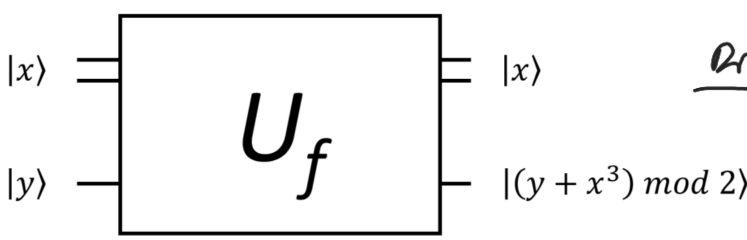
Exercise set #3

Exercise 1:

Consider the boolean function

$$f(x) = x^3 mod\ 2,$$

with $x = x_1x_0$ a 2-bit number (x_0 is the LSB, x_1 is the MSB). We would like to encode this function in a quantum unitary, using a top register with 2 qubits and a bottom register with 1 qubit:



Dr [illegible]

$$|x\rangle = \frac{1}{2}\left(|\overset{x_1\,x_0}{00}\rangle + |01\rangle + |10\rangle + |11\rangle \right)$$

→ Questions are below.

		x			
		0	1	2	3
y	0	0	1	~~8~~ 0	~~27~~ 1
	1	1	~~2~~ 0	~~9~~ 1	~~28~~ 0

a) $|\psi_{initial}\rangle = \frac{1}{2}\left(\overset{x}{|0\rangle}\overset{y}{|0\rangle} + \overset{x}{|1\rangle}\overset{y}{|0\rangle} + \overset{x}{|2\rangle}\overset{y}{|0\rangle} + \overset{x}{|3\rangle}\overset{y}{|0\rangle} \right)$

$|\psi_{final}\rangle = \frac{1}{2}\left(|0\rangle|0\rangle + |1\rangle|1\rangle + |2\rangle|0\rangle + |3\rangle|1\rangle \right)$

$= \frac{1}{\sqrt{2}}(|0\rangle + |2\rangle) \otimes \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}(|1\rangle + |3\rangle) \otimes \frac{1}{\sqrt{2}}|1\rangle$

b) Measurement $= +1 \rightarrow |0\rangle \longrightarrow \frac{1}{\sqrt{2}}(|0\rangle + |2\rangle)$

c) Suppose $|y_{initial}\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

$|\psi\rangle = \frac{1}{2\sqrt{2}}\left(\overset{x}{|0\rangle}\overset{y}{|0\rangle} + \overset{x}{|1\rangle}\overset{y}{|1\rangle} + |2\rangle|0\rangle + |3\rangle|1\rangle \right)$

$- \frac{1}{2\sqrt{2}}\left(\overset{x}{|0\rangle}\overset{y}{|1\rangle} + |1\rangle|0\rangle + |2\rangle|1\rangle + |3\rangle|0\rangle \right)$

$\frac{1}{2}(|0\rangle + |2\rangle - |1\rangle - |3\rangle) \otimes \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{2}(-|0\rangle - |2\rangle + |1\rangle + |3\rangle) \otimes \frac{1}{\sqrt{2}}|1\rangle$

d) Measurement $= +1 \rightarrow |0\rangle \rightarrow \frac{1}{2}(|0\rangle + |2\rangle - |1\rangle - |3\rangle)$

e) $|y\rangle \longrightarrow |(y + x^3) \bmod 2\rangle$

$x = x_1 x_0$

$x_1 . x_0 + x_1 . x_0 \rightarrow$

x_0	x_1	out
0	0	0
0	1	0
1	0	0
1	1	0 $\rightarrow$ +1

$$
\begin{array}{rcccc}
 & & & x_1 & x_0 \\
\times & & & x_1 & x_0 \\
\hline
 & & & x_1x_0 & x_0 \\
- & & x_1 & x_1x_0 & \\
\hline
\underbrace{x_1}_{A} . \underbrace{x_1x_0}_{B} & & \underbrace{x_1 \oplus x_1x_0}_{XOR} & 0 & x_0
\end{array}
$$

$\boxed{x_1x_0 \quad x_1\bar{x}_0 \quad 0 \quad x_0}$

$\overset{A}{x_1} \oplus \overset{B}{x_1x_0} = \bar{A}B + A\bar{B}$

$= \bar{x}_1 \cancel{x_1x_0} + x_1\overline{(x_1x_0)}$

$= x_1(\bar{x}_1 + \bar{x}_0) = x_1\bar{x}_0$

$A \oplus B = A\bar{B} + \bar{A}B$

$\overline{A+B} = \bar{A}.\bar{B}$

$\overline{A.B} = \bar{A} + \bar{B}$

$$
\begin{array}{rcccc}
 & x_1x_0 & x_1\bar{x}_0 & 0 & x_0 \\
 & & & x_1 & x_0 \\
\times & & & & \\
\hline
 & x_1x_0 & 0 & 0 & x_0 \\
+\quad x_1x_0 & x_1\bar{x}_0 & 0 & x_1x_0 & \\
\hline
\end{array}
$$

$\underbrace{x_1x_0 \oplus A.B}_{x_1x_0}$ $x_1x_0x_1\bar{x}_0 = \underline{\underline{0}}$ $(\underbrace{x_1x_0}_{A} \oplus \underbrace{x_1.\bar{x}_0}_{B})$ 0 x_1x_0 x_0

x_1x_0	x_1	0	x_1x_0	x_0

$\downarrow$

$|y+x^3\rangle$

	$x_0 \oplus y$

- - - - -

↪ For mod2 only the last bit is important.

$$
\begin{array}{rcccc}
 & & & 1 & 1 \\
\times & & & 1 & 1 \\
\hline
 & & & 1 & 1 \\
+ & & A\,1 & 1\,B & \\
\hline
 & 1 & 0 & 0 & 1 \\
 & A.B & A\oplus B & 0 & B
\end{array}
$$

$(\underbrace{x_1x_0}_{A} \oplus \underbrace{x_1\bar{x}_0}_{B})$

$A \oplus B = \bar{A}B + A\bar{B}$

$= (\overline{x_1x_0}).x_1\bar{x}_0 + x_1x_0\,(\overline{x_1\bar{x}_0})$

$= (\bar{x}_1 + \bar{x}_0)x_1\bar{x}_0 + x_1x_0\,(\bar{x}_1 + x_0)$

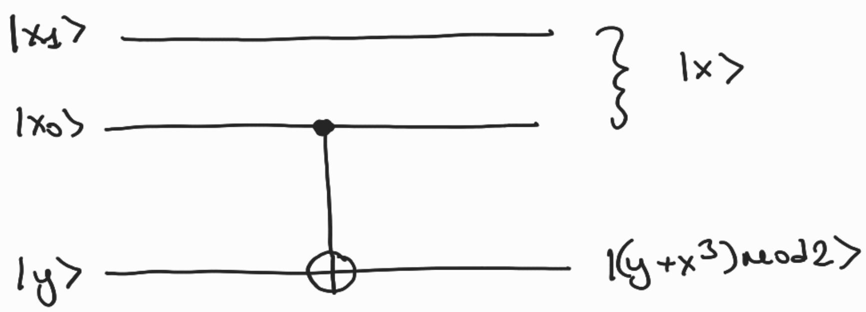
$= 0 + x_1\bar{x}_0 + 0 + x_1x_0$

$= x_1\,(\underbrace{\bar{x}_0 + x_0}_{1}) = \underline{\underline{x_1}}$

$x_1 \longrightarrow x_1$

$x_0 \longrightarrow x_0$

$y \longrightarrow x_0 \oplus y_0$ ⇝ CNOT



a) Suppose we prepare the top register in the maximal superposition state

$$\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$$

and the bottom register in $|0\rangle$. What will be the state of all the qubits after the application of U_f?

b) Suppose we obtain the result $m = +1$ (projection onto $|0\rangle$) by measuring the lower qubit in the computational basis. What is the final state of the top two qubits?

c)Suppose instead that we prepare the bottom register in $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$. What will be the state of all the qubits after the application of U_f?

d)Suppose we obtain the result $m = +1$ (projection onto $|0\rangle$) by measuring the lower qubit in the computational basis. What is the final state of the top two qubits?

e) Draw the quantum circuit that implements the unitary.

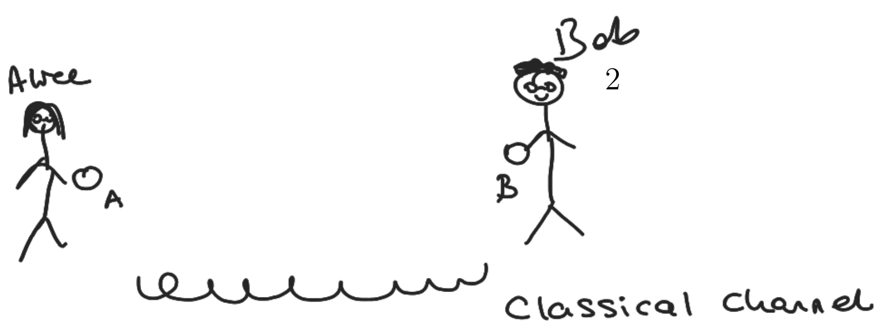
De Morgen Rules

Exercise 2:

Alice wants to teleport to Bob a qubit $|\Phi\rangle = \alpha|0\rangle + \beta|1\rangle$ using an entangled qubit pair $|e\rangle = \frac{1}{\sqrt{2}}(|0_A\rangle|0_B\rangle + |1_A\rangle|1_B\rangle)$ that they already share (Alice has qubit A and Bob has qubit B) and a classical communications channel.

a) How can Alice and Bob prepare $|e\rangle$ from $|00\rangle$?

b) Write the resulting three qubit state $|\Psi\rangle$ where Alice has the first two



qubits and Bob the last one.

c) Alice applies a CNOT gate on her two qubits, followed by a Hadamard gate on the first qubit. What is the resulting state $|\Psi'\rangle$?

d) Alice measures her two qubits in the computational basis. What state will Bob's qubit $|\Psi_B\rangle$ be in after each one of Alice's measurement outcomes?

e) Finally, Alice sends her measurement results to Bob. What correction does Bob need to apply to his qubit in each of the four cases so that he ends up with $|\Phi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$?

f) Does this instantaneous teleportation of a qubit from Alice to Bob violate the special theory of relativity that nothing can travel faster than light? $\rightarrow$ No

measurement result is sent via classical communication.

a) $|0_A\rangle|0_B\rangle + |1_A\rangle|1_B\rangle)$

$\sqrt{2}$ $\qquad$ $\sqrt{2}\,(\;\; + |11\rangle)$

$\otimes I)\;|00\rangle$

b) $|\psi_1\rangle \rightarrow$ three qubits

$$|\psi_1\rangle = |\phi\rangle \otimes |e\rangle = \frac{1}{\sqrt{2}}\left(\alpha\,|000\rangle + \alpha\,|001\rangle + \beta\,|100\rangle + \beta\,|111\rangle\right)$$

c) $|\psi_2\rangle = (H \otimes I \otimes I)(\text{CNOT} \otimes I)\,|\psi_1\rangle$

$$= \frac{1}{2}\Big(\alpha\big(|000\rangle + |011\rangle + |100\rangle + |111\rangle\big) + \beta\big(|010\rangle + |001\rangle - |110\rangle - |110\rangle\big)$$

d) $$= \frac{1}{2}\Big[|00\rangle\big(\alpha|0\rangle + \beta|1\rangle\big) + |01\rangle\big(\alpha|1\rangle + \beta|0\rangle\big)$$
$$+ |10\rangle\big(\alpha|0\rangle - \beta|1\rangle\big) + |11\rangle\big(\alpha|1\rangle - \beta|0\rangle\big)\Big]$$

e)

M	$\lvert\psi_B\rangle$	Operation Bob should apply to achive $\lvert\hat{\Phi}\rangle$
00	$\alpha\lvert 0\rangle + \beta\lvert 1\rangle$	I
01	$\alpha\lvert 1\rangle + \beta\lvert 0\rangle$	X
10	$\alpha\lvert 0\rangle - \beta\lvert 1\rangle$	Z
11	$\alpha\lvert 1\rangle - \beta\lvert 0\rangle$	ZX

$\alpha|0\rangle + \beta|1\rangle$

3.

Using CNOTs, Toffoli gates and single qubit gates implement the circuit that results in the following unitary:

Input

output →

$$U = \begin{array}{c|c} & \begin{matrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{matrix} \\ \hline \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{matrix} & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Input

$$|\psi_{out}\rangle = U|\psi_{in}\rangle$$

$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$TOFFOLI = \left(\begin{array}{c|c} I & 0 \\ \hline 0 & CNOT \end{array}\right)$$

x_2	x_1	x_0	x_2	x_1	x_0
0	0	0	1	1	0
0	0	1	1	1	1
0	1	0	1	0	0
0	1	1	1	0	1
1	0	0	0	1	0
1	0	1	0	1	1
1	1	0	0	0	1
1	1	1	0	0	0

TOFFOLI

Apply TOFFOLI

$x_2x_1x_0$	$x_2x_1x_0$	x_1
000	000	1
001	001	1
010	010	0
011	011	0
100	100	1
101	101	1
110	11$\cancel{0}$1	0
111	110	0

X Gate

Apply TOFFOLI

$x_2x_1x_0$	$x_2x_1x_0$	x_2	x_1	x_0	x_2
000	000	0	1	0	1
001	001	0	1	1	1
010	010	0	0	0	1
011	011	0	0	1	1
100	100	1	1	0	0
101	101	1	1	1	0
110	11$\cancel{0}$1	1	0	1	0
111	110	1	0	0	0

X Gate

TOFFOLI $\longrightarrow$ X $\longrightarrow$ X

$$U = (X \otimes X \otimes I) \underset{?}{\otimes} (\text{TOFFOLI}) \longrightarrow \text{Check}$$