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Problem Set 1 : Driven Dipoles and Atomic Clocks

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At the core of research in cold atoms is the interaction between atoms and light. We propose here to treat the simple case of a two-level system (typically an atom) coupled to a near-resonant classical light field (typically laser light or microwave radiation). We then apply our results to the understand the principle of atomic clocks.

I. REMINDER : A 2-LEVEL DIPOLE DRIVEN BY A CLASSICAL FIELD

The Hamiltonian of a two-level atom coupled to a field can be written (in the dipolar approximation) as :

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{int} = \frac{\hbar\omega_0}{2}\left(|1\rangle\langle 1| - |0\rangle\langle 0|\right) - \hat{\vec{d}}\cdot\vec{E} \tag{1}$$

with $\hat{\vec{d}} = d\vec{e}_z\left(|0\rangle\langle 1| + |1\rangle\langle 0|\right)$ and $\vec{E} = \mathcal{E}\vec{e}_z.\cos(\omega t - \phi)$.

1. By applying to the atomic state $|\psi\rangle$ the right unitary transformation $U_{rot}(t)$, write $\mathcal{H}$ in the rotating frame of the driving field as :

$$\mathcal{H}_{rot} = \frac{\hbar(\omega-\omega_0)}{2}\left(|0\rangle\langle 0| - |1\rangle\langle 1|\right) - \frac{d\mathcal{E}}{2}\cdot\left(e^{i(\omega t-\phi)} + e^{-i(\omega t-\phi)}\right)\left(e^{-i\omega t}|0\rangle\langle 1| + e^{i\omega t}|1\rangle\langle 0|\right) \tag{2}$$

2. Apply the “rotating wave approximation” to remove the fast-oscillating terms of Hamiltonian (2), to get :

$$\mathcal{H}_{rot} = -\frac{\hbar}{2}\cdot\begin{pmatrix} -\delta & e^{-i\phi}\Omega \\ e^{i\phi}\Omega & \delta \end{pmatrix}. \tag{3}$$

   What are the values and meanings of δ and Ω ? Write this Hamiltonian in terms of the Pauli matrices.

A. Solving the driven-dipole problem in the rotating frame

In order to understand what happens, we can start by diagonalizing the rotating frame Hamiltonian $\mathcal{H}_{rot}$ as a function of δ and Ω.

1. What are the eigenvalues $E_\pm$ of $\mathcal{H}_{rot}$? You can introduce the value $\delta' = \sqrt{\delta^2 + \Omega^2}$.
2. What are the corresponding (normalized) eigenvectors $|\pm\rangle$ of $\mathcal{H}_{rot}$? Check that $|+\rangle$ and $|-\rangle$ are othogonal. Introduce the ”mixing angle” η such that $|+\rangle = -\sin\eta\,|0\rangle + e^{i\alpha}\cos\eta\,|1\rangle$ and $|-\rangle = \cos\eta\,|0\rangle + e^{i\alpha}\sin\eta\,|1\rangle$. Give the expressions for α, $\sin\eta$, $\cos\eta$ and $\tan\eta$. there'll be change
3. What do $E_\pm$ and $|\pm\rangle$ become when $\delta = 0$? In the limit of large detunings (explicitly at 2nd order in Ω/δ) ?
4. Draw and comment a cursary energy-level diagram of the driven dipole as a function of δ.

B. Time evolution of the driven dipole

To solve the time evolution of an atom coupled to a field, we can use the time-evolution operator U :

$$|\psi(t)\rangle = U(t)\,|\psi(0)\rangle = U(t)\,|0\rangle$$

and decompose $|0\rangle$ onto the relevant basis.

1. What is the time evolution of states $|\pm\rangle$?

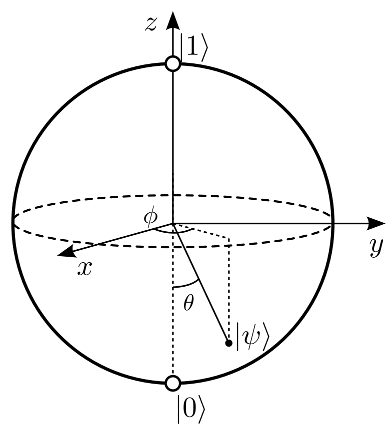
2. Write $|\psi(t)\rangle$ in a more explicit form given by the expression of $U(t)$ and the decomposition of $|0\rangle$.
3. What simple operation will provide the expression for $\mathcal{P}_1(t)$, the probability of the atom to be in state $|1\rangle$ at a given time t ? The oscillations of $\mathcal{P}_1(t)$ are called **"Rabi oscillations"**.
4. Comment on the maximum value and on the period of $\mathcal{P}_1(t)$.
5. In the case $\delta = 0$, what value of Ωt makes a balanced superposition of $|0\rangle$ and $|1\rangle$ ("$\pi/2$-**pulse**") ? a state inversion $|0\rangle \to |1\rangle$ ("π-**pulse**") ?
6. Draw $\mathcal{P}_1(t)$ for $\Omega t \in [0; 3\pi]$ in three different cases : $\delta = 0$, $\delta = \Omega$ and $\delta = 3\Omega$.

C. Graphical representation : the Bloch sphere

The state of the atomic dipole can always be written

$$|\psi\rangle = \cos(\theta/2)\,|0\rangle + e^{i\phi}(\sin\theta/2)\,|1\rangle \tag{4}$$

with $\theta \in [0; \pi]$ and $\phi \in [0; 2\pi]$. We can then represent $|\psi\rangle$ on a sphere defined by these two angles : the **Bloch sphere**.



1. What is the evolution of the Bloch vector $\vec{a}$ under Hamiltonian (3) when $\delta = 0$?
2. What is the evolution $\vec{a}$ under Hamiltonian (3) when $\Omega = 0$, for a state lying on the equator ?
3. Relate this to the Pauli matrices form of the Hamiltonian.
4. What becomes of state $|0\rangle$ after a π-pulse ? after a $\pi/2$-pulse ?

D. Phase of the driving field

1. How does the phase of the field map on the Bloch sphere ?
2. Represent on the Bloch sphere the following sequence, starting from state $|0\rangle$:
   - a $\pi/2$-pulse with $\delta = 0$ and $\phi = 0$;
   - free evolution ($\Omega = 0$) with $\delta \neq 0$ for a time $T = 2\pi/\delta$;
   - a second $\pi/2$-pulse, with a phase $\phi_0 = 0$ or $\phi_1 = \pi/2$ or $\phi_2 = \pi$.

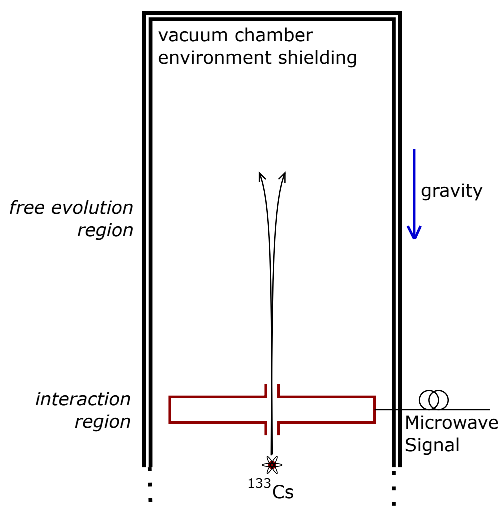
Such a sequence can be used to measure very precisely what happens when the dipole evolves freely between the two $\pi/2$-pulses, in what is called a **Ramsey interference** experiment. This type of sequence is the core principle of atomic clocks.

II. ATOMIC CLOCKS

Atomic clocks are the most precise measurement instruments. They are designed to produce a purely harmonic signal at a given frequency. That frequency is compared to an absolute reference given by the energy difference between two levels of an atom, for example the two lowest energy levels of caesium. The precision of state of the art atomic clocks is at the moment close to 10^{-19}, which means that the clock would be delayed by 0.1 s over the age of the universe.

There are different possible designs for atomic clocks. We will study here the so called caesium fountain clock. It is the current design used for the definition of the second (the second is by definition the duration between 9 192 631 770 oscillations of the caesium clock signal).

The design of the atomic clock is shown on figure 1. The atoms are prepared at the bottom. They are then launched in the vertical direction at a given velocity v. They travel through an interaction region, where they interact with a oscillating magnetic field (the microwave field) at the angular frequency ω. They then move freely in the upper part of the fountain, where no field is present. Then, due to gravity, they fall back through the same interaction region, where they interact once again with the microwave field. The state of the atoms is then measured in the preparation region. As you will see, the number of atoms in a given state at the output of the clock gives a very sensitive indication of the frequency difference between the microwave signal and the atomic transition frequency that can be used for example to regulate the microwave frequency in a feedback loop.



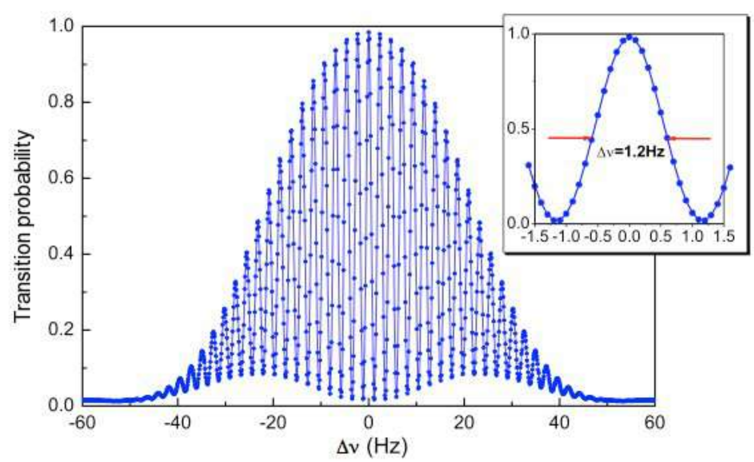


FIG. 1. Left : Simplified scheme of a Caesium fountain clock. Right : Ramsey fringes obtained experimentaly on the Rubidium clock of Observatoire de Paris. Each point corresponds to a single measurement.

A. Frequency measurement and clock signal

In a clock the measurement is done in three phases similar to the ones described in I D.2. In order to simplify the analysis, we will first neglect the effect of the angular frequency difference $\delta = \omega - \omega_0$ during the interaction phases. But we will keep it for the description of the free evolution. The state of an atom on the Bloch sphere will be represented as the Bloch vector $\vec{a}$. $\vec{a} = (\langle\sigma_x\rangle, \langle\sigma_y\rangle, \langle\sigma_z\rangle)$.

1. The atom is prepared initially in state $|0\rangle$. At t = 0, the atom enters the interaction region and takes a time τ to cross it. The driving field is homogeneous in that region with a Rabi frequency Ω, and $\omega = \omega_0$. Show that after a time τ :

$$\vec{a}_\tau = \begin{pmatrix} 0 \\ \sin(\Omega\tau) \\ -\cos(\Omega\tau) \end{pmatrix} \tag{5}$$

2. After the interaction region, the atom moves freely in the upper part of the clock during a time $T \gg \tau$. Then due to gravity, they fall back in the interaction region. Show that after that free evolution :

$$\vec{a}_{\tau+T} = \begin{pmatrix} -\sin(\delta T)\sin(\Omega\tau) \\ \cos(\delta T)\sin(\Omega\tau) \\ -\cos(\Omega\tau) \end{pmatrix} \tag{6}$$

3. After the free evolution, the atom passes again through the interaction region. In that case, compute the coordinates of the Bloch vector when the Rabi frequency was chosen such that $\Omega\tau = \frac{\pi}{2}$. Compute the probability $\mathcal{P}_1$ to detect the atom in state $|1\rangle$ as a function of the frequency difference $\delta/2\pi$. This pattern is called "Ramsey fringes".

4. *Effects of the detuning during the interaction.* So far we did not consider the effect of the detuning of the microwave compared to the atomic frequency in the interaction region. Identify each of the 3 steps of the experiment as a rotation of the Bloch vector around a certain axis depending on the Rabi frequency Ω and the detuning δ. Use rotation matrices in Python Scipy (by using Jupyter notebook at `https://mybinder.org/v2/git/https%3A%2F%2Fc4science.ch%2Fsource%2Fquantum_optics2021.git/master`) to compute the exact expectation value of $\langle\sigma_z\rangle$ including the effects of detuning. Plot the obtained fringe signal as a function of detuning for the case where $\Omega/2\pi = 14\,\text{Hz}$, $\Omega\tau = \frac{\pi}{2}$ and $T = 0.3\,\text{s}$.

5. *Effects of velocity distribution of atoms.* In a real atomic clock, the experiment is done on many atoms at the same time (typically 10^6). However, the atoms do not all have the same velocity, and therefore the travel time T is not the same for all the atoms. Suppose that the we have atoms at a temperature of $10\,\mu\text{K}$, and that the velocity distribution follows the Boltzmann law (the mass of a cesium atom is $2.2 \times 10^{-25}\,\text{kg}$), with a center velocity chosen so that $T = 0.3\,\text{s}$. Use the Python program to compute what portion of the atoms will be detected in state $|1\rangle$ in the final measurement and plot the corresponding population as a function of the frequency difference. You should get a signal resembling the plot on figure 1. What happens when the temperature of the atoms is larger (say around $1\,\text{mK}$) ?

B. Further analysis : Precision and fundamental noise

1. Using the Heisenberg uncertainty relation for time and frequency, what is the limit of precision in frequency for one measurement of a single atom in the Caesium clock defined previously ? What is the ratio of this precision limit to the transition frequency ? Does the number you obtain seem like a reasonable precision limit knowing that the best fountain clocks now reach in the 10^{-17} relative precision ?

2. What explains the difference ? What would you suggest in order to improve the precision of the atomic clock ?

3. *Quantum projection noise and Standard Quantum Limit (**SQL**).* There is actually a fundamental limit to the accuracy at which one can measure the position of the Ramsey fringes, and therefore to the precision of the clock. Suppose that we are now working with N independent atoms (of the order of 10^6 in real experiments), close to the point of maximum sensitivity of the clock. Show that the probability to detect n atoms in state $|1\rangle$ in the final measurement follows the Poisson distribution. What is the standard deviation ? How does it convert into standard deviation of the frequency ?

This noise comes from the fact that at the end of the measurement, each atom is projected in one or the other internal state, with a certain probability. The randomness that is added in that step leads to a fundamental noise which is called quantum projection noise. How could we hope to beat the $1/\sqrt{N}$ scaling imposed by the Standard Quantum Limit ?

The Hamiltonian of a two-level atom coupled to a field can be written (in the dipolar approximation) as :

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{int} = \frac{\hbar\omega_0}{2}\overbrace{(|1\rangle\langle 1| - |0\rangle\langle 0|)}^{-\sigma_z} - \hat{\vec{d}}\cdot\vec{E} \tag{1}$$

with $\hat{\vec{d}} = d\vec{e}_z\,\overbrace{(|0\rangle\langle 1| + |1\rangle\langle 0|)}^{\sigma_x}$ and $\vec{E} = \mathcal{E}\vec{e}_z.\cos(\omega t - \phi)$. $\quad \vec{d}\cdot\vec{E} = d\mathcal{E}\cos(\omega t - \phi)\,\sigma_x$

1. By applying to the atomic state $|\psi\rangle$ the right unitary transformation $U_{rot}(t)$, write $\mathcal{H}$ in the rotating frame of the driving field as :

$$\mathcal{H}_{rot} = \frac{\hbar(\omega-\omega_0)}{2}(|0\rangle\langle 0| - |1\rangle\langle 1|) - \frac{d\mathcal{E}}{2}\cdot(e^{i(\omega t-\phi)} + e^{-i(\omega t-\phi)})(e^{-i\omega t}|0\rangle\langle 1| + e^{i\omega t}|1\rangle\langle 0|) \tag{2}$$

In rotating frame, H becomes (H_{rot}) time-independent

$$H = H_0 + H_{int} = -\frac{\hbar\omega_0\sigma_z}{2} - d\mathcal{E}\cos(\omega t-\phi)\sigma_x$$

driven-field frequency

$$U_{rot}(t) = e^{-i\frac{\omega}{2}\sigma_z t}$$

Derivation of H_{rot}:

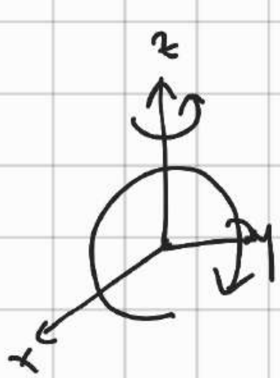
$$|\tilde{\Psi}(t)\rangle = \overset{U_{rot}(t)}{\overset{\parallel}{U(t)}}\,|\Psi(t)\rangle$$

$$i\hbar\frac{\partial}{\partial t}|\tilde{\Psi}(t)\rangle = i\hbar\frac{\partial}{\partial t}U(t)\underbrace{|\Psi(t)\rangle}_{U^{+}(t)|\tilde{\Psi}(t)\rangle} + U(t)\overbrace{i\hbar\frac{d}{dt}|\Psi(t)\rangle}^{H|\Psi(t)\rangle \,\to\, U^{+}|\tilde{\Psi}(t)\rangle}$$

$$i\hbar\frac{\partial}{\partial t}|\tilde{\Psi}(t)\rangle = i\hbar\frac{\partial}{\partial t}U(t)U^{+}(t)|\tilde{\Psi}(t)\rangle + U(t)HU^{+}|\tilde{\Psi}(t)\rangle$$

$$i\hbar\frac{\partial}{\partial t}|\tilde{\Psi}(t)\rangle = \Big[\overbrace{\Big(i\hbar\frac{\partial}{\partial t}U\Big)U^{+} + UHU^{+}}^{H_{rot}}\Big]|\tilde{\Psi}(t)\rangle$$

$$\boxed{H_{rot} = i\hbar\dot{U}U^{+} + UHU^{+}}$$



$$U_{rot(t)} = e^{-i\frac{\omega}{2}\sigma_z t} = \cos\left(\frac{\omega}{2}t\right) I - i\sin\left(\frac{\omega}{2}t\right)\sigma_z = \begin{pmatrix} e^{-\frac{i\omega t}{2}} & 0 \\ 0 & e^{+\frac{i\omega t}{2}} \end{pmatrix}$$

$$\dot{U}_{rot(t)} = -\frac{\omega}{2}\sin\left(\frac{\omega}{2}t\right)I - i\frac{\omega}{2}\cos\left(\frac{\omega}{2}t\right)\sigma_z = \frac{i\omega}{2}\begin{pmatrix} -e^{-\frac{i\omega t}{2}} & 0 \\ 0 & e^{\frac{i\omega t}{2}} \end{pmatrix}$$

$$U^{+}_{rot(t)} = e^{+i\frac{\omega}{2}\sigma_z t} = \begin{pmatrix} e^{+i\frac{\omega t}{2}} & 0 \\ 0 & e^{-\frac{i\omega t}{2}} \end{pmatrix}$$

$$H_{rot} = i\hbar\dot{U}U^{+} + UHU^{+}$$

$$= \underbrace{i\hbar\,\frac{i\omega}{2}\begin{pmatrix} -e^{-i\omega t/2} & 0 \\ 0 & e^{i\omega t/2} \end{pmatrix}\begin{pmatrix} e^{\frac{i\omega t}{2}} & 0 \\ 0 & e^{-i\frac{\omega t}{2}} \end{pmatrix}}_{-\frac{\hbar\omega}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}}$$

$$\begin{pmatrix} e^{-i\omega t/2} & 0 \\ 0 & e^{i\omega t/2} \end{pmatrix}\left[\begin{pmatrix} -\frac{\hbar\omega_0}{2} & 0 \\ 0 & +\frac{\hbar\omega_0}{2} \end{pmatrix} - \begin{pmatrix} 0 & d\mathcal{E}\cos(\omega t-\phi) \\ d\mathcal{E}\cos(\omega t-\phi) & 0 \end{pmatrix}\right]\begin{pmatrix} e^{i\omega t/2} & 0 \\ 0 & e^{-i\omega t/2} \end{pmatrix}$$

$$\begin{pmatrix} e^{-i\omega t/2} & 0 \\ 0 & e^{i\omega t/2} \end{pmatrix}\begin{pmatrix} -\frac{\hbar\omega_0}{2} & 0 \\ 0 & +\frac{\hbar\omega_0}{2} \end{pmatrix}\begin{pmatrix} e^{i\omega t/2} & 0 \\ 0 & e^{-i\omega t/2} \end{pmatrix} = \begin{pmatrix} -\hbar\frac{\omega_0}{2} & 0 \\ 0 & \frac{\hbar\omega_0}{2} \end{pmatrix}$$

$$+ \begin{pmatrix} e^{-i\omega t/2} & 0 \\ 0 & e^{i\omega t/2} \end{pmatrix}\begin{pmatrix} 0 & \frac{d\mathcal{E}}{2}\left(e^{i(\omega t-\phi)} + e^{-i(\omega t-\phi)}\right) \\ \frac{d\mathcal{E}}{2}\left(e^{i(\omega t-\phi)} + e^{-i(\omega t-\phi)}\right) & 0 \end{pmatrix}\begin{pmatrix} e^{i\omega t/2} & 0 \\ 0 & e^{-i\omega t/2} \end{pmatrix}$$

$$= \begin{pmatrix} e^{-i\omega t/2} & 0 \\ 0 & e^{i\omega t/2} \end{pmatrix}\begin{pmatrix} 0 & \frac{d\mathcal{E}}{2}\left(e^{i(\omega t-\phi)} + e^{-i(\omega t-\phi)}\right)e^{-i\omega t/2} \\ \frac{d\mathcal{E}}{2}\left(e^{i(\omega t-\phi)} + e^{-i(\omega t-\phi)}\right)e^{i\omega t/2} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \frac{d\mathcal{E}}{2}\left(e^{-i\phi} + \cancel{e^{-i(2\omega t-\phi)}}\right) \\ \frac{d\mathcal{E}}{2}\left(e^{+i\phi} + \cancel{e^{i(2\omega t-\phi)}}\right) & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{d\mathcal{E}}{2}e^{-i\phi} \\ \frac{d\mathcal{E}}{2}e^{i\phi} & 0 \end{pmatrix}$$

RWA

RWA

$$H_{rot} = -\frac{\hbar\omega}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -\frac{\hbar\omega_0}{2} & 0 \\ 0 & \frac{\hbar\omega_0}{2} \end{pmatrix} + \begin{pmatrix} 0 & \frac{d\mathcal{E}}{2}e^{-i\phi} \\ \frac{d\mathcal{E}}{2}e^{i\phi} & 0 \end{pmatrix}$$

With applied RWA: $H_{rot} = -\frac{\hbar}{2}\begin{pmatrix} -\omega+\omega_0 & \frac{d\mathcal{E}}{2}e^{-i\phi} \\ \frac{d\mathcal{E}}{2}e^{i\phi} & \omega-\omega_0 \end{pmatrix}$

$$\boxed{H_{rot} = -\frac{\hbar}{2}\begin{pmatrix} -\delta & e^{-i\phi}\Omega \\ \Omega e^{i\phi} & \delta \end{pmatrix}}$$

Here $\delta = \omega - \omega_0$ = DETUNING

$$\Omega = \frac{d\mathcal{E}}{\hbar}$$

$$H_{rot} = \frac{\hbar}{2}\left(\delta\sigma_z - \Omega\cos(\phi)\sigma_x - \Omega\sin(\phi)\sigma_y\right)$$

A. Solving the driven-dipole problem in the rotating frame

In order to understand what happens, we can start by diagonalizing the rotating frame Hamiltonian $\mathcal{H}_{rot}$ as a function of δ and Ω.

✓1. What are the eigenvalues $E_\pm$ of $\mathcal{H}_{rot}$? You can introduce the value $\delta' = \sqrt{\delta^2 + \Omega^2}$.

✓2. What are the corresponding (normalized) eigenvectors $|\pm\rangle$ of $\mathcal{H}_{rot}$? Check that $|+\rangle$ and $|-\rangle$ are othogonal. Introduce the "mixing angle" η such that $|+\rangle = -\sin\eta\,|0\rangle + e^{i\alpha}\cos\eta\,|1\rangle$ and $|-\rangle = \cos\eta\,|0\rangle + e^{i\alpha}\sin\eta\,|1\rangle$. Give the expressions for α, $\sin\eta$, $\cos\eta$ and $\tan\eta$.

✓3. What do $E_\pm$ and $|\pm\rangle$ become when $\delta = 0$? In the limit of large detunings (explicitly at 2nd order in Ω/δ) ?

4. Draw and comment a cursary energy-level diagram of the driven dipole as a function of δ.

1.) $$H_{rot} = -\frac{\hbar}{2}\begin{pmatrix} -\delta & e^{-i\phi}\Omega \\ e^{i\phi}\Omega & \delta \end{pmatrix}$$

$$H_{rot}|\psi\rangle = a|\psi\rangle$$

$$\det(\underbrace{H_{rot} - Ia}_{B}) = 0$$

$$B = \begin{pmatrix} +\frac{\hbar}{2}\delta - a & -\frac{\hbar}{2}e^{-i\phi}\Omega \\ -\frac{\hbar}{2}e^{i\phi}\Omega & -\frac{\hbar}{2}\delta - a \end{pmatrix}$$

$$\det B = \left(-\frac{\hbar^2}{4}\delta^2 + a^2\right) - \left(\frac{\hbar^2}{4}\Omega^2\right) = 0$$

$$\sqrt{a^2} = \sqrt{\frac{\hbar^2}{4}(\delta^2+\Omega^2)}$$

$$\boxed{E_\pm = a_\pm = \pm\frac{\hbar}{2}\sqrt{(\underbrace{\delta^2+\Omega^2}))} = \pm\frac{\hbar}{2}\delta'}$$

2) $$H_{rot}|+\rangle = E_+|+\rangle$$

$$|+\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$+\cancel{\frac{\hbar}{2}}\begin{pmatrix} +\delta & -e^{-i\phi}\Omega \\ -e^{i\phi}\Omega & -\delta \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = +\cancel{\frac{\hbar}{2}}\delta'\begin{pmatrix} a \\ b \end{pmatrix}$$

$$a\delta - be^{-i\phi}\Omega = \delta' a$$

$$a(\delta-\delta') = be^{-i\phi}\Omega$$

$a \to \Omega e^{-i\phi}$, $b \to \delta-\delta'$

$$\Omega e^{-i\phi}\delta - e^{-i\phi}\Omega(\delta-\delta') = +\overbrace{e^{-i\phi}}^{a\ ✓}\Omega\delta'$$

$$-\cancel{e^{i\phi}}\Omega\Omega\cancel{e^{-i\phi}} - \delta(\delta-\delta') \stackrel{?}{=} \delta'\overbrace{(\delta-\delta')}^{b\ ✓}$$

$$\hookrightarrow -\Omega^2 = \underbrace{(\delta+\delta')(\delta-\delta')}_{\delta'-\delta'^2 = -\Omega^2}$$

$$\boxed{|+\rangle = \left(\Omega e^{-i\phi}|0\rangle + (\delta-\delta')|1\rangle\right)\cdot\frac{1}{\sqrt{\Omega^2+(\delta-\delta')^2}}} \quad (*)$$

To find $|-\rangle$.

$$\cancel{\frac{\hbar}{2}}\begin{pmatrix} -\delta & e^{-i\phi}\Omega \\ e^{i\phi}\Omega & \delta \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix} = \cancel{\frac{\hbar}{2}}\delta'\begin{pmatrix} c \\ d \end{pmatrix}$$

$$-\delta c + e^{-i\phi}\Omega d = \delta' c$$

$$e^{-i\phi}\Omega d = (\delta'+\delta)c$$

$d \to (\delta'+\delta)$, $c \to e^{-i\phi}\Omega$

$$-\delta e^{-i\phi}\Omega + e^{-i\phi}\Omega(\delta'+\delta) = \delta' \underbrace{e^{-i\phi}\Omega}_{c} \quad ✓$$

$$e^{i\phi}\Omega \cdot e^{-i\phi}\Omega + \delta(\delta'+\delta) = \delta'\underbrace{(\delta'+\delta)}_{d}$$

$$\Omega^2 = \delta'^2 + \cancel{\delta'\delta} - \cancel{\delta\delta'} - \delta^2$$

$$\Omega^2 = \cancel{\delta^2} + \Omega^2 - \cancel{\delta^2} \quad ✓$$

$$\boxed{|-\rangle = \left(e^{-i\phi}\Omega|0\rangle + (\delta'+\delta)|1\rangle\right)\frac{1}{\sqrt{\Omega^2+(\delta-\delta')^2}}}$$

$$\langle +|-\rangle = \underbrace{(\Omega e^{+i\phi})}_{\langle +|}\underbrace{(e^{-i\phi}\Omega)}_{|-\rangle} + (\delta-\delta')(\delta'+\delta) \stackrel{?}{=} 0$$

$$\Omega^2 + \delta^2 - \delta'^2 = 0 \quad ✓$$

OR

$$\cancel{+\frac{\hbar}{2}}\begin{pmatrix} +\delta & -e^{-i\phi}\Omega \\ -e^{i\phi}\Omega & -\delta \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \cancel{+\frac{\hbar}{2}}\delta'\begin{pmatrix} a \\ b \end{pmatrix}$$

$$-e^{i\phi}\Omega a - \delta b = \delta' b \qquad a = -(\delta'+\delta) \qquad b = e^{i\phi}\Omega$$

$$-e^{i\phi}\Omega a = b(\delta'+\delta)$$

$$|+\rangle = \left(-(\delta'+\delta)|0\rangle + e^{i\phi}\Omega|1\rangle\right)\cdot\frac{1}{(\dots)}$$

$$\frac{\hbar}{2}\begin{pmatrix} -\delta & e^{-i\phi}\Omega \\ e^{i\phi}\Omega & \delta \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix} = \frac{\hbar}{2}\delta'\begin{pmatrix} c \\ d \end{pmatrix}$$

$$e^{i\phi}\Omega c + \delta d = \delta' d$$

$$e^{i\phi}\Omega c = d(\delta'-\delta)$$

$$c \to (\delta'-\delta) \qquad d \to e^{i\phi}\Omega$$

$$\boxed{|-\rangle = \left((\delta'-\delta)|0\rangle + e^{i\phi}\Omega|1\rangle\right)\cdot\frac{1}{(\ldots)}}$$

(*)

Take two (*) & (*) as $|+\rangle$, $|-\rangle$

$$\boxed{|+\rangle = \left(\Omega e^{-i\phi}|0\rangle + (\delta-\delta')|1\rangle\right)\cdot\frac{1}{\sqrt{\Omega^2+(\delta-\delta')^2}}} \quad (*)$$

$$\boxed{|-\rangle = \left((\delta'-\delta)|0\rangle + e^{i\phi}\Omega|1\rangle\right)\cdot\frac{1}{(\ldots)}}$$

(*)

$$\langle +|-\rangle = \Omega e^{i\phi}(\delta'-\delta) + (\delta-\delta')e^{i\phi}\Omega = 0 \checkmark$$

$$\rightsquigarrow \left.\begin{aligned} |+\rangle &= -\sin\eta|0\rangle + e^{i\alpha}\cos\eta|1\rangle \\ |-\rangle &= \cos\eta|0\rangle + e^{i\alpha}\sin\eta|1\rangle \end{aligned}\right\}$$

what are $\alpha, \sin\eta, \cos\eta, \tan\eta$

$$\alpha = \phi$$

$$\tan(\eta) = \frac{\Omega}{\delta'-\delta}\begin{cases} \sin(\eta) = \dfrac{\Omega}{\sqrt{\Omega^2+(\delta'-\delta)^2}} \\ \cos(\eta) = \dfrac{\delta'-\delta}{\sqrt{\Omega^2+(\delta'-\delta)^2}} \end{cases}$$

$$|+\rangle = \left(-\Omega|0\rangle + e^{i\phi}(\delta'-\delta)|1\rangle\right)\frac{1}{(\ldots)}$$

$$|-\rangle = \left((\delta'-\delta)|0\rangle + e^{i\phi}\Omega|1\rangle\right)\frac{1}{(\ldots)}$$

3) When $\delta = 0$ $\quad (\delta' = \Omega)$

$$E_+ = \frac{\hbar}{2}\delta' = \frac{\hbar}{2}\sqrt{\delta^2+\Omega^2} = \frac{\hbar}{2}\Omega$$

$$E_- = -\frac{\hbar}{2}\delta' = -\frac{\hbar}{2}\Omega$$

$$|+\rangle = \left(-\cancel{\Omega}|0\rangle + e^{i\phi}\cancel{\Omega}|1\rangle\right)\cdot\frac{1}{\sqrt{2}\,\cancel{\Omega}} = \frac{-|0\rangle + e^{i\phi}|1\rangle}{\sqrt{2}}$$

$$|-\rangle = \left(\Omega|0\rangle + e^{i\phi}\Omega|1\rangle\right)\frac{1}{\sqrt{2}\,\Omega} = \frac{|0\rangle + e^{i\phi}|1\rangle}{\sqrt{2}}$$

$|+\rangle$ & $|-\rangle$ are in "equal superposition" where $\sin(\eta) = \frac{1}{\sqrt{2}} = \cos(\eta)$

In the off-resonance limit $(\delta \gg 0)$, $\delta' = \sqrt{\delta^2+\Omega^2} = \delta\left(1+\frac{\Omega^2}{\delta^2}\right)$

$$E_+ = \frac{\hbar}{2}\delta' \simeq \frac{\hbar}{2}\delta$$

$$E_- = -\frac{\hbar}{2}\delta$$

$$\cos(\eta) = \frac{\delta' - \delta}{\sqrt{\Omega^2+(\delta'-\delta)^2}} = \frac{\frac{\cancel{\delta}\Omega^2}{\delta^2}}{\sqrt{\Omega^2+\left(\cancel{\delta}+\frac{\Omega^2}{\delta^2}\delta-\cancel{\delta}\right)^2}} = \frac{\frac{\Omega^2}{\delta}}{\sqrt{\Omega^2+\cancel{\frac{\Omega^4}{\delta^2}}}} = \frac{\Omega}{\delta} \simeq 0$$

$$\sin(\eta) = \frac{\Omega}{\sqrt{\Omega^2+(\delta'-\delta)^2}} \simeq \frac{\Omega}{\Omega} = 1$$

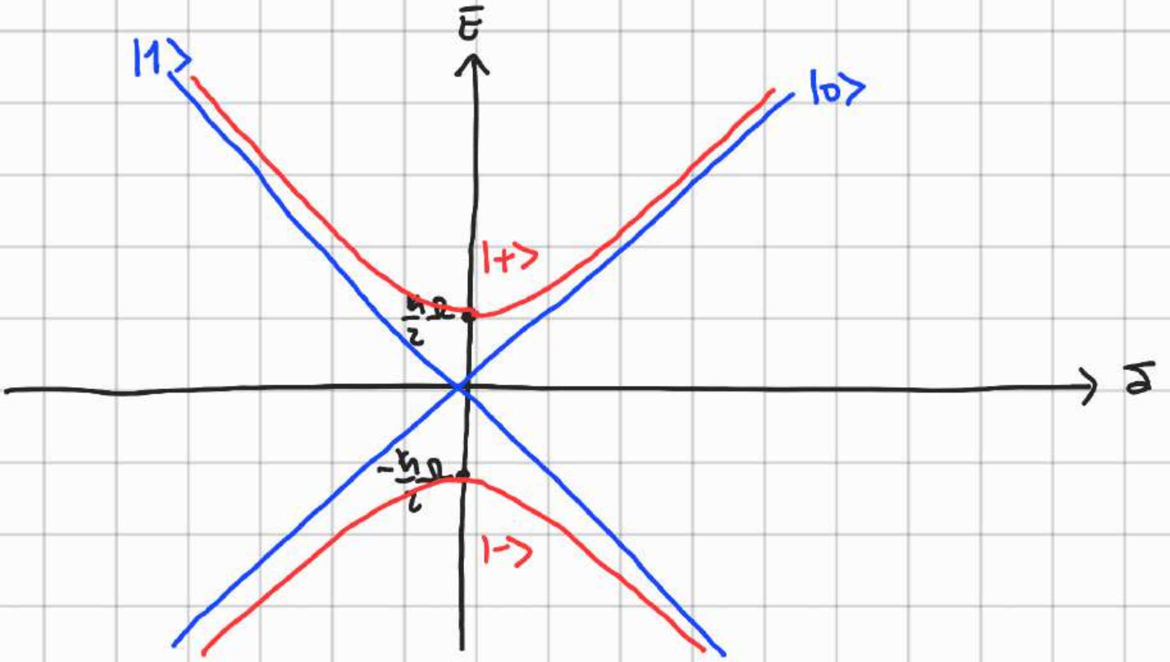
$\alpha = \pi/2$

$$|+\rangle = \overset{\text{global phase}}{-}|0\rangle = |0\rangle$$

$$|-\rangle = e^{i\frac{\pi}{2}}|1\rangle = i|1\rangle = |1\rangle$$

$*$ In the limit of large detuning, the effect of the driving field IS NOT FELT from the 2-level system. The states goes back into initial states.

4)



$$E_{\pm} = \pm \frac{\hbar}{2} \delta' = \frac{\hbar}{2} \sqrt{\delta^2 + \Omega^2}$$

B. Time evolution of the driven dipole

To solve the time evolution of an atom coupled to a field, we can use the time-evolution operator U :

$$|\psi(t)\rangle = U(t)\,|\psi(0)\rangle = U(t)\,|0\rangle$$

and decompose $|0\rangle$ onto the relevant basis.

✓ 1. What is the time evolution of states $|\pm\rangle$?

✓ 2. Write $|\psi(t)\rangle$ in a more explicit form given by the expression of $U(t)$ and the decomposition of $|0\rangle$.

✓ 3. What simple operation will provide the expression for $\mathcal{P}_1(t)$, the probability of the atom to be in state $|1\rangle$ at a given time t ? The oscillations of $\mathcal{P}_1(t)$ are called **"Rabi oscillations"**.

✓ 4. Comment on the maximum value and on the period of $\mathcal{P}_1(t)$.

✓ 5. In the case $\delta = 0$, what value of Ωt makes a balanced superposition of $|0\rangle$ and $|1\rangle$ ("**$\pi/2$-pulse**") ? a state inversion $|0\rangle \to |1\rangle$ ("**π-pulse**") ?

✓ 6. Draw $\mathcal{P}_1(t)$ for $\Omega t \in [0; 3\pi]$ in three different cases : $\delta = 0$, $\delta = \Omega$ and $\delta = 3\Omega$.

1) $|+\rangle = -\sin\eta\,|0\rangle + e^{i\alpha}\cos\eta\,|1\rangle$
$|-\rangle = \cos\eta\,|0\rangle + e^{i\alpha}\sin\eta\,|1\rangle$

$\alpha = \phi$

$\tan(\eta) = \frac{\Omega}{\delta' - \delta}$ $\left\{ \begin{array}{l} \sin(\eta) = \frac{\Omega}{\sqrt{\Omega^2 + (\delta'-\delta)^2}} \\ \cos(\eta) = \frac{\delta'-\delta}{\sqrt{\Omega^2 + (\delta'-\delta)^2}} \end{array} \right.$

$|+\rangle = \left(-\Omega|0\rangle + e^{i\phi}(\delta'-\delta)|1\rangle\right)\frac{1}{(\ldots)}$

$|-\rangle = \left((\delta'-\delta)|0\rangle + e^{i\phi}\Omega|1\rangle\right)\frac{1}{(\ldots)}$

$$\boxed{\begin{array}{l} |+(t)\rangle = e^{-iE_+t}|+\rangle = e^{-i\frac{\hbar}{2}\delta' t}|+\rangle \\ |-(t)\rangle = e^{-iE_-t}|-\rangle = e^{+i\frac{\hbar}{2}\delta' t}|-\rangle \end{array}}$$

2) $\langle 0|+\rangle = -\sin(\eta)$
$\langle 0|-\rangle = \cos(\eta)$

$|0\rangle = -\sin(\eta)|+\rangle + \cos(\eta)|-\rangle$

Initilly $|\psi(0)\rangle = |0\rangle \rightsquigarrow$ $\boxed{|\psi(t)\rangle = -\sin(\eta)e^{-i\frac{\hbar}{2}\delta' t}|+\rangle + \cos(\eta)e^{+i\frac{\hbar}{2}\delta' t}|-\rangle}$

3) $P_1(t) = |\langle 1|\psi(t)\rangle|^2$

$-\underline{\sin(\eta)}e^{-i\frac{\hbar}{2}\delta' t}\underline{e^{i\phi}}\underline{\cos\eta}|1\rangle + \underline{\cos(\eta)}e^{+i\frac{\hbar}{2}\delta' t}\underline{e^{i\phi}}\underline{\sin(\eta)}|1\rangle + \ldots \cancel{|0\rangle}$

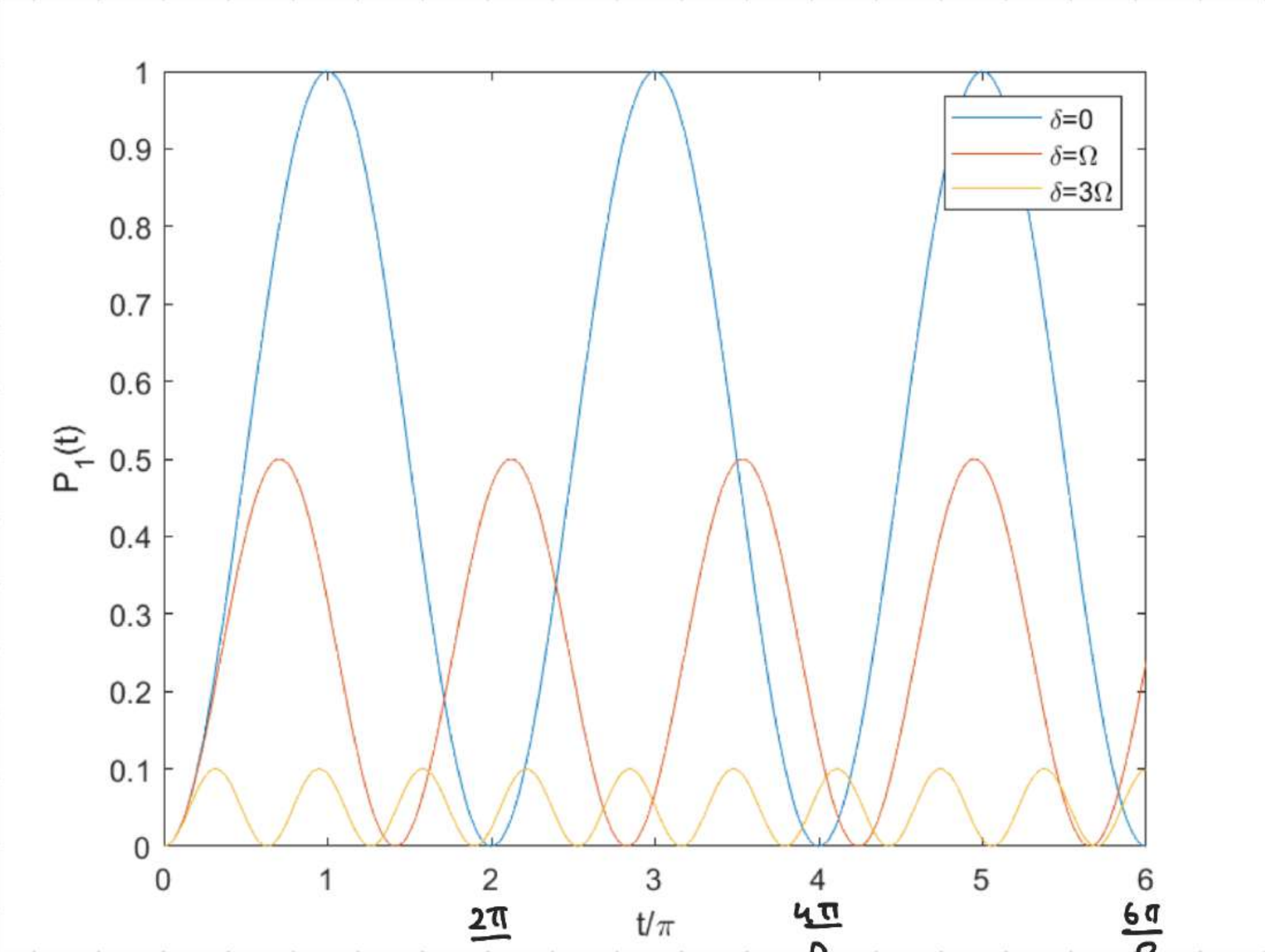
$$P_1(t) = \left|\underbrace{\left(e^{i\frac{\delta'}{2}\hbar t} - e^{-i\frac{\delta'}{2}\hbar t}\right)}_{2i\sin\left(\frac{\delta'}{2}\hbar t\right)}\sin(\eta)\cos(\eta)\,e^{i\phi}\right|^2$$

$$P_1(t) = 4.\left(\frac{\Omega.(\delta'-\delta)}{\Omega^2+(\delta'-\delta)^2}\right)^2.\sin^2\left(\frac{\delta'\hbar t}{2}\right)$$

$$\frac{\Omega(\delta'-\delta)}{\Omega^2+\delta'^2-2\delta'\delta+\delta^2} = \frac{\Omega(\delta'-\delta)}{2\delta'^2-2\delta'\delta} = \left(\frac{\Omega\cancel{(\delta'-\delta)}}{2\delta'\cancel{(\delta'-\delta)}}\right)^2$$

$$= \Omega^2/4\delta'^2$$

$$P_1(t) = \cancel{4}\,\frac{\Omega^2}{\cancel{4}\delta'^2}\sin^2\left(\frac{\delta'\hbar t}{2}\right) = \frac{\Omega^2}{\Omega^2+\delta^2}\sin^2\left(\frac{\sqrt{\Omega^2+\delta^2}\,\hbar t}{2}\right)$$



$\frac{2\pi}{\Omega}$ $\frac{4\pi}{\Omega}$ $\frac{6\pi}{\Omega}$

T

4) The amplitude is given by $\frac{\Omega^2}{\Omega^2+\delta^2}$

Max. value can be achieved when $\delta = 0$ (tuned $\omega = \omega_0$ resonance)

For $\delta = 0$, $T = \frac{2\pi}{\delta'} = \frac{2\pi}{\sqrt{\delta^2+\Omega^2}} = \frac{2\pi}{\Omega}$

Increasing detuning decreases the period, faster oscillations

5) For $\delta = 0$ $\quad P_1(t) = \sin\left(\frac{\Omega}{2}\hbar t\right)$ where $\hbar = 1$

For balanced superposition $P_1(t) = \frac{1}{2}$

thus $\frac{\Omega t}{\cancel{2}} = \frac{\pi}{\cancel{4}2} \Rightarrow \Omega t = \frac{\pi}{2}$ "$\frac{\pi}{2}$ Pulse"

Inversion $(|0\rangle \to |1\rangle)$ $P_1(t) = 1$ $\quad \frac{\Omega t}{2} = \frac{\pi}{2} \Rightarrow \Omega t = \pi$

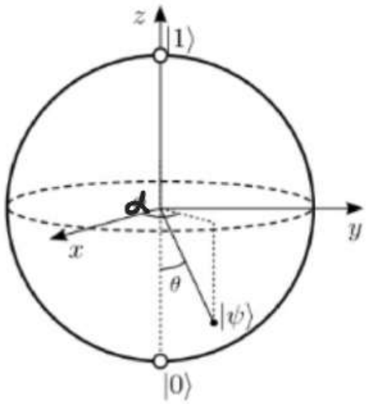
"π-Pulse"

C. Graphical representation : the Bloch sphere

The state of the atomic dipole can always be written

$$|\psi\rangle = \cos(\theta/2)\,|0\rangle + e^{i\alpha}(\sin\theta/2)\,|1\rangle \tag{4}$$

with $\theta \in [0;\pi]$ and $\phi \in [0;2\pi]$. We can then represent $|\psi\rangle$ on a sphere defined by these two angles : the **Bloch sphere**.



✓ 1. What is the evolution of the Bloch vector $\vec{a}$ under Hamiltonian (3) when $\delta = 0$?

✓ 2. What is the evolution $\vec{a}$ under Hamiltonian (3) when $\Omega = 0$, for a state lying on the equator ?

3. Relate this to the Pauli matrices form of the Hamiltonian.

4. What becomes of state $|0\rangle$ after a π-pulse ? after a $\pi/2$-pulse ?

1) $$\vec{a} = \begin{bmatrix} \sin(\theta)\cos\alpha \\ \sin(\theta)\sin\alpha \\ \cos\theta \end{bmatrix}$$ "Generic Bloch Vector"

$$H_{rot} = \frac{\hbar}{2}\Big(\underbrace{\delta}_{h_3}\sigma_z \underbrace{- \Omega\cos(\phi)}_{h_1}\sigma_x \underbrace{- \Omega\sin(\phi)}_{h_2}\sigma_y \Big) = -\vec{h}\vec{\sigma}$$

The evolution of a generic vector under the Hamiltonian H is given by its precession around $\vec{h}$:

$$\dot{\vec{a}} = \frac{2}{\hbar}\vec{h}\times\vec{a}$$

with a frequency of $\frac{2|\vec{h}|}{\hbar}$ How?

From vector product $\vec{a}\times\vec{b}$

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$\vec{h}\times\vec{a}$

$$\vec{a}\times\vec{b} = (a_2b_3 - a_3b_2)\sigma_x + (a_3b_1 - a_1b_3)\sigma_y + (a_1b_2 - a_2b_1)\sigma_z$$

$$\dot{\vec{a}} = \begin{bmatrix} -\Omega\sin(\phi)\cos\theta - \cancel{\delta}\sin(\theta)\sin\alpha \\ \cancel{\delta}\sin(\theta)\cos\alpha + \Omega\cos(\phi)\cos\theta \\ -\Omega\cos(\phi)\sin(\theta)\sin\alpha + \Omega\sin(\phi)\sin(\theta)\cos\alpha \end{bmatrix}$$

$\delta = 0$

$\delta = 0$

$$\dot{\vec{a}} = \begin{bmatrix} -\Omega\sin(\phi)\cos\theta \\ +\Omega\cos(\phi)\cos\theta \\ -\Omega\cos(\phi)\underbrace{\sin(\theta)\sin\alpha}_{a_y} + \Omega\sin(\phi)\underbrace{\sin(\theta)\cos\alpha}_{a_x} \end{bmatrix} = \Omega\begin{bmatrix} -\sin\phi\, a_z \\ \cos\phi\, a_z \\ a_x\sin(\phi) - a_y\cos\phi \end{bmatrix}$$

2) For $\Omega = 0$, no external field :

$$H_{rot} = -\frac{\hbar}{2}\left[-\delta \sigma_z\right]$$

Rotation around the equator

$$\dot{\vec{a}} = \delta \begin{bmatrix} a_y \\ -a_x \\ 0 \end{bmatrix}$$

4) Under $\frac{\pi}{2}$ pulse $(\Omega t = \frac{\pi}{2})$ with $\delta = 0$, $(\delta' = \Omega)$

$$|\psi(t)\rangle = -\sin(\eta) e^{-i\frac{\hbar}{2}\delta' t} |+\rangle + \cos(\eta) e^{+i\frac{\hbar}{2}\delta' t} |-\rangle$$

$$= \frac{1}{\sqrt{2}} e^{-i\frac{\hbar}{2}\frac{\pi}{2}} |+\rangle + \frac{1}{\sqrt{2}} e^{+i\frac{\hbar}{2}\frac{\pi}{2}} |-\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle - i e^{i\phi} |1\rangle\right]$$

lying on the equator with a phase contribution

D. Phase of the driving field

1. How does the phase of the field map on the Bloch sphere ?
2. Represent on the Bloch sphere the following sequence, starting from state $|0\rangle$:
   - a $\pi/2$-pulse with $\delta = 0$ and $\phi = 0$;
   - free evolution $(\Omega = 0)$ with $\delta \neq 0$ for a time $T = 2\pi/\delta$;
   - a second $\pi/2$-pulse, with a phase $\phi_0 = 0$ or $\phi_1 = \pi/2$ or $\phi_2 = \pi$.

Such a sequence can be used to measure very precisely what happens when the dipole evolves freely between the two $\pi/2$-pulses, in what is called a **Ramsey interference** experiment. This type of sequence is the core principle of atomic clocks.

1)

$$\vec{h} = \frac{\hbar}{2}\begin{bmatrix} \Omega\cos(\phi) \\ \Omega\sin(\phi) \\ -\Delta \end{bmatrix}$$

2) $|\psi(0)\rangle = |0\rangle$

1). Under $\pi/2$ pulse the state evolve as:

$$|0\rangle \rightarrow \frac{1}{\sqrt{2}}\left[|0\rangle + i e^{i\phi}|1\rangle\right]$$

$$|1\rangle \rightarrow \frac{1}{\sqrt{2}}\left[-e^{-i\phi}|0\rangle + i|1\rangle\right]$$

2). Free evolution : Rotation around the z-axis

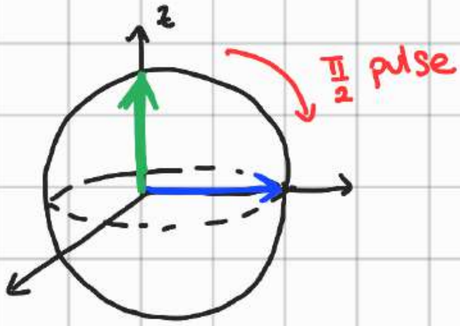
3). Under $\frac{\pi}{2}$ pulse

a) $\phi=0$ $\quad |\psi\rangle = i|1\rangle$

b) $\phi=\pi/2$ $\quad |\psi\rangle = \frac{(1+i)}{2}\left[|0\rangle + i|1\rangle\right]$ → Rot. around the equator

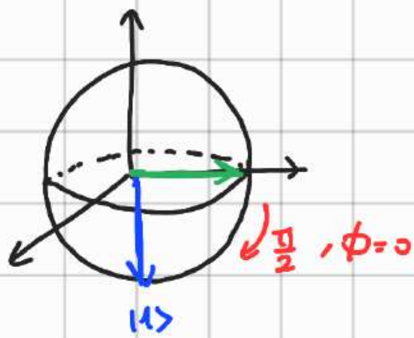
c) $\phi=\pi$ $\quad |\psi\rangle = |0\rangle$

1)

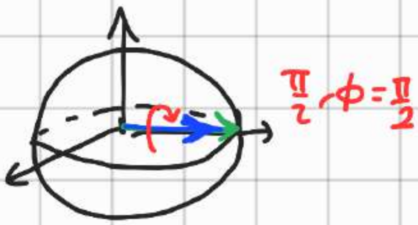


2)

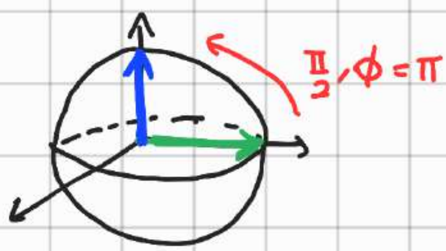
3.a)



3.b)



3.c)



II. ATOMIC CLOCKS

Atomic clocks are the most precise measurement instruments. They are designed to produce a purely harmonic signal at a given frequency. That frequency is compared to an absolute reference given by the energy difference between two levels of an atom, for example the two lowest energy levels of caesium. The precision of state of the art atomic clocks is at the moment close to 10^{-19}, which means that the clock would be delayed by 0.1 s over the age of the universe.

There are different possible designs for atomic clocks. We will study here the so called caesium fountain clock. It is the current design used for the definition of the second (the second is by definition the duration between 9 192 631 770 oscillations of the caesium clock signal).

The design of the atomic clock is shown on figure 1. The atoms are prepared at the bottom. They are then launched in the vertical direction at a given velocity v. They travel through an interaction region, where they interact with a oscillating magnetic field (the microwave field) at the angular frequency ω. They then move freely in the upper part of the fountain, where no field is present. Then, due to gravity, they fall back through the same interaction region, where they interact once again with the microwave field. The state of the atoms is then measured in the preparation region. As you will see, the number of atoms in a given state at the output of the clock gives a very sensitive indication of the frequency difference between the microwave signal and the atomic transition frequency that can be used for example to regulate the microwave frequency in a feedback loop.

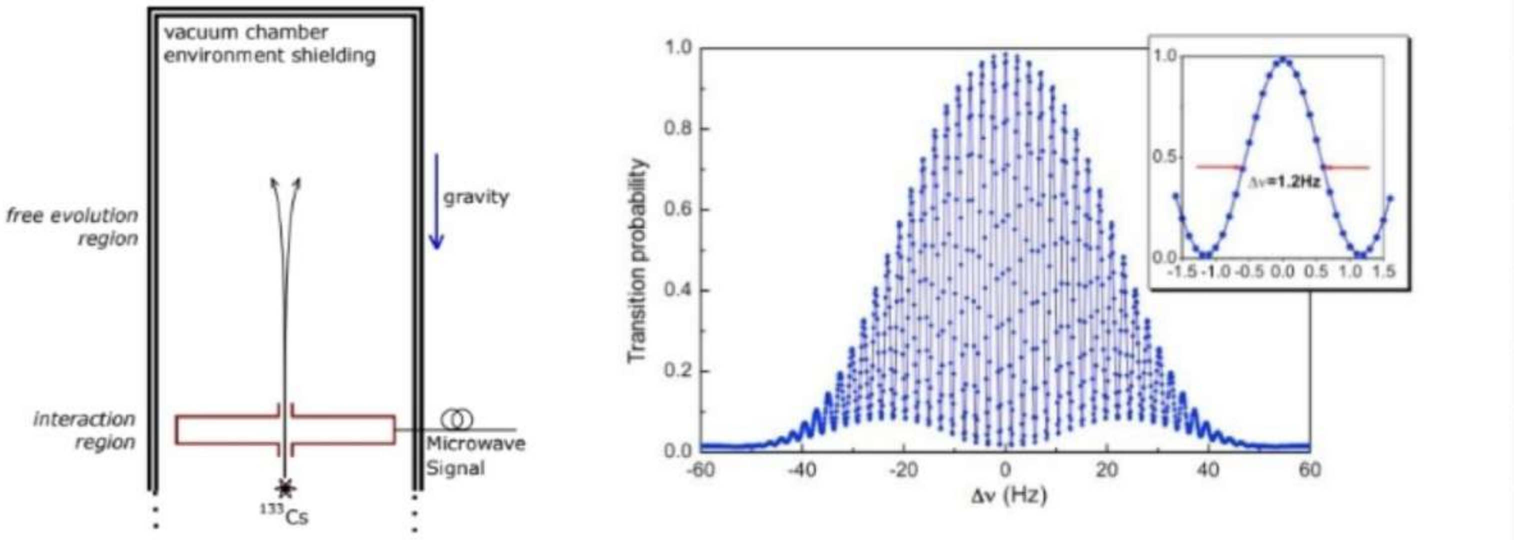


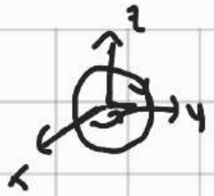
FIG. 1. Left : Simplified scheme of a Caesium fountain clock. Right : Ramsey fringes obtained experimentaly on the Rubidium clock of Observatoire de Paris. Each point corresponds to a single measurement.

A. Frequency measurement and clock signal

In a clock the measurement is done in three phases similar to the ones described in I D.2. In order to simplify the analysis, we will first neglect the effect of the angular frequency difference $\delta = \omega - \omega_0$ during the interaction phases. But we will keep it for the description of the free evolution. The state of an atom on the Bloch sphere will be represented as the Bloch vector $\vec{a}$. $\vec{a} = (\langle\sigma_x\rangle, \langle\sigma_y\rangle, \langle\sigma_z\rangle)$.

1. The atom is prepared initially in state $|0\rangle$. At t = 0, the atom enters the interaction region and takes a time τ to cross it. The driving field is homogeneous in that region with a Rabi frequency Ω, and $\omega = \omega_0$. Show that after a time τ :

$$\vec{a}_\tau = \begin{pmatrix} 0 \\ \sin(\Omega\tau) \\ -\cos(\Omega\tau) \end{pmatrix} \tag{5}$$



$$\begin{array}{cccc} & \sigma_x & \sigma_y & \sigma_z \\ 1)\quad |0\rangle = & [0, & 0, & 1] \end{array} \quad \text{(Bloch vector)}$$

For $\phi = 0$ $\quad \dot{\vec{a}} = \Omega \begin{bmatrix} \sin\phi\, a_z \\ -\cos\phi\, a_z \\ -a_x \sin(\phi) + a_y \cos\phi \end{bmatrix} = \Omega \begin{bmatrix} 0 \\ -a_z \\ a_y \end{bmatrix} = \begin{bmatrix} 0 \\ -\cos\theta \\ \sin\theta \sin\alpha \end{bmatrix}$

$\Downarrow$ $\alpha = \frac{\pi}{2}$ rot. around yz

$$\vec{a}(t) = \begin{bmatrix} 0 \\ +\sin(\Omega t) \\ -\cos(\Omega t) \end{bmatrix} + C \qquad \vec{a}(0) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \cancel{C} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \vec{a_\tau} = \begin{bmatrix} 0 \\ +\sin(\Omega\tau) \\ -\cos(\Omega\tau) \end{bmatrix}$$

$C = 0$

2. After the interaction region, the atom moves freely in the upper part of the clock during a time $T \gg \tau$. Then due to gravity, they fall back in the interaction region. Show that after that free evolution :

$$\vec{a}_{\tau+T} = \begin{pmatrix} -\sin(\delta T)\sin(\Omega\tau) \\ \cos(\delta T)\sin(\Omega\tau) \\ -\cos(\Omega\tau) \end{pmatrix} \tag{6}$$

Free evolution $(\Omega = 0)$: Rotation around z-axis

$$\dot{\vec{a}} = \delta \begin{bmatrix} a_y \\ -a_x \\ 0 \end{bmatrix} = \delta \begin{bmatrix} \sin\theta \sin\alpha \\ -\sin\theta\cos\alpha \\ 0 \end{bmatrix} = \delta \begin{bmatrix} \sin\alpha \\ -\cos\alpha \\ 0 \end{bmatrix}$$

$\Downarrow$

$\theta = \frac{\pi}{2}$

Here initial state is $\vec{a_\tau} = \begin{bmatrix} 0 \\ +\sin\Omega\tau \\ -\cos(\Omega\tau) \end{bmatrix}$

$$\int_\tau^{T+\tau} \dot{\vec{a}}\, dt = \vec{a} \Big|_\tau^{T+\tau} = \begin{bmatrix} -\cos(\delta t) \\ -\sin(\delta t) \\ 0 \end{bmatrix}$$

$$\vec{a}(t) = \begin{bmatrix} -\cos(\delta t) \\ -\sin(\delta t) \\ 0 \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$\vec{a}(\tau) = \begin{bmatrix} -\cos(\delta\tau) + c_1 \\ -\sin(\delta\tau) + c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ +\sin\Omega\tau \\ -\cos(\Omega\tau) \end{bmatrix}$$

$$c_3 = \cos(\Omega\tau)$$

$$c_2 = -\sin(\Omega\tau) + \sin(\delta\tau)$$

$$c_1 = +\cos(\delta\tau)$$

.... consider $(T - \tau \simeq T)$:

$$\vec{a}_{\tau+T} = \begin{bmatrix} \sin(\delta T)\sin(\Omega\tau) \\ \cos(\delta T)\sin(\Omega\tau) \\ -\cos(\Omega\tau) \end{bmatrix}$$

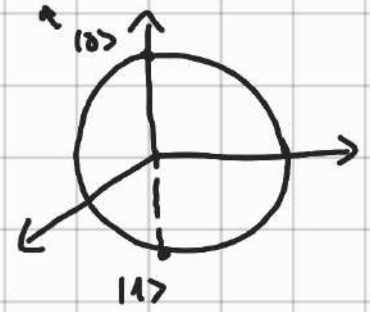
3. After the free evolution, the atom passes again through the interaction region. In that case, compute the coordinates of the Bloch vector when the Rabi frequency was chosen such that $\Omega\tau = \frac{\pi}{2}$. Compute the probability $\mathcal{P}_1$ to detect the atom in state $|1\rangle$ as a function of the frequency difference $\delta/2\pi$. This pattern is called "Ramsey fringes".

After free evolution, the state will evolve again in the interaction region.

$$\vec{a}_{fin} = \begin{bmatrix} \sin(\Omega\tau)\sin(\delta T) \\ \cos(\Omega\tau)\sin(\Omega(\underbrace{t-T}_{\tau})) + \sin(\Omega\tau)\cos(\delta T)\cos(\Omega(\underbrace{t-T}_{\tau})) \\ -\cos(\Omega\tau)\cos(\Omega(t-T)) + \sin(\Omega\tau)\cos(\delta T)\sin(\Omega(t-T)) \end{bmatrix}$$

$$\vec{a}_{fin} = \begin{bmatrix} \sin(\delta T) \\ \cos(\delta T)\cos(\Omega J) \\ \cos(\delta T)\sin(\Omega J) \end{bmatrix} \xrightarrow{\text{for } \Omega J = \frac{\pi}{2}} \begin{bmatrix} \sin(\delta T) \\ 0 \\ \cos(\delta T) \end{bmatrix}$$

eigenvalue +1



eigenvalue −1

$$\langle \sigma_z \rangle = p_0 \cdot (+1) + p_1(-1) = p_0 - p_1$$

$p_0 + p_1 = 1$ (from normalization)

$$\begin{aligned} -/\ p_0 - p_1 &= \langle \sigma_z \rangle \\ p_0 + p_1 &= 1 \\ \hline 2p_1 &= 1 - \langle \sigma_z \rangle \end{aligned}$$

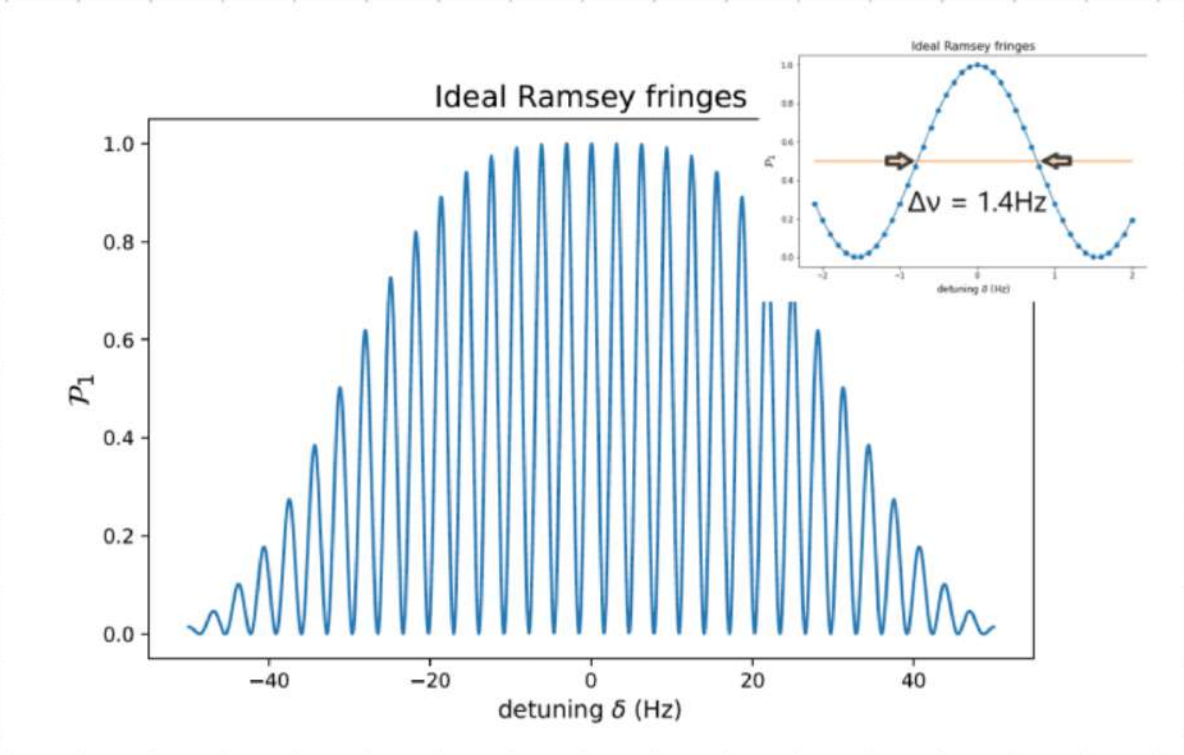
$$p_1 = \frac{1}{2}\left(1 - \langle \sigma_z \rangle\right) = \boxed{\frac{1}{2}\left(1 - \cos(\delta T)\right)}$$

4. *Effects of the detuning during the interaction.* So far we did not consider the effect of the detuning of the microwave compared to the atomic frequency in the interaction region. Identify each of the 3 steps of the experiment as a rotation of the Bloch vector around a certain axis depending on the Rabi frequency Ω and the detuning δ. Use rotation matrices in Python Scipy (by using Jupyter notebook at `https://mybinder.org/v2/git/https%3A%2F%2Fc4science.ch%2Fsource%2Fquantum_optics2021.git/master`) to compute the exact expectation value of $\langle\sigma_z\rangle$ including the effects of detuning. Plot the obtained fringe signal as a function of detuning for the case where $\Omega/2\pi = 14\,\mathrm{Hz}$, $\Omega\tau = \frac{\pi}{2}$ and $T = 0.3\,\mathrm{s}$.

The evolution of the Bloch vector in the rotating frame is a precession around the vector $-\frac{\hbar}{2}[\Omega, 0, -\delta]$ with frequency of $\sqrt{\Omega^2+\delta^2} = \delta'$

We know that free evolution corresponds to a rotation of δT around the z-axis

$$\vec{a_f} = R(\delta'\tau)\, R_0(\delta T)\, R(\delta'\tau)\, \vec{a_i}$$



By computing the portion of atoms in the excited state, one can estimate the freq. difference between the atoms and the driving field. This info. is used for feedback to regulate the microwave frequency.