4 The Dirac equation

This chapter provides an introduction to the Dirac equation, which is the relativistic formulation of quantum mechanics used to describe the fundamental fermions of the Standard Model. Particular emphasis is placed on the free-particle solutions to the Dirac equation that will be used to describe fermions in the calculations of cross sections and decay rates in the following chapters.

4.1 The Klein–Gordon equation

One of the requirements for a relativistic formulation of quantum mechanics is that the associated wave equation is Lorentz invariant. The Schrödinger equation, introduced in Section 2.3.1, is first order in the time derivative and second order in the spatial derivatives. Because of the different dependence on the time and space coordinates, the Schrödinger equation is clearly not Lorentz invariant, and therefore cannot provide a description of relativistic particles. The non-invariance of the Schrödinger equation under Lorentz transformations is a consequence its construction from the non-relativistic relationship between the energy of a free particle and its momentum

$$E = \frac{\mathbf{p}^2}{2m}.$$

The first attempt at constructing a relativistic theory of quantum mechanics was based on the Klein–Gordon equation. The Klein–Gordon wave equation is obtained by writing the Einstein energy–momentum relationship,

$$E^2 = \mathbf{p}^2 + m^2,$$

in the form of operators acting on a wavefunction,

$$\hat{E}^2\psi(\mathbf{x},t) = \hat{\mathbf{p}}^2\psi(\mathbf{x},t) + m^2\psi(\mathbf{x},t).$$

Using the energy and momentum operators identified in Section 2.3.1,

$$\hat{\mathbf{p}} = -i\boldsymbol{\nabla} \quad \text{and} \quad \hat{E} = i\frac{\partial}{\partial t},$$

this leads to the Klein–Gordon wave equation,

$$\boxed{\frac{\partial^2 \psi}{\partial t^2} = \boldsymbol{\nabla}^2 \psi - m^2 \psi.} \tag{4.1}$$

The Klein–Gordon equation, which is second order in both space and time derivatives, can be expressed in the manifestly Lorentz-invariant form

$$(\partial^\mu \partial_\mu + m^2)\psi = 0, \tag{4.2}$$

where

$$\partial^\mu \partial_\mu \equiv \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2},$$

is the Lorentz-invariant scalar product of two four-vectors.

The Klein–Gordon equation has plane wave solutions,

$$\psi(\mathbf{x}, t) = N e^{i(\mathbf{p}\cdot\mathbf{x} - Et)}, \tag{4.3}$$

which when substituted into (4.2) imply that

$$E^2 \psi = \mathrm{p}^2 \psi + m^2 \psi,$$

and thus (by construction) the plane wave solutions to the Klein–Gordon equation satisfy the Einstein energy–momentum relationship, where the energy of the particle is related to its momentum by

$$E = \pm\sqrt{\mathrm{p}^2 + m^2}.$$

In classical mechanics, the negative energy solutions can be dismissed as being unphysical. However, in quantum mechanics all solutions are required to form a complete set of states, and the negative energy solutions simply cannot be discarded. Whilst it is not clear how the negative energy solutions should be interpreted, there is a more serious problem with the associated probability densities. The expressions for the probability density and probability current for the Klein–Gordon equation can be identified following the procedure used in Section 2.3.2. Taking the difference $\psi^* \times (4.1) - \psi \times (4.1)^*$ gives

$$\psi^* \frac{\partial^2 \psi}{\partial t^2} - \psi \frac{\partial^2 \psi^*}{\partial t^2} = \psi^*(\nabla^2 \psi - m^2 \psi) - \psi(\nabla^2 \psi^* - m^2 \psi^*)$$

$$\Rightarrow \quad \frac{\partial}{\partial t}\underbrace{\left(\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t}\right)}_{\rho/i} = \boldsymbol{\nabla} \cdot \overbrace{(\psi^* \boldsymbol{\nabla}\psi - \psi \boldsymbol{\nabla}\psi^*)}^{j/i}. \tag{4.4}$$

Comparison with the continuity equation of (2.20) leads to the identification of the probability density and probability current for solutions to the Klein–Gordon equation as

$$\rho = i\left(\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t}\right) \quad \text{and} \quad \mathbf{j} = -i(\psi^* \boldsymbol{\nabla}\psi - \psi \boldsymbol{\nabla}\psi^*), \tag{4.5}$$

where the factor of i is included to ensure that the probability density is real. For a plane wave solution, the probability density and current are

$$\rho = 2|N|^2 E \quad \text{and} \quad \mathbf{j} = 2|N|^2 \mathbf{p},$$

which can be written as a four-vector $j^{\mu}_{KG} = 2|N|^2 p^{\mu}$. The probability density is proportional to the energy of the particle, which is consistent with the discussion of relativistic length contraction of Section 3.2.1. However, this implies that the negative energy solutions have unphysical negative probability densities. From the presence of negative probability density solutions, it can be concluded that the Klein–Gordon equation does not provide a consistent description of single particle states for a relativistic system. It should be noted that this problem does not exist in quantum field theory, where the Klein–Gordon equation is used to describe *multi-particle* excitations of a spin-0 quantum field. ?

4.2 The Dirac equation

The apparent problems with the Klein–Gordon equation led Dirac (1928) to search for an alternative formulation of relativistic quantum mechanics. The resulting wave equation not only solved the problem of negative probability densities, but also provided a natural description of the intrinsic spin and magnetic moments of spin-half fermions. Its development represents one of the great theoretical breakthroughs of the twentieth century.

The requirement that relativistic particles satisfy $E^2 = \mathbf{p}^2 + m^2$ results in the Klein–Gordon wave equation being second order in the derivatives. Dirac looked for a wave equation that was first order in both space and time derivatives,

$$\hat{E}\psi = (\boldsymbol{\alpha} \cdot \hat{\mathbf{p}} + \beta m)\psi, \tag{4.6}$$

which in terms of the energy and momentum operators can be written

$$i\frac{\partial}{\partial t}\psi = \left(-i\alpha_x \frac{\partial}{\partial x} - i\alpha_y \frac{\partial}{\partial y} - i\alpha_z \frac{\partial}{\partial z} + \beta m\right)\psi. \tag{4.7}$$

If the solutions of (4.7) are to represent relativistic particles, they must *also* satisfy the Einstein energy–momentum relationship, which implies they satisfy the Klein–Gordon equation. This requirement places strong constraints on the possible nature of the constants $\boldsymbol{\alpha}$ and β in (4.6). The conditions satisfied by $\boldsymbol{\alpha}$ and β can be obtained by "squaring" (4.7) to give

$$-\frac{\partial^2\psi}{\partial t^2} = \left(i\alpha_x \frac{\partial}{\partial x} + i\alpha_y \frac{\partial}{\partial y} + i\alpha_z \frac{\partial}{\partial z} - \beta m\right)\left(i\alpha_x \frac{\partial}{\partial x} + i\alpha_y \frac{\partial}{\partial y} + i\alpha_z \frac{\partial}{\partial z} - \beta m\right)\psi,$$

which, when written out in gory detail, is

$$\begin{aligned}\frac{\partial^2\psi}{\partial t^2} &= \alpha_x^2\frac{\partial^2\psi}{\partial x^2} + \alpha_y^2\frac{\partial^2\psi}{\partial y^2} + \alpha_z^2\frac{\partial^2\psi}{\partial z^2} - \beta^2 m^2\psi \\ &\quad + (\alpha_x\alpha_y + \alpha_y\alpha_x)\frac{\partial^2\psi}{\partial x\partial y} + (\alpha_y\alpha_z + \alpha_z\alpha_y)\frac{\partial^2\psi}{\partial y\partial z} + (\alpha_z\alpha_x + \alpha_x\alpha_z)\frac{\partial^2\psi}{\partial z\partial x} \\ &\quad + i(\alpha_x\beta + \beta\alpha_x)m\frac{\partial\psi}{\partial x} + i(\alpha_y\beta + \beta\alpha_y)m\frac{\partial\psi}{\partial y} + i(\alpha_z\beta + \beta\alpha_z)m\frac{\partial\psi}{\partial z}. \qquad (4.8)\end{aligned}$$

In order for (4.8) to reduce to the Klein–Gordon equation,

$$\frac{\partial^2\psi}{\partial t^2} = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} - m^2\psi,$$

the coefficients $\boldsymbol{\alpha}$ and β must satisfy

$$\alpha_x^2 = \alpha_y^2 = \alpha_z^2 = \beta^2 = I, \qquad (4.9)$$

$$\alpha_j\beta + \beta\alpha_j = 0, \qquad (4.10)$$

$$\alpha_j\alpha_k + \alpha_k\alpha_j = 0 \quad (j \neq k), \qquad (4.11)$$

?

where I represents unity. The anticommutation relations of (4.10) and (4.11) cannot be satisfied if the α_i and β are normal numbers. The simplest mathematical objects that can satisfy these anticommutation relations are matrices. From the cyclic property of traces, $\mathrm{Tr}(ABC) = \mathrm{Tr}(BCA)$, and the requirements that $\beta^2 = I$ and $\alpha_i\beta = -\beta\alpha_i$, it is straightforward to show that the α_i and β matrices must have trace zero:

$$\mathrm{Tr}(\alpha_i) = \mathrm{Tr}(\alpha_i\beta\beta) = \mathrm{Tr}(\beta\alpha_i\beta) = -\mathrm{Tr}(\alpha_i\beta\beta) = -\mathrm{Tr}(\alpha_i).$$

Furthermore, it can be shown that the eigenvalues of the α_i and β matrices are ±1. This follows from multiplying the eigenvalue equation,

$$\alpha_i X = \lambda X,$$

by α_i and using $\alpha_i^2 = I$, which implies

$$\alpha_i^2 X = \lambda\alpha_i X \quad \Rightarrow \quad X = \lambda^2 X,$$

and therefore $\lambda = \pm1$. Because the sum of the eigenvalues of a matrix is equal to its trace, and here the matrices have eigenvalues of either $+1$ or -1, the only way the trace can be zero is if the α_i and β matrices are of even dimension. Finally, because the Dirac Hamiltonian operator of (4.6),

?

$$\hat{H}_D = (\boldsymbol{\alpha}\cdot\hat{\mathbf{p}} + \beta m),$$

must be Hermitian in order to have real eigenvalues, the $\boldsymbol{\alpha}$ and β matrices also must be Hermitian,

$$\alpha_x = \alpha_x^\dagger, \quad \alpha_y = \alpha_y^\dagger, \quad \alpha_z = \alpha_z^\dagger \quad \text{and} \quad \beta = \beta^\dagger. \tag{4.12}$$

Hence α_x, α_y, α_z and β are four mutually anticommuting Hermitian matrices of even dimension and trace zero. Because there are only three mutually anticommuting 2×2 traceless matrices, for example the Pauli spin-matrices, the lowest dimension object that can represent α_x, α_y, α_z and β are 4×4 matrices. Therefore, the Dirac Hamiltonian of (4.6) is a 4×4 matrix of operators that must act on a four-component wavefunction, known as a *Dirac spinor*,

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}.$$

The consequence of requiring the quantum-mechanical wavefunctions for a relativistic particle satisfy the Dirac equation and be consistent with the Klein–Gordon equation, is that the wavefunctions are forced to have four degrees of freedom. Before leaving this point, it is worth noting that, if all particles were massless, there would be no need for the β term in (4.7) and the $\boldsymbol{\alpha}$ matrices could be represented by the three Pauli spin-matrices. In this Universe without mass, it would be possible to describe a particle by a two-component object, known as a Weyl spinor.

The algebra of the Dirac equation is fully defined by the relations of (4.9)–(4.11) and (4.12). Nevertheless, it is convenient to introduce an explicit form for α_x, α_y, α_z and β. The conventional choice is the *Dirac–Pauli representation*, based on the familiar Pauli spin-matrices,

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad \text{and} \quad \alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \tag{4.13}$$

with

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

This is only one possible representation of the $\boldsymbol{\alpha}$ and β matrices. The matrices $\alpha_i' = U\alpha_i U^{-1}$ and $\beta = U\beta U^{-1}$, generated by any 4×4 unitary matrix U, are Hermitian and also satisfy the necessary anticommutation relations. The physical predictions obtained from the Dirac equation will not depend on the specific representation used; the physics of the Dirac equation is defined by the algebra satisfied by α_x, α_y, α_z and β, not by the specific representation.

4.3 Probability density and probability current

The expressions for the probability density and probability current for solutions of the Dirac equation can be obtained following a similar procedure to that used for the Schrödinger and Klein–Gordon equations. Since the wavefunctions are now four-component spinors, the complex conjugates of wavefunctions have to be replaced by Hermitian conjugates, $\psi^* \to \psi^\dagger = (\psi^*)^T$. The Hermitian conjugate of the Dirac equation,

$$-i\alpha_x \frac{\partial \psi}{\partial x} - i\alpha_y \frac{\partial \psi}{\partial y} - i\alpha_z \frac{\partial \psi}{\partial z} + m\beta\psi = +i\frac{\partial \psi}{\partial t}, \tag{4.14}$$

is simply

$$+i\frac{\partial \psi^\dagger}{\partial x}\alpha_x^\dagger + i\frac{\partial \psi^\dagger}{\partial y}\alpha_y^\dagger + i\frac{\partial \psi^\dagger}{\partial z}\alpha_z^\dagger + m\psi^\dagger\beta^\dagger = -i\frac{\partial \psi^\dagger}{\partial t}. \tag{4.15}$$

Using the fact that the α and β matrices are Hermitian, the combination of $\psi^\dagger \times$ (4.14) – (4.15) $\times \psi$ gives

$$\begin{aligned} &\psi^\dagger\left(-i\alpha_x \frac{\partial \psi}{\partial x} - i\alpha_y \frac{\partial \psi}{\partial y} - i\alpha_z \frac{\partial \psi}{\partial z} + \beta m\psi\right) \\ &-\left(i\frac{\partial \psi^\dagger}{\partial x}\alpha_x + i\frac{\partial \psi^\dagger}{\partial y}\alpha_y + i\frac{\partial \psi^\dagger}{\partial z}\alpha_z + m\psi^\dagger\beta\right)\psi = i\psi^\dagger\frac{\partial \psi}{\partial t} + i\frac{\partial \psi^\dagger}{\partial t}\psi. \end{aligned} \tag{4.16}$$

Equation (4.16) can be simplified by writing

$$\psi^\dagger\alpha_x\frac{\partial \psi}{\partial x} + \frac{\partial \psi^\dagger}{\partial x}\alpha_x\psi \equiv \frac{\partial(\psi^\dagger\alpha_x\psi)}{\partial x} \quad \text{and} \quad \psi^\dagger\frac{\partial \psi}{\partial t} + \frac{\partial \psi^\dagger}{\partial t}\psi \equiv \frac{\partial(\psi^\dagger\psi)}{\partial t},$$

giving

$$\nabla\cdot(\psi^\dagger\boldsymbol{\alpha}\psi) + \frac{\partial(\psi^\dagger\psi)}{\partial t} = 0,$$

where $\psi^\dagger = (\psi_1^*, \psi_2^*, \psi_3^*, \psi_4^*)$. By comparison with the continuity equation of (2.20), the probability density and probability current for solutions of the Dirac equation can be identified as

$$\boxed{\rho = \psi^\dagger\psi \quad \text{and} \quad \mathbf{j} = \psi^\dagger\boldsymbol{\alpha}\psi.} \tag{4.17}$$

In terms of the four components of the Dirac spinors, the probability density is

$$\rho = \psi^\dagger\psi = |\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2 + |\psi_4|^2,$$

and thus, all solutions of the Dirac equation have positive probability density. By requiring that the wavefunctions satisfy a wave equation linear in both space and time derivatives, in addition to being solutions of the Klein–Gordon equation, Dirac solved the perceived problem with negative probability densities. The price is that particles now have to be described by four-component wavefunctions. The Dirac equation could have turned out to be a purely mathematical construction without physical relevance. However, remarkably, it can be shown that the additional degrees of freedom of the four-component wavefunctions naturally describe the intrinsic angular momentum of spin-half particles and antiparticles. The proof that the Dirac equation provides a natural description of spin-half particles is given in the following starred section. It is fairly involved and the details are not essential to understand the material that follows.

4.4 *Spin and the Dirac equation

In quantum mechanics, the time dependence of an observable corresponding to an operator $\hat{O}$ is given by (2.29),

$$\frac{\mathrm{d}O}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{O}\rangle = i\langle\psi|[\hat{H},\hat{O}]|\psi\rangle.$$

Therefore, if the operator for an observable commutes with the Hamiltonian of the system, it is a constant of the motion. The Hamiltonian of the free-particle Schrödinger equation,

$$\hat{H}_{SE} = \frac{\hat{\mathbf{p}}^2}{2m},$$

commutes with the angular momentum operator $\hat{\mathbf{L}} = \hat{\mathbf{r}} \times \hat{\mathbf{p}}$, and thus angular momentum is a conserved quantity in non-relativistic quantum mechanics. For the free-particle Hamiltonian of the Dirac equation,

$$\hat{H}_D = \boldsymbol{\alpha}\cdot\hat{\mathbf{p}} + \beta m, \tag{4.18}$$

the corresponding commutation relation is

$$[\hat{H}_D, \hat{\mathbf{L}}] = [\boldsymbol{\alpha}\cdot\hat{\mathbf{p}} + \beta m, \hat{\mathbf{r}}\times\hat{\mathbf{p}}] = [\boldsymbol{\alpha}\cdot\hat{\mathbf{p}}, \hat{\mathbf{r}}\times\hat{\mathbf{p}}]. \tag{4.19}$$

This can be evaluated by considering the commutation relation for a particular component of $\hat{\mathbf{L}}$, for example

$$[\hat{H}_D, \hat{L}_x] = [\boldsymbol{\alpha}\cdot\hat{\mathbf{p}}, (\hat{\mathbf{r}}\times\hat{\mathbf{p}})_x] = [\alpha_x\hat{p}_x + \alpha_y\hat{p}_y + \alpha_z\hat{p}_z,\ \hat{y}\hat{p}_z - \hat{z}\hat{p}_y]. \tag{4.20}$$

The only terms in (4.20) that are non-zero arise from the non-zero position–momentum commutation relations

$$[\hat{x},\hat{p}_x] = [\hat{y},\hat{p}_y] = [\hat{z},\hat{p}_z] = i,$$

giving

$$[\hat{H}_D, \hat{L}_x] = \alpha_y[\hat{p}_y, \hat{y}]\hat{p}_z - \alpha_z[\hat{p}_z, \hat{z}]\hat{p}_y$$
$$= -i(\alpha_y \hat{p}_z - \alpha_z \hat{p}_y)$$
$$= -i(\boldsymbol{\alpha} \times \hat{\mathbf{p}})_x,$$

where $(\boldsymbol{\alpha} \times \hat{\mathbf{p}})_x$ is the x-component of $\boldsymbol{\alpha} \times \hat{\mathbf{p}}$. Generalising this result to the other components of $\hat{\mathbf{L}}$ gives

$$\boxed{[\hat{H}_D, \hat{\mathbf{L}}] = -i\boldsymbol{\alpha} \times \hat{\mathbf{p}}.} \tag{4.21}$$

Hence, for a particle satisfying the Dirac equation, the "orbital" angular momentum operator $\hat{\mathbf{L}}$ does not commute with the Dirac Hamiltonian, and therefore does not correspond to a conserved quantity.

Now consider the 4×4 matrix operator $\hat{\mathbf{S}}$ formed from the Pauli spin-matrices

$$\hat{\mathbf{S}} \equiv \tfrac{1}{2}\hat{\boldsymbol{\Sigma}} \equiv \tfrac{1}{2}\begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}, \tag{4.22}$$

with

$$\hat{\Sigma}_x = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \hat{\Sigma}_y = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} \quad \text{and} \quad \hat{\Sigma}_z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Because the $\boldsymbol{\alpha}$-matrices in the Dirac–Pauli representation and the Σ-matrices are both derived from the Pauli spin-matrices, they have well-defined commutation relations. Consequently, the commutator $[\alpha_i, \hat{\Sigma}_x]$ can be expressed in terms of the commutators of the Pauli spin-matrices. Writing the 4×4 matrices in 2×2 block form,

$$[\alpha_i, \hat{\Sigma}_x] = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}\begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix} - \begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix}\begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & [\sigma_i, \sigma_x] \\ [\sigma_i, \sigma_x] & 0 \end{pmatrix}. \tag{4.23}$$

The commutation relations,

$$[\sigma_x, \sigma_x] = 0, \quad [\sigma_y, \sigma_x] = -2i\sigma_z \quad \text{and} \quad [\sigma_z, \sigma_x] = 2i\sigma_y,$$

imply that (4.23) is equivalent to

$$[\alpha_x, \Sigma_x] = 0, \tag{4.24}$$
$$[\alpha_y, \Sigma_x] = \begin{pmatrix} 0 & -2i\sigma_z \\ -2i\sigma_z & 0 \end{pmatrix} = -2i\alpha_z, \tag{4.25}$$
$$[\alpha_z, \Sigma_x] = \begin{pmatrix} 0 & 2i\sigma_y \\ 2i\sigma_y & 0 \end{pmatrix} = 2i\alpha_y. \tag{4.26}$$

Now consider the commutator of $\hat{\Sigma}_x$ with the Dirac Hamiltonian

$$[\hat{H}_D, \Sigma_x] = [\boldsymbol{\alpha}\cdot\hat{\mathbf{p}} + \beta m, \Sigma_x].$$

It is straightforward to show that $[\beta, \hat{\Sigma}_x] = 0$ and hence

$$\begin{aligned}[\hat{H}_D, \hat{\Sigma}_x] &= [\boldsymbol{\alpha}\cdot\hat{\mathbf{p}}, \hat{\Sigma}_x] = [\alpha_x\hat{p}_x + \alpha_y\hat{p}_y + \alpha_z\hat{p}_z, \hat{\Sigma}_x] \\ &= \hat{p}_x[\alpha_x, \hat{\Sigma}_x] + \hat{p}_y[\alpha_y, \hat{\Sigma}_x] + \hat{p}_z[\alpha_z, \hat{\Sigma}_x].\end{aligned} \tag{4.27}$$

Using the commutation relations of (4.24)–(4.26) implies that

$$\begin{aligned}[\hat{H}_D, \hat{\Sigma}_x] &= -2i\hat{p}_y\alpha_z + 2i\hat{p}_z\alpha_y \\ &= 2i(\boldsymbol{\alpha}\times\hat{\mathbf{p}})_x.\end{aligned}$$

Generalising this derivation to the y and z components of $[\hat{H}_D, \hat{\boldsymbol{\Sigma}}]$ and using $\hat{\mathbf{S}} = \frac{1}{2}\hat{\boldsymbol{\Sigma}}$ gives the result

$$\boxed{[\hat{H}_D, \hat{\mathbf{S}}] = i\,\boldsymbol{\alpha}\times\hat{\mathbf{p}}.} \tag{4.28}$$

Because $\hat{\mathbf{S}}$ does not commute with the Dirac Hamiltonian, the corresponding observable is not a conserved quantity. However, from (4.21) and (4.28) it can be seen that the sum $\hat{\mathbf{J}} = \hat{\mathbf{L}} + \hat{\mathbf{S}}$ commutes with the Hamiltonian of the Dirac equation,

$$\boxed{\left[\hat{H}_D, \hat{\mathbf{J}}\right] \equiv \left[\hat{H}_D, \hat{\mathbf{L}} + \hat{\mathbf{S}}\right] = 0.}$$

Hence $\hat{\mathbf{S}}$ can be identified as the operator for the intrinsic angular momentum (the spin) of a particle. The total angular momentum of the particle, associated with the operator $\hat{\mathbf{J}} = \hat{\mathbf{L}} + \hat{\mathbf{S}}$, is a conserved quantity.

Because the 4×4 matrix operator $\hat{\mathbf{S}}$ is defined in terms of the Pauli spin-matrices,

$$\hat{\mathbf{S}} = \tfrac{1}{2}\hat{\boldsymbol{\Sigma}} = \frac{1}{2}\begin{pmatrix}\boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma}\end{pmatrix}, \tag{4.29}$$

its components have the same commutation relations as the Pauli spin-matrices, for example $[\hat{S}_x, \hat{S}_y] = i\hat{S}_z$. These are the same commutation relations satisfied by the operators for orbital angular momentum, $[\hat{L}_x, \hat{L}_y] = i\hat{L}_z$, etc. Therefore, from the arguments of Section 2.3.5, it follows that spin is quantised in exactly the same way as orbital angular momentum. Consequently, the total spin s can be identified from the eigenvalue of the operator,

$$\hat{\mathbf{S}}^2 = \frac{1}{4}(\hat{\Sigma}_x^2 + \hat{\Sigma}_y^2 + \hat{\Sigma}_z^2) = \frac{3}{4}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},$$

for which $\hat{\mathbf{S}}^2|s, m_s\rangle = s(s+1)|s, m_s\rangle$. Hence, for any Dirac spinor ψ,

$$\hat{\mathbf{S}}^2\psi = s(s+1)\psi = \tfrac{3}{4}\psi,$$

and thus a particle satisfying the Dirac equation has intrinsic angular momentum $s = \frac{1}{2}$. Furthermore, it can be shown (see Appendix B.1) that the operator $\hat{\boldsymbol{\mu}}$ giving the intrinsic magnetic moment of a particle satisfying the Dirac equation is given by

$$\hat{\boldsymbol{\mu}} = \frac{q}{m}\hat{\mathbf{S}}, \tag{4.30}$$

where q and m are respectively the charge and mass of the particle. Hence $\hat{\mathbf{S}}$ has all the properties of the quantum-mechanical spin operator for a Dirac spinor. The Dirac equation therefore provides a natural description of spin-half particles. This is a profound result, spin emerges as a direct consequence of requiring the wavefunction to satisfy the Dirac equation.

4.5 Covariant form of the Dirac equation

Up to this point the Dirac equation has been expressed in terms of the α- and β-matrices. This naturally brings out the connection with spin. However, the Dirac equation is usually expressed in the form which emphasises its covariance. This is achieved by first pre-multiplying the Dirac equation of (4.7) by β to give

$$i\beta\alpha_x\frac{\partial\psi}{\partial x} + i\beta\alpha_y\frac{\partial\psi}{\partial y} + i\beta\alpha_z\frac{\partial\psi}{\partial z} + i\beta\frac{\partial\psi}{\partial t} - \beta^2 m\psi = 0. \tag{4.31}$$

By defining the four Dirac γ-matrices as

$$\gamma^0 \equiv \beta, \quad \gamma^1 \equiv \beta\alpha_x, \quad \gamma^2 \equiv \beta\alpha_y \quad \text{and} \quad \gamma^3 \equiv \beta\alpha_z,$$

and using $\beta^2 = I$, equation (4.31) becomes

$$i\gamma^0\frac{\partial\psi}{\partial t} + i\gamma^1\frac{\partial\psi}{\partial x} + i\gamma^2\frac{\partial\psi}{\partial y} + i\gamma^3\frac{\partial\psi}{\partial z} - m\psi = 0.$$

By labelling the four γ-matrices by the index μ, such that $\gamma^\mu = (\gamma^0, \gamma^1, \gamma^2, \gamma^3)$, and using the definition of the covariant four-derivative

$$\partial_\mu \equiv (\partial_0, \partial_1, \partial_2, \partial_3) \equiv \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right),$$

the Dirac equation can be expressed in the covariant form

$$(i\gamma^\mu\partial_\mu - m)\psi = 0, \tag{4.32}$$

with the index μ being *treated* as the Lorentz index of a four-vector and, as usual, summation over repeated indices is implied. Despite the suggestive way in which (4.32) is written, it is important to realise that the Dirac γ-matrices are not four-vectors; they are constant matrices which are invariant under Lorentz transformations. Hence, the Lorentz covariance of the Dirac equation, which means that it applies in all rest frames, is not immediately obvious from Equation (4.32). The proof of the covariance of the Dirac equation and the derivation of the Lorentz transformation properties of Dirac spinors is quite involved and is deferred to Appendix B.2.

The properties of the γ-matrices can be obtained from the properties of the α- and β-matrices given in (4.9), (4.11) and (4.12). For example, using $\beta^2 = I$, $\alpha_x^2 = I$ and $\beta\alpha_x = -\alpha_x\beta$, it follows that

$$(\gamma^1)^2 = \beta\alpha_x\beta\alpha_x = -\alpha_x\beta\beta\alpha_x = -\alpha_x^2 = -I.$$

Similarly, it is straightforward to show that the products of two γ-matrices satisfy

$$\begin{aligned} (\gamma^0)^2 &= I, \\ (\gamma^k)^2 &= -I, \\ \text{and}\quad \gamma^\mu\gamma^\nu &= -\gamma^\nu\gamma^\mu \quad \text{for} \quad \mu \neq \nu, \end{aligned}$$

where the convention used here is that the index $k = 1$, 2 or 3. The above expressions can be written succinctly as the anticommutation relation

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}. \tag{4.33}$$

The γ^0 matrix, which is equivalent to β, is Hermitian and it is straightforward to show that the other three gamma matrices are anti-Hermitian, for example,

$$\gamma^{1\dagger} = (\beta\alpha_x)^\dagger = \alpha_x^\dagger\beta^\dagger = \alpha_x\beta = -\beta\alpha_x = -\gamma^1,$$

and hence

$$\gamma^{0\dagger} = \gamma^0 \quad \text{and} \quad \gamma^{k\dagger} = -\gamma^k. \tag{4.34}$$

Equations (4.33) and (4.34) fully define the algebra of the γ-matrices, which in itself is sufficient to define the properties of the solutions of the Dirac equation. Nevertheless, from a practical and pedagogical perspective, it is convenient to consider a particular representation of the γ-matrices. In the Dirac–Pauli representation, the γ-matrices are

$$\gamma^0 = \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad \text{and} \quad \gamma^k = \beta\alpha_k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix},$$

where the α- and β-matrices are those defined previously. Hence in the Dirac–Pauli representation,

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$
$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{4.35}$$

4.5.1 The adjoint spinor and the covariant current

In Section 4.3, it was shown that the probability density and the probability current for a wavefunction satisfying the Dirac equation are respectively given by $\rho = \psi^\dagger\psi$ and $\mathbf{j} = \psi^\dagger\boldsymbol{\alpha}\psi$. These two expressions can be written compactly as

$$j^\mu = (\rho, \mathbf{j}) = \psi^\dagger\gamma^0\gamma^\mu\psi, \tag{4.36}$$

which follows from $(\gamma^0)^2 = 1$ and $\gamma^0\gamma^k = \beta\beta\alpha_k = \alpha_k$. By considering the Lorentz transformation properties of the four components of j^μ, as defined in (4.36), it can be shown (see Appendix B.3) that j^μ is a four-vector. Therefore, the continuity equation (2.20), which expresses the conservation of particle probability,

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{j} = 0,$$

can be written in the manifestly Lorentz-invariant form of a four-vector scalar product

$$\partial_\mu j^\mu = 0.$$

The expression for the four-vector current, $j^\mu = \psi^\dagger\gamma^0\gamma^\mu\psi$, can be simplified by introducing the *adjoint spinor* $\overline{\psi}$, defined as

$$\overline{\psi} \equiv \psi^\dagger\gamma^0.$$

The definition of the adjoint spinor allows the four-vector current j^μ to be written compactly as

$$j^\mu = \overline{\psi}\gamma^\mu\psi. \tag{4.37}$$

For completeness, it is noted that in the Dirac–Pauli representation of the γ-matrices, the adjoint spinor is simply

$$\overline{\psi} = \psi^\dagger\gamma^0 = (\psi^*)^T\gamma^0 = (\psi_1^*, \psi_2^*, \psi_3^*, \psi_4^*)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = (\psi_1^*, \psi_2^*, -\psi_3^*, -\psi_4^*).$$

4.6 Solutions to the Dirac equation

The ultimate aim of this chapter is to identify explicit forms for the wavefunctions of spin-half particles that will be used in the matrix element calculations that follow. It is natural to commence this discussion by looking for free-particle plane wave solutions of the form

$$\psi(\mathbf{x}, t) = u(E, \mathbf{p})e^{i(\mathbf{p}\cdot\mathbf{x} - Et)}, \tag{4.38}$$

where $u(E, \mathbf{p})$ is a four-component Dirac spinor and the overall wavefunction $\psi(\mathbf{x}, t)$ satisfies the Dirac equation

$$(i\gamma^\mu\partial_\mu - m)\psi = 0. \tag{4.39}$$

The position and time dependencies of the plane waves described by (4.38) occur solely in exponent; the four-component spinor $u(E, \mathbf{p})$ is a function of the energy and momentum of the particle. Hence the derivatives $\partial_\mu\psi$ act only on the exponent and therefore,

$$\partial_0\psi \equiv \frac{\partial\psi}{\partial t} = -iE\psi, \quad \partial_1\psi \equiv \frac{\partial\psi}{\partial x} = ip_x\psi, \quad \partial_2\psi = ip_y\psi \quad \text{and} \quad \partial_3\psi = ip_z\psi. \tag{4.40}$$

Substituting the relations of (4.40) back into (4.39) gives

$$(\gamma^0 E - \gamma^1 p_x - \gamma^2 p_y - \gamma^3 p_z - m)u(E, \mathbf{p})e^{i(\mathbf{p}\cdot\mathbf{x} - Et)} = 0,$$

and therefore the four-component Dirac spinor $u(E, \mathbf{p})$ satisfies

$$\boxed{(\gamma^\mu p_\mu - m)\, u = 0,} \tag{4.41}$$

where, because of the covariance of the Dirac equation, the index μ on the γ-matrices can be treated as a four-vector index. Equation (4.41), which contains no derivatives, is the free-particle Dirac equation for the *spinor* u written in terms of the four-momentum of the particle.

4.6.1 Particles at rest

For a particle at rest with $\mathbf{p} = \mathbf{0}$, the free-particle wavefunction is simply

$$\psi = u(E, 0)e^{-iEt},$$

and thus (4.41) reduces to

$$E\gamma^0 u = mu.$$

This can be expressed as an eigenvalue equation for the components of the spinor

$$E\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} = m\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix}.$$

Because γ^0 is diagonal, this yields four orthogonal solutions. The first two,

$$u_1(E, 0) = N\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad u_2(E, 0) = N\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \tag{4.42}$$

have positive energy eigenvalues, $E = +m$. The other two solutions,

$$u_3(E, 0) = N\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad u_4(E, 0) = N\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \tag{4.43}$$

have negative energy eigenvalues, $E = -m$. In all cases N determines the normalisation of the wavefunction. These four states are also eigenstates of the $\hat{S}_z$ operator, as defined in Section 4.4. Hence $u_1(E, 0)$ and $u_2(E, 0)$ represent spin-up and spin-down positive energy solutions to the Dirac equation, and $u_3(E, 0)$ and $u_4(E, 0)$ represent spin-up and spin-down negative energy solutions. The four solutions to the Dirac equation for a particle at rest, including the time dependence, are therefore

$$\psi_1 = N\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}e^{-imt}, \ \psi_2 = N\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}e^{-imt}, \ \psi_3 = N\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}e^{+imt} \ \text{and} \ \psi_4 = N\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}e^{+imt}.$$

4.6.2 General free-particle solutions

The general solutions of the free-particle Dirac equation for a particle with momentum $\mathbf{p}$ can be obtained from the solutions for a particle at rest, using the Lorentz transformation properties of Dirac spinors derived in Appendix B.2. However, it is more straightforward to solve directly the Dirac equation for the general plane wave solution of (4.38). The Dirac equation for the spinor $u(E, \mathbf{p})$ given in (4.41) when written in full is

$$(E\gamma^0 - p_x\gamma^1 - p_y\gamma^2 - p_z\gamma^3 - m)u = 0.$$

This can be expressed in matrix form using the Dirac–Pauli representation of the γ-matrices, giving

$$\left[\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} E - \begin{pmatrix} 0 & \boldsymbol{\sigma}\cdot\mathbf{p} \\ -\boldsymbol{\sigma}\cdot\mathbf{p} & 0 \end{pmatrix} - m\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}\right] u = 0, \tag{4.44}$$

where the 4×4 matrix multiplying the four-component spinor u has been expressed in 2×2 block matrix form with

$$\boldsymbol{\sigma}\cdot\mathbf{p} \equiv \sigma_x p_x + \sigma_x p_y + \sigma_x p_z = \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}.$$

Writing the spinor u in terms of two two-component column vectors, u_A and u_B,

$$u = \begin{pmatrix} u_A \\ u_B \end{pmatrix},$$

allows (4.44) to be expressed as

$$\begin{pmatrix} (E-m)I & -\boldsymbol{\sigma}\cdot\mathbf{p} \\ \boldsymbol{\sigma}\cdot\mathbf{p} & -(E+m)I \end{pmatrix}\begin{pmatrix} u_A \\ u_B \end{pmatrix} = 0,$$

giving coupled equations for u_A in terms of u_B,

$$u_A = \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E-m} u_B, \tag{4.45}$$

$$u_B = \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+m} u_A. \tag{4.46}$$

Two solutions to the free-particle Dirac equation, u_1 and u_2, can be found by taking the two simplest orthogonal choices for u_A,

$$u_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad u_A = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{4.47}$$

The corresponding components of u_B, given by (4.46), are

$$u_B = \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+m} = \frac{1}{E+m}\begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix} u_A,$$

and thus the first two solutions of the free-particle Dirac equation are

$$u_1(E,\mathbf{p}) = N_1\begin{pmatrix}1\\0\\\frac{p_z}{E+m}\\\frac{p_x+ip_y}{E+m}\end{pmatrix} \quad\text{and}\quad u_2(E,\mathbf{p}) = N_2\begin{pmatrix}0\\1\\\frac{p_x-ip_y}{E+m}\\\frac{-p_z}{E+m}\end{pmatrix},$$

where N_1 and N_2 determine the wavefunction normalisation. It should be noted that whilst the choice of the two orthogonal forms for u_A is arbitrary, any other orthogonal choice would have been equally valid, since a general $(E > 0)$ spinor can be expressed as a linear combination of u_1 and u_2. Choosing the forms of u_A of (4.47) is analogous to choosing a particular basis for spin where conventionally the z-axis is chosen to label the states. The two other solutions of the Dirac equation can be found by writing

$$u_B = \begin{pmatrix}1\\0\end{pmatrix} \quad\text{and}\quad u_B = \begin{pmatrix}0\\1\end{pmatrix},$$

and using (4.45) to give the corresponding components for u_A. The four orthogonal plane wave solutions to the free-particle Dirac equation of the form

$$\psi_i = u_i(E,\mathbf{p})e^{i(\mathbf{p}\cdot\mathbf{x}-Et)}$$

are therefore

$$u_1 = N_1\begin{pmatrix}1\\0\\\frac{p_z}{E+m}\\\frac{p_x+ip_y}{E+m}\end{pmatrix},\quad u_2 = N_2\begin{pmatrix}0\\1\\\frac{p_x-ip_y}{E+m}\\\frac{-p_z}{E+m}\end{pmatrix},\quad u_3 = N_3\begin{pmatrix}\frac{p_z}{E-m}\\\frac{p_x+ip_y}{E-m}\\1\\0\end{pmatrix} \quad\text{and}$$
$$u_4 = N_4\begin{pmatrix}\frac{p_x-ip_y}{E-m}\\\frac{-p_z}{E-m}\\0\\1\end{pmatrix}. \tag{4.48}$$

If any one of these four spinors is substituted back into the Dirac equation, the Einstein energy–momentum relation $E^2 = p^2 + m^2$ is recovered. In the limit $\mathbf{p} = \mathbf{0}$, the spinors u_1 and u_2 reduce to the $E > 0$ spinors for a particle at rest given in (4.42). Hence u_1 and u_2 can be identified as the positive energy spinors with

$$E = +\left|\sqrt{\mathrm{p}^2 + m^2}\right|,$$

and u_3 and u_4 are the negative energy particle spinors with

$$E = -\left|\sqrt{\mathrm{p}^2 + m^2}\right|.$$

The same identification of u_1 and u_2 as being the positive energy spinors, and u_3 and u_4 as the negative energy spinors, can be reached by transforming the solutions for a particle at rest into the frame where the particle has momentum $\mathbf{p}$ (see Appendix B.2).

At this point it is reasonable to ask whether it is possible to interpret all four solutions of (4.48) as having $E > 0$. The answer is no, as if this were the case, the exponent of the wavefunction,

$$\psi(\mathbf{x}, t) = u(E, \mathbf{p})e^{i(\mathbf{p}\cdot\mathbf{x}-Et)},$$

would be the same for all four solutions. In this case the four solutions no longer would be independent since, for example, it would be possible to express u_1 as the linear combination

$$u_1 = \frac{p_z}{E+m}u_3 + \frac{p_x + ip_y}{E+m}u_4.$$

Hence, there are only four *independent* solutions to the Dirac equation when two are taken to have $E < 0$; it is not possible to avoid the need for the negative energy solutions. The same conclusion can be reached from the fact that the Dirac Hamiltonian is a 4×4 matrix with trace zero, and therefore the sum of its eigenvalues is zero, implying equal numbers of positive and negative energy solutions.

4.7 Antiparticles

The Dirac equation provides a beautiful mathematical framework for the relativistic quantum mechanics of spin-half fermions in which the properties of spin and magnetic moments emerge naturally. However, the presence of negative energy solutions is unavoidable. In quantum mechanics, a complete set of basis states is required to span the vector space, and therefore the negative energy solutions cannot simply be discarded as being unphysical. It is therefore necessary to provide a physical interpretation for the negative energy solutions.

4.7.1 The Dirac sea interpretation

If negative energy solutions represented accessible negative energy particle states, one would expect that all positive energy electrons would fall spontaneously into these lower energy states. Clearly this does not occur. To avoid this apparent contradiction, Dirac proposed that the vacuum corresponds to the state where all negative energy states are occupied, as indicated in Figure 4.1. In this "Dirac sea" picture, the Pauli exclusion principle prevents positive energy electrons from falling into the fully occupied negative energy states. Furthermore, a photon with energy $E > 2m_e$ could excite an electron from a negative energy state, leaving a hole in

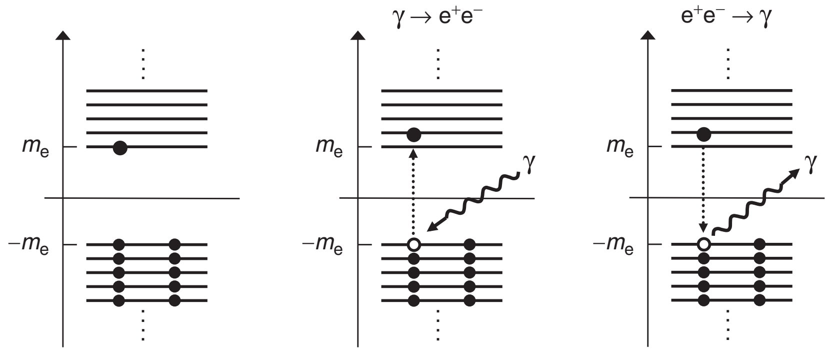

Fig. 4.1 The Dirac interpretation of negative energy solutions as holes in the vacuum that correspond to antiparticle states.

the vacuum. A hole in the vacuum would correspond to a state with more energy (less negative energy) and a positive charge relative to the fully occupied vacuum. In this way, holes in the Dirac sea correspond to positive energy antiparticles with the opposite charge to the particle states. The Dirac sea interpretation thus provides a picture for e^+e^- pair production and also particle–antiparticle annihilation (shown in Figure 4.1). The discovery of positively charged electrons in cosmic-ray tracks in a cloud chamber, Anderson (1933), provided the experimental confirmation that the antiparticles predicted by Dirac corresponded to physical observable states.

Nowadays, the Dirac sea picture of the vacuum is best viewed in terms of historical interest. It has a number of conceptual problems. For example, antiparticle states for bosons are also observed and in this case the Pauli exclusion principle does not apply. Furthermore, the fully occupied Dirac sea implies that the vacuum has infinite negative energy and it is not clear how this can be interpreted physically. The negative energy solutions are now understood in terms of the Feynman–Stückelberg interpretation.

4.7.2 The Feynman–Stückelberg interpretation

It is an experimentally established fact that for each fundamental spin-half particle there is a corresponding antiparticle. The antiparticles produced in accelerator experiments have the opposite charges compared to the corresponding particle. Apart from possessing different charges, antiparticles behave very much like particles; they propagate forwards in time from the point of production, ionise the gas in tracking detectors, produce the same electromagnetic showers in the calorimeters of large collider particle detectors, and undergo many of the same interactions as particles. It is not straightforward to reconcile these physical observations with the negative energy solutions that emerge from the abstract mathematics of the Dirac equation.

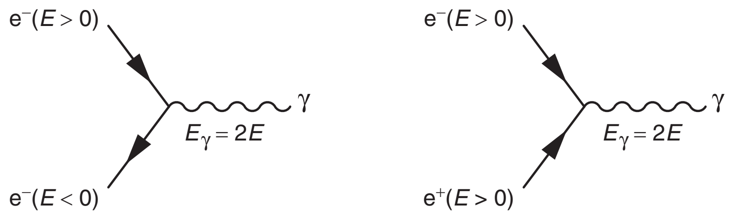

Fig. 4.2 (left) The process of e^+e^- annihilation in terms of a positive energy electron producing a photon and a negative energy electron propagating backwards in time. (right) The Feynman–Stückelberg interpretation with a positive energy positron propagating forwards in time. In both diagrams, time runs from the left to right.

The modern interpretation of the negative energy solutions, due to Stückelberg and Feynman, was developed in the context of quantum field theory. The $E<0$ solutions are interpreted as *negative* energy *particles* which propagate *backwards* in time. These negative energy particle solutions correspond to physical *positive* energy *antiparticle* states with opposite charge, which propagate *forwards* in time. Since the time dependence of the wavefunction, $\exp(-iEt)$, is unchanged under the simultaneous transformation $E \to -E$ and $t \to -t$ these two pictures are mathematically equivalent,

$$\exp\{-iEt\} \equiv \exp\{-i(-E)(-t)\}.$$

To illustrate this idea, Figure 4.2 shows the process of electron–positron annihilation in terms of negative energy particle solutions and in the Feynman–Stückelberg interpretation of these solutions as positive energy antiparticles. In the left plot, an electron of energy E emits a photon with energy $2E$ and, to conserve energy, produces a electron with energy $-E$, which being a negative energy solution of the Dirac equation propagates backwards in time. In the Feynman–Stückelberg interpretation, shown on the right, a positive energy positron of energy E annihilates with the electron with energy E to produce a photon of energy $2E$. In this case, both the particle and antiparticle propagate forwards in time. It should be noted that although antiparticles propagate forwards in time, in a Feynman diagram they are still drawn with an arrow in the “backwards in time” sense, as shown in the left plot of Figure 4.2.

4.7.3 Antiparticle spinors

In principle, it is possible to perform calculations with the negative energy particle spinors u_3 and u_4. However, this necessitates remembering that the energy which appears in the definition of the spinor is the negative of the physical energy. Furthermore, because u_3 and u_4 are interpreted as propagating backwards in time, the momentum appearing in the spinor is the negative of the physical momentum. To avoid this possible confusion, it is more convenient to work with antiparticle spinors written in terms of the physical momentum and physical energy, $E = +|\sqrt{\mathbf{p}^2 + m^2}|$. Following the Feynman–Stückelberg interpretation, the negative energy particle spinors, u_3 and u_4, can be rewritten in terms of the physical positive energy *antiparticle spinors*, v_1 and v_2, simply by reversing the signs of E and $\mathbf{p}$ to give

$$v_1(E, \mathbf{p})e^{-i(\mathbf{p}\cdot\mathbf{x}-Et)} = u_4(-E, -\mathbf{p})e^{i[-\mathbf{p}\cdot\mathbf{x}-(-E)t]}$$
$$v_2(E, \mathbf{p})e^{-i(\mathbf{p}\cdot\mathbf{x}-Et)} = u_3(-E, -\mathbf{p})e^{i[-\mathbf{p}\cdot\mathbf{x}-(-E)t]}.$$

A more formal approach to identifying the antiparticle spinors is to look for solutions of the Dirac equation of the form

$$\psi(\mathbf{x}, t) = v(E, \mathbf{p})e^{-i(\mathbf{p}\cdot\mathbf{x}-Et)}, \tag{4.49}$$

where the signs in the exponent are reversed with respect to those of (4.38). For $E > 0$, the wavefunctions of (4.49) still represent negative energy solutions in the sense that

$$i\frac{\partial}{\partial t}\psi = -E\psi.$$

Substituting the wavefunction of (4.49) into the Dirac equation, $(i\gamma^\mu\partial_\mu - m)\psi = 0$, gives

$$(-\gamma^0 E + \gamma^1 p_x + \gamma^2 p_y + \gamma^3 p_z - m)v = 0,$$

which can be written as

$$\boxed{(\gamma^\mu p_\mu + m)v = 0.}$$

This is the Dirac equation in terms of momentum for the v spinors. Proceeding as before and writing

$$v = \begin{pmatrix} v_A \\ v_B \end{pmatrix},$$

leads to

$$v_A = \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+m}v_B \quad \text{and} \quad v_B = \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E-m}v_A,$$

giving the solutions

$$v_1 = N\begin{pmatrix} \frac{p_x - ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix}, \quad v_2 = N\begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \\ 1 \\ 0 \end{pmatrix}, \quad v_3 = N\begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E-m} \\ \frac{p_x+ip_y}{E-m} \end{pmatrix} \text{ and}$$

$$v_4 = N\begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E-m} \\ \frac{-p_z}{E-m} \end{pmatrix}, \tag{4.50}$$

where

$$E = +\left|\sqrt{\mathbf{p}^2 + m^2}\right|$$

for v_1 and v_2, and

$$E = -\left|\sqrt{\mathbf{p}^2 + m^2}\right|$$

for v_3 and v_4. Hence we have now identified eight solutions to the free particle Dirac equation, given in (4.48) and (4.50). Of these eight solutions, only four are independent. In principle it would be possible to perform calculations using only the u-spinors, or alternatively using only the v-spinors. Nevertheless, it is more natural to work with the four solutions for which the energy that appears in the spinor is the positive physical energy of the particle/antiparticle, namely $\{u_1, u_2, v_1, v_2\}$.

To summarise, in terms of the physical energy, the two *particle* solutions to the Dirac equation are

$$\psi_i = u_i e^{+i(\mathbf{p}\cdot\mathbf{x} - Et)}$$

with

$$u_1(p) = \sqrt{E+m}\begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix} \quad \text{and} \quad u_2(p) = \sqrt{E+m}\begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}, \tag{4.51}$$

and the two *antiparticle* solutions are

$$\psi_i = v_i e^{-i(\mathbf{p}\cdot\mathbf{x} - Et)}$$

with

$$v_1(p) = \sqrt{E+m}\begin{pmatrix} \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad v_2(p) = \sqrt{E+m}\begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \\ 1 \\ 0 \end{pmatrix}. \tag{4.52}$$

Wavefunction normalisation

The spinors in (4.51) and (4.52) have been normalised to the conventional $2E$ particles per unit volume. This required normalisation factor can be found from the definition of probability density, $\rho = \psi^\dagger\psi$, which for $\psi = u_1(p)\exp i(\mathbf{p}\cdot\mathbf{x} - Et)$ is

$$\rho = \psi^\dagger\psi = (\psi^*)^T\psi = u_1^\dagger u_1.$$

Using the explicit form for u_1 of (4.48) gives

$$u_1^\dagger u_1 = |N|^2\left(1 + \frac{p_z^2}{(E+m)^2} + \frac{p_x^2+p_y^2}{(E+m)^2}\right) = |N|^2\frac{2E}{E+m}.$$

Hence, to normalise the wavefunctions to $2E$ particles per unit volume implies

$$N = \sqrt{E+m}.$$

The same normalisation factor is obtained for the u and v spinors.

4.7.4 Operators and the antiparticle spinors

There is a subtle, but nevertheless important, point related to using the antiparticle spinors written in terms of the physical energy and momenta,

$$\psi = v(E,\mathbf{p})e^{-i(\mathbf{p}\cdot\mathbf{x}-Et)}.$$

The action of the normal quantum mechanical operators for energy and momentum do not give the physical quantities,

$$\hat{H}\psi = i\frac{\partial\psi}{\partial t} = -E\psi \quad \text{and} \quad \hat{\mathbf{p}}\psi = -i\boldsymbol{\nabla}\psi = -\mathbf{p}\psi.$$

The minus signs should come as no surprise; the antiparticle spinors are still the negative energy particle solutions of the Dirac equation, albeit expressed in terms of the physical (positive) energy E and physical momentum $\mathbf{p}$ of the antiparticle. The operators which give the *physical* energy and momenta of the antiparticle spinors are therefore

$$\hat{H}^{(v)} = -i\frac{\partial}{\partial t} \quad \text{and} \quad \hat{\mathbf{p}}^{(v)} = +i\boldsymbol{\nabla},$$

where the change of sign reflects the Feynman–Stückelberg interpretation of the negative energy solutions. Furthermore, with the replacement $(E, \mathbf{p}) \to (-E, -\mathbf{p})$, the orbital angular momentum of a particle

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} \to -\mathbf{L}.$$

In order for the commutator $[\hat{H}_D, \hat{\mathbf{L}} + \hat{\mathbf{S}}]$ to remain zero for the antiparticle spinors, the operator giving the physical spin states of the v spinors must be

$$\hat{\mathbf{S}}^{(v)} = -\hat{\mathbf{S}},$$

where $\hat{\mathbf{S}}$ is defined in (4.29). Reverting (very briefly) to the the Dirac sea picture, a spin-up hole in the negative energy particle sea, leaves the vacuum in a net spin-down state.

4.7.5 *Charge conjugation

Charge conjugation is an important example of a discrete symmetry transformation that will be discussed in depth in Chapter 14. The effect of charge conjugation is to replace particles with the corresponding antiparticles and vice versa. In classical dynamics, the motion of a charged particle in an electromagnetic field $A^\mu = (\phi, \mathbf{A})$ can be obtained by making the minimal substitution

$$E \to E - q\phi \quad \text{and} \quad \mathbf{p} \to \mathbf{p} - q\mathbf{A}, \tag{4.53}$$

where ϕ and $\mathbf{A}$ are the scalar and vector potentials of electromagnetism and q is the charge of the particle. In four-vector notation, (4.53) can be written

$$p_\mu \to p_\mu - qA_\mu. \tag{4.54}$$

Following the canonical procedure for moving between classical physics and quantum mechanics and replacing energy and momentum by the operators $\hat{\mathbf{p}} = -i\nabla$ and $\hat{E} = i\partial/\partial t$, Equation (4.54) can be written in operator form as

$$i\partial_\mu \to i\partial_\mu - qA_\mu. \tag{4.55}$$

The Dirac equation for an electron with charge $q = -e$ (where $e \equiv +|e|$ is the magnitude of the electron charge) in the presence of an electromagnetic field can be obtained by making the minimal substitution of (4.55) in the free-particle Dirac equation, giving

$$\gamma^\mu(\partial_\mu - ieA_\mu)\psi + im\psi = 0. \tag{4.56}$$

The equivalent equation for the positron can be obtained by first taking the complex conjugate of (4.56) and then pre-multiplying by $-i\gamma^2$ to give

$$-i\gamma^2(\gamma^\mu)^*(\partial_\mu + ieA_\mu)\psi^* - m\gamma^2\psi^* = 0. \tag{4.57}$$

In the Dirac–Pauli representation of the γ-matrices, $(\gamma^0)^* = \gamma^0$, $(\gamma^1)^* = \gamma^1$, $(\gamma^2)^* = -\gamma^2$ and $(\gamma^3)^* = \gamma^3$. Using these relations and $\gamma^2\gamma^\mu = -\gamma^\mu\gamma^2$ for $\mu \neq 2$, Equation (4.57) becomes

$$\gamma^\mu(\partial_\mu + ieA_\mu)i\gamma^2\psi^* + im\, i\gamma^2\psi^* = 0. \tag{4.58}$$

If ψ' is defined as

$$\psi' = i\gamma^2\psi^*,$$

then (4.58) can be written

$$\gamma^\mu(\partial_\mu + ieA_\mu)\psi' + im\psi' = 0. \tag{4.59}$$

The equation satisfied by ψ' is the same as that for ψ (4.56), except that the ieA_μ term now appears with the opposite sign. Hence, ψ' is a wavefunction describing a particle which has the same mass as the original particle but with opposite charge; ψ' can be interpreted as the antiparticle wavefunction. In the Dirac–Pauli representation, the charge conjugation operator $\hat{C}$, which transforms a particle wavefunction into the corresponding antiparticle wavefunction, therefore can be identified as

$$\boxed{\psi' = \hat{C}\psi = i\gamma^2\psi^*.}$$

The identification of $\hat{C}$ as the charge conjugation operator can be confirmed by considering its effect on the particle spinor

$$\psi = u_1 e^{i(\mathbf{p}\cdot\mathbf{x}-Et)}.$$

The corresponding charge-conjugated wavefunction ψ' is

$$\psi' = \hat{C}\psi = i\gamma^2\psi^* = i\gamma^2 u_1^* e^{-i(\mathbf{p}\cdot\mathbf{x}-Et)}.$$

The spinor part of ψ' is

$$i\gamma^2 u_1^* = i\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}\sqrt{E+m}\begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix}^* = \sqrt{E+m}\begin{pmatrix} \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix},$$

which is the antiparticle spinor v_1 identified in Section 4.7.3. The effect of the charge-conjugation operator on the u_1 particle spinor is

$$\psi = u_1 e^{i(\mathbf{p}\cdot\mathbf{x}-Et)} \xrightarrow{\hat{C}} \psi' = v_1 e^{-i(\mathbf{p}\cdot\mathbf{x}-Et)},$$

and likewise (up to a unobservable overall complex phase) the effect on u_2 is

$$\psi = u_2 e^{i(\mathbf{p}\cdot\mathbf{x}-Et)} \xrightarrow{\hat{C}} \psi' = v_2 e^{-i(\mathbf{p}\cdot\mathbf{x}-Et)}.$$

Therefore, the effect of the charge-conjugation operator on the particle spinors u_1 and u_2 is to transform them respectively to the antiparticle spinors v_1 and v_2.

4.8 Spin and helicity states

For particles at rest, the spinors $u_1(E,0)$ and $u_2(E,0)$ of (4.42) are clearly eigenstates of

$$\hat{S}_z = \frac{1}{2}\Sigma_z = \frac{1}{2}\begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

and therefore represent "spin-up" and "spin-down" eigenstates of the z-component of the spin operator. However, from the forms of the u and v spinors, given in (4.51) and (4.52), it is immediately apparent that the u_1, u_2, v_1 and v_2 spinors are not in general eigenstates of $\hat{S}_z$. Nevertheless, for particles/antiparticles travelling in the $\pm z$-direction ($\mathbf{p} = \pm\,\mathrm{p}\hat{\mathbf{z}}$), the u and v spinors are

$$u_1 = N\begin{pmatrix} 1 \\ 0 \\ \frac{\pm \mathrm{p}}{E+m} \\ 0 \end{pmatrix}, \quad u_2 = N\begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{\mp \mathrm{p}}{E+m} \end{pmatrix}, \quad v_1 = N\begin{pmatrix} 0 \\ \frac{\mp \mathrm{p}}{E+m} \\ 0 \\ 1 \end{pmatrix} \text{ and } v_2 = N\begin{pmatrix} \frac{\pm \mathrm{p}}{E+m} \\ 0 \\ 1 \\ 0 \end{pmatrix},$$

and therefore

$$\hat{S}_z\, u_1(E,0,0,\pm\mathrm{p}) = +\tfrac{1}{2}u_1(E,0,0,\pm\mathrm{p}),$$
$$\hat{S}_z\, u_2(E,0,0,\pm\mathrm{p}) = -\tfrac{1}{2}u_2(E,0,0,\pm\mathrm{p}).$$

For antiparticle spinors, the *physical* spin is given by the operator $\hat{S}_z^{(v)} = -\hat{S}_z$ and therefore

$$\hat{S}_z^{(v)} v_1(E,0,0,\pm\mathrm{p}) \equiv -\hat{S}_z v_1(E,0,0,\pm\mathrm{p}) = +\tfrac{1}{2}v_1(E,0,0,\pm\mathrm{p}),$$
$$\hat{S}_z^{(v)} v_2(E,0,0,\pm\mathrm{p}) \equiv -\hat{S}_z v_2(E,0,0,\pm\mathrm{p}) = -\tfrac{1}{2}v_2(E,0,0,\pm\mathrm{p}).$$

Hence for a particle/antiparticle with momentum $\mathbf{p} = (0,0,\pm\mathrm{p})$, the u_1 and v_1 spinors represent spin-up states and the u_2 and v_2 spinors represent spin-down states, as indicated in Figure 4.3.

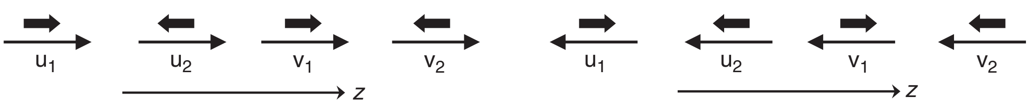

Fig. 4.3 The u_1, u_2, v_1 and v_2 spinors for particles/antiparticles travelling in the $\pm z$-direction.

4.8.1 Helicity

In the chapters that follow, interaction cross sections will be analysed in terms of the spin states of the particles involved. Since the u_1, u_2, v_1 and v_2 spinors only map onto easily identified spin states for particles travelling in the z-direction, their use for this purpose is limited. Furthermore, since $\hat{S}_z$ does not commute with the Dirac Hamiltonian, $[\hat{H}_D, \hat{S}_z] \neq 0$, it is not possible to define a basis of simultaneous eigenstates of $\hat{S}_z$ and $\hat{H}_D$. Rather than defining basis states in terms of an external axis, it is more natural to introduce to concept of helicity. As illustrated in Figure 4.4, the helicity h of a particle is defined as the normalised component of its spin along its direction of flight,

$$h \equiv \frac{\mathbf{S} \cdot \mathbf{p}}{\mathrm{p}}. \tag{4.60}$$

For a four-component Dirac spinor, the helicity operator is

$$\hat{h} = \frac{\hat{\boldsymbol{\Sigma}} \cdot \hat{\mathbf{p}}}{2\mathrm{p}} = \frac{1}{2\mathrm{p}} \begin{pmatrix} \boldsymbol{\sigma} \cdot \hat{\mathbf{p}} & 0 \\ 0 & \boldsymbol{\sigma} \cdot \hat{\mathbf{p}} \end{pmatrix}, \tag{4.61}$$

where $\hat{\mathbf{p}}$ is the momentum operator. From the form of the Dirac Hamiltonian (4.18), it follows that $[\hat{H}_D, \hat{\boldsymbol{\Sigma}} \cdot \hat{\mathbf{p}}] = 0$ and therefore $\hat{h}$ commutes with the free-particle Hamiltonian. Consequently, it is possible to identify spinor states which are simultaneous eigenstates of the free particle Dirac Hamiltonian and the helicity operator. For a spin-half particle, the component of spin measured along any axis is quantised to be either $\pm 1/2$. Consequently, the eigenvalues of the helicity operator acting on a Dirac spinor are $\pm 1/2$. The two possible helicity states for a spin-half fermion are termed *right-handed* and *left-handed* helicity states, as shown in Figure 4.5. Whilst helicity is an important concept in particle physics, it is important to remember that helicity is not Lorentz invariant; for particles with mass, it is always possible to transform into a frame in which the direction of the particle is reversed. The related Lorentz-invariant concept of chirality is introduced in Chapter 6.

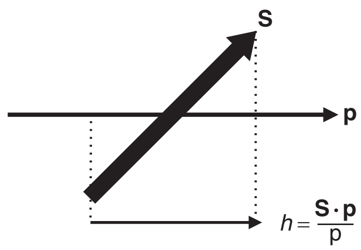

Fig. 4.4 The definition of helicity as the projection of the spin of a particle along its direction of motion.

Fig. 4.5 The two helicity eigenstates for a spin-half fermion. The $h = +1/2$ and $h = -1/2$ states are respectively referred to as right-handed (RH) and left-handed (LH) helicity states.

The simultaneous eigenstates of the free particle Dirac Hamiltonian and the helicity operator are solutions to the Dirac equation which also satisfy the eigenvalue equation,

$$\hat{h}u = \lambda u.$$

Writing the spinor in terms of two two-component column vectors u_A and u_B, and using the helicity operator defined above, this eigenvalue equation can be written

$$\frac{1}{2\mathrm{p}}\begin{pmatrix} \boldsymbol{\sigma}\cdot\mathbf{p} & 0 \\ 0 & \boldsymbol{\sigma}\cdot\mathbf{p} \end{pmatrix}\begin{pmatrix} u_A \\ u_B \end{pmatrix} = \lambda\begin{pmatrix} u_A \\ u_B \end{pmatrix},$$

implying that

$$(\boldsymbol{\sigma}\cdot\mathbf{p})u_A = 2\mathrm{p}\,\lambda u_A, \tag{4.62}$$

$$(\boldsymbol{\sigma}\cdot\mathbf{p})u_B = 2\mathrm{p}\,\lambda u_B. \tag{4.63}$$

The eigenvalues of the helicity operator can be obtained by multiplying (4.62) by $\boldsymbol{\sigma}\cdot\mathbf{p}$ and noting (see Problem 4.10) that $(\boldsymbol{\sigma}\cdot\mathbf{p})^2 = \mathrm{p}^2$, from which it follows that

$$\mathrm{p}^2 u_A = 2\mathrm{p}\lambda(\boldsymbol{\sigma}\cdot\mathbf{p})u_A = 4\mathrm{p}^2\lambda^2 u_A,$$

and therefore, as anticipated, $\lambda = \pm 1/2$. Because the spinors corresponding to the two helicity states are also eigenstates of the Dirac equation, u_A and u_B are related by (4.46),

$$(\boldsymbol{\sigma}\cdot\mathbf{p})u_A = (E+m)u_B,$$

which when combined with (4.62) gives

$$u_B = 2\lambda\left(\frac{\mathrm{p}}{E+m}\right)u_A. \tag{4.64}$$

Therefore for a helicity eigenstate, u_B is proportional to u_A and once (4.62) is solved to obtain u_A, the corresponding equation for u_B (4.63) is automatically satisfied.

Equation (4.62) is most easily solved by expressing the helicity states in terms of spherical polar coordinates where

$$\mathbf{p} = (\mathrm{p}\sin\theta\cos\phi,\ \mathrm{p}\sin\theta\sin\phi,\ \mathrm{p}\cos\theta),$$

and the helicity operator can be written as

$$\frac{1}{2\mathrm{p}}(\boldsymbol{\sigma}\cdot\mathbf{p}) = \frac{1}{2\mathrm{p}}\begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}$$
$$= \frac{1}{2}\begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}.$$

Writing the components of u_A as

$$u_A = \begin{pmatrix} a \\ b \end{pmatrix},$$

the eigenvalue equation of (4.62) becomes

$$\begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 2\lambda\begin{pmatrix} a \\ b \end{pmatrix},$$

and therefore the ratio of b/a is equal to

$$\frac{b}{a} = \frac{2\lambda - \cos\theta}{\sin\theta}e^{i\phi}.$$

For the right-handed helicity state with $\lambda = +1/2$,

$$\frac{b}{a} = \frac{1-\cos\theta}{\sin\theta}e^{i\phi} = \frac{2\sin^2\left(\frac{\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}e^{i\phi} = e^{i\phi}\frac{\sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)}.$$

Using the relation between u_A and u_B from (4.64), the right-handed helicity particle spinor, denoted $u_\uparrow$, then can be identified as

$$u_\uparrow = N\begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ e^{i\phi}\sin\left(\frac{\theta}{2}\right) \\ \frac{\mathrm{p}}{E+m}\cos\left(\frac{\theta}{2}\right) \\ \frac{\mathrm{p}}{E+m}e^{i\phi}\sin\left(\frac{\theta}{2}\right) \end{pmatrix},$$

where $N = \sqrt{E+m}$ is the overall normalisation factor. The left-handed helicity spinor with $h = -1/2$, denoted $u_\downarrow$, can be found in the same manner and thus the right-handed and left-handed helicity particle spinors, normalised to $2E$ particles per unit volume, are

$$u_\uparrow = \sqrt{E+m}\begin{pmatrix} c \\ se^{i\phi} \\ \frac{\mathrm{p}}{E+m}c \\ \frac{\mathrm{p}}{E+m}se^{i\phi} \end{pmatrix} \quad u_\downarrow = \sqrt{E+m}\begin{pmatrix} -s \\ ce^{i\phi} \\ \frac{\mathrm{p}}{E+m}s \\ -\frac{\mathrm{p}}{E+m}ce^{i\phi} \end{pmatrix}, \tag{4.65}$$

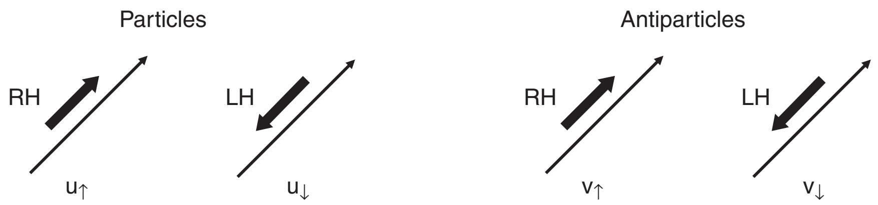

Fig. 4.6 The helicity eigenstates for spin-half particles and antiparticles.

where $s = \sin\left(\frac{\theta}{2}\right)$ and $c = \cos\left(\frac{\theta}{2}\right)$. The corresponding antiparticle states, $v_\uparrow$ and $v_\downarrow$, are obtained in the same way remembering that the physical spin of an antiparticle spinor is given by $\hat{\mathbf{S}}^{(v)} = -\hat{\mathbf{S}}$, and hence for the $h = +1/2$ antiparticle state

$$\left(\frac{\boldsymbol{\Sigma}\cdot\mathbf{p}}{2\mathrm{p}}\right)v_\uparrow = -\tfrac{1}{2}v_\uparrow .$$

The resulting normalised antiparticle helicity spinors are

$$v_\uparrow = \sqrt{E+m}\begin{pmatrix} \frac{\mathrm{p}}{E+m}s \\ -\frac{\mathrm{p}}{E+m}ce^{i\phi} \\ -s \\ ce^{i\phi} \end{pmatrix} \quad v_\downarrow = \sqrt{E+m}\begin{pmatrix} \frac{\mathrm{p}}{E+m}c \\ \frac{\mathrm{p}}{E+m}se^{i\phi} \\ c \\ se^{i\phi} \end{pmatrix}. \tag{4.66}$$

The four helicity states of (4.65) and (4.66), which correspond to the states shown in Figure 4.6, form the helicity basis that is used to describe particles and antiparticles in the calculations that follow. In many of these calculations, the energies of the particles being considered are much greater than their masses. In this ultra-relativistic limit ($E \gg m$) the helicity eigenstates can be approximated by

$$u_\uparrow \approx \sqrt{E}\begin{pmatrix} c \\ se^{i\phi} \\ c \\ se^{i\phi} \end{pmatrix}, \; u_\downarrow \approx \sqrt{E}\begin{pmatrix} -s \\ ce^{i\phi} \\ s \\ -ce^{i\phi} \end{pmatrix}, \; v_\uparrow \approx \sqrt{E}\begin{pmatrix} s \\ -ce^{i\phi} \\ -s \\ ce^{i\phi} \end{pmatrix} \; \text{and} \; v_\downarrow \approx \sqrt{E}\begin{pmatrix} c \\ se^{i\phi} \\ c \\ se^{i\phi} \end{pmatrix}. \tag{4.67}$$

It should be remembered that the above spinors all can be multiplied by an overall complex phase with no change in any physical predictions.

4.9 Intrinsic parity of Dirac fermions

Charge conjugation, discussed in Section 4.7.5, is one example of a discrete symmetry transformation, particle ↔ antiparticle. Another example is the parity transformation, which corresponds to spatial inversion through the origin,

$$x' = -x, \quad y' = -y, \quad z' = -z \quad \text{and} \quad t' = t.$$

Parity is an important concept in particle physics because both the QED and QCD interactions always conserve parity. To understand why this is the case (which is explained in Chapter 11), we will need to use the parity transformation properties of Dirac spinors and will need to identify the corresponding parity operator which acts on solutions of the Dirac equation.

Suppose ψ is a solution of the Dirac equation and ψ' is the corresponding solution in the "parity mirror" obtained from the action of the parity operator $\hat{P}$ such that

$$\psi \to \psi' = \hat{P}\psi.$$

From the definition of the parity operation, the effect of two successive parity transformations is to recover the original wavefunction. Consequently $\hat{P}^2 = I$ and thus

$$\psi' = \hat{P}\psi \quad \Rightarrow \quad \hat{P}\psi' = \psi.$$

The form of the parity operator can be deduced by considering a wavefunction $\psi(x, y, z, t)$ which satisfies the free-particle Dirac equation,

$$i\gamma^1\frac{\partial\psi}{\partial x} + i\gamma^2\frac{\partial\psi}{\partial y} + i\gamma^3\frac{\partial\psi}{\partial z} - m\psi = -i\gamma^0\frac{\partial\psi}{\partial t}. \tag{4.68}$$

The parity transformed wavefunction $\psi'(x', y', z', t') = \hat{P}\psi(x, y, z, t)$ must satisfy the Dirac equation in the new coordinate system

$$i\gamma^1\frac{\partial\psi'}{\partial x'} + i\gamma^2\frac{\partial\psi'}{\partial y'} + i\gamma^3\frac{\partial\psi'}{\partial z'} - m\psi' = -i\gamma^0\frac{\partial\psi'}{\partial t'}. \tag{4.69}$$

Writing $\psi = \hat{P}\psi'$, equation (4.68) becomes

$$i\gamma^1\hat{P}\frac{\partial\psi'}{\partial x} + i\gamma^2\hat{P}\frac{\partial\psi'}{\partial y} + i\gamma^3\hat{P}\frac{\partial\psi'}{\partial z} - m\hat{P}\psi' = -i\gamma^0\hat{P}\frac{\partial\psi'}{\partial t}.$$

Premultiplying by γ^0 and expressing the derivatives in terms of the primed system (which introduces minus signs for all the space-like coordinates) gives

$$-i\gamma^0\gamma^1\hat{P}\frac{\partial\psi'}{\partial x'} - i\gamma^0\gamma^2\hat{P}\frac{\partial\psi'}{\partial y'} - i\gamma^0\gamma^3\hat{P}\frac{\partial\psi'}{\partial z'} - m\gamma^0\hat{P}\psi' = -i\gamma^0\gamma^0\hat{P}\frac{\partial\psi'}{\partial t'},$$

which using $\gamma^0\gamma^k = -\gamma^k\gamma^0$ can be written

$$i\gamma^1\gamma^0\hat{P}\frac{\partial\psi'}{\partial x'} + i\gamma^2\gamma^0\hat{P}\frac{\partial\psi'}{\partial y'} + i\gamma^3\gamma^0\hat{P}\frac{\partial\psi'}{\partial z'} - m\gamma^0\hat{P}\psi' = -i\gamma^0\gamma^0\hat{P}\frac{\partial\psi'}{\partial t'}. \tag{4.70}$$

In order for (4.70) to reduce to the desired form of (4.69), $\gamma^0\hat{P}$ must be proportional to the 4×4 identity matrix,

$$\gamma^0\hat{P} \propto I.$$

In addition, $\hat{P}^2 = I$ and therefore the parity operator for Dirac spinors can be identified as either

$$\hat{P} = +\gamma^0 \quad \text{or} \quad \hat{P} = -\gamma^0.$$

It is conventional to choose $\hat{P} = +\gamma^0$ such that under the parity transformation, the form of the Dirac equation is unchanged provided the Dirac spinors transform as

$$\psi \rightarrow \hat{P}\psi = \gamma^0\psi. \tag{4.71}$$

The intrinsic parity of a fundamental particle is defined by the action of the parity operator $\hat{P} = \gamma^0$ on a spinor for a particle at rest. For example, the u_1 spinor for a particle *at rest* given by (4.42), is an eigenstate of the parity operator with

$$\hat{P}u_1 = \gamma^0 u_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \sqrt{2m} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = +u_1.$$

Similarly, $\hat{P}u_2 = +u_2$, $\hat{P}v_1 = -v_1$ and $\hat{P}v_2 = -v_2$. Hence the *intrinsic* parity of a fundamental spin-half particle is opposite to that of a fundamental spin-half antiparticle.

The conventional choice of $\hat{P} = +\gamma^0$ rather than $\hat{P} = -\gamma^0$, corresponds to defining the intrinsic parity of particles to be positive and the intrinsic parity of antiparticles to be negative,

$$\hat{P}u(m,0) = +u(m,0) \quad \text{and} \quad \hat{P}v(m,0) = -v(m,0).$$

Since particles and antiparticles are always created and destroyed in pairs, this choice of sign has no physical consequence. Finally, it is straightforward to verify that the action of the parity operator on Dirac spinors corresponding to a particle with momentum $\mathbf{p}$ reverses the momentum but does not change the spin state, for example

$$\hat{P}u_1(E,\mathbf{p}) = +u_1(E,-\mathbf{p}).$$

Summary

This chapter described the foundations of relativistic quantum mechanics and it is worth reiterating the main points. The formulation of relativistic quantum mechanics in terms of the Dirac equation, which is linear in both time and space derivatives,

$$\hat{H}_D\psi = (\boldsymbol{\alpha}\cdot\hat{\mathbf{p}} + \beta m)\psi = i\frac{\partial\psi}{\partial t},$$

implies new degrees of freedom of the wavefunction. Solutions to the Dirac equation are represented by four-component Dirac spinors. These solutions provide a natural description of the spin of the fundamental fermions and antifermions. The $E < 0$ solutions to the Dirac equation are interpreted as negative energy particles propagating backwards in time, or equivalently, the physical positive energy antiparticles propagating forwards in time.

The Dirac equation is usually expressed in terms of four γ-matrices,

$$(i\gamma^\mu\partial_\mu - m)\psi = 0.$$

The properties of the solutions to the Dirac equation are fully defined by the algebra of the γ-matrices. Nevertheless, explicit free-particle solutions were derived using the Dirac–Pauli representation. The four-vector probability current can be written in terms of the γ-matrices

$$j^\mu = \psi^\dagger\gamma^0\gamma^\mu\psi = \overline{\psi}\gamma^\mu\psi,$$

where $\overline{\psi}$ is the adjoint spinor defined as $\overline{\psi} = \psi^\dagger\gamma^0$. The four-vector current will play a central role in the description of particle interactions through the exchange of force-carrying particles.

The solutions to the Dirac equation provide the relativistic quantum mechanical description of spin-half particles and antiparticles. In particular the states $u_\uparrow$, $u_\downarrow$, $v_\uparrow$ and $v_\downarrow$, which are simultaneous eigenstates of the Dirac Hamiltonian and the helicity operator, form a suitable basis for the calculations of cross sections and decay rates that follow.

Finally, two discrete symmetry transformations were introduced, charge conjugation and parity, with corresponding operators

$$\psi \rightarrow \hat{C}\psi = i\gamma^2\psi^* \quad \text{and} \quad \psi \rightarrow \hat{P}\psi = \gamma^0\psi.$$

The transformation properties of the fundamental interactions under parity and charge-conjugation operations will be discussed in detail in the context of the weak interaction.