03.10.22

① Qubit

Bit – basic unit of classical computation
– 2 level classical system → as on-off switch
– either "0" or "1"

Qubit – basic unit of Quantum Info
– 2 level Quantum System eg. spin ↑/↓
2 energy levels of atom / spin / superconductor
H/V polarized photon
– Can be in $|0\rangle$ or $|1\rangle$ or superposition of the two

② Braket Notation

– Ket $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{pmatrix}\alpha \\ \beta\end{pmatrix}$

– Bra $\langle\psi| = \alpha^*\langle 0| + \beta^*\langle 1| = (\alpha^* \ \ \beta^*)$

– Braket $\langle\psi|\psi\rangle = |\alpha|^2 + |\beta|^2$ → inner product
(=1 for normalized state)

– ket bra $|\psi\rangle\langle\psi| = (\alpha^*\langle 0| + \beta^*\langle 1|)(\alpha|0\rangle + \beta|1\rangle)$

$$= |\alpha|^2|0\rangle\langle 0| + \beta^*\alpha\,|0\rangle\langle 1| + \alpha^*\beta\,|1\rangle\langle 0| + |\beta|^2|1\rangle\langle 1|$$

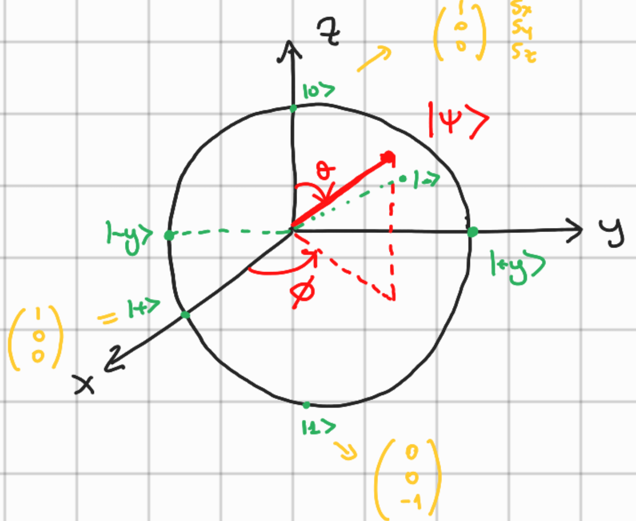
$$= \begin{pmatrix}|\alpha|^2 & \beta^*\alpha \\ \alpha^*\beta & |\beta|^2\end{pmatrix}$$

③ Bloch Sphere

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \rightarrow |\alpha|^2 + |\beta|^2 = 1$$

$$= \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\frac{\theta}{2}\,|1\rangle$$

$$\cos^2\left(\frac{\theta}{2}\right) + \sin^2\frac{\theta}{2} = 1$$



$\theta = 0 \quad \phi = 0 \quad |\psi\rangle = 0$

$\theta = \pi \quad \phi = 0 \quad |\psi\rangle = |1\rangle$

$\theta = \frac{\pi}{2} \quad \phi = 0 \quad |\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |+\rangle$

$\theta = \frac{\pi}{2} \quad \phi = \pi \quad |\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = |-\rangle$

$\theta = \frac{\pi}{2} \quad \phi = \frac{\pi}{2} \quad |\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle) = |+y\rangle$

$\theta = \frac{\pi}{2} \quad \phi = -\frac{\pi}{2} \quad |\psi\rangle = |-y\rangle$

$$\rho = |\psi\rangle\langle\psi| = \begin{pmatrix} |\alpha|^2 & \alpha\beta^* \\ \beta\alpha^* & |\beta|^2 \end{pmatrix} = \begin{pmatrix} \cos^2\left(\frac{\theta}{2}\right) & \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)e^{-i\phi} \\ \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)e^{i\phi} & \sin^2\left(\frac{\theta}{2}\right) \end{pmatrix}$$

Any 2x2 matrices can be expanded in Pauli basis

$$\{\sigma_i\}_{i=0}^{3} = \{I, X, Y, Z\}$$

$$\begin{pmatrix}1&0\\0&1\end{pmatrix} \quad \begin{pmatrix}0&1\\1&0\end{pmatrix} \quad \begin{pmatrix}0&-i\\i&0\end{pmatrix} \quad \begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

Super useful properties of Pauli Matrices

① $Tr(I) = 2$, $Tr(x) = Tr(y) = Tr(z) = 0$

② $\sigma_j \sigma_k = \delta_{jk} I + i\epsilon_{jkl}\sigma_l$

③ $Tr(\sigma_i \sigma_j) = 2\delta_{ij}$

what is it for

$$\rho = \frac{1}{2} I + \frac{1}{2} \sum_{i=1}^{3} S_i \sigma_i$$

① $Tr(\rho\, \sigma_i) = \frac{1}{2} Tr(S_i\, \underbrace{\sigma_i^2}_{I}) = S_i$

$= Tr(|\psi\rangle\langle\psi| \sigma_i) = \langle\psi| \sigma_i |\psi\rangle$

② $\underline{S}$ is vector of state on Bloch Sphere

$$\underline{S} = \begin{pmatrix} S_x \\ S_y \\ S_z \end{pmatrix} = \begin{pmatrix} \sin\theta \cos\phi \\ \sin\theta \sin\phi \\ \cos\theta \end{pmatrix} ?$$

Mixed States

$$\rho = \sum_k p_k |\psi_k\rangle\langle\psi_k|$$

$0 \le p_k \le 1$ $\qquad \sum_k p_k = 1$

$$\rho = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| \neq |+\rangle\langle +|$$

$$= \frac{1}{2}|+\rangle\langle +| + \frac{1}{2}|-\rangle\langle -|$$

4 Basic Gates

$$i\hbar \frac{\partial |\psi\rangle}{\partial t} = H|\psi\rangle$$ Schrödinger's Equation

$$|\psi\rangle = \underbrace{e^{-iH}}_{\text{Unitary Matrix}} |\psi\rangle$$

$U^{+}U = I$

* Reversible $\quad U^{+}(U|\psi\rangle) = I|\psi\rangle$

* Length preserving $\quad |\phi\rangle = U|\psi\rangle \Rightarrow \langle\phi|\phi\rangle = \langle\psi|U^{+}U|\psi\rangle = \langle\psi|\psi\rangle$

Basic Single Gates

Identity $= I$ $\quad |0\rangle \to |0\rangle \quad |1\rangle \to |1\rangle$

Not $= X$ $\quad |0\rangle \to |1\rangle \quad |1\rangle \to |0\rangle$

Hadamard $= H$ $\quad |0\rangle \to |+\rangle \quad |1\rangle \to |-\rangle$

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$H|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$e^{i\theta\sigma_i/2} = \cos\left(\frac{\theta}{2}\right) I - i\sin\left(\frac{\theta}{2}\right)\sigma_i$$

rotation angle

rotation Matrix

What's the effect of under $e^{-i\theta\sigma_i/2}$?

$$e^{-i\pi\frac{\sigma_z}{2}}|0\rangle = \overset{0}{\cancel{\cos\left(\frac{\pi}{2}\right)}} I\,|0\rangle - i\,\overbrace{\sin\left(\frac{\pi}{2}\right)}^{1}\sigma_z|0\rangle$$

$$= -i\,|1\rangle$$

global phase

5) Measurements

☆ Born Rule $p(i) = |\langle i|\psi\rangle|^2$

≡ in density operator formalism $p(i) = \mathrm{Tr}(\underbrace{|i\rangle\langle i|}_{\text{projective meas.}}\ \rho)$ → state

$$= \langle i|\psi\rangle\langle\psi|i\rangle = |\langle i|\psi\rangle|^2$$

for mixed states:

$$p_i = \mathrm{Tr}\left(|i\rangle\langle i|\sum_k p_k|\psi_k\rangle\langle\psi_k|\right) = \sum_k p_k\left(|i\rangle\langle i|\,|\psi_k\rangle\langle\psi_k|\right)$$

$$= \sum_k p_k\,|\langle i|\psi\rangle|^2$$

☆ Most Q.C measure in $\{|0\rangle, |1\rangle\}$ basis → often called computational basis

☆ What if we want to measure in $\{|+\rangle, |-\rangle\}$ basis?

$$p_+ = |\langle +|\psi\rangle|^2 \qquad (|+\rangle)^\dagger = (H|0\rangle)^\dagger$$

$$= |\langle 0|(H^\dagger|\psi\rangle)|^2 \qquad \langle +| = \langle 0|H^\dagger$$

$H^\dagger \to H$ $(H^\dagger = H)$

Key idea

To perform a measurement in generic basis first rotate into that basis ...

① Composite Systems

Use tensor product notation to describe multi qubit systems

$$\underbrace{|\psi\rangle \otimes |\phi\rangle}_{} = (a|0\rangle + b|1\rangle) \otimes (c|0\rangle + d|1\rangle)$$

$$= ac \underbrace{|0\rangle \otimes |0\rangle}_{|00\rangle} + bc|1\rangle \otimes |0\rangle + ad|0\rangle \otimes |1\rangle + bd|1\rangle \otimes |1\rangle$$

Seperable state
Qubit 1 in state $|\psi\rangle$
Qubit 2 in state $|\phi\rangle$

$$= \begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} ac \\ ad \\ bc \\ bd \end{pmatrix}$$

But not all the states can be written in this form

states can not be written in this form = entangled states

② Two Qubit Gates

$$\underbrace{X \otimes X}_{|X\rangle^{\otimes 2}} \; \underbrace{|0\rangle \otimes |0\rangle}_{|0\rangle^{\otimes 2}} = (X|0\rangle) \otimes (X|0\rangle) = |1\rangle \otimes |1\rangle$$

CNOT = "Apply NOT to 2^{nd} Qubit iff 1^{st} qubit is in $|1\rangle$ state

$$CNOT\,|00\rangle = |00\rangle$$
$$CNOT\,|01\rangle = |01\rangle$$
$$CNOT\,|10\rangle = |11\rangle$$
$$CNOT\,|11\rangle = |10\rangle$$

$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$(CNOT)(H \otimes I)|00\rangle = CNOT\,|+0\rangle = CNOT\,\frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)$$

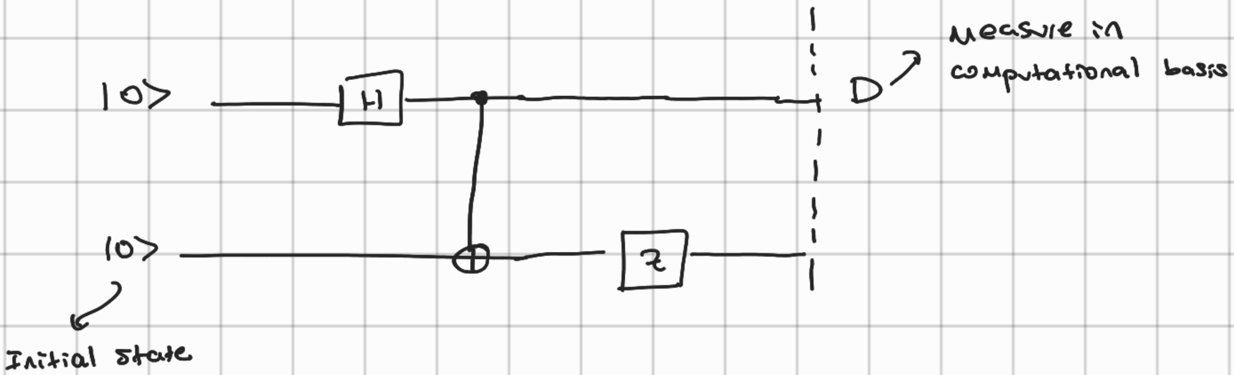
$$= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

Bell States

$$|\Phi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$$ → perfectly correlated

$$|\Psi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$$ → perfectly anti-correlated

③ Drawing Circuits



$$|\psi\rangle = (I \otimes Z)\,CNOT\,(H \otimes I)|00\rangle$$

$$= (I \otimes Z)\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$= \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$