Exercise set #2

September 29, 2021

Exercise 1:

Given the following qubits:

$$
\begin{aligned}
|\Psi_1\rangle &= \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle) \\
|\Psi_2\rangle &= \frac{1}{5}(4\,|00\rangle + 2\,|01\rangle + 2\,|10\rangle + |11\rangle) \\
|\Psi_3\rangle &= \frac{1}{2}(i\,|00\rangle - i\,|01\rangle - |10\rangle - i\,|11\rangle) \\
|\Psi_4\rangle &= \frac{1}{2}(|0\rangle + i\,|1\rangle + i\,|2\rangle - |3\rangle) \\
|\Psi_5\rangle &= \frac{1}{\sqrt{2}}(|00\rangle + i\,|10\rangle) - \frac{i}{\sqrt{2}}\,|11\rangle
\end{aligned}
$$

a) Are the qubits entangled?

b) Determine the tensor product $X \otimes H$.

c) Determine the tensor product $Z \otimes X$.

d) Determine the tensor product $X \otimes Y$.

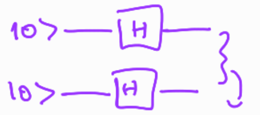
e) What is the result of applying $X \otimes H$ to $|\Psi_1\rangle$?

$\langle \phi |\ |0\rangle\langle 0| \otimes I \otimes I\, |0\rangle$

$|0 \psi_1 \psi_2\rangle$

$|0\rangle\langle 0|0\rangle \otimes I|\psi_1\rangle \otimes I|\psi_2\rangle$

a) $|\psi_1\rangle = \frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right)$



no gate between them

$\frac{1}{\sqrt{2}}|0\rangle \otimes \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) + \frac{1}{\sqrt{2}}|1\rangle \otimes \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$

$= \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \otimes \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$ → separable not entangled

$|\psi_2\rangle = \frac{1}{5}\left(4|00\rangle + 2|01\rangle + 2|10\rangle + |11\rangle\right)$

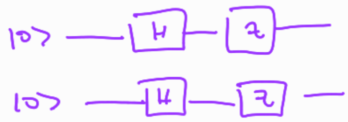
$\frac{1}{\sqrt{5}}|0\rangle \otimes \frac{2}{\sqrt{5}}\left(2|0\rangle + |1\rangle\right) + \frac{1}{\sqrt{5}}|1\rangle \otimes \frac{1}{\sqrt{5}}\left(2|0\rangle + |1\rangle\right)$

$\frac{1}{\sqrt{5}}\left(2|0\rangle + |1\rangle\right) \otimes \frac{1}{\sqrt{5}}\left(2|0\rangle + |1\rangle\right)$ → not entangled separable

$|\psi_3\rangle = \frac{1}{2}\left(i|00\rangle - i|01\rangle - |10\rangle - i|11\rangle\right)$ → No symmetry

$\frac{1}{\sqrt{2}}|0\rangle \otimes \frac{i}{\sqrt{2}}\left(|0\rangle - |1\rangle\right) + \frac{1}{\sqrt{2}}|1\rangle \otimes \frac{1}{\sqrt{2}}\left(-|0\rangle - i|1\rangle\right)$ → entangled (non-separable)

$|\psi_4\rangle = \frac{1}{2}\left(|0\rangle + i|1\rangle + i\overset{10}{|2\rangle} - \underset{i^2}{\overset{11}{|3\rangle}}\right)$



$|\psi_4\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + i|1\rangle\right) \otimes \frac{1}{\sqrt{2}}\left(|0\rangle + i|1\rangle\right)$ → Not entangled

$|\psi_5\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + i|10\rangle\right) - \frac{i}{\sqrt{2}}|11\rangle$ → No symmetry → Entangled

b) $X \otimes H = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} 1/\sqrt{2}$

$X \to \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $H \to \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

c) $Z \otimes X = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$

d) $X \otimes Y = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix}$

e) $X \otimes H\,|\psi_1\rangle = \frac{1}{2\sqrt{2}}\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

$(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$

or

$= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}\left(|00\rangle + |10\rangle\right)$ → Bell State

$X \otimes H\ |\psi_1\rangle = X \otimes H\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right)$

$H|0\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$

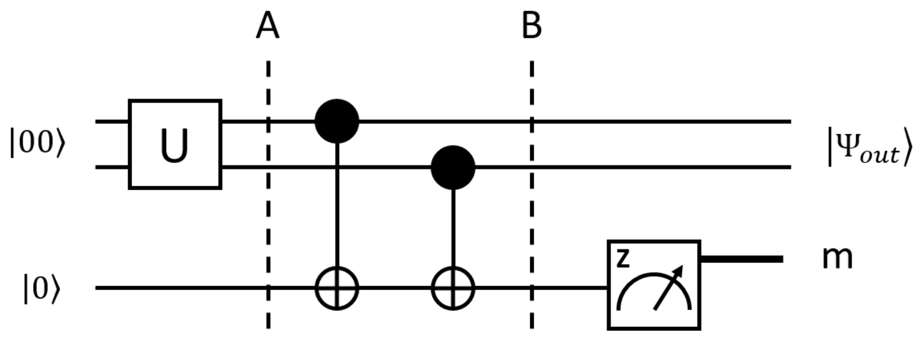
$X|0\rangle = |1\rangle$

$= \frac{1}{2\sqrt{2}}\left(|1\rangle\left(|0\rangle + |1\rangle\right) + |1\rangle\left(|0\rangle - |1\rangle\right) + |0\rangle\left(|0\rangle + |1\rangle\right) + |0\rangle\left(|0\rangle - |1\rangle\right)\right)$

$\frac{1}{2\sqrt{2}}\left(|10\rangle + \cancel{|11\rangle} + |10\rangle - \cancel{|11\rangle} + |00\rangle + \cancel{|01\rangle} + |00\rangle - \cancel{|01\rangle}\right)$

Exercise 2:

Given the following circuit:



a) Determine U so that at point A the top register will be in the maximum superposition state $|\Psi_A^{top}\rangle = \frac{1}{2}(|0\rangle + |1\rangle + |2\rangle + |3\rangle)$.

b) What is the state of the three cubit system at point A?

c) What is the state of the three cubit system at point B?

d) What is the probability we obtain $m = +1$ (outcome $|0\rangle$) when we measure the bottom qubit in the computational basis?

e) What is the final state $|\Psi_{out}\rangle$ of the top register when we obtained $m = +1$ (outcome $|0\rangle$) when we measured the bottom qubit in the computational basis?

f) Is the final state obtained at e) entangled?

a) $H \otimes H = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$

b) $|\psi_A\rangle = \frac{1}{2}\left(|00\rangle|0\rangle + |01\rangle|0\rangle + |10\rangle|0\rangle + |11\rangle|0\rangle\right)$

$= \frac{1}{2}\left(|0\rangle + |2\rangle + |4\rangle + |6\rangle\right)$

$H \otimes H \otimes I\, |000\rangle = |\psi_A\rangle$

c) $|\psi_A\rangle = \frac{1}{2}\left(|\overset{12}{000}\rangle + |010\rangle + |101\rangle + |111\rangle\right)$

after 1'st CNOT

CNOT

after 2'nd

$\frac{1}{2}\left(|000\rangle + |011\rangle + |101\rangle + |110\rangle\right)$

$|\psi_A\rangle = \frac{1}{2}\left(|0\rangle + |3\rangle + |5\rangle + |6\rangle\right)$

d) $|\psi_B\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) \otimes \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right) \otimes \frac{1}{\sqrt{2}}|1\rangle \longrightarrow$ prob $= 1/2$

e) $|\psi_{out}\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$

f) ψ_{out} is entangled.

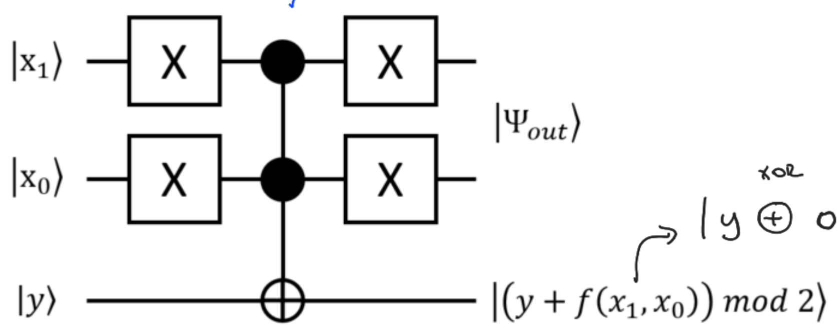
b) $|\psi\rangle = (X \otimes X \otimes I)\ \text{TOFFOLI}\ (X \otimes X \otimes I)\,|000\rangle$

$= (X \otimes X \otimes I)\ \underbrace{\text{TOFFOLI}\ |110\rangle}_{|111\rangle}$

$= |001\rangle$

Exercise 3:

Given the following circuit: Toffoli Gate



XOR $|y \oplus 0\rangle$

a) What is the value of $|\Psi_{out}\rangle$? $|x_1 x_0\rangle$ → no change $(XX = I)$

b) Suppose $|x_1x_0\rangle = |00\rangle$ and $|y\rangle = |0\rangle$, what is the final three qubit state? $|001\rangle$

c) Repeat for all the other input states and write the final value of the bottom qubit into the following table:

only change occurs here

x_1	x_0	y	$(y + f(x_1, x_0))mod2$
0	0	0	1
0	1	0	0
1	0	0	0
1	1	0	1
0	0	1	0
0	1	1	1
1	0	1	0
1	1	1	1

f=0

d) This function corresponds to that of a search problem with a specific solution x^*. What is x^*? = 00

$-|000\rangle + |001\rangle - |010\rangle + |011\rangle - |100\rangle$
$+ |101\rangle - |110\rangle + |111\rangle$

=

$|\psi_A\rangle = H \otimes H \otimes H \; |001\rangle = \frac{1}{\sqrt{8}} \left(|0\rangle - |1\rangle + |2\rangle - |3\rangle + |4\rangle - |5\rangle + |6\rangle + |7\rangle \right)$

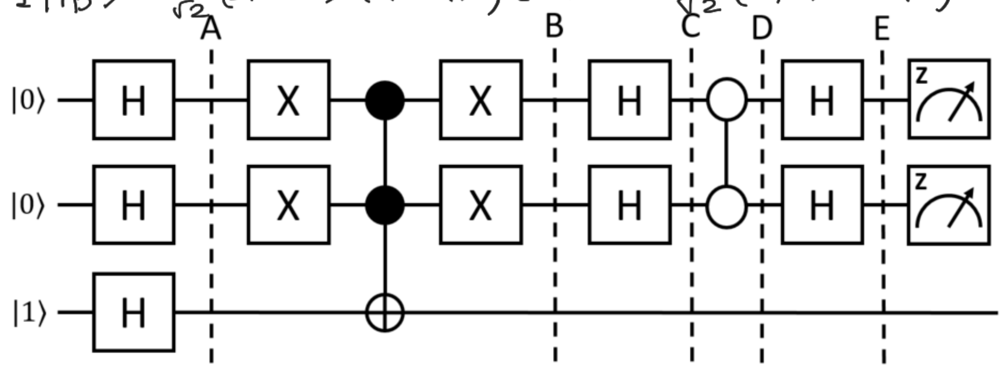
$|\psi_B\rangle = (X \otimes X \otimes I) \text{ Toffoli } (X \otimes X \otimes I) \, |\psi_A\rangle = \frac{1}{\sqrt{8}} \left(-|0\rangle + |1\rangle + |2\rangle - |3\rangle + |4\rangle - |5\rangle + |6\rangle - |7\rangle \right)$

$|\psi_C\rangle = (H \otimes H \otimes I) \; |\psi_B\rangle = \frac{1}{\sqrt{8}} \left(|0\rangle - |1\rangle - |2\rangle + |3\rangle - |4\rangle + |5\rangle - |6\rangle + |7\rangle \right)$

$|\psi_D\rangle = \frac{1}{\sqrt{8}} \left(-|0\rangle + |1\rangle - |2\rangle + |3\rangle - |4\rangle + |5\rangle - |6\rangle + |7\rangle \right)$

We will implement Grover's algorithm as shown in the following figure:

$|\psi_E\rangle = H \otimes H \otimes I \, |\psi_D\rangle = \frac{1}{\sqrt{2}} \left(-|000\rangle + |001\rangle \right) = |00\rangle \otimes \frac{1}{\sqrt{2}} \left(-|0\rangle + |1\rangle \right)$



e) Determine the state of the system at points A,B,C,D and E, knowing that the gate between C and D has the following unitary:

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

→ look at symmetries

f) Which of the five states from e) have entanglement in the top two qubits?

g) Did the algorithm succeed in finding $x^* = 00$?