Solid state systems for quantum information, Session 1

Assistants : guillaume.beaulieu@epfl.ch, vincent.jouanny@epfl.ch

1 Exercises

Exercise 1 : Pauli algebra and the Bloch sphere representation

A qubit (short for quantum bit) is the basic unit of information in quantum computing. Quantum mechanical systems in which we can isolate two energy levels, labeled $|0\rangle$ (ground state) and $|1\rangle$ (excited state), can be used as qubits. The two eigenstates associated with these two energy levels span a 2D space. In this space, the behavior of the qubit can be described by using a set of 2D matrices called Pauli Spin Matrices. As we will see throughout the course, Pauli Spin matrices are useful to describe the Hamiltonian of two level systems, construct quantum gates and represent the qubit state. The goal of this exercise is to introduce the Pauli algebra and understand how it can be used to represent a qubit state.

The three Pauli Spin matrices σ_x, σ_y and σ_z are defined as:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

1. Show that these matrices are Hermitian (i.e $A = A^\dagger$) and unitary (i.e $I = A^\dagger A = A^\dagger A$) . Find their eigenvectors and eigenvalues.

2. Show that the Pauli matrices satisfy the following commutation relation: $[\sigma_x, \sigma_y] = 2i\sigma_z$, and find similar expressions for $[\sigma_y, \sigma_z]$ and $[\sigma_z, \sigma_x]$.

3. The three Pauli matrices can be used to define a Pauli vector $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)^T$ in a 3D space. The Pauli vector pointing in a direction $\boldsymbol{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)^T$ in spherical coordinates is defined as :

$$\sigma_n = \boldsymbol{\sigma} \cdot \boldsymbol{n} = \sin\theta\cos\phi\sigma_x + \sin\theta\sin\phi\sigma_y + \cos\theta\sigma_z$$

   Show that the eigenvalues of σ_n are $\lambda = \pm 1$ and that the eigenstate associated to the eigenvalue $\lambda = 1$ can be written as $|\psi_n\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)e^{i\phi}|1\rangle$ where $|0\rangle = (1,0)^T$ and $|1\rangle = (0,1)^T$. Useful identities : $\sin^2\theta = \frac{1-\cos 2\theta}{2}$ and $\sin 2\theta = 2\sin\theta\cos\theta$.

4. The previous question showed that every point (θ, ϕ) on the surface of a unit sphere could represent a unique state $|\psi_n\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)e^{i\phi}|1\rangle$ of a two dimensional Hilbert space. The unit sphere is commonly called the Bloch sphere and each point (θ, ϕ) is associated to a unit vector called a Bloch vector. The Bloch vector in Cartesian coordinates is expressed by $\boldsymbol{r} = \langle\boldsymbol{\sigma}\rangle = (\langle\sigma_x\rangle, \langle\sigma_y\rangle, \langle\sigma_z\rangle)^T$. Graphically visualize the Bloch vector on the Bloch sphere for $\theta = \pi/4$ and $\phi = \pi/2$. How would you represent the eigenstate of σ_x on the bloch sphere ?

1. $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_y^{\dagger} = \begin{pmatrix} 0 & (i)^* \\ (-i)^* & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \sigma_y$

$$\left.\begin{array}{l} \sigma_x = \sigma_x^{\dagger} \\ \sigma_z = \sigma_z^{\dagger} \end{array}\right\} \text{HERMITIAN}$$

$$I = A^{\dagger}A = AA^{\dagger} = \sigma_x^2 = \sigma_y^2 = \sigma_z^2$$

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$\det(A - \lambda I) = 0$ for eigenvalues

$$\det(\sigma_x - \lambda I) = 0$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} -\lambda & 0 \\ 0 & -\lambda \end{pmatrix} = \begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} \xrightarrow{\det} \lambda^2 - 1 = 0$$

$$\lambda^2 = 1$$

$$\lambda_{\pm} = \pm 1$$

For λ_+, find v_+

$\sigma_x v_+ = \lambda_+ v_+ \qquad v_+ = \begin{pmatrix} a \\ b \end{pmatrix} \quad v_+ = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}}_{\begin{pmatrix} b \\ a \end{pmatrix}} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{pmatrix} b \\ a \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \rightarrow \begin{array}{c} a = b \\ 1 \quad 1 \end{array}$$

$\sigma_x v_- = \lambda_- v_-$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = -1 \cdot \begin{pmatrix} a \\ b \end{pmatrix} \longrightarrow \begin{pmatrix} b \\ a \end{pmatrix} = \begin{pmatrix} -a \\ -b \end{pmatrix} \qquad \cancel{v_- = \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

or

$$\left.\begin{matrix} b = -a \\ a = -b \end{matrix}\right\} \begin{matrix} a = 1 \\ b = -1 \end{matrix} \qquad v_- = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Eigenvectors of σ_x are $\underset{\lambda=-1}{\cancel{\begin{pmatrix} -1 \\ 1 \end{pmatrix}}}$ & $\underset{\lambda=1}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}$

$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ (?)

$$\langle v_- | v_+ \rangle = (-1 \;\; 1)\begin{pmatrix} 1 \\ 1 \end{pmatrix} = -1 + 1 = 0$$

★ These eigenvectors must be orthogonal. Since they're eigenvectors of a Hermitian Matrix corresponding to distinct eigenvalues.

$\sigma_y U_+ = 1 \cdot U_+$

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ i \end{pmatrix} \text{ or } \begin{pmatrix} -i \\ 1 \end{pmatrix}$$

$$\begin{matrix} -i\overset{i}{b} = \overset{1}{a} \\ i\underset{1}{a} = \underset{i}{b} \end{matrix} \quad \text{or} \quad \begin{matrix} b = 1 \\ a = -i \end{matrix}$$

$\sigma_y U_- = -U_-$

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -a \\ -b \end{pmatrix} = \begin{pmatrix} 1 \\ -i \end{pmatrix} \text{ or } \begin{pmatrix} i \\ 1 \end{pmatrix}$$

$$\begin{matrix} -ib = -a^{1} \\ i\underset{1}{a} = -\underset{-i}{b} \end{matrix} \quad \text{or} \quad \begin{matrix} a = i \\ b = 1 \end{matrix}$$

$$\langle U_+ | U_- \rangle = 0$$

$$(1 \;\; -i)\begin{pmatrix} 1 \\ -i \end{pmatrix} = 1 + i^2 = 0 \checkmark \quad \Big| \quad (+i \;\; 1)\begin{pmatrix} i \\ 1 \end{pmatrix} = i^2 + 1 = 0 \checkmark$$

$\lambda = +1 \; \begin{pmatrix} 1 \\ i \end{pmatrix}$

$\lambda = -1 \; \begin{pmatrix} 1 \\ -i \end{pmatrix}$

Eigenvectors of σ_y are $\underset{\lambda=-1}{\begin{pmatrix} i \\ 1 \end{pmatrix}}$ & $\underset{\lambda=+1}{\begin{pmatrix} -i \\ 1 \end{pmatrix}}$

$\sigma_z \psi_+ = 1 \psi_+$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 1.\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$a = a \qquad b = 0$

$-b = b \qquad a = 1$

$\sigma_z \psi_- = -1 . \psi_-$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -a \\ -b \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} a \\ -b \end{pmatrix} = \begin{pmatrix} -a \\ -b \end{pmatrix} \rightarrow \begin{matrix} a=0 \\ b=1 \end{matrix}$$

Eigenvectors of σ_z are $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ & $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\lambda = +1 \qquad \lambda = -1$

3. The three Pauli matrices can be used to define a Pauli vector $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)^T$ in a 3D space. The Pauli vector pointing in a direction $\boldsymbol{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)^T$ in spherical coordinates is defined as :

$$\sigma_n = \boldsymbol{\sigma}\cdot\boldsymbol{n} = \sin\theta\cos\phi\sigma_x + \sin\theta\sin\phi\sigma_y + \cos\theta\sigma_z$$

Show that the eigenvalues of σ_n are $\lambda = \pm 1$ and that the eigenstate associated to the eigenvalue $\lambda = 1$ can be written as $|\psi_n\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)e^{i\phi}|1\rangle$ where $|0\rangle = (1,0)^T$ and $|1\rangle = (0,1)^T$. Useful identities : $\sin^2\theta = \frac{1-\cos 2\theta}{2}$ and $\sin 2\theta = 2\sin\theta\cos\theta$.

$$\sigma_n - I.a = \begin{pmatrix} \cos\theta - a & \sin\theta\cos\phi - i\sin\theta\sin\phi \\ \sin\theta\cos\phi + i\sin\theta\sin\phi & -\cos\theta - a \end{pmatrix}$$

$\det(..) = 0$

$(c-id)(c+id) = c^2 - (id)^2 = c^2 + d^2$

$$(\cos\theta - a).(\underbrace{-\cos\theta - a}_{-(\cos\theta + a)}) - (\overset{c}{\sin\theta\cos\phi} - i\overset{d}{\sin\theta\sin\phi})(\overset{c}{\sin\theta\cos\phi} + i\overset{d}{\sin\theta\sin\phi})$$

$$-(\cos^2\theta - a^2) - (\sin\theta\cos\phi)^2 - (\sin\theta\sin\phi)^2 = 0$$

$$-\cos^2\theta + a^2 - \underbrace{\sin^2\theta\cos^2\phi - \sin^2\theta\sin^2\phi}_{\sin^2\theta} = 0$$

$$-\cos^2\theta + a^2 - \sin^2\theta = 0$$

$$a^2 = 1$$

$$\downarrow$$

$$\lambda^2 = 1 \longrightarrow \lambda = \pm 1$$

For $\lambda = 1$ $\quad \sigma_n |\psi\rangle = |\psi\rangle$ where $|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$

$$|\psi\rangle = \begin{pmatrix} \cos\theta/2 \\ \sin\theta/2\, e^{i\phi} \end{pmatrix}$$

$$\begin{pmatrix} \cos\theta & \overbrace{\sin\theta\cos\phi - i\sin\theta\sin\phi}^{\sin\theta\, e^{-i\phi}} \\ \underbrace{\sin\theta\cos\phi + i\sin\theta\sin\phi}_{\sin\theta\, e^{i\phi}} & -\cos\theta \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\cos\theta . \cos\frac{\theta}{2} + \sin\theta\, \cancel{e^{-i\phi}} \sin\frac{\theta}{2}\, \cancel{e^{i\phi}} \stackrel{?}{=} \cos\left(\frac{\theta}{2}\right)$$

$$\cos\theta . \cos\frac{\theta}{2} + 2\sin^2\frac{\theta}{2}\cos\frac{\theta}{2} = \cos\frac{\theta}{2}\left(\underset{1-2\sin^2\frac{\theta}{2}}{\underset{\downarrow}{\cos\theta}} + 2\sin^2\frac{\theta}{2}\right) = \cos\frac{\theta}{2} \checkmark$$

4. The previous question showed that every point (θ, ϕ) on the surface of a unit sphere could represent a unique state $|\psi_n\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)e^{i\phi}|1\rangle$ of a two dimensional Hilbert space. The unit sphere is commonly called the Bloch sphere and each point (θ, ϕ) is associated to a unit vector called a Bloch vector. The Bloch vector in Cartesian coordinates is expressed by $\boldsymbol{r} = \langle\boldsymbol{\sigma}\rangle = (\langle\sigma_x\rangle, \langle\sigma_y\rangle, \langle\sigma_z\rangle)^T$. Graphically visualize the Bloch vector on the Bloch sphere for $\theta = \pi/4$ and $\phi = \pi/2$. How would you represent the eigenstate of σ_x on the bloch sphere ?

$$\langle\sigma_x\rangle = \langle\psi|\sigma_x|\psi\rangle = \left(\cos\frac{\theta}{2} \quad \sin\frac{\theta}{2}e^{-i\phi}\right)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}e^{i\phi} \end{pmatrix}$$

$$\langle\sigma_x\rangle = \left(\cos\frac{\theta}{2} \quad \sin\frac{\theta}{2}e^{-i\phi}\right)\begin{pmatrix} \sin\frac{\theta}{2}e^{i\phi} \\ \cos\frac{\theta}{2} \end{pmatrix} =$$

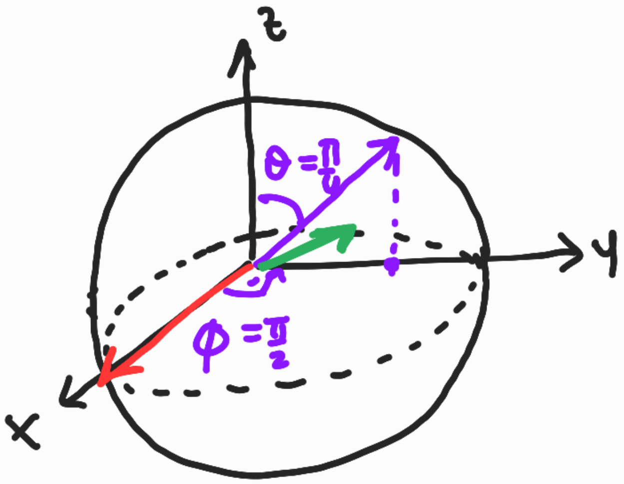
$$\langle\sigma_x\rangle = \cos\frac{\theta}{2}\sin\frac{\theta}{2}e^{i\phi} + \cos\frac{\theta}{2}\sin\frac{\theta}{2}e^{-i\phi}$$

$$= \frac{1}{2}\sin\theta\left(\underbrace{e^{i\phi} + e^{-i\phi}}_{2\cos\phi}\right) = \sin\theta\cos\phi$$

$$\langle\sigma_y\rangle = \sin\theta\sin\phi$$

$$\langle\sigma_z\rangle = \cos\theta$$

$$|r| = \sqrt{\langle\sigma_x\rangle^2 + \langle\sigma_y\rangle^2 + \langle\sigma_z\rangle^2} = 1$$



Eigenvector of σ_x

$$\nearrow \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}\cos\underbrace{\theta/2}_{\pi/2}\\ \sin\theta/2\, e^{i\underbrace{\phi}_{\pi}}\end{pmatrix}$$

$$\searrow \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}\cos\theta/2\\ \sin\underbrace{\theta/2}_{\pi/2}\, e^{i\underbrace{\phi}_{0}}\end{pmatrix}$$

5. Calculate the length Bloch vector for a mixed state with density matrix $\rho = (1-\varepsilon)|\psi\rangle\langle\psi| + \frac{\varepsilon}{2}\mathbf{1}$.

$$\langle\hat{O}\rangle = \mathrm{Tr}(\rho\hat{O})$$

From part 4, $|r_\psi|^2 = 1 = \mathrm{Tr}(\sigma_x|\psi\rangle\langle\psi|)^2 + \mathrm{Tr}(\sigma_y|\psi\rangle\langle\psi|)^2 + \mathrm{Tr}(\sigma_z|\psi\rangle\langle\psi|)^2$

* Note that: $\mathrm{Tr}(\sigma_x.\mathbb{I}) = \mathrm{Tr}(\sigma_y\mathbb{I}) = \mathrm{Tr}(\sigma_z\mathbb{I}) = 0$

$$\mathrm{Tr}(\sigma_i\rho) = (1-\varepsilon)\,\mathrm{Tr}(\sigma_i|\psi\rangle\langle\psi|) + \frac{\varepsilon}{2}\underbrace{\cancel{\mathrm{Tr}(\sigma_i\mathbb{I})}}_{0}$$

$$= (1-\varepsilon)\,\mathrm{Tr}(\langle\psi|\sigma_i|\psi\rangle)$$

$$|r_\rho| = \sqrt{(1-\varepsilon)^2\langle\psi|\sigma_x|\psi\rangle^2 + (1-\varepsilon)^2\langle\psi|\sigma_y|\psi\rangle^2 + (1-\varepsilon)^2\langle\psi|\sigma_z|\psi\rangle^2}$$

$$|r_\rho| = (1-\varepsilon)\underbrace{|r_\psi|}_{1} = (1-\varepsilon) \in [0,1]$$

(b) Calculate the state after a time t assuming that at $t = 0$ the state was in $|\psi(0)\rangle = |1\rangle$.What happens at times $t = \pi/\omega$ and $t = \pi/2\omega$?

Exercise 2: Projective measurements of qubits

One way to determine the state of a qubit is to perform a projective measurement. When a projective measurement is performed on a quantum system, the system is projected onto an eigenstate of the observable being measured. For example, performing a measurement of the observable σ_z, projects the state vector in the state $|0\rangle$ or $|1\rangle$ (projection along the z axis of the bloch sphere). In other words, a projective measurement causes the quantum state to "collapse" into a single eigenstate of the measured observable.

1. After applying the proper magnetic pulses, a spin qubit is brought to the superposition state :

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

we perform a projective measurement in the σ_z basis. For each possible measurement result, determine the probablity of obtaining that result as well as the state in which the qubit will be after the measurement.

2. Discuss what happens in the previous question when the same observable is measured a second time right after the first measurement. How would the situation change if the measurement apparatus was imperfect and misclassified the states with finite probability ?

An $R_y(\theta)$ gate transforms the qubit state, $|\psi\rangle \to R_y(\theta)|\psi\rangle$, with the unitary

$$R_y(\theta) = \begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}.$$

3. We apply an $R_y(-\pi/2)$ gate to a qubit that was initialized in a state $|\psi\rangle$, and then measure it in the σ_z basis. Argue why this procedure is equivalent to performing a measurement in the σ_x basis. Moreoever, determine the probabilities of all possible measurement results for the following specific cases:

   - $|\psi\rangle = |+\rangle$
   - $|\psi\rangle = |0\rangle$

4. For a qubit in a mixed state with the density matrix

$$\rho = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|),$$

determine the probabilities of all possible measurement results when the qubit is measured in

   - the σ_x basis.
   - the σ_y basis.
   - the σ_z basis.

Interpret the results with the help of the Bloch sphere representation of ρ.

1. After applying the proper magnetic pulses, a spin qubit is brought to the superposition state :

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

we perform a projective measurement in the σ_z basis. For each possible measurement result, determine the probablity of obtaining that result as well as the state in which the qubit will be after the measurement.

$$\lambda_+ = +1 \quad |\lambda_+\rangle = \begin{pmatrix}1\\0\end{pmatrix} = |0\rangle \; ; \quad \lambda_- = -1 \quad |\lambda_-\rangle = \begin{pmatrix}0\\1\end{pmatrix} = |1\rangle$$

$P_i = |\lambda_i\rangle\langle\lambda_i|$ → Projection Operator

$|\lambda_1\rangle = |0\rangle$

$$P(1) = \langle +|\overbrace{\lambda_1\rangle\langle\lambda_1}^{P_1}|+\rangle = \langle +|0\rangle\langle 0|+\rangle = |\langle +|0\rangle|^2 = \frac{1}{2}$$

$$P(-1) = 1 - P(1) = 1 - \frac{1}{2} = \frac{1}{2}$$

$$\rho_i = \frac{P_i\,|+\rangle\langle +|\,P_i}{\langle +|P_i|+\rangle} = \begin{cases} \dfrac{|0\rangle\overbrace{\langle 0|+\rangle\langle +|0\rangle}^{1/2}\langle 0|}{\underbrace{P(1)}_{1/2}} = |0\rangle\langle 0| & \text{for result } \lambda = +1 \\[2ex] \dfrac{|1\rangle\overbrace{\langle 1|+\rangle\langle +|1\rangle}^{P(-1)}\langle 1|}{P(-1)} = |1\rangle\langle 1| & \text{for result is } -1 \end{cases}$$

The qubit state is $|0\rangle$ if the result was $+1$

" $|1\rangle$ " -1

2. Discuss what happens in the previous question when the same observable is measured a second time right after the first measurement. How would the situation change if the measurement apparatus was imperfect and misclassified the states with finite probability ?

For projective measurements, the second measurement yields the same result as the first under idealized conditions.

if error occurs with P_e

Ex $\quad \rho_1 = (1 - P_e)|0\rangle\langle 0| + P_e|1\rangle\langle 1|$

First meas. result is $+1$, the state should be $|0\rangle$

Result Matching (First meas $= |0\rangle$, Second meas $|0\rangle$ with prob = ?)

$|0\rangle \to \boxed{\nearrow}$ → $|0\rangle$ $(1-P_e)^2$; → $|1\rangle$ $(P_e)^2$

$(1-P_e)^2$

$|1\rangle \to \boxed{\nearrow}$ → $|0\rangle$ $(P_e)^2$; → $|1\rangle$ $(1-P_e)^2$

$(P_e)^2$

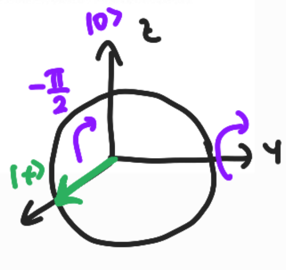
For the second measurement, the results match with probability $(1-P_e)^2 + P_e^2$

An $R_y(\theta)$ gate transforms the qubit state, $|\psi\rangle \to R_y(\theta)\,|\psi\rangle$, with the unitary

$$R_y(\theta) = \begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}.$$

3. We apply an $R_y(-\pi/2)$ gate to a qubit that was initialized in a state $|\psi\rangle$, and then measure it in the σ_z basis. Argue why this procedure is equivalent to performing a measurement in the σ_x basis. Moreoever, determine the probabilities of all possible measurement results for the following specific cases:

- $|\psi\rangle = |+\rangle$
- $|\psi\rangle = |0\rangle$



$|\psi\rangle \to \boxed{R_y(-\pi/2)} \xrightarrow{|\psi'\rangle} \underset{\sigma_z}{\boxed{\nearrow}} = \underset{\sigma_x}{\boxed{\nearrow}}$

For $|\psi\rangle = |+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$

$$R_y(-\pi/2) = \begin{pmatrix} \cos(-\pi/4) & -\sin(-\pi/4) \\ \sin(-\pi/4) & \cos(-\pi/4) \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$$= 1/\sqrt{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$|\psi'\rangle = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}2\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} = |0\rangle$$

$|0\rangle \to \boxed{\nearrow} \to |0\rangle$ with prob = 1

Eigenvalues and eigenvectors of σ_x are given by :

$$\lambda_{\sigma_x,1} = 1 = \lambda_1 \qquad |\lambda_{\sigma_x,1}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = |+\rangle$$

$$\lambda_{\sigma_x,2} = -1 = \lambda_2 \qquad |\lambda_{\sigma_x,2}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} = |-\rangle$$

For $|0\rangle, |1\rangle$: (σ_z eigenvectors)

$$\langle \overset{\lambda_1}{0}|\, R_y(-\pi/2) = (1\ \ 0)\,\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix} = \frac{1}{\sqrt{2}}(1\ \ 1) = \langle\lambda_{\sigma_x,1}|$$

$$\langle \overset{\lambda_2}{1}|\, R_y(-\pi/2) = (0\ \ 1)\,\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix} = \frac{1}{\sqrt{2}}(-1\ \ 1) = \langle\lambda_{\sigma_x,2}|$$

global phase

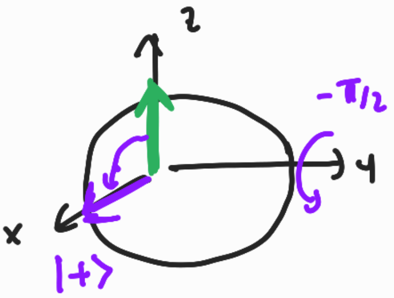
$$\frac{1}{\sqrt{2}}(1\ \ -1)$$?

$$\left|\langle\lambda_i|\, R_y(-\pi/2)\,|\psi\rangle\right|^2 = \boxed{\left|\langle\lambda_{\sigma_x,i}|\psi\rangle\right|^2}$$

corresponds to measurement with σ_x

For $|\psi\rangle = |+\rangle$, $\overset{|0\rangle}{P(1)} = 1$ and $\overset{|1\rangle}{P(-1)} = 0$

For $|\psi\rangle = |0\rangle$, $\underset{|0\rangle}{P(1) = \frac{1}{2}}$ and $\underset{|1\rangle}{P(-1) = \frac{1}{2}}$



4. For a qubit in a mixed state with the density matrix

$$\rho = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|),$$

determine the probabilities of all possible measurement results when the qubit is measured in

- the σ_x basis.
- the σ_y basis.
- the σ_z basis.

Interpret the results with the help of the Bloch sphere representation of ρ.

$* \; \underbrace{\langle\psi|P_i|\psi\rangle}_{\text{projector}} = \underbrace{P(i)}_{\text{probability}}$

$P_i = |\lambda_i\rangle\langle\lambda_i| \qquad Tr(P_i \rho)$

. For σ_z basis, $P_1 = |0\rangle\langle 0| \quad P_{-1} = |1\rangle\langle 1|$

$$P(1) = Tr\left[|0\rangle\langle 0| \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)\right] = Tr\left[\frac{1}{2}\left(|0\rangle\overbrace{\langle 0|0\rangle}^{1}\langle 0| + |0\rangle\underbrace{\langle 0|1\rangle}_{0}\langle 1|\right)\right]$$

$$= \frac{1}{2} Tr\left[|0\rangle\langle 0|\right] = \frac{1}{2}\cdot 1 = \frac{1}{2}$$

$$P(-1) = Tr\left[|1\rangle\langle 1| \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|\right] = 1 - P(1) = \frac{1}{2}$$

. For σ_y basis, $P_1 = \frac{1}{2}(|0\rangle + i|1\rangle)(\langle 0| - i\langle 1|)$

$$P_1 = \frac{1}{2}(|0\rangle\langle 0| - i|0\rangle\langle 1| + i|1\rangle\langle 0| + |1\rangle\langle 1|)$$

$$P(1) = \frac{1}{4} Tr\left(\left[|0\rangle\langle 0| - i|0\rangle\langle 1| + i|1\rangle\langle 0| + |1\rangle\langle 1|\right]\left[|0\rangle\langle 0| + |1\rangle\langle 1|\right]\right)$$

$$= \frac{1}{4} Tr\left(|0\rangle\langle 0| - i|0\rangle\langle 1| + i|1\rangle\langle 0| + |1\rangle\langle 1|\right) = \frac{1}{4}(2) = \frac{1}{2}$$

$$\begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}$$

$$P(-1) = 1 - P(1) = \frac{1}{2}$$

· For σ_x basis, $P_1 = \frac{1}{2}(|0\rangle + |1\rangle)(\langle 0| + \langle 1|)$

$$P(1) = \frac{1}{4} Tr\left(\left[|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1| \right] \left[|0\rangle\langle 0| + |1\rangle\langle 1| \right] \right)$$

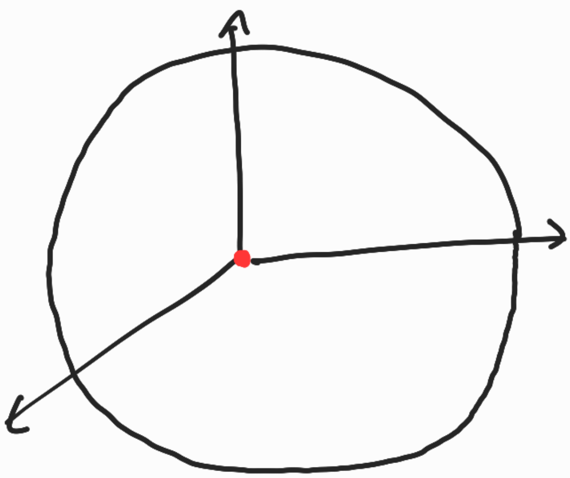
$$\frac{1}{4} Tr\left(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1| \right) = \frac{1}{4} \cdot 2 = \frac{1}{2}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$P(-1) = 1 - P(1) = 1 - \frac{1}{2} = \frac{1}{2}$$

$$\rho = \frac{1}{2}\left[\mathbb{I} + r_x \sigma_x + r_y \sigma_y + r_z \sigma_z \right]$$

$$\rho = \frac{1}{2}\left(\mathbb{I} + \vec{r} \cdot \vec{\sigma} \right)$$



5. Calculate the length Bloch vector for a mixed state with density matrix $\rho = (1-\varepsilon)\left|\psi\right\rangle\left\langle\psi\right| + \frac{\varepsilon}{2}\mathbb{1}$.

Exercise 3 : Time evolution of a spin-1/2 particle in a magnetic field

One common two-level system is the electron spin. The electron spin can point in two directions, which we commonly refer to as "spin-up" ($|0\rangle$) and "spin-down" ($|1\rangle$). In the presence of a magnetic field, the two spin states of the electron can be separated which creates a two-level system. As we will see later in the course, spin qubits are realized by isolating single electrons by shaping the potential of a two dimensional electron gas obtained by stacking different semiconductors. The goal of this exercise is to show how different quantum states can be created by applying a pulsed magnetic field (equivalent to applying a quantum gate).

The Hamiltonian H describing a spin in a magnetic field is given by :

$$H = -\frac{\hbar\gamma}{2}\boldsymbol{B}\cdot\boldsymbol{\sigma} = -\frac{\hbar\gamma}{2}(B_x\sigma_x + B_y\sigma_y + B_z\sigma_z), \tag{1}$$

γ is the gyromagnetic ratio. If H is constant in time. The time evolution of this quantum

2. Let's assume we have a constant magnetic field in the x-direction:

$$\boldsymbol{B} = \begin{pmatrix} B_x \\ 0 \\ 0 \end{pmatrix}$$

(a) Calculate the state after a time t assuming that at $t=0$ the state was in $|\psi(0)\rangle = |0\rangle$. What happens at times $t = \pi/\omega$ and $t = \pi/2\omega$? Draw the time evolution of the state on the Bloch sphere.

(b) Calculate the state after a time t assuming that at $t=0$ the state was in $|\psi(0)\rangle = |1\rangle$.What happens at times $t = \pi/\omega$ and $t = \pi/2\omega$?

(2)

time evolution of as follows :

(3)

1. Consider now the case where we have a constant magnetic field in the z-direction:

$$\boldsymbol{B} = \begin{pmatrix} 0 \\ 0 \\ B_z \end{pmatrix} \tag{4}$$

   (a) If at time $t=0$ the state is given by $|\psi(0)\rangle = \cos\theta/2\,|0\rangle + \sin\theta/2 e^{i\phi}\,|1\rangle$, calculate the state $|\psi(t)\rangle$ after a given time $t = t_1$. Draw the time evolution of state on the Bloch sphere.
   Hint: Use the following identity $e^{i\frac{\theta}{2}(n_x\sigma_x+n_y\sigma_y+n_z\sigma_z)} = \cos\frac{\theta}{2}\mathbb{1} + i\sin\frac{\theta}{2}(n_x\sigma_x+n_y\sigma_y+n_z\sigma_z)$.

2. Let's assume we have a constant magnetic field in the x-direction:

$$\boldsymbol{B} = \begin{pmatrix} B_x \\ 0 \\ 0 \end{pmatrix}$$

   (a) Calculate the state after a time t assuming that at $t=0$ the state was in $|\psi(0)\rangle = |0\rangle$. What happens at times $t = \pi/\omega$ and $t = \pi/2\omega$? Draw the time evolution of the state on the Bloch sphere.

   (b) Calculate the state after a time t assuming that at $t=0$ the state was in $|\psi(0)\rangle = |1\rangle$.What happens at times $t = \pi/\omega$ and $t = \pi/2\omega$?

1. Consider now the case where we have a constant magnetic field in the z-direction:

$$\boldsymbol{B} = \begin{pmatrix} 0 \\ 0 \\ B_z \end{pmatrix} \tag{4}$$

(a) If at time $t = 0$ the state is given by $|\psi(0)\rangle = \cos\theta/2\,|0\rangle + \sin\theta/2 e^{i\phi}\,|1\rangle$, calculate the state $|\psi(t)\rangle$ after a given time $t = t_1$. Draw the time evolution of state on the Bloch sphere.
Hint: Use the following identity $e^{i\frac{\theta}{2}(n_x\sigma_x+n_y\sigma_y+n_z\sigma_z)} = \cos\frac{\theta}{2}\mathbb{1} + i\sin\frac{\theta}{2}(n_x\sigma_x+n_y\sigma_y+n_z\sigma_z)$.

$$|\Psi(t)\rangle = e^{\frac{-iHt}{\hbar}} \, |\Psi(0)\rangle$$

$$H = -\frac{\hbar\gamma}{2}\vec{B}\cdot\vec{\sigma} = -\frac{\hbar\gamma B_z\sigma_z}{2}$$

$$e^{-iHt/\hbar} = e^{+i\frac{\cancel{\hbar}\gamma B_z \cdot \sigma_z}{2}\cdot\frac{1}{\cancel{\hbar}}t} = e^{i\frac{\gamma B_z}{2}\sigma_z t} = \cos\left(\frac{\gamma B_z t}{2}\right)I + i\sin\left(\frac{\gamma B_z t}{2}\right)\sigma_z$$

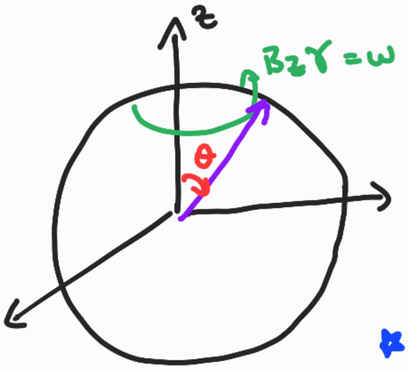
$$|\Psi(t)\rangle = \begin{bmatrix} \underbrace{\cos\left(\frac{\gamma B_z t}{2}\right) + i\sin\left(\frac{\gamma B_z t}{2}\right)}_{e^{i\frac{B_z}{2}\gamma t}} & 0 \\ 0 & \underbrace{\cos\left(\frac{\gamma B_z t}{2}\right) - i\sin\left(\frac{\gamma B_z t}{2}\right)}_{e^{-i\frac{B_z}{2}\gamma t}} \end{bmatrix} \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}\, e^{i\phi} \end{pmatrix}$$

global phase factor

$$|\Psi(t)\rangle = \begin{pmatrix} e^{i\frac{B_z\gamma}{2}} \cos\theta/2 \\ e^{-i\frac{B_z}{2}} \cdot e^{i\phi} \sin\theta/2 \end{pmatrix} = \begin{pmatrix} \cos\theta/2 \\ e^{-iB_z\gamma t} \cdot e^{i\phi} \sin\theta/2 \end{pmatrix}$$

$$e^{i(\phi - B_z\gamma t)}$$

$$\omega = B_z\gamma$$



Larmor Frequency (?)

* B field along the z-axis, makes the spin precess along the z-axis → Rotation in the xy plane

2. Let's assume we have a constant magnetic field in the x-direction:

$$\boldsymbol{B} = \begin{pmatrix} B_x \\ 0 \\ 0 \end{pmatrix}$$

(a) Calculate the state after a time t assuming that at $t = 0$ the state was in $|\psi(0)\rangle = |0\rangle$. What happens at times $t = \pi/\omega$ and $t = \pi/2\omega$? Draw the time evolution of the state on the Bloch sphere.

(b) Calculate the state after a time t assuming that at $t = 0$ the state was in $|\psi(0)\rangle = |1\rangle$.What happens at times $t = \pi/\omega$ and $t = \pi/2\omega$?

$$|\psi(t)\rangle = e^{-iHt/\hbar}\,|\psi(0)\rangle$$

$$H = \frac{-\hbar\gamma}{2} B_x \sigma_x$$

$$e^{-iHt/\hbar} = e^{i\frac{\cancel{\hbar}}{2}\gamma B_x t \sigma_x \cdot \frac{1}{\cancel{\hbar}}} = e^{i\frac{\gamma}{2}B_x t\sigma_x}$$

$$|\psi(t)\rangle = \cos\left(\frac{\gamma}{2}B_x t\right)I + i\sin\left(\frac{\gamma}{2}B_x t\right)\sigma_x$$

$$= \cos\left(\frac{\omega t}{2}\right)I + i\sin\left(\frac{\omega t}{2}\right)\sigma_x$$

$$|\psi(t)\rangle = \begin{pmatrix} \cos\left(\frac{\omega t}{2}\right) & i\sin\left(\frac{\omega t}{2}\right) \\ i\sin\left(\frac{\omega t}{2}\right) & \cos\left(\frac{\omega t}{2}\right) \end{pmatrix} |\psi(0)\rangle$$

a) $$|\psi(t)\rangle = \begin{pmatrix} \cos\left(\frac{\omega t}{2}\right) & i\sin\left(\frac{\omega t}{2}\right) \\ i\sin\left(\frac{\omega t}{2}\right) & \cos\left(\frac{\omega t}{2}\right) \end{pmatrix} \overset{|0\rangle}{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} = \begin{pmatrix} \cos\left(\frac{\omega t}{2}\right) \\ i\sin\left(\frac{\omega t}{2}\right) \end{pmatrix}$$

$$|\psi(t)\rangle = \cos\left(\frac{\omega t}{2}\right)|0\rangle + i\sin\left(\frac{\omega t}{2}\right)|1\rangle$$

At $t = \frac{\pi}{\omega}$ $\rightsquigarrow$ $$|\psi(t)\rangle = \underbrace{\cancel{\cos\left(\frac{\omega}{2}\cdot\frac{\pi}{\omega}\right)}}_{0}|0\rangle + i\sin\left(\frac{\cancel{\omega}}{2}\cdot\frac{\pi}{\cancel{\omega}}\right)|1\rangle$$

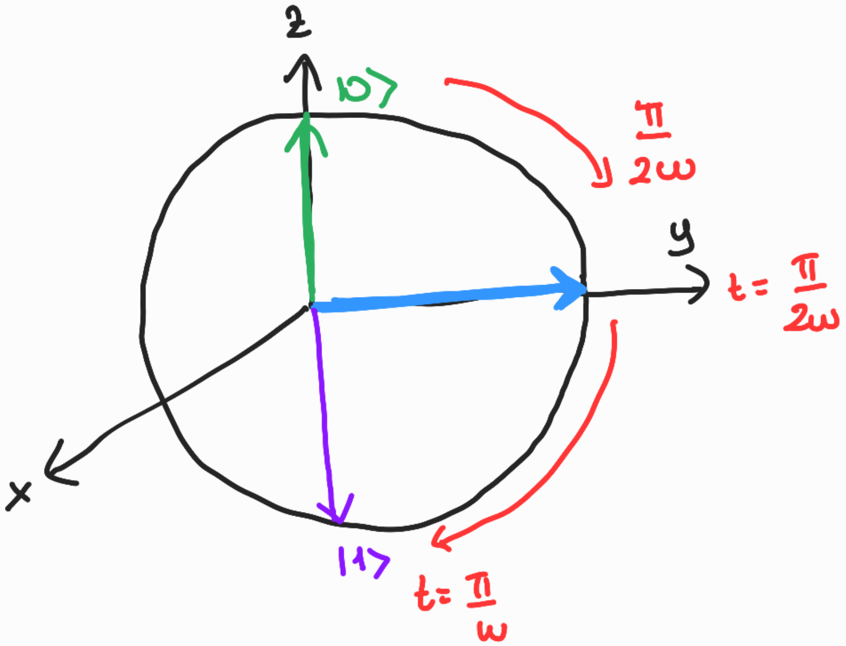
$$|\psi(t)\rangle = i|1\rangle$$

global phase

At: $t = \frac{\pi}{2\omega}$ $\rightsquigarrow$ $$|\psi(t)\rangle = \cos\frac{\pi}{4}|0\rangle + i\sin\left(\frac{\pi}{4}\right)|1\rangle$$

$$= \frac{1}{\sqrt{2}}\left(|0\rangle + i|1\rangle\right)$$

eigenstate of y



b) $|\psi(0)\rangle = |1\rangle$

$$|\psi(t)\rangle = \begin{pmatrix}\cos(\frac{\omega t}{2}) & i\sin(\frac{\omega t}{2})\\ i\sin(\frac{\omega t}{2}) & \cos(\frac{\omega t}{2})\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix} i\sin(\frac{\omega t}{2})\\ \cos(\frac{\omega t}{2})\end{pmatrix}$$

$$|\psi(t)\rangle = i\sin\left(\frac{\omega t}{2}\right)|0\rangle + \cos\left(\frac{\omega t}{2}\right)|1\rangle$$

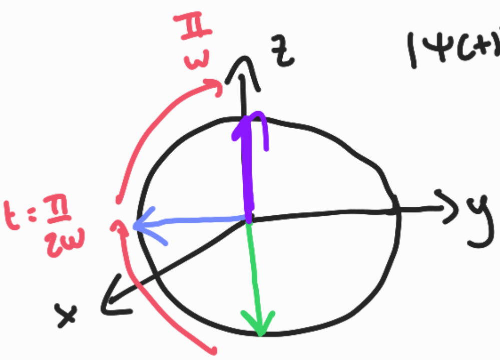
At $t = \frac{\pi}{\omega}$ $\quad |\psi(t)\rangle = i\sin\left(\frac{\pi}{2}\right)|0\rangle + \underbrace{\cancel{\cos\left(\frac{\pi}{2}\right)}}_{0}|1\rangle$

$$|\psi(t)\rangle = \cancel{i}|0\rangle$$

At $t = \frac{\pi}{2\omega}$ $\quad |\psi(t)\rangle = i\sin\left(\frac{\pi}{4}\right)|0\rangle + \cos\left(\frac{\pi}{4}\right)|1\rangle$

$$|\psi(t)\rangle = i\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle = \frac{1}{\sqrt{2}}\left(i|0\rangle + |1\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\, i \left(|0\rangle \underbrace{-i}_{\phi = \frac{3\pi}{2}}|1\rangle\right)$$



2 Reminder: Dirac Notations

Dirac notation is an abstract notation for describing quantum states introduced by Dirac in 1939. We will remind ourselves of the notation using few examples. We will restrict ourselves to 2 dimensional Hilbert spaces, as they are the relevant size for single qubits. Multi-qubit systems will be handled at a later time.

1. ket vector
$$|\psi\rangle \doteq \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix} \doteq \alpha |0\rangle + \beta |1\rangle, \tag{5}$$
where $\alpha^2 + \beta^2 = 1$.

2. bra vector
$$\langle\psi| = |\psi\rangle^\dagger \doteq \begin{pmatrix} \alpha^\star & \beta^\star \end{pmatrix} \doteq \alpha^\star \langle 0| + \beta^\star \langle 1|. \tag{6}$$

3. Inner product
Let
$$|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \text{ and } |\phi\rangle = \begin{pmatrix} \gamma \\ \delta \end{pmatrix}, \tag{7}$$
the inner product is defined as
$$\langle\phi|\psi\rangle = (\langle\psi|\phi\rangle)^\star = \gamma^\star\alpha + \delta^\star\beta. \tag{8}$$
Normalization condition
$$\langle\psi|\psi\rangle = 1. \tag{9}$$
Orthogonality
$$\langle\phi|\psi\rangle = 0. \tag{10}$$

4. Outer product
The products of the form $\langle\phi|\psi\rangle$ are called outer-products and result in matrices. Using the same definition of $|\phi\rangle$ and $|\psi\rangle$ as above we get,
$$|\phi\rangle\langle\psi| = (|\psi\rangle\langle\phi|)^\dagger = \begin{pmatrix} \alpha^\star\gamma & \beta^\star\gamma \\ \alpha^\star\delta & \beta^\star\delta \end{pmatrix}. \tag{11}$$
Outer products can be used to represent/create operators.
Completeness
$$|0\rangle\langle 0| + |1\rangle\langle 1| = \mathbb{1}. \tag{12}$$

3 Reminder: Single qubit rotations

3.1 Introduction: The Bloch Sphere

The general qubit state can be written as follows:

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle ,$$

where α abd β are complex numbers which can be written as

$$\alpha = r_\alpha e^{i\phi_\alpha},$$
$$\beta = r_\beta e^{i\phi_\beta}.$$

The overall phase is not observable we can only measure the probabilities $|\alpha|^2$ and $|\beta|^2$. This means that we can multiply the $|\psi\rangle$ with an overall phase factor $e^{-i\phi_\alpha}$, leaving only a real factor in front of $|0\rangle$

$$\begin{aligned} |\psi'\rangle &= r_\alpha |0\rangle + r_\beta e^{i(\phi_\beta - \phi_\alpha)} |1\rangle \\ &= r_\alpha |0\rangle + r_\beta e^{i\phi} |1\rangle . \end{aligned}$$

If we now apply the normalisation condition $|\alpha|^2 + |\beta|^2 = 1$ and we use $\beta = r_\beta e^{i\phi} = x - iy$ we get the following relationship:

$$|\alpha|^2 + |\beta|^2 = r_\alpha^2 + (x + iy)(x - iy) \tag{13}$$
$$= r_\alpha^2 + x^2 + y^2 = 1. \tag{14}$$

This last equation defines a sphere in the cartesian coordinate system (x, y, r_α). We can switch to the polar coordinate system using the following relations, see fig. 1:

$$\begin{aligned} x &= r \sin(\theta') \cos(\phi), \\ y &= r \sin(\theta') \sin(\phi), \\ z &= r \cos(\theta') = r_\alpha, \end{aligned}$$

with $0 \leq \theta' \leq \pi$ and $0 \leq \phi \leq 2\pi$. As $r = 1$ we can derive the following relations

$$\begin{aligned} r_\alpha &= \cos(\theta'), \\ \beta &= x + iy = \sin(\theta')(\cos(\phi) + i \sin(\phi)), \\ &= \sin(\theta'))e^{i\phi}, \\ \Rightarrow |\psi'\rangle &= \cos(\theta') |0\rangle + \sin(\theta') e^{i\phi} |1\rangle . \end{aligned}$$

Note that $0 \leq \theta' \leq \pi/2$(the upper hemisphere) is enough to cover all possible states, with the north pole being the $|0\rangle$ state and the equator being the $|1\rangle$ state. Going further south means evolving to the $|0\rangle$ state again. It can be shown that 2 opposite points on the polar coordinates sphere give the same physical state (neglecting an overall phase factor of $-\pi$). For convenience we will now map the upper hemisphere on a new sphere (the Bloch sphere, see fig2) by defining $\theta = 2\theta'$, where θ is the angle in the Bloch sphere and θ' is the angle in polar coordinates (as in FIG.1 and FIG.2). If we then let θ go from 0 to π in the Bloch sphere, θ' goes from 0 to $\pi/2$ in polar coordinates, covering all possible states. With this, we finally have

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right) |0\rangle + \sin\left(\frac{\theta}{2}\right) e^{i\phi} |1\rangle .$$

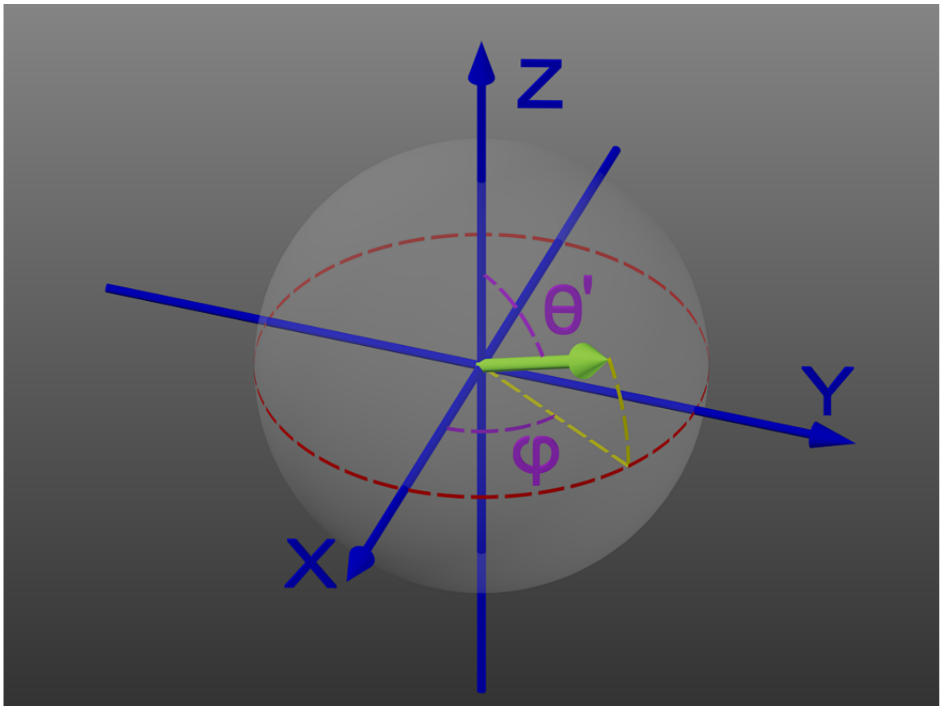


Figure 1: Eq.14 defines a sphere with the radius r equal to 1. The angles in the polar coordinate system are also shown.

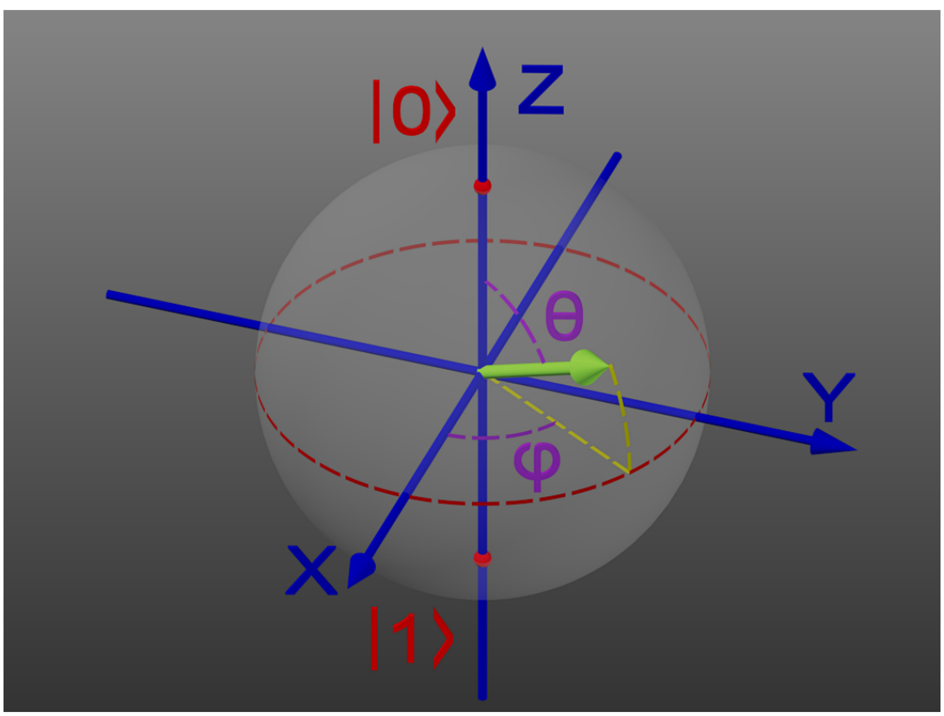


Figure 2: The Bloch sphere. Note that the $|0\rangle$ state is found on the north pole, while the $|1\rangle$ state is found on the southpole, unlike in the polar coordinate sphere (where it lies on the equator.)

4 Reminder: Density matrix

A more general way to describe a quantum state is to make use of the *density matrix* representation.

Pure states VS mixed states It allows us to describe situations where the state vector is not precisely known, and states other than pure quantum states. It will be useful later when we will discuss quantum decoherence.
What is the difference between saying that a two-state system is in a pure state $|\psi(\pi/2, 0)\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and that it has a probability $P_0 = 0.5$ to be in state $|0\rangle$ and a probability $P_1 = 0.5$ to be in state $|1\rangle$.
An example of how these states are different can be seen by rotating by $\pi/2$ around the y axis:

$$R_y(-\pi/2)\,|\psi(\pi/2, 0)\rangle = |\psi(0, 0)\rangle = |0\rangle \Rightarrow P_0 = 1$$

$$\begin{cases} R_y(-\pi/2)\,|0\rangle = |\psi(-\pi/2, 0)\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}} \\ R_y(-\pi/2)\,|1\rangle = |\psi(\pi/2, 0)\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \end{cases} \Rightarrow P_0 = P_1 = 1/2$$

In general:

- Pure states: $P(m) = \langle\psi|\, M\, |\psi\rangle$.
- Mixed states: $P(m) = \sum_j p_j \langle\psi_j|\, M\, |\psi_j\rangle$.

Choosing a basis $\{|b_i\rangle\} : \mathbb{1} = \sum_j |b_j\rangle\langle b_j|$.

$$\begin{aligned} P(m) &= \sum_i \langle\psi_i|\, M\, |\psi_i\rangle \\ &= \sum_{i,j,k} \langle\psi_i|b_j\rangle \overbrace{\langle b_j|\, M\, |b_k\rangle}^{M_{jk}} \langle b_k|\psi_i\rangle \\ &= \sum_{j,k} \sum_i p_i \langle b_j|\psi_i\rangle \langle\psi_i|b_k\rangle M_{jk} \\ &= \sum_{j,k} \rho_{kj} M_{jk} = Tr(\rho M) \end{aligned}$$

where $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$. And we recall the properties of the trace:

$$Tr(A) = \sum_i a_{ii},$$

$$Tr(AB) = \sum_k \sum_i a_{ik} b_{ki}.$$

So, for a $50-50\%$ mixture of $|0\rangle$ and $|1\rangle$ we have $\rho = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) = \frac{1}{2}\mathbb{1}$.

How to recognize a pure state ? The density operator of a pure state can be written $\rho = |\psi\rangle\langle\psi|$, so it's clear that $\rho^2 = |\psi\rangle\langle\psi|\psi\rangle\langle\psi| = |\psi\rangle\langle\psi| = \rho$, so $Tr(\rho^2) = 1$.
It can be shown that for a mixed state $0 < Tr(\rho^2) < 1$. For the $50-50$ mixed state we have $Tr(\rho^2) = Tr\left(\frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \cdot \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right) = \frac{1}{4}Tr(\mathbb{1}) = \frac{1}{2}$.

The Bloch-sphere for mixed states The set of matrices $\{\mathbb{1}, \sigma_x, \sigma_y, \sigma_z\}$ form a complete basis for 2×2 Hermitian matrices. An arbitrary single qubit density matrix can be written as,

$$\rho = \frac{1}{2}\left[\mathbb{1} + r_x\sigma_x + r_y\sigma_y + r_z\sigma_z\right]] =\equiv \frac{1}{2}\left(\mathbb{1} + \boldsymbol{r}\cdot\boldsymbol{\sigma}\right),$$

where $\boldsymbol{r} = (r_x, r_y, r_z)$ is an arbitrary real vector of length $\boldsymbol{r} \leq 1$. $Tr(\rho) = 1$ (Since $Tr(\sigma_i) = 0$). For the projection of the state into the x-axis,

$$\begin{aligned}\langle\sigma_x\rangle = Tr(\rho\sigma_x) &= Tr\left(\frac{1}{2}\left[\mathbb{1}\sigma_x, r_x\sigma_x^2 + r_y\sigma_y\sigma_x + r_z\sigma_z\sigma_x\right]\right)\\ &= Tr(\frac{1}{2}r_x\mathbb{1}) = r_x, \text{ since } Tr(\sigma_i\sigma_j) = 0 \text{ for } i \neq j.\end{aligned}$$

Similarly $\langle\sigma_y\rangle = r_y$ and $\langle\sigma_z\rangle = r_z$.
Therefore, for a pure state $|\boldsymbol{r}| = 1$, i.e. the points are on the surface of the Bloch sphere. In general, we see that,

$$\begin{aligned}Tr(\rho^2) &= Tr\left(\frac{1}{4}\left[\mathbb{1}^2 + 2(r_x\sigma_x + r_y\sigma_y + r_z\sigma_z) + (r_x\sigma_x + r_y\sigma_y + r_z\sigma_z)^2\right]\right)\\ &= Tr\left(\frac{1 + r_x^2 + r_y^2 + r_z^2}{4}\mathbb{1}\right)\\ &= \frac{1 + |\boldsymbol{r}|^2}{2}\end{aligned}$$

so that only the pure states are on the surface of the sphere, whereas mixed states are inside the sphere.

Interpretation of the density matrix What is the probability to find the qubit in the state $|0\rangle$ when it is described by a density matrix ρ ?

- This probability is given by the expectation value of the projection operator $P_o = |0\rangle\langle 0|$. For a pure state, $|\psi\rangle = c_0|0\rangle + c_1|1\rangle$ this gives the usual

$$\langle\psi| P_0 |\psi\rangle = (c_0^\star\langle 0| + c_1\langle 1|)|0\rangle\langle 0|(c_0|0\rangle + c_1|1\rangle) = |c_0|^2$$

- For an arbitrary single qubit density matrix

$$P_0 = Tr(P_o\rho) = Tr\left(\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\cdot\begin{pmatrix}\rho_{00} & \rho_{01}\\ \rho_{10} & \rho_{11}\end{pmatrix}\right) = \rho_{00}$$

  and similarly $P_1 = \rho_{11}$.

- So the probability to find the qubit in its computational basis states is given by the corresponding diagonal elements

- The off-diagonal elements give the amount of coherence between the states. The interaction with the environment tries to make the density matrix diagonal in some basis. On the Bloch sphere this corresponds to r_x and $r_y \to 0$.

?

Density matrices give a more general postulates of quantum mechanics:

	State vectors	**Density matrices**

1. Quantum states can be described by:

$$|\psi\rangle \qquad\qquad \rho$$

$$\langle\psi|\psi\rangle = 1 \qquad\qquad Tr(\rho) = 1$$

$$\text{e.g. } |\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \qquad\qquad \rho = |\psi\rangle\langle\psi| = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

$$\rightarrow Tr(\rho^2) = 1 \rightarrow \text{Pure state}$$

$$\rho = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

$$\rightarrow Tr(\rho^2) < 1 \rightarrow \text{Mixed state}$$

2. Operators describe physical quantities
3. Eigenvalues of the operators are the measurable quantities
4. Measurements:

$$\langle\psi| M |\psi\rangle \qquad\qquad Tr(\rho M)$$

5. Time evolution:

$$i\hbar\left|\dot{\psi}(t)\right\rangle = H|\psi\rangle(t) \qquad\qquad i\hbar\dot{\rho}(t) = [H, \rho]$$