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MICRO-435 Hw 3

$$|GHZ\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle_A |0\rangle_B |0\rangle_C + |1\rangle_A |1\rangle_B |1\rangle_C \right)$$

a) Alice applies Hadamard gate to her qubit

$$\frac{1}{\sqrt{2}}\left(H|0\rangle_A |0\rangle_B |0\rangle_C + H|1\rangle_A |1\rangle_B |1\rangle_C \right)$$

$$= \frac{1}{2}\Big((|0\rangle_A + |1\rangle_A) |0\rangle_B |0\rangle_C + (|0\rangle_A - |1\rangle_A) |1\rangle_B |1\rangle_C \Big)$$

$$|\psi\rangle = \frac{1}{2}\left(|0\rangle_A |0\rangle_B |0\rangle_C + |1\rangle_A |0\rangle_B |0\rangle_C + |0\rangle_A |1\rangle_B |1\rangle_C - |1\rangle_A |1\rangle_B |1\rangle_C \right)$$

b) Alice measures her qubit

Meas: $|0\rangle \longrightarrow \frac{1}{\sqrt{2}}\left(|0\rangle_B |0\rangle_C + |1\rangle_B |1\rangle_C \right)$

Meas: $|1\rangle \longrightarrow \frac{1}{\sqrt{2}}\left(|0\rangle_B |0\rangle_C - |1\rangle_B |1\rangle_C \right)$

c) Bob and Chuck share the positive even parity Bell state $|\Phi_+\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle \right)$.

Protocol

m	Applied operation
0	I
1	$I \otimes Z$

$$\frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) \longrightarrow \frac{1}{\sqrt{2}}\left(|00\rangle - |11\rangle\right)$$

$$\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \quad \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \qquad\qquad \begin{pmatrix}1\\0\\0\\0\end{pmatrix} + \begin{pmatrix}0\\0\\0\\-1\end{pmatrix}$$

$$\underset{4\times 1}{\begin{pmatrix}1\\0\\0\\1\end{pmatrix}} = \underset{4\times 4}{\begin{bmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{bmatrix}} \begin{pmatrix}1\\0\\0\\-1\end{pmatrix}$$

$$I \otimes Z = \begin{bmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{bmatrix}$$

$$I = \begin{pmatrix}1&0\\0&1\end{pmatrix} \qquad Z = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

2)

Input

$$U = \begin{array}{c} \\ 0\\1\\2\\3\\4\\5\\6\\7 \end{array} \begin{array}{c} \begin{array}{cccccccc}0&1&2&3&4&5&6&7\end{array} \\ \begin{bmatrix}1&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&1\\0&1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&1&0\end{bmatrix} \end{array} \underset{\text{input}}{\begin{bmatrix} \\ \\ \\ \\ \end{bmatrix}}$$

$$|\psi_{out}\rangle = U|\psi_{in}\rangle$$

Start from x_2, think zero controlled Toffoli gate

Input x_2	Input x_1	Input x_0	Output x_2	Output x_1	Output x_0
0	0	0	0	0	0
0	0	1	0	1	0
0	1	0	0	1	1
0	1	1	1	0	0
1	0	0	1	0	1
1	0	1	1	1	0
1	1	0	1	1	1
1	1	1	0	0	1

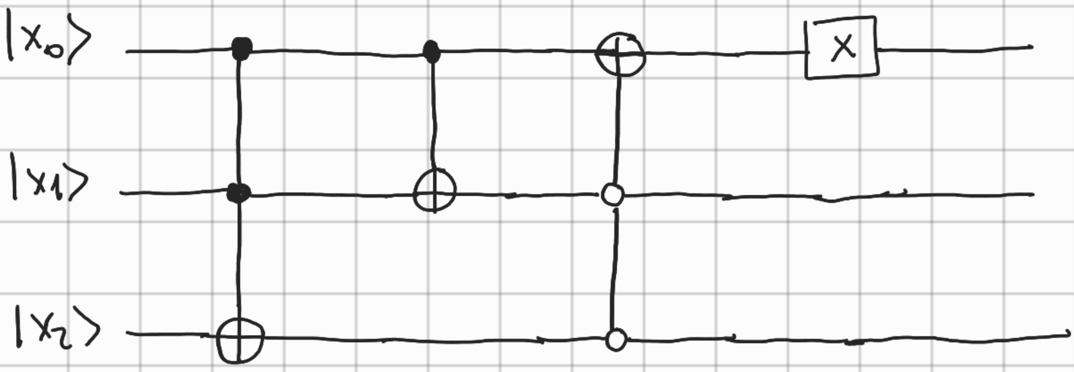
↳ TOFFOLI gate

x_2	x_1	x_0	x_1
0	0	0	0
0	0	1	1
0	1	0	1
1	1	1	0
1	0	0	0
1	0	1	1
1	1	0	1
0	1	1	0

→ CNOT gate

x_2	x_1	x_0	x_0
0	0	0	0
0	1	1	0
0	1	0	1
1	0	1	0
1	0	0	1
1	1	1	0
1	1	0	1
0	0	1	1

→ Zero-Controlled Toffoli + X gate



$$U = (I \otimes I \otimes X)(Z\text{TOFFOLI})(I \otimes \text{CNOT})(\text{TOFFOLI})\,|x_2\, x_1\, x_0\rangle$$

$x_2x_1x_0$

$$\text{TOFFOLI} = \left[\begin{array}{cccc|cccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\hline
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0
\end{array}\right]$$

$x_2x_1x_0$	$x_2x_1x_0$	y
000	000	0
001	001	1
010	010	2
011	111	7
100	100	4
101	101	5
110	110	6
111	011	3

$$
\mathrm{CNOT} = \begin{array}{c|cccc}
 & \overset{(0)}{00} & \overset{(3)}{11} & \overset{(2)}{10} & \overset{(1)}{01} \\
 & 00 & 01 & 10 & 11 \\
\hline
00 & 1_0 & 0 & 0 & 0 \\
01 & 0 & 0 & 0 & 1_1 \\
10 & 0 & 0 & 1_2 & 0 \\
11 & 0 & 1_3 & 0 & 0
\end{array}
$$

$|x_1 x_0\rangle$

$$
I \otimes \mathrm{CNOT} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \left[\begin{array}{cccc|cccc}
1 & 0 & 0 & 0 & & & & \\
0 & 0 & 0 & 1 & & \multicolumn{2}{c}{0} & \\
0 & 0 & 1 & 0 & & & & \\
0 & 1 & 0 & 0 & & & & \\
\hline
 & & & & 1 & 0 & 0 & 0 \\
 & \multicolumn{2}{c}{0} & & 0 & 0 & 0 & 1 \\
 & & & & 0 & 0 & 1 & 0 \\
 & & & & 0 & 1 & 0 & 0
\end{array}\right]
$$

$$
(I \otimes \mathrm{CNOT}) \otimes \mathrm{TOFFOLI} = \left[\begin{array}{cccc|cccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{array}\right]
\left[\begin{array}{cccc|cccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\hline
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0
\end{array}\right]
= \left[\begin{array}{cccc|cccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{array}\right]
$$

$$
Z\mathrm{TOFFOLI} = \begin{array}{c}
x_2 x_1 x_0 \\
\left[\begin{array}{cccc|cccc}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right]
\end{array}
$$

$x_2 x_1 x_0$	$x_2 x_1 x_0$	y
000	001	1
001	000	0
010	010	2
011	011	3
100	100	4
101	101	5
110	110	6
111	111	7

$$
\underbrace{Z\mathrm{TOFFOLI}\,(I \otimes \mathrm{CNOT})\,(\mathrm{TOFFOLI})}_{A} = \left[\begin{array}{cccc|cccc}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right]
\left[\begin{array}{cccc|cccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{array}\right]
$$

$$= \left[\begin{array}{cccc|cccc}
0&0&0&0&0&0&0&1\\
1&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0\\
0&1&0&0&0&0&0&0\\
\hline
0&0&0&0&1&0&0&0\\
0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&1&0\\
0&0&0&0&0&1&0&0
\end{array}\right]$$

$$I \otimes I \otimes X = \left[\begin{array}{cccc|cccc}
0&1&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0\\
0&0&1&0&0&0&0&0\\
\hline
0&0&0&0&0&1&0&0\\
0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&1&0
\end{array}\right]$$

$$\begin{pmatrix}1&0\\0&1\end{pmatrix} \quad \underbrace{\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}}_{\left[\begin{array}{c|c}\begin{smallmatrix}0&1\\1&0\end{smallmatrix} & 0\\ \hline 0 & \begin{smallmatrix}0&1\\1&0\end{smallmatrix}\end{array}\right]}$$

$$(I \otimes I \otimes X)\, A = \left[\begin{array}{cccc|cccc}
0&1&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0\\
0&0&1&0&0&0&0&0\\
\hline
0&0&0&0&0&1&0&0\\
0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&1&0
\end{array}\right]
\left[\begin{array}{cccc|cccc}
0&0&0&0&0&0&0&1\\
1&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0\\
0&1&0&0&0&0&0&0\\
\hline
0&0&0&0&1&0&0&0\\
0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&1&0\\
0&0&0&0&0&1&0&0
\end{array}\right]$$

$$U = \left[\begin{array}{cccc|cccc}
1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1\\
0&1&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0\\
\hline
0&0&0&1&0&0&0&0\\
0&0&0&0&1&0&0&0\\
0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&1&0
\end{array}\right] \quad ✓$$