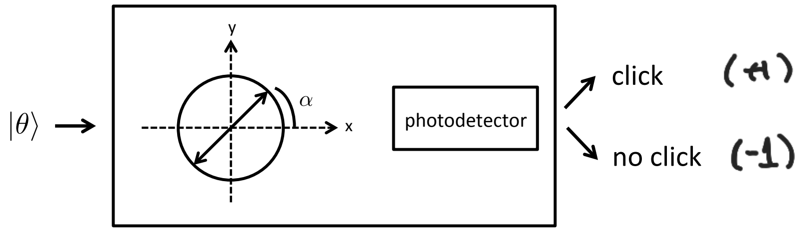
14/10/22

Homework 4
Traitement Quantique de l'Information

Exercise 1 *Polarization observable and measurement principle*

Consider the "measurement apparatus" (in the below figure) constituted of "an analyzer and a detector". The incoming (initial) state of the photon is linearly polarized:

$$|\theta\rangle = \cos\theta\,|x\rangle + \sin\theta\,|y\rangle\,.$$



When the photodetector clicks we record $+1$ and when it does not click we record -1. Thus the "polarization observable" is represented by the 2×2 matrix

$$P_\alpha = (+1)\,|\alpha\rangle\langle\alpha| + (-1)\,|\alpha_\perp\rangle\langle\alpha_\perp|$$

where $|\alpha\rangle = \cos\alpha\,|x\rangle + \sin\alpha\,|y\rangle$ and $|\alpha_\perp\rangle = -\sin\alpha\,|x\rangle + \cos\alpha\,|y\rangle$ are the two vectors of the measurement basis. Note that the two orthogonal projectors of the measurement basis are $\Pi_\alpha = |\alpha\rangle\langle\alpha|$ and $\Pi_{\alpha_\perp} = |\alpha_\perp\rangle\langle\alpha_\perp|$.

1) Show that $\Pi_\alpha^2 = \Pi_\alpha$, $\Pi_{\alpha_\perp}^2 = \Pi_{\alpha_\perp}$ and $\Pi_\alpha\Pi_{\alpha_\perp} = \Pi_{\alpha_\perp}\Pi_\alpha = 0$.

2) Check the following formulas:

$$\text{Prob}(p=+1)\quad |\langle\theta|\alpha\rangle|^2 = \langle\theta|\,\Pi_\alpha\,|\theta\rangle\,,$$
$$\text{Prob}(p=-1)\quad |\langle\theta|\alpha_\perp\rangle|^2 = \langle\theta|\,\Pi_{\alpha_\perp}\,|\theta\rangle$$

3) Let $p=\pm 1$ the random variable corresponding to the event click / no-click of the detector. Express $\text{Prob}(p=\pm 1)$ with simple trigonometric functions and check that the two probabilities sum to one.

4) Deduce from 3) $\mathbb{E}(p)$ and $\text{Var}(p)$ and check that you find the same expressions by directly computing $\langle\theta|\,P_\alpha\,|\theta\rangle$ and $\langle\theta|\,P_\alpha^2\,|\theta\rangle - \langle\theta|\,P_\alpha\,|\theta\rangle^2$ in Dirac notation.

Exercise 2 *Product versus entangled states*

Prove whether the following states are product or entangled states ? (check also they are correctly normalized)

1. $\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$
2. $\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle - |11\rangle)$
3. $\frac{1}{\sqrt{6}}|00\rangle + \frac{1}{\sqrt{3}}|01\rangle + \frac{1}{6}|10\rangle - \frac{1}{\sqrt{3}}|11\rangle$
4. $|\beta_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, $|\beta_{01}\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$, $|\beta_{10}\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$, $|\beta_{11}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$, .
5. $\frac{1}{\sqrt{1+\epsilon^2}}(|00\rangle + \epsilon|11\rangle)$, for $0 \leq \epsilon \leq 1$
6. $\frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$
7. $\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$
8. $\frac{1}{2\sqrt{2}}(|000\rangle + |100\rangle + |010\rangle + |001\rangle + |110\rangle + |101\rangle + |011\rangle + |111\rangle)$
9.

Exercise 3 *Unitary transformations*

Verify that the following transformations are unitary (check also the identities between matrix tables and Dirac notation):

1. Simple time evolution of the type $|\psi_t\rangle = e^{i\omega t}|\psi_0\rangle$. This is for example the time evolution of a free photon of frequency $\nu = \omega/2\pi$ or energy $E = h\nu = \hbar\omega$.
2. The Hadamard gate.
$$H = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = |0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1| \tag{1}$$
Check how the basis $|0\rangle$, $|1\rangle$ is transformed. Remark: in interferometers models for example a semi-transparent mirror.
3. The X or NOT gate
$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = |0\rangle\langle 1| + |1\rangle\langle 0|$$
Check how the basis $|0\rangle$, $|1\rangle$ is transformed. Remark: in interfreometers it models for example a reflecting mirror.
4. $U_1 \otimes U_2$ if U_1 and U_2 are unitary. Remark: if U_i, $i = 1, 2$ act each on a one-qubit Hilbert space $\mathbb{C}^2$ then the tensor product acts on the two-qubit space $\mathbb{C}^2 \otimes \mathbb{C}^2$.

5. The control-NOT gate. This gate flips the control bit (the second) if the target bit (the first) is 1.

$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = |00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 11| + |11\rangle\langle 10| = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X$$

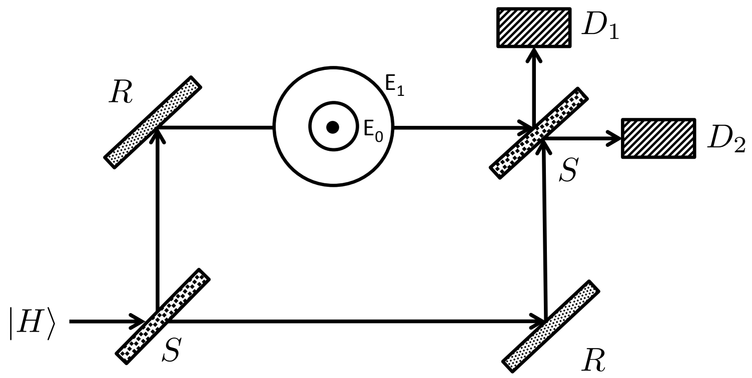
Verify the following identity:

$$|\beta_{ij}\rangle = CNOT \otimes H|i\rangle \otimes |j\rangle, \quad i, j = 0, 1$$

Deduce that $\{|\beta\rangle, i, j = 0, 1\}$ is an orthonormal basis. This identity shows that $CNOT$ entangles the two qubits.

Exercise 4 *Interferometer with an atom on the upper path*

Consider the following set-up where an atom may absorb the photon on the upper arm of the interferometer.



The Hilbert space of the photon is here $\mathbb{C}^3$ with basis states

$$|H\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, |V\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, |\text{abs}\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

the semi-transparent and reflecting mirrors are modeled by the unitary matrices

$$S = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad R = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and the "absorption-reemission" process[1] is modeled by the unitary matrix

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

[1] On the picture E_0 and E_1 are two energy levels of the atom corresponding to ground state and excited state; but you can ignore this aspect in this problem.

Note that in Dirac notation

$$A = |H\rangle\langle \text{abs}| + |V\rangle\langle V| + |\text{abs}\rangle\langle H|$$

This models three possible transitions: $A|H\rangle = |\text{abs}\rangle$ (absorption); $A|\text{abs}\rangle = |H\rangle$ (emission); and $A|V\rangle = |V\rangle$ (nothing happens).

1) Write down all matrices in Dirac notation and then compute the unitary operator $U = SARS$ representing the total evolution process of this interferometer.

2) Given that the initial state is $|H\rangle$, what is the state after the second semi-transparent mirror? What are the probabilities of the following three events: click in D_1; or click in D_2; or no clicks in D_1 nor D_2? Verify the probabilities sum to to 1.

3) Suppose the photon-atom interaction is not absorption-reemission but some other process modeled by a matrix. Which of the two following matrices would be legitimate in QM?

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

and why?

1) Show that $\Pi_\alpha^2 = \Pi_\alpha$, $\Pi_{\alpha_\perp}^2 = \Pi_{\alpha_\perp}$ and $\Pi_\alpha \Pi_{\alpha_\perp} = \Pi_{\alpha_\perp}\Pi_\alpha = 0$.

$$\Pi_a = |a\rangle\langle a|$$

$$\Pi_a^2 = |a\rangle\underbrace{\langle a|a\rangle}_{1}\langle a| = |a\rangle\langle a| = \Pi_a$$

$$\Pi_{a_\perp} = |a_\perp\rangle\langle a_\perp|$$

$$\Pi_{a_\perp}^2 = |a_\perp\rangle\underbrace{\langle a_\perp|a_\perp\rangle}_{1}\langle a_\perp| = |a_\perp\rangle\langle a_\perp| = \Pi_{a_\perp}$$

$$\Pi_a \Pi_{a_\perp} = |a\rangle\underbrace{\langle a|a_\perp\rangle}_{0}\langle a_\perp| = 0$$

$$\Pi_{a_\perp}\Pi_a = |a_\perp\rangle\underbrace{\langle a_\perp|a\rangle}_{0}\langle a| = 0$$

2) Check the following formulas:

$$\checkmark \quad |\langle\theta|\alpha\rangle|^2 = \langle\theta|\,\Pi_\alpha\,|\theta\rangle \;\rightarrow\; \langle\theta|a\rangle\langle a|\theta\rangle,$$
$$\checkmark \quad |\langle\theta|\alpha_\perp\rangle|^2 = \langle\theta|\,\Pi_{\alpha_\perp}\,|\theta\rangle \;\rightarrow\; \langle\theta|a_\perp\rangle\langle a_\perp|\theta\rangle$$

$$|\langle\theta|\alpha\rangle|^2 = \langle\theta|\alpha\rangle\langle\alpha|\theta\rangle$$

$$|\langle\theta|\alpha_\perp\rangle|^2 = \langle\theta|\alpha_\perp\rangle\langle\alpha_\perp|\theta\rangle$$

3) Let $p = \pm 1$ the random variable corresponding to the event click / no-click of the detector. Express Prob$(p = \pm 1)$ with simple trigonometric functions and check that the two probabilities sum to one.

$$\text{Prob}(p=+1) = \overset{|\langle\alpha|\theta\rangle|^2}{|\langle\theta|\alpha\rangle|^2} = \left|(\cos\theta\langle x| + \sin\theta\langle y|)(\cos\alpha|x\rangle + \sin\alpha|y\rangle)\right|^2$$

$$= \Big(\cos\theta\cos\alpha\underbrace{\langle x|x\rangle}_{1} + \sin\theta\sin\alpha\underbrace{\langle y|y\rangle}_{1}\Big)^2$$

$$= \cos^2\theta\cos^2\alpha + \underbrace{2\cos\theta\cos\alpha\sin\theta\sin\alpha}_{\frac{\sin 2\theta\,\sin 2\alpha}{2}} + \sin^2\theta\sin^2\alpha$$

$$= (\cos(\theta-\alpha))^2$$

$$\text{Prob}(p=-1) = |\langle\theta|\alpha_\perp\rangle|^2 = |(\cos\theta\langle x| + \sin\theta\langle y|)(-\sin\alpha|x\rangle + \cos\alpha|y\rangle)|^2$$

$$= \left|-\cos\theta\sin\alpha\underbrace{\langle x|x\rangle}_{1} + \sin\theta\cos\alpha\underbrace{\langle y|y\rangle}_{1}\right|^2$$

$$= (-\cos\theta\sin\alpha + \sin\theta\cos\alpha)^2$$

$$= \cos^2\theta\sin^2\alpha - \overbrace{2\cos\theta\sin\alpha\sin\theta\cos\alpha}^{\sin 2\theta\sin 2\alpha/2} + \sin^2\theta\cos^2\alpha$$

$= [\sin(\theta-\alpha)]^2$

$$\text{Prob}(p=+1) + \text{Prob}(p=-1) \stackrel{?}{=} 1$$

$$\underbrace{\boxed{\cos^2\theta\cos^2\alpha}}_{\cos^2\theta} + \cancel{\frac{\sin 2\theta\sin 2\alpha}{2}} - \cancel{\frac{\sin 2\theta\sin 2\alpha}{2}} + \boxed{\sin^2\theta\sin^2\alpha} + \boxed{\cos^2\theta\sin^2\alpha} + \boxed{\sin^2\theta\cos^2\alpha}$$

$\sin^2\theta$

$$= \cos^2\theta + \sin^2\theta = 1 \checkmark$$

$\rightarrow E(p) \quad Ep^2 \quad ?$

4) Deduce from 3) $\mathbb{E}(p)$ and $\text{Var}(p)$ and check that you find the same expressions by directly computing $\langle\theta|\,P_\alpha\,|\theta\rangle$ and $\langle\theta|\,P_\alpha^2\,|\theta\rangle - \langle\theta|\,P_\alpha\,|\theta\rangle^2$ in Dirac notation.

$$E(p) = (+1)\,\text{Prob}(p=+1) + (-1)\,\text{Prob}(p=-1)$$

$$= \cos^2\theta\cos^2\alpha + \frac{\sin 2\theta\sin 2\alpha}{2} + \sin^2\theta\sin^2\alpha$$

$$- \cos^2\theta\sin^2\alpha + \frac{\sin 2\theta\sin 2\alpha}{2} - \sin^2\theta\cos^2\alpha$$

$$= \cos^2\alpha\,(\underbrace{\cos^2\theta - \sin^2\theta}_{\cos 2\theta}) + \sin^2\alpha\,(\underbrace{\sin^2\theta - \cos^2\theta}_{-\cos 2\theta})$$

$$= \cos 2\theta\,(\cos^2\alpha - \sin^2\alpha) = \boxed{\cos 2\theta\,\cos 2\alpha}$$

From Solution:

$$(\cos(\theta-\alpha))^2 - (\sin(\theta-\alpha))^2$$

$$= \cos(2(\theta-\alpha))$$

It is given that $P_\alpha = (+1)\,|\alpha\rangle\langle\alpha| + (-1)\,|\alpha_\perp\rangle\langle\alpha_\perp|$

↳ polarization observable

$\langle \theta | \underset{\sim}{P_a} | \theta \rangle$

$$P_a|\theta\rangle = (+1)\,|a\rangle\langle a|\theta\rangle + (-1)\,|a_\perp\rangle\langle a_\perp|\theta\rangle$$

$$\langle\theta|P_a|\theta\rangle = (+1)\underbrace{\langle\theta|a\rangle\langle a|\theta\rangle}_{Prob(p=+1)} + (-1)\underbrace{\langle\theta|a_\perp\rangle\langle a_\perp|\theta\rangle}_{Prob(p=-1)} = E[p]$$

From the solution

$$Var(p) = \underset{\sim}{E[p^2]} - (E[p])^2$$
$$= 1 - E[p]^2$$
$$= 1 - \cos(2(\theta-\alpha))^2$$
$$= \sin^2(2(\theta-\alpha))$$

$(\Pi_a - \Pi_{a_\perp})$

$$\langle\theta|P_a^2|\theta\rangle = \langle\theta|((+1)\Pi_a + (-1)\Pi_{a_\perp})^2|\theta\rangle$$

$$= \langle\theta|(\Pi_a^2 - \cancel{\Pi_a\Pi_{a_\perp}}_{0} + \cancel{\Pi_{a_\perp}\Pi_a}_{0} + \Pi_{a_\perp}^2)|\theta\rangle$$

$$= \langle\theta|\Pi_a + \Pi_{a_\perp}|\theta\rangle$$

$$= (+1)^2\underbrace{\langle\theta|\Pi_a|\theta\rangle}_{Prob(p=+1)} + (-1)^2\underbrace{\langle\theta|\Pi_{a_\perp}|\theta\rangle}_{Prob(p=-1)}$$

$$= E[p^2] \;//$$

Prove whether the following states are product or entangled states ? (check also they are correctly normalized)

can be looked to determinant
if $\det(A) = 0 \rightarrow$ product state
$\neq 0 \rightarrow$ entangled

✓ 1. $\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$

$$= \frac{1}{2}\left(|0\rangle \otimes (|0\rangle + |1\rangle) + |1\rangle \otimes (|0\rangle + |1\rangle)\right)$$

$$= \frac{1}{2}\left((|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle)\right) \rightarrow \text{Product state, Normalized}$$

✓ 2. $\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle - |11\rangle)$

$$= \frac{1}{2}\left(|0\rangle \otimes (|0\rangle + |1\rangle) + |1\rangle \otimes |0\rangle - |1\rangle\right) \rightarrow \text{Entangled State, Normalized}$$

✓ 3. $\frac{1}{\sqrt{6}}|00\rangle + \frac{1}{\sqrt{3}}|01\rangle + \frac{1}{6}|10\rangle - \frac{1}{\sqrt{3}}|11\rangle$

No symmetry $\rightarrow$ entangled

$$\underset{(6)}{\frac{1}{6}} + \underset{(12)}{\frac{1}{3}} + \frac{1}{36} + \underset{(12)}{\frac{1}{3}} = \frac{18+1+12}{36} = \frac{31}{36} \neq 1 \rightarrow \text{not normalized}$$

✓ 4. $|\beta_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, $|\beta_{01}\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$, $|\beta_{10}\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$, $|\beta_{11}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$, .

Entangled States $\rightarrow$ Bell States

Normalized ✓

5. $\frac{1}{\sqrt{1+\epsilon^2}}(|00\rangle + \epsilon|11\rangle)$, for $0 \leq \epsilon \leq 1$

No symmetry $\rightarrow$ Entangled

Normalized $\rightarrow \frac{1}{1+\epsilon^2} + \frac{\epsilon^2}{1+\epsilon^2} = 1$ ✓

! For $\epsilon = 0 \rightarrow$ it is product state
otherwise entangled

✓ 6. $\frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$

No symmetry $\rightarrow$ Entangled

Normalized $\rightarrow \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$ ✓

7. $\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$

No symmetry → Entangled

Normalized ✓ $\frac{1}{2}+\frac{1}{2}=1$ ✓

8. $\frac{1}{2\sqrt{2}}(|000\rangle + |100\rangle + |010\rangle + |001\rangle + |110\rangle + |101\rangle + |011\rangle + |111\rangle)$

Normalized ✓

Symmetric → Product state

$$\left[\, |0\rangle \otimes (\underbrace{|00\rangle + |10\rangle + |01\rangle + |11\rangle}_{\text{from 1}}) + |1\rangle \otimes (|00\rangle + |10\rangle + |01\rangle + |11\rangle) \,\right]$$

$$(|0\rangle+|1\rangle) \otimes (|0\rangle+|1\rangle) \otimes (|0\rangle+|1\rangle)$$

Verify that the following transformations are unitary (check also the identities between matrix tables and Dirac notation):

1. Simple time evolution of the type $|\psi_t\rangle = e^{i\omega t}|\psi_0\rangle$. This is for example the time evolution of a free photon of frequency $\nu = \omega/2\pi$ or energy $E = h\nu = \hbar\omega$. $\mathbb{I}$

$$U_t = \begin{pmatrix} e^{i\omega t} & 0 \\ 0 & e^{i\omega t} \end{pmatrix} \qquad U_t U_t^{\dagger} \overset{?}{=} \mathbb{1}$$

$U = e^{i\omega t}|\psi_0\rangle\langle\psi_0|$

$U^{\dagger} = e^{-i\omega t}|\psi_0\rangle\langle\psi_0|$

$$U_t^{\dagger} = \begin{pmatrix} e^{-i\omega t} & 0 \\ 0 & e^{-i\omega t} \end{pmatrix} \qquad \begin{pmatrix} e^{i\omega t} & 0 \\ 0 & e^{i\omega t} \end{pmatrix}\begin{pmatrix} e^{-i\omega t} & 0 \\ 0 & e^{-i\omega t} \end{pmatrix} = \begin{pmatrix} e^{0} & 0 \\ 0 & e^{0} \end{pmatrix} = \mathbb{1} \checkmark$$

2. The Hadamard gate.

$$H = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = |0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1| \tag{1}$$

Check how the basis $|0\rangle$, $|1\rangle$ is transformed. Remark: in interferometers models for example a semi-transparent mirror.

$$\left.\begin{aligned} H &= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ H^{\dagger} &= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \end{aligned}\right\} HH^{\dagger} \qquad \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 2\mathbb{I} \checkmark$$

$H = H^{\dagger}$

$$(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|)(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|)$$

$$|0\rangle\overbrace{\langle 0|0\rangle}^{1}\langle 0| + |0\rangle\overbrace{\langle 0|0\rangle}^{1}\langle 1| + |0\rangle\overbrace{\langle 1|1\rangle}^{1}\langle 0| - |0\rangle\overbrace{\langle 1|1\rangle}\langle 1| + |1\rangle\langle 0|0\rangle\langle 0|$$

$$+ |1\rangle\langle 0|0\rangle\langle 1| - |1\rangle\langle 1|1\rangle\langle 0| + |1\rangle\langle 1|1\rangle\langle 1|$$

$$|0\rangle\langle 0| + \cancel{|0\rangle\langle 1|} + |0\rangle\langle 0| - \cancel{|0\rangle\langle 1|} + \cancel{|1\rangle\langle 0|} + |1\rangle\langle 1|$$

$$- \cancel{|1\rangle\langle 0|} + |1\rangle\langle 1| = 2\Big(\underbrace{|0\rangle\langle 0|}_{\binom{1}{0}(1\ 0)} + \underbrace{|1\rangle\langle 1|}_{\binom{0}{1}(0\ 1)}\Big)$$

$$\begin{pmatrix}1&0\\0&0\end{pmatrix} + \begin{pmatrix}0&0\\0&1\end{pmatrix} = \begin{pmatrix}1&0\\0&1\end{pmatrix} = I$$

$$H|i\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + (-1)^i|1\rangle\right) \text{ for } i \in \{0,1\}$$

$$\langle j|\underbrace{H^\dagger H}_{I}|i\rangle = \frac{1}{2}\left(\left(\langle 0| + (-1)^j\langle 1|\right)\left(|0\rangle + (-1)^i|1\rangle\right)\right) = \frac{1}{2}\left(1 + (-1)^{i+j}\right) = \delta_{ij}$$

3. The X or NOT gate

$I = |0\rangle\langle 0| + |1\rangle\langle 1|$

$$X = \begin{pmatrix}0&1\\1&0\end{pmatrix} = |0\rangle\langle 1| + |1\rangle\langle 0|$$

Check how the basis $|0\rangle$, $|1\rangle$ is transformed. Remark: in interfreometers it models for example a reflecting mirror.

$$\left.\begin{array}{l} X = \begin{pmatrix}0&1\\1&0\end{pmatrix} \\ X^\dagger = \begin{pmatrix}0&1\\1&0\end{pmatrix}\end{array}\right\} \begin{array}{l} XX^\dagger = \begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix} = \begin{pmatrix}1&0\\0&1\end{pmatrix} = I \ \checkmark \\ X = X^\dagger \end{array}$$

Unitary check

$$\delta_{ij} \equiv \begin{cases} 0 & \text{for } i \neq j \\ 1 & \text{for } i = j \end{cases}$$

$$\left(|0\rangle\langle 1| + |1\rangle\langle 0|\right)\left(|0\rangle\langle 1| + |1\rangle\langle 0|\right)$$

$$= |0\rangle\underbrace{\langle 1|1\rangle}_{1}\langle 0| + |1\rangle\underbrace{\langle 0|0\rangle}_{1}\langle 1| = |0\rangle\langle 0| + |1\rangle\langle 1|$$

$X|i\rangle = |i \oplus 1\rangle$ thus $\langle j|X^\dagger X|i\rangle = \langle j\oplus 1|i \oplus 1\rangle = \delta_{ij}$

$\oplus$ ↘ XOR

4. $U_1 \otimes U_2$ if U_1 and U_2 are unitary. Remark: if U_i, $i = 1, 2$ act each on a one-qubit Hilbert space $\mathbb{C}^2$ then the tensor product acts on the two-qubit space $\mathbb{C}^2 \otimes \mathbb{C}^2$.

$$U_1 = \begin{pmatrix}a&b\\c&d\end{pmatrix}_{2\times 2} \quad U_2 = \begin{pmatrix}e&f\\g&h\end{pmatrix}_{2\times 2}$$

$$U_1 \otimes U_2 = \begin{pmatrix} ae & af & be & bf \\ ag & ah & bg & bh \\ ce & cf & de & df \\ cg & ch & dg & dh \end{pmatrix}_{4\times 4}$$

$$(U_1 \otimes U_2)^\dagger (U_1 \otimes U_2)$$

$$= (U_1^\dagger \otimes U_2^\dagger)(U_1 \otimes U_2)$$

$$= (U_1^\dagger U_1) \otimes (U_2^\dagger U_2)$$

$$= I \otimes I$$

$$= I$$

$(a+b)(c+d)$

$ac + ad + bc + bd$

?

5. The control-NOT gate. This gate flips the control bit (the second) if the target bit (the first) is 1.

$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = |00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 11| + |11\rangle\langle 10| = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X$$

Verify the following identity:

$$|\beta_{ij}\rangle = CNOT \otimes H|i\rangle \otimes |j\rangle, \quad i,j = 0,1$$

Deduce that $\{|\beta\rangle, i,j = 0,1\}$ is an orthonormal basis. This identity shows that $CNOT$ entangles the two qubits.

$CNOT \otimes H|0\rangle \otimes |1\rangle$

$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |1\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |11\rangle)$

$\frac{1}{\sqrt{2}}(CNOT\, |\underset{\text{target}}{01}\rangle + CNOT\, |\underset{\text{target change}}{11}\rangle) = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$ ↘ Entangled state

$(|0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X) \otimes H|0\rangle \otimes |1\rangle$

$|0\rangle\langle 0| \otimes I \otimes H|0\rangle \otimes |1\rangle + |1\rangle\langle 1| \otimes X \otimes H|0\rangle|1\rangle$

$\langle k, l | CNOT^{\dagger} CNOT | i,j\rangle = \langle k, l \oplus k | i, i \oplus j\rangle = \delta_{i,k}\, \delta_{(l \oplus k),(i \oplus j)} = \delta_{i,k}\, \delta_{l,j}$

$|\psi_1\rangle = H|i\rangle \otimes |j\rangle$

$= \frac{1}{\sqrt{2}}(|0,j\rangle + (-1)^i |1,j\rangle)$

$CNOT\, |\psi_1\rangle = \frac{1}{\sqrt{2}}(|0,j\rangle + (-1)^i |1, j \oplus 1\rangle) = |\beta_{ij}\rangle$

since O is unitary, $|\beta_{ij}\rangle$ forms an orthonormal basis.

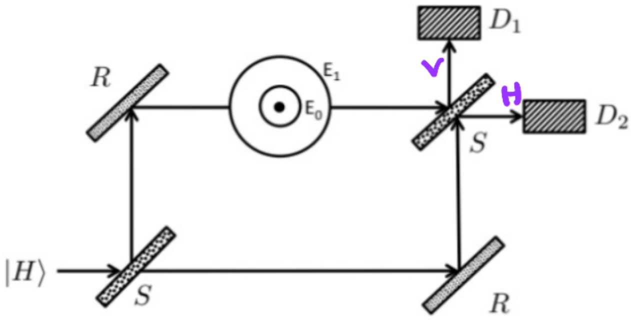
$O = CNOT . (H \otimes I) \longrightarrow$ unitary

O: unitary operator

unitary

Exercise 4 *Interferometer with an atom on the upper path*

Consider the following set-up where an atom may absorb the photon on the upper arm of the interferometer.



The Hilbert space of the photon is here $\mathbb{C}^3$ with basis states

$$|H\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix}, |V\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix}, |\text{abs}\rangle = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

the semi-transparent and reflecting mirrors are modeled by the unitary matrices

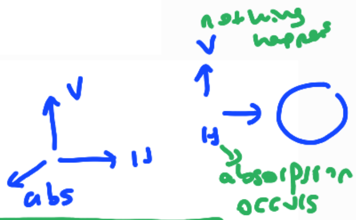
$$S = \begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0\\ 0 & 0 & 1\end{pmatrix}, \qquad R = \begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}$$

and the "absorption-reemission" process[1] is modeled by the unitary matrix

$$A = \begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}$$

Note that in Dirac notation

$$A = |H\rangle\langle\text{abs}| + |V\rangle\langle V| + |\text{abs}\rangle\langle H|$$



This models three possible transitions: $A|H\rangle = |\text{abs}\rangle$ (absorption); $\boxed{A|\text{abs}\rangle = |H\rangle \text{ (emission)};}$ and $A|V\rangle = |V\rangle$ (nothing happens).

1) Write down all matrices in Dirac notation and then compute the unitary operator $U = SARS$ representing the total evolution process of this interferometer.

$$S|H\rangle \quad \begin{pmatrix}1/\sqrt{2} & 1/\sqrt{2} & 0\\ 1/\sqrt{2} & -1/\sqrt{2} & 0\\ 0&0&1\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}(|H\rangle + |V\rangle)$$

$$S|V\rangle = \begin{pmatrix}1/\sqrt{2} & 1/\sqrt{2} & 0\\ 1/\sqrt{2} & -1/\sqrt{2} & 0\\ 0&0&1\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}(|H\rangle - |V\rangle)$$

$$S|\text{abs}\rangle \quad \begin{pmatrix}1/\sqrt{2} & 1/\sqrt{2} & 0\\ 1/\sqrt{2} & -1/\sqrt{2} & 0\\ 0&0&1\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix} = |\text{abs}\rangle$$

$$S = \frac{1}{\sqrt{2}}\Big((|H\rangle + |V\rangle)\langle H| + (|H\rangle - |V\rangle)\langle V| + |\text{abs}\rangle\langle\text{abs}|\Big)$$

$$= \frac{1}{\sqrt{2}}\Big(|H\rangle\langle H| + |V\rangle\langle H| + |H\rangle\langle V| - |V\rangle\langle V|\Big) + |\text{abs}\rangle\langle\text{abs}|$$

$$R|H\rangle = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = |V\rangle$$

$$R|V\rangle = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = |H\rangle$$

$$R|abs\rangle = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = |abs\rangle$$

$$R = |V\rangle\langle H| + |H\rangle\langle V| + |abs\rangle\langle abs|$$

SARS $\rightarrow$ Even the order is reversed, the result is same! since they're unitary and reversible?

$$SA = \frac{1}{\sqrt{2}}\Big(|H\rangle\langle H|H\rangle\langle abs| + |V\rangle\langle H|H\rangle\langle abs|$$

$$+ |H\rangle\langle V|V\rangle\langle V| - |V\rangle\langle V|V\rangle\langle V| \Big) + |abs\rangle\langle abs|abs\rangle\langle H|$$

$$SA = \frac{1}{\sqrt{2}}\Big(|H\rangle\langle abs| + |V\rangle\langle abs| + |H\rangle\langle V| - |V\rangle\langle V| \Big) + |abs\rangle\langle H|$$

$$SAR = \frac{1}{\sqrt{2}}\Big(|H\rangle\langle abs| + |V\rangle\langle abs| + |H\rangle\langle H| - |V\rangle\langle H| \Big) + |abs\rangle\langle V|$$

$$SARS = \frac{1}{\sqrt{2}}\Big(|H\rangle\langle abs| + |V\rangle\langle abs| + \frac{1}{\sqrt{2}}|H\rangle\langle H| + \frac{1}{\sqrt{2}}|H\rangle\langle V|$$

$$- \frac{1}{\sqrt{2}}|V\rangle\langle H|H\rangle\langle H| - \frac{1}{\sqrt{2}}|V\rangle\langle H|H\rangle\langle V| \Big) +$$

$$\frac{1}{\sqrt{2}}|abs\rangle\langle V|V\rangle\langle H| - \frac{1}{\sqrt{2}}|abs\rangle\langle V|V\rangle\langle V|$$

$$SARS = \frac{1}{\sqrt{2}}|H\rangle\langle abs| + \frac{1}{\sqrt{2}}|V\rangle\langle abs| + \frac{1}{2}|H\rangle\langle H| + \frac{1}{2}|H\rangle\langle V|$$

$$+ \; -\frac{1}{2}|V\rangle\langle H| - \frac{1}{2}|V\rangle\langle V| + \frac{1}{\sqrt{2}}|abs\rangle\langle H| - \frac{1}{\sqrt{2}}|abs\rangle\langle V|$$

2) Given that the initial state is $|H\rangle$, what is the state after the second semi-transparent mirror? What are the probabilities of the following three events: click in D_1; or click in D_2; or no clicks in D_1 nor D_2? Verify the probabilities sum to to 1.

$$SARS|H\rangle = \frac{1}{2}|H\rangle - \frac{1}{2}|V\rangle + \frac{1}{\sqrt{2}}|abs\rangle$$

$$\left.\begin{aligned} Prob(D_1) &= |\langle V|SARS|H\rangle|^2 = \frac{1}{4} \\ Prob(D_2) &= |\langle H|SARS|H\rangle|^2 = \frac{1}{4} \\ Prob(abs) &= |\langle abs|SARS|H\rangle|^2 = \frac{1}{2} \end{aligned}\right\} \quad \frac{1}{4}+\frac{1}{4}+\frac{1}{2}=1$$

3) Suppose the photon-atom interaction is not absorption-reemission but some other process modeled by a matrix. Which of the two following matrices would be legitimate in QM?

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

and why?

Unitary? ✓

It is not unitary

$$U = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix} \qquad U^\dagger = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix}$$

$$UU^\dagger = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I \checkmark$$

The second matrix may model. This matrix acts like a Hadamard matrix on the subspace $\{|H\rangle, |abs\rangle\}$