Annals of Physics 387 (2017) 121–134

Contents lists available at ScienceDirect

Annals of Physics

journal homepage: www.elsevier.com/locate/aop

Two-mode generalization of the Jaynes–Cummings and Anti-Jaynes–Cummings models

E. Choreño [a], D. Ojeda-Guillén [b,*], M. Salazar-Ramírez [b], V.D. Granados [a]

[a] *Escuela Superior de Física y Matemáticas, Instituto Politécnico Nacional, Ed. 9, Unidad Profesional Adolfo López Mateos, Delegación Gustavo A. Madero, C.P. 07738, Ciudad de México, Mexico*

[b] *Escuela Superior de Cómputo, Instituto Politécnico Nacional, Av. Juan de Dios Bátiz esq. Av. Miguel Othón de Mendizábal, Col. Lindavista, Delegación Gustavo A. Madero, C.P. 07738, Ciudad de México, Mexico*

ARTICLE INFO

Article history:
Received 19 April 2017
Accepted 7 October 2017
Available online 16 October 2017

Keywords:
Coherent state
Jaynes–Cummings model
Lie algebra

ABSTRACT

We introduce two generalizations of the Jaynes–Cummings (JC) model for two modes of oscillation. The first model is formed by two Jaynes–Cummings interactions, while the second model is written as a simultaneous Jaynes–Cummings and Anti-Jaynes–Cummings (AJC) interactions. We study some of its properties and obtain the energy spectrum and eigenfunctions of these models by using the tilting transformation and the Perelomov number coherent states of the two-dimensional harmonic oscillator. Moreover, as physical applications, we connect these new models with two important and novel problems: The relativistic non-degenerate parametric amplifier and the relativistic problem of two coupled oscillators.

1. Introduction

The Dirac oscillator was introduced as an instance of a relativistic wave equation such that its non-relativistic limit leads to the well-known Schrödinger equation for the harmonic oscillator. This relativistic oscillator was introduced for the first time by Ito et al. [1] and Cook [2], and reintroduced later by Moshinsky and Szczepaniak [3]. They added the linear term $-imc\omega\beta\alpha \cdot \mathbf{r}$ to the relativistic momentum $\mathbf{p}$ of the free-particle Dirac equation.

* Corresponding author.
E-mail addresses: quike_choreno@hotmail.com (E. Choreño), dojedag@ipn.mx (D. Ojeda-Guillén).

https://doi.org/10.1016/j.aop.2017.10.009

The Dirac–Moshinsky oscillator has been extensively studied in the last thirty years. The exact solution, symmetry algebra and other important properties of the Dirac–Moshinsky oscillator in two and tree dimensions have been studied in references [4–8]. More recently, the Dirac–Moshinsky oscillator has been studied in a non-commutative space, in the presence of an external magnetic field and it has been used to study the graphene in two dimensions. These results are reported in references [9–16].

On the other hand, the Dirac–Moshinsky oscillator also has been applied, among other problems, to quark confinement models in quantum chromodynamics, hexagonal lattices, and the emulation of graphene in electromagnetic billiards [17]. Moreover, the $1+1$ Dirac–Moshinsky oscillator has been mapped to the Anti-Jaynes–Cummings model by using the usual creation and annihilation operators [18–21]. The exact solution of this problem was obtained by using the theory of the non-relativistic harmonic oscillator. Similarly, the $2+1$-dimensional Dirac–Moshinsky oscillator can be related to quantum optics via the Jaynes–Cummings and Anti-Jaynes–Cummings models by using the chiral creation and annihilation operators [22,23]. This connection between the Dirac–Moshinsky oscillator and the Jaynes–Cummings model have been successfully driven even if we consider a non-commutative space [24] and a q-deformed scenario [25].

The Jaynes–Cummings model [26] plays an important role in quantum optics, since it describes the atom-field interaction. This is a fully quantized model and yet analytically solvable. It describes the interaction between a two-level atom and a quantized field. The exact solution of this model has been found by using the rotating wave approximation in the Hamiltonian of the atom-field interaction [27]. These solutions yield quantum collapse and revival of atomic inversion [28], squeezing of the radiation field [29], among other quantum effects. All these effects have been corroborated experimentally, as can be seen in references [30–32]. Also, the Jaynes–Cummings model has been studied in the Bargmann–Segal representation [33] and in terms of field quadrature operators, instead of the typically used boson ladder operators [34].

However, two of the more interesting applications of the Jaynes–Cummings model have been appeared recently in the relativistic treatment for trapped ions and in circuit quantum electrodynamics technology. In references [35–37], the authors made simulations of Dirac equation using different physical systems. These simulations allow to study relevant quantum relativistic effects, like the simulation of Unruh effect in trapped ions, Zitterbewegung and Klein's paradox in controllable physical systems. In reference [38], an extensive overview of current theoretical proposals and experiments for such quantum simulations with trapped ions is given. In quantum information science, there have been enormous efforts in the design of devices with a high level of quantum control and coherence. One of these devices is the circuit QED, which performance is approaching that of trapped ions and all optical implementations. For the circuit-QED architectures, the Jaynes–Cummings model has been used to describe the implement of ultrafast two-qubit gates valid for the ultrastrong coupling and deep strong coupling regimes of light–matter interaction [39]. The entanglement and interference effects of photons in different circuit-QED systems has been analyzed in references [40–42].

Therefore, it has been shown that the Jaynes–Cummings model is of vital importance in the study of relativistic quantum simulations. In this sense, in reference [21] it has been introduced a generalization of the Jaynes–Cummings model in terms of the creation and annihilation operators of the one-dimensional harmonic oscillator. For a particular choice of the model parameters, the generalized Jaynes–Cummings model was connected with two problems: the relativistic degenerate parametric amplifier and the Hamiltonian of the quantum simulation of a single trapped ion. The aim of the present work is to introduce two generalizations of the Jaynes–Cummings model with two modes of oscillation. We relate these new models with the relativistic non-degenerate parametric amplifier and the relativistic problem of two coupled oscillators.

This work is organized as follows. In Section 2, we review the main properties of $2+1$ Dirac oscillator and its exact mapping onto the Jaynes–Cummings model. In Sections 3 and 4, we introduce two generalizations of the Jaynes–Cummings model with two modes of oscillation. We use the Jordan–Schwinger realizations of the $su(1,1)$ and $su(2)$ Lie algebra to obtain the energy spectrum and eigenfunctions of these new JC-type models. We connect these models with the non-degenerate parametric amplifier and the problem of two coupled oscillators in the non-relativistic limit. Finally, we give some concluding remarks.

2. The 2 + 1 Dirac oscillator and Jaynes–Cummings model

The time-independent Dirac equation for the Dirac–Moshinsky oscillator is given by the Hamiltonian

$$H_D\Psi = \left[c\alpha\cdot(\mathbf{p} - im\omega\mathbf{r}\beta) + mc^2\beta\right]\Psi = E\Psi, \tag{1}$$

where α and β are usual Dirac matrices, m is the rest mass of the particle and c is the speed of light. In the non-relativistic limit, this system reduces to a quantum simple harmonic oscillator (with mass m and angular frequency ω) with a strong spin–orbit coupling [3]. The matrices α and β must obey a Clifford algebra given by the anticommutation relations

$$\alpha_j\alpha_k + \alpha_k\alpha_j = 2\delta_{jk}I, \qquad \alpha_j\beta + \beta\alpha_j = 0. \tag{2}$$

In 2 + 1 dimensions, the solution to the Clifford algebra is given by the 2×2 Pauli matrices $\alpha_x = \sigma_x, \alpha_y = \sigma_y$ and $\beta = \sigma_z$. In this case, the eigenfunction $|\Psi\rangle$ is described by a two-component spinor, where these components satisfy the coupled equations

$$\left(E - mc^2\right)|\Psi_1\rangle = c\left[(p_x + im\omega x) - i(p_y + im\omega y)\right]|\Psi_2\rangle, \tag{3}$$

$$\left(E + mc^2\right)|\Psi_2\rangle = c\left[(p_x - im\omega x) - i(p_y - im\omega y)\right]|\Psi_1\rangle. \tag{4}$$

The solutions of this problem can be found by introducing the chiral creation and annihilation operators (written in terms of the usual creation and annihilation operators $a_x, a_x^\dagger$ and $a_y, a_y^\dagger$)

$$a_r := \frac{1}{\sqrt{2}}\left(a_x - ia_y\right), \qquad a_r^\dagger := \frac{1}{\sqrt{2}}\left(a_x^\dagger + ia_y^\dagger\right), \tag{5}$$

$$a_l := \frac{1}{\sqrt{2}}\left(a_x + ia_y\right), \qquad a_l^\dagger := \frac{1}{\sqrt{2}}\left(a_x^\dagger - ia_y^\dagger\right). \tag{6}$$

These operators create a right or left quantum of angular momentum, respectively, and can be used to write the coupled equations (3) and (4) as [22]

$$|\Psi_1\rangle = i\frac{2mc^2\sqrt{\xi}}{E - mc^2}a_l^\dagger|\Psi_2\rangle, \qquad |\Psi_2\rangle = -i\frac{2mc^2\sqrt{\xi}}{E + mc^2}a_l|\Psi_1\rangle, \tag{7}$$

where the parameter $\xi = \hbar\omega/mc^2$ controls the non-relativistic limit.

The uncoupled equations for the two components $|\Psi_1\rangle$ and $|\Psi_2\rangle$ result in

$$(E^2 - m^2c^4)|\Psi_1\rangle = 4m^2c^4\xi a_l^\dagger a_l|\Psi_1\rangle, \tag{8}$$

$$(E^2 - m^2c^4)|\Psi_2\rangle = 4m^2c^4\xi(1 + a_l^\dagger a_l)|\Psi_2\rangle. \tag{9}$$

Thus, the energy spectrum in terms of the quantum number n_l is [22]

$$E_{n_l} = \pm mc^2\sqrt{1 + 4\xi n_l}. \tag{10}$$

The 2+1 Dirac–Moshinsky oscillator can be mapped onto the Jaynes–Cummings model of quantum optics as [22]

$$\begin{aligned} H &= 2imc^2\sqrt{\xi}(a_l^\dagger|\Psi_2\rangle\langle\Psi_1| - a_l|\Psi_1\rangle\langle\Psi_2|) + mc^2\sigma_z \\ &= \hbar(g\sigma_- a_l^\dagger + g^*\sigma_+ a_l) + mc^2\sigma_z, \end{aligned} \tag{11}$$

where $g = 2imc^2\sqrt{\xi}/\hbar$ is the coupling strength between orbital and spin degrees of freedom, and $\sigma_\pm$ are the spin raising and lowering operators. Formally, Eq. (11) is the Hamiltonian of the Jaynes–Cummings model of the quantum optics. Similarly, the 2 + 1-dimensional Dirac–Moshinsky oscillator can be mapped onto the Anti-Jaynes–Cummings model by substituting $\omega \longrightarrow -\omega$ into Eq. (1). This change leads to regard the excited state as the ground state, with a simultaneous change of sign in the detuning.

3. The two-mode Jaynes–Cummings–Anti-Jaynes–Cummings model

In order to connect the JC and AJC models with more general problems, we can introduce two modes of oscillation a and b to these models as follows [21]

$$H_{JC} = \hbar(g\sigma_- a^\dagger + g^*\sigma_+ a) + mc^2\sigma_z, \tag{12}$$

$$H_{AJC} = \hbar(f\sigma_- b + f^*\sigma_+ b^\dagger) + mc^2\sigma_z. \tag{13}$$

With these two definitions we can propose a generalized JC–AJC model as a linear combination of Eqs. (12) and (13) as follows

$$H = \hbar\left[\sigma_-(ga^\dagger + fb) + \sigma_+(g^*a + f^*b^\dagger)\right] + mc^2\sigma_z, \tag{14}$$

where g and f are two general complex parameters. This model will be connected later with a particular system by choosing suitably these complex parameters.

The coupled equations for the spinor components $|\Psi_1\rangle$ and $|\Psi_2\rangle$ are

$$\hbar(ga^\dagger + fb)|\Psi_2\rangle = (E - mc^2)|\Psi_1\rangle, \tag{15}$$

$$\hbar(g^*a + f^*b^\dagger|\Psi_1\rangle = (E + mc^2)|\Psi_2\rangle. \tag{16}$$

Uncoupling the above equations we found that the equations for $|\Psi_1\rangle$ and $|\Psi_2\rangle$ result to be

$$\hbar^2(|g|^2a^\dagger a + |f|^2b^\dagger b + fg^*ab + f^*ga^\dagger b^\dagger + |f|^2)|\Psi_1\rangle = (E^2 - m^2c^4)|\Psi_1\rangle, \tag{17}$$

$$\hbar^2(|g|^2a^\dagger a + |f|^2b^\dagger b + fg^*ab + f^*ga^\dagger b^\dagger)|\Psi_2\rangle = (E^2 - m^2c^4)|\Psi_2\rangle. \tag{18}$$

Since these equations have the same mathematical structure, we will focus only on the equation for $|\Psi_1\rangle$. Then, the uncoupled equation for $|\Psi_1\rangle$ can be written as

$$\hbar^2\left[\frac{1}{2}\left(|g|^2 + |f|^2\right)(a^\dagger a + b^\dagger b + 1) + fg^*ab + f^*ga^\dagger b^\dagger + \frac{1}{2}\left(|f|^2 - |g|^2\right)(b^\dagger b - a^\dagger a + 1)\right]|\Psi_1\rangle$$
$$= (E^2 - m^2c^4)|\Psi_1\rangle. \tag{19}$$

Thus, by using the Jordan–Schwinger realization of the $su(1,1)$ algebra (see Eq. (79) of Appendix, the above equation for $|\Psi_1\rangle$ becomes

$$\hbar^2\left[\left(|g|^2 + |f|^2\right)K_0 + fg^*K_- + f^*gK_+ + \frac{1}{2}\left(|f|^2 - |g|^2\right)(b^\dagger b - a^\dagger a + 1)\right]|\Psi_1\rangle$$
$$= (E^2 - m^2c^4)|\Psi_1\rangle. \tag{20}$$

The ladder operators $K_\pm$ can be removed in the Klein–Gordon-type Hamiltonian $H_{KG}|\Psi_1\rangle = \left(E^2 - m^2c^4\right)|\Psi_1\rangle$ by using the tilting transformation with the $SU(1,1)$ displacement operator $D(\xi)$. Therefore, if we apply the tilting transformation to both sides of Eq. (20) and we proceed as in references [43–45], we obtain

$$D^\dagger(\xi)H_{KG}D(\xi)D^\dagger(\xi)|\Psi_1\rangle = (E^2 - m^2c^4)D^\dagger(\xi)|\Psi_1\rangle, \tag{21}$$

$$H'|\Psi_1'\rangle = (E^2 - m^2c^4)|\Psi_1'\rangle. \tag{22}$$

Notice that in these expressions $H' = D^\dagger(\xi)H_{KG}D(\xi)$ is the tilted Hamiltonian and $|\Psi_1'\rangle$ its wave function. Thus, by using Eqs. (76)–(78) of Appendix, the tilted Hamiltonian can be written as

$$H' = \hbar^2\left[\left((|f|^2 + |g|^2)(2\beta + 1) + \frac{f^*g\xi^*\alpha}{|\xi|} + \frac{fg^*\xi\alpha}{|\xi|}\right)K_0\right.$$
$$+ \left(\frac{(|f|^2 + |g|^2)\alpha\xi}{2|\xi|} + \frac{fg^*\beta\xi}{\xi^*} + gf^*(\beta + 1)\right)K_+$$

$$+\left(\frac{(|f|^2+|g|^2)\alpha\xi^*}{2|\xi|}+fg^*(\beta+1)+\frac{f^*g\beta\xi^*}{\xi}\right)K_-\Bigg]$$
$$+\frac{\hbar^2}{2}(|f|^2-|g|^2)(b^\dagger b-a^\dagger a+1). \tag{23}$$

In this expression, the term $N_d+1=b^\dagger b-a^\dagger a+1$ remains unchanged under the tilting transformation, since it commutes with the $K_\pm$ operators (see Appendix). By choosing the coherent state parameters θ and φ as [21]

$$\theta=\tanh^{-1}\left(\frac{2|f||g|}{|f|^2+|g|^2}\right),\qquad \varphi=i\ln\left[\frac{(|f|^2+|g|^2)\alpha}{2fg^*(2\beta+1)}\right], \tag{24}$$

the tilted Hamiltonian of Eq. (23) is reduced to

$$H'=\hbar^2\left(\frac{1}{2}(|f|^2-|g|^2)(N_d+1)+\sqrt{(|g|^2+|f|^2)^2-4|g|^2|f|^2}K_0\right). \tag{25}$$

Let us now look the eigenfunctions $|\Psi_1'\rangle = D(\xi)^\dagger|\Psi_1\rangle$ of H'. Since the operator K_0 is the Hamiltonian of the two-dimensional harmonic oscillator and commutes with N_d+1, we have that the eigenfunctions of H' are given by

$$\psi'_{n_l,m_n}(\rho,\phi)=\frac{1}{\sqrt{\pi}}e^{im_n\phi}(-1)^{n_l}\sqrt{\frac{2(n_l)!}{(n_l+m_n)!}}\rho^{m_n}L_{n_l}^{m_n}(\rho^2)e^{-1/2\rho^2}, \tag{26}$$

where n_l is the left chiral quantum number.

From the action of the operators a, $a^\dagger$, b and $b^\dagger$ on the basis $|n,m_n\rangle$, we have

$$K_0|n,m_n\rangle=\left(n_l+\frac{m_n}{2}+\frac{1}{2}\right)|n,m_n\rangle,$$
$$N_d|n,m_n\rangle=(b^\dagger b-a^\dagger a+1)|n,m_n\rangle=-(m_n-1)|n,m_n\rangle. \tag{27}$$

In this $SU(1,1)$ representation, the group numbers n, k are related with the physical numbers n_l, m_n as $n=n_l$ and $k=\frac{1}{2}(m_n+1)$ [44]. Thus, from these results we can obtain that the energy spectrum of the generalized JC–AJC model is

$$E=\pm\sqrt{\hbar^2\left(\sqrt{(|g|^2+|f|^2)^2-4|g|^2|f|^2}\left(n_l+\frac{m_n}{2}+\frac{1}{2}\right)-\frac{1}{2}(|f|^2-|g|^2)(m_n-1)\right)+m^2c^4}. \tag{28}$$

Analogously, if we apply the same procedure to the uncoupled equation for the other spinor component $|\Psi_2\rangle$ we obtain

$$E=\pm\sqrt{\hbar^2\left(\sqrt{(|g|^2+|f|^2)^2-4|g|^2|f|^2}\left(n_l'+\frac{m_n'}{2}+\frac{1}{2}\right)-\frac{1}{2}(|f|^2-|g|^2)(m_n'+1)\right)+m^2c^4}. \tag{29}$$

Since both energies belong to the same solution, the energies must be the same. This leads to the fact that the spinor components $|\Psi_1\rangle$ and $|\Psi_2\rangle$ satisfy the relationships $n_l'\Rightarrow n_l+1$ and $m_n'\Rightarrow m_n-2$. It is worthwhile to mention that the energy spectrum (28) can also be rewritten as

$$E_{n_l}=\pm mc^2\sqrt{1+\frac{2\hbar^2}{m^2c^4}(|f|^2-|g|^2)(n_l+1)}. \tag{30}$$

The JC–AJC model of Eq. (14) can be connected with the standard $1+1$ Dirac–Moshinsky oscillator if we set $g=0$ and $f=\sqrt{\omega mc^2/\hbar}$. With this definition of the model parameters f and g the energy spectrum of Eq. (30) matches with those presented in references [19–21].

The eigenfunctions of the generalized JC–AJC model are obtained from the relationship $|\Psi_1\rangle = D(\xi)|\Psi_1'\rangle$. Thus, by using the Eqs. (26) and (74) of Appendix, we find that the action of the displacement operator $D(\xi)$ on $|\Psi_1'\rangle$ are the Perelomov number coherent states for the two-dimensional harmonic oscillator [44]

$$\Psi^{(1)}_{\zeta,k,n_l} = \langle\rho,\varphi|\zeta,k,n_l\rangle = \frac{(-1)^{n_l}}{\sqrt{\pi}} e^{i(l-1/2)\phi} \sum_{s=0}^{\infty} \frac{\zeta^s}{s!} \sum_{j=0}^{n_l} \frac{(-\zeta^*)^j}{j!} e^{\eta(k+n_l-j)}$$
$$\times \frac{\sqrt{2\Gamma(n_l+1)\Gamma\left(n_l+l+\frac{1}{2}\right)}}{\Gamma\left(n_l-j+l+\frac{1}{2}\right)}$$
$$\times \frac{\Gamma(n_l-j+s+1)}{\Gamma(n_l-j+1)} e^{-\rho^2/2}\rho^{l-1/2} L^{l-1/2}_{n_l-j+s}(\rho^2), \tag{31}$$

with $l = m_n + \frac{1}{2}$. The above expression can also be rewritten as

$$\psi^{(1)}_{n_l,m_n} = \sqrt{\frac{2\Gamma(n_l+1)}{\Gamma(n_l+m_n+1)}} \frac{(-1)^{n_l}}{\sqrt{\pi}} e^{im_n\phi} \frac{(-\zeta^*)^{n_l}(1-|\zeta|^2)^{\frac{m_n}{2}+\frac{1}{2}}(1+\sigma)^{n_l}}{(1-\zeta)^{m_n+1}}$$
$$\times e^{-\frac{\rho^2(\zeta+1)}{2(1-\zeta)}} \rho^{m_n} L^{m_m}_{n_l}\left(\frac{\rho^2\sigma}{(1-\zeta)(1-\sigma)}\right), \tag{32}$$

where we have used $m_n = l - \frac{1}{2}$ and defined σ as

$$\sigma = \frac{1-|\zeta|^2}{(1-\zeta)(-\zeta^*)}. \tag{33}$$

A similar result is obtained for the other spinor component $|\Psi_2\rangle$. Hence, we are able to construct the normalized spinor $|\Psi\rangle$ for the generalized JC–AJC model

$$|\Psi_{n_l,m_n}\rangle = \begin{pmatrix} \sqrt{\dfrac{E\pm mc^2}{2E}}\psi^{(1)}_{n_l,m_n} \\ \mp i\sqrt{\dfrac{E\mp mc^2}{2E}}\psi^{(2)}_{n_l+1,m_n-2} \end{pmatrix}. \tag{34}$$

Therefore, we have introduced a generalized JC–AJC model with two modes of oscillation. This new model was solved by using the tilting transformation and the $SU(1,1)$ Perelomov number coherent states for the two-dimensional harmonic oscillator.

3.1. Special case: the relativistic non-degenerate parametric amplifier

Now, we shall use the theory developed in Section 3 to give a connection between the generalized JC–AJC model and the relativistic non-degenerate parametric amplifier. Thus, for convenience we set the complex parameters f and g of the JC–AJC model as [21]

$$g = i\sqrt{\frac{2mc^2\omega_1}{\hbar}}e^{-i\omega}, \qquad f = \sqrt{\frac{2mc^2\omega_2}{\hbar}}e^{i\omega}. \tag{35}$$

If we substitute these parameters into the uncoupled equation (19) for the component $|\Psi_1\rangle$ we obtain

$$H|\Psi_1\rangle = 2mc^2\hbar\left[\frac{1}{2}(\omega_1+\omega_2)(a^\dagger a + b^\dagger b + 1) + 2i\sqrt{\omega_1\omega_2}\left(a^\dagger b^\dagger e^{-2i\omega} - abe^{2i\omega}\right)\right.$$
$$\left. + \frac{1}{2}(\omega_2-\omega_1)(b^\dagger b - a^\dagger a + 1)\right]|\Psi_1\rangle = (E^2 - m^2c^4)|\Psi_1\rangle. \tag{36}$$

A similar equation holds for the component $|\Psi_2\rangle$. The tilted Hamiltonian of Eq. (25) is now given by

$$H'|\Psi_1'\rangle = \hbar mc^2\left[(\omega_2-\omega_1)(b^\dagger b - a^\dagger a + 1) + 2\sqrt{(\omega_1+\omega_2)^2 - 4\omega_1\omega_2 K_0}\right]|\Psi_1'\rangle$$
$$= (E^2 - m^2c^4)|\Psi_1'\rangle. \tag{37}$$

The energy spectrum of this problem for f and g defined as in Eq. (35) is obtained from the expression (28)

$$E = \pm\sqrt{\hbar mc^2\left(4\sqrt{(\omega_1+\omega_2)^2 - 4\omega_1\omega_2}\left(n_l + \frac{m}{2} + \frac{1}{2}\right) - (\omega_2-\omega_1)(m-1)\right) + m^2c^4}, \tag{38}$$

with the corresponding eigenfunctions given by Eqs. (32) and (34).

If we write the energy as $E = mc^2 + \varepsilon$ in (36), we obtain that in the non-relativistic limit ($\epsilon \ll mc^2$) the uncoupled equation for $|\Psi_1\rangle$ becomes

$$H|\Psi_1\rangle = \left[\hbar\omega_1 a^\dagger a + \hbar\omega_2 b^\dagger b + i\hbar\sqrt{\omega_1\omega_2}(a^\dagger b^\dagger e^{-2i\omega} - abe^{2i\omega})\right]|\Psi_1\rangle = \varepsilon|\Psi_1\rangle. \tag{39}$$

If we identify the coupling constant χ with the term $\sqrt{\omega_1\omega_2}$, we obtain the time-independent Hamiltonian of the non-degenerate parametric amplifier [46]. Therefore, if we set the complex parameters f and g as those given by Eq. (35), our generalized JC–AJC model is reduced, in the non-relativistic limit ($\epsilon \ll mc^2$), to the time-independent Hamiltonian of the non-degenerate parametric amplifier. Moreover, the energy spectrum of Eq. (38), in the non-relativistic limit, is in full agreement with the energy spectrum of the non-degenerate parametric amplifier [44]. Also, it is important to note that the definition of the parameters f and g of Eq. (35) is not unique, since they can be introduced in other form and the connection holds.

4. The two-mode Jaynes–Cummings model

In a similar way to the model proposed in Section 3, we can consider two Jaynes–Cummings models H_{JC_1} and H_{JC_2} with two different modes of oscillation

$$H_{JC_1} = \hbar(g\sigma_- a^\dagger + g^*\sigma_+ a) + mc^2\sigma_z, \tag{40}$$

and

$$H_{JC_2} = \hbar(f\sigma_- b^\dagger + f^*\sigma_+ b) + mc^2\sigma_z. \tag{41}$$

Therefore, we can propose a generalization of the Jaynes–Cummings model as follows

$$H = \hbar\left[\sigma_-(ga^\dagger + fb^\dagger) + \sigma_+(g^*a + f^*b)\right] + mc^2\sigma_z, \tag{42}$$

where g and f are again two general complex parameters.

The coupled equations for the spinor components $|\Psi_1\rangle$ and $|\Psi_2\rangle$ are

$$\hbar(ga^\dagger + fb^\dagger)|\Psi_2\rangle = (E - mc^2)|\Psi_1\rangle, \tag{43}$$

$$\hbar(g^*a + f^*b|\Psi_1\rangle = (E + mc^2)|\Psi_2\rangle. \tag{44}$$

Hence, the uncoupled equations for the spinor components are easily obtained from above expressions, which result to be

$$\hbar^2(|g|^2aa^\dagger + |f|^2bb^\dagger + g^*fab^\dagger + f^*gba^\dagger)|\Psi_2\rangle = (E^2 - m^2c^4)|\Psi_2\rangle, \tag{45}$$

$$\hbar^2(|g|^2a^\dagger a + |f|^2b^\dagger b + fg^*b^\dagger a + f^*ga^\dagger b)|\Psi_1\rangle = (E^2 - m^2c^4)|\Psi_1\rangle. \tag{46}$$

In this manner, by using the Jordan–Schwinger realization of the $su(2)$ Lie algebra (see Eq. (95) of Appendix, in addition to the number operator $N_s = a^\dagger a + b^\dagger b$, we can write the uncoupled equation for $|\Psi_1\rangle$ as

$$\hbar^2\left[(|g|^2-|f|^2)J_0+f^*gJ_++fg^*J_-+\frac{1}{2}(|f|^2+|g|^2)N_s\right]|\Psi_1\rangle=(E^2-m^2c^4)|\Psi_1\rangle. \tag{47}$$

Now, we can apply the tilting transformation to the above eigenvalue equation, in order to remove the ladder operators $J_\pm$

$$\begin{aligned}&\hbar^2D^\dagger(\xi)\left[(|g|^2-|f|^2)J_0+f^*gJ_++fg^*J_-+\frac{1}{2}(|f|^2+|g|^2)N_s\right]D(\xi)D^\dagger(\xi)|\Psi_1\rangle\\&=(E^2-m^2c^4)D^\dagger(\xi)|\Psi_1\rangle,\end{aligned} \tag{48}$$

where $D(\xi)$ is the $SU(2)$ displacement operator and $\xi=-\frac{1}{2}\theta e^{-i\varphi}$ (see Appendix. If we define the tilted Hamiltonian $H'=D^\dagger(\xi)HD(\xi)$ and the wave function $|\Psi_1'\rangle=D^\dagger(\xi)|\Psi_1\rangle$, this equation can be written as $H'|\Psi_1'\rangle=(E^2-m^2c^4)|\Psi_1'\rangle$. Moreover, since the operator N_s commutes with J_+ and J_-, it remains unchanged under the tilting transformation. Therefore, we find that the tilted Hamiltonian results to be

$$\begin{aligned}H'=&\hbar^2\left[\left((|g|^2-|f|^2)(2\epsilon+1)-\frac{f^*g\xi^*\delta}{|\xi|}-\frac{fg^*\xi\delta}{|\xi|}\right)J_0\right.\\&+\left(\frac{(|g|^2-|f|^2)\delta\xi^*}{2|\xi|}+\frac{f^*g\epsilon\xi^*}{\xi}+fg^*(\epsilon+1)\right)J_-\\&\left.+\left(\frac{(|g|^2-|f|^2)\delta\xi}{2|\xi|}+f^*g(\epsilon+1)+\frac{fg^*\epsilon\xi}{\xi^*}\right)J_+\right]\\&+\frac{\hbar^2}{2}(|f|^2+|g|^2)N_s.\end{aligned} \tag{49}$$

By choosing the coherent state parameters θ and φ as

$$\theta=\arctan\left(\frac{2|f||g|}{|f|^2-|g|^2}\right),\qquad \varphi=i\ln\left[\frac{(|f|^2-|g|^2)\delta}{2fg(2\epsilon+1)}\right], \tag{50}$$

the tilted Hamiltonian H' (49) is now written as

$$\begin{aligned}H'=|\Psi_1'\rangle&=\hbar^2\left(\frac{1}{2}(|f|^2+|g|^2)N_s+\sqrt{(|g|^2-|f|^2)^2+4|g|^2|f|^2}J_0\right)|\Psi_1'\rangle\\&=(E^2-m^2c^4)|\Psi_1'\rangle.\end{aligned} \tag{51}$$

Since the operators J_0 and N_s commute, they share common eigenfunctions. Thus, by using that N_s is the Hamiltonian of the two-dimensional harmonic oscillator, the eigenfunctions of the tilted Hamiltonian H' are

$$\psi'_{n_l,m_n}(\rho,\phi)=\frac{1}{\sqrt{\pi}}e^{im_n\phi}(-1)^{n_l}\sqrt{\frac{2(n_l)!}{(n_l+m_n)!}}\rho^{m_n}L_{n_l}^{m_n}(\rho^2)e^{-1/2\rho^2}. \tag{52}$$

From the action of the operators a, $a^\dagger$, b and $b^\dagger$ on the basis $|n,m_n\rangle$, we have

$$\begin{aligned}J_0|n,m_n\rangle&=\frac{m_n}{2}|n,m_n\rangle,\\N_s|n,m_n\rangle&=(b^\dagger b+a^\dagger a)|n,m_n\rangle=(2n_l+m_n)|n,m_n\rangle.\end{aligned} \tag{53}$$

In this $SU(2)$ representation, the group numbers j, μ are related with the physical numbers n_l, m_n as $j=n_l+\frac{m_n}{2}$ and $\mu=\frac{m_n}{2}$ [45]. Thus, the energy spectrum of the generalized Jaynes–Cummings model is

$$E=\pm\sqrt{\hbar^2\left((|f|^2+|g|^2)\left(n_l+\frac{m_n}{2}\right)\pm\frac{1}{2}\sqrt{(|g|^2-|f|^2)^2+4|g|^2|f|^2}m_n\right)+m^2c^4}. \tag{54}$$

If we apply the same procedure to the uncoupled equation for the other spinor component $|\Psi_2\rangle$ we obtain the energy spectrum

$$E = \pm\sqrt{\hbar^2 \left((|f|^2 + |g|^2)(n_l' + \frac{m_n'}{2} + 1) \pm \frac{1}{2}\sqrt{(|g|^2 - |f|^2)^2 + 4|g|^2|f|^2 m_n'} \right) + m^2c^4}. \tag{55}$$

Since both energies belong to the same eigenstate, this leads to take $n_l' \Rightarrow n_l - 1$ in the spinor component $|\Psi_2\rangle$.

The energy spectrum of Eq. (54) can also be written as

$$E = \pm mc^2 \sqrt{\frac{\hbar^2}{2m^2c^4} \left(|f|^2 + |g|^2\right)(N \pm m_n) + 1}, \tag{56}$$

where N is the eigenvalue of the operator N_s. It is important to note that the JC–JC model of Eq. (42) can be connected with the $1+1$ Dirac–Moshinsky oscillator if we set the model parameters as $g = 0$ and $f = \sqrt{2mc^2\omega/\hbar}$. Similarly, the connection with the $2+1$ Dirac–Moshinsky oscillator can be established if we set $f = g = \sqrt{2mc^2\omega/\hbar}$.

The eigenfunctions of the generalized Jaynes–Cummings model are obtained from the relationship $|\Psi_1\rangle = D(\xi)|\Psi_1'\rangle$. Thus, by using Eq. (52) we find that the action of the displacement operator $D(\xi)$ on $|\Psi_1'\rangle$ is

$$\begin{aligned}\Psi^{(1)}_{n_l,m_n,\zeta} = \langle \rho, \varphi | \zeta, n_l, m_n\rangle = & \frac{e^{-\frac{\rho^2}{2}}}{\sqrt{\pi}} \sum_{s=0}^{n_l+n} \frac{\zeta^s}{s!} \sum_{n=0}^{n_l+m_n} \frac{(-\zeta^*)^n}{n!} e^{\frac{\eta}{2}(m_n-2n)} e^{i(m_n-2n+2s)\varphi}(-1)^{n_l+n-s} \\ & \times \frac{\Gamma(n_l+n+1)}{\Gamma(n_l+m_n-n+1)} \left[\frac{2\Gamma(n_l+m_n+1)}{\Gamma(n_l+1)}\right]^{1/2} \\ & \times \rho^{(m_n-2n+2s)} L^{(m_n-2n+2s)}_{n_l+n-s}(\rho^2).\end{aligned} \tag{57}$$

A similar result holds for the other spinor component $\Psi^{(2)}_{n_l,m_n,\zeta}$. Thus, we are able to construct the normalized spinor $|\Psi\rangle$ in the form

$$|\Psi_{n_l,m_n,\zeta}\rangle = \begin{pmatrix} \sqrt{\frac{E \pm mc^2}{2E}}\Psi^{(1)}_{n_l,m_n,\zeta} \\ \mp i\sqrt{\frac{E \mp mc^2}{2E}}\Psi^{(2)}_{n_l-1,m_n,\zeta} \end{pmatrix}. \tag{58}$$

In this way, we have introduced a generalized JC–JC model with two modes of oscillation. As in the case of the JC–AJC model, the solution was obtained by using the tilting transformation and the $SU(2)$ Perelomov number coherent states for the two-dimensional harmonic oscillator.

4.1. Special case: the relativistic problem of two coupled oscillators

In this section, we shall give a physical application of the theory developed in Section 4 related to the two-mode generalization of the Jaynes–Cummings model. In particular, we shall give the connection between this generalized Jaynes–Cummings model and the relativistic problem of two coupled oscillators.

A particular case of the Hamiltonian proposed in Eq. (42) is obtained if we set the complex parameters g and f as

$$g = \sqrt{\frac{2mc^2\omega_1}{\hbar}} e^{-i\phi}, \qquad f = \sqrt{\frac{2mc^2\omega_2}{\hbar}} e^{i\phi}. \tag{59}$$

Thus, the uncoupled equation for the upper component $|\Psi_1\rangle$ takes the form (see Eq. (46))

$$H|\Psi_1\rangle = (E^2 - m^2c^4)|\Psi_1\rangle, \tag{60}$$

where the Hamiltonian written in terms of the Jordan–Schwinger realization of the $su(2)$ Lie algebra of Eq. (95) is

$$H = \hbar\left[2mc^2\left(\omega_1 - \omega_2\right)J_0 + 2mc^2\sqrt{\omega_1\omega_2}(J_+ + J_-) + mc^2\left(\omega_1 + \omega_2\right)N_s\right]. \tag{61}$$

A similar equation holds for the lower component $|\Psi_2\rangle$. The transformations performed to the general case (Eqs. (47)–(51)), lead us to obtain the following tilted Hamiltonian H'

$$H'|\Psi_1'\rangle = 2mc^2\hbar\left[\left(\omega_1 + \omega_2\right)N_s + \sqrt{(\omega_1 - \omega_2)^2 + 4\omega_1\omega_2}J_0\right]|\Psi_1'\rangle = (E^2 - m^2c^4)|\Psi_1'\rangle. \tag{62}$$

The exact energy spectrum for g and f defined as in Eq. (59) is obtained from Eq. (54)

$$E = \pm mc^2\sqrt{\hbar\left(2(\omega_1 + \omega_2)j + \sqrt{(\omega_1 - \omega_2)^2 + 4\omega_1\omega_2}\mu\right) + 1}, \tag{63}$$

and the corresponding eigenfunctions are given by Eq. (58).

By writing $E = mc^2 + \epsilon$, we obtain that the non-relativistic limit (NR) ($\epsilon \ll mc^2$) of the uncoupled equation (60) becomes

$$H_{NR}|\Psi_1\rangle = \hbar\left[\omega_1 a^\dagger a + \omega_2 b^\dagger b + \sqrt{\omega_1\omega_2}(b^\dagger a + a^\dagger b)\right]|\Psi_1\rangle. \tag{64}$$

Formally, this is the Hamiltonian of the time-independent problem of two coupled oscillators [47,48], where we have identified the term $\sqrt{\omega_1\omega_2}$ with the coupling constant λ of the oscillators. Moreover, the relativistic energy spectrum of Eq. (63), in the non-relativistic limit is in full agreement with the energy spectrum of this problem [45]. Therefore, the generalized Jaynes–Cummings model of Eq. (42) can be considered a relativistic version of the two coupled oscillators, with a particular choice of the complex parameters f and g.

5. Concluding remarks

We introduced two generalizations of the Jaynes–Cummings and Anti-Jaynes–Cummings models with two modes of oscillation. In the first generalization, we considered a linear combination of the JC and AJC models and showed that this system possesses the $SU(1, 1)$ symmetry. By using the Jordan–Schwinger realization of this algebra, the tilting transformation and the $SU(1, 1)$ number coherent states of the two-dimensional harmonic oscillator, we obtained the energy spectrum and eigenfunctions of this new model. As a physical application we connected this generalized JC–AJC model, in the non-relativistic limit, with the non-degenerate parametric amplifier.

For the second generalization, we considered a linear combination of two JC models and showed that this system possesses the $SU(2)$ symmetry. By using the appropriate Jordan–Schwinger realization and the tilting transformation, we were able to obtain the energy spectrum. Also, it has been shown that the eigenfunctions of this model are the $SU(2)$ number coherent states of the two-dimensional harmonic oscillator. In the non-relativistic limit, we connected this model with the problem of two coupled oscillators.

The relevance of the JC–AJC model studied in this work has been proven, since this model was used to describe the $2 + 1$ Dirac oscillator (with frequency ω) in the presence of a constant magnetic field B with vector potential $A = \frac{B}{2}[x, -y, 0]$ [23].

Acknowledgments

The authors thank the referee's suggestions to improve our paper. This work was partially supported by SNI-México, COFAA-IPN, EDI-IPN, EDD-IPN, SIP-IPN project number 20170632.

Appendix

A.1. $SU(1,1)$ Perelomov number coherent states

Three operators $K_\pm, K_0$ close the $su(1,1)$ Lie algebra if they satisfy the commutation relations [49]

$$[K_0, K_\pm] = \pm K_\pm, \qquad [K_-, K_+] = 2K_0. \tag{65}$$

The Casimir operator K^2 for any irreducible representation of this group is given by

$$K^2 = K_0^2 - \frac{1}{2}\left(K_+K_- + K_-K_+\right). \tag{66}$$

The action of these four operators on the Fock space states $\{|k,n\rangle, n = 0, 1, 2, \ldots\}$ is

$$K_+|k,n\rangle = \sqrt{(n+1)(2k+n)}|k,n+1\rangle, \tag{67}$$

$$K_-|k,n\rangle = \sqrt{n(2k+n-1)}|k,n-1\rangle, \tag{68}$$

$$K_0|k,n\rangle = (k+n)|k,n\rangle, \tag{69}$$

$$K^2|k,n\rangle = k(k-1)|k,n\rangle. \tag{70}$$

Thus, a representation of the $su(1,1)$ algebra is determined by the number k, called the Bargmann index. The discrete series are those for which $k > 0$.

The displacement operator $D(\xi)$ is defined in terms of the creation and annihilation operators K_+, K_- as

$$D(\xi) = \exp(\xi K_+ - \xi^* K_-), \tag{71}$$

where $\xi = -\frac{1}{2}\tau e^{-i\varphi}$, $-\infty < \tau < \infty$ and $0 \leq \varphi \leq 2\pi$. The so-called normal form of the displacement operator is given by

$$D(\xi) = \exp(\zeta K_+)\exp(\eta K_0)\exp(-\zeta^* K_-), \tag{72}$$

where $\zeta = -\tanh(\frac{1}{2}\tau)e^{-i\varphi}$ and $\eta = -2\ln\cosh|\xi| = \ln(1-|\zeta|^2)$ [50].

The $SU(1,1)$ Perelomov coherent states are defined as the action of the displacement operator $D(\xi)$ onto the lowest normalized state $|k,0\rangle$ as [51]

$$|\zeta\rangle = D(\xi)|k,0\rangle = (1-|\zeta|^2)^k \sum_{n=0}^{\infty} \sqrt{\frac{\Gamma(n+2k)}{n!\Gamma(2k)}}\zeta^n|k,n\rangle. \tag{73}$$

The $SU(1,1)$ Perelomov number coherent state $|\zeta,k,n\rangle$ is defined as the action of the displacement operator $D(\xi)$ onto an arbitrary excited state $|k,n\rangle$ [44,52]

$$|\zeta,k,n\rangle = \sum_{s=0}^{\infty}\frac{\zeta^s}{s!}\sum_{j=0}^{n}\frac{(-\zeta^*)^j}{j!}e^{\eta(k+n-j)}\frac{\sqrt{\Gamma(2k+n)\Gamma(2k+n-j+s)}}{\Gamma(2k+n-j)} \times \frac{\sqrt{\Gamma(n+1)\Gamma(n-j+s+1)}}{\Gamma(n-j+1)}|k,n-j+s\rangle. \tag{74}$$

The similarity transformations $D^\dagger(\xi)K_+D(\xi)$, $D^\dagger(\xi)K_-D(\xi)$, and $D^\dagger(\xi)K_0D(\xi)$ of the $su(1,1)$ Lie algebra generators are computed by using the displacement operator $D(\xi)$ and the Baker–Campbell–Hausdorff identity

$$e^A B e^{-A} = B + [B,A] + \frac{1}{2!}[[B,A],A] + \frac{1}{3!}[[[B,A]A]A] + \cdots. \tag{75}$$

These results explicitly are

$$D^\dagger(\xi)K_+D(\xi) = \frac{\xi^*}{|\xi|}\alpha K_0 + \beta\left(K_+ + \frac{\xi^*}{\xi}K_-\right) + K_+, \tag{76}$$

$$D^\dagger(\xi)K_-D(\xi) = \frac{\xi}{|\xi|}\alpha K_0 + \beta\left(K_- + \frac{\xi}{\xi^*}K_+\right) + K_-, \tag{77}$$

$$D^\dagger(\xi)K_0D(\xi) = (2\beta+1)K_0 + \frac{\alpha\xi}{2|\xi|}K_+ + \frac{\alpha\xi^*}{2|\xi|}K_-, \tag{78}$$

where $\alpha = \sinh(2|\xi|)$ and $\beta = \frac{1}{2}\left[\cosh(2|\xi|) - 1\right]$.

A particular realization of the $su(1,1)$ Lie algebra is given by the Jordan–Schwinger operators

$$K_0 = \frac{1}{2}\left(a^\dagger a + b^\dagger b + 1\right), \quad K_+ = a^\dagger b^\dagger, \quad K_- = ba, \tag{79}$$

where the two sets of operators $(a, a^\dagger)$ and $(b, b^\dagger)$ satisfy the bosonic algebra

$$[a, a^\dagger] = [b, b^\dagger] = 1, \qquad [a, b^\dagger] = [a, b] = 0. \tag{80}$$

If N_d is the difference of the number operators of the two oscillators, then N_d commutes with all the generators of this algebra and the Casimir operator is given by [53]

$$K^2 = \frac{1}{4}N_d^2 - \frac{1}{4}, \qquad N_d = b^\dagger b - a^\dagger a,$$

$$[N_d, K_0] = [N_d, K_+] = [N_d, K_-] = 0. \tag{81}$$

A.2. $SU(2)$ Perelomov number coherent states

The $su(2)$ Lie algebra is spanned by the generators J_+, J_- and J_0, which satisfy the commutation relations [49]

$$[J_0, J_\pm] = \pm J_\pm, \qquad [J_+, J_-] = 2J_0. \tag{82}$$

The Casimir operator J^2 in this representation is defined as

$$J^2 = J_0^2 + \frac{1}{2}\left(J_+J_- + J_-J_+\right). \tag{83}$$

Dicke space states (angular momentum states)
$\{|j,\mu\rangle, -j \le \mu \le j\}$ is

$$J_+|j,\mu\rangle = \sqrt{(j-\mu)(j+\mu+1)}|j,\mu+1\rangle, \tag{84}$$

$$J_-|j,\mu\rangle = \sqrt{(j+\mu)(j-\mu+1)}|j,\mu-1\rangle, \tag{85}$$

$$J_0|j,\mu\rangle = \mu|j,\mu\rangle, \tag{86}$$

$$J^2|j,\mu\rangle = j(j+1)|j,\mu\rangle. \tag{87}$$

The displacement operator $D(\xi)$ for this Lie algebra is given by

$$D(\xi) = \exp(\xi J_+ - \xi^* J_-), \tag{88}$$

where $\xi = -\frac{1}{2}\theta e^{-i\varphi}$. By means of Gaussian decomposition we can obtain the normal form of this operator

$$D(\xi) = \exp(\zeta J_+)\exp(\eta J_0)\exp(-\zeta^* J_-), \tag{89}$$

where $\zeta = -\tanh(\frac{1}{2}\theta)e^{-i\varphi}$ and $\eta = -2\ln\cosh|\xi| = \ln(1-|\zeta|^2)$.

The $SU(2)$ Perelomov coherent states $|\zeta\rangle = D(\xi)|j,-j\rangle$ are defined as [51,54]

$$|\zeta\rangle = \sum_{\mu=-j}^{j}\sqrt{\frac{(2j)!}{(j+\mu)!(j-\mu)!}}(1+|\zeta|^2)^{-j}\zeta^{j+\mu}|j,\mu\rangle. \tag{90}$$

Hence, we can define the $SU(2)$ Perelomov number coherent states $|\zeta, j, \mu\rangle$ as the action of the displacement operator $D(\xi)$ onto an arbitrary excited state $|j,\mu\rangle$ [45,52]

$$\begin{aligned} |\zeta, j, \mu\rangle = &\sum_{s=0}^{j-\mu+n}\frac{\zeta^s}{s!}\sum_{n=0}^{\mu+j}\frac{(-\zeta^*)^n}{n!}e^{\eta(\mu-n)}\frac{\Gamma(j-\mu+n+1)}{\Gamma(j+\mu-n+1)} \\ &\times\left[\frac{\Gamma(j+\mu+1)\Gamma(j+\mu-n+s+1)}{\Gamma(j-\mu+1)\Gamma(j-\mu+n-s+1)}\right]^{\frac{1}{2}}|j,\mu-n+s\rangle. \end{aligned} \tag{91}$$

The similarity transformations $D^\dagger(\xi)J_+D(\xi)$, $D^\dagger(\xi)J_-D(\xi)$, and $D^\dagger(\xi)J_0D(\xi)$ of the $su(2)$ Lie algebra generators are computed by using the displacement operator $D(\xi)$ and the Baker–Campbell–Hausdorff identity

$$D^\dagger(\xi)J_+D(\xi) = -\frac{\xi^*}{|\xi|}\delta J_0 + \epsilon\left(J_+ + \frac{\xi^*}{\xi}J_-\right) + J_+, \tag{92}$$

$$D^\dagger(\xi)J_-D(\xi) = -\frac{\xi}{|\xi|}\delta J_0 + \epsilon\left(J_- + \frac{\xi}{\xi^*}J_+\right) + J_-, \tag{93}$$

$$D^\dagger(\xi)J_0D(\xi) = (2\epsilon+1)J_0 + \frac{\delta\xi}{2|\xi|}J_+ + \frac{\delta\xi^*}{2|\xi|}J_-, \tag{94}$$

where $\delta = \sin(2|\xi|)$ and $\epsilon = \frac{1}{2}[\cos(2|\xi|)-1]$.

The Jordan–Schwinger realization of the $su(2)$ algebra is

$$J_0 = \frac{1}{2}\left(a^\dagger a - b^\dagger b\right), \quad J_+ = a^\dagger b, \quad J_- = b^\dagger a, \tag{95}$$

where again, the two sets of operators $(a, a^\dagger)$ and $(b, b^\dagger)$ satisfy the bosonic algebra. It is important to note that, besides the Casimir operator, there is another operator N_s (called the number operator) which commutes with all the generators of the $su(2)$ algebra. The Casimir operator J^2 for this realization is written in terms of this operator N_s and is given by [53]

$$\begin{aligned} &J^2 = \frac{N_s}{2}\left(\frac{N_s}{2}+1\right), \qquad N_s = a^\dagger a + b^\dagger b, \\ &[N_s, J_+] = [N_s, J_-] = [N_s, J_z] = 0. \end{aligned} \tag{96}$$

References

[1] D. Ito, K. Mori, E. Carrieri, Nuovo Cimento A 51 (1967) 1119.
[2] P.A. Cook, Lett. Nuovo Cimento 1 (1971) 419.
[3] M. Moshinsky, A. Szczepaniak, J. Phys. A 22 (1989) L817.
[4] M. Moreno, A. Zentella, J. Phys. A: Math. Gen. 22 (1989) L821.
[5] C. Quesne, M. Moshinsky, J. Phys. A 23 (1990) 2263.
[6] J. Benítez, R.P. Martínez y Romero, H.N. Núñez Yépez, A.L. Salas-Brito, Phys. Rev. Lett. 64 (1990) 1643.

[7] R.P. Martínez y Romero, A.L. Salas-Brito, J. Math. Phys. 33 (1992) 1831.
[8] V.M. Villalba, Phys. Rev. A 49 (1994) 586.
[9] A. Boumali, H. Hassanabadi, Eur. Phys. J. Plus. 128 (2013) 124.
[10] C. Quimbay, P. Strange, Quantum phase transition in the chirality of the (2+1)-dimensional Dirac oscillator. arXiv preprint arXiv:1312.5251, (2013).
[11] C. Quimbay, P. Strange, Graphene physics via the Dirac oscillator in (2 + 1) dimensions. arXiv preprint arXiv:1311.2021, (2013).
[12] M.H. Pacheco, R.V. Maluf, C.A.S. Almeida, R.R. Landim, Europhys. Lett. 108 (2014) 10005.
[13] O. Panella, P. Roy, Phys. Rev. A 90 (2014) 042111.
[14] L. Menculini, O. Panella, P. Roy, Phys. Rev. D 91 (2015) 045032.
[15] D. Ojeda-Guillén, R.D. Mota, V.D. Granados, Commun. Theor. Phys. 63 (2015) 271.
[16] A. Boumali, Phys. Scr. 90 (2015) 109501.
[17] E. Sadurni, AIP Conf. Proc. 1334 (2010) 249.
[18] Y. Nogami, F.M. Toyama, Can. J. Phys. 74 (1996) 114.
[19] F.M. Toyama, Y. Nogami, F.A.B. Coutinho, J. Phys. A: Math. Gen. 30 (1997) 2585.
[20] R. Szmytkowski, M. Gruchowski, J. Phys. A: Math. Gen. 34 (2001) 4991.
[21] D. Ojeda-Guillén, R.D. Mota, V.D. Granados, J. Math. Phys. 57 (2016) 062104.
[22] A. Bermudez, M.A. Martin-Delgado, E. Solano, Phys. Rev. A 76 (2007) 041801.
[23] A. Bermudez, M.A. Martin-Delgado, A. Luis, Phys Rev. A 77 (2008) 063815.
[24] Y. Luo, Y. Cui, Z. Long, J. Jing, Int. J. Theor. Phys. 50 (2011) 2992.
[25] P. Majari, A. Luis, M.R. Setare, Mapping of the 2 + 1 q-deformed Dirac oscillator onto the q-deformed Jaynes–Cummings model: It's non-relativistic limit and Zitterbewegung effect. arXiv preprint arXiv:1702.03077, (2017).
[26] E.T. Jaynes, F.W. Cummings, Proc. IEEE 51 (1963) 89.
[27] S. Haroche, J.M. Raimond, Exploring the Quantum: Atoms, Cavities and Photons, Oxford University Press, Oxford, 2007.
[28] J.H. Eberly, N.B. Narozhny, J.J. Sanchez-Mondragon, Phys. Rev. Lett. 44 (1980) 1323.
[29] J.R. Kuklinski, J. Madajczyk, Phys. Rev. A 37 (1988) 3175.
[30] P. Goy, J.M. Raimond, M. Gross, S. Haroche, Phys. Rev. Lett. 50 (1983) 1903.
[31] M. Brune, F. Schmidt-Kaler, A. Maali, J. Dreyer, E. Hagley, J.M. Raimond, S. Haroche, Phys. Rev. Lett. 76 (1996) 1800.
[32] C. Guerlin, J. Bernu, S. Deleglise, C. Sayrin, S. Gleyzes, S. Kuhr, M. Brune, J.M. Raimond, S. Haroche, Nature 448 (2007) 889.
[33] H. Panahi, S. Asghari Rad, Int. J. Theor. Phys. 52 (2013) 4068.
[34] J. Larson, Phys. Scr. 76 (2007) 146.
[35] L. Lamata, J. León, T. Schätz, E. Solano, Phys. Rev. Lett. 98 (2007) 253005.
[36] R. Gerritsma, G. Kirchmair, F. Zähringer, E. Solano, R. Blatt, C.F. Roos, Nature (London) 463 (2010) 68.
[37] L. Lamata, J. Casanova, R. Gerritsma, C.F. Roos, J.J. García-Ripoll, E. Solano, New J. Phys. 13 (2011) 095003.
[38] M. Johanning, A.F. Varón, C. Wunderlich, J. Phys. B: At. Mol. Opt. Phys. 42 (2009) 154009.
[39] G. Romero, D. Ballester, Y.M. Wang, V. Scarani, E. Solano, Phys. Rev. Lett. 108 (2012) 120501.
[40] I.M. Mirza, S.J. van Enk, Opt. Commun. 343 (2015) 172.
[41] I.M. Mirza, Phys. Lett. A 379 (2015) 1643.
[42] I.M. Mirza, J.C. Schotland, Phys. Rev. A 94 (2016) 012309.
[43] C.C. Gerry, Phys. Rev. A 39 (1989) 3204.
[44] D. Ojeda-Guillén, R.D. Mota, V.D. Granados, J. Math. Phys. 55 (2014) 042109.
[45] D. Ojeda-Guillén, R.D. Mota, V.D. Granados, Commun. Theor. Phys. 64 (2015) 34.
[46] D.F. Walls, G.J. Milburn, Quantum Optics, Springer-Verlag, Berlin, 1994.
[47] W.H. Louisell, Quantum Statistical Properties of Radiation, Wiley, New York, 1973.
[48] H. Fan, C. Li, Z. Jiang, Phys. Lett. A 327 (2004) 416.
[49] A. Vourdas, Phys. Rev. A 41 (1990) 1653.
[50] C.C. Gerry, Phys. Rev. A 31 (1985) 2721.
[51] A.M. Perelomov, Generalized Coherent States and their Applications, Springer, Berlin, 1986.
[52] D. Ojeda-Guillén, M. Salazar-Ramírez, R.D. Mota, V.D. Granados, J. Nonlinear Math. Phys. 23 (2016) 607.
[53] A. Vourdas, J. Phys. A 39 (2006) R65.
[54] F.T. Arecchi, E. Courtens, R. Gilmore, H. Thomas, Phys. Rev. A 6 (1972) 2211.