Muddheth, Cnelen, Faller

Klein-Gordon Equation

$$E^2 = \vec{p}^{\,2} + m^2 \qquad \text{(Einstein energy - momentum)}$$

$$\hat{E}^2 \psi(x,t) = \hat{\vec{p}}^{\,2} \psi(x,t) + m^2 \psi(x,t)$$

$$\hat{E} = i \frac{\partial}{\partial t} \qquad \hat{p} = -i \nabla$$

$$-\frac{\partial^2}{\partial t^2} \psi = -\nabla^2 \psi(x,t) + m^2 \psi(x,t)$$

$$\frac{\partial^2 \psi}{\partial t^2} = \nabla^2 \psi - m^2 \psi$$

$$\frac{\partial^2 \psi}{\partial t^2} - \nabla^2 \psi + m^2 \psi = 0$$

$$\partial^\mu \partial_\mu = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} \rightsquigarrow \text{Lorentz Invariant}$$

$$(\partial^\mu \partial_\mu + m^2) \psi = 0$$

Negative energy solutions!

$$E = \pm \sqrt{p^2 + m^2}$$

The Dirac Equation

$$E^2 = \vec{p}^{\,2} + m^2$$

$$\hat{E}\psi = (\alpha \cdot \hat{\vec{p}} + \beta m)\psi$$

$$-i\frac{\partial}{\partial t}\psi = \left(+i\alpha_x \frac{\partial}{\partial x} + i\alpha_y \frac{\partial}{\partial y} + i\alpha_z \frac{\partial}{\partial z} \mp \beta m\right)\psi$$

$\downarrow$ Square this

$$-\frac{\partial^2}{\partial t^2}\psi = \left(+i\alpha_x \frac{\partial\psi}{\partial x} + i\alpha_y \frac{\partial\psi}{\partial y} + i\alpha_z \frac{\partial\psi}{\partial z} - \beta m\right)\left(+i\alpha_x \frac{\partial\psi}{\partial x} + i\alpha_y \frac{\partial\psi}{\partial y} + i\alpha_z \frac{\partial\psi}{\partial z} - \beta m\right)$$

$$+\frac{\partial^2}{\partial t^2}\psi = +\alpha_x^2 \frac{\partial^2\psi}{\partial x^2} + \alpha_y^2 \frac{\partial^2\psi}{\partial y^2} + \alpha_z^2 \frac{\partial^2\psi}{\partial z^2} - \beta^2 m^2 \psi$$

$$(+\alpha_x\alpha_y + \alpha_y\alpha_x)\frac{\partial\psi}{\partial x\partial y} + (+\alpha_y\alpha_z + \alpha_z\alpha_y)\frac{\partial\psi}{\partial y\partial z} + (+\alpha_z\alpha_x + \alpha_x\alpha_z)\frac{\partial\psi}{\partial x\partial z}$$

$$+ (+i)(\alpha_x\beta + \beta\alpha_x) m \frac{\partial\psi}{\partial x} + (+i)(\alpha_y\beta + \beta\alpha_y) m\frac{\partial\psi}{\partial y} + (+i)(\alpha_z\beta + \beta\alpha_z) m \frac{\partial\psi}{\partial z}$$

$\downarrow$ Reduce to the Klein-Gordon Eqn.

$$\frac{\partial^2\psi}{\partial t^2} = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} - m^2\psi$$

$$\boxed{\begin{gathered} \alpha_x^2 = \alpha_y^2 = \alpha_z^2 = \beta^2 = I \\ \alpha_j\beta + \beta\alpha_j = 0 \\ \alpha_j\alpha_k + \alpha_k\alpha_j = 0 \quad (k \neq j) \end{gathered}}$$

Anticommutation Relations

$\downarrow$

α_i and β should be matrices

$Tr(ABC) = Tr(BCA)$ "Cyclic property"

$$Tr(\alpha_i) = Tr(\alpha_i \underbrace{\beta\beta}_{I}) = Tr(\beta \underbrace{\alpha_i \beta}_{-\beta\alpha_i}) = -Tr(\underbrace{\beta\beta}_{I}\alpha_i)$$

$$\boxed{Tr(\alpha_i) = -Tr(\alpha_i)}$$

$\downarrow$

Trace of α and β should be "0"

Eigenvalues of α_i and β $= \pm 1$

$$\alpha_i X = \lambda X$$

since $\alpha_i^2 = I$

$$X = \alpha_i^2 X = \alpha_i \underbrace{\alpha_i X}_{\lambda X} = \lambda \underbrace{\alpha_i X}_{\lambda X} = \lambda^2 X$$

Thus $\lambda^2 = 1 \rightarrow \boxed{\lambda = \pm 1}$

$$\lambda_1 + \lambda_2 = 0 = \text{Trace}$$

$$\hat{E}\Psi = (\alpha \cdot \hat{\vec{p}} + \beta m)\,\Psi$$

$$\underbrace{\hat{H}_D}_{\text{Dirac Hamiltonian}} = \alpha \cdot \hat{p} + \beta m$$

To have real eigenvalues α_i and β should be Hermitian

$$\underbrace{\alpha_x = \alpha_x^{\dagger} \quad \alpha_y = \alpha_y^{\dagger} \quad \alpha_z = \alpha_z^{\dagger} \quad \beta = \beta^{\dagger}}$$

4 mutually anticommuting Hermitian matrices of even dimension and trace zero

Therefore, the Dirac Hamiltonian is a 4x4 matrix of operators that must act on a 4-component wavefunction

Dirac Spinor

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

Conventional Choice: $\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$ and $\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Probability density and probability current

$$\psi^* \longrightarrow \psi^\dagger = (\psi^*)^T$$

$$-i\alpha_x \frac{\partial \psi}{\partial x} - i\alpha_y \frac{\partial \psi}{\partial y} - i\alpha_z \frac{\partial \psi}{\partial z} + m\beta\psi = +i\frac{\partial \psi}{\partial z} \quad (1)$$

$$\Big\downarrow \text{ Hermitian conjugate}$$

$$+i\frac{\partial \psi^\dagger}{\partial x}\alpha_x^\dagger + i\frac{\partial \psi^\dagger}{\partial y}\alpha_y^\dagger + i\frac{\partial \psi^\dagger}{\partial z}\alpha_z^\dagger + m\psi^\dagger\beta^\dagger = -i\frac{\partial \psi^\dagger}{\partial t} \quad (2)$$

$$\psi^\dagger(1) - (2)\psi$$

$$= \psi^\dagger\left(-i\alpha_x \frac{\partial \psi}{\partial x} - i\alpha_y \frac{\partial \psi}{\partial y} - i\alpha_z \frac{\partial \psi}{\partial z} + \cancel{m\beta\psi}\right)$$

$$-\left(+i\frac{\partial \psi^\dagger}{\partial x}\alpha_x^\dagger + i\frac{\partial \psi^\dagger}{\partial y}\alpha_y^\dagger + i\frac{\partial \psi^\dagger}{\partial z}\alpha_z^\dagger + \cancel{m\psi^\dagger\beta^\dagger}\right)\psi$$

$$= i\psi^\dagger\frac{\partial \psi}{\partial z} + i\frac{\partial \psi^\dagger}{\partial t}\psi$$

$$\psi^\dagger\alpha_x\frac{\partial \psi}{\partial x} + \frac{\partial \psi^\dagger}{\partial x}\alpha_x\psi = \frac{\partial(\psi^\dagger\alpha_x\psi)}{\partial x}$$

$$\psi^\dagger\frac{\partial \psi}{\partial t} + \frac{\partial \psi^\dagger}{\partial z}\psi = \frac{\partial(\psi^\dagger\psi)}{\partial t}$$

$$\nabla \cdot (\underbrace{\psi^+ \vec{\alpha} \psi}_{\vec{j}}) + \frac{\partial (\overbrace{\psi^+ \psi}^{\rho})}{\partial t} = 0$$

"Continuity Equation"

prob. current

$$\vec{j} = \psi^+ \vec{\alpha} \psi$$

prob. density

$$\rho = \psi^+ \psi$$

$$\rho = \psi^+ \psi = |\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2 + |\psi_4|^2$$

"Probability density"

Spin and the Dirac Equation

$$\frac{d\langle \hat{O} \rangle}{dt} = i \langle \psi | [\hat{H}, \hat{O}] | \psi \rangle$$

If the operator $\hat{O}$ commutes with the Hamiltonian of the system, it's a constant of the motion [conserved quantity]

Hamiltonian of the free particle

$$\underbrace{\hat{H}_{SE}}_{\text{Schrodinger}} = \frac{\hat{\vec{p}}^2}{2m}$$

$$[\hat{H}_{SE}, \underbrace{\hat{L}}_{\hat{r} \times \hat{p}}] = 0$$

$\hat{L}$ is conserved quantity in non-relativistic QM

$$\hat{H}_D = \alpha.\hat{\vec{p}} + \beta m$$

$$[\hat{H}_D, \hat{L}] = [\alpha.\hat{\vec{p}} + \beta m \,,\, \hat{\vec{r}} \times \hat{\vec{p}}] = [\alpha.\hat{p} \,,\, \hat{\vec{r}} \times \hat{\vec{p}}]$$

$$[\hat{H}_D, \hat{L}_x] = [\alpha.\hat{\vec{p}} \,,\, \underbrace{(\hat{\vec{r}} \times \hat{\vec{p}})_x}_{\hat{y}\hat{p}_z - \hat{z}\hat{p}_y}] = [\alpha_x \hat{p}_x + \alpha_y \hat{p}_y + \alpha_z \hat{p}_z \,,\, \hat{y}\hat{p}_z - \hat{z}\hat{p}_y]$$

$$[\hat{H}_D, \hat{L}_x] = \alpha_y \underbrace{[\hat{p}_y, \hat{y}]}_{-i} \hat{p}_z - \alpha_z \underbrace{[\hat{p}_z, \hat{z}]}_{-i} \hat{p}_y$$

$$= -i(\alpha_y \hat{p}_z - \alpha_z \hat{p}_y) = -i(\alpha \times \hat{\vec{p}})_x$$

$\downarrow$ Generalize this result to L

$$[\hat{H}_D, \hat{L}] = -i\alpha \times \hat{\vec{p}}$$

$\hat{L}$ does not commute with $\hat{H}_D$

Thus, $\hat{L}$ is not a conserved quantity let's define $\hat{S} = \frac{1}{2}\hat{\Sigma} = \frac{1}{2}\begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}_{4\times 4}$

$\begin{pmatrix} \vdots & \vdots \end{pmatrix}$

Pauli-spin matrices

$$[\alpha_i, \hat{\Sigma}_x] = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}\begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix} - \begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix}\begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & \sigma_i\sigma_x \\ \sigma_i\sigma_x & 0 \end{pmatrix} - \begin{pmatrix} 0 & \sigma_x\sigma_i \\ \sigma_x\sigma_i & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & [\sigma_i, \sigma_x] \\ [\sigma_i, \sigma_x] & 0 \end{pmatrix}$$

$[\sigma_x, \sigma_x] = 0$ $\quad$ $[\sigma_y, \sigma_x] = -2i\sigma_z$ $\quad$ $[\sigma_z, \sigma_x] = 2i\sigma_y$

$$[\alpha_x, \Sigma_x] = 0$$

$$[\alpha_y, \Sigma_x] = \begin{pmatrix} 0 & -2i\sigma_z \\ -2i\sigma_z & 0 \end{pmatrix} = -2i\alpha_z$$

$$[\alpha_z, \Sigma_x] = \begin{pmatrix} 0 & 2i\sigma_y \\ 2i\sigma_y & 0 \end{pmatrix} = 2i\alpha_y$$

$$[\hat{H}_D, \Sigma_x] = [\alpha \cdot \hat{\vec{p}} + \cancel{\beta m}, \Sigma_x] \qquad [\beta, \hat{\Sigma}_x] = 0$$

$$= [\alpha \cdot \hat{\vec{p}}, \Sigma_x]$$

$$= [\alpha_x \hat{p}_x + \alpha_y \hat{p}_y + \alpha_z \hat{p}_z, \Sigma_x]$$

$$= \hat{p}_x \underbrace{[\alpha_x, \hat{\Sigma}_x]}_{0} + \hat{p}_y \underbrace{[\alpha_y, \hat{\Sigma}_x]}_{-2i\sigma_z} + \hat{p}_z \underbrace{[\alpha_z, \hat{\Sigma}_x]}_{2i\sigma_y}$$

$$[\hat{H}_D, \hat{\Sigma}_x] = 2i(\alpha \times \hat{\vec{p}})_x$$

$$\Big\downarrow \text{ Generalize where } \hat{S} = \frac{1}{2}\hat{\Sigma}$$

$$[\hat{H}_D, \hat{S}] = i\alpha \times \hat{\vec{p}}$$

$\hat{S}$ also does not commute

$$\hat{J} = \hat{L} + \hat{S}$$

$$[\hat{H}_D, \hat{J}] = [\hat{H}_D, \hat{L} + \hat{S}] = -i\alpha \times \hat{\vec{p}} + i\alpha \times \hat{\vec{p}} = 0$$

conserved quantity

orbital angular mom.

intrinsic angular momentum (SPIN)

$$\hat{S}^2 = \frac{1}{4}\left(\hat{\Sigma}_x^2 + \hat{\Sigma}_y^2 + \hat{\Sigma}_z^2\right) = \frac{3}{4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\hat{S}^2 |s, m_s\rangle = s(s+1)\,|s, m_s\rangle$$

$$\hat{S}^2 \psi = s(s+1)\psi = \frac{3}{4}\psi$$

$$\hat{\mu} = \frac{q}{m}\hat{S}$$

intrinsic magnetic momentum $s = \frac{1}{2}$

Covariant form of the Dirac Equation

Pre-multiply the Dirac Eqn with β

$$\hat{E}\psi = (\alpha \cdot \hat{\vec{p}} + \beta m)\psi$$

$$i\frac{\partial}{\partial t}\psi = \left(-i\alpha_x \frac{\partial}{\partial x} - i\alpha_y \frac{\partial}{\partial y} - i\alpha_z \frac{\partial}{\partial z} + \beta m\right)\psi$$

$$i\underbrace{\beta\alpha_x}_{\gamma^1}\frac{\partial\psi}{\partial x} + i\underbrace{\beta\alpha_y}_{\gamma^2}\frac{\partial\psi}{\partial y} + i\underbrace{\beta\alpha_z}_{\gamma^3}\frac{\partial\psi}{\partial z} + i\underbrace{\beta}_{\gamma^0}\frac{\partial\psi}{\partial t} - \underbrace{\beta^2}_{I} m\psi = 0$$

The four-Dirac $\boxed{\text{matrices}}$ $\rightsquigarrow$ $\gamma^\mu = (\beta, \beta\alpha_x, \beta\alpha_y, \beta\alpha_z)$

↳ They're not 4 vectors !

$$i\gamma^0 \frac{\partial \Psi}{\partial t} + i\gamma^1 \frac{\partial \Psi}{\partial x} + i\gamma^2 \frac{\partial \Psi}{\partial y} + i\gamma^3 \frac{\partial \Psi}{\partial z} - m\Psi = 0$$

$\partial_0 \qquad \partial_1 \qquad \partial_2 \qquad \partial_3$

$$(i\gamma^\mu \partial_\mu - m)\,\psi = 0$$

Properties

$$(\gamma^0)^2 = I$$

from anti-commutation

$$(\gamma^1)^2 = (\beta\alpha_x)(\beta\alpha_x) = \underbrace{(-\alpha_x\beta)(\beta}_{I}\alpha_x) = -\underbrace{\alpha_x^2}_{I} = -I$$

$$(\gamma^k)^2 = -I \qquad (k \neq 0)$$

$$\gamma^\mu \gamma^\nu = -\gamma^\nu \gamma^\mu \qquad (\mu \neq \nu)$$

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}$$

$$(\gamma^1)^\dagger = (\beta\alpha_x)^\dagger = \alpha_x^\dagger \beta^\dagger = \alpha_x \beta = -\beta\alpha_x = -\gamma^1$$

Hermitian $\qquad\qquad$ anti-Hermitian

$$(\gamma^0)^\dagger = \gamma^0 \quad \text{and} \quad (\gamma^k)^\dagger = -\gamma^k$$

$$\gamma^0 = \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

$$\gamma^k = \beta\alpha_k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}$$

The adjoint spinor and the covariant current

$$\left.\begin{aligned} \rho &= \psi^\dagger\psi \\ j &= \psi^\dagger\alpha\psi \end{aligned}\right\} \quad j^\mu = (\rho, j) = \psi^\dagger\gamma^0\gamma^\mu\psi$$

$$\boxed{\begin{aligned} \frac{\partial\rho}{\partial t} + \nabla\cdot\vec{j} &= 0 \\ \partial_\mu j^\mu &= 0 \end{aligned}}$$

"Charge Conservation"

$$j^\mu = \underbrace{\psi^\dagger\gamma^0}_{\overline{\psi}\text{ (adjoint spinor)}}\gamma^\mu\psi \xrightarrow{\text{simplify}} j^\mu = \overline{\psi}\gamma^\mu\psi$$

$$\overline{\psi} = \psi^\dagger\gamma^0 = (\psi^*)^T\gamma^0 = (\psi_1^*, \psi_2^*, \psi_3^*, \psi_4^*)\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

$$= (\psi_1^*, \psi_2^*, -\psi_3^*, -\psi_4^*)$$

Solutions to the Dirac Equation

$$\psi(x,t) = \underbrace{u(E,p)}_{\text{4-component Dirac spinor } (u_0, u_1, u_2, u_3)} e^{i(p.x - Et)}$$

$$\boxed{(i\gamma^\mu \partial_\mu - m)\psi = 0}$$ "Dirac Equation"

Substitude

$$\partial_0 \psi = \frac{\partial \psi}{\partial t} = -iE\psi$$

$$\partial_1 \psi = \frac{\partial \psi}{\partial x} = ip_x \psi$$

$$\partial_2 \psi = \frac{\partial \psi}{\partial y} = ip_y \psi$$

$$\partial_3 \psi = \frac{\partial \psi}{\partial z} = ip_z \psi$$

$$(\gamma^0 E - \gamma^1 p_x - \gamma^2 p_y - \gamma^3 p_z - m)\, \overbrace{u(E,p) e^{i(p.x - Et)}}^{\psi} = 0$$

$$\boxed{(\gamma^\mu p_\mu - m)\, u = 0}$$

No derivative is included

→ Particle at rest

$$\psi = u(E,0)\, e^{-iEt}$$

$$\gamma^0 E u = m u$$

$$E \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} = m \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix}$$

$$\left.\begin{aligned} u_1(E,0) &= N \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\ u_2(E,0) &= N \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \end{aligned}\right\} \text{Eigenvalues } E = +m$$

$$\left.\begin{aligned} u_3(E,0) &= N \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \\ u_4(E,0) &= N \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \end{aligned}\right\} \text{Eigenvalues } E = -m$$

It is not possible to avoid negative energy solutions

$$\psi_1 = N \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} e^{-imt} \qquad \psi_2 = N \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} e^{-imt} \qquad \psi_3 = N \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} e^{+imt}$$

$$\psi_4 = N \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} e^{+imt}$$

→ General free-particle solutions

$$\left(E\gamma^0 - p_x\gamma^1 - p_y\gamma^2 - p_z\gamma^3 - m\right) u = 0$$

$$\left[E\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} - \begin{pmatrix} 0 & \vec{\sigma}.\vec{p} \\ \vec{\sigma}.\vec{p} & 0 \end{pmatrix} - m\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}\right] u = 0$$

$$\vec{\sigma}.\vec{p} = \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}$$

$$u = \begin{pmatrix} u_A \\ u_B \end{pmatrix} \qquad u_A \to \begin{pmatrix} \vdots \end{pmatrix}, \quad u_B \to \begin{pmatrix} \vdots \end{pmatrix}$$

$$\begin{pmatrix} (E-m)I & -\vec{\sigma}.\vec{p} \\ \vec{\sigma}.\vec{p} & -(E+m)I \end{pmatrix} \begin{pmatrix} u_A \\ u_B \end{pmatrix} = 0$$

$$(E-m)\,u_A - \vec{\sigma}.\vec{p}\,u_B = 0$$

$$u_A = \frac{\vec{\sigma}.\vec{p}}{E-m}\,u_B$$

$$\vec{\sigma}.\vec{p}\,u_A - (E+m)\,u_B = 0$$

$$u_B = \frac{\vec{\sigma}.\vec{p}}{E+m}\,u_A$$

① u_1 and u_2

Assume $u_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $u_A = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$u_B = \frac{\vec{\sigma}\cdot\vec{p}}{(E+m)} = \frac{1}{(E+M)} \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix} u_A$$

$$u_1(E,\vec{p}) = N_1 \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+M} \\ \frac{p_x + ip_y}{E+m} \end{pmatrix} \qquad u_2(E,\vec{p}) = N_2 \begin{pmatrix} 0 \\ 1 \\ \frac{p_x - ip_y}{E+M} \\ \frac{-p_z}{E+M} \end{pmatrix}$$

② u_3 and u_4

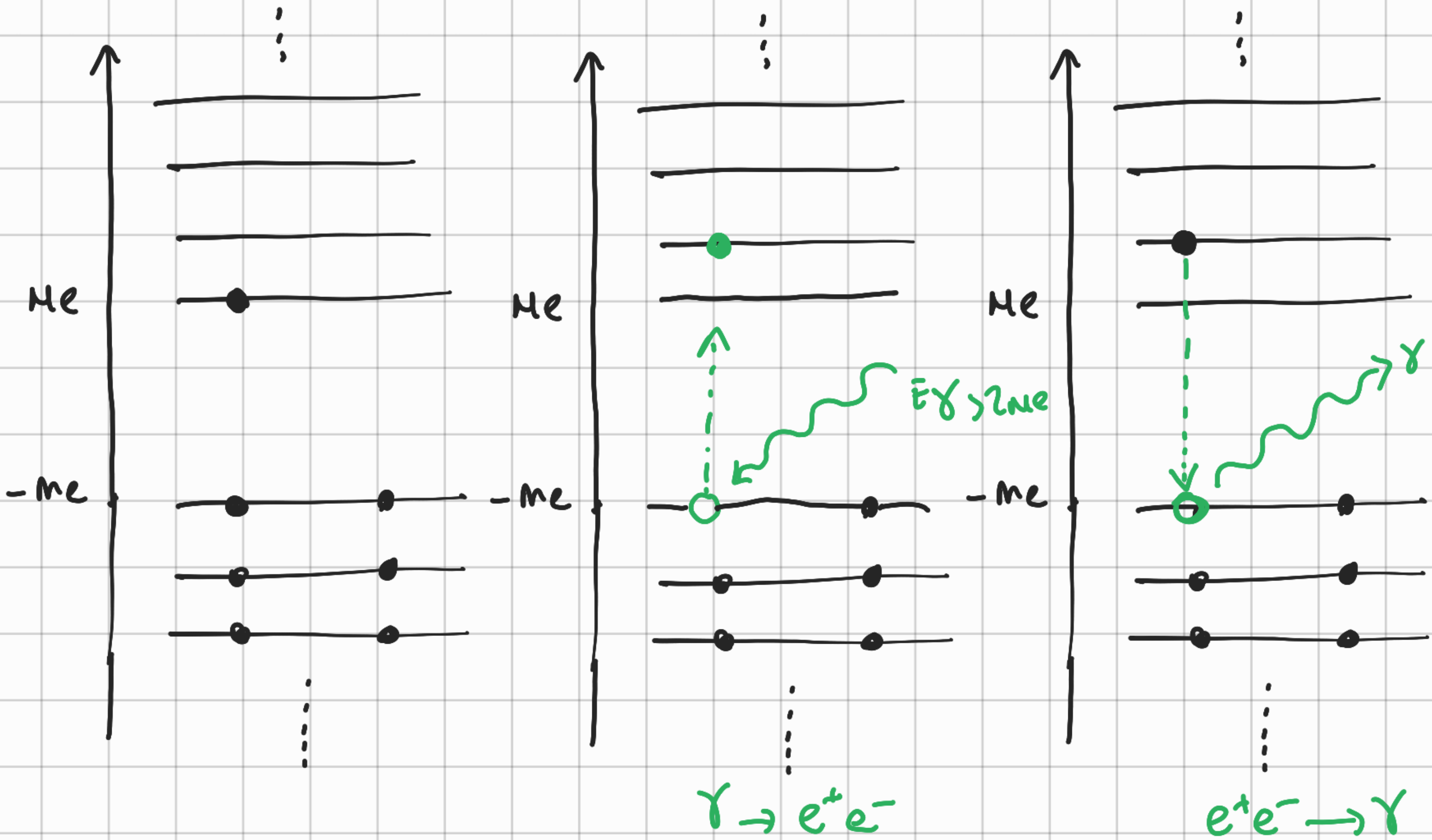
$u_B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $u_B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$u_A = \frac{\vec{\sigma}\cdot\vec{p}}{E-M} u_B$$

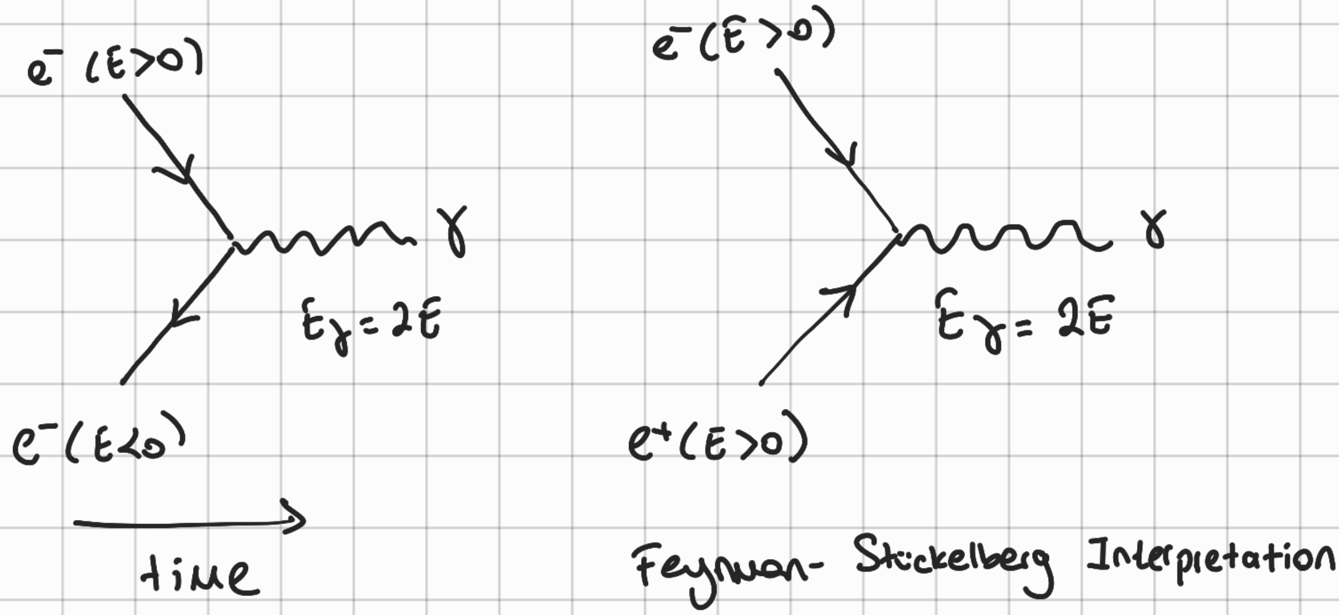
$$v_3(E,\vec{p}) = N_3 \begin{pmatrix} \frac{p_z}{E-m} \\ \frac{p_x + ip_y}{E-M} \\ 1 \\ 0 \end{pmatrix} \qquad u_4(E,\vec{p}) = N_4 \begin{pmatrix} \frac{p_x - ip_y}{E-M} \\ \frac{-p_z}{E-M} \\ 0 \\ 1 \end{pmatrix}$$

Antiparticles

The Dirac sea interpretation



The Feynman - Stückelberg Interpretation



Feynman- Stückelberg Interpretation

Time dependence of the wavefunction $\exp(-iEt)$

$$E \to -E \qquad t \longrightarrow -t$$

$$\exp(-iEt) = \exp(-i(-E)(-t))$$

Antiparticle Spinors

Write $(E<0)$ solutions v_3 and v_4 in terms of $(E>0)$ antiparticle spinors v_1 and v_2

↳ Simply reverse the sign E and $\vec{p}$

$$v_1(E,\vec{p})\, e^{-i(p\cdot x - Et)} = v_4(-E,-\vec{p})\, e^{i[(-\vec{p})\cdot\vec{x} - (-E)t]}$$

$$v_2(E,\vec{p})\, e^{-i(px - Et)} = v_3(-E,-\vec{p})\, e^{i[(-\vec{p})\cdot\vec{x} - (-E)t]}$$

$$\psi(x,t) = v(E,\vec{p})\, e^{-i(\vec{p}\cdot x - Et)}$$

↑ exp. is reversed

$$\psi(x,t) = v(\vec{E},\vec{p})\, e^{+i(\vec{p}\cdot x - Et)}$$

$$i\frac{\partial\psi}{\partial t} = -E\psi$$

Substitute the ψ into Dirac Eqn
$(i\gamma^\mu \partial_\mu - m)\psi = 0$

$$(-\gamma^0 E + \gamma^1 p_x + \gamma^2 p_y + \gamma^3 p_z - m)\, v = 0$$

$$(\gamma^\mu p_\mu + m)\, v = 0$$

$$v = \begin{pmatrix} v_a \\ v_b \end{pmatrix}$$

$$v_a = \frac{\sigma \cdot \vec{p}}{E+m} v_b \quad \text{and} \quad v_b = \frac{\vec{\sigma} \cdot \vec{p}}{E-m} v_a$$

$$v_b = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \leftarrow v_1, v_2 \rightarrow v_b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$v_a = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \leftarrow v_3, v_4 \rightarrow v_a = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$v_1 = N \begin{pmatrix} \frac{p_x - ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix} \qquad v_3 = N \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E-m} \\ \frac{p_x + ip_y}{E-m} \end{pmatrix}$$

$E = +\left|\sqrt{p^2+m^2}\right|$ $\qquad$ $E = -\left|\sqrt{p^2+m^2}\right|$

$$v_2 = N \begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x + ip_y}{E+m} \\ 1 \\ 0 \end{pmatrix} \qquad v_4 = N \begin{pmatrix} 0 \\ 1 \\ \frac{p_x - ip_y}{E-m} \\ \frac{-p_z}{E-m} \end{pmatrix}$$

Solutions : $\{ u_1, u_2, v_1, v_2 \}$ with $E>0$

$$\psi_i = u_i \, e^{+i(\vec{p}.x - Et)} \qquad (i=1,2)$$

$$u_1(p) = \sqrt{E+m} \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix} \qquad u_2(p) = \sqrt{E+m} \begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}$$

$$\psi_i = v_i \, e^{-i(\vec{p}.x - Et)}$$

$$v_1(p) = \sqrt{E+m} \begin{pmatrix} \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix} \qquad v_2(p) = \sqrt{E+m} \begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \\ 1 \\ 0 \end{pmatrix}$$

Wavefunction Normalization

* Spinors are normalized to the conventional $2E$ particles per unit volume.

<u>Ex</u>. $\psi = u_1(p) \exp i(p.x - Et) \longrightarrow \rho = \psi^{+}\psi = u_1^{+} u_1$

$$u_1^\dagger u_1 = |N|^2 \left(1 + \frac{p_z^2}{(E+m)^2} + \frac{p_x^2 + p_y^2}{(E+m)^2} \right) = |N|^2 \frac{2E}{E+m}$$

$$\frac{(E+m)^2 + \vec{p}^2}{(E+m)^2} = \frac{E^2 + 2Em + \underbrace{m^2 + \vec{p}^2}^{E^2}}{(E+m)^2} = \frac{2E^2 + 2Em}{(E+m)^2}$$

$$= \frac{2E\,\cancel{(E+m)}}{(E+m)^{\cancel{2}}}$$

Thus to normalize ψ to $2E$ particles per unit volume

$$\hookrightarrow N = \sqrt{E+m}$$

Operators and the antiparticle spinors

$$\psi = v(E,p)\, e^{-i(p \cdot x - Et)}$$

$$\hat{H}\psi = i\frac{\partial \psi}{\partial t} = -E\psi \qquad \hat{p}\psi = -i\nabla\psi = -\vec{p}\,\psi$$

Thus, define new operators for anti-particle spinors

$$\hat{H}^{(v)} = -i\frac{\partial}{\partial t} \qquad \hat{\vec{p}}^{(v)} = +i\nabla$$

$$\hat{L} = \vec{r} \times \vec{p} \longrightarrow -\hat{L} = \hat{L}^{(v)}$$

$$\hat{S}^{(v)} = -S$$

Charge Conjugation

Electromagnetic field $A^{\mu} = (\phi, \vec{A})$

$$E \to E - q\phi \qquad \vec{p} = \vec{p} - q\vec{A}$$

(energy) $\quad$ charge of the particle

$$p_{\mu} \longrightarrow p_{\mu} - qA_{\mu} \qquad \text{(4-vector notation)}$$

QM version

$$i\partial_{\mu} \longrightarrow i\partial_{\mu} - qA_{\mu} \qquad \text{where } q = -e \text{ (electron charge)}$$

$$(i\gamma^{\mu}\partial_{\mu} - m)\psi = 0 \rightarrow \text{Dirac Equation}$$

$$\gamma^{\mu}(i\partial_{\mu})\psi - m\psi = 0$$

$$(-i) \times \gamma^{\mu}(i\partial_{\mu} - qA_{\mu})\psi - m\psi = 0$$

$$\gamma^{\mu}(+\partial_{\mu} - ieA_{\mu})\psi + im\psi = 0 \rightarrow \text{Dirac Eqn for } e^-$$

take the complex conjugate

$$(-i\gamma^2)\left[(\gamma^{\mu})^*(\partial_{\mu} + ieA_{\mu})\psi^* - im\psi\right] = 0$$

$$-i\gamma^2(\gamma^{\mu})^*(\partial_{\mu} + ieA_{\mu})\psi^* - m\gamma^2\psi = 0$$

$$(\gamma^0)^* = \gamma^0 \qquad (\gamma^1)^* = \gamma^1 \qquad (\gamma^2)^* = -\gamma^2 \qquad (\gamma^3)^* = \gamma^3$$

due to σ_y

$$\gamma^2 \gamma^\mu = -\gamma^\mu \gamma^2 \quad \text{for } \mu \neq 2$$

$$-i\gamma^2(\gamma^\mu)^*(\partial_\mu + ieA_\mu)\psi^* - m\gamma^2\psi = 0$$

for $\mu = 2$ $\quad -i\gamma^2\underbrace{(\gamma^2)^*}_{-\gamma^2}(\partial_\mu + ieA_\mu)\psi^* - m\gamma^2\psi = 0$

$$+i(-I)(\partial_\mu + ieA_\mu)\psi^* - m\gamma^2\psi = 0$$

for $\mu \neq 2$ $\quad -i\gamma^2\gamma^\mu(\partial_\mu + ieA_\mu)\psi^* - m\gamma^2\psi = 0$

$$\gamma^\mu(\partial_\mu + ieA_\mu)\underbrace{(i\gamma^2)\psi^*}_{\psi'} + im\underbrace{i\gamma^2\psi^*}_{\psi'} = 0$$

Define $\psi' = i\gamma^2\psi^*$

$$\boxed{\gamma^\mu(\partial_\mu + ieA_\mu)\psi' + im\psi' = 0}$$

anti-particle wavefunction

$$\boxed{\psi' = \hat{C}\psi = i\gamma^2\psi^*}$$

charge conjugation operator

Ex

$$\psi = u_1 e^{i(\vec{p}\cdot x - Et)}$$

$$\psi' = \hat{C}\psi = i\gamma^2\psi^* = i\gamma^2 u_1^* e^{-i(\vec{p}\cdot x - Et)}$$

$$i\gamma^2 u_1^* = \cancel{i}\begin{pmatrix}0&0&0&\overset{+1}{\cancel{-i}}\\0&0&\overset{-1}{\cancel{i}}&0\\0&\overset{-1}{\cancel{i}}&0&0\\\underset{+1}{\cancel{-i}}&0&0&0\end{pmatrix}\sqrt{E+m}\begin{pmatrix}1\\0\\\frac{p_z}{E+m}\\\frac{p_x+ip_y}{E+m}\end{pmatrix}^* = \underbrace{\sqrt{E+m}\begin{pmatrix}\frac{p_x-ip_y}{E+m}\\\frac{-p_z}{E+m}\\0\\1\end{pmatrix}}_{v_1}$$

(anti-particle spinor)

$$\psi = u_1 e^{i(\vec{p}\cdot x - Et)} \xrightarrow{\hat{C}} v_1 e^{-i(\vec{p}\cdot x - Et)}$$

$$\psi = u_2 e^{i(\vec{p}\cdot x - Et)} \xrightarrow{\hat{C}} v_2 e^{-i(\vec{p}\cdot x - Et)}$$

Spin and Helicity States

$$\hat{S}_z = \frac{1}{2}\Sigma_z = \frac{1}{2}\begin{pmatrix}\sigma_z & 0\\ 0 & \sigma_z\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1&&&\\&-1&&\\&&1&\\&&&-1\end{pmatrix}$$

→ For particle at rest $\underset{\text{spin-up}}{u_1(E,0)}$ and $\underset{\text{spin-down}}{u_2(E,0)}$ are eigenstates of $\hat{S}_z$.

→ However general solutions $\{u_1, u_2, v_1, v_2\}$ spinors are not eigenstates of $\hat{S}_z$.

For the particles/antiparticles travelling in the $\pm \hat{z}$ direction $\vec{p} = \pm p\hat{z}$

$$u_1(p) = \sqrt{E+m}\begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \cancel{\frac{p_x+ip_y}{E+m}} \to 0 \end{pmatrix} \qquad u_2(p) = \sqrt{E+m}\begin{pmatrix} 0 \\ 1 \\ \cancel{\frac{p_x-ip_y}{E+m}} \to 0 \\ \frac{-p_z}{E+m} \end{pmatrix}$$

$$v_1(p) = \sqrt{E+m}\begin{pmatrix} \cancel{\frac{p_x-ip_y}{E+m}} \to 0 \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix} \qquad v_2(p) = \sqrt{E+m}\begin{pmatrix} \frac{p_z}{E+m} \\ \cancel{\frac{p_x+ip_y}{E+m}} \to 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\hat{S}_z\, u_1(E,0,0,\pm p) = +\frac{1}{2}\, u_1(E,0,0,\pm p) \rightarrow \text{SPIN-UP}$$

$$\hat{S}_z\, u_2(E,0,0,\pm p) = -\frac{1}{2}\, u_2(E,0,0,\pm p) \rightarrow \text{SPIN DOWN}$$

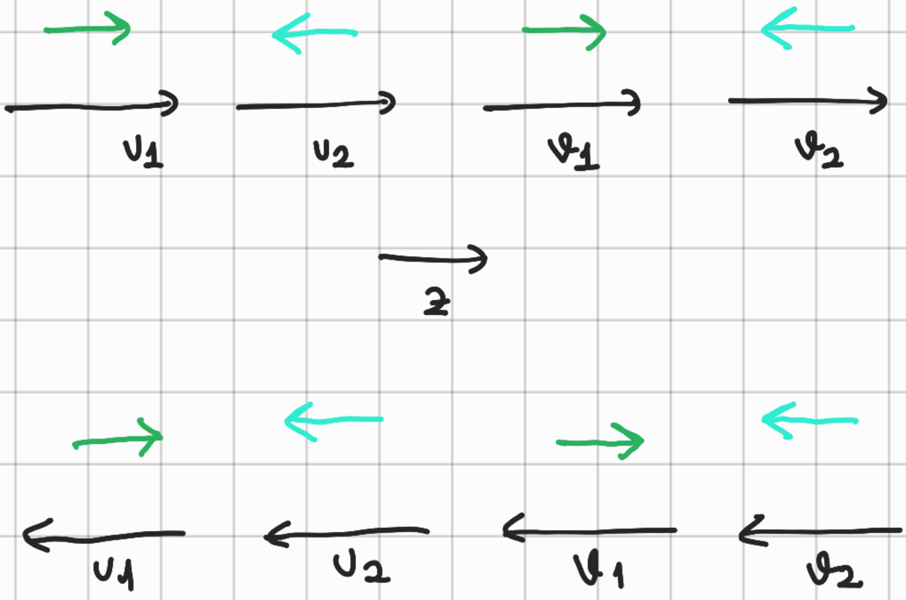
for the anti-particles

↘

$$S_z^{(v_1)}\, v_1(E,0,0,\pm p) = -\hat{S}_z\, v_1(E,0,0,\pm p) = +\frac{1}{2}\, v_1(E,0,0,\pm p) \rightarrow \text{SPIN-UP}$$

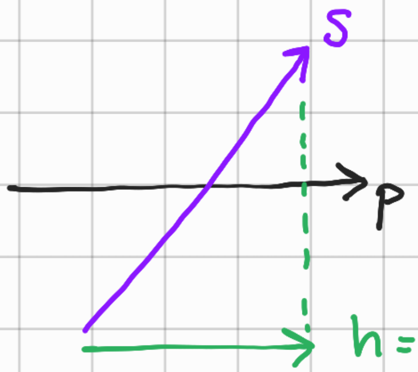
$$S_z^{(v_2)}\, v_2(\qquad) = -\hat{S}_z\, v_2(\qquad) = -\frac{1}{2}\, v_2(\ldots\ldots)$$

↘ SPIN DOWN



Helicity

$[\hat{H}_D, \hat{S}_z] \neq 0$ ⇝ It is NOT possible to define a basis of simultaneous eigenstates of $\hat{S}_z$ and $\hat{H}_D$.

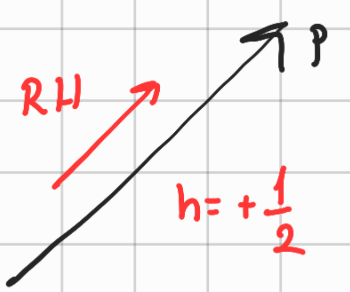


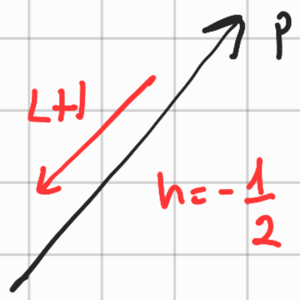
$h = \dfrac{\vec{S}\cdot\vec{p}}{p}$

$$[\hat{H}_D, \hat{\vec{\Sigma}}\cdot\hat{\vec{p}}] = 0$$

commutes with $\hat{H}_D$

$$\hat{h} = \frac{\hat{\vec{\Sigma}}\cdot\hat{\vec{p}}}{2p} = \frac{1}{2p}\begin{pmatrix}\vec{\sigma}\cdot\vec{p} & 0 \\ 0 & \vec{\sigma}\cdot\vec{p}\end{pmatrix}$$





$$\hat{h}\, u = \lambda u$$

$$\frac{1}{2p}\begin{pmatrix} \vec{\sigma}.\vec{p} & 0 \\ 0 & \vec{\sigma}.\vec{p} \end{pmatrix}\begin{pmatrix} u_A \\ u_B \end{pmatrix} = \lambda \begin{pmatrix} u_A \\ u_B \end{pmatrix}$$

$$(\vec{\sigma}.\vec{p})\times \boxed{(\vec{\sigma}.\vec{p})\, u_A = 2p\lambda u_A}$$

$$(\vec{\sigma}\times\vec{p})\times \; (\vec{\sigma}.\vec{p})\, u_B = 2p\lambda u_B$$

$$(\vec{\sigma}.\vec{p})^2 = p^2$$

$$(\vec{\sigma}.\vec{p})^2 = \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}\begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}$$

$$= \begin{pmatrix} p_z^2 + p_x^2 + p_y^2 & 0 \\ 0 & p_z^2 + p_x^2 + p_y^2 \end{pmatrix} = p^2 . I = p^2$$

$$p^2 u_A = 2p\lambda \underbrace{(\vec{\sigma}.\vec{p})\, u_A}_{2p\lambda u_A} = 4p^2\lambda^2 u_A$$

$$4\lambda^2 = 1 \rightarrow \lambda = \pm 1/2$$

$$\hat{h}\, u = \pm \frac{1}{2} u \quad \rightarrow h = \pm \frac{1}{2}$$

* Since the spinors corresponding to the two helicity states are also eigenstates of the Dirac Eqn.

$$\underbrace{(\vec{\sigma}\cdot\vec{p})\,u_A}_{2p\lambda u_A} = (E+m)\,u_B$$

$$u_B = 2\lambda\left(\frac{p}{E+m}\right)u_A$$

$$\vec{p} = (p\sin\theta\cos\phi\,,\ p\sin\theta\sin\phi\,,\ p\cos\theta)$$

$$\frac{1}{2p}(\vec{\sigma}\cdot\vec{p}) = \frac{1}{2p}\begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta \end{pmatrix}$$

Let $u_A = \begin{pmatrix} a \\ b \end{pmatrix}$ $\rightsquigarrow$ $u_B = 2\lambda\left(\frac{p}{E+m}\right)u_A$

$$\frac{1}{2p}(\vec{\sigma}\cdot\vec{p})\,u_A = \lambda u_A$$

$$\frac{1}{2}\begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \lambda\begin{pmatrix} a \\ b \end{pmatrix}$$

$$a\cos\theta + b\sin\theta e^{-i\phi} = 2\lambda a$$

$$a(2\lambda - \cos\theta) = b\sin\theta e^{-i\phi}$$

$$\frac{b}{a} = \frac{(2\lambda - \cos\theta)}{\sin\theta} e^{i\phi}$$

For RH $\rightarrow h = +\frac{1}{2} \rightarrow \lambda = +\frac{1}{2}$

$$\frac{b}{a} = \frac{1-\cos\theta}{\sin\theta} e^{i\phi} = \frac{2\sin^2\left(\frac{\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)} e^{i\phi} = e^{i\phi}\frac{\sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)}$$

$$u_\uparrow = \sqrt{E+m}\begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ e^{i\phi}\sin\left(\frac{\theta}{2}\right) \\ \frac{p}{E+m}\cos\left(\frac{\theta}{2}\right) \\ \frac{p}{E+m}e^{i\phi}\sin\left(\frac{\theta}{2}\right) \end{pmatrix}$$

$$a\sin\theta e^{i\phi} - b\cos\theta = 2\lambda b$$

$$b(2\lambda + \cos\theta) = a\sin\theta e^{i\phi}$$

$$\frac{b}{a} = \frac{\sin\theta e^{i\phi}}{(2\lambda + \cos\theta)} \xrightarrow{\text{for } \lambda = -\frac{1}{2}} \frac{b}{a} = \frac{\sin\theta e^{i\phi}}{-1+\cos\theta}$$

$h = -\frac{1}{2}$

$$\frac{b}{a} = \frac{\cancel{2\sin\left(\frac{\theta}{2}\right)}\cos\left(\frac{\theta}{2}\right)}{-\cancel{2}\sin^{\cancel{2}}\left(\frac{\theta}{2}\right)} e^{i\phi} = -\frac{\cos\left(\frac{\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)} e^{i\phi}$$

$$u_B = 2\lambda \left(\frac{p}{E+m}\right)^{-\frac{1}{2}} u_A = -\left(\frac{p}{E+m}\right) u_A$$

$$u_\downarrow = \sqrt{E+m} \begin{pmatrix} -\sin\left(\frac{\theta}{2}\right) \\ \cos\left(\frac{\theta}{2}\right) e^{i\phi} \\ \left(\frac{p}{E+m}\right) \sin\left(\frac{\theta}{2}\right) \\ -\frac{p}{E+m} \cos\left(\frac{\theta}{2}\right) e^{i\phi} \end{pmatrix}$$

For antiparticle $\hat{S}^{(v)} = -\hat{S}$, hence for the $h = +1/2$ antiparticle state

$$\left(\frac{\Sigma \cdot p}{2p}\right) v_\uparrow = -\frac{1}{2} v_\uparrow$$

$$v_\uparrow = \sqrt{E+m} \begin{pmatrix} \frac{p}{E+m} \sin\left(\frac{\theta}{2}\right) \\ -\frac{p}{E+m} \cos\left(\frac{\theta}{2}\right) e^{i\phi} \\ -\sin\left(\frac{\theta}{2}\right) \\ \cos\left(\frac{\theta}{2}\right) e^{i\phi} \end{pmatrix} \qquad v_\downarrow = \sqrt{E+m} \begin{pmatrix} \frac{p}{E+m} \cos\left(\frac{\theta}{2}\right) \\ \frac{p}{E+m} \sin\left(\frac{\theta}{2}\right) e^{i\phi} \\ \cos\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right) e^{i\phi} \end{pmatrix}$$

For the ultra-relativistic limit ($E>>m$)

helicity eigenstates become: $\frac{p}{E+\cancel{m}} = \frac{\sqrt{E^2-\cancel{m^2}}}{E} = \frac{E}{E} \simeq 1$

$$u_\uparrow \approx \sqrt{E}\begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\left(\frac{\theta}{2}\right)e^{i\phi} \\ \cos\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right)e^{i\phi} \end{pmatrix} \qquad u_\downarrow \approx \sqrt{E}\begin{pmatrix} -\sin\left(\frac{\theta}{2}\right) \\ \cos\left(\frac{\theta}{2}\right)e^{i\phi} \\ \sin\left(\frac{\theta}{2}\right) \\ -\cos\left(\frac{\theta}{2}\right)e^{i\phi} \end{pmatrix}$$

$$v_\uparrow \approx \sqrt{E}\begin{pmatrix} \sin\left(\frac{\theta}{2}\right) \\ -\cos\left(\frac{\theta}{2}\right)e^{i\phi} \\ -\sin\left(\frac{\theta}{2}\right) \\ \cos\left(\frac{\theta}{2}\right)e^{i\phi} \end{pmatrix} \qquad v_\downarrow \approx \sqrt{E}\begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right)e^{i\phi} \\ \cos\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right)e^{i\phi} \end{pmatrix}$$

Intrinsic Parity of Dirac fermions

$$x' = -x \quad , \quad y' = -y \, , \quad z' = -z \, , \quad t' = t$$

$$\psi \longrightarrow \psi' = \hat{P}\psi$$

From the definition $\hat{P}^2 = I$

$$\psi' = \hat{P}\psi \implies \hat{P}\psi' = \psi$$

Consider a wavefunction $\psi'(x',y',z',t') = \hat{P}\psi(x,y,z,t)$

$$i\gamma^1 \frac{\partial \psi}{\partial x} + i\gamma^2 \frac{\partial \psi}{\partial x} + i\gamma^3 \frac{\partial \psi}{\partial z} - m\psi = -i\gamma^0 \frac{\partial \psi}{\partial t}$$

$$\downarrow$$

$$i\gamma^1 \frac{\partial \psi'}{\partial x'} + i\gamma^2 \frac{\partial \psi'}{\partial y'} + i\gamma^3 \frac{\partial \psi'}{\partial z'} - m\psi' = -i\gamma^0 \frac{\partial \psi'}{\partial t'}$$

$\psi = \hat{P}\psi'$

$x' = -x \quad y' = -y \quad z' = -z \quad t' = t$

$$\gamma^0 \times \left(-i\gamma^1 \hat{P} \frac{\partial \psi'}{\partial x'} - i\gamma^2 \hat{P} \frac{\partial \psi'}{\partial y'} - i\gamma^3 \hat{P} \frac{\partial \psi'}{\partial z'} - m\hat{P}\psi' \right) = \left(-i\gamma^0 \hat{P} \frac{\partial \psi'}{\partial t'} \right) \times \gamma^0$$

$(\gamma^0\gamma^k = -\gamma^k\gamma^0)$

$$-i\underbrace{\gamma^0\gamma^1}_{-\gamma^1\gamma^0} \hat{P} \frac{\partial \psi'}{\partial x'} - i\underbrace{\gamma^0\gamma^2}_{-\gamma^2\gamma^0} \hat{P} \frac{\partial \psi'}{\partial y'} - i\underbrace{\gamma^0\gamma^3}_{-\gamma^3\gamma^0} \hat{P} \frac{\partial \psi'}{\partial z'} - m\gamma^0\hat{P}\psi' = -i\underbrace{\gamma^0\gamma^0}_{I} \hat{P} \frac{\partial \psi'}{\partial t'}$$

Thus $\gamma^0 \hat{P} \propto I$
proportional

Also by definition $\hat{P}^2 = I$

$\hat{P} = +\gamma^0$ or $\hat{P} = -\gamma^0$

by convention

$$\psi \rightarrow \hat{P}\psi = \gamma^0 \psi$$

For example u_1 spinor for a particle at rest

$$u_1 = \sqrt{E+m}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \sqrt{2m}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$$

$p=0 \rightarrow E=m \quad \sqrt{E+m} = \sqrt{2m}$

$$\hat{P}u_1 = \gamma^0 u_1 = \begin{pmatrix}1&&&\\&1&&\\&&-1&\\&&&-1\end{pmatrix}\underbrace{\sqrt{2m}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}}_{u_1} = +u_1$$

$$\left.\begin{aligned}\hat{P}u_2 &= +u_2\\ \hat{P}v_1 &= -v_1\\ \hat{P}v_2 &= -v_2\end{aligned}\right\} \quad \begin{aligned}\hat{P}u(m,0) &= +u(m,0)\\ \hat{P}v(m,0) &= -v(m,0)\end{aligned}$$

} Intrinsic parity of a fundamental spin-half particle is opposite of antiparticles

$$\hat{P}u_1(E,\vec{p}) = +u_1(E,-\vec{p})$$

$\hat{P}$ operator reverses the $\vec{p}$ but does not revert the spin state