05/10/22

Cauchy-Schwarz Inequality

$$|\langle\phi|\psi\rangle| \leq \sqrt{\langle\phi|\psi\rangle} \cdot \sqrt{\langle\psi|\psi\rangle}$$

$$|\langle\phi|\psi\rangle| \leq \|\phi\| . \|\psi\|$$

$$\| \xrightarrow{?} \text{ Heisenberg Uncertainty Principle}$$

Notion of Tensor Product

Hilbert Space $\mathcal{H}_1$ $\dim \mathcal{H}_1 = D_1$ $\rightarrow$ ortho-basis $\{|\varphi_1\rangle, |\varphi_2\rangle, \dots, |\varphi_{D_1}\rangle\}$

" " $\mathcal{H}_2$ $\dim \mathcal{H}_2 = D_2$ $\rightarrow$ ortho-basis $\{|\chi_1\rangle, |\chi_2\rangle, \dots, |\chi_{D_2}\rangle\}$

$\mathcal{H}_1 \otimes \mathcal{H}_2$ has orthonormal of dim $D_1 D_2$ given by all pairs.

$$|\varphi_i\rangle \otimes |\chi_j\rangle \,, \quad \begin{array}{l} i = 1, \dots, D_1 \\ j = 1, \dots, D_2 \end{array}$$

$$\mathcal{H}_1 \otimes \mathcal{H}_2 \ni |\psi\rangle = \sum_{i=1}^{D_1} \sum_{j=1}^{D_2} \underbrace{c_{ij}}_{\in \mathbb{C}} |\varphi_i\rangle \otimes |\chi_j\rangle$$

$$(\langle\varphi_k| \otimes \langle\chi_l|)(|\varphi_i\rangle \otimes |\chi_j\rangle) = \delta_{ki}\,\delta_{lj} = \begin{cases} 1 & \text{if } (k,l) = (i,j) \\ 0 & \text{if } (k,l) \neq (i,j) \end{cases}$$

umm ....

Let $|\varphi\rangle \otimes |\chi\rangle$ and $|\varphi'\rangle \otimes |\chi'\rangle$ not necessarily orthonormal vectors.

$$\left(\langle\varphi'|_1 \otimes \langle\chi'|_2 \right) \left(|\varphi\rangle_1 \otimes |\chi\rangle_2 \right) = \langle \varphi' | \varphi \rangle_1 \ \langle \chi' | \chi \rangle_2$$

Remarks functions $f(x)$ $x \in R^{N_1}$ $g(y)$ $y \in R^{N_2}$

$$f(x)\, g(y) = \Psi(x,y)$$

$$\Psi(x,y) = \sum_{k,\ell} c \underbrace{\phi_k(x)}_{e^{i\vec{k}\cdot\vec{x}}} \underbrace{\chi_\ell(y)}_{e^{i\vec{\ell}\cdot\vec{y}}}$$

Tensor prod. Representation in Components

$\mathcal{H}_1 \in \mathbb{C}^2$ dim=2 $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}$

$\mathcal{H}_2 \in \mathbb{C}^2$ dim=2 $|\chi\rangle = \gamma|0\rangle + \delta|1\rangle = \begin{pmatrix}\gamma\\ \delta\end{pmatrix}$

$$\left.\begin{aligned} \mathcal{H}_1 \otimes \mathcal{H}_2 &= \mathbb{C}^2 \otimes \mathbb{C}^2 \\ &= (\mathbb{C}^2)^{\otimes 2} \end{aligned}\right\} \text{dim} = 4$$

orthonormal basis of $(\mathbb{C}^2)^{\otimes 2}$

$|0\rangle \otimes |0\rangle = |00\rangle$
$|0\rangle \otimes |1\rangle = |01\rangle$
$|1\rangle \otimes |0\rangle = |10\rangle$
$|1\rangle \otimes |1\rangle = |11\rangle$

$$|\psi\rangle = c_0 \underset{\begin{pmatrix}1\\0\\0\\0\end{pmatrix}}{|00\rangle} + c_1 \underset{\begin{pmatrix}0\\1\\0\\0\end{pmatrix}}{|01\rangle} + c_2 \underset{\begin{pmatrix}0\\0\\1\\0\end{pmatrix}}{|10\rangle} + c_3 \underset{\begin{pmatrix}0\\0\\0\\1\end{pmatrix}}{|11\rangle}$$

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \otimes \begin{pmatrix} \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} \alpha\gamma \\ \alpha\delta \\ \beta\gamma \\ \beta\delta \end{pmatrix}$$

PRINCIPLES OF QM

Principle 1 The state of isolated system is a vector in a complex Hilbert space.

$|\psi\rangle \longrightarrow$ state vector
"wave function"

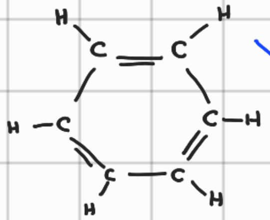
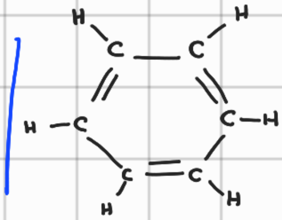
Examples: One qubit $\mathcal{H} \in \mathbb{C}^2$, $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, $|\alpha|^2 + |\beta|^2 = 1$

Two qubits $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2$, $|\psi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$

$$\sum_{i,j=0}^{1} |\alpha_{ij}|^2 = 1$$

M-Z interferometer $\mathcal{H} = \mathbb{C}^2$, $|\psi\rangle = \alpha|H\rangle + \beta|V\rangle$

Benzene molecule

$\frac{1}{\sqrt{2}}\Big|$  $\Big\rangle + \frac{1}{\sqrt{2}}\Big|$  $\Big\rangle \to$ qubit

Principle 2 Evolution of a state of isolated system is unitary

$$U|\psi\rangle_{t=0} = |\psi\rangle_t \quad \text{s.t.} \quad \underbrace{UU^{T*}}_{U^\dagger} = U^{T*}U = \mathbb{1}$$

$U^\dagger$ → adjoint of U

Example: Photon with polarization state $|\theta, \delta_x, \delta_y\rangle$

$$\begin{pmatrix} \cos\theta \, e^{i\delta_x} \\ \sin\theta \, e^{i\delta_y} \end{pmatrix} \in \mathbb{C}^2$$

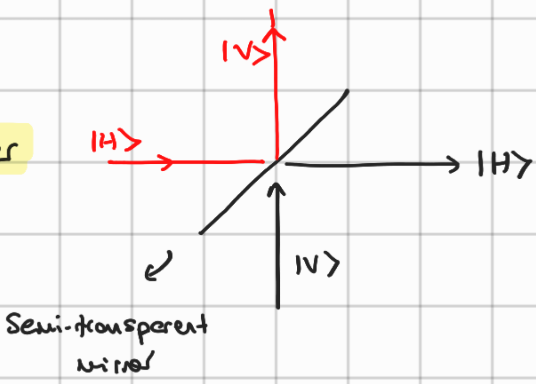
→ one-qubit state

For a monochromatic photon of frequency $\omega = 2\pi \cdot \nu$

Time evolution $\underbrace{e^{i\omega t}}\, |\theta, \delta_x, \delta_y\rangle$

$$\begin{pmatrix} e^{i\omega t} & 0 \\ 0 & e^{i\omega t} \end{pmatrix} = e^{i\omega t}\mathbb{1} = U_t \qquad e^{i\omega t} e^{-i\omega t} = 1$$

M-Z Interferometer



$$U(\alpha|H\rangle + \beta|V\rangle) = \alpha|V\rangle + \beta|H\rangle \iff U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Also called NOT gate

Dirac Notation of this matrix

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}(0 \;\; 1) + \begin{pmatrix} 0 \\ 1 \end{pmatrix}(1 \;\; 0)$$

$$= |H\rangle\langle V| + |V\rangle\langle H|$$

$U|H\rangle = |V\rangle$

$\left(|H\rangle\langle V| + |V\rangle\langle H| \right) |H\rangle = |V\rangle$

Examples

CNOT $|i,j\rangle = |i, j \oplus i\rangle \qquad i,j = 0,1$

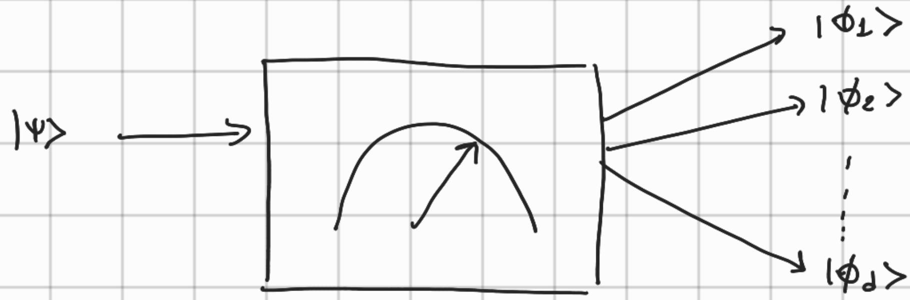
4x4 $\qquad \mathbb{C}^2 \otimes \mathbb{C}^2$

$$\left(\begin{array}{cc|cc} 1 & 0 & & \\ 0 & 1 & \multicolumn{2}{c}{0} \\ \hline & & 0 & 1 \\ \multicolumn{2}{c|}{0} & 1 & 0 \end{array}\right)$$

Principle 3 Measurement principle

- System prepared in state $|\psi\rangle = \mathcal{H}$

- We observe the system "with Meas. Appratus which is modelled by same orthonormal basis $|\phi_1\rangle, |\phi_2\rangle, \ldots |\phi_d\rangle$



$\text{prob}(1) = |\langle \phi_1 | \psi \rangle|^2$

$\text{prob}(2) = |\langle \phi_2 | \psi \rangle|^2$

$\vdots$

$\text{prob}(d) = |\langle \phi_d | \psi \rangle|^2$

$$\sum_{i=1}^{d} |\langle \phi_i | \psi \rangle|^2 = \| \psi \|^2 = 1$$

1st principle

Properties

1) $$\sum_{i=1}^{d} \text{prob}(i) = \underbrace{\sum_{i=1}^{d} |\langle \varphi_i | \psi \rangle|^2}_{\|\psi\|^2} = 1$$

2) Many different basis with Meas. Appratus.

3) You can collect measurement outcomes into an "observable" i.e. a matrix which is hermitian.

$$A = \sum_{i=1}^{d} a_i \, |\varphi_i\rangle\langle\varphi_i|$$

physical quantity

$A = A^{\dagger}$ → self adjoint

$$\left(|\varphi_i\rangle\langle\varphi_i| \right)^{T,*} = \left(\langle\varphi_i| \right)^{T,*} \left(|\varphi_i\rangle \right)^{T,*}$$

$$E(a_i) = \sum_{i=1}^{d} a_i \, prob(i) = \sum_{i=1}^{d} a_i \left| \, |\varphi_i\rangle\langle\psi| \, \right|^2$$

$$\langle\psi|\varphi_i\rangle\langle\varphi_i|\psi\rangle$$

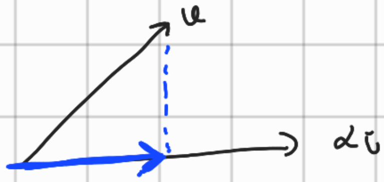
$$= \langle\psi| \Big(\underbrace{\sum_{i=1}^{d} a_i \, |\varphi_i\rangle\langle\varphi_i|}_{A} \Big) |\psi\rangle = \langle\psi|A|\psi\rangle$$

Uncertinity of Meas.

$$\Delta A = \sqrt{\quad}$$

$$|\varphi_i\rangle\langle\varphi_i| = P_i \rightarrow \text{projection Matrix}$$

$$P_i|u\rangle = |\varphi_i\rangle\langle\varphi_i|u\rangle$$



$$prob(i) = |\langle\varphi_i|\psi\rangle|^2$$

$$= \langle\psi|\underbrace{\varphi_i\rangle\langle\varphi_i}_{P_i}|\psi\rangle$$

$$\boxed{prob(i) = \langle\psi|P_i|\psi\rangle}$$

$$|\varphi_i\rangle \overset{?}{=} \frac{P_i|\psi\rangle}{\| P_i|\psi\rangle\|^2} = \frac{|\varphi_i\rangle\langle\varphi_i|\psi\rangle}{\sqrt{\underbrace{\langle\psi|P_i\,P_i|\psi\rangle}_{\langle\psi|\varphi_i\rangle\langle\varphi_i|\psi\rangle}}}$$

$$= \frac{|\varphi_i\rangle\cancel{\langle\varphi_i|\psi\rangle}}{\underbrace{\cancel{|\langle\varphi_i|\psi\rangle|}}_{e^{i\gamma}}} = |\varphi_i\rangle$$

$$\sum_{i=1}^{d} \text{prob}(i) = \langle\psi|\underbrace{\sum_{i=1}^{d} P_i}_{Id}|\psi\rangle = \|\psi\|^2 = 1$$

Principle 4 Composition of Q-systems

$\mathcal{H}_1$ one photon
$\mathcal{H}_2$ second photon

$\Big\}$ composed system is described by $\mathcal{H}_1 \otimes \mathcal{H}_2$

State of composed system $|\psi\rangle = \sum_{i,j} c_{i,j} \underbrace{|\varphi_i\rangle_1 \otimes |\chi_j\rangle_2}_{\to |\varphi_i, \chi_j\rangle_{12}}$

$\mathbb{C}^2 \otimes \mathbb{C}^2 \qquad |\psi\rangle = c_{00}|00\rangle + c_{01}|01\rangle + c_{10}|10\rangle + c_{11}|11\rangle$

$$|\psi\rangle = \sum c_{b_1 \dots b_n} |b_1 b_2 \dots\dots\dots b_n\rangle$$

For composed systems

product states versus entangled states

it can not be factorized

$\mathcal{H}_1 \otimes \mathcal{H}_2$

$|\psi\rangle = |v_1\rangle \otimes |v_2\rangle$

Example: $\frac{1}{\sqrt{2}}\left(|00\rangle + |10\rangle \right)$

$\frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle \right) \otimes |0\rangle$

$|Bell\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle \right) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

$\neq \left(\alpha|0\rangle + \beta|1\rangle \right) \otimes \left(\gamma|0\rangle + \delta|1\rangle \right)$