Circuit Quantum Electrodynamics

Superconducting platform

(second Lecture)

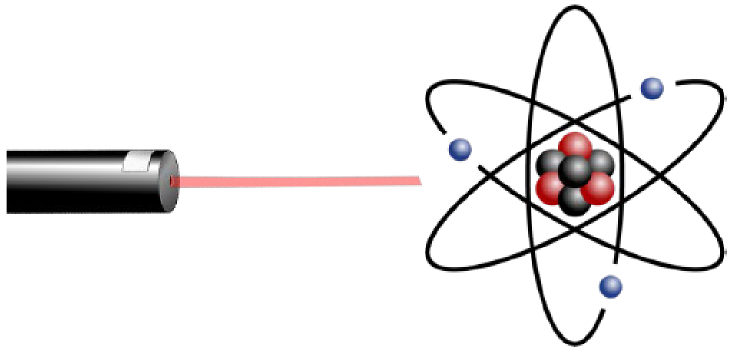
Covering: basic concepts, measurement techniques, implementations, qubit approaches, current trends

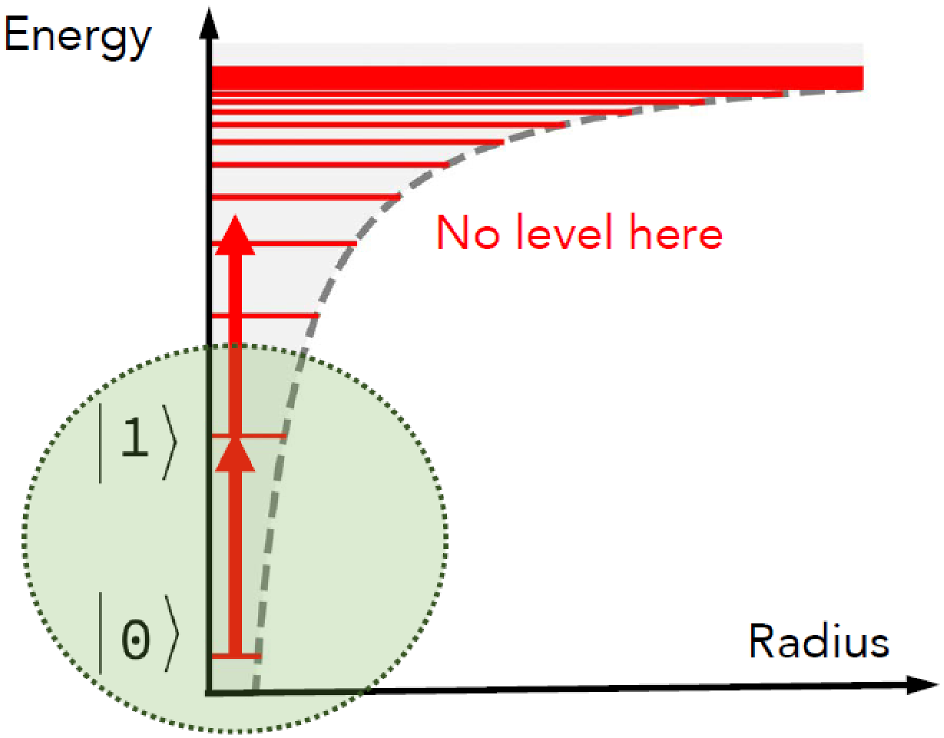
With figures and slides borrowed from

A. Wallraff (ETH-Zurich), P. Bertet (CEA Saclay)

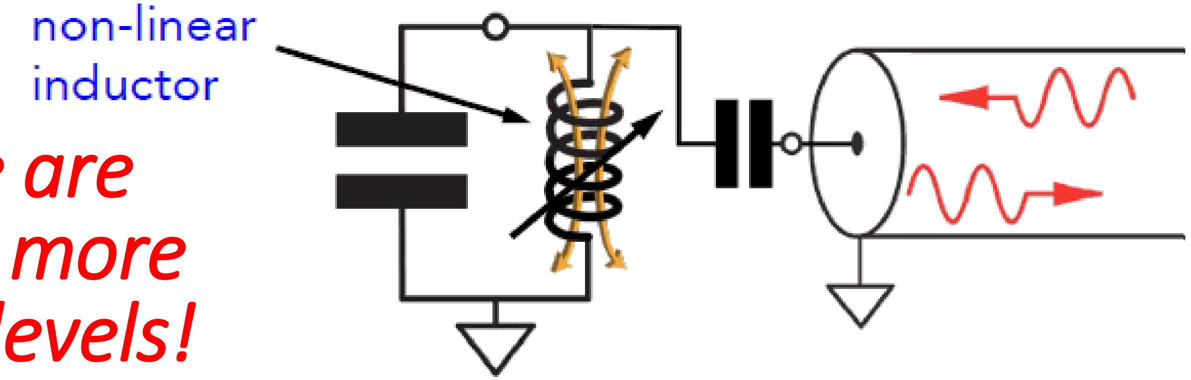
II. Josephson Junctions and Superconducting Qubits

Real Atoms

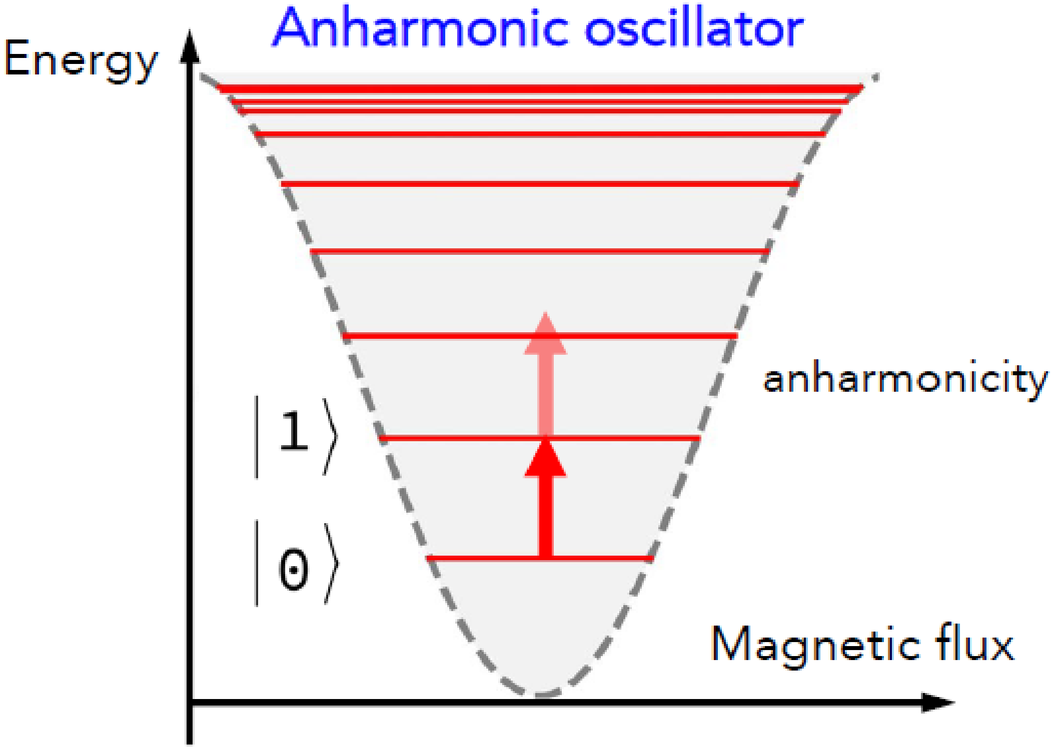




Artificial Systems

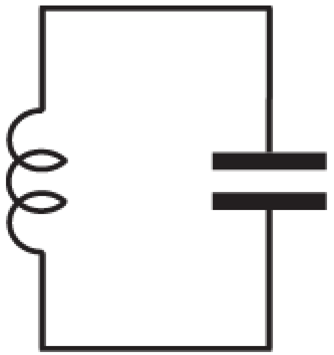


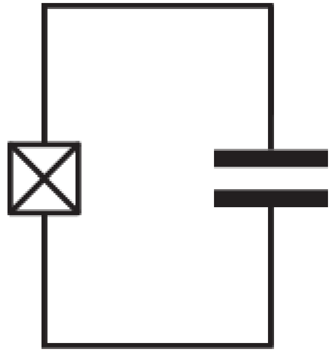
There are always more than 2 levels!



Linear vs. Nonlinear Superconducting Oscillators

LC resonator:

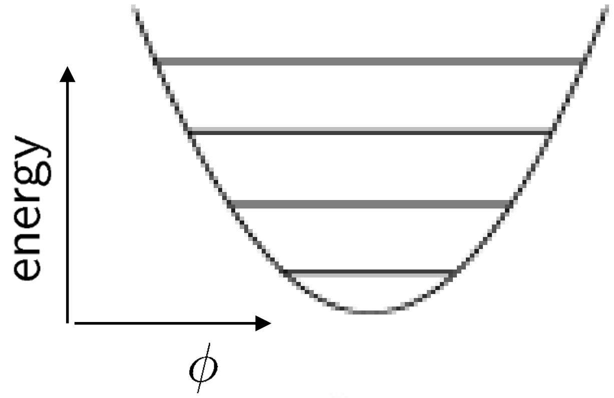


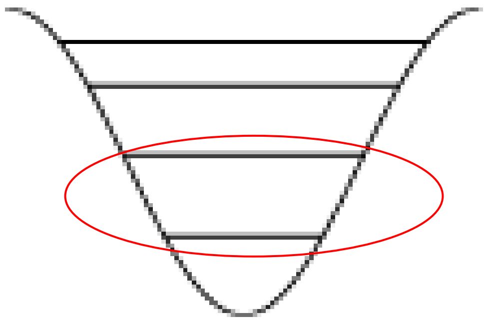


Josephson junction resonator:
Josephson junction = nonlinear inductor

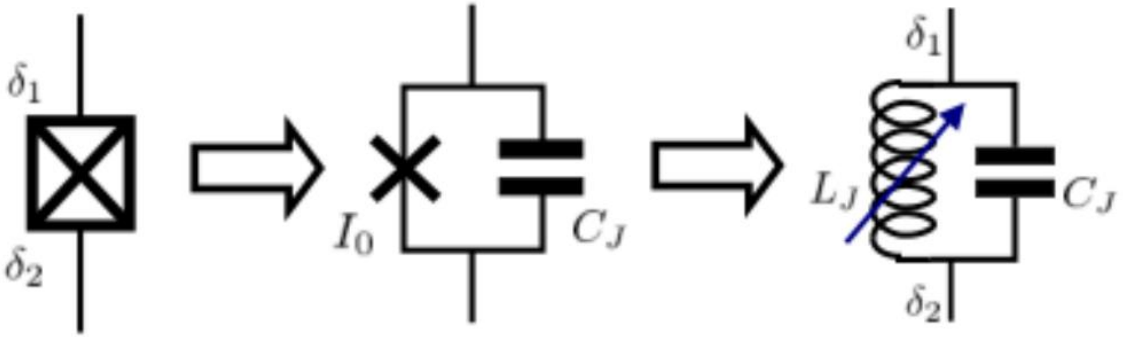
anharmonicity defines effective two-level system

$$U(\phi) = \frac{\Phi^2}{2L}$$





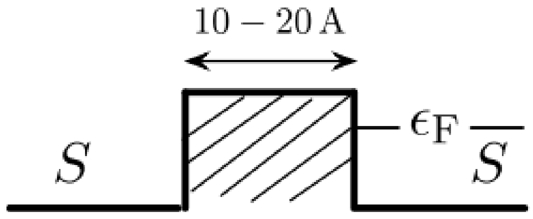
$$U(\phi) = \frac{I_0 \Phi_0}{2\pi}(1 - \cos\Phi)$$



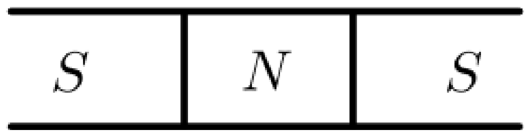
A Low-Loss Nonlinear Element: The Josephson Tunnel Junction

Examples:

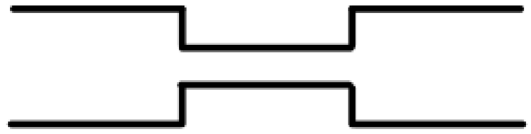
(1) tunnel oxide (S-I-S) junction: schematically,



(2) proximity (S-N-S) junction



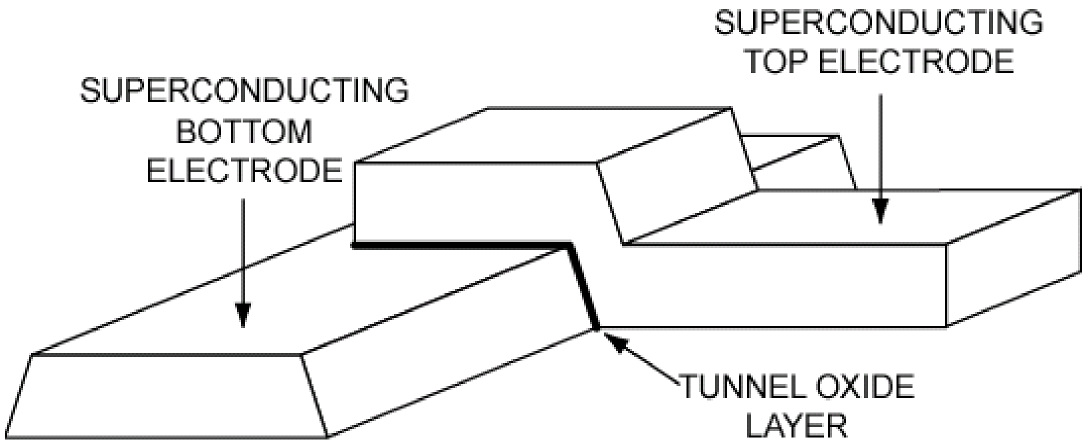
(3) constriction (‘microbridge’)



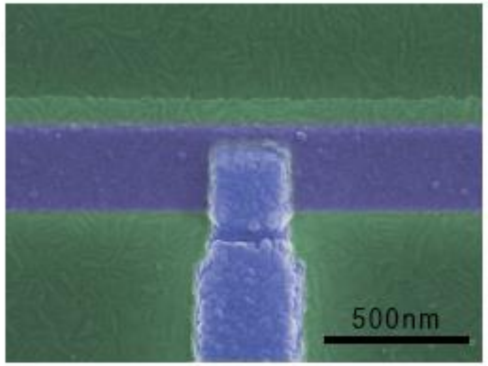
(4) point contact

Josephson effect occurs whenever two bulk superconductors are connected via ‘weak link’, i.e., region which allows passage of electrons but with increased difficulty

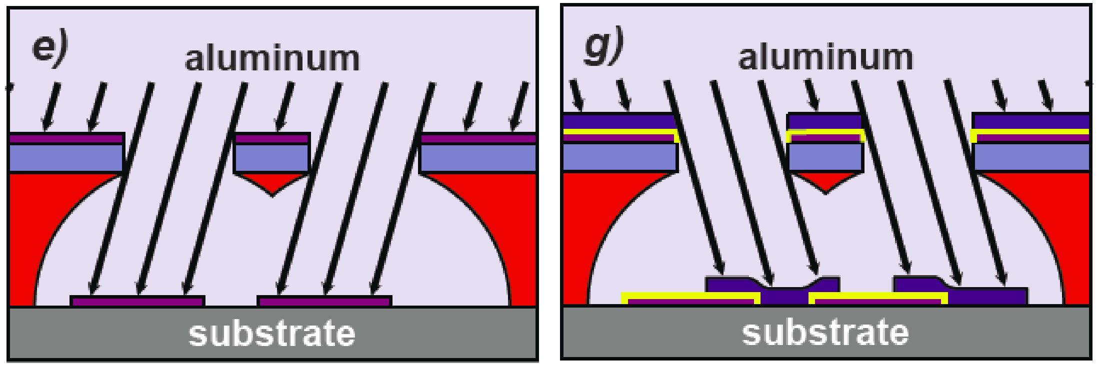
A Low-Loss Nonlinear Element: The Josephson Tunnel Junction



superconductors: Nb, Al
tunnel barrier: AlO_x



Josephson junction fabricated by shadow evaporation:



Procedure:

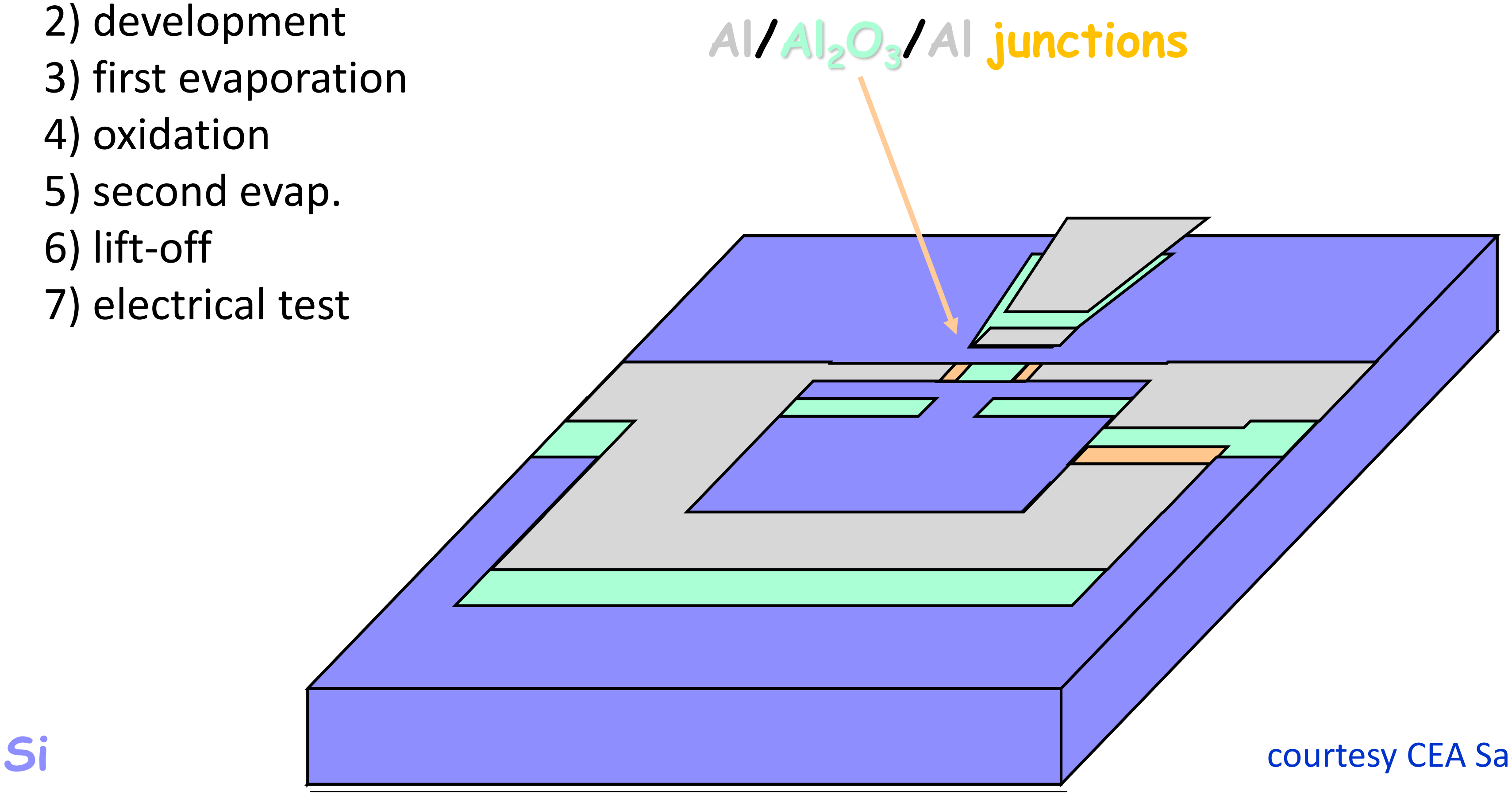
- Mask fabricated from two-layer resist stack (red & blue) exposed by electron beam lithography (EBL) and developed.
- Undercut allows to form suspended bridge (Dolan bridge).
- Deposit two layers (purple & dark blue) of Al under different angles. Form oxide barrier before 2nd evaporation.
- Lift off Al on top of resist by dissolving resist.

M. Tinkham, Introduction to Superconductivity, McGraw-Hill

Fabrication techniques

small junctions → e-beam lithography

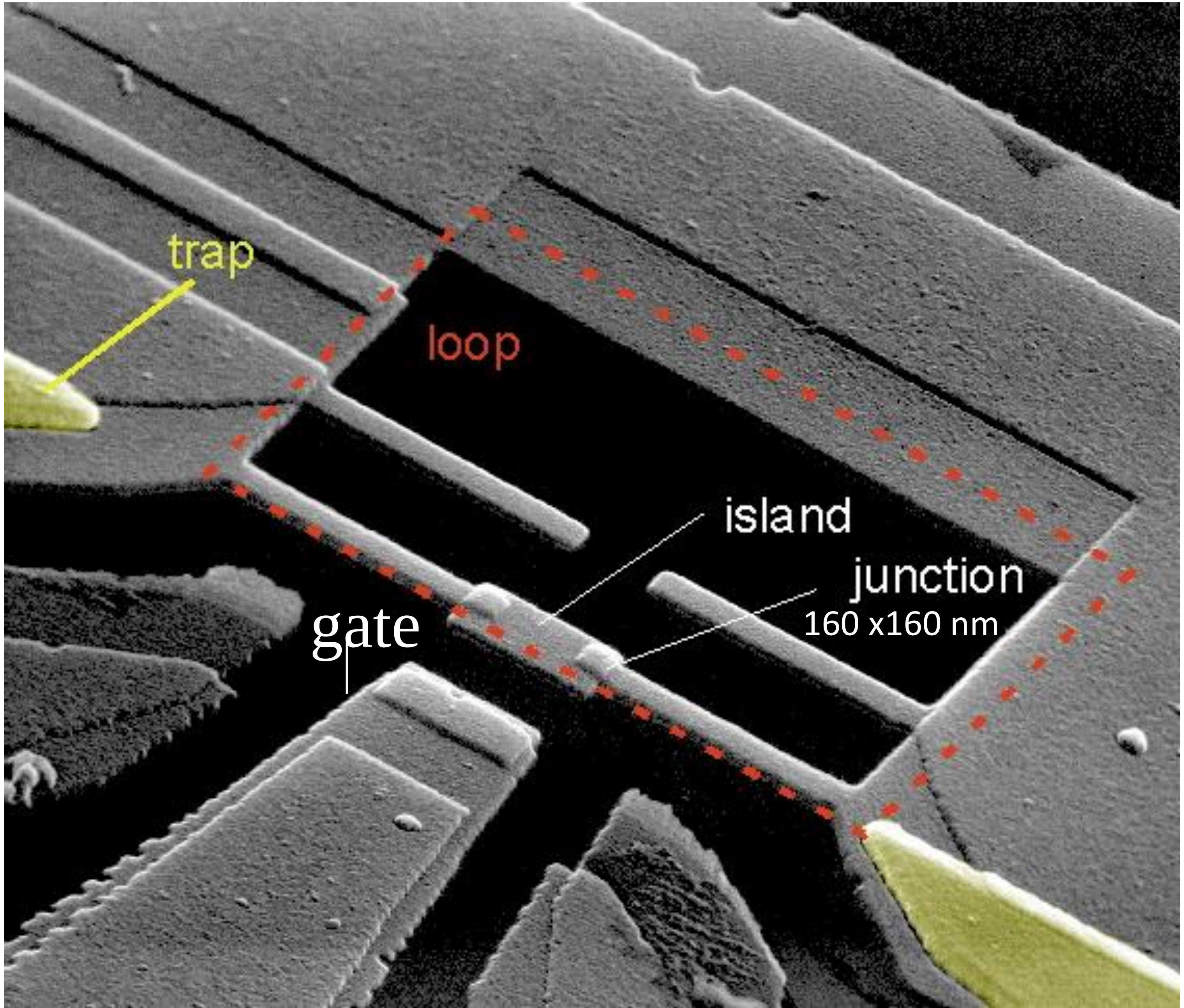
1) e-beam patterning
2) development
3) first evaporation
4) oxidation
5) second evap.
6) lift-off
7) electrical test



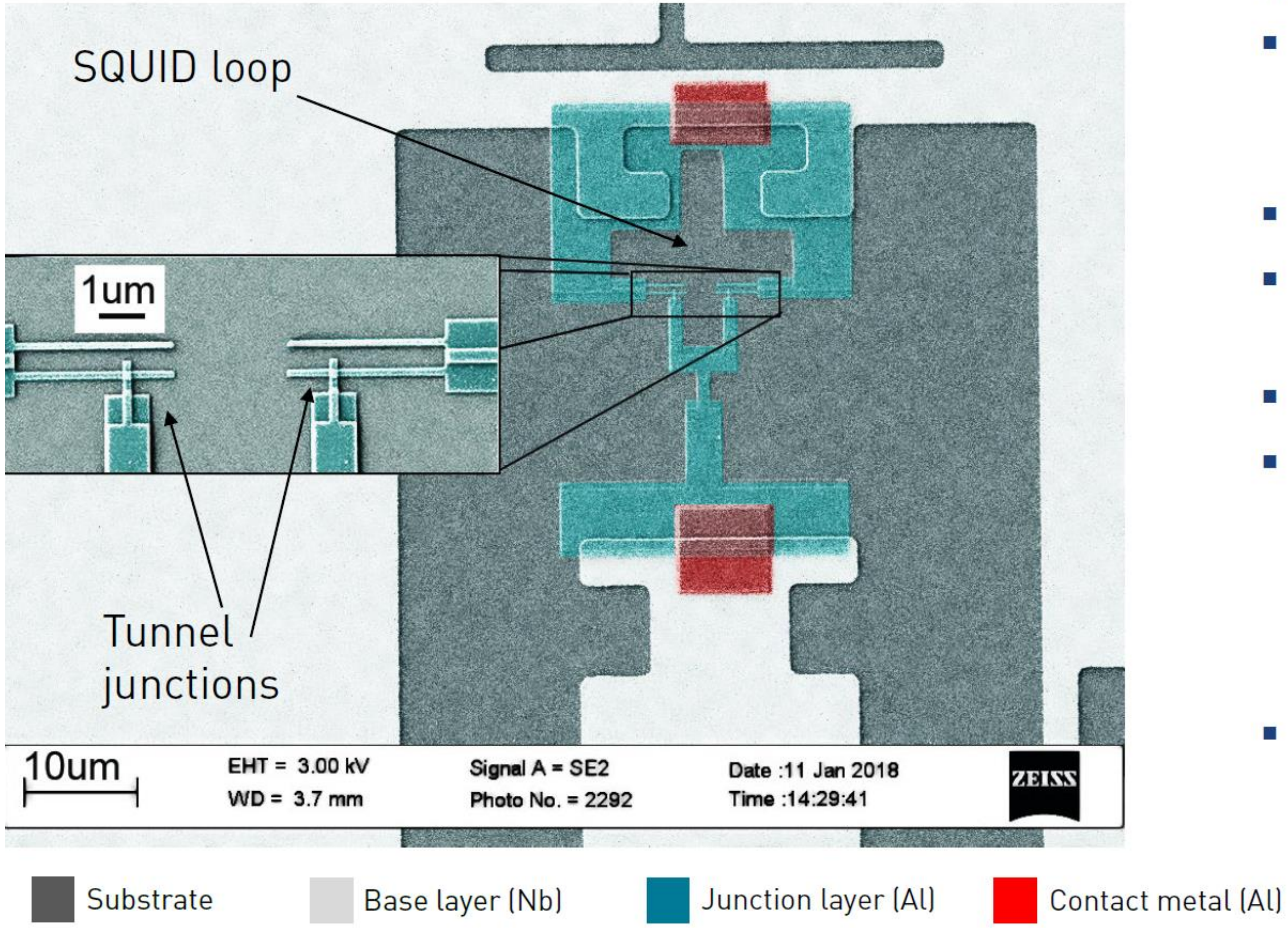
courtesy CEA Saclay

small junctions → Multi angle shadow evaporation

QUANTRONIUM (Saclay group)



Fabrication of Josephson Tunnel Junctions



Properties

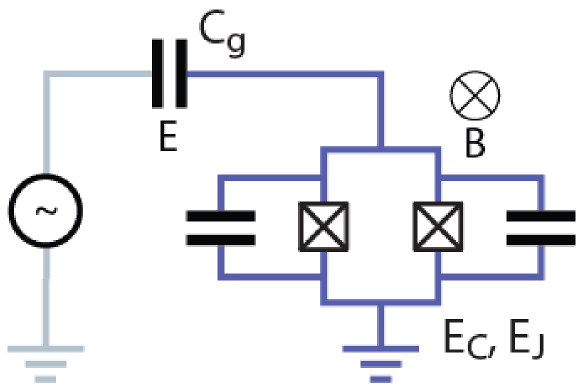
- The critical current I_0 is proportional to the overlap area of the junction A and decreases with the thickness d of the barrier.
- Junction size A controlled lithographically.
- Thickness d of barrier controlled by oxidation parameters (pressure, time, ...)
- Typical values: $A \sim 100 \times 100\ \mathrm{nm}^2$, $d \sim 1\ \mathrm{nm}$
- Critical current related to normal state resistance R, which can be probed at room temperature, by Ambegaokar relation (1963)

$$I_0 = \frac{\pi}{2}\frac{\Delta/e}{R}$$

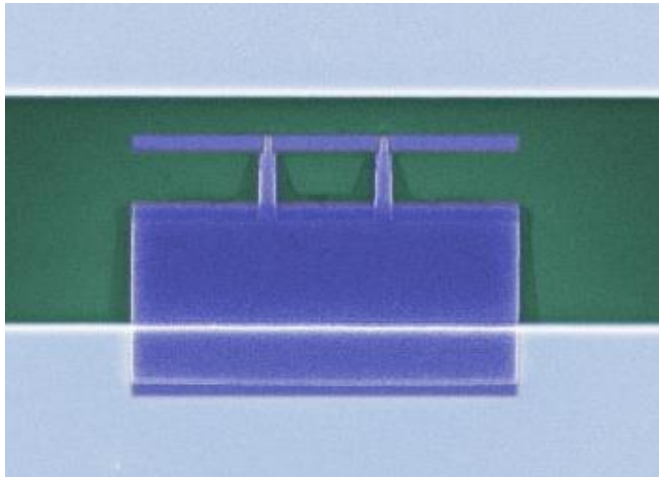
- Coupling to external field determined by geometry and size of the SQUID loop.

Flavors of Superconducting Quantum Bits

The same electrical circuit can be realized in different geometries

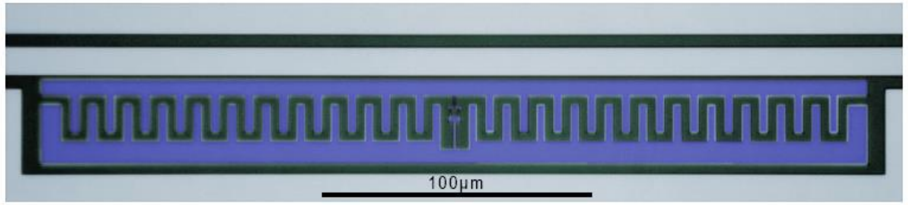


Cooper pair box:

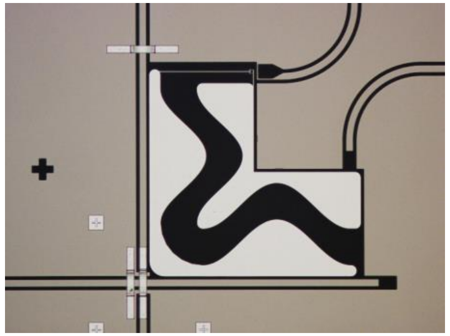


Bouchiat et al., *Physica Scripta* **T76**, 165 (1998).

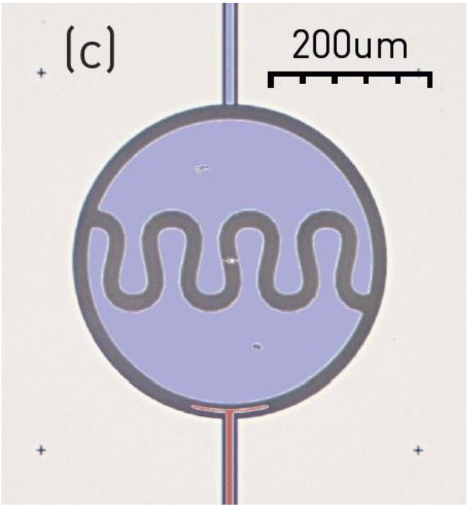
Transmon:



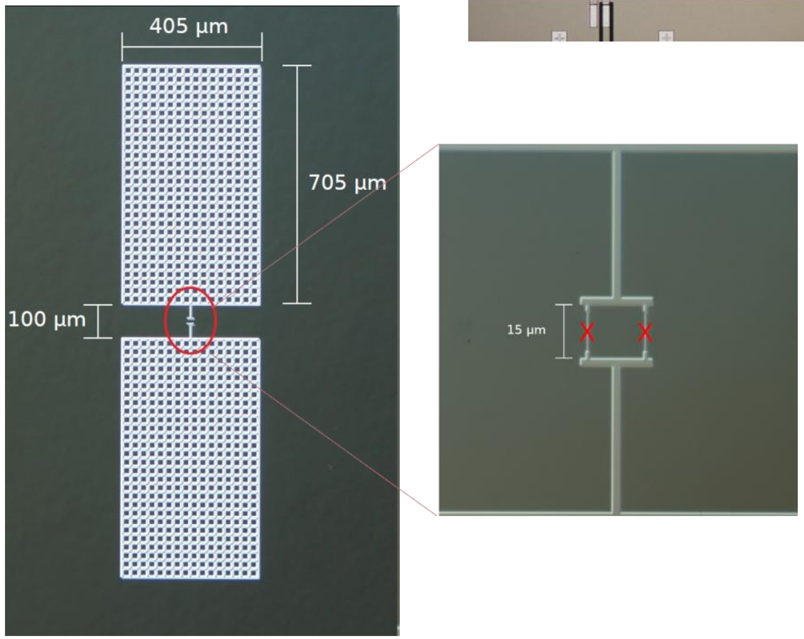
J. Koch *et al.*, PRA **76**, 042319 (2007)



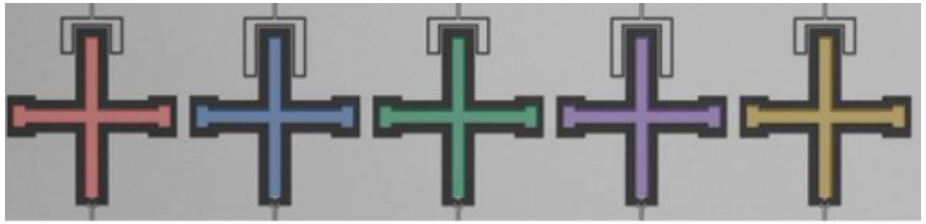
(Jellymon):



M. Pechal et al., *Phys. Rev. Applied* **6**, 024009 (2016)

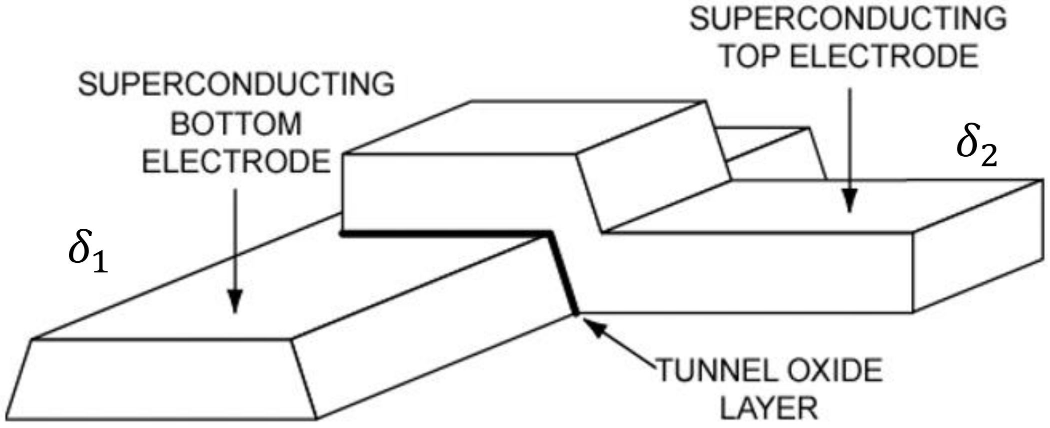


Xmons:



Barends *et al.*, *Phys. Rev. Lett.* **111**, 080502 (2013)

The Josephson Junction



$$\phi = \delta_2 - \delta_1$$

$\phi_0 = \frac{h}{2e}$ is the flux quantum

Nobel Prize in Physics 1973

"for his theoretical predictions of the properties of a supercurrent through a tunnel barrier, in particular those phenomena which are generally known as the Josephson effects".

Non-linear current-phase relation

The potential energy is not harmonic

No dissipation

Josephson relations:

$$I = I_c \sin\phi$$

$$V = \frac{\Phi_0}{2\pi}\dot{\phi}$$

Josephson inductance

Specific Josephson inductance

$$V = \boxed{\frac{\Phi_0}{2\pi I_0}}\frac{1}{\cos\phi}\dot{I} = L_J \dot{I}$$

Josephson energy

Specific Josephson energy

$$E_J = \int VI\,dt = \boxed{\frac{I_0\Phi_0}{2\pi}}(1-\cos\phi)$$

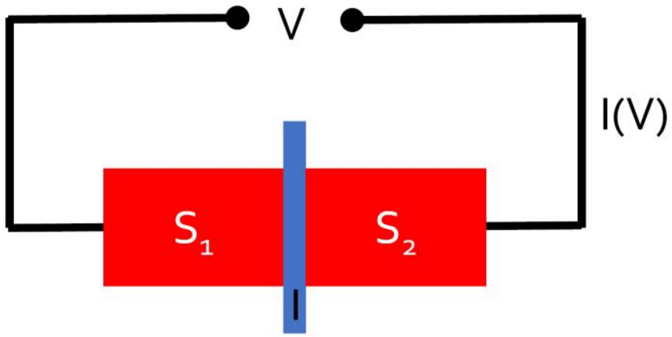
I_0 = 100 nA corresponds to L_{J0} ~ 3 nH

I_0 = 100 nA corresponds to E_{J0}/h ~ 50 GHz

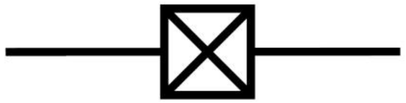
Inductive energy $U(\phi) = \frac{\phi^2}{2L}$ *for an harmonic oscillator*

The Josephson Junction

Josephson effect (Brian Josephson, 1962):



circuit symbol

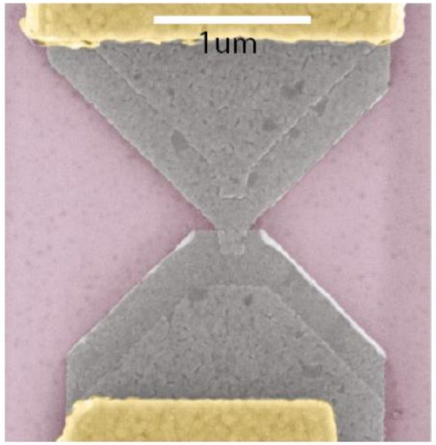


$\psi_1 = |\psi_1| e^{i\phi_1}$ $\psi_2 = |\psi_2| e^{i\phi_2}$

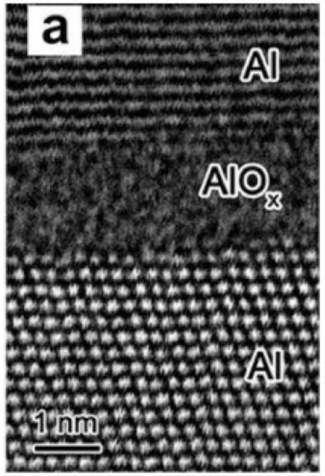
1/3 Nobel prize in physics (1973)

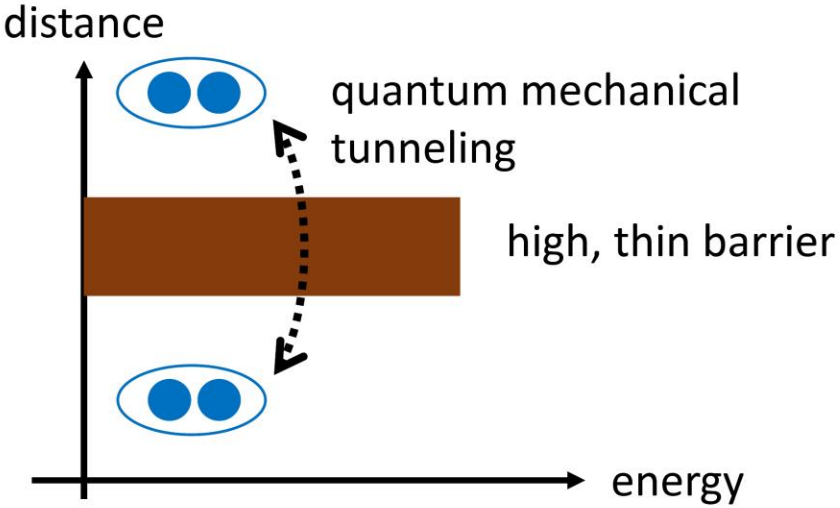
"for his theoretical predictions of the properties of a supercurrent through a tunnel barrier, in particular those phenomena which are generally known as the Josephson effect"

Typical top view



Cross-sectional view:

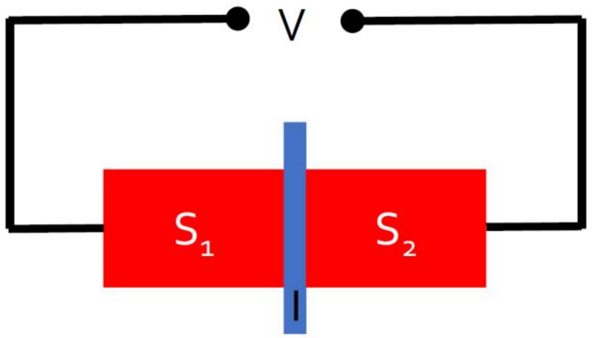




Zeng et al, European Microscopy Congress 2016: Proceedings, p696
Eva Olsson group, Chalmers

The Josephson Junction

Josephson effect (Brian Josephson, 1962):



circuit symbol

$\psi_1 = |\psi_1| e^{i\phi_1}$ $\psi_2 = |\psi_2| e^{i\phi_2}$

Time-dependent Schrödinger-equations for each side:

$$i\hbar \frac{\partial \psi_1}{\partial t} = \mu_1 \psi_1 + T\psi_2$$

$$\underbrace{i\hbar \frac{\partial \psi_2}{\partial t} = \mu_2 \psi_2}_{\text{Evolution of standalone parts}} + \underbrace{T\psi_1}_{\text{tunnel coupling}}$$

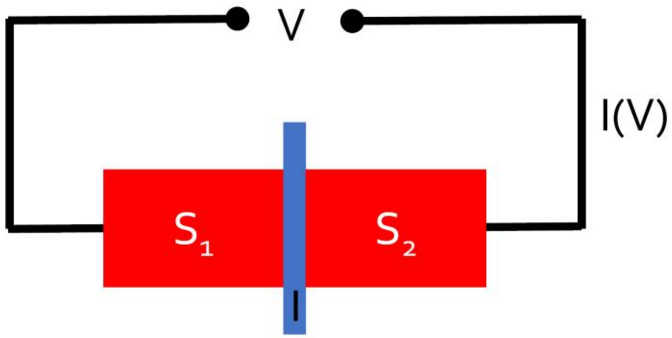
Note: $2eV = \mu_1 - \mu_2$

1/3 Nobel prize in physics (1973)

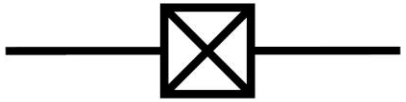
"for his theoretical predictions of the properties of a supercurrent through a tunnel barrier, in particular those phenomena which are generally known as the Josephson effect"

The Josephson Junction

Josephson effect (Brian Josephson, 1962):



circuit symbol



1/3 Nobel prize in physics (1973)

$\psi_1 = |\psi_1| e^{i\phi_1}$ $\psi_2 = |\psi_2| e^{i\phi_2}$

Time-dependent Schrödinger-equations for each side:

$$i\hbar \frac{\partial \psi_1}{\partial t} = \mu_1 \psi_1 + T\psi_2$$

We use $\frac{\partial \psi_{1,2}}{\partial t} = \frac{\partial |\psi_{1,2}|}{\partial t} e^{i\phi_{1,2}} + |\psi_{1,2}| i \frac{\partial \phi_{1,2}}{\partial t} e^{i\phi_{1,2}}$

$$i\hbar \frac{\partial \psi_2}{\partial t} = \mu_2 \psi_2 + T\psi_1$$

Evolution of standalone parts

tunnel coupling

to get $\frac{\partial |\psi_1|}{\partial t} = \frac{T}{\hbar} |\psi_2| \sin(\phi_2 - \phi_1)$

$$\frac{\partial |\psi_2|}{\partial t} = \frac{T}{\hbar} |\psi_1| \sin(\phi_1 - \phi_2)$$

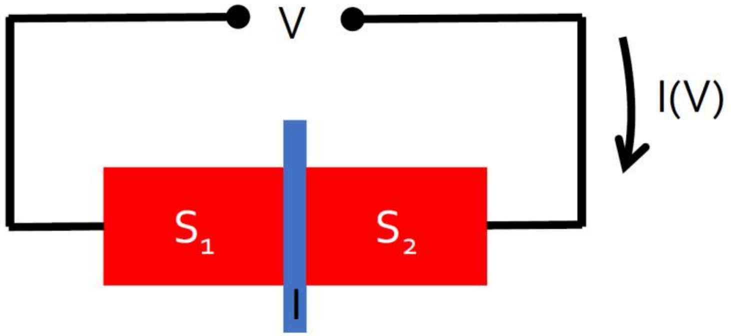
$$\Rightarrow \quad j = -2e \frac{\partial |\psi_1|^2}{\partial t} = -\frac{4e}{\hbar} T |\psi_1||\psi_2| \sin(\phi_2 - \phi_1)$$

$$\boxed{I_S(\phi) = I_c \sin \phi}$$

DC Josephson relation

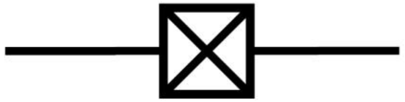
Note: $2eV = \mu_1 - \mu_2$

The AC Josephson effect



$\psi_1 = |\psi_1| e^{i\phi_1}$ $\psi_2 = |\psi_2| e^{i\phi_2}$

circuit symbol



1/3 Nobel prize in physics (1973)

Time-dependent Schrödinger-equations for each side:

$$\underbrace{i\hbar \frac{\partial \psi_1}{\partial t} = \mu_1 \psi_1}_{\text{Evolution of standalone parts}} + \underbrace{T\psi_2}_{\text{tunnel coupling}}$$

$$i\hbar \frac{\partial \psi_2}{\partial t} = \mu_2 \psi_2 + T\psi_1$$

we also get

$$\frac{\partial \phi_1}{\partial t} = -\frac{\mu_1}{\hbar} - \frac{T}{\hbar} \frac{|\psi_2|}{|\psi_1|} \cos(\phi_2 - \phi_1)$$

$$\frac{\partial \phi_2}{\partial t} = -\frac{\mu_2}{\hbar} - \frac{T}{\hbar} \frac{|\psi_1|}{|\psi_2|} \cos(\phi_1 - \phi_2)$$

now we assume that the two leads have the same condensate density

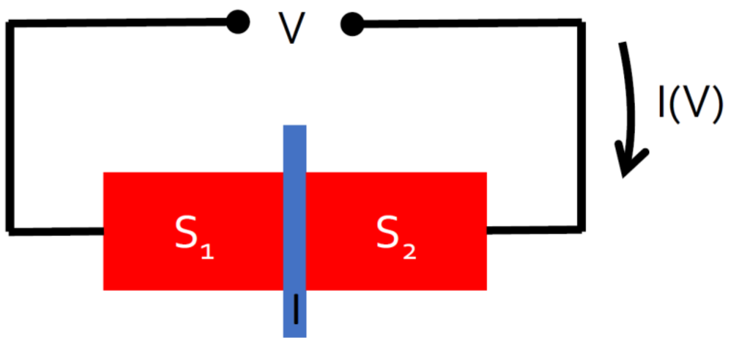
$$\frac{\partial \phi_1}{\partial t} - \frac{\partial \phi_2}{\partial t} = \frac{\mu_2}{\hbar} - \frac{\mu_1}{\hbar} \rightarrow \boxed{\frac{\partial \phi}{\partial t} = \frac{2e}{\hbar} V}$$

AC Josephson relation

$1\ \mu V \rightarrow 483.6\ MHz$

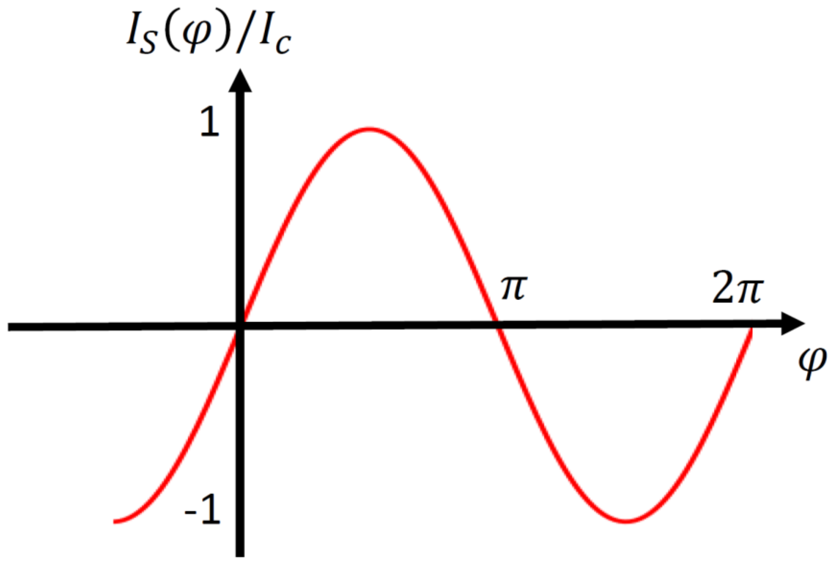
Note: $2eV = \mu_1 - \mu_2$

Current-Phase relationship



Let's summarize:

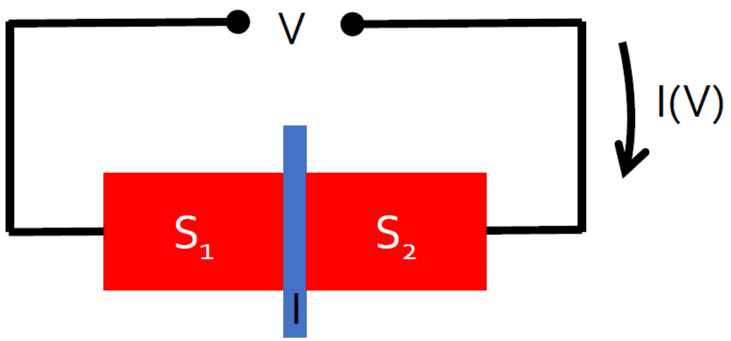
$$I_S(\phi) = I_c \sin\phi$$



There is a correspondence between current and phase (unique apart from global phase shift)

Thus, the critical current tells how strongly the two sides are coupled.

The Josephson inductance

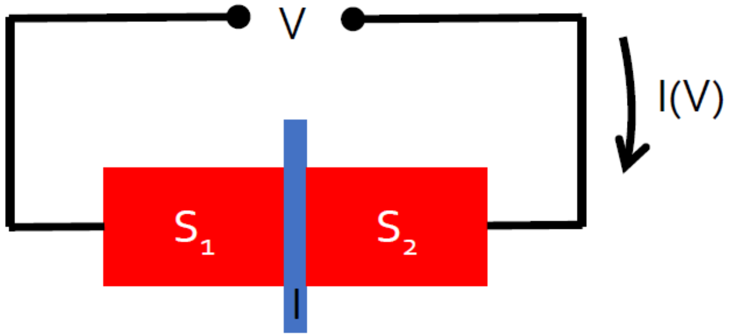


Let's summarize:

$$I_S(\phi) = I_c \sin\phi$$

$$\frac{\partial\phi}{\partial t} = \frac{2e}{\hbar}V$$

The Josephson inductance



This is a **non-linear inductor**: inductance depends on the phase, which in turn, depends on the current.

Let's summarize:

$$\left.\begin{aligned} I_S(\phi) &= I_c \sin\phi \\ \frac{\partial\phi}{\partial t} &= \frac{2e}{\hbar}V \end{aligned}\right\} \quad \frac{\partial I_S}{\partial t} = I_C \frac{\partial\phi}{\partial t}\cos(\phi)$$

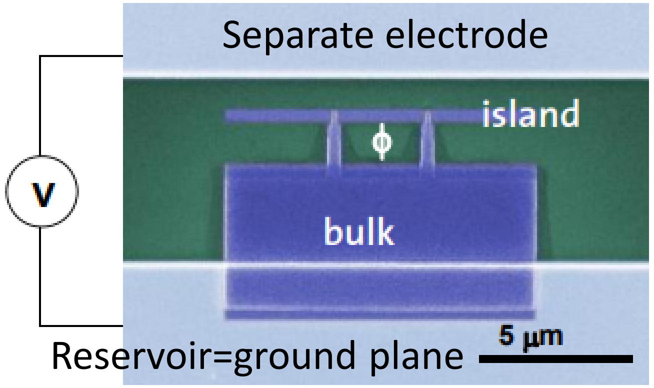
$$V = \underbrace{\frac{\hbar}{2eI_C\cos(\phi)}}_{\text{Josephson inductance}} \frac{\partial I_S}{\partial t}$$

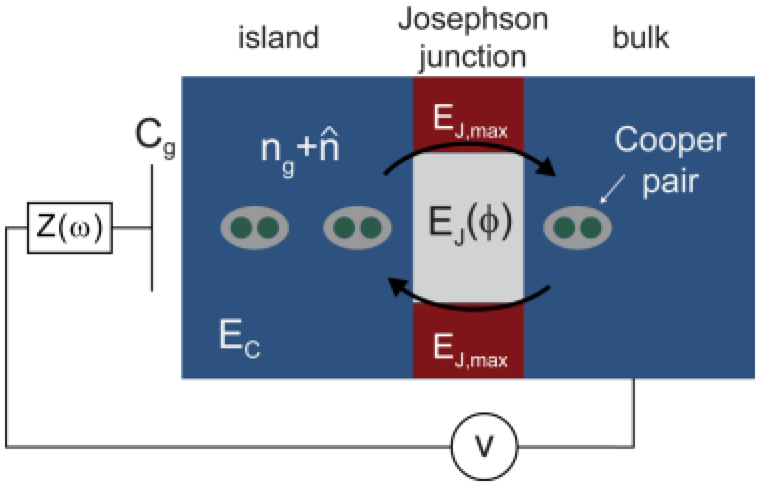
which can be written as $V = L\frac{\partial I}{\partial t}$

inductor

III. The Cooper Pair Box Qubit

Cooper Pair Box Qubit





superconducting island connected via Josephson junctions to grounded reservoir (bulk)

Cooper pairs can tunnel onto island

relevant degree of freedom: number of Cooper pairs on island (N)

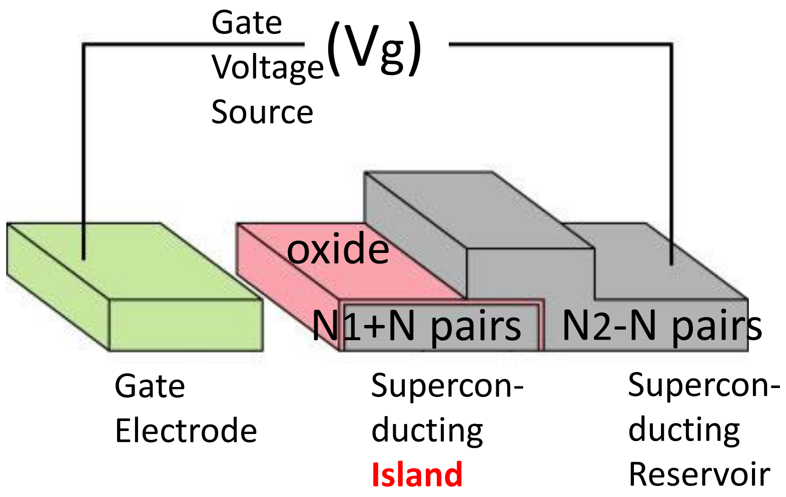
polarization charge adjustable via voltage bias

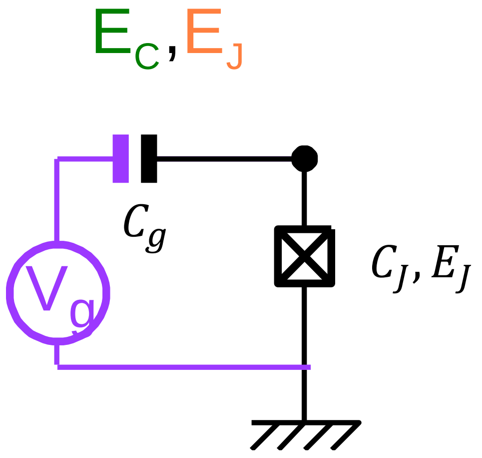
energy scales: charging energy E_C (energy to add another Cooper pair)
Josephson energy E_J (coupling energy)

[Bouchiat, Vion, Joyez, Esteve, Devoret, Physica Scripta T76, 165 (1998).]

courtesy CEA Saclay

The Cooper-Pair Box





$$\hat{\mathbf{N}} = \frac{\hat{Q}}{2e}$$

charge operator
With discrete eigenstates

$$N_g = C_g V_g/(2e)$$

Reduced Gate Charge
(Continuum classic variable)

$$Q_\Sigma = Q_{CP} - Q_g$$

1 degree of freedom

1 knob

$$\left[\hat{\theta},\hat{N}\right] = i$$

$$E_c = \frac{(2e)^2}{2C_\Sigma}$$

$$C_\Sigma = C_g + C_J + C_{ext}$$

« charging energy »

$$\boxed{\hat{H} = E_C(\hat{N} - N_g)^2 - E_J \cos\hat{\theta}}$$

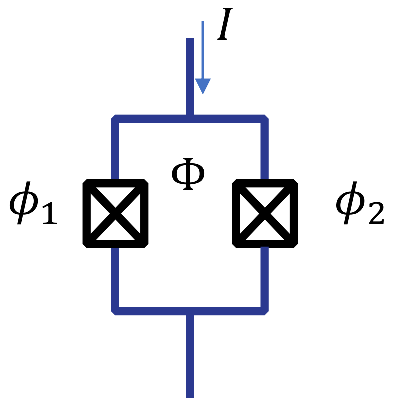
- The Cooper pairs can tunnel through the Josephson junction and the only remaining degree of freedom of the system is the integer number N of Cooper pairs in excess or deficit on the island.
- The variable θ ∈ [0, 2π[, which is the phase of the Cooper pair condensate in the island.

Charge basis is a complete basis $\left|\psi\right\rangle = \sum_N c_N \left|N\right\rangle$

$$\hat{\mathbf{N}}\,|\mathbf{N}\rangle_c = \mathbf{N}\,|\mathbf{N}\rangle_c\,,\ \mathbf{N} \in \mathbb{Z}$$

$$\exp(\pm \mathbf{i}\hat{\theta})\,|\mathbf{N}\rangle_c = |\mathbf{N} \pm 1\rangle_c$$

DC SQUID - qubit frequency tunability



Condition on currents:

$$I = I_1 + I_2$$

Now use Josephson equation:

$$I = I_c \sin\phi_1 + I_c \sin\phi_2$$

This is equal to

$$I = I_c(\sin\phi_1 + \sin\phi_2) = 2I_c \cos\frac{\phi_1 - \phi_2}{2} \sin\frac{\phi_1 + \phi_2}{2}$$

Use flux quantization condition $\phi_1 - \phi_2 = 2\pi\Phi/\Phi_0$ and $\phi = \frac{\phi_1+\phi_2}{2}$.

We have

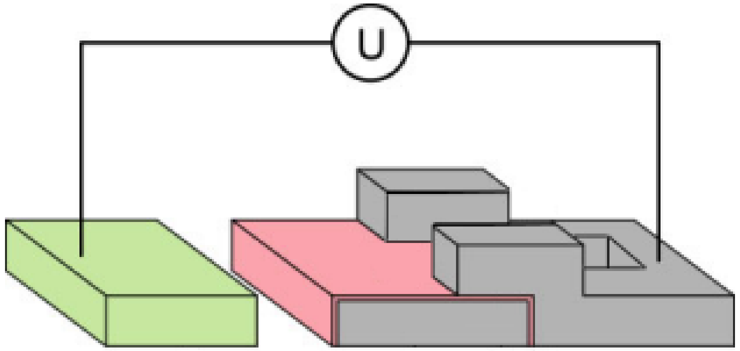
$$I = 2I_c\, cos\, \pi\Phi/\Phi_0 \sin\phi = I_c(\Phi)\sin\phi$$

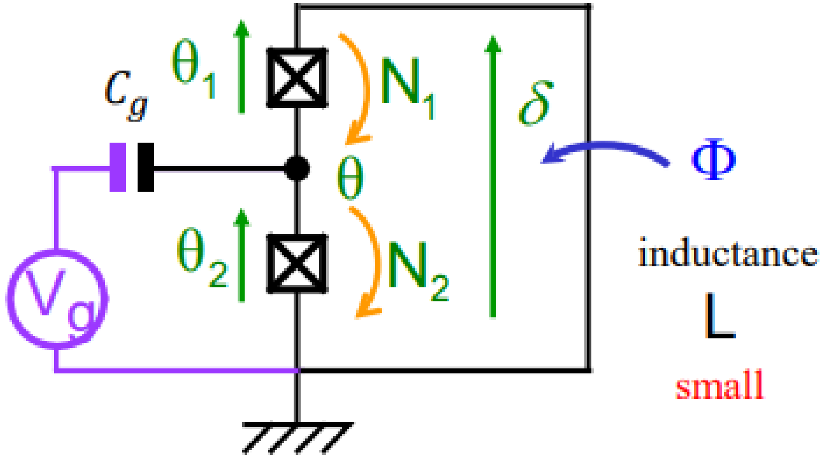
DC SQUID is equivalent to a single JJ but

with the critical current tunable by the external magnetic field!

The split CPB

courtesy CEA Saclay





2 d° of freedom

$$\left[\hat{\theta}_1,\hat{N}_1\right]=i$$
$$\left[\hat{\theta}_2,\hat{N}_2\right]=i$$

or

$$\left[\hat{\theta}=\frac{\hat{\theta}_2-\hat{\theta}_1}{2},\hat{N}=\hat{N}_1-\hat{N}_2\right]=i$$
$$\left[\hat{\delta}=\hat{\theta}_1+\hat{\theta}_2,\hat{K}=\frac{\hat{N}_1+\hat{N}_2}{2}\right]=i$$

2 knobs

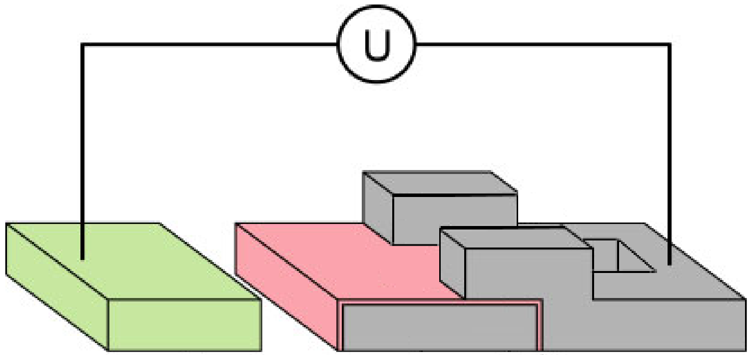
The conjugate variable of δ is the integer number K of Cooper pairs which tunneled through both junctions.

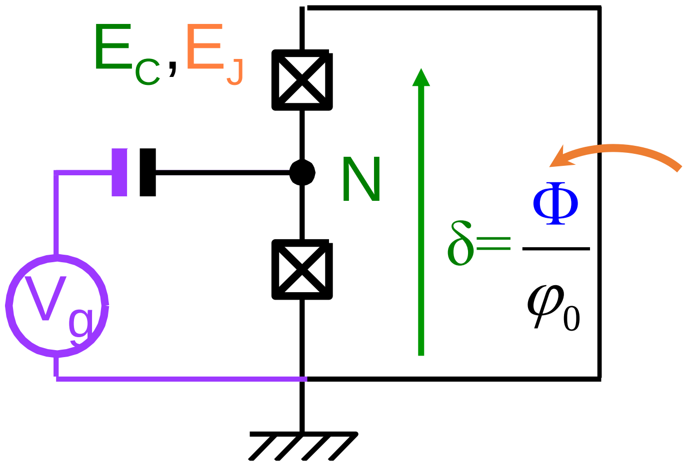
$$\boxed{\hat{H}=E_C(\hat{N}-N_g)^2-E_J\cos\frac{\cancel{\hat{\delta}}}{2}\cos\hat{\theta}+\cancel{\frac{(\Phi-\varphi_0\hat{\delta})^2}{2L}}}$$

$$L \ll \phi_0^2/E_J$$

The split CPB

courtesy CEA Saclay





$$\hat{\theta} = \frac{\hat{\theta}_2 - \hat{\theta}_1}{2}$$

$$\hat{\delta} = \hat{\theta}_1 + \hat{\theta}_2$$

1 d° of freedom $\quad \left[\hat{\theta}, \hat{N}\right] = i$

2 knobs

$$\boxed{\hat{H} = E_C(\hat{N} - N_g)^2 - \underbrace{E_J \cos\frac{\delta}{2}}_{\text{tunable } E_J} \cos\hat{\theta}}$$

The split CPB

courtesy CEA Saclay

Polarization: $N_g = C_g V_g / 2e$

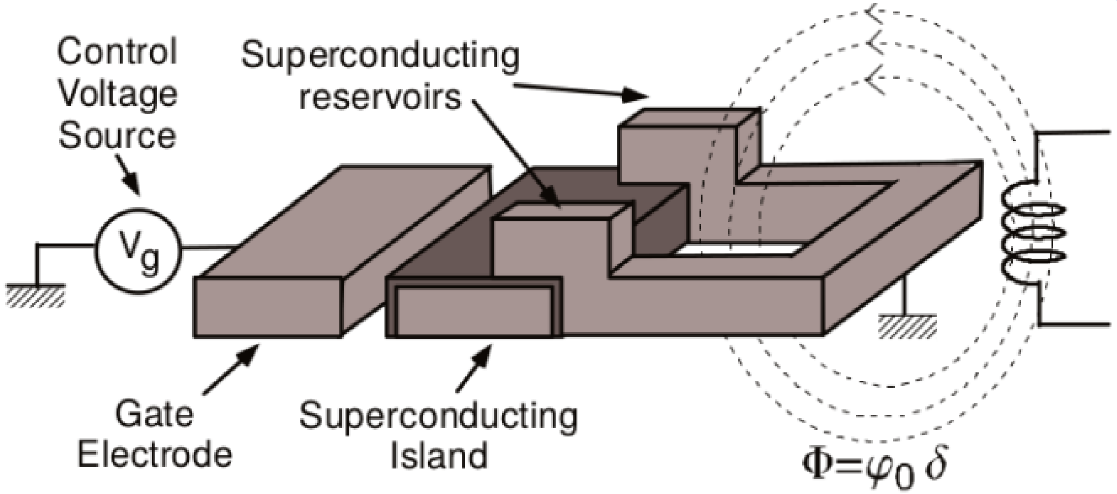
or $Q_g = C_g V_g$

$$H_C = E_C(\hat{N} - N_g)^2$$

Applied flux: $\delta = 2\pi \Phi_{ext} / \Phi_0$

$$H_J = E_J \cos\frac{\delta}{2} \cos\hat{\varphi}$$
$$= E'_J \cos\hat{\varphi}$$

$$\widehat{H}^*_J = -E_J \frac{1+d}{2}\cos(\widehat{\delta}_1) - E_J \frac{1-d}{2}\cos(\widehat{\delta}_2)$$
$$= -E_J \cos(\frac{\widehat{\delta}}{2})\cos(\widehat{\theta}) + dE_J \sin(\frac{\widehat{\delta}}{2})\sin(\widehat{\theta})$$



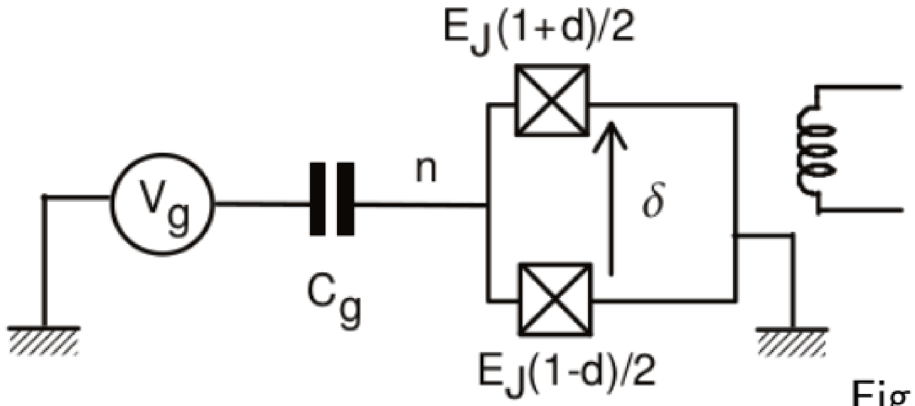


Fig. A. Cottet

$$H = \frac{(\hat{Q} - Q_g)^2}{2C} - E'_J \cos 2\pi \frac{\hat{\Phi}}{\Phi_0} \longleftrightarrow H = \frac{(\hat{p} - A)^2}{2m} - V_0 \cos 2\pi \frac{\hat{x}}{a}$$

Tunable spectrum controlled by V_g and Φ_{ext}.

Similar to the problem of a particle of mass m in the presence of a magnetic field and embedded in a periodic potential.

Hamilton Operator of the Cooper Pair Box

Hamiltonian: $$\hat{H} = \hat{H}_{\mathrm{el}} + \hat{H}_{\mathrm{mag}} = E_C(\hat{N} - N_g)^2 - E_J \cos\hat{\delta}$$

$$\cos\hat{\delta} = \frac{1}{2}(e^{i\hat{\delta}} + e^{-i\hat{\delta}})$$

commutation relation: $$[\hat{\delta}, \hat{N}] = i$$

It would be ideal to put this equation in terms of one of the operators and use that operator as a basis state

charge number operator: $$\hat{N}|N\rangle = N|N\rangle$$ eigenvalues, eigenfunctions

$$\sum_N |N\rangle\langle N| = 1$$ completeness

$$\langle N|M\rangle = \delta_{NM}$$ orthogonality

Being conjugate variables we can transform between the charge number basis and the phase basis by

We expect the number operator N to return the number of Cooper-pairs on the island in a particular state

phase basis: $$|\delta\rangle = \frac{1}{\sqrt{2\pi}} \sum_N e^{iN\delta}|N\rangle$$ basis transformation

$$e^{\pm i\hat{\delta}}|N\rangle = |N \pm 1\rangle$$

Solving the Cooper Pair Box Hamiltonian

Hamilton operator in the charge basis N:

$$\hat{H} = \sum_N \left[E_C (N - N_g)^2 |N\rangle\langle N| - \frac{E_J}{2} (|N\rangle\langle N+1| + |N+1\rangle\langle N|) \right]$$

solutions in the charge basis:

$$\hat{H}|\psi_n(N)\rangle = E_n|\psi_n(N)\rangle$$

If insteadd we want to write the electronic term in the phase basis:

Hamilton operator in the phase basis δ:

$$\hat{H} = E_C(\hat{N} - N_g)^2 - E_J \cos\hat{\delta} = E_C(-i\frac{\partial}{\partial\delta} - N_g)^2 - E_J \cos\hat{\delta}$$

transformation of the number operator:

$$\hat{N} = \frac{\hat{Q}}{2e} = -i\hbar\frac{1}{2e}\frac{\partial}{\partial\phi} = -i\frac{\partial}{\partial\delta}$$

solutions in the phase basis:

$$\hat{H}|\psi_n(\delta)\rangle = E_n|\psi_n(\delta)\rangle$$

Energy levels of the CPB

courtesy CEA Saclay

$$\hat{H}(N_g,\Phi) = E_C(\hat{N}-N_g)^2 - E_J(\Phi)\cos\hat{\theta} \longrightarrow \left\{E_k(N_g,\Phi), |\psi_k\rangle(N_g,\Phi)\right\}$$

Solve **either** in charge basis |N> $\quad$ *N integer* $\quad |\psi_k\rangle = \sum_N c_{k,N}|N\rangle$

$$\hat{H} = E_C(N-N_g)^2|N\rangle\langle N| - \frac{E_J}{2}\sum_{N\in\mathbb{Z}}\left(|N+1\rangle\langle N| + |N\rangle\langle N+1|\right)$$

Diagonalize

$$\begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & E_C(-1-N_g)^2 & -E_J/2 & 0 & \cdots \\ \cdots & -E_J/2 & E_C(0-N_g)^2 & -E_J/2 & \cdots \\ \cdots & 0 & -E_J/2 & E_C(1-N_g)^2 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}_{\ldots|N=-1\rangle|N=0\rangle|N=1\rangle\ldots}$$

Energy levels of the CPB

courtesy CEA Saclay

$$\hat{H}(N_g,\Phi) = E_C(\hat{N}-N_g)^2 - E_J(\Phi)\cos\hat{\theta} \longrightarrow \left\{E_k(N_g,\Phi), |\psi_k\rangle(N_g,\Phi)\right\}$$

... **or** in phase basis $|\theta>$ $(\theta \in [0,2\pi])$ $|\psi_k\rangle = \int_0^{2\pi} d\theta \psi_k(\theta)|\theta\rangle$

$$\hat{H}(N_g,\Phi) = E_C\left(\frac{1}{i}\frac{\partial}{\partial\theta} - N_g\right)^2 - E_J(\Phi)\cos\theta$$

Solve Mathieu equation

$$E_C\left(\frac{1}{i}\frac{\partial}{\partial\theta} - N_g\right)^2 \psi_k(\theta) - E_J(\Phi)\cos\theta\,\psi_k(\theta) = E_k\psi_k(\theta)$$

Equivalent solution to the Hamiltonian can be found in both representations, e.g. by numerically solving the Schrödinger equation for the charge (N) representation or analytically solving the Schrödinger equation for the phase (θ) representation

Two simple limits : (1)

$$\hat{H}\,(N_g,\Phi) = E_C(\hat{N}-N_g)^2 - E_J(\Phi)\cos\hat{\theta}$$

courtesy CEA Saclay

$E_J(\Phi) = 0$

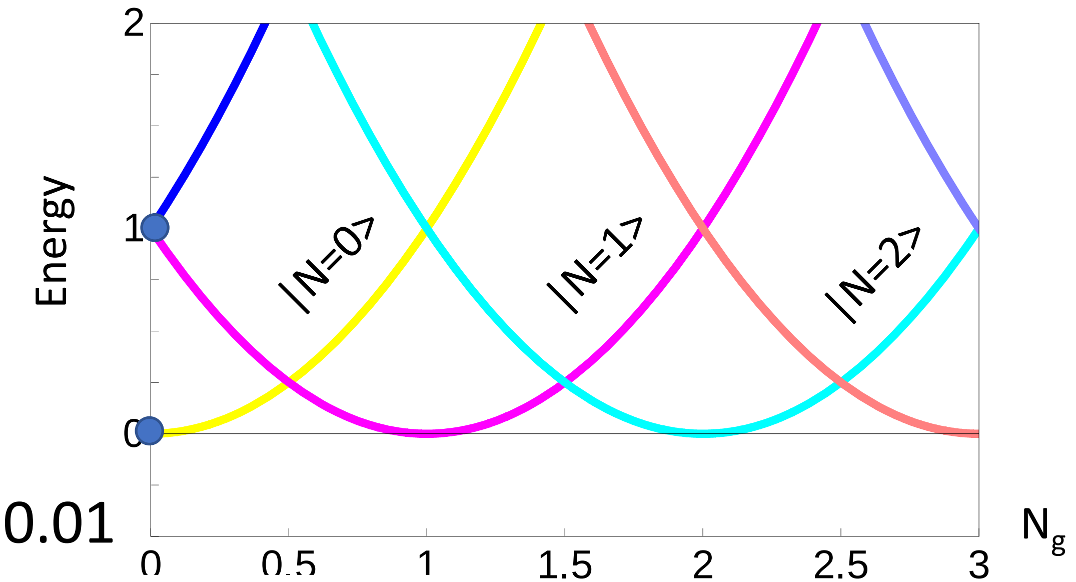
(charge regime) $E_J/E_c=0$

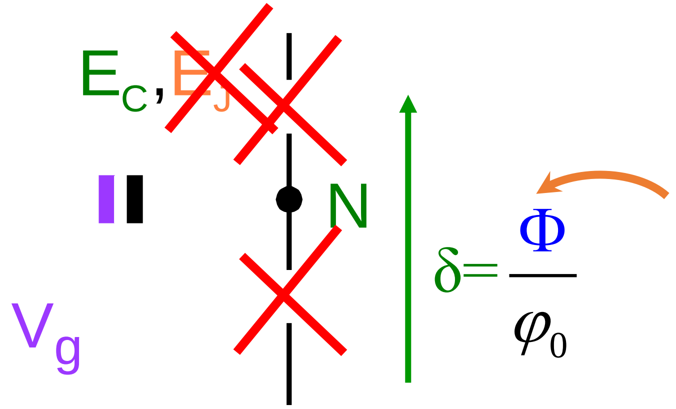
in **charge** basis |N>

$$|\psi\rangle = \sum_N c_N |N\rangle$$

N integer

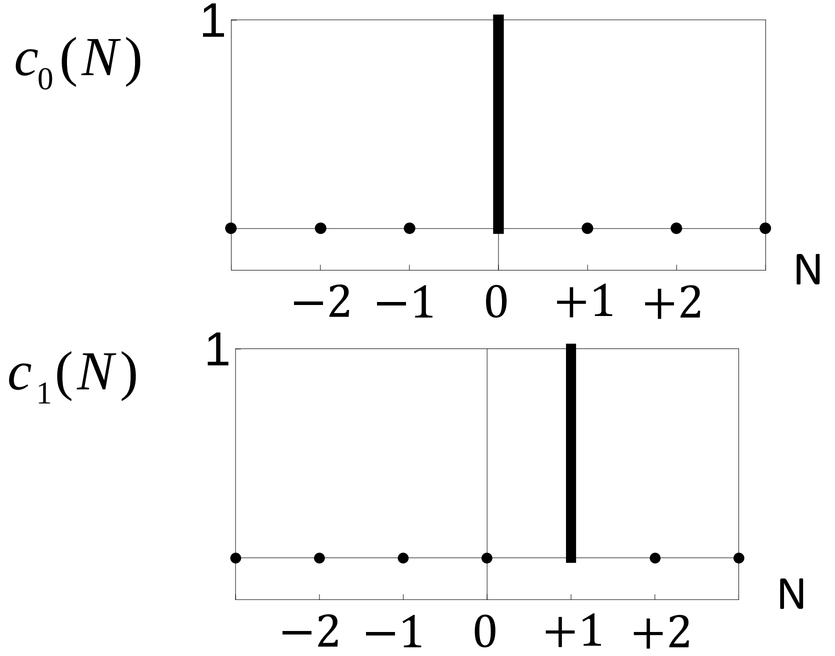
N_g=0.01

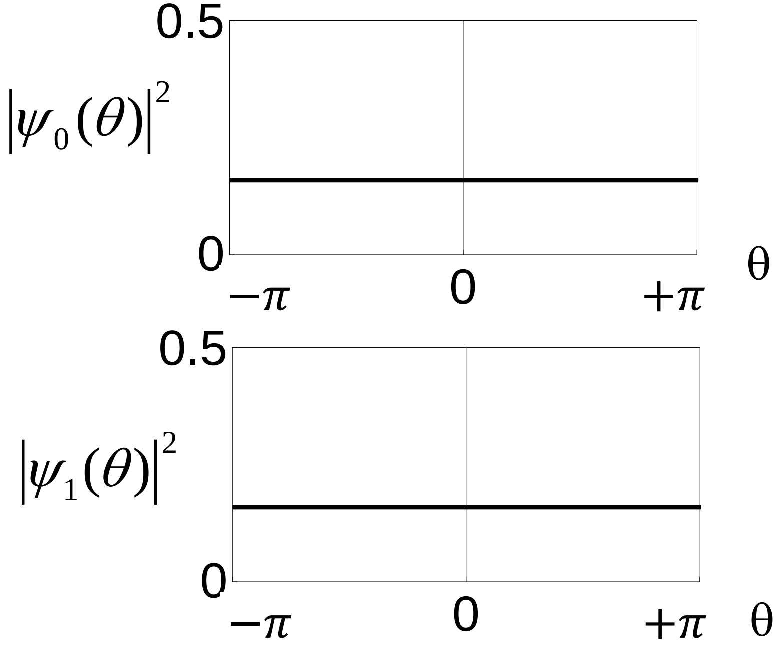




$$\hat{\theta} = \frac{\hat{\theta}_2 - \hat{\theta}_1}{2}$$

$$\hat{\delta} = \hat{\theta}_1 + \hat{\theta}_2$$





Two simple limits : (2)

$$\hat{H}\,(N_g,\Phi) = E_C(\hat{N}-N_g)^2 - E_J(\Phi)\cos\hat{\theta}$$

courtesy CEA Saclay

$E_J(\Phi) \ll E_C$

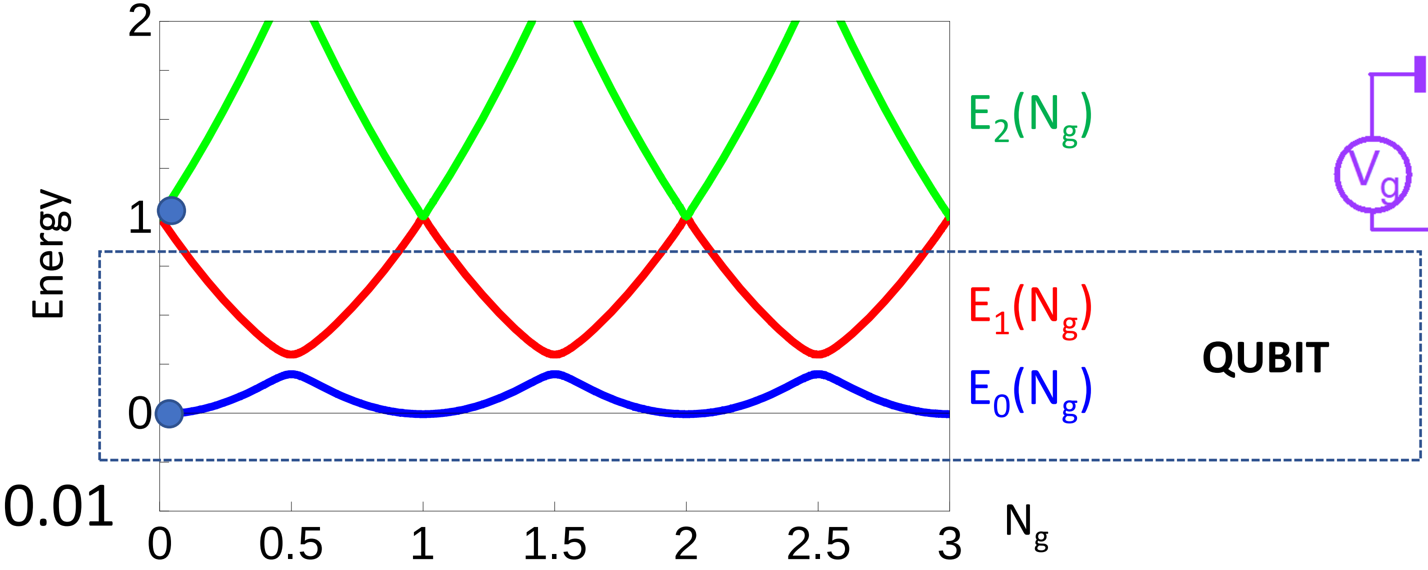
(charge regime) $E_J/E_c=0.1$

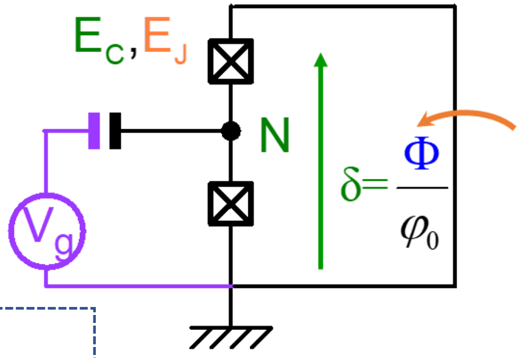
in **charge** basis |N>

$$|\psi\rangle = \sum_N c_N |N\rangle$$

N *integer*

$N_g=0.01$

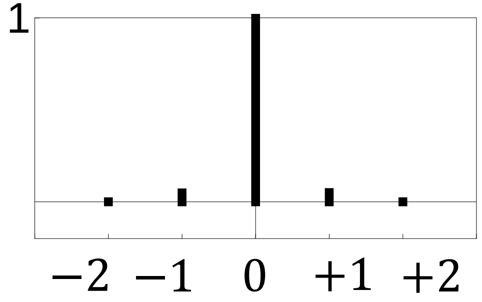




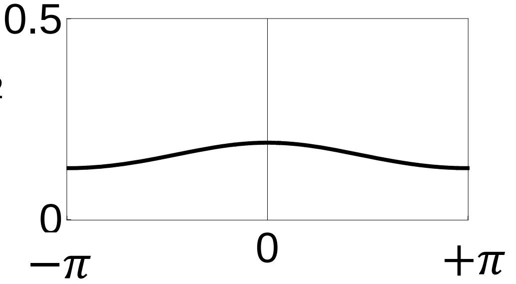
$$\hat{\theta} = \frac{\hat{\theta}_2 - \hat{\theta}_1}{2}$$

$$\hat{\delta} = \hat{\theta}_1 + \hat{\theta}_2$$

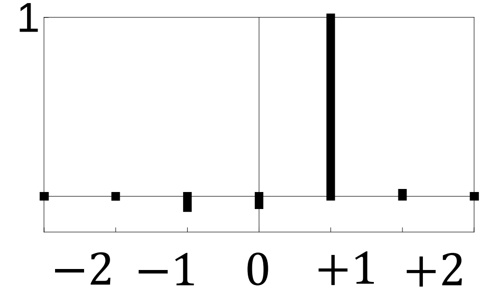
$c_0(N)$



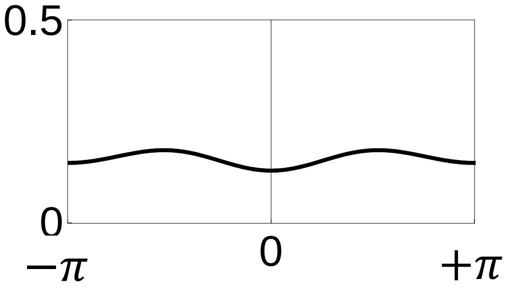
$|\psi_0(\theta)|^2$



$c_1(N)$



$|\psi_1(\theta)|^2$



Two simple limits : (2)

$$\hat{H}\,(N_g,\Phi) = E_C(\hat{N}-N_g)^2 - E_J(\Phi)\cos\hat{\theta}$$

courtesy CEA Saclay

$E_J(\Phi) \ll E_C$

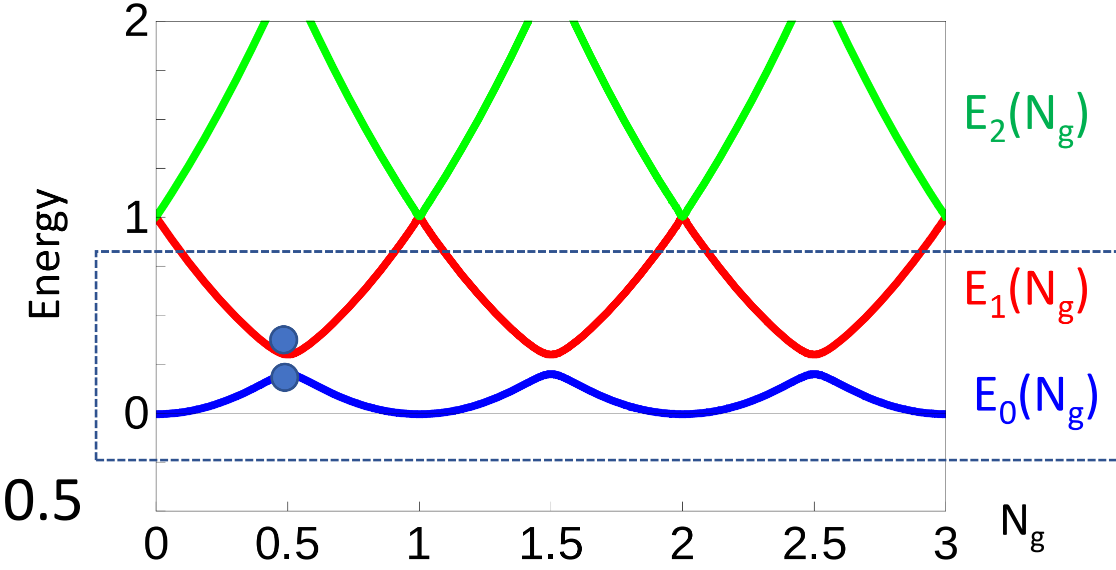
(charge regime) $E_J/E_c=0.1$

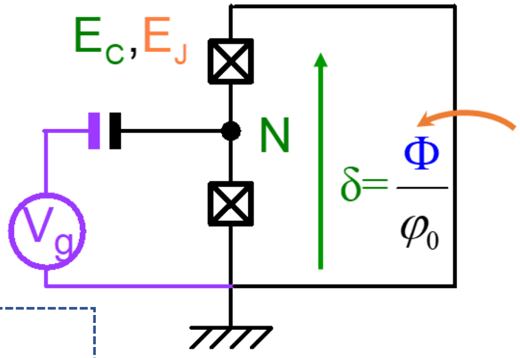
in **charge** basis |N>

$$|\psi\rangle = \sum_N c_N |N\rangle$$

N *integer*

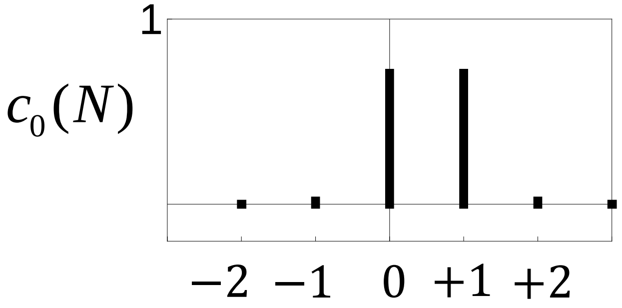
$N_g=0.5$

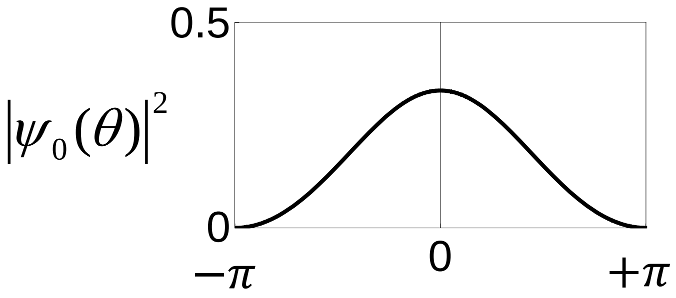


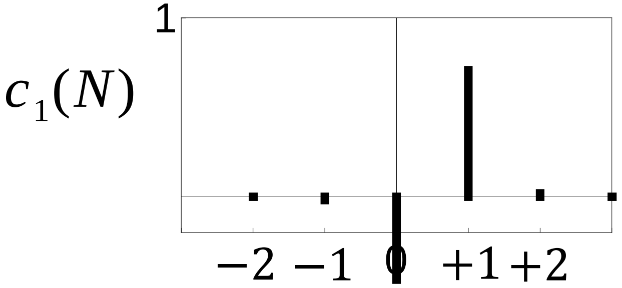


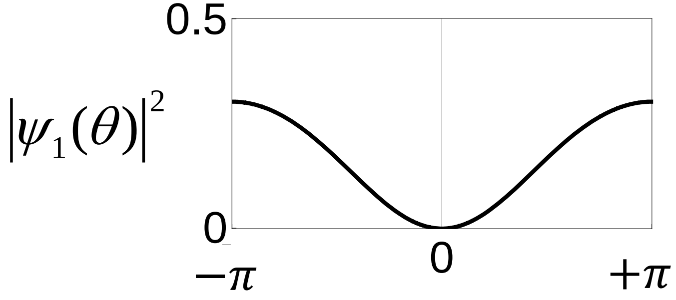
$$\hat{\theta} = \frac{\hat{\theta}_2 - \hat{\theta}_1}{2}$$

$$\hat{\delta} = \hat{\theta}_1 + \hat{\theta}_2$$



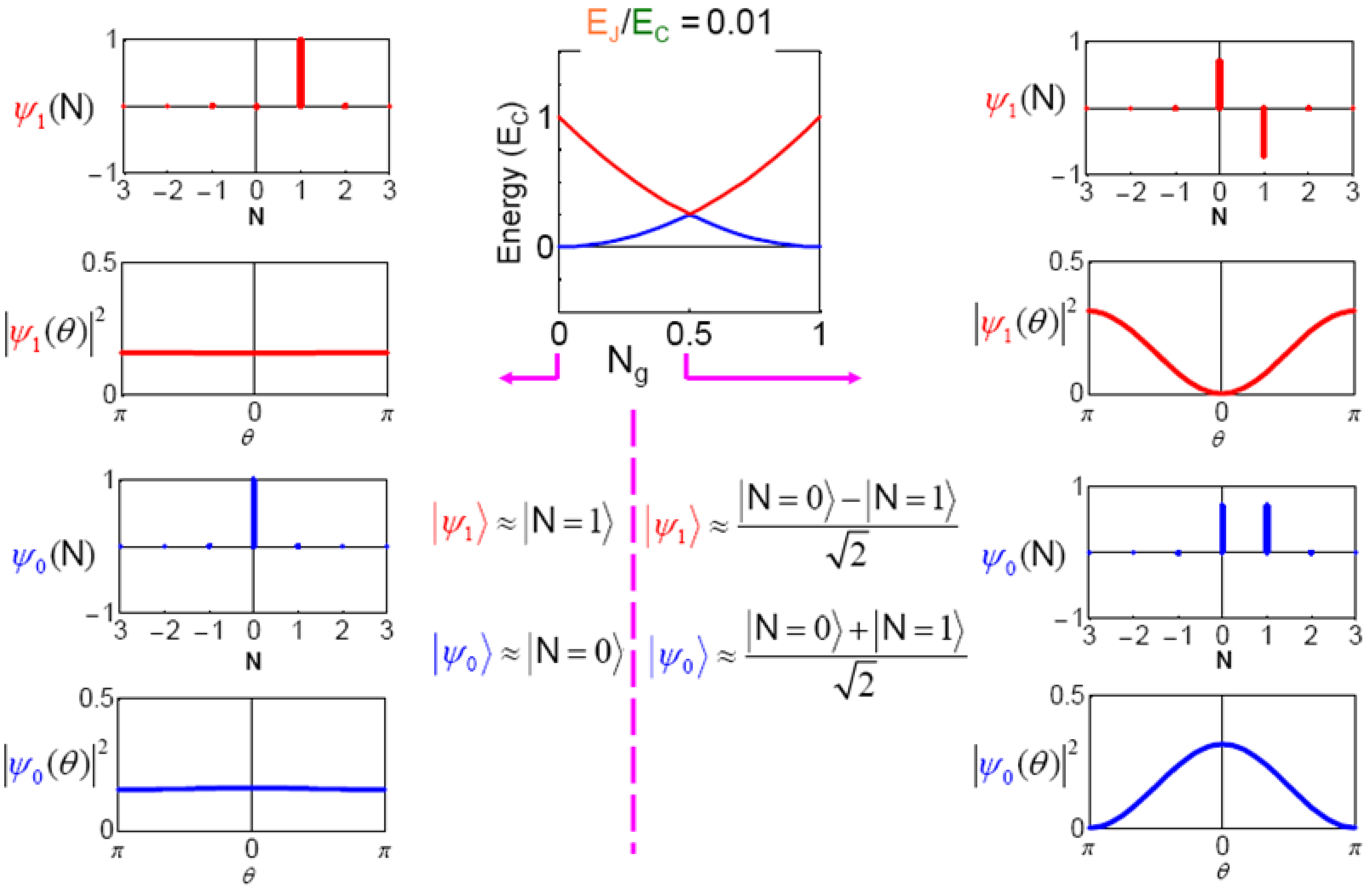






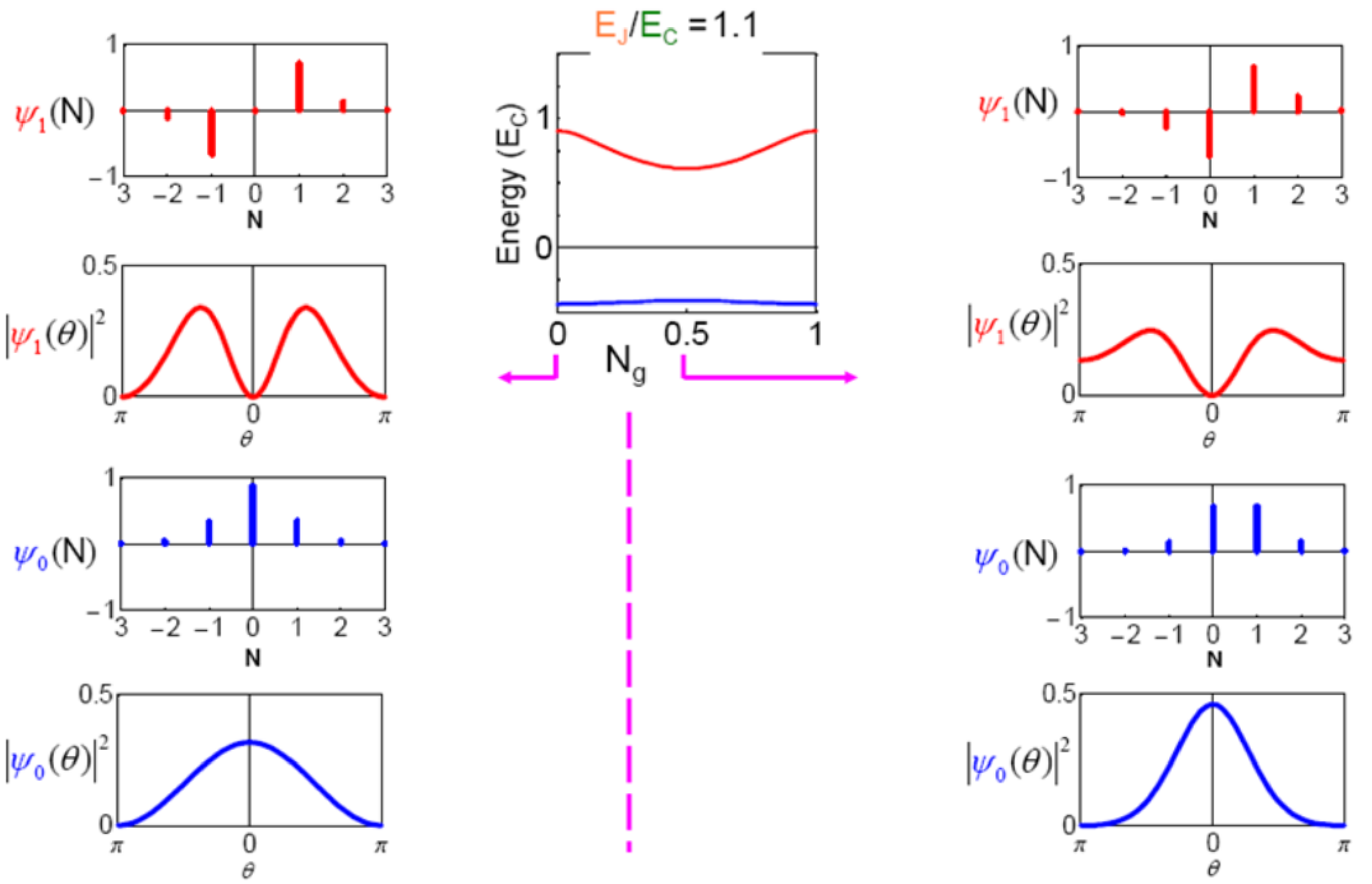
courtesy CEA Saclay

Charge and Phase Wave Functions ($E_J \ll E_C$)



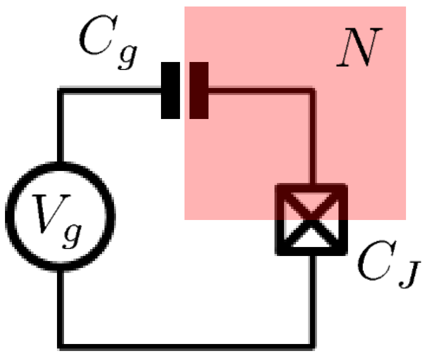
Charge and Phase Wave Functions ($E_J \sim E_C$)

courtesy CEA Saclay



Charge Qubits

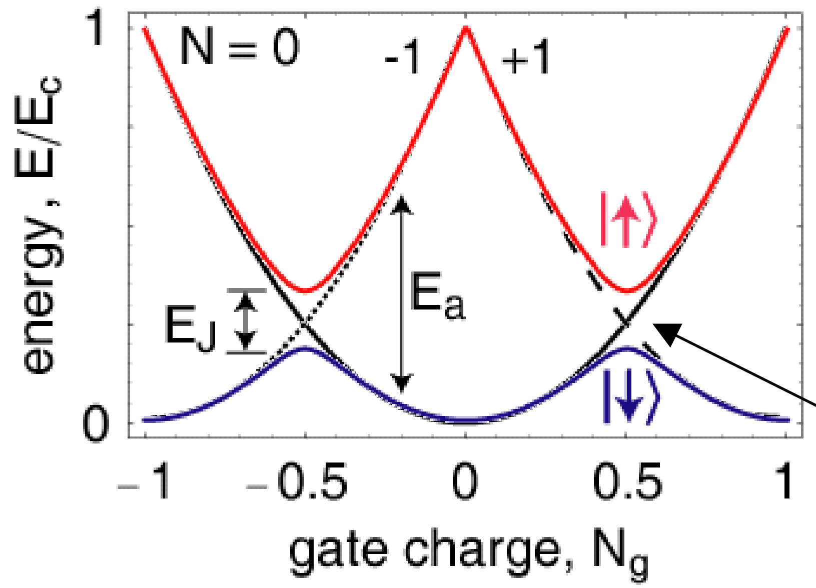
Cooper pair box



charging energy $$E_C = \frac{(2e)^2}{2(C_J + C_g)}$$

gate charge $$N_g = \frac{C_g V_g}{2e}$$

Josepshon energy $$E_J = \frac{I_0 \Phi_0}{2\pi} = \frac{h\Delta}{8e^2 R_J}$$



$$H = E_C\,(N - N_g)^2 - E_J \cos\theta$$

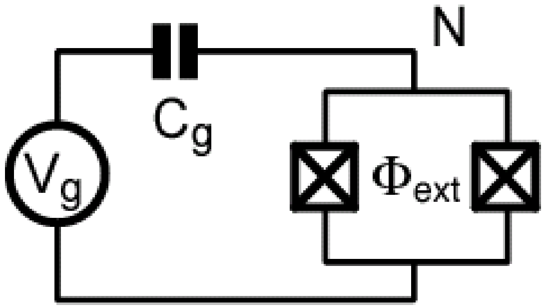
electrostatic energy Josephson energy

charge degeneracy

V. Bouchiat, D. Vion, P. Joyez, D. Esteve, and M. H. Devoret,
Physica Scripta **T76,** 165 (1998).

Tuning the Josephson Energy

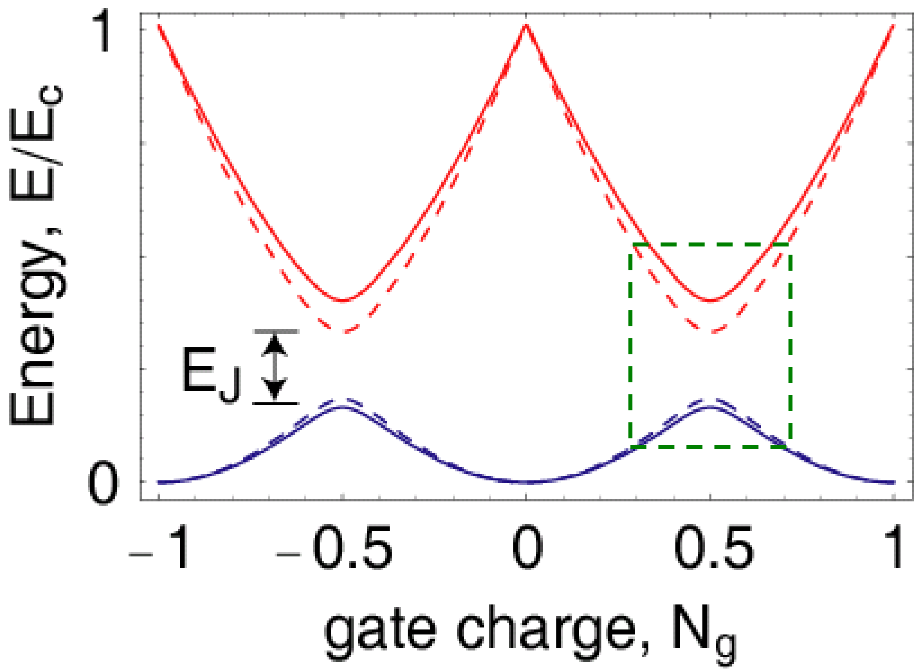
split Cooper pair box in perpendicular field



$$H = E_C\,(N - N_g)^2 - E_{J,\max} \cos\left(\pi \frac{\phi_{\text{ext}}}{\phi_0}\right) \cos\theta$$

SQUID modulation of Josephson energy

$$E_J = E_{J,\max} \cos\left(\pi \frac{\phi_{\text{ext}}}{\phi_0}\right)$$



consider two state approximation

J. Clarke, *Proc. IEEE* **77**, 1208 (1989)

Two-State Approximation

$$\hat{H}_{\mathrm{CPB}} = \hat{H}_{\mathrm{el}} + \hat{H}_{\mathrm{J}} = E_C(\hat{N} - N_g)^2 - E_J \cos\hat{\delta}$$

$$\hat{H}_{\mathrm{CPB}} = \sum_N \left[E_C(N - N_g)^2 \left|N\right\rangle \left\langle N\right| - \frac{E_{\mathrm{J}}}{2} \left(\left|N\right\rangle \left\langle N+1\right| + \left|N+1\right\rangle \left\langle N\right| \right) \right]$$

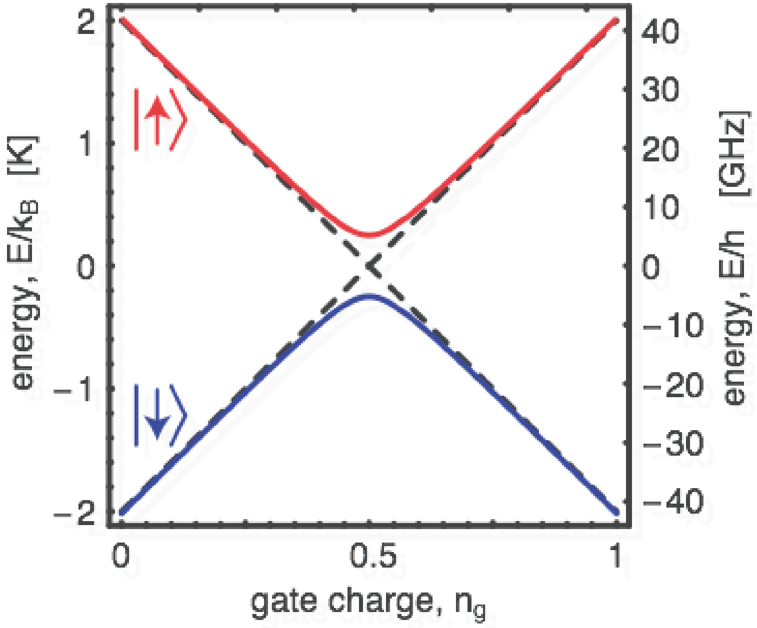
Restricting to a two-charge Hilbert space:

$$\hat{H}_{\mathrm{JC_G N_{basis}}} = E_C N_G^2 \left|0\right\rangle \left\langle 0\right| + E_C \left(1 - N_G\right)^2 \left|1\right\rangle \left\langle 1\right| - \frac{E_{\mathrm{J}_0}}{2} \left(\left|0\right\rangle \left\langle 1\right| + \left|1\right\rangle \left\langle 0\right| + \left|1\right\rangle \left\langle 0\right| + \left|0\right\rangle \left\langle 1\right| \right)$$

$$\hat{N} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1 - \hat{\sigma}_z}{2}$$

$$\cos\hat{\delta} = \frac{\hat{\sigma}_x}{2}$$

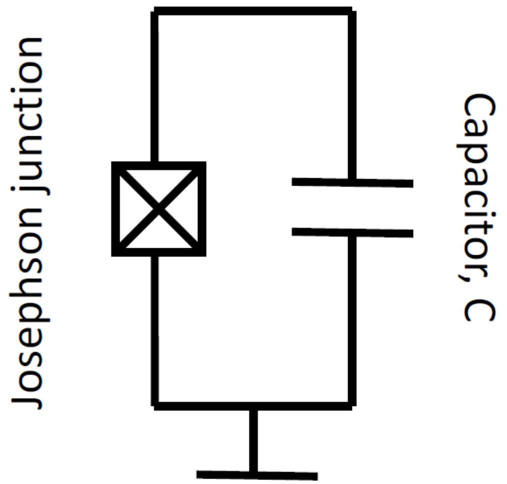
$$\begin{aligned} \hat{H} &= -\frac{E_C}{2}(1 - 2N_g)\hat{\sigma}_z - \frac{E_J}{2}\hat{\sigma}_x \\ &= -\frac{1}{2}(E_{\mathrm{el}}\hat{\sigma}_z + E_J\hat{\sigma_x}) \end{aligned}$$



Shnirman *et al.*, *Phys. Rev. Lett.* **79**, 2371 (1997)

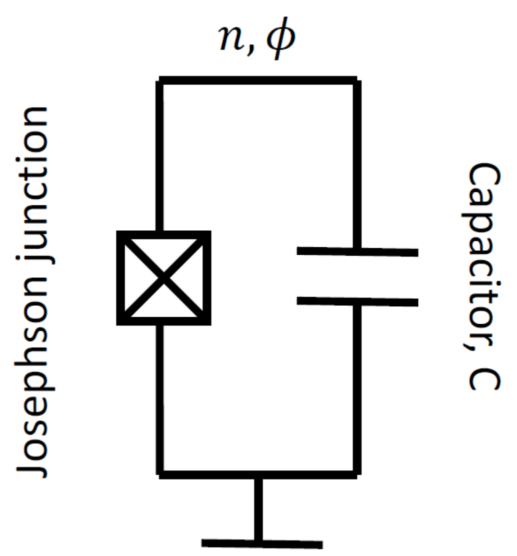
Cooper-Pair Box Brief Summary

imagine the following circuit



Cooper-Pair Box Brief Summary

imagine the following circuit



Capacitive energy: $W_c = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}\frac{(2e)^2 n^2}{C} = E_C n^2$

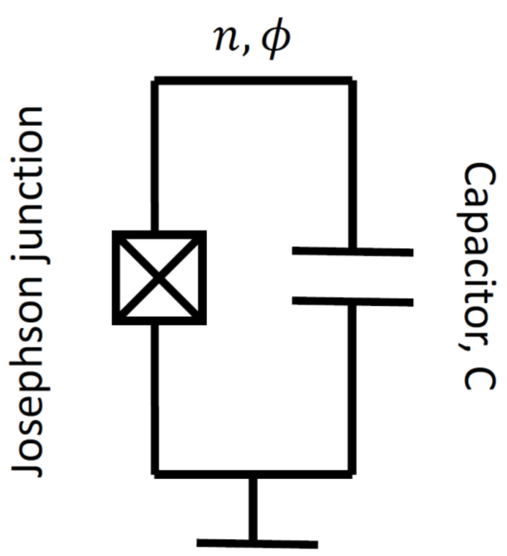
charging energy

number of Cooper pairs

Cooper-Pair Box

Brief Summary

imagine the following circuit



Capacitive energy: $W_c = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}\frac{(2e)^2 n^2}{C} = E_C n^2$

charging energy

number of Cooper pairs

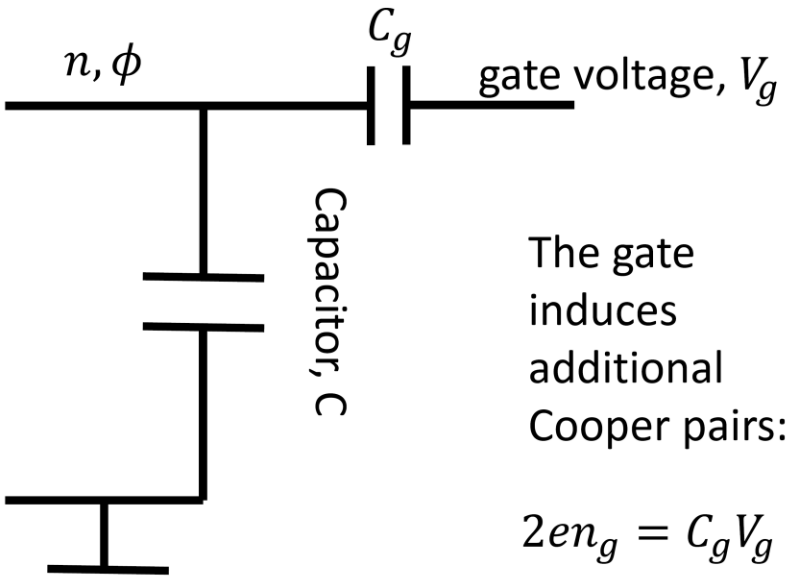
Inductive energy: $W_I = \int_0^I L(I) I dI = -\frac{\Phi_0}{2\pi} I_C \cos\phi = -E_J \cos\phi$

$$\boxed{E_J = \frac{\Phi_0}{2\pi} I_C}$$

Josephson energy

Cooper-Pair Box   Brief Summary

imagine the following circuit



The gate induces additional Cooper pairs:

$$2en_g = C_g V_g$$

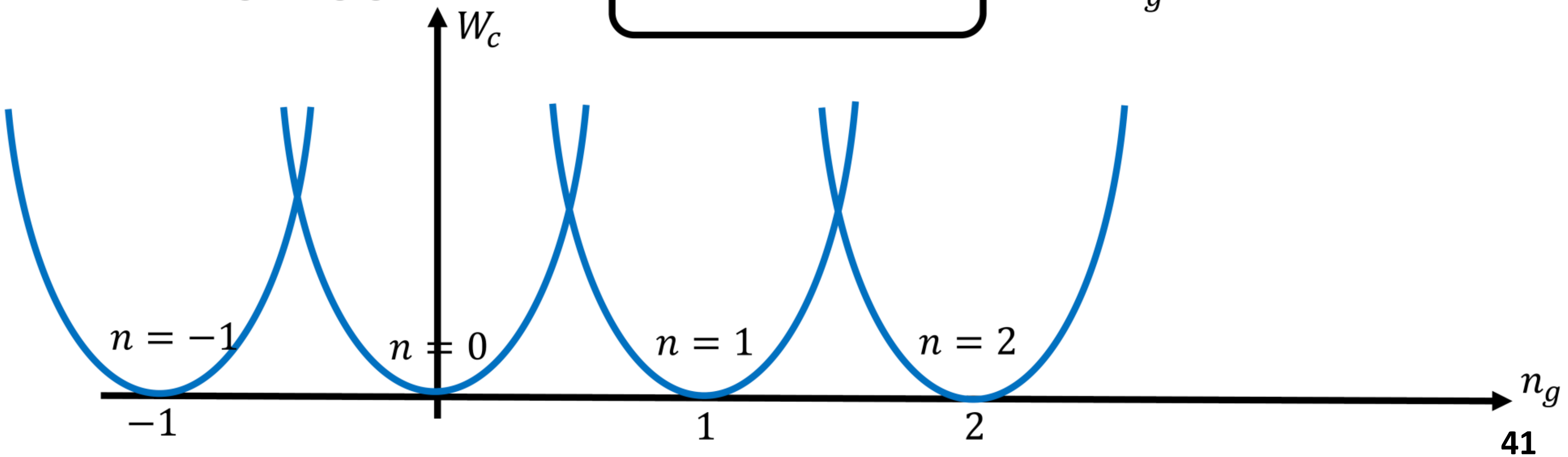
$$W_c = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}\frac{(2e)^2 n^2}{C} = E_C n^2$$

charging energy

number of Cooper pairs

$$\boxed{W_c = E_C(n - n_g)^2}$$

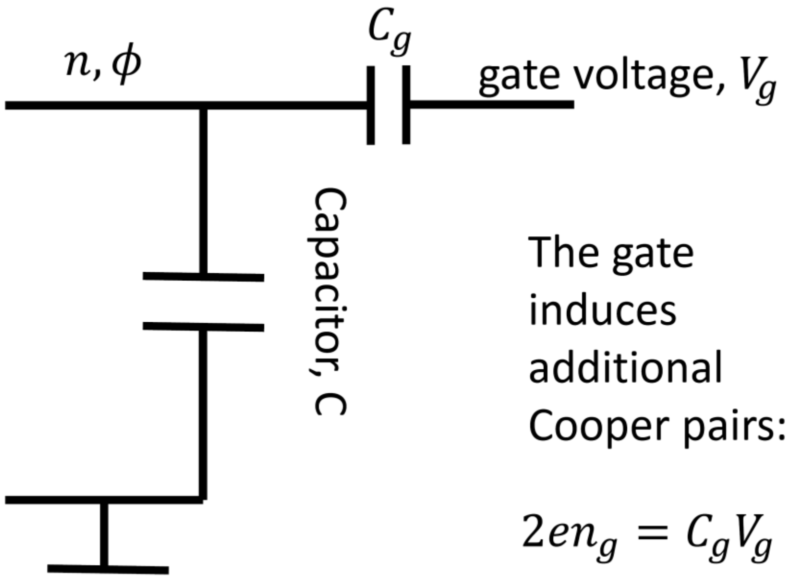
n is quantized
n_g is continuous



Cooper-Pair Box

Brief Summary

imagine the following circuit



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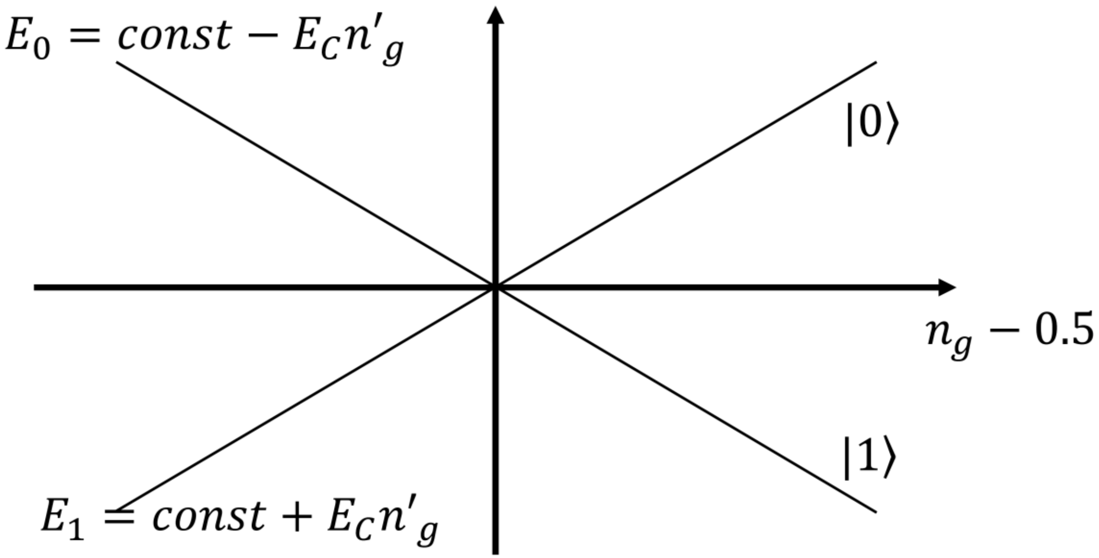
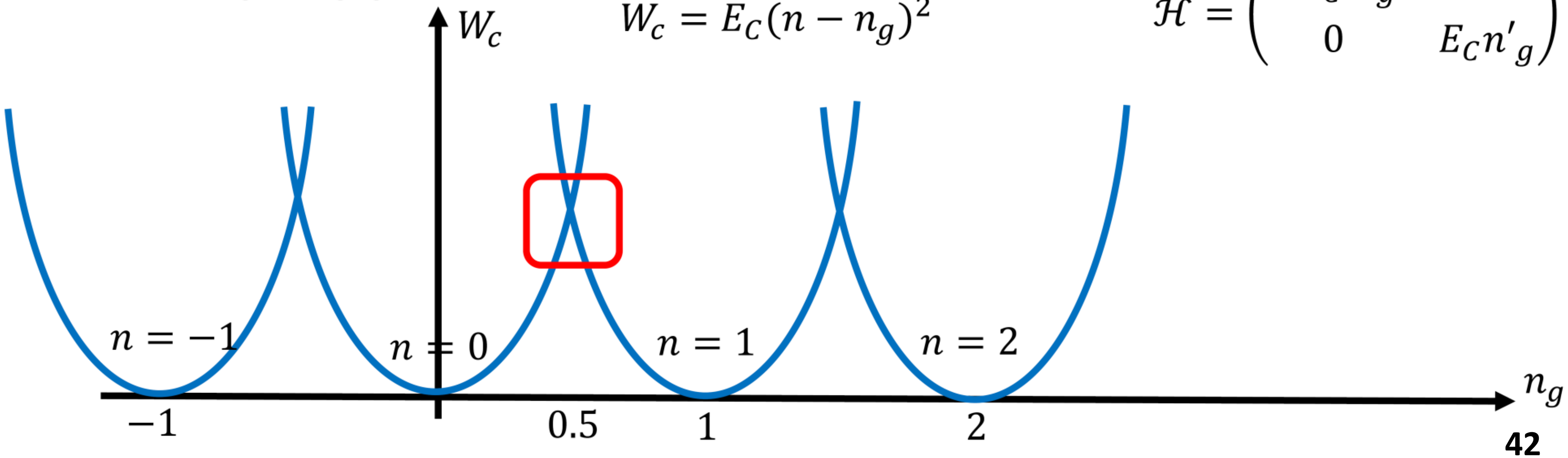


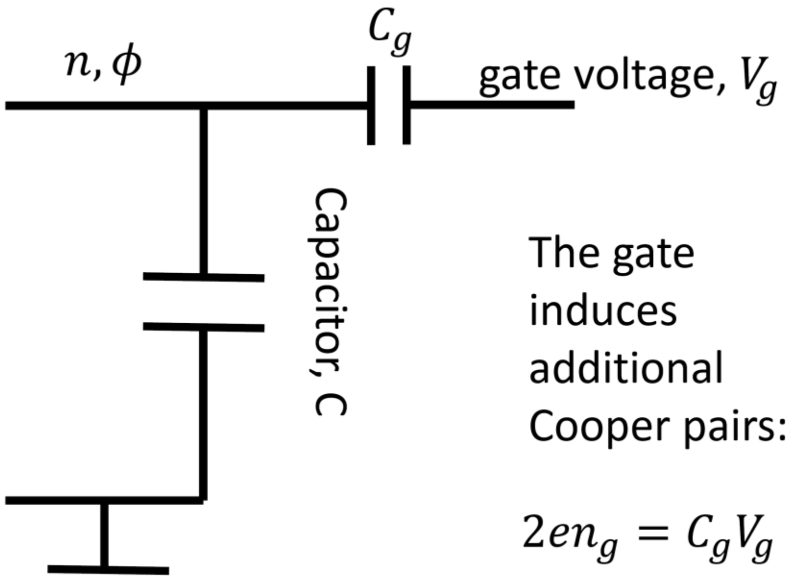
$$\mathcal{H} = \begin{pmatrix} -E_C n'_g & 0 \\ 0 & E_C n'_g \end{pmatrix}$$

$$W_c = E_C(n - n_g)^2$$



Cooper-Pair Box Brief Summary

imagine the following circuit



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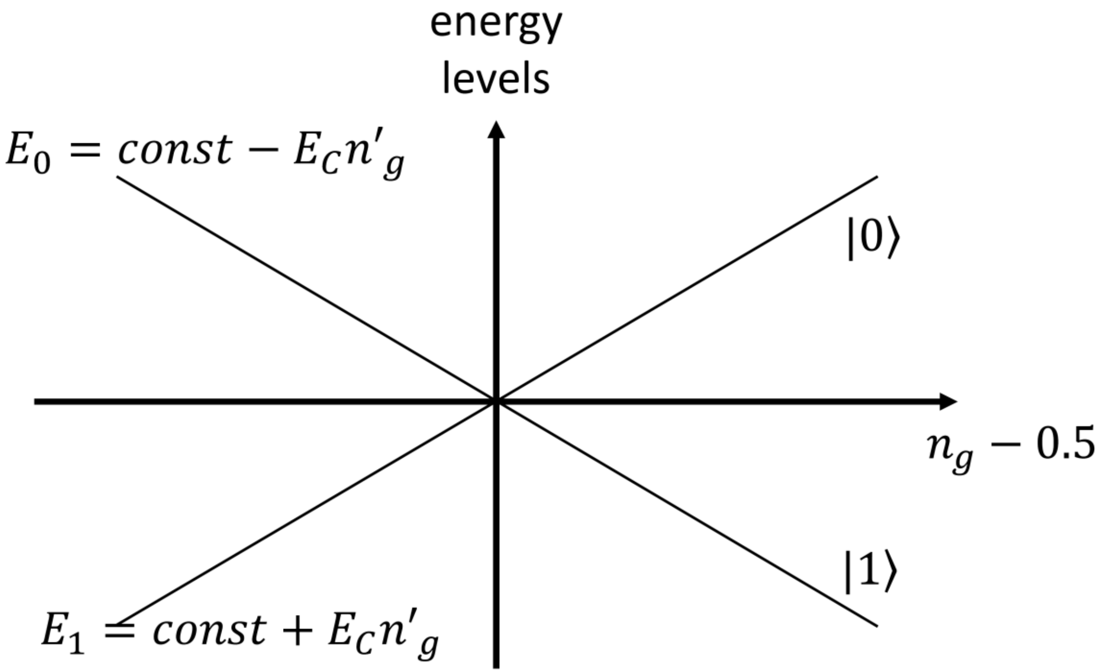
$$2en_g = C_g V_g$$

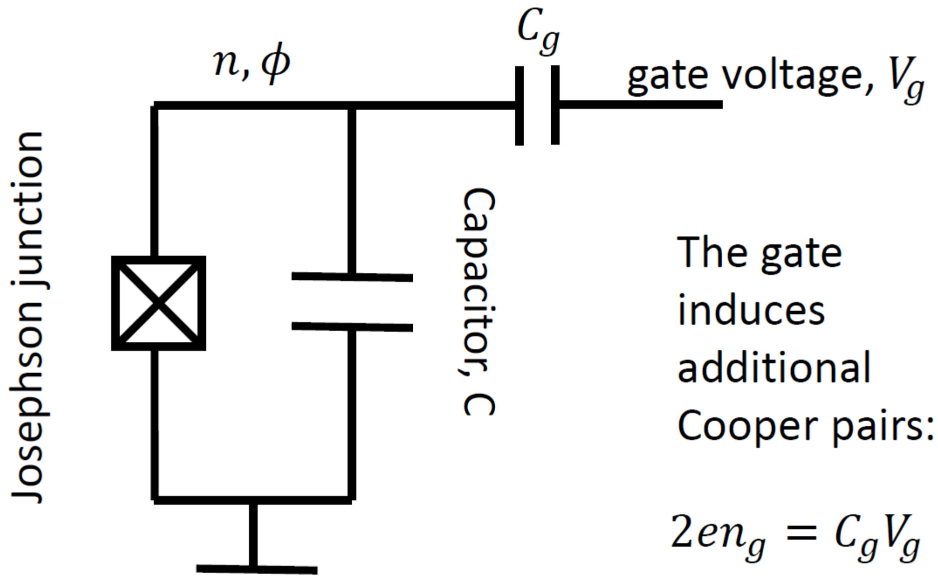


$$\mathcal{H} = \begin{pmatrix} -E_C n'_g & 0 \\ 0 & E_C n'_g \end{pmatrix}$$

What will be the off-diagonal coupling term?

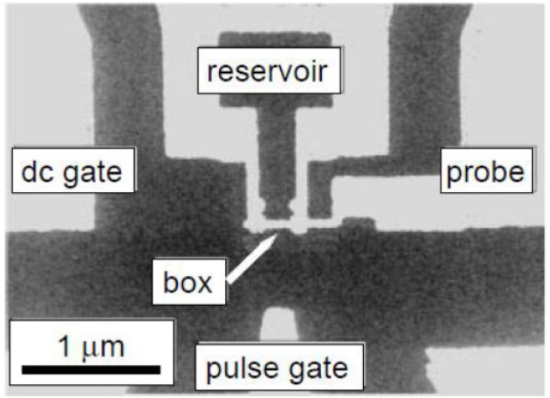
Cooper-Pair Box Brief Summary

imagine the following circuit

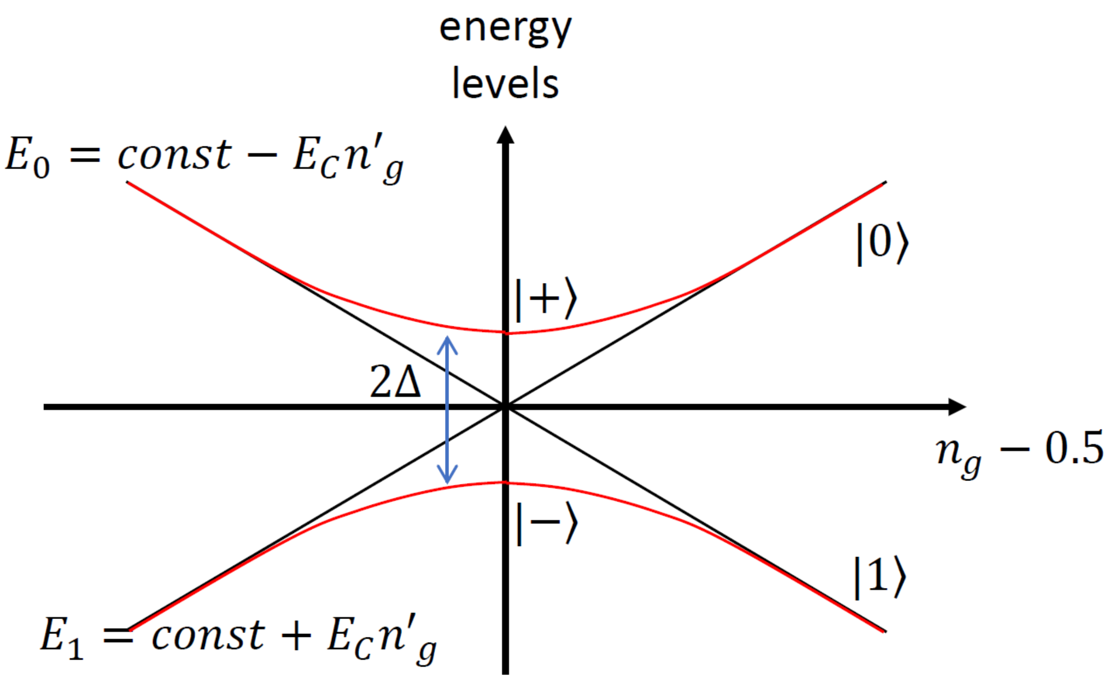


The gate induces additional Cooper pairs:

$$2en_g = C_g V_g$$



Nakamura et al, *Nature* 1999

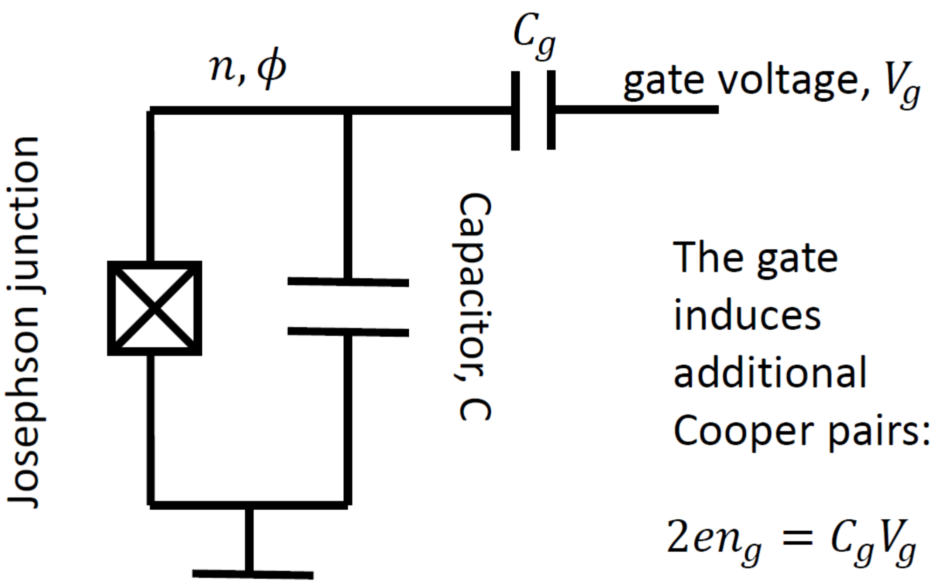


$$\mathcal{H} = \begin{pmatrix} -E_C n'_g & E_J/2 \\ E_J/2 & E_C n'_g \end{pmatrix}$$

Josephson coupling enables the hopping of Cooper pairs on and off the box

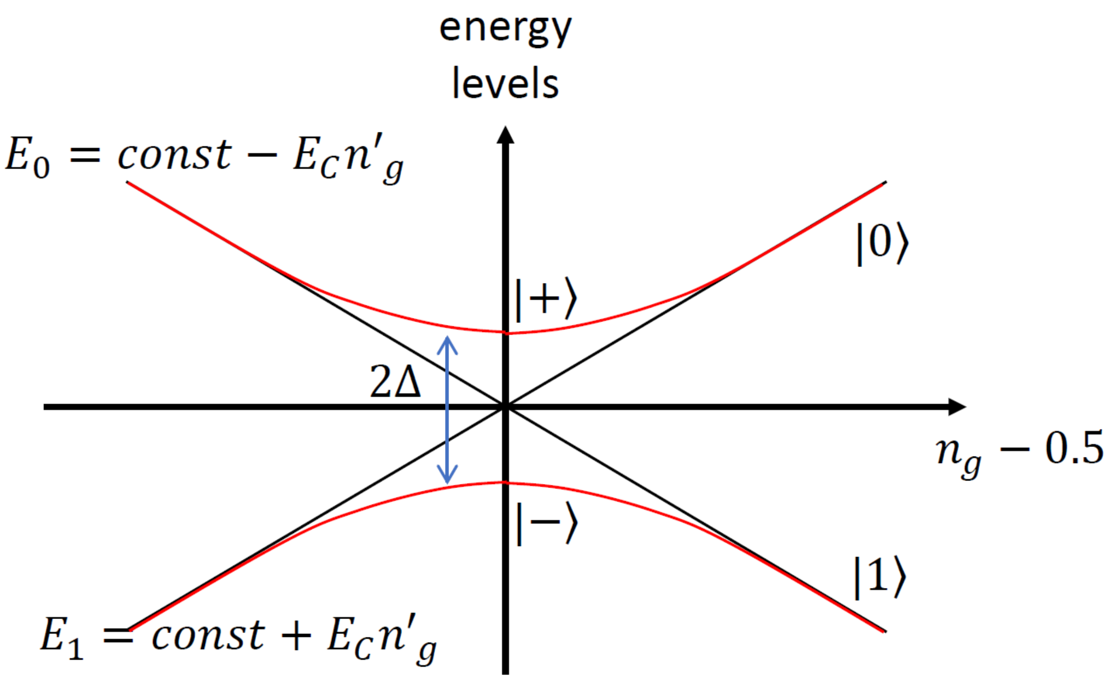
Cooper-Pair Box   Brief Summary

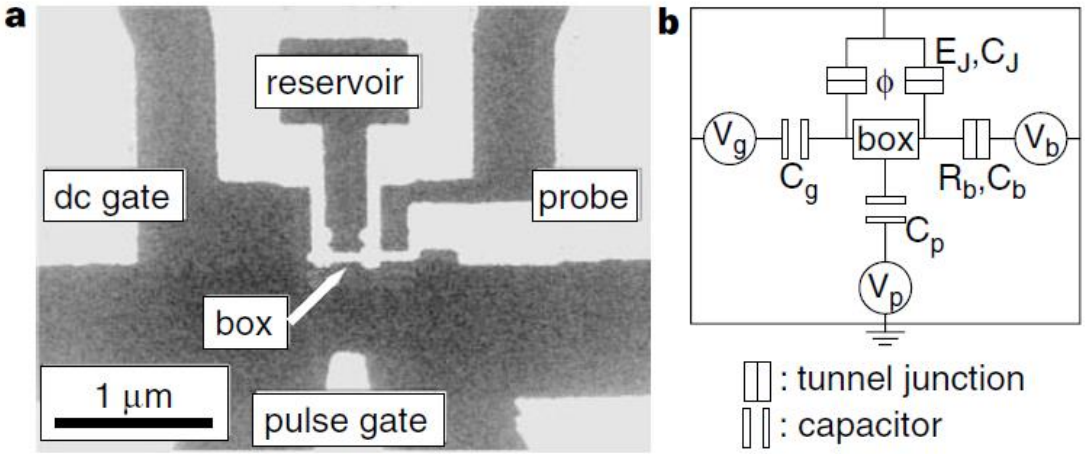
imagine the following circuit



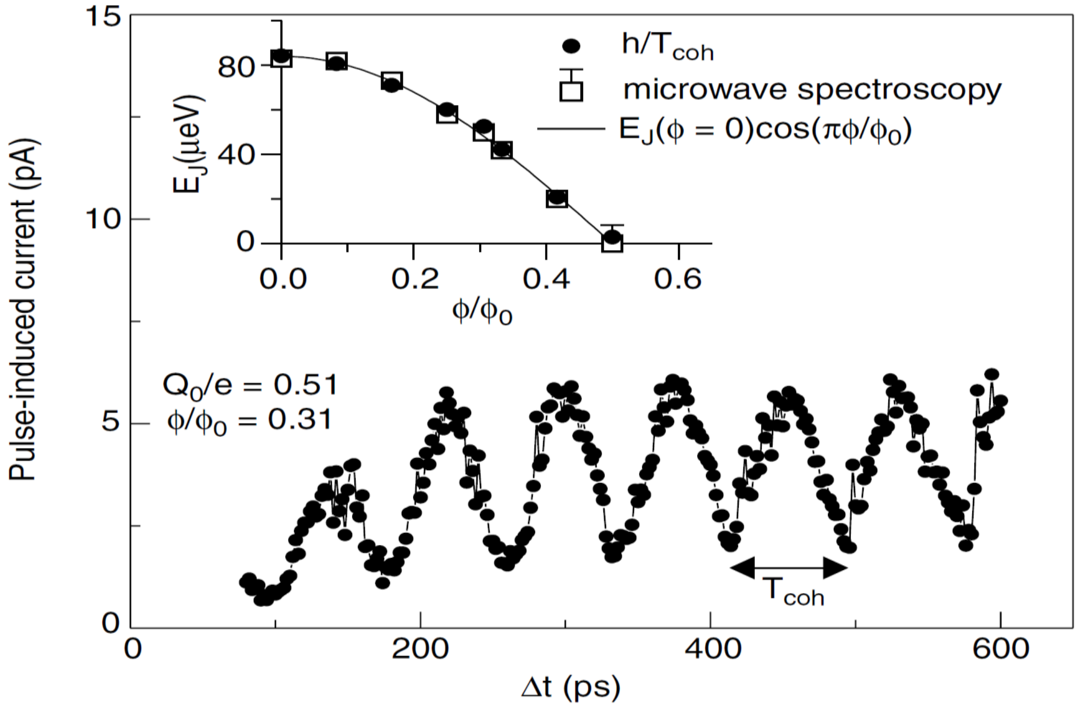
The gate induces additional Cooper pairs:

$$2en_g = C_g V_g$$





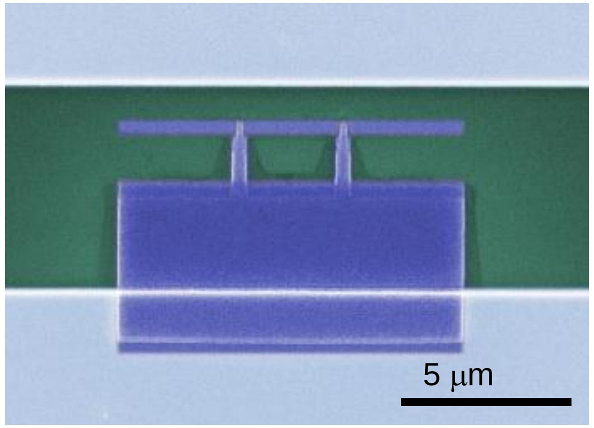
Nakamura et al, *Nature* 1999



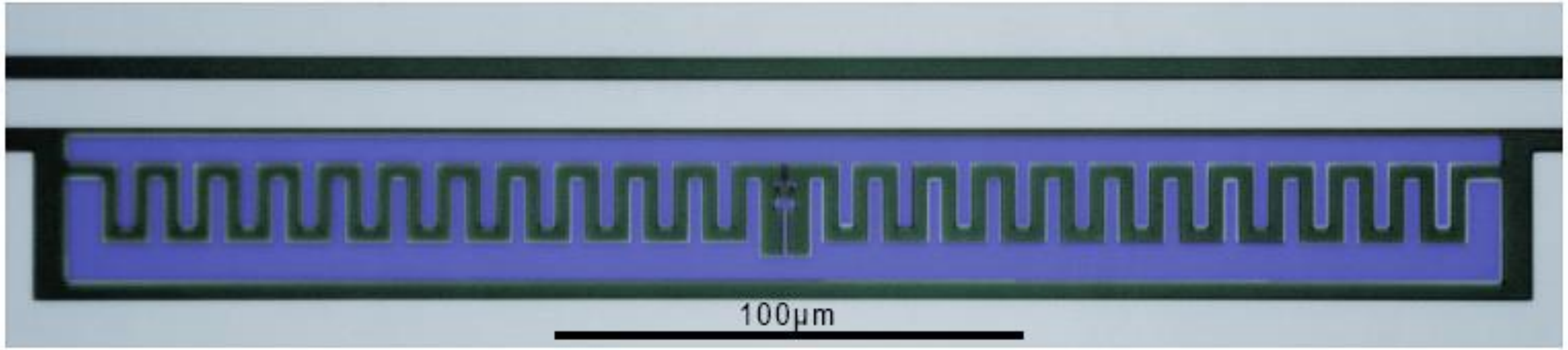
IV. Transmon Limit

A Variant of the Cooper Pair Box with Small Charging Energy

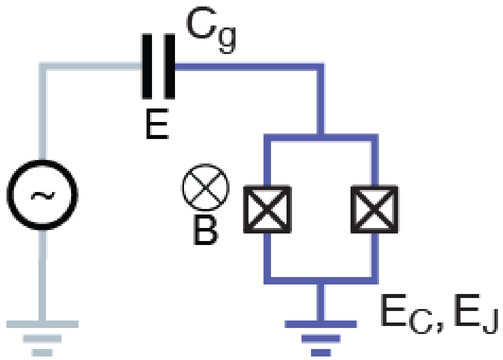
standard CPB:



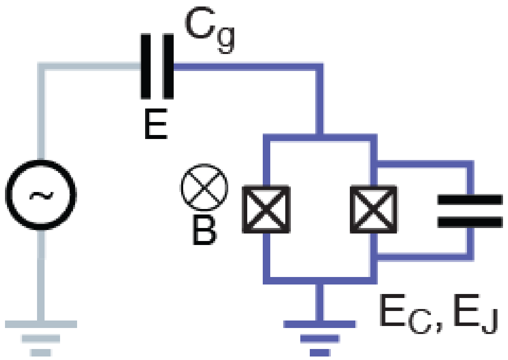
Transmon qubit:



circuit diagram:

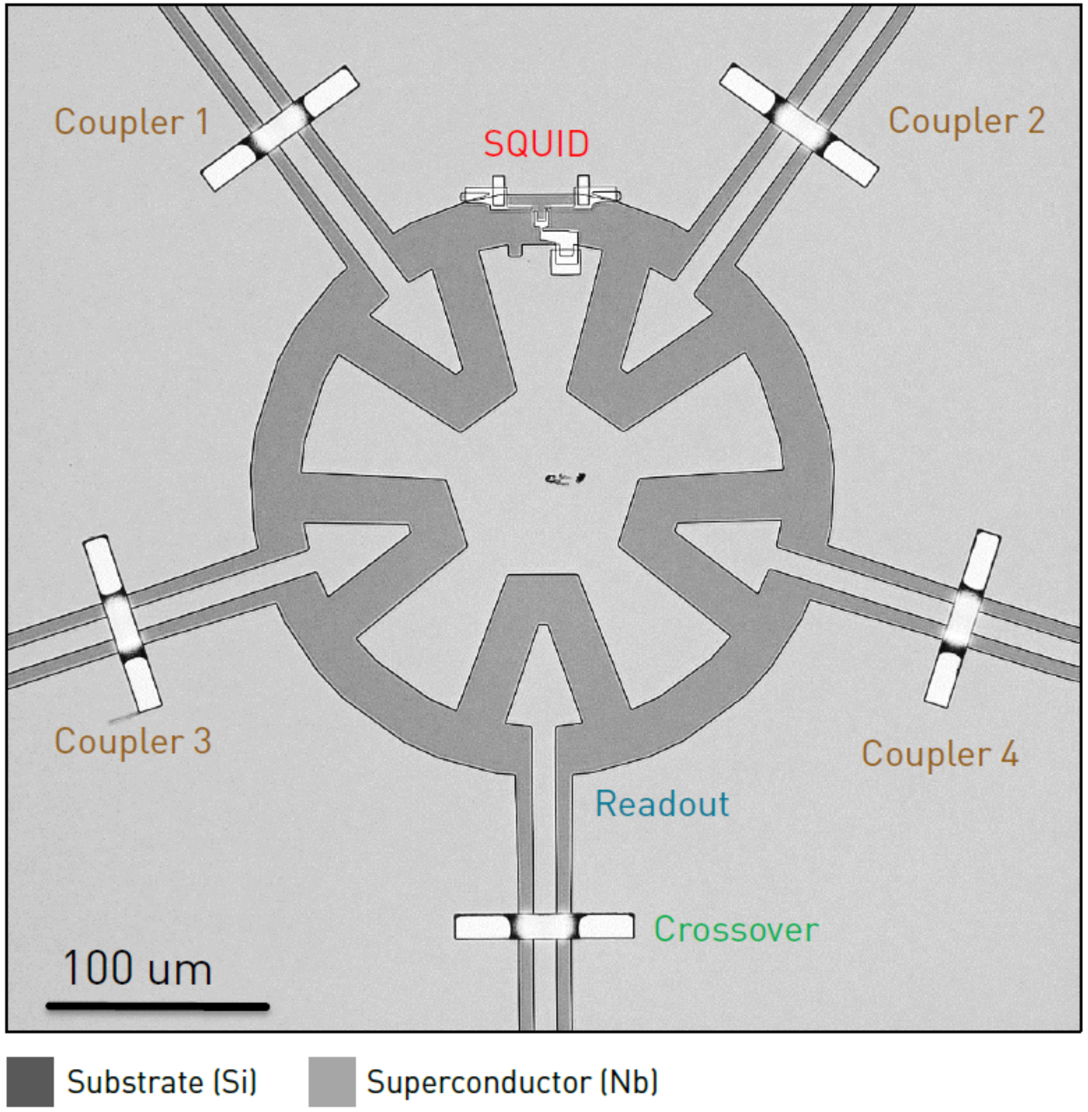


- Large shunt capacitance
- Large physical size
- Large dipole coupling to microwave resonator
- Split Jospehson junction (SQUID) for magnetic flux tuning



J. Koch *et al.*, *Phys. Rev. A* **76**, 042319 (2007)
J. Schreier *et al.*, *Phys. Rev. B* **77**, 180502 (2008)

Design and fabrication of Transmon



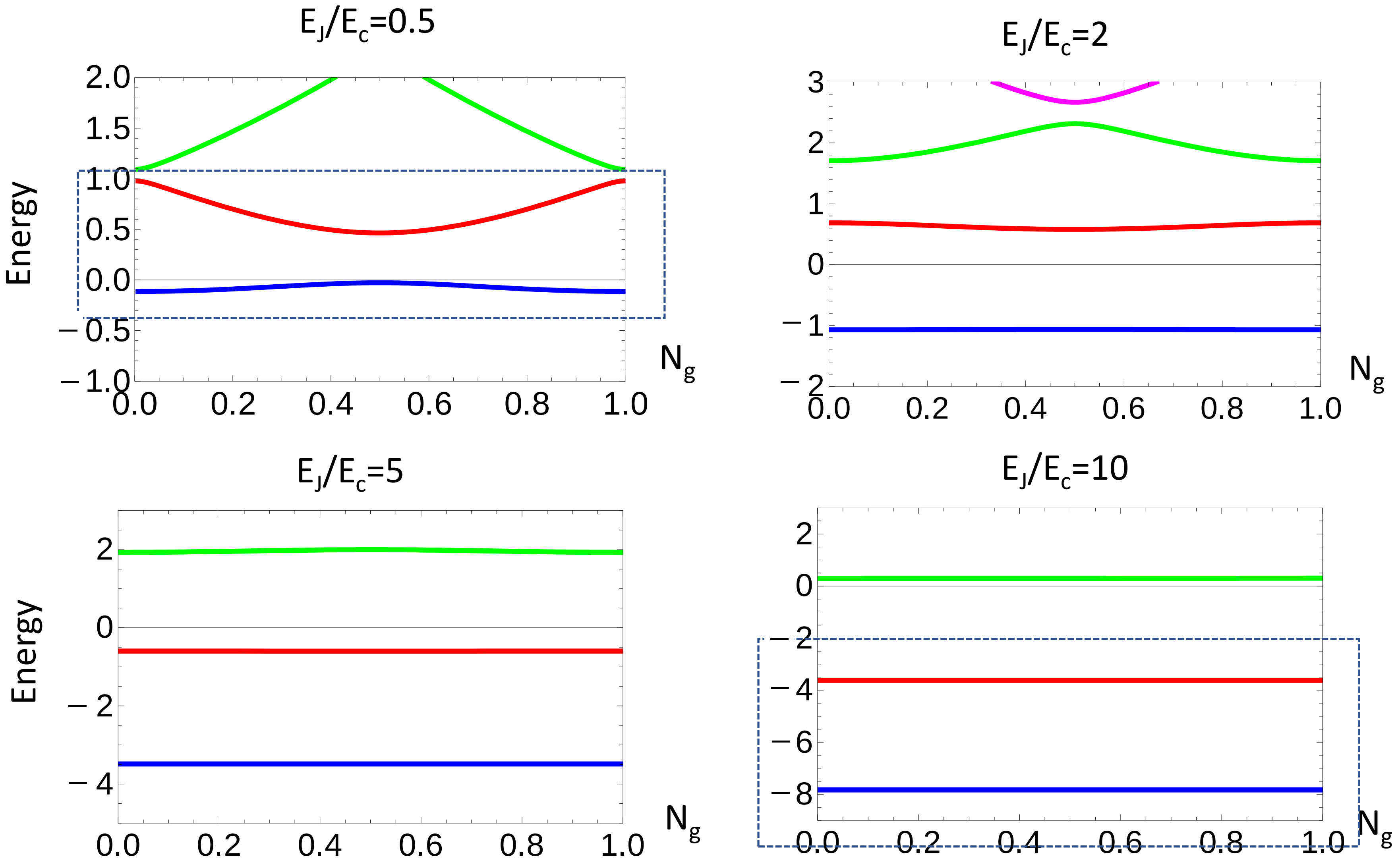
Properties

- SQUID connects transmon island to ground plane.
- Total capacitance between islands and all other elements C_Σ sets set charging energy $E_C = {e^2}/{2C_\Sigma}$. Typical values $\frac{E_C}{\hbar} \approx 300$ MHz.
- Capacitance to coupling elements used to mediate coupling to neighboring qubits, control lines, and the readout circuit.
- Crossovers establish connection between different parts of the ground plane.

Quantum Device Lab , ETHZ (2019)

From $E_J(\Phi) \ll E_C$ to $E_J(\Phi) > E_C$

courtesy CEA Saclay

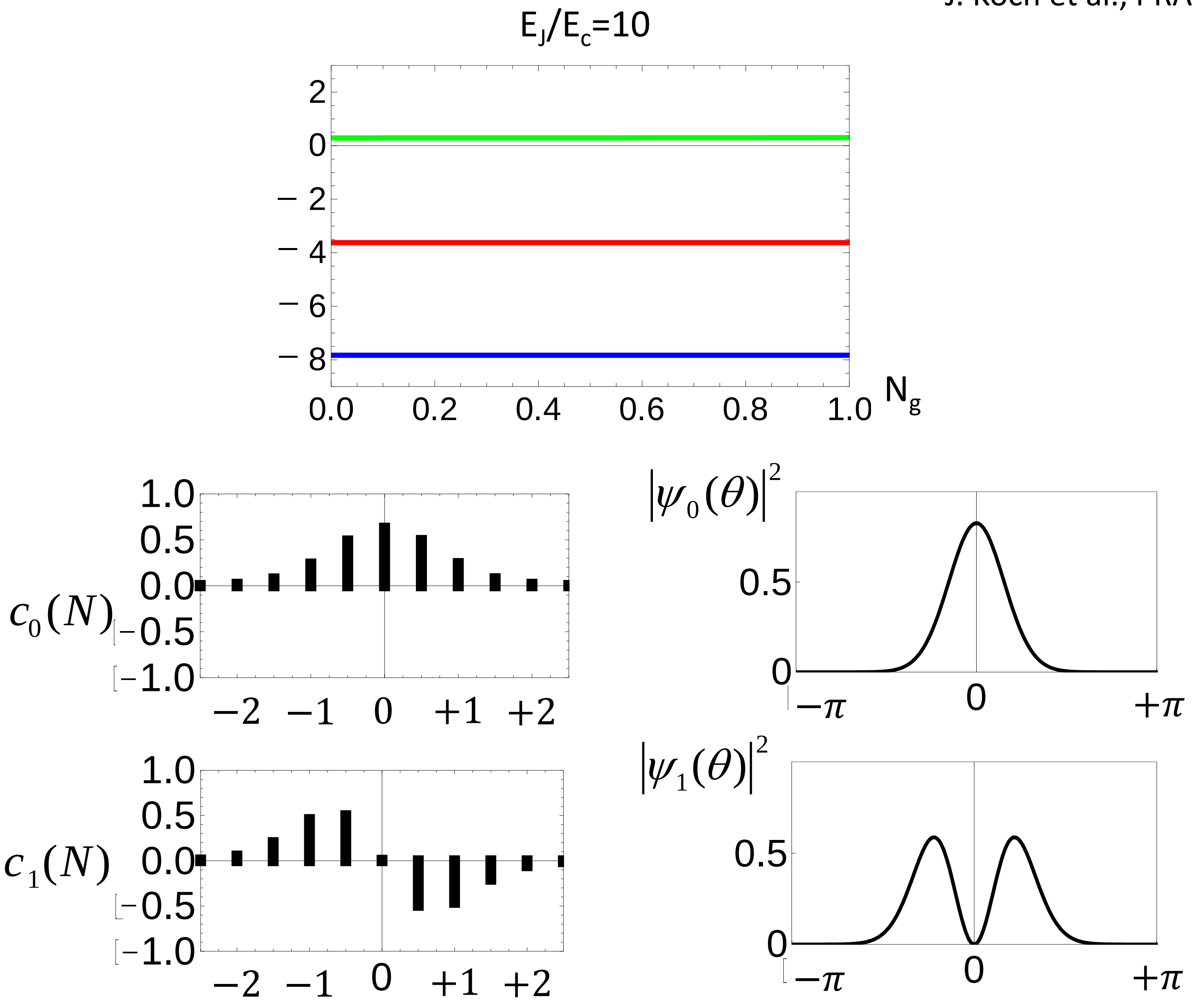


STILL A QUBIT !

Two simple limits : (2) $E_J(\Phi) > E_C$ (phase regime)

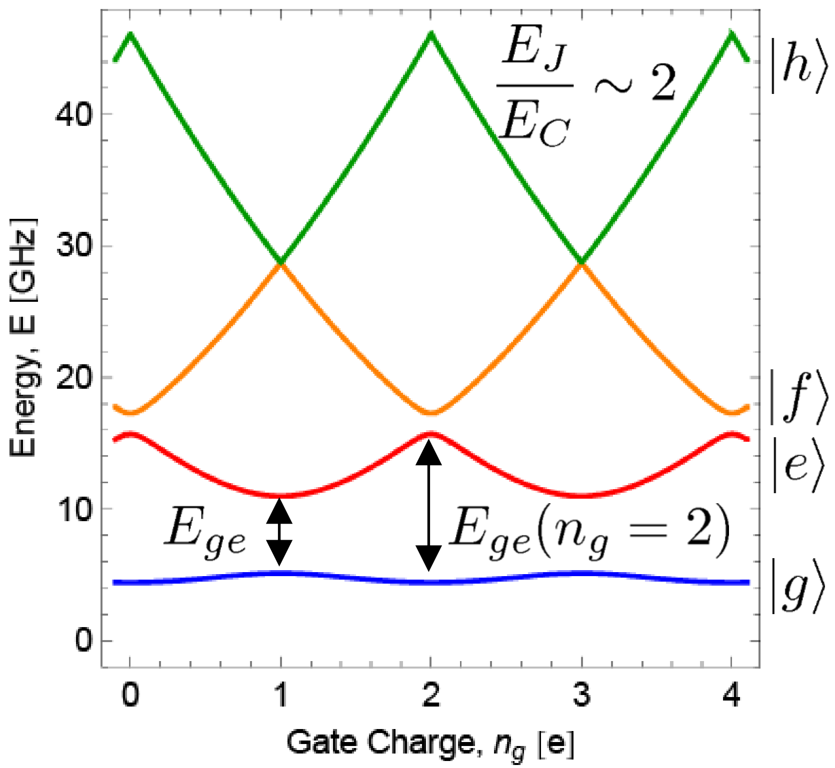
courtesy CEA Saclay

J. Koch et al., PRA (2008)



The Transmon: A Charge Noise Insensitive Qubit

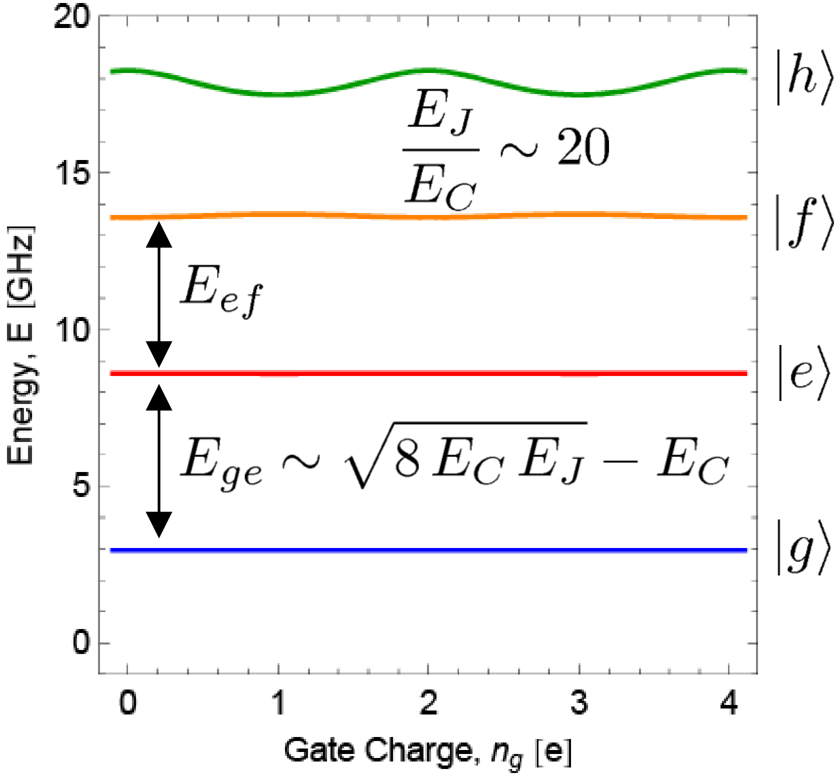
Cooper pair box energy levels:



dispersion:

$$\epsilon = E_{ge}(n_g = 1) - E_{ge}(n_g = 2)$$

Transmon energy levels:



relative anharmonicity:

$$\alpha_r = \frac{E_{ef} - E_{ge}}{E_{ge}}$$

Reduced sensitivity to gate charge:

- Disables tuning by gate charge
- Reduces sensitivity to charge noise
- Use magnetic flux for tuning instead

Reduced anharmonicity

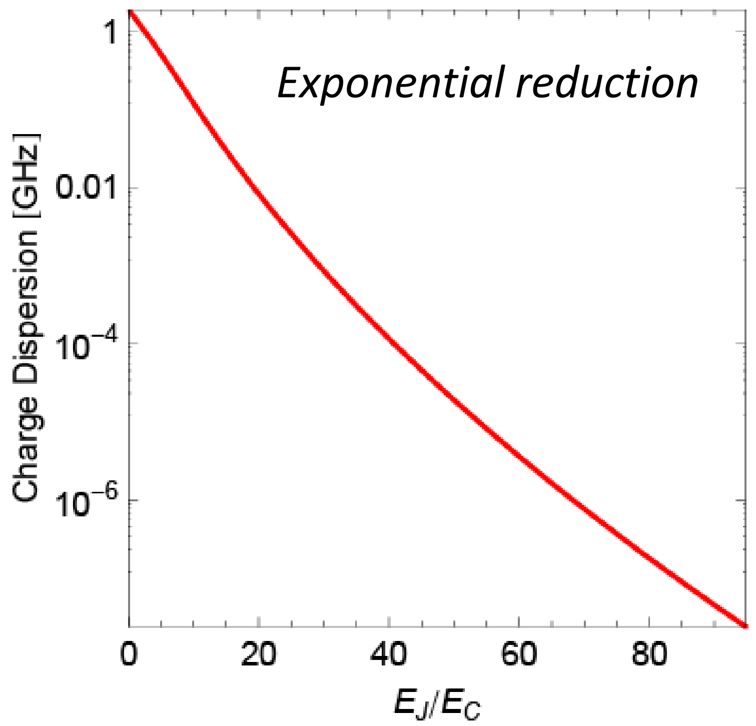
- Creates channel for leakage
- Limits shortest possible single qubit gate times
- Use optimized pulse shapes

J. Koch *et al., Phys. Rev. A* **76**, 042319 (2007)

Dispersion and Anharmonicity of the Transmon Qubit

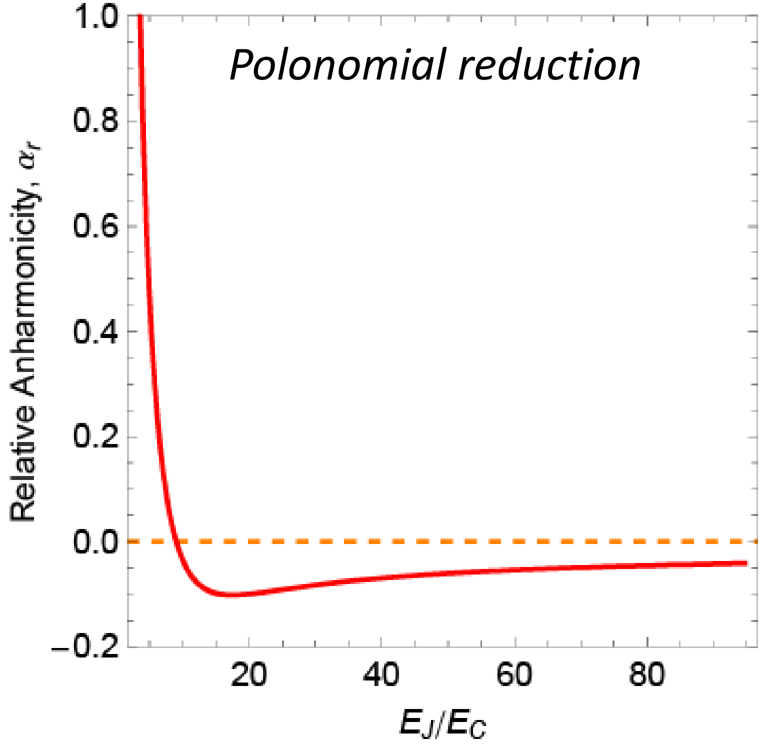
Charge dispersion:

$$\epsilon = E_{ge}(n_g = 1) - E_{ge}(n_g = 2)$$



Anharmonicity:

$$\alpha_r = \frac{E_{ef} - E_{ge}}{E_{ge}}$$

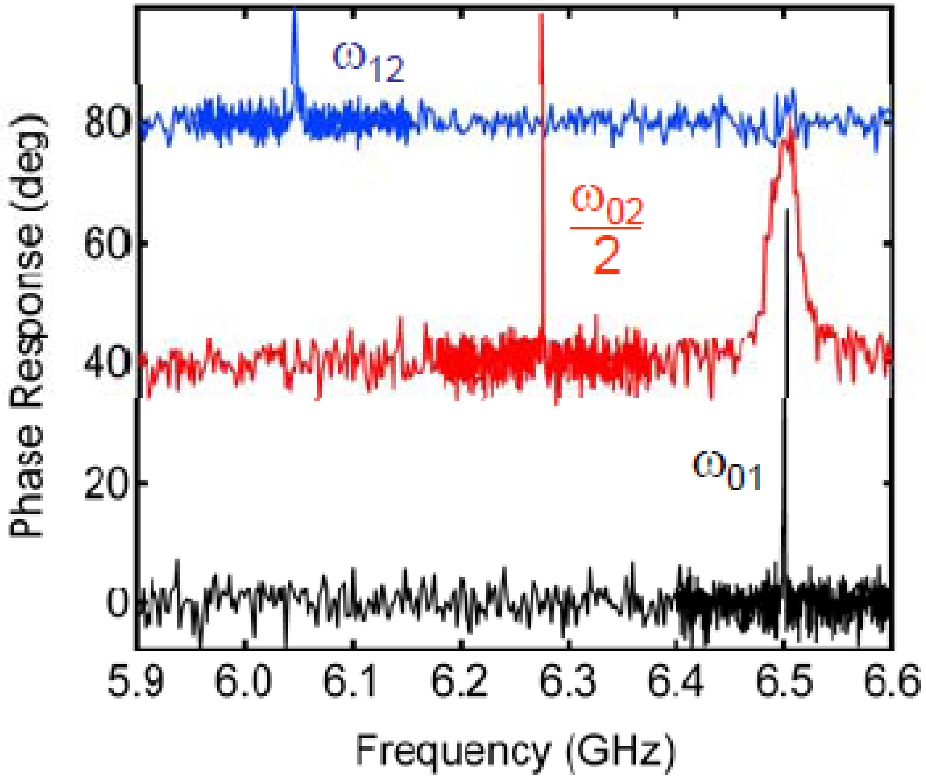


Good coherence ($\sim 100\ \mu s$), simple design -> most popular qubit and the main platform for superconducting quantum processors

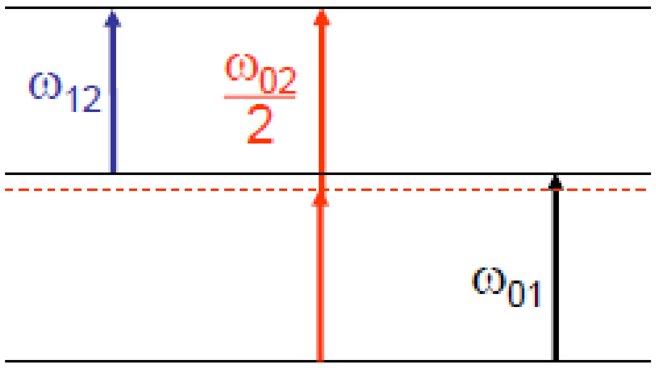
J. Koch *et al., Phys. Rev. A* **76**, 042319 (2007)

Experimental spectrum of a transmon

$$E_{n+1} - E_n = \sqrt{2E_J E_C} - (n+1)E_C/4$$



J. Schreier et al., PRB (2008)

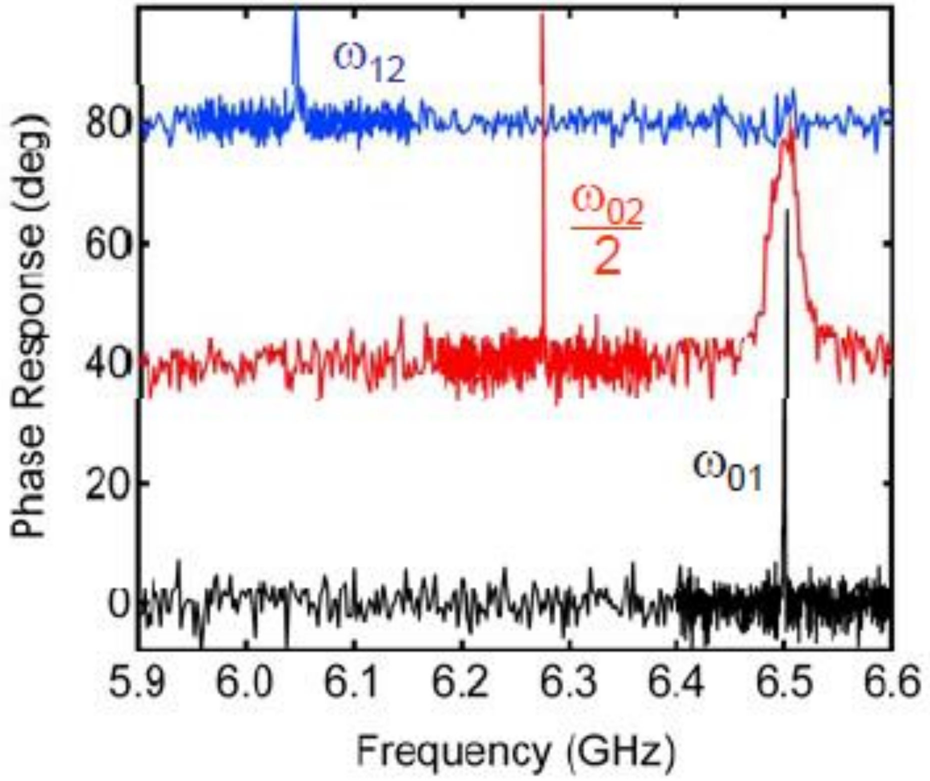


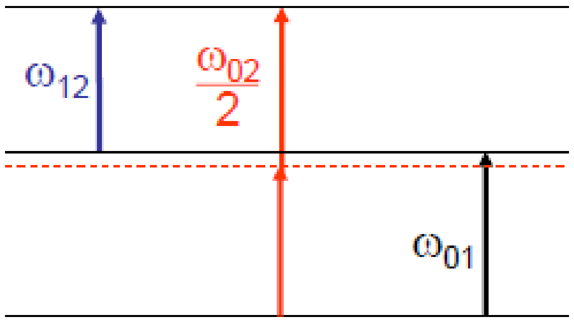
$$\frac{E_C}{h} \simeq 1.8 \text{ GHz}$$

$$\frac{1}{h}\sqrt{2E_C E_J} \simeq 6.5 \text{ GHz} \sim 300\,\text{mK}$$

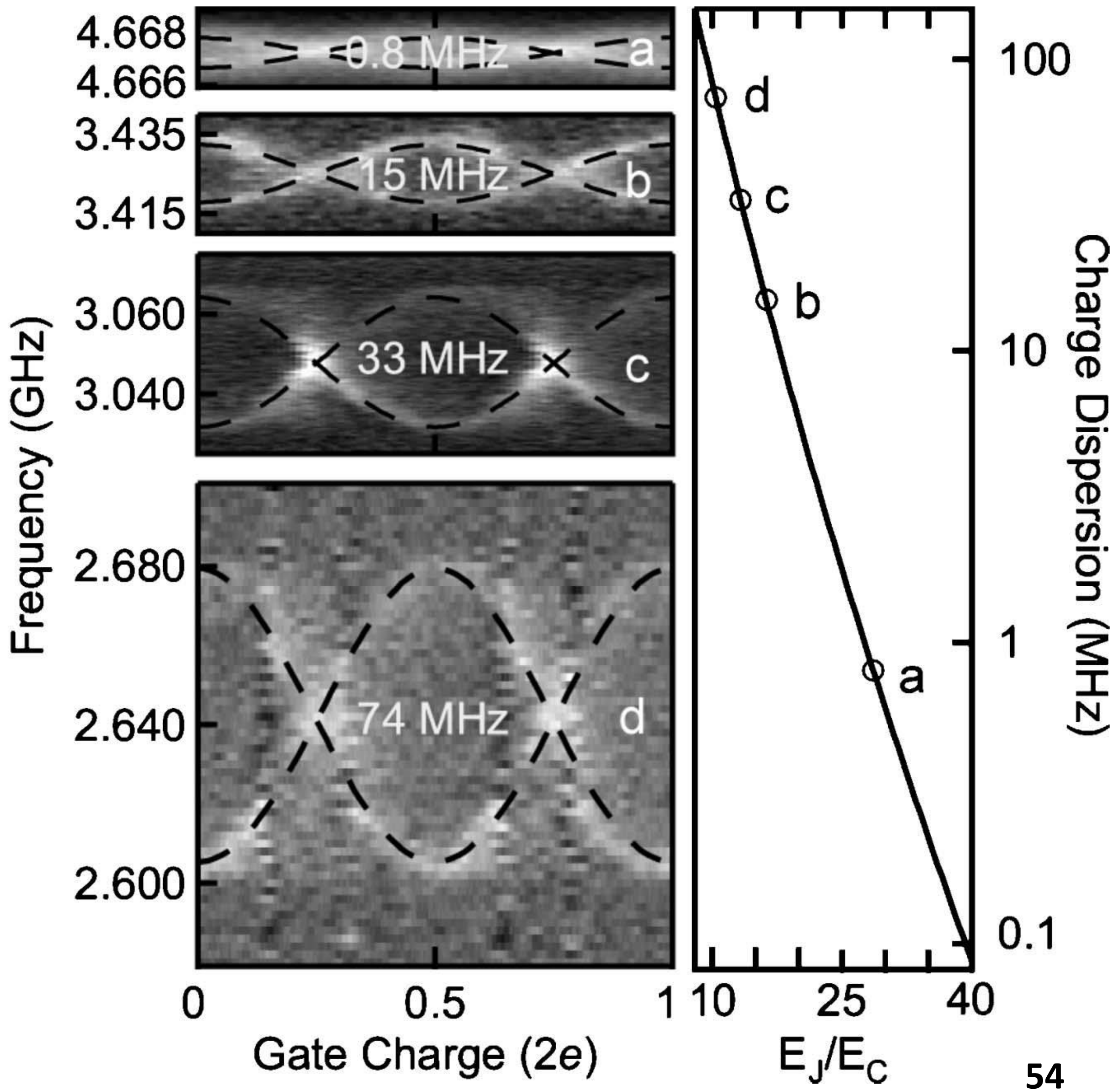
$\Rightarrow$ dilution fridge!

Experimental spectrum of a transmon

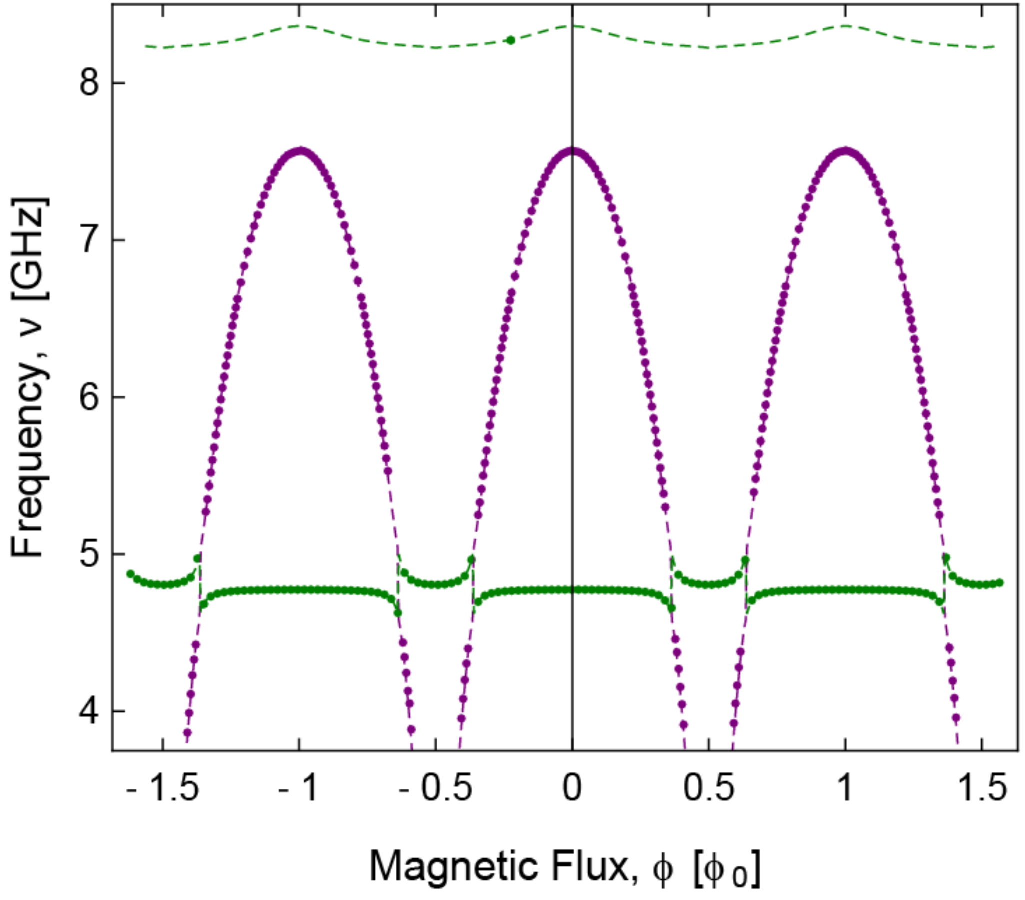




J. Schreier et al., PRB (2008)



Tuning Transmon Qubits with Applied Magnetic Flux



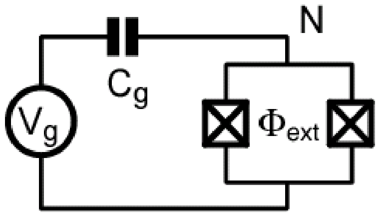
Measured qubit/resonator freq. (*,*)

Fit to full Hamiltonian (-,-)

Changing Qubit Transition Frequency:

- Make qubits with superconducting quantum interference device (SQUID) loop

M. Tinkham, Introduction to Superconductivity, McGraw-Hill



$$E_J = E_{J,\max} \cos\left(\pi \frac{\phi_{\text{ext}}}{\phi_0}\right)$$

- Apply global magnetic field using off-chip coil
- Apply “local” magnetic field using on-chip flux line

Reminder: Resonator and Qubit Spectroscopy:

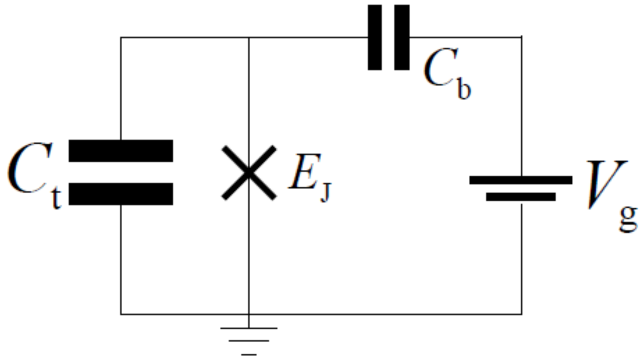
- Probe resonator
- Drive qubit
- When drive matches the qubit transition, change in transmission of resonator probe observed.

55

Transmon Limit

Recap

$$\hat{H} = -E_{\mathrm{J}}\cos\left(2\pi\frac{\hat{\Phi}_{\mathrm{t}}}{\Phi_o}\right) + \frac{\left(\hat{Q}_{\mathrm{t}} + C_{\mathrm{b}}V_{\mathrm{g}}\right)^2}{2\left(C_{\mathrm{t}} + C_{\mathrm{b}}\right)}$$



Is commonly written using other variables:

$$\hat{\varphi}_{\mathrm{t}} \equiv 2\pi\frac{\hat{\Phi}_{\mathrm{t}}}{\Phi_o} \qquad \hat{N}_{\mathrm{t}} \equiv -\frac{\hat{Q}_{\mathrm{t}}}{2e} \qquad N_{\mathrm{g}} \equiv \frac{C_{\mathrm{b}}V_{\mathrm{g}}}{2e} \qquad E_C \equiv \frac{e^2}{2\left(C_{\mathrm{t}} + C_{\mathrm{b}}\right)}$$

Phase operator — Cooper-pair number operator — Number offset — Charging energy

Quantum operators (Phase operator, Cooper-pair number operator)

These are numbers (not operators!) (Number offset, Charging energy)

$$\boxed{\hat{H} = -E_{\mathrm{J}}\cos\left(\hat{\varphi}_{\mathrm{t}}\right) + 4E_{\mathrm{C}}\left(\hat{N}_{\mathrm{t}} - N_{\mathrm{g}}\right)^2} \qquad \left[\hat{\varphi}_{\mathrm{t}}, \hat{N}_{\mathrm{t}}\right] = -i$$

Hamiltonian in charge basis

$$\boxed{\hat{H} = \sum_{N=-\infty}^{\infty} -\frac{E_{\mathrm{J}}}{2}\left(|N\rangle\langle N+1| + |N+1\rangle\langle N|\right) + 4E_{\mathrm{C}}\left(N - N_{\mathrm{g}}\right)^2 |N\rangle\langle N|}$$

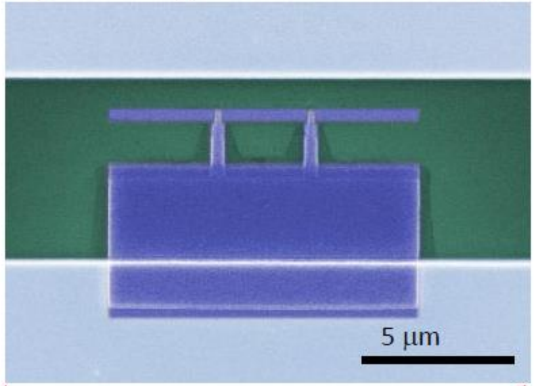
CPB and transmon regimes

$$\hat{H} = \sum_{N=-\infty}^{\infty} -\frac{E_J}{2}\left(|N\rangle\langle N+1| + |N+1\rangle\langle N|\right) + 4E_C\left(N - N_g\right)^2 |N\rangle\langle N|$$

CPB and Transmon differ only in the regimes of Josephson and charging energies used.

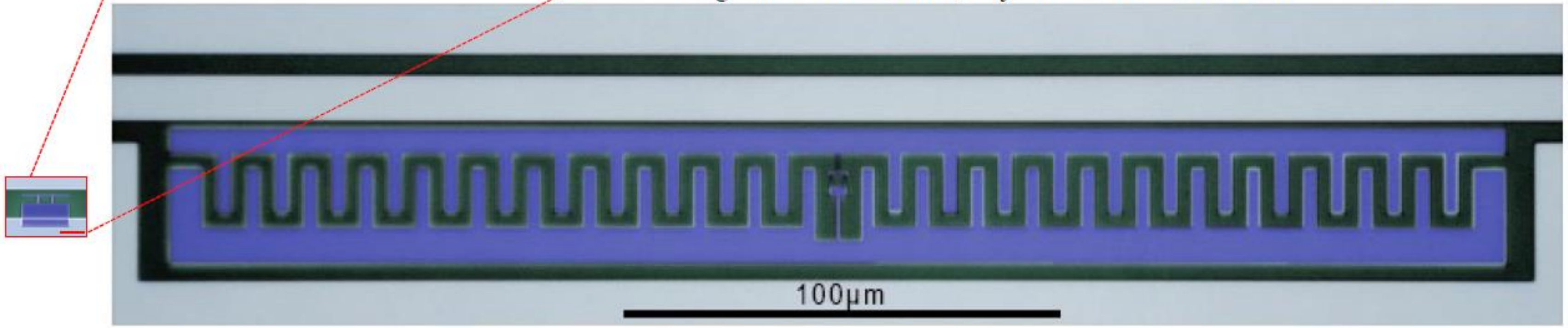
Cooper-Pair Box $\frac{E_J}{E_C} \sim 1$

$E_C / h \sim E_J / h \sim 5\ \text{GHz}$



Transmon $\frac{E_J}{E_C} >\sim 30$

$E_C / h \approx 0.3\ \text{GHz};\ E_J / h \sim 10-30\ \text{GHz}$



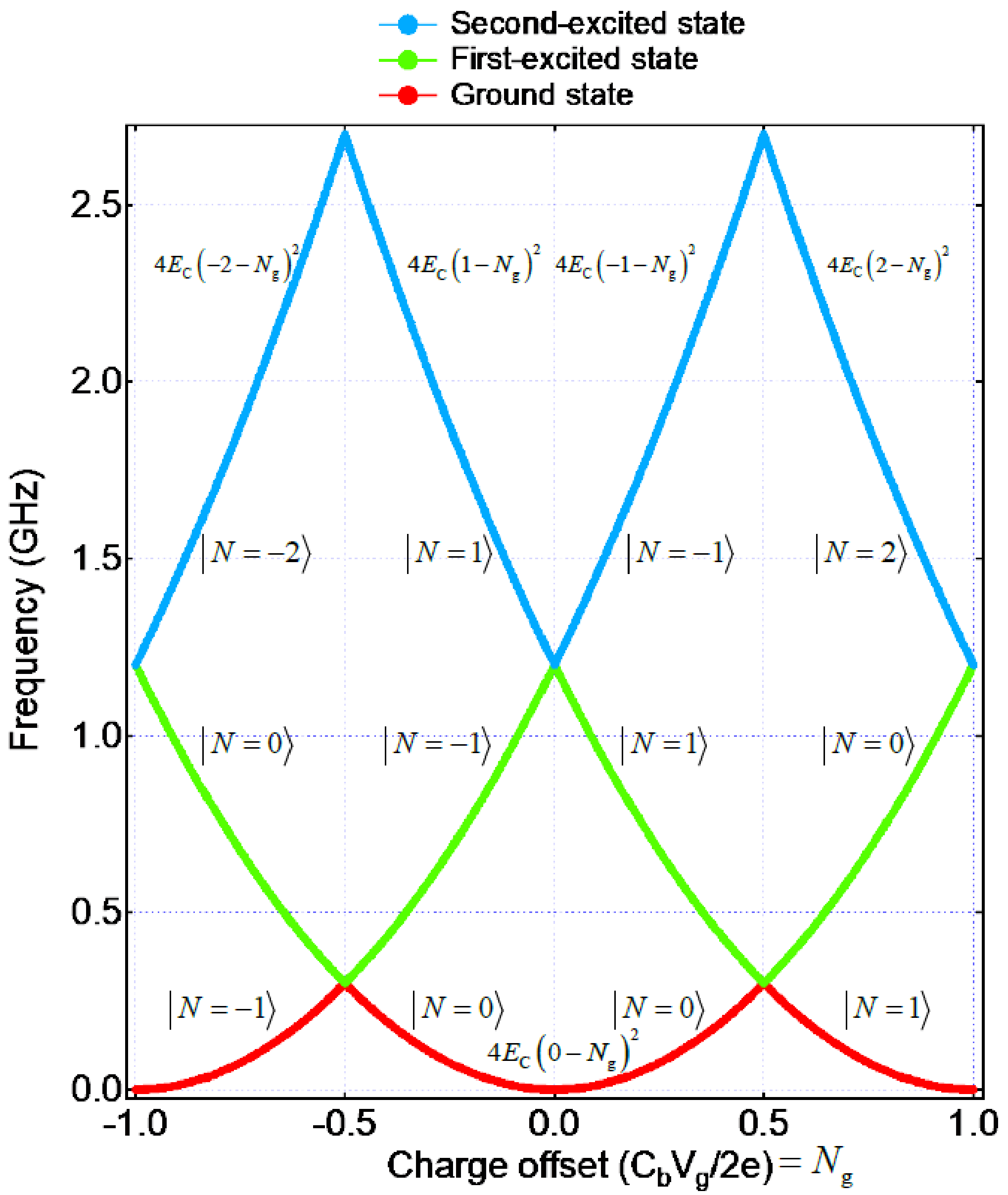
$E_J = 0$

$$\boxed{\hat{H} = \sum_{N=-\infty}^{\infty} -\frac{E_J}{2}\left(|N\rangle\langle N+1| + |N+1\rangle\langle N|\right) + 4E_C\left(N - N_g\right)^2 |N\rangle\langle N|}$$

0

- The Hamiltonian is diagonal in the charge basis.
- What are the eigenstates? $|N\rangle$
- What are the eigenvalues? $4E_C\left(N - N_g\right)^2$

E_J = 0 (Energy spectrum)



$E_C / h = 0.3$ GHz

$E_J = 0$

Second-excited state

First-excited state

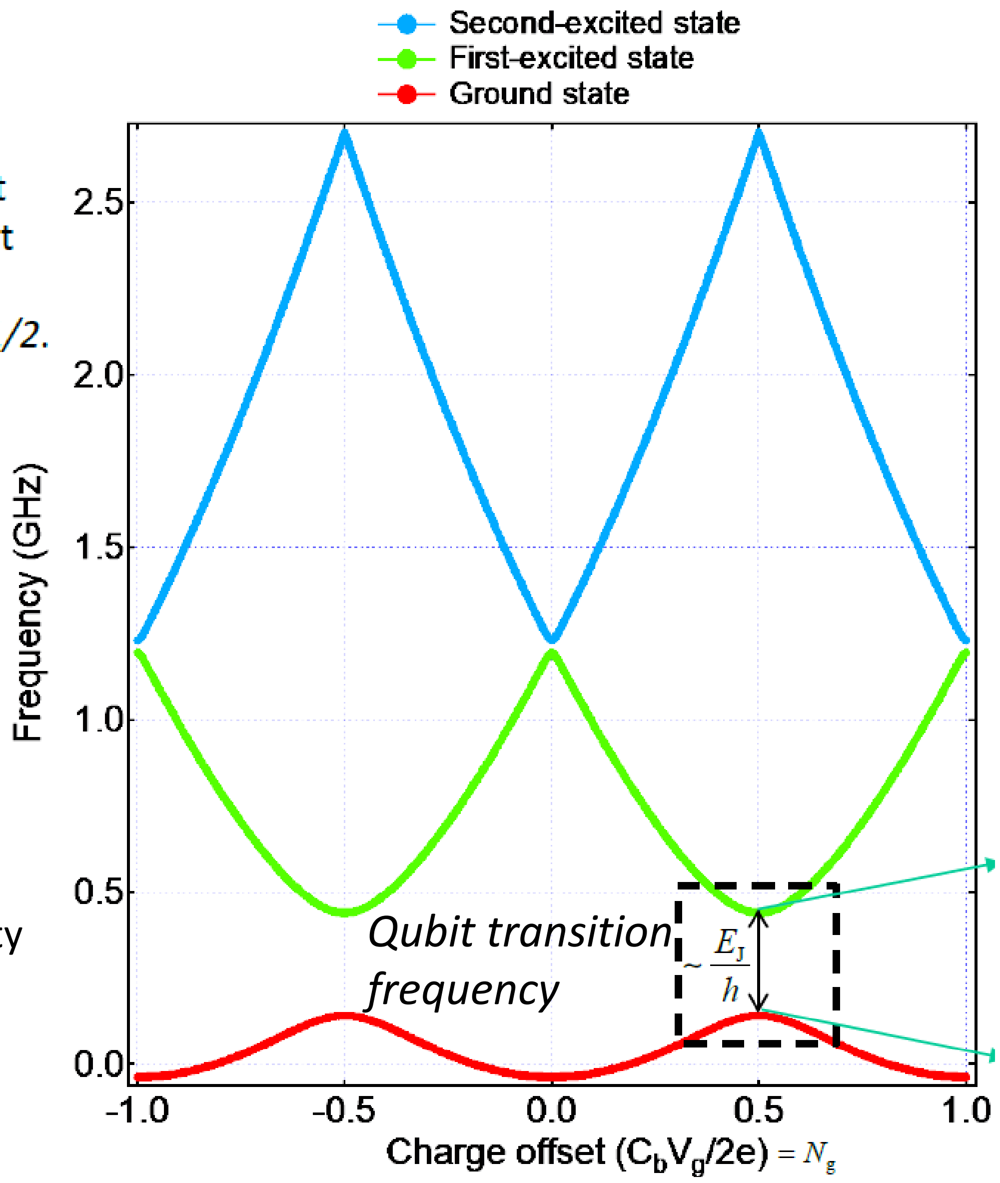
Ground state

CPB: $E_J = E_C$

The CPB quantum bit consists of the lowest two energy states at charge offset N_g=*1/2*.

Large anharmonicity

$$\alpha \equiv f_{12} - f_{01}$$



$E_C / h = 0.3$ GHz

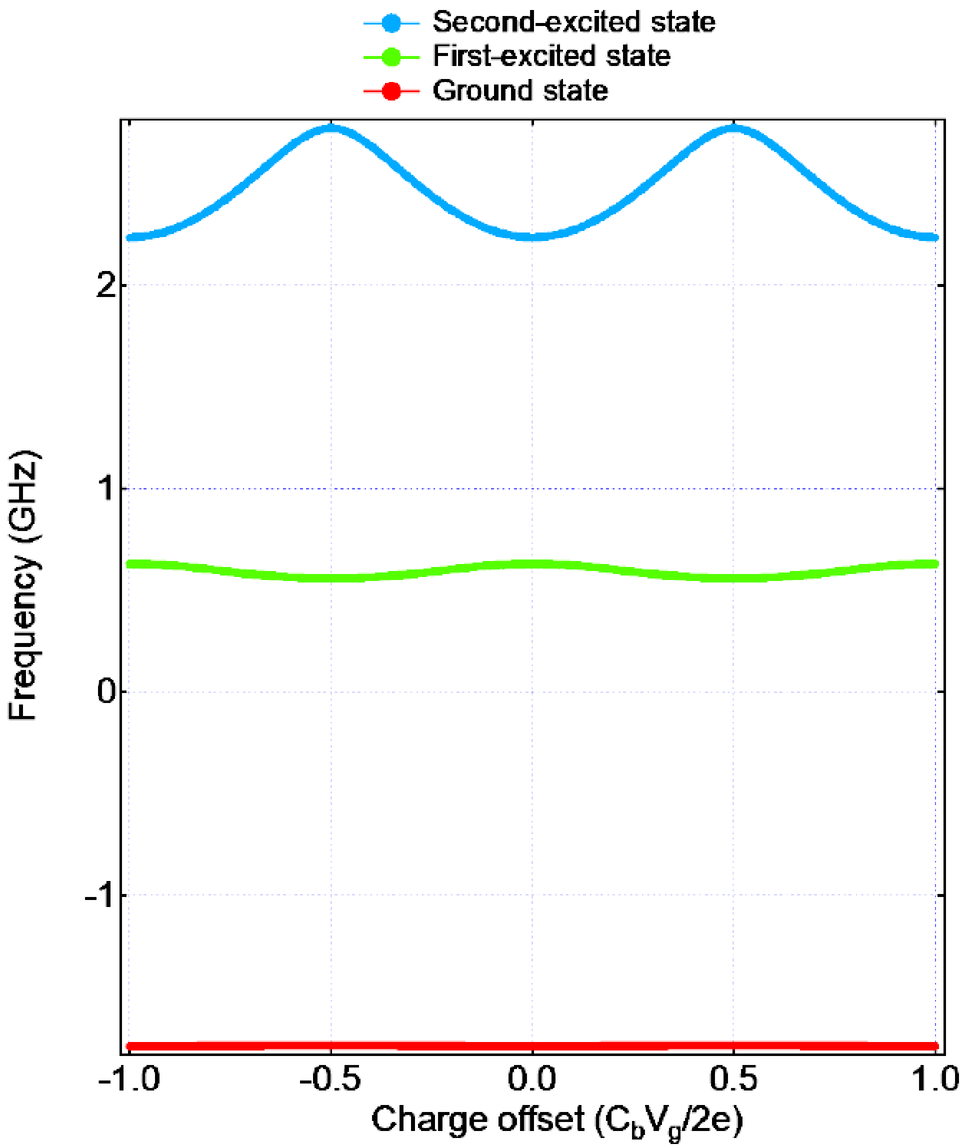
$E_J / h = 0.3$ GHz

At this charge offset, the ground and first excited states are, to a good approximation, the symmetric and antisymmetric maximal superposition of charge states $|N=0\rangle$ and $|N=1\rangle$.

$$|\psi_1\rangle \approx \frac{1}{\sqrt{2}}|N=0\rangle - \frac{1}{\sqrt{2}}|N=1\rangle$$

$$|\psi_0\rangle \approx \frac{1}{\sqrt{2}}|N=0\rangle + \frac{1}{\sqrt{2}}|N=1\rangle$$

Intermediate regime: $E_J > E_C$



$$E_C / h = 0.3 \text{ GHz}$$

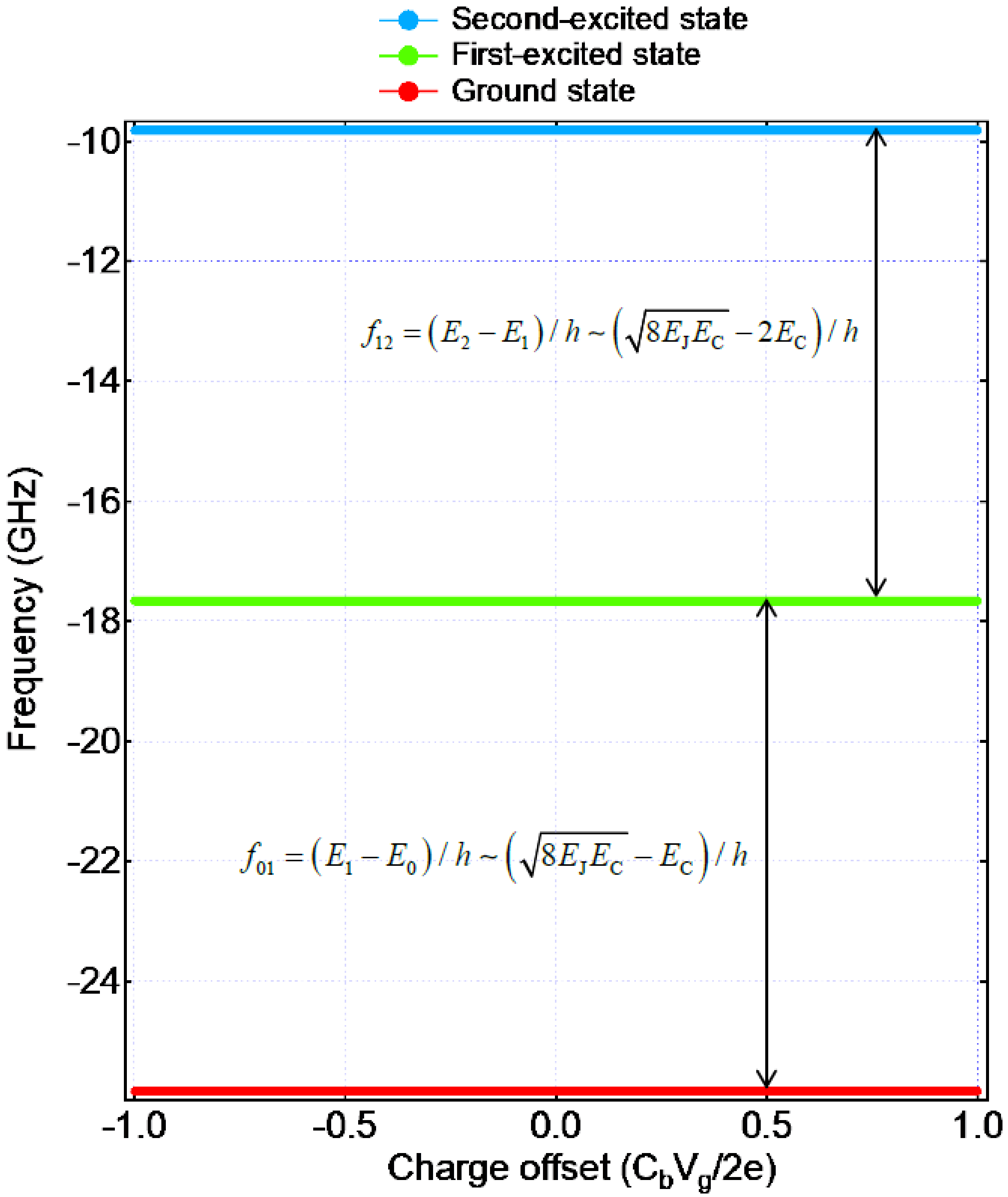
$$E_J / h = 3 \text{ GHz}$$

Transmon regime: $E_J \gg E_C$

In the transmon regime, the energy levels become insensitive to the charge offset.

The anharmonicity is reduced but not fully eliminated

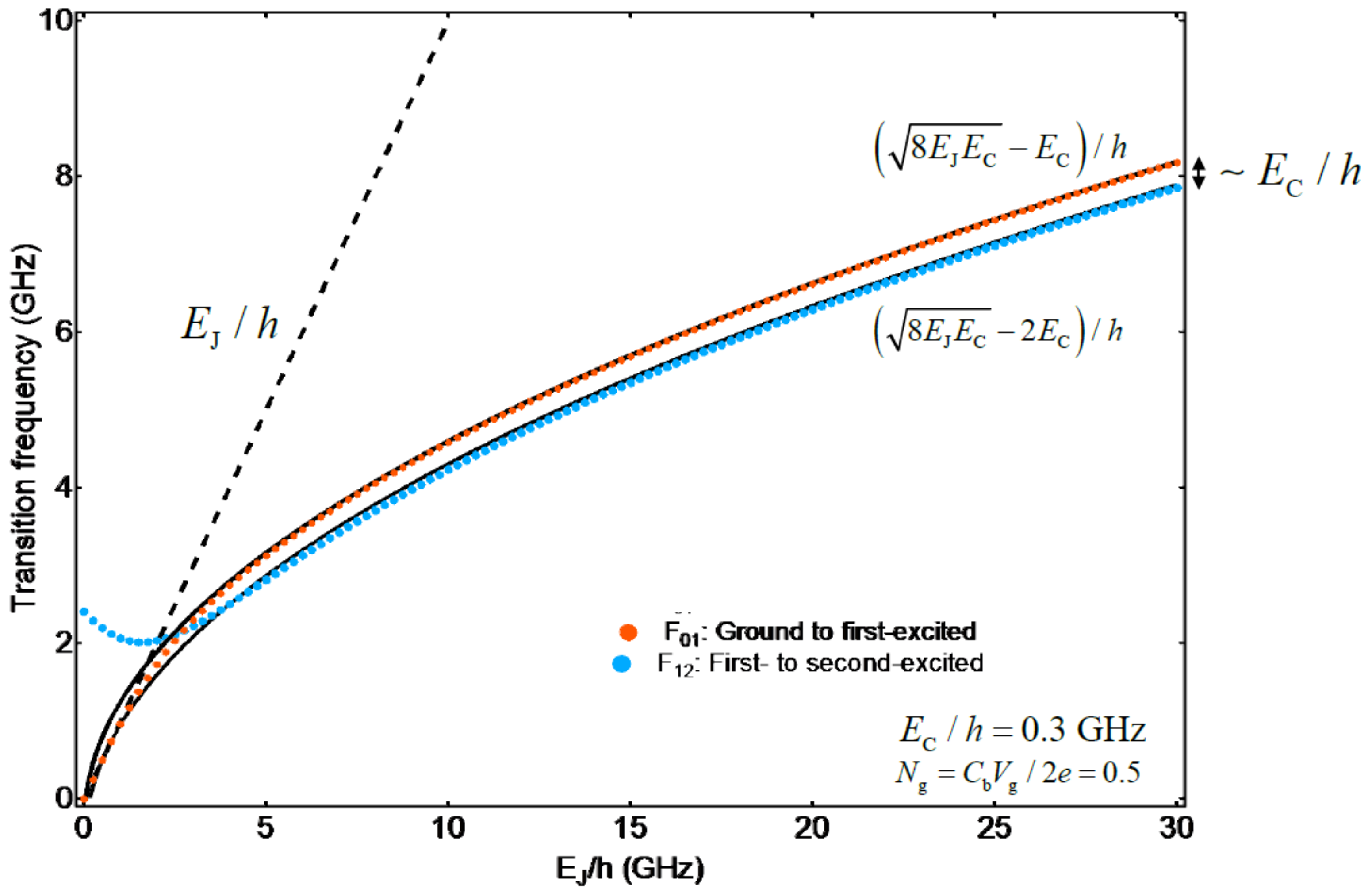
$$\alpha \equiv f_{12} - f_{01} \sim -E_C/h$$



$$E_C/h = 0.3 \text{ GHz}$$

$$E_J/h = 30 \text{ GHz}$$

Qubit transition frequency and anharmonicity



CPB regime

Very anharmonic but
charge sensitive spectrum

transmon regime

charge insensitive and
weakly anharmonic spectrum

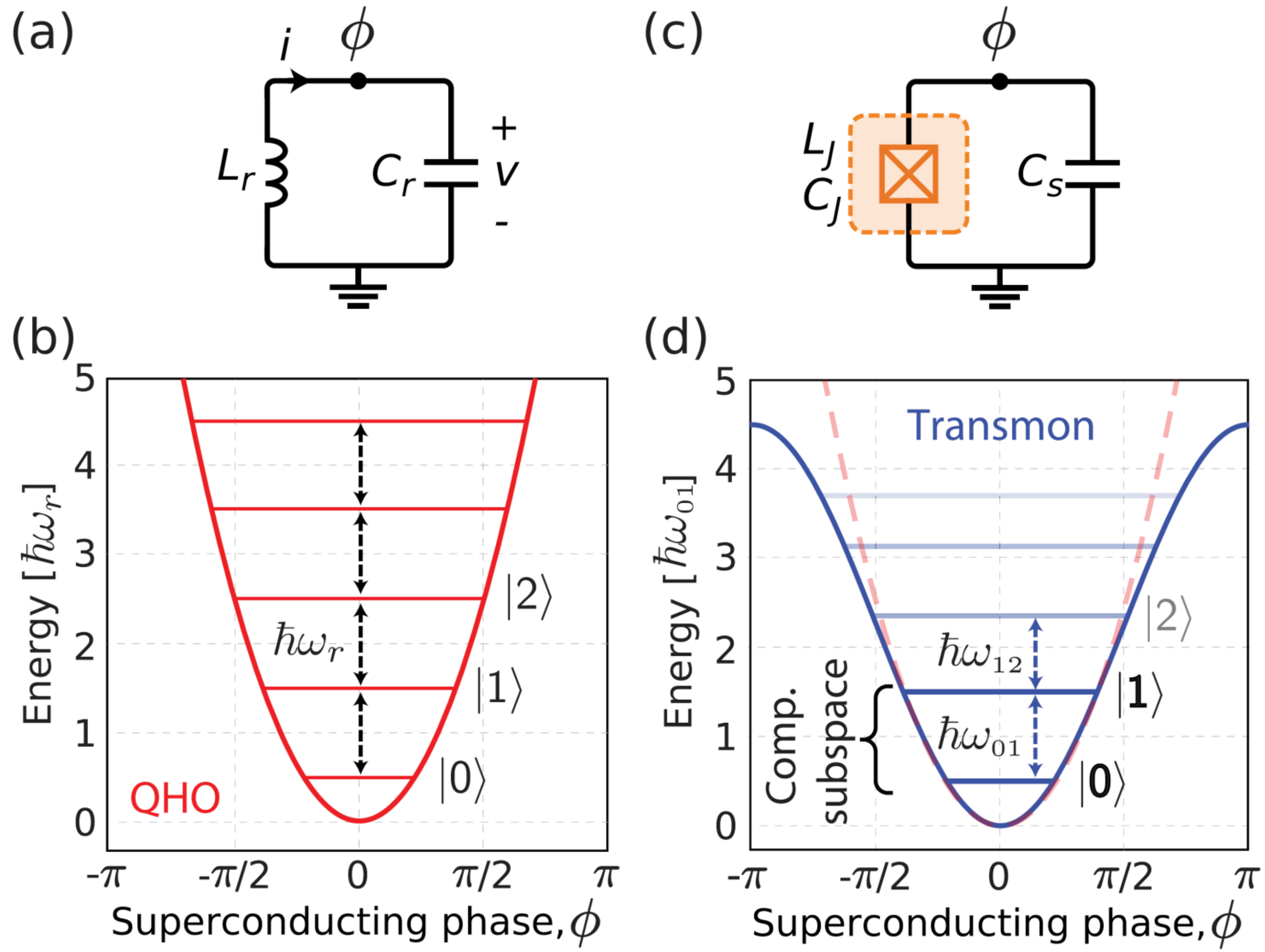
Important messages

● The two relevant energy scales of the charge qubit Hamiltonian are the Josephson coupling energy E_J and the charging energy E_C . In the Cooper-pair box regime, $E_J/E_C \sim 1$. In the transmon regime, $E_J/E_C > \sim 30$.

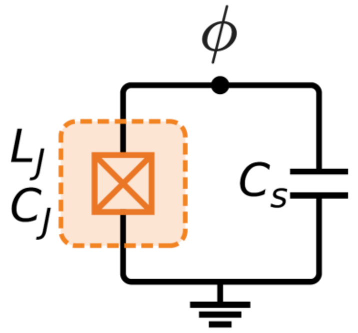
● In the CPB regime, a voltage bias is required. The qubit transition frequency at the typical bias point is approximately E_J / h.

● In the transmon regime, the energy levels are insensitive to voltage bias (and also to charge noise!). The qubit transition frequency is approximately $\left(\sqrt{8E_J(\Phi_{ext})E_C} - E_C\right)/h$.
The anharmonicity is reduced but not eliminated, it is approximately $-E_C / h$.

Transmon as an anharmonic oscillator



Transmon as an anharmonic oscillator



- Step 1: Write down Lagrangian

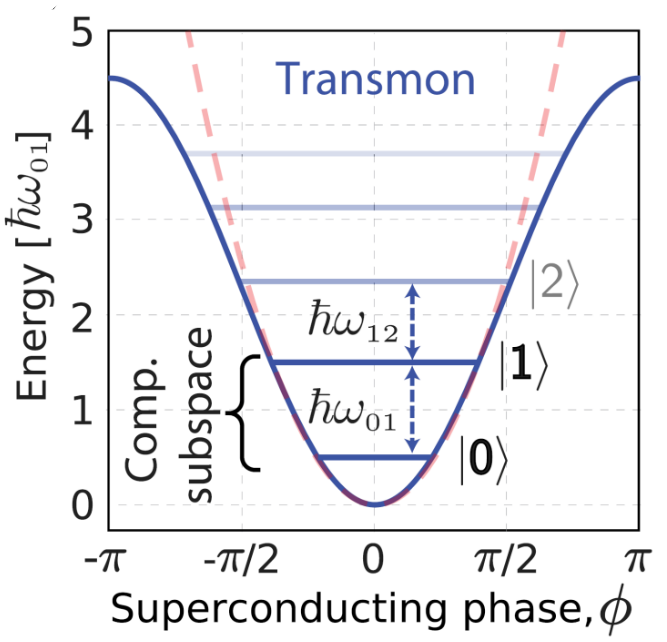
$$\mathcal{L} = E_{cap} - E_{ind} = \frac{1}{2} C\,\dot{\Phi}^2 + E_J \cos\left(2\pi \frac{\Phi}{\Phi_0}\right)$$
$$C = C_S + C_J$$

- Step 2: Find conjugate variable

$$Q = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}} = C\dot{\Phi}$$

- Step 3: Calculate classical Hamiltonian

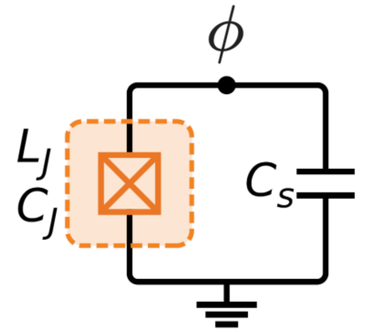
$$\mathcal{H}(\Phi, Q) = C\dot{\Phi}^2 - \mathcal{L}(\Phi, \dot{\Phi}) = \frac{1}{2C} Q^2 - E_J \cos\left(2\pi \frac{\Phi}{\Phi_0}\right)$$



- Step 4: Quantize the Hamiltonian

$$[\hat{\Phi}, \hat{Q}] = i\hbar \qquad H = -E_J \cos\left(2\pi \frac{\hat{\Phi}}{\Phi_0}\right) + \frac{1}{2C} \hat{Q}^2 = -E_J \cos\left(2\pi \frac{\hat{\Phi}}{\Phi_0}\right) + 4E_C\, n^2 \qquad n = Q/2e$$

Transmon as an anharmonic oscillator



- What if Φ is very very small?
  - Question: when does that happen?

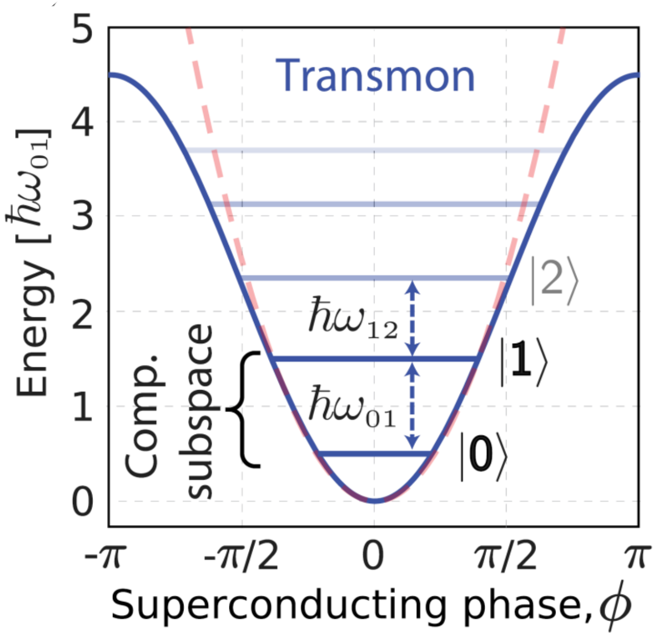
$$H = -E_J \cos\left(2\pi \frac{\hat{\Phi}}{\Phi_0}\right) + \frac{1}{2C}\hat{Q}^2 \approx -E_J\left(1 - 0.5\left(2\pi\frac{\hat{\Phi}}{\Phi_0}\right)^2\right) + \frac{1}{2C}\hat{Q}^2$$

$$H = \frac{E_J}{2}\left(\frac{2\pi}{\Phi_0}\right)^2 \hat{\Phi}^2 + \frac{1}{2C}\hat{Q}^2 = \frac{E_J}{2}\hat{\varphi}^2 \quad + 4E_C\,\hat{n}^2$$

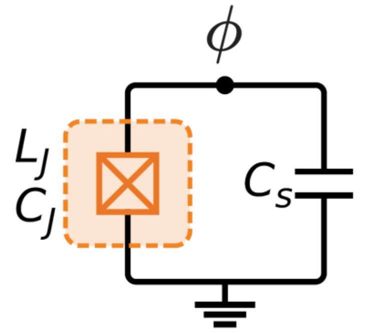
Harmonic Oscillator

- What if Φ is small?

$$H = \frac{E_J}{2}\left(\frac{2\pi}{\Phi_0}\right)^2 \hat{\Phi}^2 \underbrace{- \frac{E_J}{24}\left(\frac{2\pi}{\Phi_0}\right)^4}_{\text{Non-linear energy term}} \hat{\Phi}^4 + \frac{1}{2C}\hat{Q}^2$$



Transmon as an anharmonic oscillator



- Consider the Hamiltonian term:

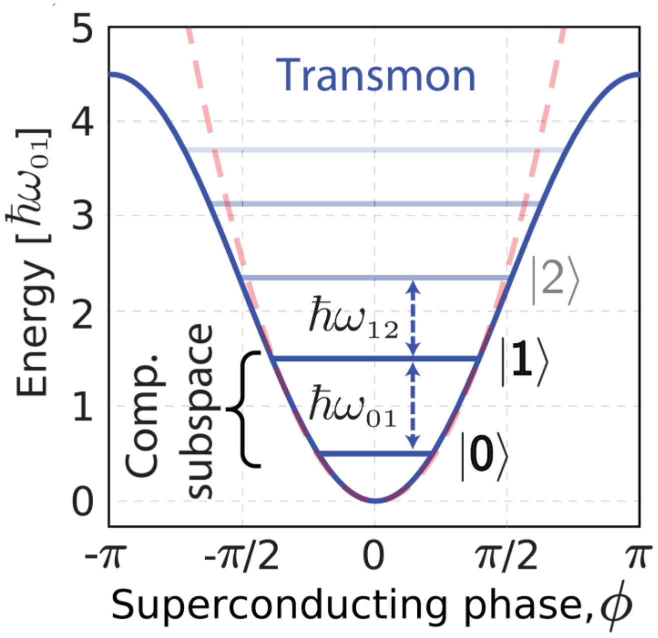
$$-\frac{E_J}{24}\left(\frac{2\pi}{\Phi_0}\right)^4\hat{\Phi}^4$$

- Rewrite this term to:

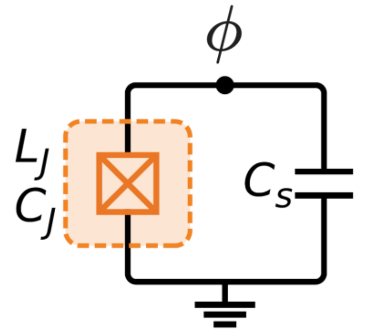
$$-Ka^\dagger a^\dagger aa-\delta a^\dagger a+\Delta_0$$

- Hints:
  - Keep only "energy preserving" terms (same number of creation and annihilation operators)

$$\hat{\Phi}=\sqrt{\frac{\hbar}{2C\omega}}(a^\dagger+a),\qquad [a,a^\dagger]=1$$



Anharmonic energies

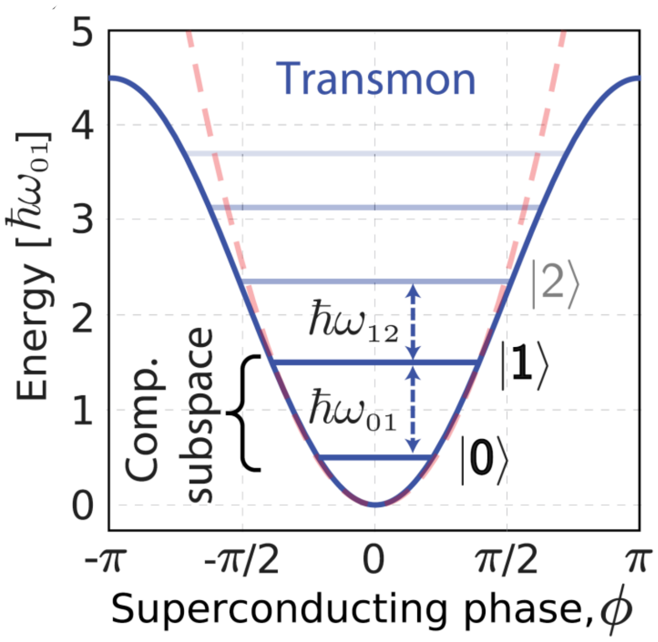


- With the Hamiltonian $\hbar\omega = \sqrt{8E_J E_C}$

$$\color{red}{H} = \hbar\omega a^\dagger a - E_c a^\dagger a - \frac{E_c}{2} a^\dagger a^\dagger a a =$$

$$\color{red}{= \hbar\omega_T a^\dagger a - \frac{\hbar\alpha}{2} a^\dagger a^\dagger a a}$$

$$\color{red}{\boldsymbol{\hbar\omega_T = \sqrt{8E_J E_C} - E_C \qquad \hbar\alpha = E_C}}$$



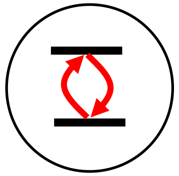
- Energies:

$$E_1 - E_0 = \hbar\omega - E_c$$
$$E_2 - E_1 = \hbar\omega - 2E_c$$

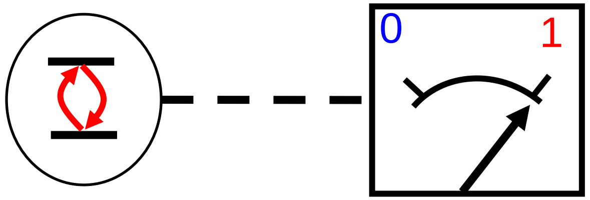
- Because $E_2 - E_1 \neq E_1 - E_0$ we can use as a qubit

Requirements for QC

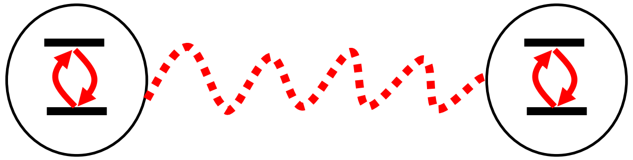
**High-Fidelity
Single Qubit Operations**



**High-Fidelity Readout
of Individual Qubits**

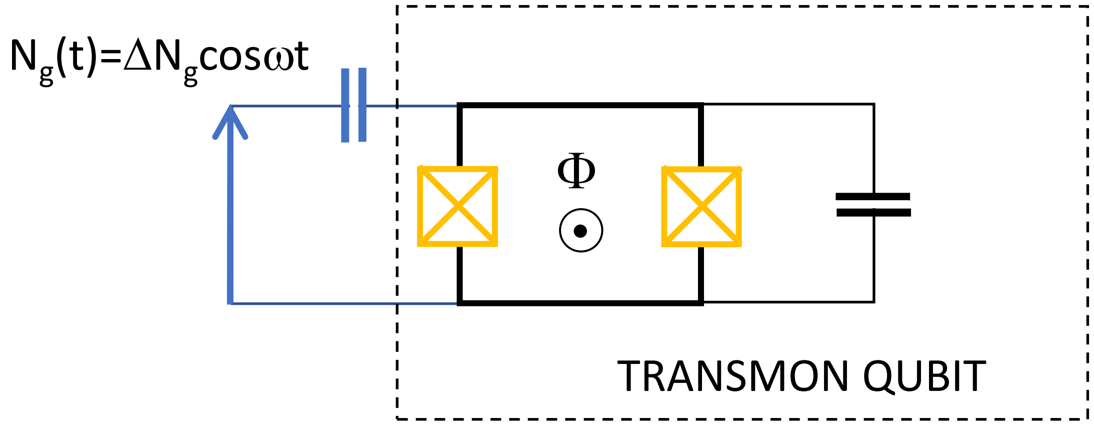


**Deterministic, On-Demand
Entanglement between Qubits**



Single Qubit Operations

Single-qubit gates: coupling a qubit to microwave field



$$\hat{H} = E_C\left(\hat{N} - N_g(t)\right)^2 - E_J\cos\hat{\theta}$$

$$\hat{H} = \underbrace{E_C\hat{N}^2 - E_J\cos\hat{\theta}}_{\text{transmon}} - \boxed{\underbrace{2E_C\Delta N_g\cos\omega t\hat{N}}_{\text{drive}}}$$

Two-level approximation

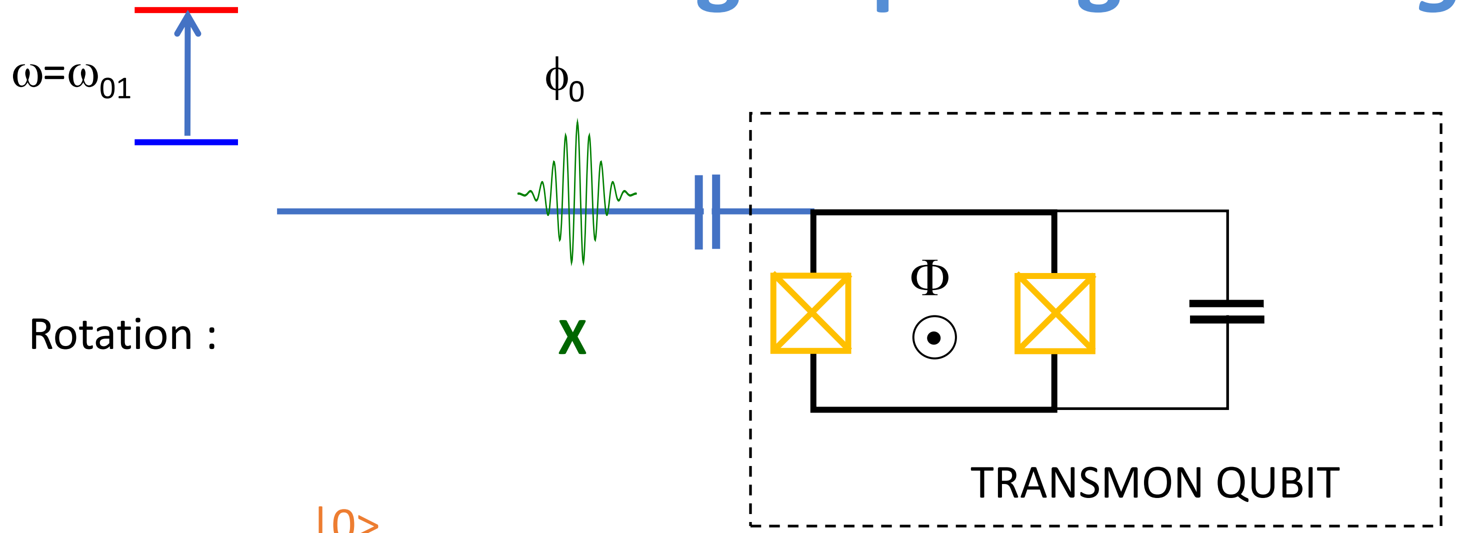
$$= -\hbar\frac{\omega_{01}(\Phi)}{2}\sigma_z \quad + \hbar\Omega_R\cos\omega t\sigma_x \qquad \hbar\Omega_R = 2E_C\langle 0|N|1\rangle\Delta N_g$$

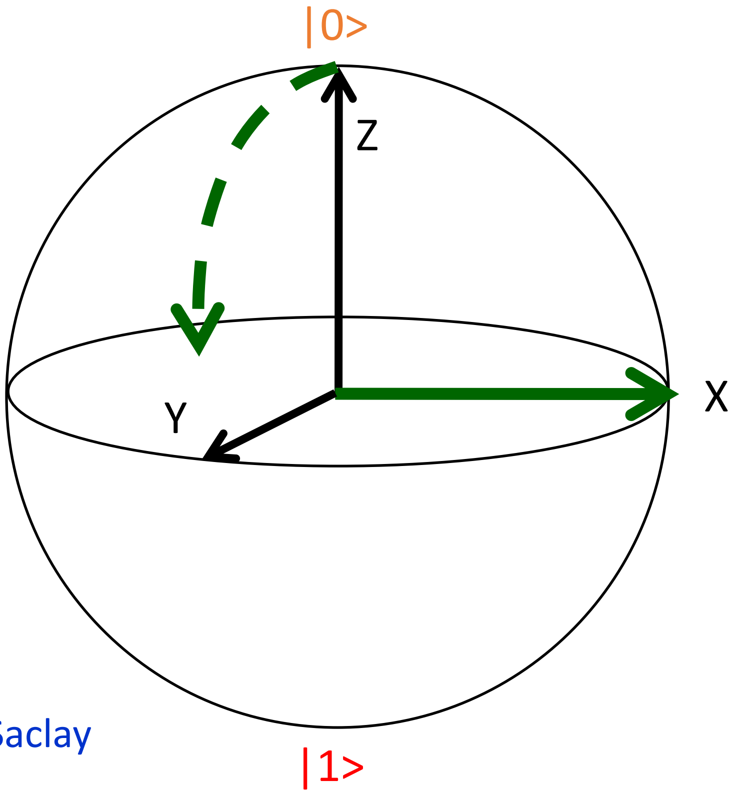
Advantage: Ω large, $2\pi/\Omega \sim 30$ ns

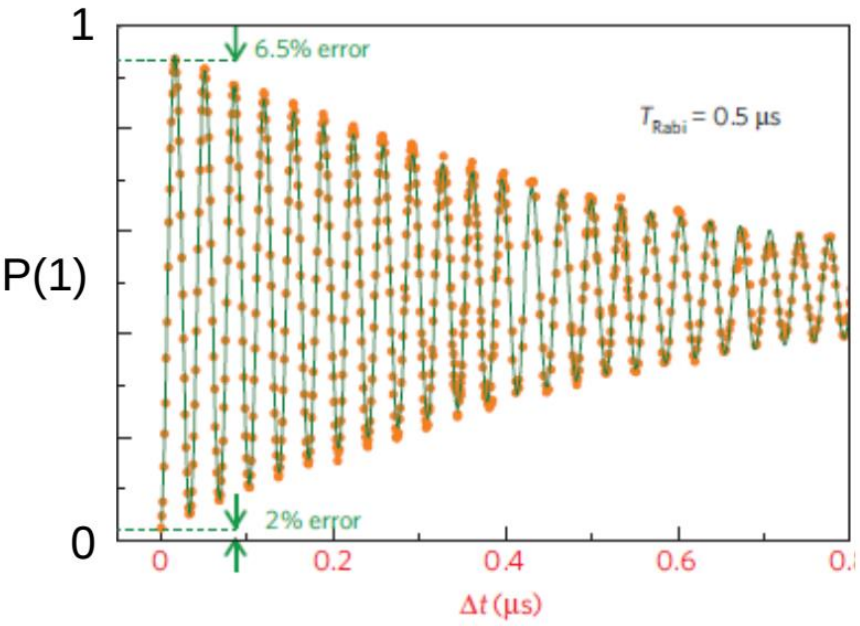
Chose rotation axis with phase $\cos(\omega t + \varphi_0)$

courtesy CEA Saclay

Single-qubit gates: *X-gate*

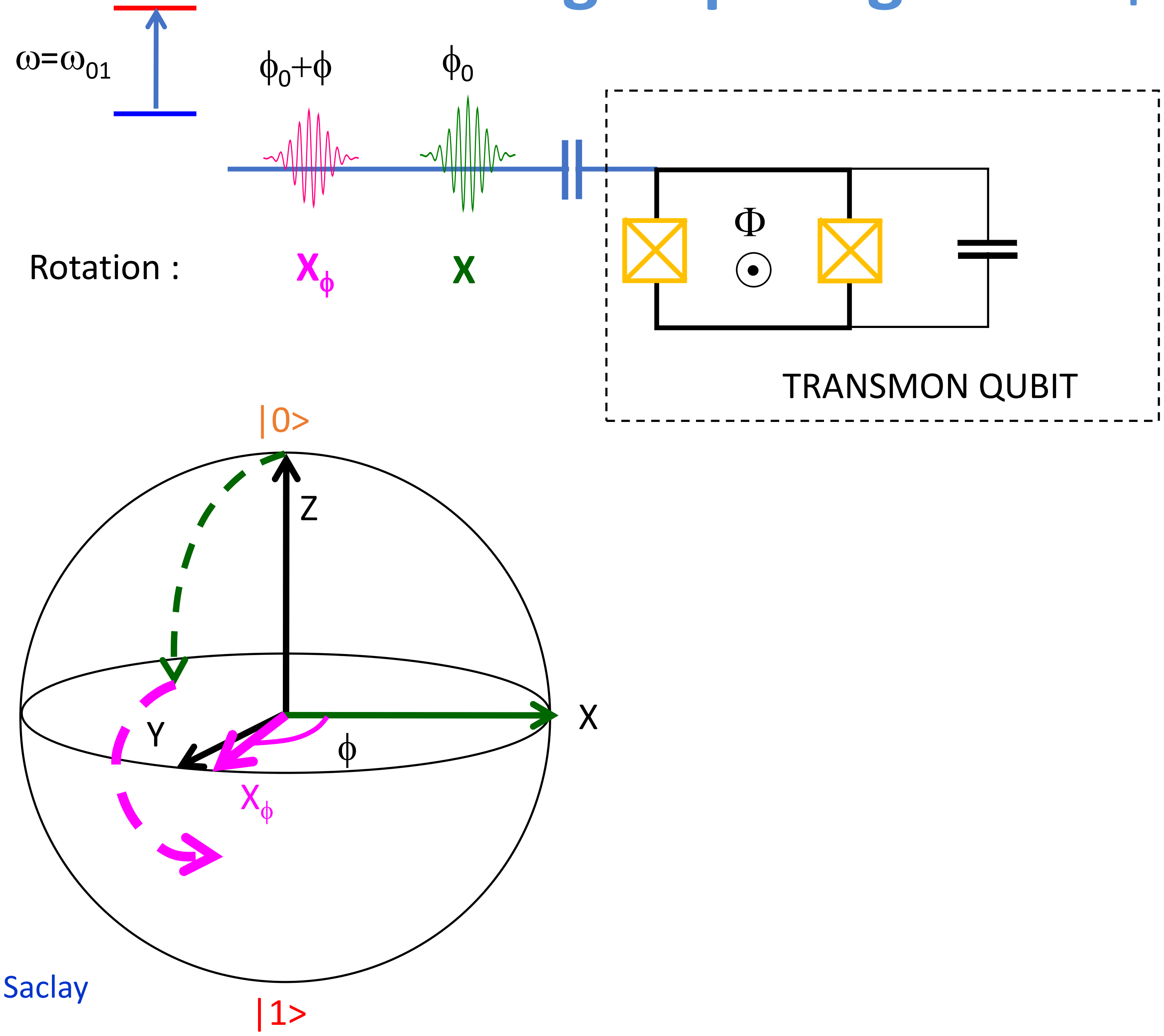






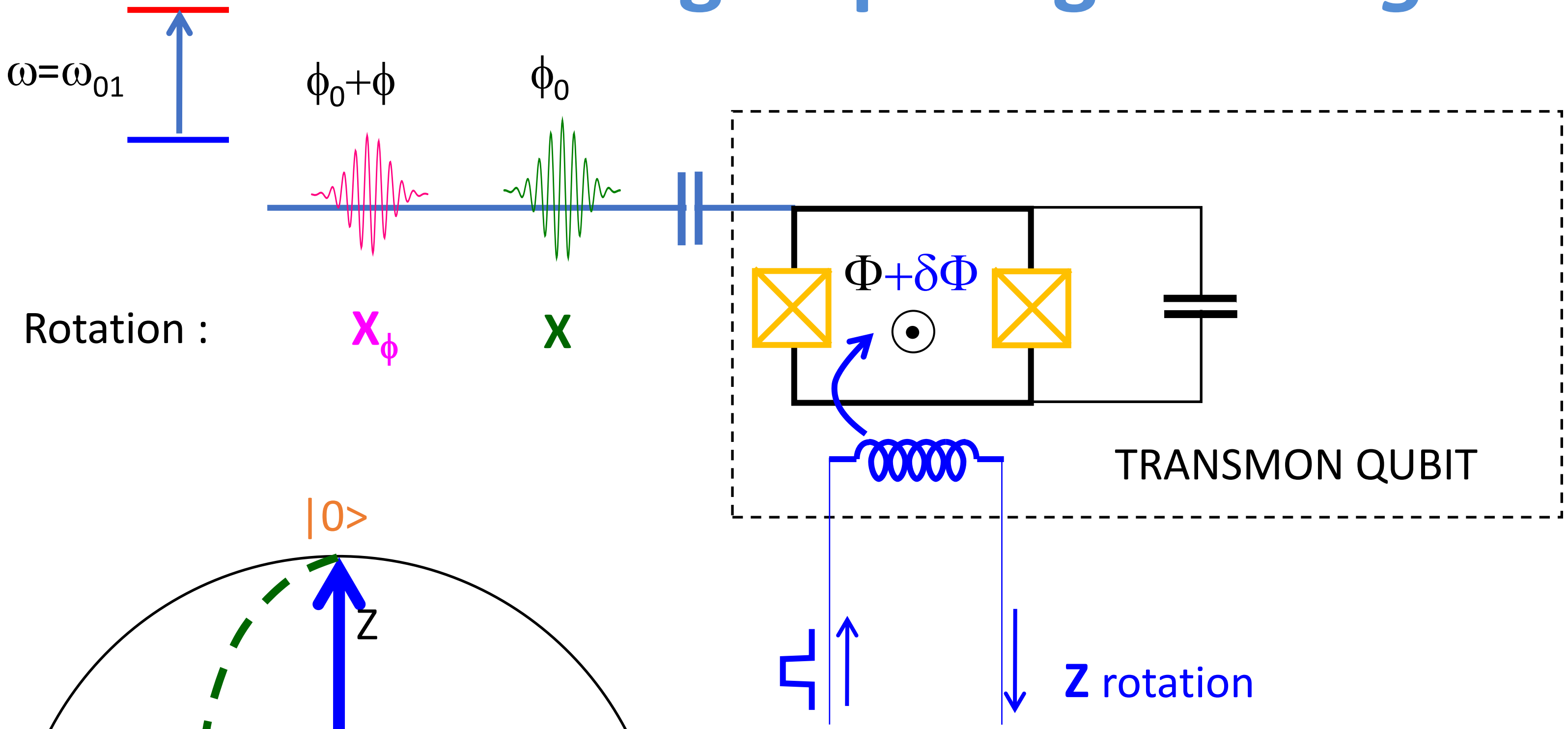
courtesy CEA Saclay

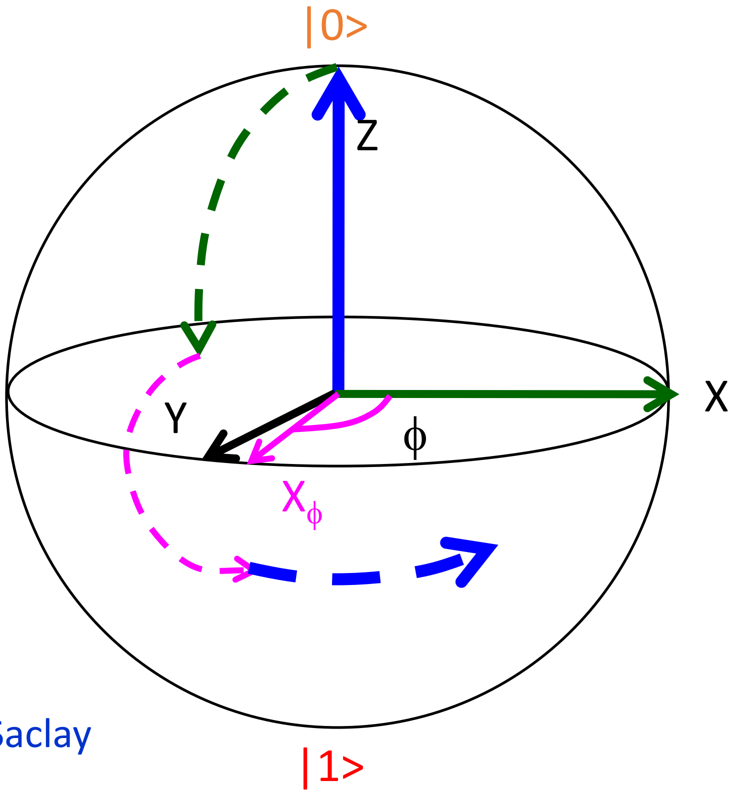
Single-qubit gates: X_ϕ*-gate*



courtesy CEA Saclay

Single-qubit gates: *Z-gate*





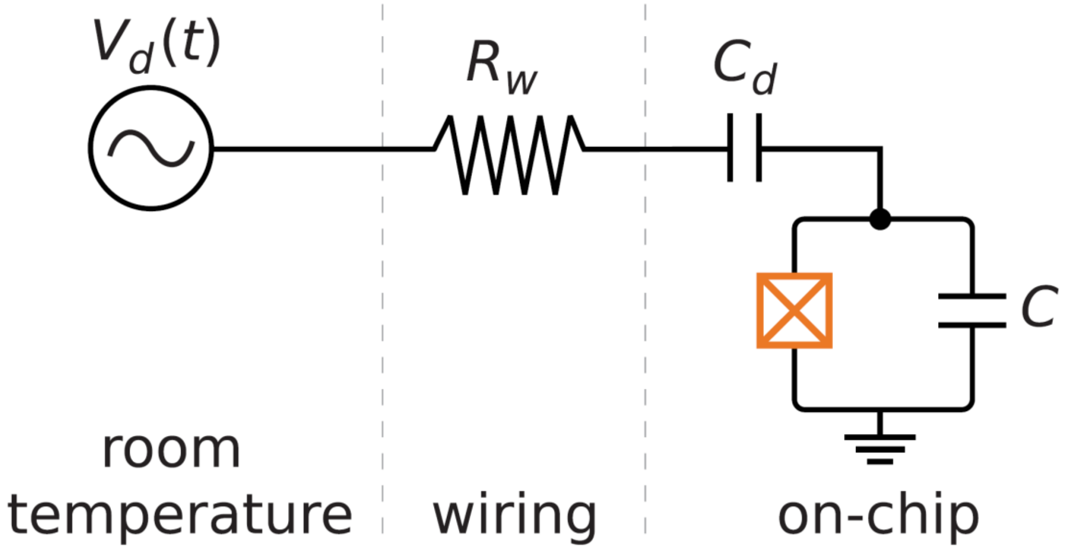
All rotations on Bloch sphere

Fidelity ? $> 99\%$

J.M. Chow et al., PRL 102, 090502 (2009)

courtesy CEA Saclay

One-qubit gates



$$H = \frac{\tilde{Q}(t)^2}{2C_\Sigma} + \frac{\Phi^2}{2L} + \frac{C_\mathrm{d}}{C_\Sigma} V_\mathrm{d}(t)\tilde{Q},$$

$$C_\Sigma = C + C_\mathrm{d} \quad \tilde{Q} = C_\Sigma \dot{\Phi} - C_\mathrm{d} V_\mathrm{d}(t)$$

$$H = \underbrace{-\frac{\omega_\mathrm{q}}{2}\sigma_z}_{H_0} + \underbrace{\Omega V_\mathrm{d}(t)\sigma_y}_{H_\mathrm{d}}$$

where $\Omega = (C_\mathrm{d}/C_\Sigma)Q_\mathrm{zpf}$ and $\omega_\mathrm{q} = (E_1 - E_0)/\hbar$

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To elucidate the role of the drive, we move into a frame rotating with the qubit at frequency ω_q (also denoted "the rotating frame" or the "the interaction frame"). To see the usefulness of this rotating frame, consider a state $|\psi_0\rangle = (1 \quad 1)^T/\sqrt{2}$. By the time-dependent Schrödinger equation this state evolves according to

$$|\psi_0(t)\rangle = U_{H_0}|\psi_0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{i\omega_q t/2} \\ e^{-i\omega_q t/2} \end{pmatrix}, \tag{79}$$

where U_{H_0} is the propagator corresponding to H_0. By calculating, e.g., $\langle\psi_0|\sigma_x|\psi_0\rangle = \cos(\omega_q t)$, it is evident that the phase is winding with a frequency of ω_q due to the σ_z term. By going into a frame rotating with the qubit at frequency ω_q, the action of the drive can be more clearly appreciated. To this end, we define $U_{rf} = e^{iH_0 t} = U_{H_0}^\dagger$ and the new state in the rotating frame is $|\psi_{rf}(t)\rangle = U_{rf}|\psi_0\rangle$. The time-evolution in this new frame is again found from the Schrödinger equation (using the shorthand $\partial_t = \partial/\partial t$)

$$i\partial_t|\psi_{rf}(t)\rangle = i(\partial_t U_{rf})|\psi_0\rangle + iU_{rf}(\partial_t|\psi_0\rangle), \tag{80}$$

$$= i\dot{U}_{rf}U_{rf}^\dagger|\psi_{rf}\rangle + U_{rf}H_0|\psi_0\rangle, \tag{81}$$

$$= \underbrace{\left(i\dot{U}_{rf}U_{rf}^\dagger + U_{rf}H_0U_{rf}^\dagger\right)}_{\tilde{H}_0}|\psi_{rf}\rangle. \tag{82}$$

Returning to Eq. (78), the form of H_d in the rotating frame is found to be

$$\tilde{H}_d = \Omega V_d(t)\left(\cos(\omega_q t)\sigma_y - \sin(\omega_q t)\sigma_x\right). \tag{83}$$

We can in general assume that the time-dependent part of the voltage $(V_d(t) = V_0 v(t))$ has the generic form

$$v(t) = s(t)\sin(\omega_d t + \phi), \tag{84}$$

$$= s(t)(\cos(\phi)\sin(\omega_d t) + \sin(\phi)\cos(\omega_d t)), \tag{85}$$

where $s(t)$ is a dimensionless envelope function, so that the amplitude of the drive is set by $V_0 s(t)$. Adopting the definitions

$$I = \cos(\phi)(\text{the 'in} - \text{phase' component}), \tag{86}$$

$$Q = \sin(\phi)(\text{the 'out} - \text{of} - \text{phase' component}), \tag{87}$$

the driving Hamiltonian in the rotating frame takes the form

$$\tilde{H}_d = \Omega V_0 s(t)(I\sin(\omega_d t) - Q\cos(\omega_d t)) \times \left(\cos(\omega_q t)\sigma_y - \sin(\omega_q t)\sigma_x\right). \tag{88}$$

Performing the multiplication and dropping fast rotating terms that will average to zero (i.e., terms with $\omega_q + \omega_d$), known as the rotating wave approximation (RWA), we are left with

$$\tilde{H}_d = \frac{1}{2}\Omega V_0 s(t)[(-I\cos(\delta\omega t) + Q\sin(\delta\omega t))\sigma_x + (I\sin(\delta\omega t) - Q\cos(\delta\omega t))\sigma_y], \tag{89}$$

where $\delta\omega = \omega_q - \omega_d$. Finally, by reusing the definitions from Eq. (85), the driving Hamiltonian in the rotating frame using the RWA can be written as

$$\tilde{H}_d = -\frac{\Omega}{2} V_0 s(t) \begin{pmatrix} 0 & e^{i(\delta\omega t+\phi)} \\ e^{-i(\delta\omega t+\phi)} & 0 \end{pmatrix}. \tag{90}$$

Equation (90) is a powerful tool for understanding single-qubit gates in superconducting qubits. As a concrete example, assume that we apply a pulse at the qubit frequency, so that $\delta\omega = 0$, then

$$\tilde{H}_d = -\frac{\Omega}{2} V_0 s(t)(I\sigma_x + Q\sigma_y), \tag{91}$$

showing that an "in-phase" pulse ($\phi = 0$, i.e., the I-component) corresponds to rotations around the x-axis, while an out-of-phase pulse ($\phi = \pi/2$, i.e., the Q-component), corresponds to rotations about the y-axis. As a concrete example of an in-phase pulse, writing out the unitary operator yields

$$U_{\mathrm{rf,d}}^{\phi=0}(t) = \exp\left(\left[\frac{i}{2}\Omega V_0 \int_0^t s(t')\mathrm{d}t'\right]\sigma_x\right), \tag{92}$$

which depends only on the macroscopic design parameters of the circuit as well as the envelope of the baseband pulse $s(t)$ and amplitude V_0, which can both be controlled using arbitrary waveform generators (AWGs). Equation (92) is known as "Rabi driving" and can serve as a useful tool for engineering the circuit parameters needed for efficient gate operation (subject to the available output voltage V_0). To see this, we define the shorthand

$$\Theta(t) = -\Omega V_0 \int_0^t s(t')\mathrm{d}t', \tag{93}$$

which is the angle by which a state is rotated given the capacitive cou-

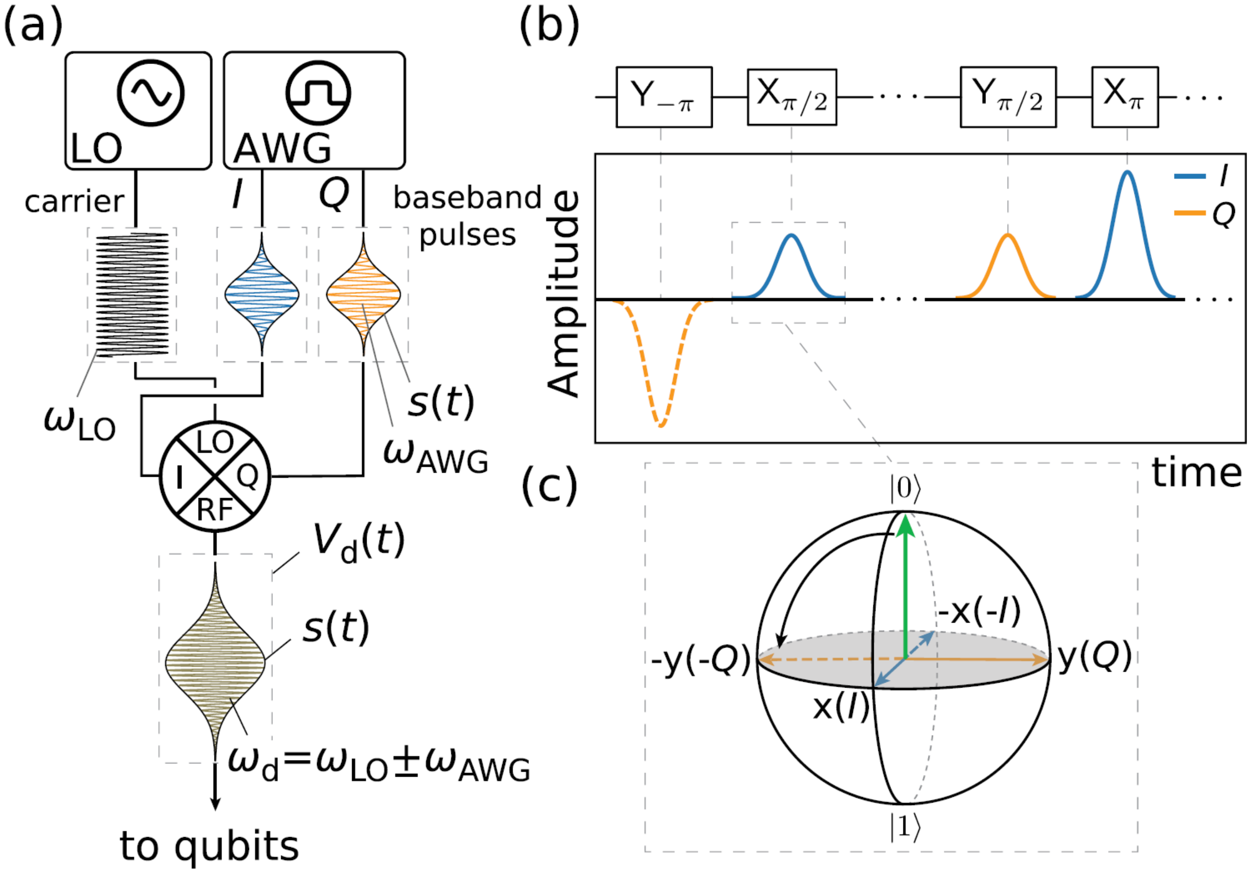

FIG. 13. (a) Schematic of a typical qubit drive setup. A microwave source supplies a high-frequency signal (ω_{LO}), while an arbitrary waveform generator (AWG) supplies a pulse-envelope ($s(t)$), sometimes with a low frequency component, ω_{AWG}, generated by the AWG. The IQ-mixer combines the two signals to generate a shaped waveform $V_d(t)$ with a frequency $\omega_d = \omega_{LO} \pm \omega_{AWG}$, typically resonant with the qubit. (b) Example of how a gate sequence is translated into a waveform generated by the AWG. Colors indicate the I and Q components. (c) The action of a $X_{\pi/2}$ pulse on a $|0\rangle$ state to produce the $|-i\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$ state.

Rabi Oscillations

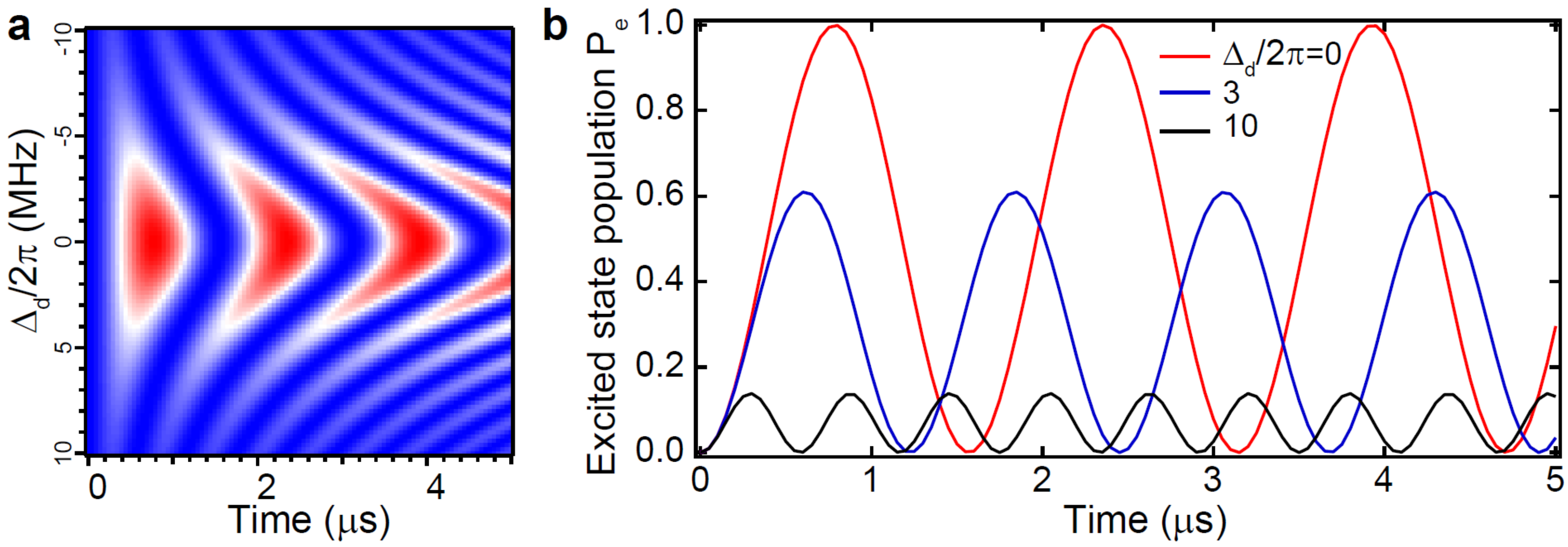

Figure 2.13: **Rabi oscillations: a**, The chevron plot. The excited state population P_e versus time for different detunings Δ_q. **b**, Three cuts from the chevron plot at different detuning values. The on-resonant drive gives the maximum contrast for the oscillations.

(Fast) Transmon driving