Solid state systems for quantum information, Session 3

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Exercise 1 : Quantization of radiation

1. The Hamiltonian of a single-mode electromagnetic field of frequency ω is defined as

$$\hat{\mathcal{H}} = \frac{1}{2}(\omega^2 \hat{q}^2 + \hat{p}^2), \tag{1}$$

with $\hat{q}$ and $\hat{p}$ describing the canonical "position" and "momentum" of the electromagnetic field.

   (a) Rewrite this Hamiltonian by using the annihilation operator $\hat{a}$ and creation operator $\hat{a}^\dagger$. Remember that $\hat{a} = \frac{1}{\sqrt{2\hbar\omega}}(\omega\hat{q} + i\hat{p})$ and $\left[\hat{a}, \hat{a}^\dagger\right] = 1$.

   (b) To what general system is this Hamiltonian formally equivalent? Give at least one example of another physical system (besides electromagnetic radiation), which is described by this type of Hamiltonian.

2. Quantized states of the radiation field
The quantized version of a single-mode electric field is written as

$$\hat{E}_x(z,t) = \sqrt{\frac{\hbar\omega}{\varepsilon_0 V}}(\hat{a} + \hat{a}^\dagger)\sin(kz) \tag{2}$$

   (a) Calculate the expectation value of this electric field for a photon number state $|n\rangle$, i.e., $\left\langle \hat{E}_x(z,t) \right\rangle = \langle n| \hat{E}_x(z,t) |n\rangle$. Could this expectation value describe the radiation emitted by a laser? Explain your answer.

   (b) Calculate the variance of this electric field for a photon number state $|n\rangle$, i.e., $(\Delta\hat{E}_x(z,t)) = \left\langle \hat{E}_x(z,t)^2 \right\rangle - \left\langle \hat{E}_x(z,t) \right\rangle^2$. How do you interpret the result for the case where the photon number is zero?

   (c) Calculate the expectation value of the electric field for a coherent state $|\alpha(t)\rangle$, i.e., $\left\langle \hat{E}_x(z,t) \right\rangle = \langle \alpha(t)| \hat{E}_x(z,t) |\alpha(t)\rangle$ with $|\alpha(t)\rangle = |e^{-i\omega t}\alpha(0)\rangle$. Remember the definition of a coherent state as $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$ and the adjoint relation $\langle\alpha| \hat{a}^\dagger = \langle\alpha| \alpha^\star$ and use $\alpha(0) = |\alpha(0)|e^{i\phi_0}$. Could this expectation value describe the radiation emitted by a laser? Explain your answer.

Exercise 2 : Hamiltonian of two coupled linear LC resonators

Consider two LC resonators, see Fig. 1, with respective inductance, capacitance values L_1, C_1 and L_2, C_2. These resonators are capacitively coupled through a capacitance C_0. The flux variable at the i-th independent node corresponds to ϕ_i.

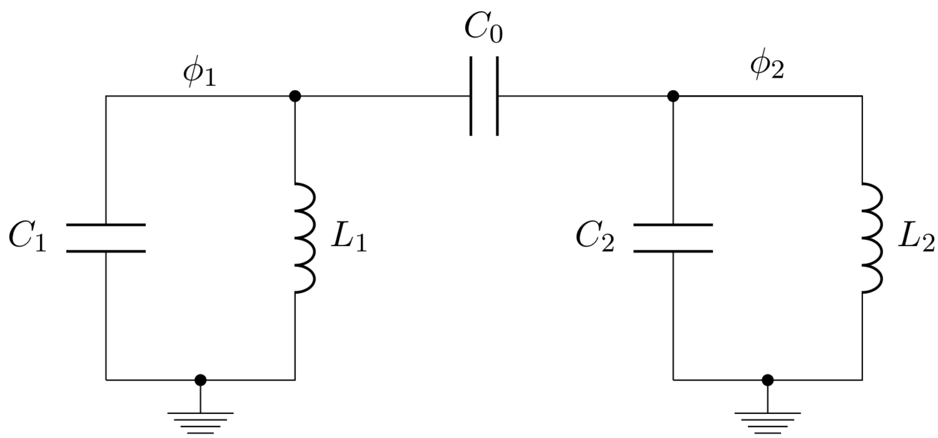

Figure 1: Circuit diagram of two capacitively coupled LC resonators.

1. Write down the Lagrangian $\mathcal{L}(\phi, \dot{\phi})$ of the system as a quadratic function of the node flux variables ϕ_i and their derivatives $\dot{\phi}_i$. Introduce the capacitance matrix $\mathbb{C}$ and the inverse of the inductance matrix $\mathbb{L}^{-1}$ and use the flux variables and their derivatives in the vector representation,

$$\vec{\phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \qquad \text{and} \qquad \dot{\vec{\phi}} = \begin{pmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \end{pmatrix}.$$

2. Perform the Legendre transformation analytically and extract the Hamiltonian H, as a function of the charge $Q_i = \partial\mathcal{L}(\phi, \dot{\phi})/\partial\dot{\phi}_i$ and the flux variables ϕ_i. Rewrite this Hamiltonian as a quadratic form, using $\mathbb{C}^{-1}$ and $\mathbb{L}^{-1}$.

   Note: A square 2×2 matrix is inverted by

$$\mathbb{A}^{-1} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^{-1} = \frac{1}{\det(\mathbb{A})} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

3. To perform quantization, first rewrite the Hamiltonian $\mathcal{H}$ in terms of both inductances L_1, L_2 and the bare (uncoupled) angular frequencies ω_i with $\omega_i^2 = \mathbb{L}_{ii}^{-1}(\mathbb{C}^{-1})_{ii}$. Summarize the capacitive coupling in a single constant $\beta = C_0/\sqrt{(C_1 + C_0)(C_2 + C_0)}$. Perform the quantization by introducing the corresponding quantum operators $\hat{Q}_i$ and $\hat{\phi}_i$ which satisfy the canonical commutation relation $[\hat{Q}_i, \hat{\phi}_i] = -i\hbar\delta_{ij}$. Subsequently, use the following definition of the charge and flux operator to write the Hamiltonian H in terms of annihilation and creation operators, $\hat{a}_i$ and $\hat{a}_i^\dagger$,

$$\hat{Q}_i = -i\sqrt{\frac{\hbar}{2L_i\omega_i}}(\hat{a}_i - \hat{a}_i^\dagger) \qquad \text{and} \qquad \hat{\phi}_i = \sqrt{\frac{\hbar L_i \omega_i}{2}}(\hat{a}_i + \hat{a}_i^\dagger)$$

   Hint: Before quantization of the Hamiltonian, you should arrive at a Hamiltonian of this shape

$$\mathcal{H} = \frac{1}{2}L_1\omega_1^2 Q_1^2 + \frac{1}{2L_1}\phi_1^2 + \frac{1}{2}L_2\omega_2^2 Q_2^2 + \frac{1}{2L_2}\phi_2^2 + \beta\sqrt{L_1 L_2}\omega_1\omega_2 Q_1 Q_2$$

For the following subproblems, we want to compare the bare mode frequencies $f_i = \omega_i/2\pi$ to the eigenmode frequencies f_i' obtained by diagonalizing the Hamiltonian $\hat{\mathcal{H}}$, which correspond to the frequencies of the coupled modes. We can determine the eigenvalues of $\hat{\mathcal{H}}$ numerically with the python package QuTip. Find the notebook "quantization coupled resonators template.ipynb" on Moodle where you can enter your Hamiltonian and perform numerical diagonalization.

4. For values of $C = C_1 = C_2 = 70\,\mathrm{fF}$ and $L_1 = 10\,\mathrm{nH}$, plot the two bare mode frequencies f_i as functions of L_2.

5. For the same values as in 4., plot the lowest two coupled mode frequencies f_i' as functions of L_2. What do you observe?

Exercise 1 : Quantization of radiation

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with $\hat{q}$ and $\hat{p}$ describing the canonical "position" and "momentum" of the electromagnetic field.

   (a) Rewrite this Hamiltonian by using the annihilation operator $\hat{a}$ and creation operator $\hat{a}^\dagger$. Remember that $\hat{a} = \frac{1}{\sqrt{2\hbar\omega}}(\omega\hat{q} + i\hat{p})$ and $[\hat{a}, \hat{a}^\dagger] = 1$.

   (b) To what general system is this Hamiltonian formally equivalent? Give at least one example of another physical system (besides electromagnetic radiation), which is described by this type of Hamiltonian.

$$\hat{a} = \frac{1}{\sqrt{2\hbar\omega}}(\omega\hat{q} + i\hat{p})$$

$$\hat{a}^+ = \frac{1}{\sqrt{2\hbar\omega}}(\omega\hat{q} - i\hat{p})$$

$$\hat{a} + \hat{a}^+ = \frac{\overset{\sqrt{2}\,\sqrt{\omega}}{\cancel{2\omega}}}{\cancel{\sqrt{2\hbar\omega}}}\hat{q} \longrightarrow \hat{q} = \sqrt{\frac{\hbar}{2\omega}}(\hat{a} + \hat{a}^+)$$

$$\hat{a} - \hat{a}^+ = \frac{\overset{\sqrt{2}}{\cancel{2}}i}{\cancel{\sqrt{2}}\hbar\omega}\hat{p} \longrightarrow \hat{p} = \frac{1}{i}\sqrt{\frac{\hbar\omega}{2}}(\hat{a} - \hat{a}^+)$$

$$\hat{q}^2 = \frac{\hbar}{2\omega}(\hat{a} + \hat{a}^+)^2$$

$$\hat{p}^2 = -\frac{\hbar\omega}{2}(\hat{a} - \hat{a}^+)^2$$

$$[\hat{a}, \hat{a}^+] = 1 \qquad aa^+ - a^+a = 1 \Rightarrow aa^+ = 1 + a^+a$$

$$\hat{H} = \frac{1}{2}\left(\omega^2 \hat{q}^2 + \hat{p}^2\right)$$

$$\hat{H} = \frac{1}{2}\left(\omega^{\cancel{2}} \frac{\hbar}{2\cancel{\omega}} (\hat{a} + \hat{a}^+)^2 + \left(-\frac{\hbar\omega}{2}\right)(\hat{a} - \hat{a}^+)^2\right)$$

$$\hat{H} = \frac{1}{2}\frac{\hbar\omega}{2}\left[(\hat{a} + \hat{a}^+)^2 - (\hat{a} - \hat{a}^+)^2 \right]$$

$$(\hat{a} + \hat{a}^+)(\hat{a} + \hat{a}^+) = \cancel{\hat{a}^2} + \hat{a}\hat{a}^+ + \hat{a}^+\hat{a} + \cancel{\hat{a}^{+2}}$$

$$-(\hat{a} - \hat{a}^+)(\hat{a} - \hat{a}^+) = -\cancel{\hat{a}^2} + \hat{a}\hat{a}^+ + \hat{a}^+\hat{a} - \cancel{\hat{a}^{+2}}$$

$+$

$$\hat{H} = \frac{1}{2}\frac{\hbar\omega}{2}\left[2\underbrace{\hat{a}\hat{a}^+}_{(1 + a^+a)} + 2a^+\hat{a} \right]$$

$$\hat{H} = \frac{\hbar\omega}{2}\left[(a^+a + 1) + a^+a \right] = \hbar\omega\left[\underbrace{a^+a}_{\hat{n}} + \frac{1}{2} \right]$$

$$\hat{H} = \hbar\omega\left(\hat{n} + \frac{1}{2}\right)$$

b) This Hamiltonian is the same as that for a simple harmonic oscillator (mechanical pendulum)

(Raising) $\hat{a}^+$
(lowering) $\hat{a}$ $\Big\} \rightarrow$ ladder operators

2. Quantized states of the radiation field

The quantized version of a single-mode electric field is written as

$$\hat{E}_x(z,t) = \sqrt{\frac{\hbar\omega}{\varepsilon_0 V}}(\hat{a} + \hat{a}^\dagger)\sin(kz) \tag{2}$$

(a) Calculate the expectation value of this electric field for a photon number state $|n\rangle$, i.e., $\left\langle \hat{E}_x(z,t) \right\rangle = \langle n | \hat{E}_x(z,t) | n \rangle$. Could this expectation value describe the radiation emitted by a laser? Explain your answer.

(b) Calculate the variance of this electric field for a photon number state $|n\rangle$, i.e., $(\Delta \hat{E}_x(z,t)) = \left\langle \hat{E}_x(z,t)^2 \right\rangle - \left\langle \hat{E}_x(z,t) \right\rangle^2$. How do you interpret the result for the case where the photon number is zero? vacuum state

(c) Calculate the expectation value of the electric field for a coherent state $|\alpha(t)\rangle$, i.e., $\left\langle \hat{E}_x(z,t) \right\rangle = \langle \alpha(t) | \hat{E}_x(z,t) | \alpha(t) \rangle$ with $|\alpha(t)\rangle = |e^{-i\omega t}\alpha(0)\rangle$. Remember the definition of a coherent state as $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$ and the adjoint relation $\langle\alpha|\hat{a}^\dagger = \langle\alpha|\alpha^\star$ and use $\alpha(0) = |\alpha(0)|e^{i\phi_0}$. Could this expectation value describe the radiation emitted by a laser? Explain your answer.

a) $\langle n | \hat{E}_x(z,t) | n \rangle =$

$$\sqrt{\frac{\hbar\omega}{\varepsilon_0 V}} \sin(kz) \underbrace{(\hat{a} + \hat{a}^+)|n\rangle}$$

$a|n\rangle = \sqrt{n}\,|n-1\rangle$

$a^+|n\rangle = \sqrt{n+1}\,|n+1\rangle$

$$\sqrt{\frac{\hbar\omega}{\varepsilon_0 V}} \sin(kz) \left[\langle n | \sqrt{n} \cancel{|n-1\rangle} + \langle n | \sqrt{n+1} \cancel{|n+1\rangle} \right] = 0$$

This means that the average electric field for a Fock state is zero, and therefore it's not possible to describe a coherent state with it.

?

b) $\Delta E_x(z,t) = \langle E_x(z,t)^2 \rangle - \cancel{\langle E_x(z,t) \rangle^2} \rightarrow 0$ from part a

$$E_x(z,t)^2 = \frac{\hbar\omega}{\varepsilon_0 V} \sin^2(kz) \underbrace{(\hat{a}+\hat{a}^+)^2}_{\hat{a}\hat{a} + \hat{a}\hat{a}^+ + \hat{a}^+\hat{a} + \hat{a}^+\hat{a}^+}$$

$$\left(\hat{a}\hat{a} + \hat{a}\hat{a}^+ + \hat{a}^+\hat{a} + \hat{a}^+\hat{a}^+\right)|n\rangle$$

$$\hat{a}\hat{a}|n\rangle = \hat{a}\sqrt{n}\,|n-1\rangle = \sqrt{n(n-1)}\,|n-2\rangle$$

$$\hat{a}\hat{a}^+|n\rangle = \hat{a}\sqrt{n+1}\,|n+1\rangle = (n+1)\,|n\rangle$$

$$\hat{a}^+\hat{a}|n\rangle = \hat{a}^+\sqrt{n}\,|n-1\rangle = n\,|n\rangle$$

$$\hat{a}^+\hat{a}^+|n\rangle = \hat{a}^+\sqrt{n+1}\,|n+1\rangle = \sqrt{(n+1)(n+2)}\,|n+2\rangle$$

$$\Delta \hat{E}_x(z,t) = \langle n|\hat{E}_x(z,t)^2|n\rangle = \frac{\hbar\omega}{\varepsilon_0 V}\sin^2(kz)\left[\underbrace{n+1+n}_{2n+1}\right]$$

For $|n\rangle = |0\rangle$ (vacuum state), variance $= \frac{\hbar\omega}{\varepsilon_0 V}\sin^2(kz)$

represents the vacuum fluctuation or the Heisenberg uncertainty principle.

(c) Calculate the expectation value of the electric field for a coherent state $|\alpha(t)\rangle$, i.e., $\left\langle \hat{E}_x(z,t) \right\rangle = \langle\alpha(t)|\, \hat{E}_x(z,t)\, |\alpha(t)\rangle$ with $|\alpha(t)\rangle = |e^{-i\omega t}\alpha(0)\rangle$. Remember the definition of a coherent state as $\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle$ and the adjoint relation $\langle\alpha|\,\hat{a}^\dagger = \langle\alpha|\,\alpha^\star$ and use $\alpha(0) = |\alpha(0)|e^{i\phi_0}$. Could this expectation value describe the radiation emitted by a laser? Explain your answer.

$$|\alpha(t)\rangle = |e^{-i\omega t}\alpha(0)\rangle$$

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$$

$$\langle\alpha|a^+ = \langle\alpha|\alpha^*$$

$$\alpha(0) = |a(0)|e^{i\phi_0}$$

$$\langle \hat{E}x(z,t)\rangle = \sqrt{\frac{\hbar\omega}{\varepsilon_0 V}}\sin(kz)\langle\alpha(t)|(\hat{a}+\hat{a}^+)|\alpha(t)\rangle$$

$$\langle\alpha(t)|\hat{a}|\alpha(t)\rangle = \langle\alpha(t)|\alpha|\alpha(t)\rangle = \alpha(t)$$

$$\langle\alpha(t)|\hat{a}^+|\alpha(t)\rangle = \langle\alpha(t)|\alpha^*|\alpha(t)\rangle = \alpha^*(t)$$

$$\langle \hat{E}x(z,t)\rangle = \sqrt{\frac{\hbar\omega}{\varepsilon_0 V}}\sin(kz)\left(\alpha(t) + \alpha^*(t)\right)$$

$$\alpha(t) \rightarrow e^{-i\omega t}\alpha(0) \rightarrow e^{-i(\omega t-\phi_0)}|a(0)|$$

$$\alpha^*(t) \rightarrow e^{+i\omega t}\alpha(0) \rightarrow e^{i(\omega t-\phi_0)}|a(0)|$$

$$\rightarrow 2\cos(\omega t-\phi_0)$$

$$\langle \hat{E}x(z,t)\rangle = 2|\alpha(0)|\sqrt{\frac{\hbar}{\varepsilon_0 V}}\sin(kz)\cos(\omega t-\phi_0)$$

* The average electric field evolve as a coherent travelling wave. This is the same field produced by a laser. i.e. "a coherent state"

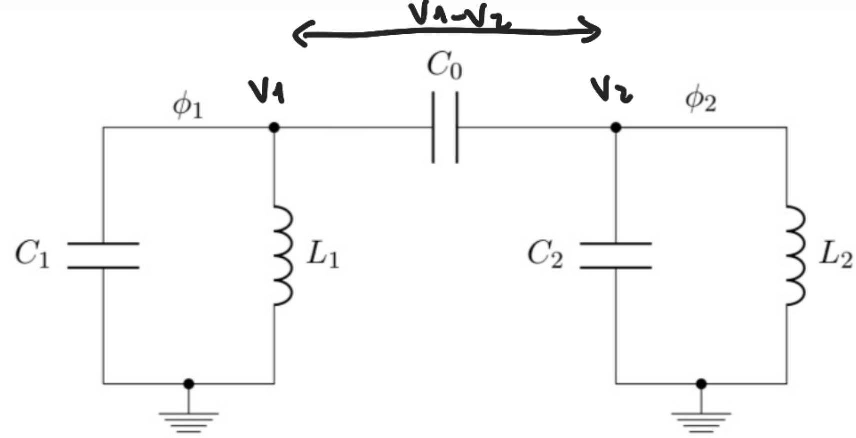

Figure 1: Circuit diagram of two capacitively coupled LC resonators.

1. Write down the Lagrangian $\mathcal{L}(\phi, \dot{\phi})$ of the system as a quadratic function of the node flux variables ϕ_i and their derivatives $\dot{\phi}_i$. Introduce the capacitance matrix $\mathbb{C}$ and the inverse of the inductance matrix $\mathbb{L}^{-1}$ and use the flux variables and their derivatives in the vector representation,

$$\vec{\phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \qquad \text{and} \qquad \vec{\dot{\phi}} = \begin{pmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \end{pmatrix}.$$

$$\mathcal{L}(\phi, \dot{\phi}) = E_{kin} - E_{pot}$$

$$\mathcal{L}(\phi, \dot{\phi}) = \left(\frac{1}{2} C_0 (\underset{\substack{\downarrow \\ V_1}}{\dot{\phi}_1} - \underset{\substack{\downarrow \\ V_2}}{\dot{\phi}_2})^2 + \frac{C_1 \dot{\phi}_1^2}{2} + \frac{C_2 \dot{\phi}_2^2}{2} - \frac{\Phi_1^2}{2L_1} - \frac{\Phi_2^2}{2L_2} \right)$$

$$= \frac{1}{2} (\dot{\phi}_1 \quad \dot{\phi}_2) \, C \begin{pmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \end{pmatrix} - \frac{1}{2} (\phi_1 \quad \phi_2) \, L^{-1} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

$$= \frac{1}{2} \dot{\phi}^T C \dot{\phi} - \frac{1}{2} \phi^T L^{-1} \phi$$

$$C_0 \dot{\phi}_1^2 - C_0 2 \dot{\phi}_1 \dot{\phi}_2 + C_0 \dot{\phi}_2^2 + C_1 \dot{\phi}_1^2 + C_2 \dot{\phi}_2^2$$

$\dot{\phi}_1^2 (C_0 + C_1)$

$\dot{\phi}_2^2 (C_0 + C_2)$

$\dot{\phi}_1 \dot{\phi}_2 (-2C_0)$

$$C = \begin{pmatrix} C_0 + C_1 & -C_0 \\ -C_0 & C_0 + C_2 \end{pmatrix}$$

$\frac{1}{L_1} \Phi_1^2 \qquad \frac{1}{L_2} \Phi_2^2$

$$L^{-1} = \begin{pmatrix} 1/L_1 & 0 \\ 0 & 1/L_2 \end{pmatrix}$$

2. Perform the Legendre transformation analytically and extract the Hamiltonian H, as a function of the charge $Q_i = \partial\mathcal{L}(\phi,\dot\phi)/\partial\dot\phi_i$ and the flux variables ϕ_i. Rewrite this Hamiltonian as a quadratic form, using $\mathbb{C}^{-1}$ and $\mathbb{L}^{-1}$.

Note: A square 2×2 matrix is inverted by

$$\mathbb{A}^{-1} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^{-1} = \frac{1}{\det(\mathbb{A})}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

$$\mathcal{L}(\phi,\dot\phi) = \frac{1}{2}\dot\phi^T C\dot\phi - \frac{1}{2}\phi^T L^{-1}\phi$$

$$Q_i = \frac{\partial\mathcal{L}(\phi,\dot\phi)}{\partial\dot\phi} = \sum_{j=1}^{2}\mathbb{C}_{ij}\dot\phi_j = C_{ij}\dot\phi_j$$

Einstein's summation convention

$$Q_1 = \underbrace{\mathbb{C}_{11}\dot\Phi_1}_{(C_0+C_1)V_1} + \underbrace{\mathbb{C}_{12}\dot\Phi_2}_{(-C_0)V_2} = C_1V_1 + C_0(V_1-V_2)$$

$$Q_2 = \underbrace{\mathbb{C}_{21}\dot\Phi_1}_{(-C_0)V_1} + \underbrace{C_{22}\dot\Phi_2}_{(C_0+C_2)V_2} = C_2V_2 + C_0(V_2-V_1)$$

$$\boxed{\vec{\dot\phi} = C^{-1}\vec{Q}}$$

$$H = \sum_{i=1}^{2} Q_i\underbrace{\dot\Phi_i}_{C^{-1}Q} - \mathcal{L}(\phi,\dot\phi) = \vec{Q}^T C^{-1}\vec{Q} - \left(\frac{1}{2}\vec{Q}^T C^{-1}\vec{Q}^T - \frac{1}{2}\vec{\Phi}^T L^{-1}\vec{\Phi}\right)$$

$$\boxed{H = \frac{1}{2}\vec{Q}^T C^{-1}\vec{Q} + \frac{1}{2}\vec{\Phi}^T L^{-1}\vec{\Phi}}$$

$$C^{-1} = \frac{1}{C_1C_2 + C_0(C_1+C_2)}\begin{pmatrix} C_0+C_2 & C_0 \\ C_0 & C_0+C_1 \end{pmatrix} \qquad L^{-1} = \begin{pmatrix} 1/L_1 & 0 \\ 0 & 1/L_2 \end{pmatrix}$$

3. To perform quantization, first rewrite the Hamiltonian $\mathcal{H}$ in terms of both inductances L_1, L_2 and the bare (uncoupled) angular frequencies ω_i with $\omega_i^2 = \mathbb{L}_{ii}^{-1}(\mathbb{C}^{-1})_{ii}$. Summarize the capacitive coupling in a single constant $\beta = C_0/\sqrt{(C_1+C_0)(C_2+C_0)}$. Perform the quantization by introducing the corresponding quantum operators $\hat{Q}_i$ and $\hat{\phi}_i$ which satisfy the canonical commutation relation $[\hat{Q}_i, \hat{\phi}_i] = -i\hbar\delta_{ij}$. Subsequently, use the following definition of the charge and flux operator to write the Hamiltonian H in terms of annihilation and creation operators, $\hat{a}_i$ and $\hat{a}_i^\dagger$,

$$\hat{Q}_i = -i\sqrt{\frac{\hbar}{2L_i\omega_i}}(\hat{a}_i - \hat{a}_i^\dagger) \qquad \text{and} \qquad \hat{\phi}_i = \sqrt{\frac{\hbar L_i\omega_i}{2}}(\hat{a}_i + \hat{a}_i^\dagger)$$

Hint: Before quantization of the Hamiltonian, you should arrive at a Hamiltonian of this shape

$$\mathcal{H} = \frac{1}{2}L_1\omega_1^2Q_1^2 + \frac{1}{2L_1}\phi_1^2 + \frac{1}{2}L_2\omega_2^2Q_2^2 + \frac{1}{2L_2}\phi_2^2 + \beta\sqrt{L_1L_2}\omega_1\omega_2Q_1Q_2$$

$$\sqrt{\omega_1^2} = \sqrt{L_{11}^{-1}C_{11}^{-1}} = \sqrt{\frac{1}{L_1}\cdot\frac{C_0+C_2}{(C_1C_2+C_0(C_1+C_2))}}$$

$$\sqrt{\omega_2^2} = \sqrt{L_{22}^{-1}C_{22}^{-1}} = \sqrt{\frac{1}{L_2}\cdot\frac{C_1+C_0}{C_1C_2+C_0C_1+C_0C_2}}$$

$$C^{-1} = \begin{pmatrix} L_1\omega_1^2 & \beta\sqrt{L_1L_2}\,\omega_1\omega_2 \\ \beta L_1L_2\omega_1\omega_2 & L_2\omega_2^2 \end{pmatrix}$$

$$H = \frac{1}{2}\vec{Q}^T C^{-1} Q + \frac{1}{2}\Phi^T L^{-1}\Phi$$

$$H = \frac{1}{2}L_1\omega_1^2\hat{Q}_1^2 + \frac{1}{2}L_2\omega_2^2\hat{Q}_2^2 + \frac{1}{2}\cdot 2\cdot\beta\sqrt{L_1L_2}\,\omega_1\omega_2\hat{Q}_1\hat{Q}_2$$

$$+\frac{1}{2L_1}\hat{\Phi}_1^2 + \frac{1}{2L_2}\hat{\Phi}_2^2$$

$$\hat{Q}_i = \sqrt{\frac{\hbar}{2 Z_{c_i}}}\,(\hat{a}_i - \hat{a}_i^{+}) \qquad Z_c = \sqrt{LC}$$

$$\hat{\Phi}_i = i\sqrt{\frac{\hbar Z_{c_i}}{2}}\,(\hat{a}_i^{+} + \hat{a}_i)$$

$$\hat{Q}_1^2 = \frac{\hbar C_1}{2 L_1}\left(a_1^{+}a_1^{+} - a_1^{+}a_1 - a_1 a_1^{+} + a_1 a_1\right)$$

$$\hat{Q}_2^2 = \frac{\hbar C_2}{2 L_2}\left(a_2^{+}a_2^{+} - a_2^{+}a_2 - a_2 a_2^{+} + a_2 a_2\right)$$

$$\hat{Q}_1\hat{Q}_2 = \frac{\hbar\sqrt{C_1 C_2}}{2\sqrt{L_1 L_2}}\,\underbrace{(a_1 - a_1^{+})(a_2 - a_2^{+})}$$

$$\Phi_1^2 = -\frac{\hbar L_1}{2C_1}\,\underbrace{(a_1^{+} + a_1)}^2 = -\frac{\hbar L_1}{2C_1}\left(a_1^{+}a_1^{+} + a_1^{+}a_1 + a_1 a_1^{+} + a_1 a_1\right)$$

look at it later