Week 1 Review

$$\hat{\rho} = \frac{1}{2}\left(\mathbb{I} + \vec{n}\cdot\vec{\sigma} \right)$$

why do we have '1' here?

$Tr(\hat{\rho}) = 1 \rightarrow$ to represent a physical state

$$Tr(\hat{\rho}) = \frac{1}{2}\Big(\underbrace{Tr(\overset{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)}{I})}_{2} + \underbrace{Tr(\vec{n}\cdot\cancel{\vec{\sigma}})}_{0} \Big) = 1$$

$Tr(\sigma_x) = Tr(\sigma_y) = Tr(\sigma_z) = 0$

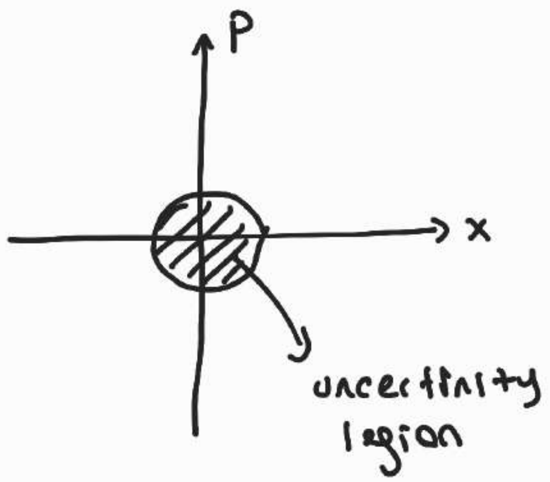
To ensure positive semi-definite. (non-zero) eigenvalues

$$i\hbar \frac{\partial \hat{\rho}}{\partial t} = [\hat{H}, \hat{\rho}] \quad \text{Von Neumann Equation}$$

$$i\hbar \frac{\partial \hat{A}}{\partial t} = [\hat{A}, \hat{H}] \quad \text{Heisenberg Equation}$$

Von Neumann	Heisenberg
. Operators are time-independent	. Operators are time-dependent
. State vectors are time-dependent	. State vectors are time-independent

Phase- Space Representation



Wigner Function

- quasi-probability distribution
- Fourier transform of the density matrix

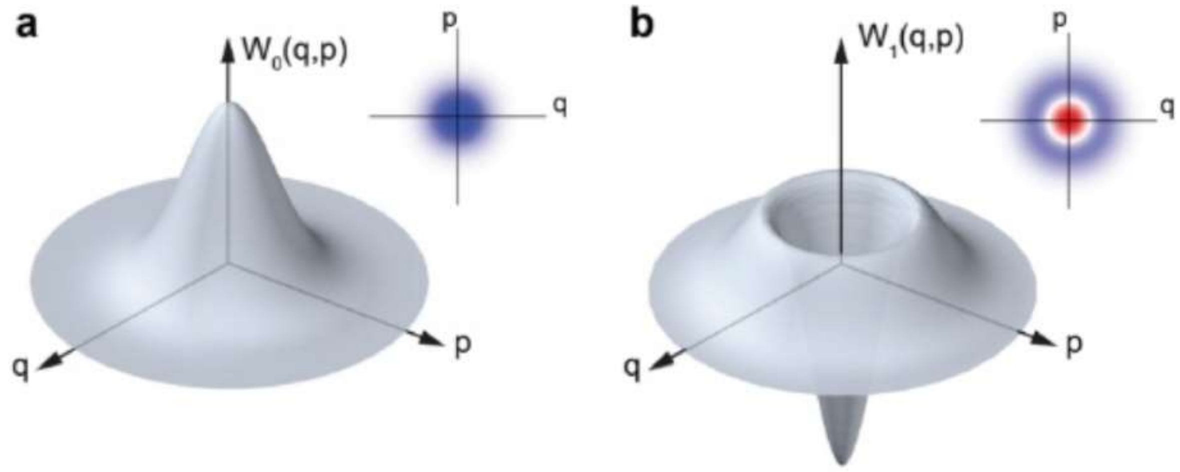


Figure 2.2: **Wigner distribution for photon-number states: a**, The vacuum state $|0\rangle$ has a Gaussian distribution centered at the origin of the phase space. **b**, The single photon state $|1\rangle$ exhibits negative probabilities around the origin.

Driven Harmonic Oscillator

$$m \frac{d^2x}{dt^2} + kx = F(t) = F_0 \cos(\omega t)$$

Example → $H = -\frac{1}{2}\hbar\omega\sigma_z - A\cos(\omega_d t)\sigma_x$

Rotating Frame:

Ex. dynamics of spin 1/2 in magnetic field

$$\vec{B} = (B_1 \cos\omega t, B_1 \sin\omega t, B_0)$$

$$H(t) = -\frac{\gamma\hbar}{2} \underbrace{\vec{B}(t)\vec{\sigma}}_{B_x(t)\sigma_x + B_y(t)\sigma_y + B_z(t)\sigma_z}$$

$$H(t) = -\underbrace{\frac{\gamma\hbar}{2}B_0}_{\hbar\omega_0}\sigma_z - \underbrace{\frac{\gamma\hbar}{2}B_1}_{\hbar\omega_1}\{\sigma_x \cos\omega t + \sigma_y \sin\omega t\}$$

Find $H_{rot} = \tilde{H}$ which does not depend on time.

$$H(t) = -\frac{\hbar\omega_0}{2}\sigma_z - \frac{\hbar\omega_1}{2}\{\sigma_+ e^{-i\omega t} + \sigma_- e^{+i\omega t}\}$$

Change of Frame:

$$|\psi\rangle_{\text{Rotating frame}} = \underbrace{e^{-i\frac{\omega t}{2}\sigma_z}}_{\text{Rotation around angle } \omega t \text{ around } z\text{-axis}} |\psi\rangle_{\text{lab frame}}$$

$$K = -\frac{\hbar\omega}{2}\sigma_z$$

$$|\psi_t\rangle_{rot} = e^{itK/\hbar}|\psi_t\rangle_{lab}$$

$$\tilde{U}_t|\tilde{\psi}_0\rangle_{rot} = e^{itK/\hbar}U_t|\psi_0\rangle_{lab}$$

$$\tilde{U}_t = U_t e^{itK/\hbar}$$

Schrödinger Equation: $i\hbar\frac{d}{dt}\tilde{U}_t = \tilde{H}\tilde{U}_t$

does not depend on time

$$\tilde{U}_t = \exp\left(\frac{-it}{\hbar}\tilde{H}\right)$$

$$i\hbar\frac{d}{dt}\tilde{U}_t = -K\underbrace{e^{itK/\hbar}U_t}_{\tilde{U}_t} + e^{itK/\hbar}\underbrace{\frac{dU_t}{dt}i\hbar}_{H(t)U_t}$$

$$i\hbar\frac{d}{dt}\tilde{U}_t = -Ke^{itK/\hbar}U_t + e^{itK/\hbar}H(t)U_t$$

$$i\hbar\frac{d}{dt}\tilde{U}_t = (\underbrace{-K + e^{itK/\hbar}H(t)e^{-itK/\hbar}}_{\tilde{H}(t)})\tilde{U}_t$$

$$\tilde{H}(t) = +\frac{\hbar\omega}{2}\sigma_z + e^{itK/\hbar}\left[-\frac{\hbar\omega_0}{2}\sigma_z - \frac{\hbar\omega_1}{2}\left\{\sigma_x\cos(\omega t) + \sigma_y\sin(\omega t)\right\}\right]e^{-itK/\hbar}$$

commutes

How?

$$\boxed{\tilde{H} = -\frac{\hbar(\omega_0-\omega)}{2}\sigma_z - \frac{\hbar\omega_1}{2}\sigma_x}$$

wrong in the LN!!

$$k = -\frac{\hbar\omega}{2}\sigma_z \qquad e^{itk/\hbar} = e^{it\left(-\frac{\hbar\omega}{2}\right)\cdot\frac{1}{\hbar}\sigma_z} = e^{-i\frac{t\omega}{2}\sigma_z}$$

$$\rightarrow e^{-i\frac{t\omega}{2}\sigma_z}\,\sigma_x\, e^{i\frac{t\omega}{2}\sigma_z}$$ cos(wt)

$$\underbrace{\begin{pmatrix} e^{-i\frac{t\omega}{2}} & 0 \\ 0 & e^{+i\frac{t\omega}{2}} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}\begin{pmatrix} e^{it\omega/2} & 0 \\ 0 & e^{-i\frac{t\omega}{2}} \end{pmatrix}$$

$$\begin{pmatrix} 0 & e^{-it\omega/2} \\ e^{it\omega/2} & 0 \end{pmatrix}\begin{pmatrix} e^{it\omega/2} & 0 \\ 0 & e^{-it\omega/2} \end{pmatrix} = \begin{pmatrix} 0 & e^{-it\omega} \\ e^{it\omega} & 0 \end{pmatrix}$$ cos(wt)

$$\rightarrow e^{-i\frac{t\omega}{2}\sigma_z}\,\sigma_y\, e^{i\frac{t\omega}{2}\sigma_z}$$ sin(wt)

$$\underbrace{\begin{pmatrix} e^{-it\omega/2} & 0 \\ 0 & e^{it\omega/2} \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}\begin{pmatrix} e^{it\omega/2} & 0 \\ 0 & e^{-it\omega/2} \end{pmatrix}$$

$$\begin{pmatrix} 0 & -ie^{-it\omega/2} \\ ie^{it\omega/2} & 0 \end{pmatrix}\begin{pmatrix} e^{it\omega/2} & 0 \\ 0 & e^{-it\omega/2} \end{pmatrix} = \begin{pmatrix} 0 & -ie^{-it\omega} \\ ie^{it\omega} & 0 \end{pmatrix}$$ sin(wt)

$$\begin{pmatrix} 0 & e^{-it\omega} \\ e^{it\omega} & 0 \end{pmatrix}\frac{(e^{i\omega t}+e^{-i\omega t})}{2} + \begin{pmatrix} 0 & -\cancel{i}e^{-it\omega} \\ \cancel{i}e^{it\omega} & 0 \end{pmatrix}\frac{(e^{i\omega t}-e^{-i\omega t})}{2\cancel{i}}$$

$$\begin{pmatrix} 0 & \frac{1+e^{-2i\omega t}}{2} \\ \frac{1+e^{2i\omega t}}{2} & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\frac{\left(1-e^{-2it\omega}\right)}{2} \\ \frac{e^{2it\omega}-1}{2} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & e^{-2i\omega t} \\ e^{2i\omega t} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \cos(2\omega t) - i\sin(2\omega t) \\ \cos(2\omega t) + i\sin(2\omega t) & 0 \end{pmatrix}$$

$$= \cos(2\omega t)\,\sigma_x + \sin(2\omega t)\,\sigma_y$$

$$H_{ROT} = U H U^{+} + i\hbar \dot{U} U^{+}$$

$$U = e^{-it\frac{\omega}{2}\sigma_z} \qquad \dot{U} = \frac{dU}{dt} = -i\frac{\omega}{2} e^{-it\frac{\omega}{2}\sigma_z}$$

$$H(t) = -\underbrace{\frac{\gamma\hbar}{2} B_0}_{\hbar\omega_0} \sigma_z - \underbrace{\frac{\gamma\hbar}{2} B_1}_{\hbar\omega_1} \left\{ \sigma_x \cos\omega t + \sigma_y \sin\omega t \right\}$$

$$= \begin{pmatrix} -\frac{\hbar\omega_0}{2} & 0 \\ 0 & +\frac{\hbar\omega_0}{2} \end{pmatrix} + \begin{pmatrix} 0 & -\frac{\hbar\omega_1}{2}\cos\omega t \\ -\frac{\hbar\omega_1}{2}\cos\omega t & 0 \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & -\frac{\hbar\omega_1}{2}\sin(\omega t)(-i) \\ -\frac{\hbar\omega_1}{2}\sin(\omega t)(i) & 0 \end{pmatrix} = \begin{pmatrix} -\frac{\hbar\omega_0}{2} & -\frac{\hbar\omega_1}{2} e^{-i\omega t} \\ -\frac{\hbar\omega_1}{2} e^{i\omega t} & +\frac{\hbar\omega_0}{2} \end{pmatrix}$$

$$H_{rot} = UHU^{\dagger} + i\hbar \dot{U} U^{\dagger}$$

$$e^{-it\frac{\omega}{2}\sigma_z} = \cos\left(\frac{\omega t}{2}\right) I - i\sin\left(\frac{\omega t}{2}\right)\sigma_z = \begin{pmatrix} \overbrace{\cos\left(\frac{\omega t}{2}\right) - i\sin\left(\frac{\omega t}{2}\right)}^{e^{-i\frac{\omega t}{2}}} & 0 \\ 0 & \underbrace{\cos\left(\frac{\omega t}{2}\right) + i\sin\left(\frac{\omega t}{2}\right)}_{e^{i\omega t/2}} \end{pmatrix}$$

$$UHU^{\dagger} = \underbrace{\begin{pmatrix} e^{-i\omega t/2} & 0 \\ 0 & e^{i\omega t/2} \end{pmatrix} \begin{pmatrix} -\frac{\hbar\omega_0}{2} & -\frac{\hbar\omega_1}{2} e^{-i\omega t} \\ -\frac{\hbar\omega_1}{2} e^{i\omega t} & \frac{\hbar\omega_0}{2} \end{pmatrix}} \begin{pmatrix} e^{i\omega t/2} & 0 \\ 0 & e^{-i\omega t/2} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{\hbar\omega_0}{2} e^{-i\omega t/2} & -\frac{\hbar\omega_1}{2} e^{-\frac{3}{2}i\omega t} \\ -\frac{\hbar\omega_1}{2} e^{\frac{3}{2}i\omega t} & \frac{\hbar\omega_0}{2} e^{i\omega t/2} \end{pmatrix} \begin{pmatrix} e^{i\omega t/2} & 0 \\ 0 & e^{-i\omega t/2} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{\hbar\omega_0}{2} & -\frac{\hbar\omega_1}{2} e^{-2i\omega t} \\ -\frac{\hbar\omega_1}{2} e^{2i\omega t} & \frac{\hbar\omega_0}{2} \end{pmatrix}$$

$$i\hbar \dot{U} U^{\dagger} = i\hbar \cdot \left(-\frac{i\omega}{2}\right) \underbrace{e^{-it\frac{\omega}{2}\sigma_z} e^{+it\frac{\omega}{2}\sigma_z}}_{I} = \frac{\hbar\omega}{2} I$$

$$H_{rot} = UHU^{\dagger} + i\hbar\dot{U}U^{\dagger} = \begin{pmatrix} -\frac{\hbar\omega_0}{2} + \frac{\hbar\omega}{2} & -\frac{\hbar\omega_1}{2} e^{-2i\omega t} \\ -\frac{\hbar\omega_1}{2} e^{2i\omega t} & \frac{\hbar\omega_0}{2} + \frac{\hbar\omega}{2} \end{pmatrix}$$

$$H_{rot} = \frac{\hbar(\omega - \omega_0)}{2}\sigma_z - \frac{\hbar\omega_1}{2}\left[\sigma_x \cos(\cancel{2\omega t}) + \sigma_y \sin(\cancel{2\omega t})\right]$$

RWA

Uncertinity and Commutation

$$\Delta x = \sqrt{\langle (x - \langle x \rangle)^2 \rangle}$$

$$\Delta p = \sqrt{\langle (p - \langle p \rangle)^2 \rangle}$$

Cauchy-Schwarz Inequality

$$(\Delta x)^2 (\Delta p)^2 \geq \frac{1}{4} \langle |[x,p]| \rangle^2$$

expectation

$$(\Delta x)^2 \, (\Delta p)^2 \geq \underbrace{\frac{1}{4} (|i\hbar|)^2}_{\frac{\hbar^2}{4}} \quad \Rightarrow \quad (\Delta x)(\Delta p) \geq \frac{\hbar}{2}$$

Quadrature and Phase Space

Classical Considerations

Classical electromagnetic field

$$E(t) = E_0 \cos(\omega t + \vartheta)$$

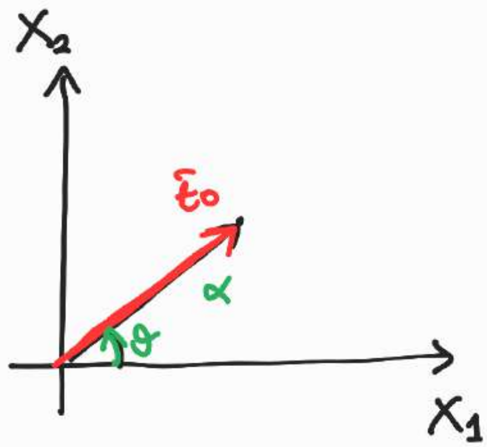
$$= E_0 \cos\vartheta \cos(\omega t) - E_0 \sin\vartheta \sin(\omega t)$$

$$= X_1 \cos(\omega t) + X_2 \sin(\omega t)$$

Phasor Representation of field

$$\underbrace{\alpha(t)}_{\text{phasor}} = E_0 e^{-i\vartheta} e^{-i\omega t} = \underbrace{\alpha}_{\substack{\text{amplitude}\\ \text{of phasor}}} e^{-i\omega t}$$

$$\alpha = X_1 + iX_2$$



$$X_1 = \mathrm{Re}(\alpha) = \frac{1}{2}(\alpha + \alpha^*)$$

$$X_2 = \mathrm{Im}(\alpha) = \frac{1}{2}(\alpha - \alpha^*)$$

Quantum

Quadrature Operators

$$\hat{X}_1 = \frac{1}{2}(\hat{a} + \hat{a}^\dagger) \hat{=} \hat{x} \qquad \hat{X}_\varphi = \frac{1}{2}(\hat{a}e^{-i\varphi} + \hat{a}^\dagger e^{i\varphi})$$

Hermition

$$\hat{X}_2 = \frac{1}{2i}(\hat{a} - \hat{a}^\dagger) \hat{=} \hat{p} \qquad \hat{X}_{\varphi+\frac{\pi}{2}} = \frac{1}{2i}(\hat{a}e^{-i\varphi} - \hat{a}^\dagger e^{i\varphi})$$

$$[\hat{X}_1, \hat{X}_2] = \frac{i}{2}$$

$$\Delta X_1 \Delta X_2 \geq \frac{1}{4}$$

Vacuum State

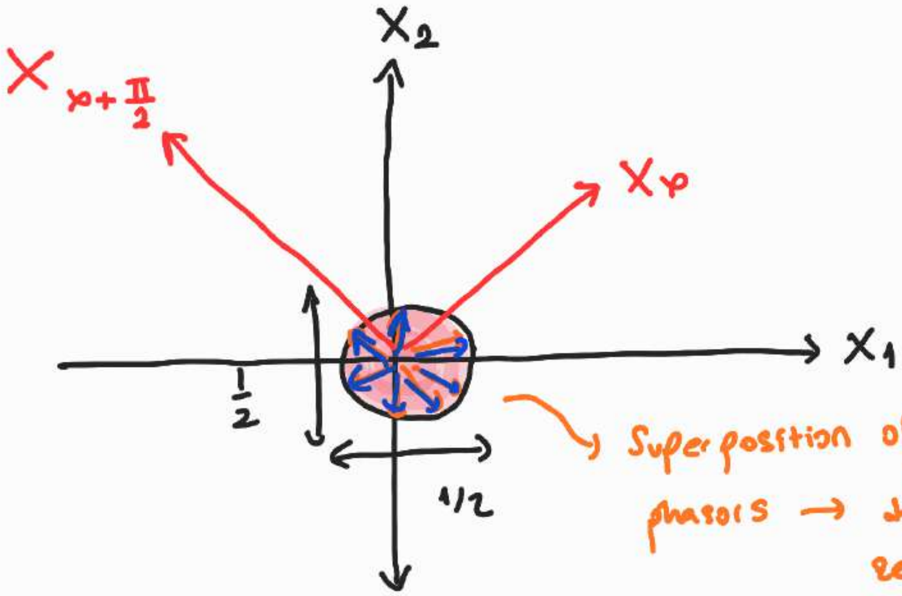
$$p^{(0)}(x_1) = |\langle x_1|0\rangle|^2 = \sqrt{\frac{2}{\pi}}\, e^{-2x_1^2}$$

$$p^{(0)}(x_2) = |\langle x_2|0\rangle|^2 = \sqrt{\frac{2}{\pi}}\, e^{-2x_2^2}$$

FT of Gaussian is again Gaussian

Fluctuations

$$\Delta X_1 = \sqrt{\langle 0|\hat{X}_1^2|0\rangle - \langle 0|\hat{X}_1|0\rangle^2} = \frac{1}{2}$$

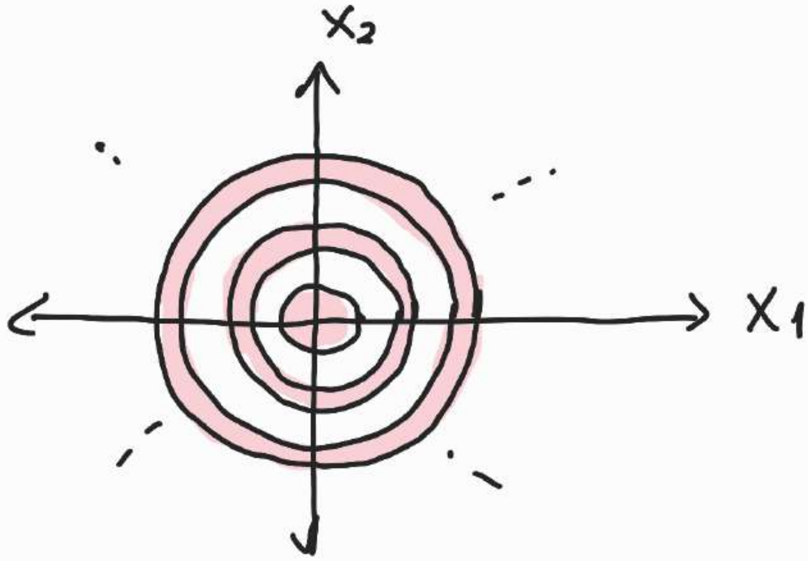


Superposition of all possible phasors → different fluctuation, zero averaged field

Fock State $|n\rangle$

$$P^{(0)}(X_1) = |\langle X_1|n\rangle|^2 = \sqrt{\frac{2}{\pi}}\,\frac{1}{2^n n!}\,e^{-2X_1^2}\left(H_n(\sqrt{2}X_1)\right)^2$$

$\Delta X_1 = \frac{1}{2}\sqrt{2n+1}$ similar to electric field fluctuations



Coherent State

$\hat{D}(\alpha) = e^{\alpha\hat{a}^+ - \alpha^*\hat{a}}$ → Displacement Operator

$\hat{D}(\alpha)\,|\beta\rangle \longrightarrow |\beta+\alpha\rangle$

$|\alpha\rangle = \hat{D}(\alpha)\,|0\rangle$