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Quantum Optics 2 – Spring semester 2023 – 30/03/2023
Problem Set 5 : Entanglement

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This problem set is based on the final exam of Quantum Optics 2 course taught in Spring semester 2022.

I. ENTANGLEMENT CARVING

In this exercise, we show that it is possible to use a realistic measurement process to produce an entangled state of a large number of particles.

Consider an ensemble of N spin 1/2 particles. We denote the two states of the spins as $|\uparrow\rangle$ and $|\downarrow\rangle$. The spin ensemble is placed in a cavity, which realizes the following operation: upon sending a photon onto the cavity, the photon is transmitted (lost) unless at least one of the spin is in the $|\uparrow\rangle$ state, in which case the photon is reflected. We describe the photon degrees of freedom using a two dimensional Hilbert space $\mathrm{Sp}\{|0\rangle, |1\rangle\}$, with $|0\rangle$ representing the photon lost (not detected) and $|1\rangle$ the photon reflected.

1. Write the action of the operation described above on the full system spins+photon, initialized in the state $|\psi\rangle_s \otimes |1\rangle$, with $|\psi\rangle_s$ any state of the spin ensemble.
2. Write the measurement operators acting on the spin system, obtained upon measuring the photon state, and show that the corresponding generalized measurement is projective.

Consider the spins initialized in the state $|\Downarrow\rangle = |\downarrow\downarrow ... \downarrow\rangle$ (all spin pointing down). We induce a global spin rotation by an angle θ around the y axis of the Bloch sphere, induced by a classical drive and identical for all spins. Let $|\theta\rangle$ be the resulting state.

3. Are the spins in state $|\theta\rangle$ entangled with each other ? Justify your answer.
4. We now apply the measurement protocol with the spins prepared in the state $|\theta\rangle$. What is the state $|\phi\rangle$ of the spin system conditioned upon detecting a reflected photon ? Express it in terms of $|\theta\rangle$ and $|\Downarrow\rangle$.
5. Let $|W\rangle = \frac{1}{\sqrt{N}} \sum_i |\downarrow\downarrow ... \downarrow\uparrow_i\downarrow ... \downarrow\rangle$, the state comprising a symmetric superposition of all states with exactly one spin up. Prove that in state $|W\rangle$, one spin is entangled with every other (*Hint: calculate the purity of the state after tracing out one of the spins.*)
6. Calculate $\mathcal{F}(W, \phi) = |\langle W|\phi\rangle|^2$, the fidelity of producing the W state using the protocol described above, as a function of θ and N for $\theta \ll 1$.
7. Comment on the possible applications and limitations of such a protocol.

II. PERMUTATIONS AND ENTANGLEMENT MEASUREMENTS

Consider a system comprising two identical ensembles of qubits (denoted 1 and 2). Let $\hat{V}$ be the SWAP operator acting on the full Hilbert space describing two ensembles together, and defined by

$$\hat{V}\left|\phi\right\rangle_1 \otimes \left|\psi\right\rangle_2 = \left|\psi\right\rangle_1 \otimes \left|\phi\right\rangle_2 , \tag{1}$$

where the notation $\left|u\right\rangle_j$ means that ensemble j is in state $\left|u\right\rangle$, and $\left|\phi\right\rangle$ and $\left|\psi\right\rangle$ are any state in the Hilbert space of one ensemble. This operator exchanges the states between the two ensembles.

1. Show that the operator $\hat{V}$ has eigenvalues ± 1, and that the two eigenspaces correspond to symmetric and antisymmetric subspaces of the full system's Hilbert space.

2. We prepare the ensembles such that both are identical copies of each other $\hat{\rho}_{\text{full}} = \hat{\rho} \otimes \hat{\rho}$. Show that we have $\langle \hat{V} \rangle_{\text{full}} = \text{Tr}(\hat{V}\hat{\rho}_{\text{full}}) = \text{Tr}(\hat{\rho}^2)$, the purity of the state of an individual ensemble. Deduce that if ρ is pure, then the state $\hat{\rho}_{\text{full}}$ is symmetric.

3. Do you know any restrictions about the possibility to prepare the state $\hat{\rho}_{\text{full}}$?

These observations suggest a practical way to measure the purity of quantum states, by preparing two identical copies, and measuring the symmetry of the combined state. We now explore this idea in the simple context where the ensembles each contain only one qubits.

4. Write explicitly the basis states of the symmetric and antisymetric subspaces of the full Hilbert space in the $\{\left|\uparrow\right\rangle, \left|\downarrow\right\rangle\}$ basis.

5. Express the probability to measure the two qubits in the Bell state $\left|\Psi^-\right\rangle = \frac{\left|\uparrow\right\rangle_1 \otimes \left|\downarrow\right\rangle_2 - \left|\uparrow\right\rangle_2 \otimes \left|\downarrow\right\rangle_1}{\sqrt{2}}$ in terms of the purity $\text{Tr}(\hat{\rho}^2)$.

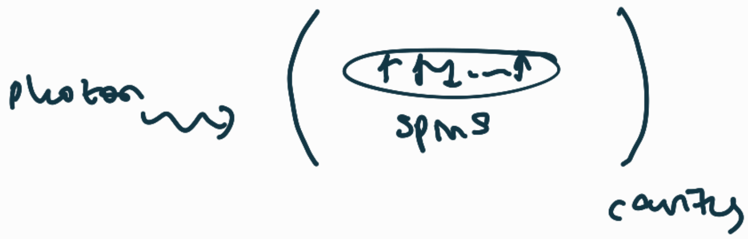
Bonus questions

For a practical implementation, we consider that the spins are realized using the ground and Rydberg state of two neutral atom, representing the states $\left|\downarrow\right\rangle$ and $\left|\uparrow\right\rangle$, respectively. For a review on Rydberg atoms see **Quantum information with Rydberg atoms**, M. Saffman, T. G. Walker, and K. Mølmer Rev. Mod. Phys. 82, 2313.

6. Show that in the Rydberg blockade regime, it is possible to map the state $\left|\Psi^+\right\rangle = \frac{\left|\uparrow\right\rangle_1 \otimes \left|\downarrow\right\rangle_2 + \left|\uparrow\right\rangle_2 \otimes \left|\downarrow\right\rangle_1}{\sqrt{2}}$ onto $\left|\downarrow\right\rangle_1 \otimes \left|\downarrow\right\rangle_2$.

7. Propose a protocol based on Rydberg blockade and single atoms operations, such that the detection of one or two atoms in the Rydberg state unambiguously indicates a loss of purity.

1. Write the action of the operation described above on the full system spins+photon, initialized in the state $|\psi\rangle_s \otimes |1\rangle$, with $|\psi\rangle_s$ any state of the spin ensemble.



let's find operator $\hat{U}$, acting on the whole system.

1) All the spins are pointing down

2) At least one of the spin is pointing up

$$|\psi_s^{(1)}\rangle = |\downarrow \dots \downarrow\rangle$$

$$|\psi_s^{(2)}\rangle = |\psi_s\rangle - \overbrace{\langle \psi_s | \downarrow \dots \downarrow \rangle |\downarrow \dots \downarrow\rangle}$$

$$|\psi_s^{(2)}\rangle \perp |\psi_s^{(1)}\rangle \Rightarrow \langle \psi_s^{(2)} | \psi_s^{(1)} \rangle = 0$$

they're orthogonal

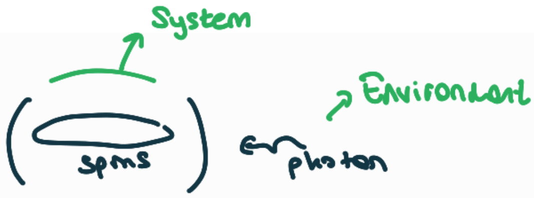
$|\psi_s\rangle$ can be in a generic superposition of $|\psi_s^{(1)}\rangle$ and $|\psi_s^{(2)}\rangle$. Then,

$$|\psi_s\rangle = \alpha |\psi_s^{(1)}\rangle + \beta |\psi_s^{(2)}\rangle \quad \text{with } |\alpha|^2 + |\beta|^2 = 1$$

$$\hat{U}(|\psi_s\rangle \otimes |1\rangle) = \alpha \hat{U}(|\psi_s^{(1)}\rangle \otimes |1\rangle) + \beta \hat{U}(|\psi_s^{(2)}\rangle \otimes 1)$$

$$= \alpha |\psi_s^{(1)}\rangle \otimes |0\rangle + \beta |\psi_s^{(2)}\rangle \otimes |1\rangle$$

2. Write the measurement operators acting on the spin system, obtained upon measuring the photon state, and show that the corresponding generalized measurement is projective.



Initial density matrix: ρ

$$\rho \to \rho' = \hat{U}\hat{\rho}\hat{U}^{\dagger}$$

$$\hat{\rho} = \hat{\rho}_s \otimes |1\rangle\langle 1| \longrightarrow \hat{\rho}_s' = \sum_N \hat{M}_N \rho_s \hat{M}_N^{\dagger}$$

total density matrix

$$\hat{\rho}_s' = Tr_E[\hat{\rho}']$$

$$\boxed{\hat{\rho}_s' = \langle 0|\hat{U}\hat{\rho}\hat{U}^{\dagger}|0\rangle + \langle 1|\hat{U}\hat{\rho}\hat{U}^{\dagger}|1\rangle}$$

$$\hat{\rho}_s' = \langle 0|\hat{U}(\hat{\rho}_s \otimes |1\rangle\langle 1|)\hat{U}^{\dagger}|0\rangle + \langle 1|\hat{U}(\hat{\rho}_s \otimes |1\rangle\langle 1|)\hat{U}^{\dagger}|1\rangle$$

$$= \underbrace{\langle 0|\hat{U}|1\rangle}_{\hat{M}_0} \hat{\rho}_s \underbrace{\langle 1|\hat{U}^{\dagger}|0\rangle}_{\hat{M}_0^{\dagger}} + \underbrace{\langle 1|\hat{U}|1\rangle}_{\hat{M}_1} \hat{\rho}_s \underbrace{\langle 1|\hat{U}^{\dagger}|1\rangle}_{\hat{M}_1^{\dagger}}$$

$$= \hat{M}_0 \hat{\rho}_s \hat{M}_0^{\dagger} + \hat{M}_1 \hat{\rho}_s \hat{M}_1^{\dagger}$$ → Krauss Representation

It is not a general manifestation

$\hat{M}_0^{\dagger}\hat{M}_0 + \hat{N}_1^{\dagger}\hat{M}_1 = I$

⇓

General version

$$\hat{M}_0 = \langle 0|\hat{U}|1\rangle$$

It creates both $|0\rangle$, $|1\rangle$ results but only $|0\rangle$ part is affected

$$\hat{M}_1 = \langle 1|\hat{U}|1\rangle = 1 - \hat{M}_0$$

Consider the spins initialized in the state $|\Downarrow\rangle = |\downarrow\downarrow \dots \downarrow\rangle$ (all spin pointing down). We induce a global spin rotation by an angle θ around the y axis of the Bloch sphere, induced by a classical drive and identical for all spins. Let $|\theta\rangle$ be the resulting state.

3. Are the spins in state $|\theta\rangle$ entangled with each other ? Justify your answer.

Global rotation can be modelled through a operator $\hat{R}_y = e^{-i\theta \hat{J}_y}$

where $\hat{J}_y = \sum_{i=1}^{N} \frac{\hat{\sigma}_i^y}{2}$

$$|\theta\rangle = e^{-i\theta \hat{J}_y} |\downarrow \dots \downarrow\rangle = e^{-i\theta \sum_{i=1}^{N} \frac{\hat{\sigma}_i^y}{2}} |\downarrow\rangle_1 \otimes \dots \otimes |\downarrow\rangle_N$$

π

$$= \bigotimes_{i=1}^{N} e^{-i\frac{\theta}{2}\hat{\sigma}_i^y} |\downarrow\rangle_i$$

?

reverses the spin

$$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Pauli Formula $\rightarrow$ $$e^{-\frac{i}{2}\theta\hat{\sigma}_i^y} = \cos\left(\frac{\theta}{2}\right)\hat{\mathbb{I}} - i\sin\left(\frac{\theta}{2}\right)\sigma_i^y$$

$$|\theta\rangle = \bigotimes_{i=1}^{N} \left[\cos\frac{\theta}{2} |\downarrow\rangle_i - \sin\left(\frac{\theta}{2}\right)|\uparrow\rangle_i\right]$$

Since it's in a product state, not entangled

4. We now apply the measurement protocol with the spins prepared in the state $|\theta\rangle$. What is the state $|\phi\rangle$ of the spin system conditioned upon detecting a reflected photon ? Express it in terms of $|\theta\rangle$ and $|\Downarrow\rangle$.

Since we detect a reflected photon, $|\theta\rangle$ evolves under the action of the measurement operator $\hat{M}_1$

$$|\phi\rangle = \frac{\hat{M}_1 |\theta\rangle}{\sqrt{\langle\theta|\hat{M}_1^\dagger \hat{M}_1|\theta\rangle}}$$

Action of M_1 on $|\theta\rangle$:

$$\underbrace{\hat{M}_1}_{1-\hat{M}_0}|\theta\rangle = |\theta\rangle - \langle\downarrow \dots \downarrow|\theta\rangle|\downarrow \dots \downarrow\rangle$$

$$= |\theta\rangle - \left(\bigotimes_{i=1}^{N} \langle\downarrow_i| \left(\cos\frac{\theta}{2}|\downarrow\rangle_i - \sin\left(\frac{\theta}{2}\right)|\uparrow\rangle_i\right)\right)|\downarrow\downarrow \dots \downarrow\rangle$$

$$\hat{M}_1|\theta\rangle = |\theta\rangle - \cos^N\left(\frac{\theta}{2}\right)|\downarrow \dots \downarrow\rangle$$

$$\langle\theta|\hat{M}_1^\dagger \hat{M}_1|\theta\rangle = \left(\langle\theta| - \cos^N\left(\frac{\theta}{2}\right)\langle\downarrow \dots \downarrow|\right)\left(|\theta\rangle - \cos^N\left(\frac{\theta}{2}\right)|\downarrow \dots \downarrow\rangle\right) = 1 - \cos^{2N}\left(\frac{\theta}{2}\right)$$

Thus,

$$|\phi\rangle = \frac{|\theta\rangle - \cos^N\left(\frac{\theta}{2}\right)|\downarrow \dots \downarrow\rangle}{\sqrt{1 - \cos^{2N}(\theta/2)}}$$

5. Let $|W\rangle = \frac{1}{\sqrt{N}} \sum_i |\downarrow\downarrow \dots \downarrow\uparrow_i\downarrow \dots \downarrow\rangle$, the state comprising a symmetric superposition of all states with exactly one spin up. Prove that in state $|W\rangle$, one spin is entangled with every other (*Hint: calculate the purity of the state after tracing out one of the spins.*)

$$|W\rangle = \frac{1}{\sqrt{N}} \sum_i |\downarrow\downarrow \dots \uparrow_i \dots \downarrow\rangle ,$$

For $N=2$, $\quad |W\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right) \rightarrow$ Bell State

$$\hat{\rho} = |W\rangle\langle W|$$

$$\hat{\rho}_{red} = Tr_1(\hat{\rho}) = \langle\downarrow_1|W\rangle\langle W|\downarrow\rangle_1 + \langle\uparrow_1|W\rangle\langle W|\uparrow_1\rangle$$

$$\hat{\rho}_{red} = \frac{1}{2}\left(|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|\right) \rightsquigarrow \text{"mixed"}$$

$$\gamma = Tr(\hat{\rho}^2_{red}) = \frac{1}{2} < 1 \Rightarrow \text{ENTANGLED}$$

Purity

For N spins :

$$|W\rangle = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} |\downarrow \dots \uparrow_i \dots \downarrow\rangle = \frac{1}{\sqrt{N}} |\uparrow\rangle_1 \otimes |\downarrow\downarrow \dots \downarrow\rangle_{N-1} + \frac{1}{\sqrt{N}} |\downarrow\rangle_1 \otimes \sum_{i=1}^{N-1} |\downarrow \dots \uparrow_i \dots \downarrow\rangle$$

similar to $N=2$ case,

$$\hat{\rho}_{red} = \frac{1}{N} |\downarrow \dots \downarrow\rangle\langle\downarrow \dots \downarrow|_{N-1} + \frac{1}{N} \sum_{i,j=1}^{N-1} |\downarrow \dots \uparrow_i \dots \downarrow\rangle\langle\downarrow \dots \uparrow_j \dots \downarrow|_{N-1}$$

$$\hat{\rho}^2_{red} = \frac{1}{N^2} |\downarrow \dots \downarrow\rangle\langle\downarrow \dots \downarrow|_{N-1} + \frac{1}{N^2} \left(\sum_{i,j=1}^{N-1} |\downarrow \dots \uparrow_i \dots \downarrow\rangle\langle\downarrow \dots \uparrow_j \dots \downarrow|\right)\left(\sum_{k,l=1}^{N-1} |\uparrow \dots \uparrow_k \dots \uparrow\rangle\langle\downarrow \dots \uparrow_l \dots\right)_{N-1}$$

$$= \frac{1}{N^2} |{\downarrow}\dots{\downarrow}\rangle\langle{\downarrow}\dots{\downarrow}|_{N-1} + \frac{1}{N^2} \sum_{i,j,k,l}^{N-1} \delta_{k,j} |{\downarrow}\dots{\uparrow_i}\dots{\downarrow}\rangle\langle{\downarrow}\dots{\uparrow_l}\dots|_{N-1}$$

$$= \frac{1}{N^2} |{\downarrow}\dots{\downarrow}\rangle\langle{\downarrow}\dots{\downarrow}|_{N-1} + \frac{N-1}{N^2} \sum_{i,l}^{N-1} |{\downarrow}\dots{\uparrow_i}\dots{\downarrow}\rangle\langle{\downarrow}\dots{\uparrow_l}\dots{\downarrow}|_{N-1}$$

Purity $\gamma = Tr(\rho_{red}^2) = \frac{1}{N^2} + \frac{(N-1)^2}{N^2} = \frac{N^2-2N+2}{N^2} < 1$

for large $N \rightarrow \quad 1 - 2/N + \cancel{2/N^2} \approx 1 - \frac{2}{N}$

6. Calculate $\mathcal{F}(W, \phi) = |\langle W|\phi\rangle|^2$, the fidelity of producing the W state using the protocol described above, as a function of θ and N for $\theta \ll 1$.

$$\langle W|\phi\rangle = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \langle{\downarrow}\dots{\uparrow_i}\dots{\downarrow}|\phi\rangle = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \frac{\langle{\downarrow}\dots{\uparrow_i}\dots{\downarrow}|\theta\rangle}{\sqrt{1-\cos^{2N}(\theta/2)}}$$

$$\langle{\downarrow}\dots{\uparrow_i}\dots{\downarrow}|{\downarrow}\dots{\downarrow}\rangle = 0$$

$$\langle{\downarrow}\dots{\uparrow_i}\dots{\downarrow}| \bigotimes_{j=1}^{N} \left[\cos\tfrac{\theta}{2} |{\downarrow}\rangle_j - \sin\tfrac{\theta}{2} |{\uparrow}\rangle_j\right]$$

$$= \left(\langle{\downarrow}|_1 \cos\tfrac{\theta}{2} |{\downarrow}\rangle_1 - \cancel{\sin\tfrac{\theta}{2}\langle{\downarrow}|{\uparrow}\rangle}\right) \dots\dots \left(\cancel{\langle{\uparrow_i}|\cos\tfrac{\theta}{2}|{\downarrow}\rangle} - \sin\tfrac{\theta}{2}\langle{\uparrow}|{\uparrow}\rangle\right) \dots\dots \left(\cos\tfrac{\theta}{2}\langle{\downarrow_N}|{\downarrow_N}\rangle\right)$$

$$= -\cos^{N-1}\frac{\theta}{2}\sin\frac{\theta}{2}$$

$$\langle W|\phi\rangle = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \frac{-\cos^{N-1}(\frac{\theta}{2})\sin(\frac{\theta}{2})}{\sqrt{1-\cos^{2N}(\theta/2)}} = -\sqrt{N}\,\frac{\cos^{N-1}(\theta/2)\sin(\theta/2)}{\sqrt{1-\cos^{2N}(\theta/2)}}$$

$$\text{Fidelity} \rightarrow F(W,\phi) = N\,\frac{\cos^{2N-2}(\frac{\theta}{2})\sin^2(\frac{\theta}{2})}{1-\cos^{2N}(\frac{\theta}{2})}$$

for the limit $\theta \to 0$ $(\theta \ll 1)$

$$F(w,\phi) \simeq N \frac{\left(1-\frac{\theta^2}{8}\right)^{2N-2} \frac{\theta^2}{4}}{1-\left(1-\frac{\theta^2}{8}\right)^{2N}} \simeq N \frac{\left[1-(2N-2)\frac{\theta^2}{8}\right]\frac{\theta^2}{4}}{1-1+2N\frac{\theta^2}{8}} = 1$$

7. Comment on the possible applications and limitations of such a protocol.

Generating entangled states

High noise ratio for measuring photons.

Consider a system comprising two identical ensembles of qubits (denoted 1 and 2). Let $\hat{V}$ be the SWAP operator acting on the full Hilbert space describing two ensembles together, and defined by

$$\hat{V}\,|\phi\rangle_1 \otimes |\psi\rangle_2 = |\psi\rangle_1 \otimes |\phi\rangle_2\,, \tag{1}$$

where the notation $|u\rangle_j$ means that ensemble j is in state $|u\rangle$, and $|\phi\rangle$ and $|\psi\rangle$ are any state in the Hilbert space of one ensemble. This operator exchanges the states between the two ensembles.

1. Show that the operator $\hat{V}$ has eigenvalues ± 1, and that the two eigenspaces correspond to symmetric and antisymmetric subspaces of the full system's Hilbert space.

$$\hat{V}^2 |\phi\rangle_1 \otimes |\psi\rangle_2 = |\phi_1\rangle \otimes |\psi\rangle_2$$

$$\hat{V}^2 = \mathbb{I} \quad \longrightarrow \quad \text{eigenvalues are } \pm 1$$

2. We prepare the ensembles such that both are identical copies of each other $\hat{\rho}_{\text{full}} = \hat{\rho} \otimes \hat{\rho}$. Show that we have $\langle \hat{V} \rangle_{\text{full}} = \text{Tr}(\hat{V}\hat{\rho}_{\text{full}}) = \text{Tr}(\hat{\rho}^2)$, the purity of the state of an individual ensemble. Deduce that if ρ is pure, then the state $\hat{\rho}_{\text{full}}$ is symmetric.

$$\hat{\rho}_{full} = \hat{\rho} \otimes \hat{\rho}$$

$$\langle \hat{V} \rangle_{full} = \text{Tr}(\hat{V}\hat{\rho}_{full}) = \text{Tr}(\hat{\rho}^2) = \gamma$$

A Expectation value of the operator $\hat{V}$ coincides with the purity γ of the state of an individual ensemble.

$$\hat{\rho} = \sum_i p_i |\phi_i\rangle\langle\phi_i|$$

$$\hat{\rho} \otimes \hat{\rho} = \sum_i p_i |\phi_i\rangle\langle\phi_i| \sum_j p_j |\phi_j\rangle\langle\phi_j| = \sum_{i,j} p_i p_j |\phi_i \phi_j\rangle\langle\phi_i \phi_j|$$

$$\hat{V} \hat{\rho} \otimes \hat{\rho} = \hat{V} \sum_{i,j} p_i p_j |\phi_i \phi_j\rangle\langle\phi_i \phi_j| = \sum_{i,j} p_i p_j |\phi_j \phi_i\rangle\langle\phi_i \phi_j|$$

$$\mathrm{Tr}(\hat{V} \hat{\rho}_{full}) = \sum_{k,l} \langle\phi_k \phi_l| \left(\sum_{i,j} p_i p_j |\phi_j \phi_i\rangle\langle\phi_i \phi_j| \right) |\phi_k \phi_l\rangle$$

Trace over the whole system

$$= \sum_{ijkl} \delta_{kl} \delta_{il} p_i p_j = \sum_i p_i^2$$

Compute $\hat{\rho}^2$

$$\rho^2 = \left(\sum_i p_i |\phi_i\rangle\langle\phi_i| \right) \left(\sum_j p_j |\phi_j\rangle\langle\phi_j| \right) = \sum_i p_i^2 |\phi_i\rangle\langle\phi_i| \xrightarrow{\text{Trace}} \sum_i p_i^2$$

$$\hat{V} = (+1) \sum_{\text{sym} |\xi\rangle} |\xi\rangle\langle\xi| + (-1) \sum_{\text{antisym} |\varphi\rangle\langle\varphi|} |\varphi\rangle\langle\varphi|$$

* If $\hat{\rho}$ is pure, $\boxed{\mathrm{Tr}(\hat{V} \hat{\rho}_{full}) = \mathrm{Tr}(\hat{\rho}^2) = 1}$ Then $\hat{\rho}_{full}$ must contain only symmetric wavefunctions

3. Do you know any restrictions about the possibility to prepare the state $\hat{\rho}_{\text{full}}$?

Difficulty is cloning quantum states. In general (for orthogonal states, we can not make a copy of it.

Initialize two identical $\hat{\rho}$ instead of the copy of it.

These observations suggest a practical way to measure the purity of quantum states, by preparing two identical copies, and measuring the symmetry of the combined state. We now explore this idea in the simple context where the ensembles each contain only one qubits.

4. Write explicitly the basis states of the symmetric and antisymetric subspaces of the full Hilbert space in the $\{|\uparrow\rangle, |\downarrow\rangle\}$ basis.

$$\left\{ |\uparrow\uparrow\rangle,\ |\downarrow\downarrow\rangle,\ \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)\right\} \rightarrow \text{Symmetric} \quad \text{Triplet}$$

$$\left\{ \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)\right\} \rightarrow \text{Antisymmetric} \quad \text{Singlet}$$

5. Express the probability to measure the two qubits in the Bell state $|\Psi^-\rangle = \frac{|\uparrow\rangle_1 \otimes |\downarrow\rangle_2 - |\uparrow\rangle_2 \otimes |\downarrow\rangle_1}{\sqrt{2}}$ in terms of the purity $\text{Tr}(\hat{\rho}^2)$.

$$\hat{\rho} = p_\uparrow |\uparrow\rangle\langle\uparrow| + p_\downarrow |\downarrow\rangle\langle\downarrow|$$

Density Matrix for one-qubit system

From previous parts, $\hat{\rho}_{full}$ is known for 4D basis of the Full Hilbert space

$\hat{\rho}_{full}$ in new basis:

$|1\rangle = |\uparrow\uparrow\rangle$

$|2\rangle = |\downarrow\downarrow\rangle$

$|3\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) = |\Psi^+\rangle$

$|4\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) = |\Psi^-\rangle$

$$\hat{\rho}_{full} = \sum_{i=1}^{4} p_i |i\rangle\langle i|$$

Antisymmetric case

$$\hat{V}\hat{\rho}_{full} = \sum_{i=1}^{3} p_i |i\rangle\langle i| - p_4 |4\rangle\langle 4|$$

$$= \sum_{i=1}^{3} p_i |i\rangle\langle i| - p_- |\Psi^-\rangle\langle\Psi^-|$$

$$\mathrm{Tr}(\hat{\mathcal{J}}\hat{\rho}_{full}) = \mathrm{Tr}\left(\sum_{i=1}^{3} p_i |i\rangle\langle i| - p_- |\psi^-\rangle\langle\psi^-|\right)$$

$$= \sum_{i=1}^{3} p_i - p_- = \underbrace{\sum_{i=1}^{4} p_i}_{1} - 2p_- = 1 - 2p_-$$

We know that $\mathrm{Tr}(\hat{\mathcal{J}}\hat{\rho}_{full}) = \mathrm{Tr}(\hat{\rho}^2) = \gamma$

$$\gamma = 1 - 2p_- \Rightarrow p_- = \frac{1-\gamma}{2}$$

probability to measure the two qubits in the Bell states