II. ENTANGLEMENT

* Traces
* Stat. mech.
* Canonical exp.
* Schmidt

Bipartite system $\hat{\rho}_A = |\psi_A\rangle\langle\psi_A|$

$$|\psi_{AB}\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$$

$$\mathrm{Tr}_B\left[\,|\psi_{AB}\rangle\langle\psi_{AB}|\,\right] = \sum_{\nu} |\psi_A\rangle\langle\nu_B|\psi_B\rangle\langle\psi_A|\langle\psi_B|\nu_B\rangle$$

$$= |\psi_A\rangle\langle\psi_A| \underbrace{\sum_{\nu} |\langle\nu_B|\psi_B\rangle|^2}_{I}$$

If $\hat{\rho}_A$ is mixed : A and B were entangled

1) Entropy

Given $\hat{\rho}_A$: → Renyi entropy $S_n(\hat{\rho}_A) = -\mathrm{Log}\left[\mathrm{Tr}(\hat{\rho}_A^{\,n})\right] \frac{1}{n-1}$ More practical

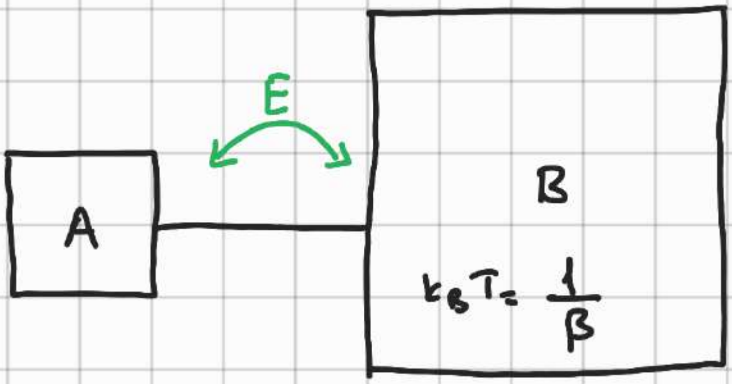
→ Entanglement entropy $S(\hat{\rho}_A) = -\mathrm{Tr}\left[\hat{\rho}_A \ln \hat{\rho}_A\right]$

$$= \lim_{n \to 1} S_n(\hat{\rho}_A)$$

$$S(|\psi\rangle\langle\psi|) = 0$$

Ex Statistical Mechanics, canonical example

$$\hat{\rho}_A = \frac{1}{Z} e^{-\beta \hat{H}_A}$$



$$S(\hat{\rho}_A) = \; .... \; = F \; \beta \langle \hat{H}_A \rangle$$

Free energy

$$\left(e^{-\beta F} = Z \right)$$

Remarks → For any bipartition of S into A/B

$$S(\hat{\rho}_A) = S(\hat{\rho}_B)$$

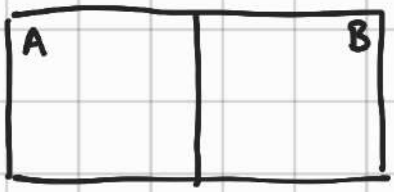
→ Maximum entropy: $\hat{\rho}_A = \frac{1}{N} \sum_i |i\rangle_A \langle i|_A$ $\quad N = \dim(H_A)$

$$S(\hat{\rho}_A) = \log N$$

$$\hat{\rho}_A = \frac{1}{N} \sum_i |i\rangle_A \langle i_A|$$ infinite temperature

→ Thermalization: Building up entanglement

S



$S \propto$ area of the boundary A/B

$S \propto$ volume of A

→ Connection between S and quantum gravity (?)

2) Schmidt Decomposition

let $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ and $|\psi\rangle \in \mathcal{H}_{AB}$

Here exists an orthonormal set in $\mathcal{H}_A$: $\{|i_A\rangle\}$

" " " $\mathcal{H}_B$: $\{|i_B\rangle\}$

Such that $|\psi\rangle = \sum_i \lambda_i |i_A\rangle \otimes |i_B\rangle$

$\{\lambda_i\}$: Schmidt coef.

λ_i's : Schmidt Rank $1 \rightarrow$ pure state

Density matrix: $\hat{\rho}_A = Tr_B[|\psi\rangle\langle\psi|]$

let $\{i_B\}$ the 'Schmidt basis'

$$= \sum_i \langle i_B | \Psi \rangle \langle \Psi | i_B \rangle$$

$$\hat{\rho}_A = \sum_i |\lambda_i|^2 \, |i_A\rangle\langle i_A|$$

$$S(\hat{\rho}_A) = - \sum_i |\lambda_i|^2 \, \mathrm{Log}\, \lambda_i^2$$

Remark: $\lambda_i = e^{-\xi_i/2}$ $\quad |\Psi\rangle = \sum_i e^{-\xi_i/2} \, |i_A\rangle \otimes |i_B\rangle$

$\{ \xi_i \}$: entanglement spectrum

Given $\hat{\rho}_A = \sum_i p_i \, |i_A\rangle\langle i_A|$

there exists system R, with Hilbert space $\mathcal{H}_R$

such that $\exists |\Psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_R$ $\quad \hat{\rho}_A = \mathrm{Tr}_R [|\Psi\rangle\langle\Psi|]$

purification of $\hat{\rho}_A$

3) Example

2 qubit system : A, $\quad B$

$$\mathcal{H}_A = \mathrm{Sp}\{ |0\rangle, |1\rangle \}$$

$$\mathcal{H}_B = \mathrm{Sp}\{ |0\rangle, |1\rangle \}$$

$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \qquad \mathcal{H} = \mathrm{Sp}\{ |00\rangle, |01\rangle, |10\rangle, |11\rangle \}$

$\left(|ij\rangle = |i\rangle_A \otimes |j\rangle_B \right)$

- $\frac{1}{\sqrt{2}} \left(|00\rangle + |01\rangle \right) \longrightarrow$ Not entangled $\left(\frac{1}{\sqrt{2}} |0\rangle \otimes (|0\rangle + |1\rangle) \right)$

$|\Phi^+\rangle =$ · $\frac{1}{\sqrt{2}} \left(|00\rangle + |11\rangle \right) \longrightarrow$ Entangled

$$\mathrm{Tr}_B \left[|\Phi^+\rangle\langle\Phi^+| \right] = \frac{1}{2} \left(|0\rangle\langle 0| + |1\rangle\langle 1| \right)$$

$\downarrow$

Maximally mixed state

$$|\phi_\varepsilon\rangle = \frac{1}{\sqrt{\varepsilon^2 + (1-\varepsilon)^2}} \left(\varepsilon |00\rangle + (1-\varepsilon) |11\rangle \right) \rightarrow \text{Yes}$$

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}} \left(|00\rangle + |11\rangle \right) \qquad |\Psi^+\rangle = \frac{1}{\sqrt{2}} \left(|01\rangle + |10\rangle \right)$$

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}} \left(|00\rangle - |11\rangle \right) \qquad |\Psi^-\rangle = \frac{1}{\sqrt{2}} \left(|01\rangle - |10\rangle \right)$$