Introduction to Quantum Algorithms

Quantum Oracles

1. Consider an oracle U_f such that $U_f|x\rangle|y\rangle = |x\rangle|y \oplus f(x)\rangle$. Given $f(x) : \{0,1\} \to \{0,1\}$, write down the explicit circuits that implement U_f for the four different possible choices in f.

Deutsch-Jozsa

Deutsch's problem. Bob has a function $f(x) : \{0, 1, \dots, 2^n - 1\} \to \{0, 1\}$. He promises Alice that the function is either *constant*, i.e. either all outputs are 0 or all outputs are 1, or *balanced*, i.e. exactly half the outputs are 0 and half the outputs are 1. Classically, Alice can query the value of $f(x)$ for one input x at a time.

2. In the worst case, how many queries will it take for Alice to determine classically with certainty whether f is constant or balanced?

The Deutsch-Jozsa algorithm is a quantum algorithm that solves Deutsch's problem using a single query. The circuit to implement the Deutsch-Jozsa algorithm is shown below:

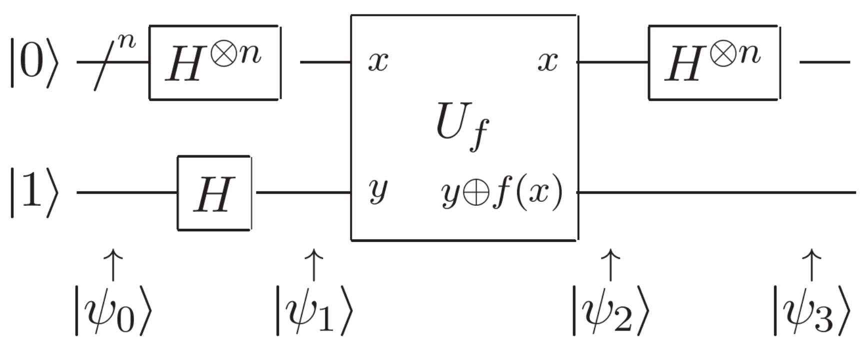


FIG. 1. **Deutsch-Jozsa Algorithm.**

3. Compute the states $|\psi_1\rangle$ and $|\psi_2\rangle$.

4. Show that $H|x\rangle = \frac{1}{\sqrt{2}} \sum_z (-1)^{xz} |z\rangle$. Hence show that $H^{\otimes n}|x_1, \dots, x_n\rangle = \frac{1}{\sqrt{2^n}} \sum_{z_1,\dots,z_n} (-1)^{x_1 z_1 + \dots + x_n z_n} |z_1, \dots, z_n\rangle$.

5. Use the previous identity to show that $|\psi_3\rangle = \frac{1}{2^n} \sum_z \sum_x (-1)^{x_1 z_1 + \dots + x_n z_n + f(x)} |z_1, \dots, z_n\rangle |-\rangle$.

6. $|\psi_3\rangle$ is measured in the computational basis. What is the probability of measuring the all zero state if f is a constant function? What is the probability of measuring the all zero state if f is a balanced function? What can we conclude from this?

1) $U_f |x\rangle |y\rangle = |x\rangle |y \oplus f(x)\rangle$

$f(x) : \{0,1\} \rightarrow \{0,1\}$

x	$f(x)$
0	0
0	1
1	0
1	1

$\|x\rangle\|y\rangle$	$f(x)$	$\|x\rangle\|y\oplus f(x)\rangle$	
0 0	0	$0\ \|0\oplus 0\rangle$	0 0
0 0	1	$0\ \|0\oplus 1\rangle$	0 1
0 1	0	$0\ \|1\oplus 0\rangle$	0 1
0 1	1	$0\ \|1\oplus 1\rangle$	0 0
1 0	0	$1\ \|0\oplus 0\rangle$	1 0
1 0	1	$1\ \|0\oplus 1\rangle$	1 1
1 1	0	$1\ \|1\oplus 0\rangle$	1 1
1 1	1	$1\ \|1\oplus 1\rangle$	1 0

If $f(x)=0$ $\quad |xy\rangle \rightarrow |xy\rangle$

no change

If $f(x)=1$ $\quad |xy\rangle \rightarrow |x\bar{y}\rangle$

y is flipped

Solution:

let choose, $f=0$, $f=1$, $f(x)=x$, $f(x)=x\oplus 1$

$U_f |x\rangle|y\rangle = |x\rangle|y\oplus f(x)\rangle$

$f=0$

$U_f = \mathbb{1} \otimes \mathbb{1}$ (identity)

$f=1$

$U_f = \mathbb{1} \otimes X$ on the second qubit

$f(x)=x$

$\left.\begin{array}{l} f(0)=0 \\ f(1)=1 \end{array}\right\}$ $U_f = CNOT$

$f(x) = x\oplus 1$

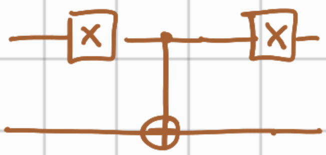
$x \Rightarrow 0 \quad f(0)=1 \rightarrow NOT$

$x \Rightarrow 1 \quad f(1)=0 \rightarrow I$

y

1st qubit

$(X\otimes I)\ CNOT\ (X\otimes I)$

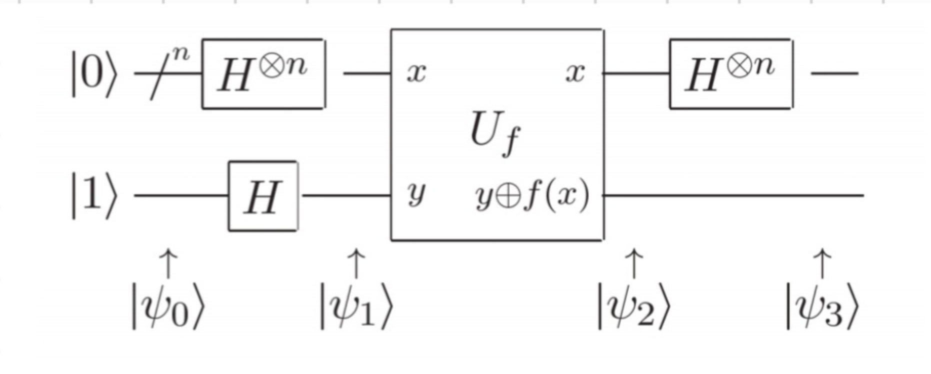


2) Deutsch's Problem:

Classically the worst case: $\frac{2^n}{2} + 1 = 2^{n-1}+1$

Ex. Half of them are 0. If the next one is 1 $\rightarrow$ balanced

0 $\rightarrow$ constant



3) $|\psi_1\rangle = \frac{1}{\sqrt{2^n}} \sum_{\cancel{x}=1}^{2^n} |\cancel{x}\rangle \otimes \frac{(|0\rangle - |1\rangle)}{\sqrt{2}} = \frac{(|0\rangle + |1\rangle + \dots\dots |2^n\rangle)}{\sqrt{2^n}} \otimes \frac{(|0\rangle - |1\rangle)}{\sqrt{2}}$

$|\psi_2\rangle = \frac{1}{\sqrt{2^n}} \sum_{i=1}^{f} |i\rangle \otimes (-1)^{f(x)} \frac{(|0\rangle - |1\rangle)}{\sqrt{2}}$

$f(x) = 0 \qquad \frac{(|0\rangle - |1\rangle)}{\sqrt{2}} \longrightarrow \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$

$U_f |x\rangle|-\rangle = (-1)^{f(x)} |x\rangle|-\rangle$

phase kickback

$f(x) = 1 \qquad \frac{(|0\rangle - |1\rangle)}{\sqrt{2}} \longrightarrow \left(\frac{|1\rangle - |0\rangle}{\sqrt{2}}\right)$

$\rightarrow |\psi_2\rangle = U_f |\psi_1\rangle = \frac{1}{\sqrt{2^n}} \sum_x (-1)^{f(x)} |x\rangle$

4) $$H|x\rangle = \frac{1}{\sqrt{2}} \sum_{z \to 0/1} (-1)^{xz} |z\rangle \quad \checkmark$$

$$\frac{1}{\sqrt{2}} \sum (-1)^{x_1 z_1} |z_1\rangle \otimes \frac{1}{\sqrt{2}} \sum (-1)^{x_2 z_2} |z_2\rangle$$

$$H|0\rangle = \frac{1}{\sqrt{2}} \sum_{z=0,1} (-1)^0 |z\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)$$

$$H|1\rangle = \frac{1}{\sqrt{2}} \sum_{z=0,1} (-1)^{1z} |z\rangle = \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle)$$

$$H^{\otimes n} |x_1 x_2 \ldots\ldots x_n\rangle = \frac{1}{\sqrt{2^n}} \sum_{z_1 \ldots z_n} (-1)^{x_1 z_1 + x_2 z_2 + \ldots + x_n z_n} |z_1 \ldots\ldots z_n\rangle$$

$$H^{\otimes n} |x_1, \ldots, x_n\rangle = \bigotimes_{i=1}^{n} \frac{1}{\sqrt{2}} \sum_{z_i} (-1)^{x_i z_i} |z_i\rangle$$

tensor product

$$\frac{1}{\sqrt{2}} (|0\rangle + (-1)^{x_1} |1\rangle) \frac{1}{\sqrt{2}} (|0\rangle + (-1)^{x_2} |1\rangle) \ldots \frac{1}{\sqrt{2}} (|0\rangle + (-1)^{x_n} |1\rangle)$$

$$= \frac{1}{2^{n/2}} \sum_{z_1, \ldots, z_n} (-1)^{x_1 z_1 + \ldots + x_n z_n} |z_1, \ldots, z_n\rangle$$

$$= H^{\otimes n} |x\rangle = \frac{1}{2^{n/2}} \sum_{z \in \{0,1\}^{\otimes n}} (-1)^{x \cdot z} |z\rangle$$

5) $$|\psi_3\rangle = H^{\otimes n} \otimes I |\psi_2\rangle = \frac{1}{2^{n/2}} \underbrace{\sum_{x \in \{0,1\}^{\otimes n}}}_{} (-1)^{f(x)} \underbrace{H^{\otimes n} |x\rangle}_{} |-\rangle$$

$$= \frac{1}{2^{n/2}} \sum_{x \in \{0,1\}^{\otimes n}} (-1)^{f(x)} \frac{1}{2^{n/2}} \sum_{z \in \{0,1\}^{\otimes n}} (-1)^{x \cdot z} |z\rangle |-\rangle$$

$$|\psi_3\rangle = \frac{1}{2^n} \sum_{x,z \in \{0,1\}^{\otimes n}} (-1)^{f(x) + xz} |z\rangle |-\rangle$$

6. $|\psi_3\rangle$ is measured in the computational basis. What is the probability of measuring the all zero state if f is a constant function? What is the probability of measuring the all zero state if f is a balanced function? What can we conclude from this?

$$|\psi_3\rangle = \frac{1}{2^n} \sum_{x,z\in\{0,1\}^{\otimes n}} (-1)^{f(x)+xz} |z\rangle|-\rangle$$

$$P(x=0) = \left(\frac{1}{2^n} \sum_{x\in\{0,1\}^{\otimes n}} (-1)^{f(x)}\right)^2$$

$f(x): \{0,1,\ldots\ldots,2^{n-1}\}$

$\downarrow$

$\{0,1\}$

constant: all 0 or all 1

balanced: half 0 half 1

If f is constant
$$P(x=0) = \frac{1}{2^n} \sum_{x\in\{0,1\}^{\otimes n}} \overbrace{(-1)^0}^{1} = \left(\frac{2^n}{2^n}\right)^2 = 1$$

$$P(x=1) = \frac{1}{2^n} \sum_{x\in\{0,1\}^{\otimes n}} (-1)^1 = \left|\frac{-2^n}{2^n}\right|^2 = 1$$

if f is balanced
$$P(x=0) = \frac{1}{2^n} \sum (-1)^x = \underbrace{\underbrace{-1-1\ldots}_{\text{half}(-1)} \; \underbrace{+1+1}_{\text{half}(+1)}}_{0}$$