bell state and a statistical mixture of $|00\rangle$ and $|11\rangle$

a) $\rho_{Bell} = ?$

$$|B_{00}\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$|B_{00}\rangle\langle B_{00}| = \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}_{4\times 4}$$

$$|B_{00}\rangle\langle B_{00}| = \frac{1}{2}\left[|00\rangle_{AB} + |11\rangle_{AB}\right]\left[\langle 00|_{BA} + \langle 11|_{BA}\right] = \frac{1}{2}\Big[|0\rangle\langle 0|_A \otimes |1\rangle\langle 1|_B$$

$$+ |0\rangle_A\langle 1|_A \otimes |0\rangle_A\langle 1|_B$$

$$+ |1\rangle_A\langle 0|_A \otimes |1\rangle_B\langle 0|_B$$

$$+ |1\rangle\langle 1| \otimes |1\rangle\langle 1|\Big]$$

b) $\rho_{stat} = \sum_i p_i |\varphi_i\rangle\langle\varphi_i| = \frac{1}{2}|00\rangle\langle 00| + \frac{1}{2}|11\rangle\langle 11|$

$$\rho_{stat} = \frac{1}{2}|0\rangle\langle 0| \otimes |0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| \otimes |1\rangle\langle 1|$$

$$= \frac{1}{2}\begin{pmatrix}1&0\\0&0\end{pmatrix}\otimes\begin{pmatrix}1&0\\0&0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}0&0\\0&1\end{pmatrix}\otimes\begin{pmatrix}0&0\\0&1\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{pmatrix}$$

$$\rho_{stat} = \frac{1}{2}\begin{bmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{bmatrix}$$

c) $X = \langle\Psi| \underbrace{A\otimes B + A\otimes B' - A'\otimes B + A'\otimes B}_{B} |\Psi\rangle$

$$X_{Bell} = \cos 2(\alpha-\beta) + \cos 2(\alpha-\beta') - \cos 2(\alpha'-\beta) + \cos 2(\alpha'-\beta')$$

for $\alpha = 0$, $\alpha' = -\frac{\pi}{4}$, $\beta = \frac{\pi}{8}$, $\beta' = -\frac{\pi}{8}$

$$X_{bell} = \cos 2.\left(0 - \frac{\pi}{8}\right) + \cos 2.\left(0 + \frac{\pi}{8}\right) - \cos 2\left(\overbrace{-\frac{\pi}{4} - \frac{\pi}{8}}^{-\frac{3\pi}{8}}\right) + \cos 2.\left(-\frac{\pi}{4} + \frac{\pi}{8}\right)$$

$$= \cos\left(-\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right) - \cos\left(-\frac{3\pi}{4}\right) + \cos\left(-\frac{\pi}{4}\right)$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - \left(-\frac{1}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}} = 4.\frac{1}{\sqrt{2}} = 2\sqrt{2}$$

$$\boxed{\text{!} \quad X = 2\sqrt{2}} \Rightarrow X > 2$$

$$\rho_{stat} = \frac{1}{2} \overbrace{|0\rangle_A\langle 0|_A \otimes |0\rangle_B\langle 0|_B}^{|00\rangle\langle 00|} + \frac{1}{2} \overbrace{|1\rangle_A\langle 1|_A \otimes |1\rangle_B\langle 1|_B}^{|11\rangle\langle 11|}$$

$$\langle \Psi_{stat} | A\otimes B | \Psi_{stat}\rangle = \mathrm{Tr}\left(A\otimes B \ \rho_{stat} \right)$$

$$= \mathrm{Tr}\left(A\otimes B \left(\frac{1}{2}|00\rangle\langle 00| + \frac{1}{2}|11\rangle\langle 11| \right)\right)$$

$$= \frac{1}{2} \mathrm{Tr}\left(A\otimes B \ |00\rangle_{BA}\langle 00|_{AB} \right) + \frac{1}{2} \mathrm{Tr}\left(A\otimes B \ |11\rangle\langle 11| \right)$$

$$= \frac{1}{2} \mathrm{Tr}\left(\langle 0|A|\underset{|\alpha\rangle}{0}\rangle \ \langle 0|B|0\rangle \right) + \frac{1}{2} \mathrm{Tr}\left(\langle \underset{|\alpha_\perp\rangle}{1}| A | 1\rangle \langle 1|B|1\rangle \right)$$

$$\underline{A = +|\alpha\rangle\langle\alpha| - |\alpha_\perp\rangle\langle\alpha_\perp|} \qquad \underline{B = +|\beta\rangle\langle\beta| - |\beta_\perp\rangle\langle\beta_\perp|}$$

$$\langle\alpha|A|\alpha\rangle = \langle\alpha|\alpha\rangle\langle\alpha|\alpha\rangle = 1$$

$$\langle\alpha|B|\alpha\rangle = \overbrace{\langle\alpha|\beta\rangle\langle\beta|\alpha\rangle}^{\cos^2(\alpha-\beta)} - \underbrace{\langle\alpha|\beta_\perp\rangle\langle\beta_\perp|\alpha\rangle}_{\cos^2(\alpha-\beta_\perp)}$$

$$\langle\alpha_\perp|A|\alpha_\perp\rangle = -\langle\alpha_\perp|\alpha_\perp\rangle\langle\alpha_\perp|\alpha_\perp\rangle = -1$$

$$\langle\alpha_\perp|B|\alpha_\perp\rangle = \overbrace{\langle\alpha_\perp|\beta\rangle\langle\beta|\alpha_\perp\rangle}^{\cos^2(\alpha_\perp-\beta)} - \underbrace{\langle\alpha_\perp|\beta_\perp\rangle\langle\beta_\perp|\alpha_\perp\rangle}_{\cos^2(\alpha_\perp-\beta_\perp)}$$

$$\langle \Psi_{stat} | A\otimes B | \Psi_{stat}\rangle = \frac{1}{2}\left[\cos^2(\alpha-\beta) - \cos^2(\alpha-\beta_\perp)\right] - \frac{1}{2}\left[\cos^2(\alpha_\perp-\beta) - \cos^2(\alpha_\perp-\beta_\perp)\right]$$

$$\langle \psi_{stat} | A \otimes B' | \psi_{stat} \rangle = Tr(A \otimes B' \rho_{stat})$$

$$= \frac{1}{2} Tr\left(\overbrace{\langle 0 | A | 0 \rangle}^{1} \langle 0 | B' | 0 \rangle \right) + \frac{1}{2} Tr\left(\overbrace{\langle 1 | A | 1 \rangle}^{-1} \langle 1 | B' | 1 \rangle \right)$$

$$B' = -|\beta\rangle\langle\beta| + |\beta_\perp\rangle\langle\beta_\perp|$$

$$\langle \alpha | B' | \alpha \rangle = -\cos^2(\alpha-\beta) + \cos^2(\alpha-\beta_\perp) \qquad \langle \alpha_\perp | B' | \alpha_\perp \rangle = -\cos^2(\alpha_\perp-\beta) + \cos^2(\alpha_\perp-\beta_\perp)$$

$$\langle \psi | A \otimes B' | \psi_{stat} \rangle = \frac{1}{2}\left[-\cos^2(\alpha-\beta) + \cos^2(\alpha-\beta_\perp)\right] - \frac{1}{2}\left[-\cos^2(\alpha_\perp-\beta) + \cos^2(\alpha_\perp-\beta_\perp)\right]$$

$$\langle \psi_{stat} | A' \otimes B | \psi_{stat} \rangle = Tr(A' \otimes B | \psi_{stat})$$

$$= \frac{1}{2} Tr\left(\underbrace{\langle 0 | A' | 0 \rangle}_{-1.} \langle 0 | B | 0 \rangle \right) + \frac{1}{2} Tr\left(\underbrace{\langle 1 | A' | 1 \rangle}_{1.} \langle 1 | B | 1 \rangle \right)$$

$$\langle \psi | A' \otimes B | \psi \rangle = +\frac{1}{2}\left[\cos^2(\alpha-\beta) - \cos^2(\alpha-\beta_\perp)\right] - \frac{1}{2}\left[\cos^2(\alpha_\perp-\beta) - \cos^2(\alpha_\perp-\beta_\perp)\right]$$

$$\langle \psi_{stat} | A' \otimes B' | \psi_{stat} \rangle = Tr\left(A' \otimes B' \rho_{stat}\right)$$

$$= \frac{1}{2} Tr\left(\underbrace{\langle 0 | A' | 0 \rangle}_{-1} \langle 0 | B' | 0 \rangle \right) + \frac{1}{2}\left(\underbrace{\langle 1 | A' | 1 \rangle}_{1} \langle 1 | B' | 1 \rangle \right)$$

$$\langle \psi | A' \otimes B' | \psi \rangle = -\frac{1}{2}\left[-\cos^2(\alpha-\beta) + \cos^2(\alpha-\beta_\perp)\right] + \frac{1}{2}\left[-\cos^2(\alpha_\perp-\beta) + \cos^2(\alpha_\perp-\beta_\perp)\right]$$

$$\langle\psi|B|\psi\rangle = X = +\frac{1}{2}\left[\cos^2(\alpha-\beta) - \cos^2(\alpha-\beta_\perp)\right] - \frac{1}{2}\left[\cos^2(\alpha_\perp-\beta) - \cos^2(\alpha_\perp-\beta_\perp)\right]$$

$$-\frac{1}{2}\left[-\cos^2(\alpha-\beta) + \cos^2(\alpha-\beta_\perp)\right] + \frac{1}{2}\left[-\cos^2(\alpha_\perp-\beta) + \cos^2(\alpha_\perp-\beta_\perp)\right]$$

$+$

$$X = \cos^2(\alpha-\beta) - \cos^2(\alpha-\beta_\perp) - \cos^2(\alpha_\perp-\beta) + \cos^2(\alpha_\perp-\beta_\perp)$$

For $\alpha = 0$, $\alpha' = -\frac{\pi}{4}$, $\beta = \frac{\pi}{8}$, $\beta' = -\frac{\pi}{8}$

$$X = \cancel{\cos^2\left(0-\frac{\pi}{8}\right)} \cancel{-\cos^2\left(0+\frac{\pi}{8}\right)} - \cos^2\underbrace{\left(-\frac{\pi}{4}-\frac{\pi}{8}\right)}_{(2)} + \cos^2\underbrace{\left(-\frac{\pi}{4}+\frac{\pi}{8}\right)}_{(2)}$$

$$-\cos^2\left(-\frac{3\pi}{8}\right) + \cos^2\left(-\frac{\pi}{8}\right)$$

$$X = -\cos^2\left(\frac{3\pi}{8}\right) + \cos^2\left(\frac{\pi}{8}\right) = -\frac{\cos\left(\frac{6\pi}{8}\right)+1}{2} + \frac{\cos\left(\frac{\pi}{4}\right)+1}{2}$$

$$\left(\cos 2\alpha = 2\cos^2\alpha - 1\right) \qquad = \frac{+\cos\frac{\pi}{4} \cancel{+1}}{2} + \frac{\cos\left(\frac{\pi}{4}\right)\cancel{+1}}{2}$$

$\downarrow$

$$= \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

$$\frac{\cos 2\alpha + 1}{2} = \cos^2\alpha$$

$$\boxed{X = \frac{1}{\sqrt{2}}}$$

$$\cos\left(\frac{3\pi}{4}\right) = -\cos\left(\frac{\pi}{4}\right)$$

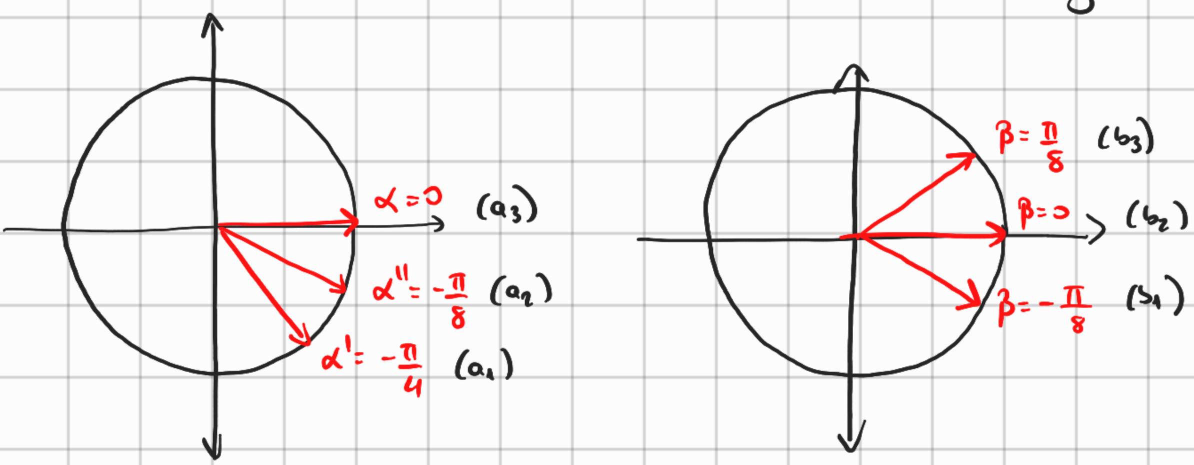
$2 \leq X \leq 2$

★ Bell State $\neq$ Statistical Mixture of $|00\rangle$ and $|11\rangle$

Exercise 3 Ekert 1991 QKD Protocol

Alice and Bob shares N Bell pairs $\frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$

Alice's Analyzer Bob's Analyzer



a) When Alice and Bob choose the same angles what can you say about the classical bits they record in their measurement. If N Bell pairs are shared how many times on average will Alice and Bob choose the same angles?

$$P(a_i = b_i) = P(a_i = 0 \;\&\; b_i = 0) + P\left(a_i = -\frac{\pi}{8} \;\&\; b_i = \frac{\pi}{8}\right)$$

Locality

$$P(a_i = 0 \;\&\; b_i = 0) = P(a_i = 0) . P(b_i = 0) = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$$

$$P\left(a_i = -\frac{\pi}{8} \;\&\; b_i = -\frac{\pi}{8}\right) = P\left(a_i = -\frac{\pi}{8}\right) . P\left(b_i = \frac{\pi}{8}\right) = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$$

$$P(a_i = b_i) = 1/9 + 1/9 = 2/9$$

	Classical bits
$\theta = 0$	00
$\theta = -\frac{\pi}{4}$	01
$\theta = -\frac{\pi}{8}$	10
$\theta = \frac{\pi}{8}$	11

When $a_i = 0$ & $b_i = 0$ → "00"

When $a_i = -\frac{\pi}{8}$ & $b_i = -\frac{\pi}{8}$ → "10"

b) Propose a scheme to generate a common string of between Alice and Bob (one-time pad)

As in the case for BB 84,

For every time n such that they used the same basis.

$$(a_3, b_2) \qquad (a_2, b_1)$$

BB84 version

$(x_i = y_i = 1)$ $\qquad a_n = b_n = 1$

or

$(x_i = y_i = -1)$ $\qquad a_n = b_n = 1$

BB84 version

Thus, they have a common subsequence of ± 1's; which they use as a shared secret key.

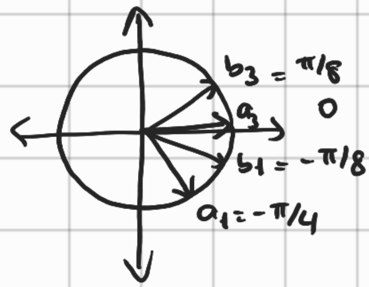
c) Security Test

Protocol :

. Alice and Bob select the time instants when the basis choices were:

(a_3, b_3), (a_3, b_1), (a_1, b_1), (a_1, b_3) $\rightarrow$ Basis for CSHS, Bell inequality

$$X_{exp} = Av(a_n(a_3) b_n(b_3)) + Av(a_n(a_3) b_n(b_1)) - Av(a_n(a_1) b_n(b_3)) + Av(a_n(a_1) b_n(b_1))$$

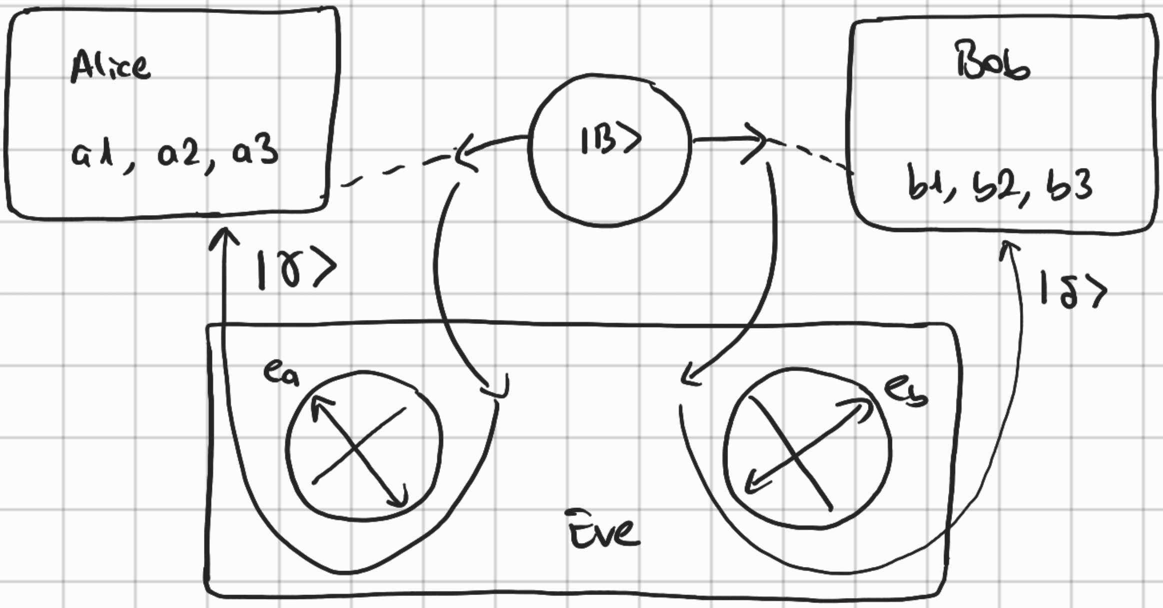


When eavesdropper is present, entanglement is destroyed.

Thus, $X_{exp} < 2$

$$\boxed{\text{Security check: } X_{exp} > 2}$$

d)



Eve collapses the pair in a tensor product state

Eve captures each photon of the EPR pair and makes measurement. Then she sends each photons to Alice and Bob. She measures Alice's photon in the basis $\{e_a, e_{a\perp}\}$ and Bob's photon in the basis $\{e_b, e_{b\perp}\}$

After measurement

γ, δ

$|e_a, e_b\rangle$, $|e_a, e_b^\perp\rangle$, $|e_a^\perp, e_b\rangle$, $|e_a^\perp, e_b^\perp\rangle$

$|\langle e_a, e_b | B_{00}\rangle|^2$ Prob, $|\langle e_a, e_b^\perp | B_{00}\rangle|^2$

$\frac{1}{2}\cos^2(e_a, e_b)$, $\frac{1}{2}\sin^2(e_a, e_b^\perp)$, $\frac{1}{2}\sin^2(e_a^\perp, e_b)$, $\frac{1}{2}\cos^2(e_a^\perp, e_b^\perp)$

$$X(e_a, e_b) = \frac{1}{2}\cos^2(e_a, e_b)\, S(e_a, e_b) + \frac{1}{2}\sin^2(e_a, e_b^\perp)\, S(e_a, e_b^\perp)$$

$$+ \frac{1}{2}\sin^2(e_a^\perp, e_b)\, S(e_a^\perp, e_b) + \frac{1}{2}\cos^2(e_a^\perp, e_b^\perp)\, S(e_a^\perp, e_b^\perp)$$

$$S(\gamma \otimes \delta) = \langle \delta \otimes \gamma \mid A(a_3) \otimes B(b_3) + A(a_3) \otimes B(b_1) - A(a_1) \otimes B(b_3)$$

$$+ A(a_1) \otimes B(b_1) \mid \gamma \otimes \delta \rangle$$

$$\underbrace{|S(\delta \otimes \gamma)|}_{\text{in product state}} \leq 2$$

$$\Downarrow$$

$$|X(e_a, e_b)| \leq 2$$