WEEK 9 Exercise

1. Rabi's Formula

Exercise 1 : Rabi's formula

A qubit in the presence of an oscillatory driving field follows the following Hamiltonian

$$H = \frac{\hbar\omega_0}{2}\sigma_z + \frac{A}{2}(\sigma_x \cos(\omega t) + \sigma_y \sin(\omega t)) \tag{1}$$

1. Show that in a rotating frame the Schrödinger equation has the following form

$$i\frac{d}{dt}\begin{pmatrix} b_0(t) \\ b_1(t) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \Delta\omega & \omega_1 \\ \omega_1 & -\Delta\omega \end{pmatrix}\begin{pmatrix} b_1(t) \\ b_0(t) \end{pmatrix} \tag{2}$$

where $\Delta\omega = \omega_0 - \omega$ and $\hbar\omega_1 = A$.
Hint: Use the unitary transform $U = e^{i\frac{\omega t}{2}\sigma_z}$
Hint: Use the fact that $e^{i\frac{\theta}{2}(n_x\sigma_x + n_y\sigma_y + n_z\sigma_z)} = \cos\frac{\theta}{2}\mathbb{1} + i\sin\frac{\theta}{2}(n_x\sigma_x + n_y\sigma_y + n_z\sigma_z)$

$$H = \frac{\hbar\omega_0}{2}\overset{\sigma_z}{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}} + \frac{\overset{\hbar\omega_1}{A}}{2}\left(\cos(\omega t)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \sin(\omega t)\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\right)$$

$$H = \frac{\hbar}{2}\begin{pmatrix} \omega_0 & \omega_1 e^{-i\omega t} \\ \omega_1 e^{i\omega t} & -\omega_0 \end{pmatrix}$$

$$i\hbar\frac{d}{dt}|\psi\rangle = H|\psi\rangle$$

$$|\psi(t)\rangle = a_0(t)|0\rangle + a_1(t)|1\rangle$$

we need to get rid of

$$i\hbar\frac{d}{dt}\begin{pmatrix} a_0(t) \\ a_1(t) \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} \omega_0 & \omega_1 e^{-i\omega t} \\ \omega_1 e^{i\omega t} & -\omega_0 \end{pmatrix}\begin{pmatrix} a_0(t) \\ a_1(t) \end{pmatrix}$$

Thus make a transformation into a rotating frame

Unitary transform $U = e^{\frac{i\omega t}{2}\sigma_z}$

$$U = e^{\frac{i\omega t}{2}\sigma_z} = \cos\left(\frac{\omega t}{2}\right) I + i\sin\left(\frac{\omega t}{2}\right)\sigma_z = \begin{pmatrix} e^{i\omega t/2} & 0 \\ 0 & e^{-i\omega t/2} \end{pmatrix}$$

$$\underbrace{|\tilde{\psi}\rangle}_{\text{rotating frame}} = U|\psi\rangle = \begin{pmatrix} a_0(t)e^{i\omega t/2} \\ a_1(t)e^{-i\omega t/2} \end{pmatrix} = \begin{pmatrix} b_0(t) \\ b_1(t) \end{pmatrix} = b_0(t)|0\rangle + b_1(t)|1\rangle$$

$$UHU^\dagger U|\psi\rangle = i\hbar U \frac{d}{dt} U^\dagger U|\psi\rangle =$$

$$= \underbrace{UHU^\dagger|\tilde{\psi}\rangle}_{i\hbar\frac{d}{dt}} = i\hbar\left(U\frac{d}{dt}U^\dagger + \underbrace{UU^\dagger}_{I}\frac{d}{dt}\right)|\tilde{\psi}\rangle$$

$$= \underbrace{\left(UHU^\dagger - i\hbar U\frac{d}{dt}U^\dagger\right)}_{\tilde{H}\ \text{(rotating Hamiltonian)}} = i\hbar\frac{d}{dt}|\tilde{\psi}\rangle$$

$$\tilde{H} = \begin{pmatrix} e^{i\omega t/2} & 0 \\ 0 & e^{-i\omega t/2} \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} \omega_0 & \omega_1 e^{-i\omega t} \\ \omega_1 e^{i\omega t} & -\omega_0 \end{pmatrix} \begin{pmatrix} e^{-i\omega t/2} & 0 \\ 0 & e^{+i\omega t/2} \end{pmatrix}$$

$$-i\hbar \begin{pmatrix} e^{i\omega t/2} & 0 \\ 0 & e^{i\omega t/2} \end{pmatrix} \begin{pmatrix} -\frac{i\omega}{2} e^{-i\omega t/2} & 0 \\ 0 & +\frac{i\omega}{2} e^{+i\omega t/2} \end{pmatrix}$$

$$\begin{pmatrix} e^{i\omega t/2} & 0 \\ 0 & e^{-i\omega t/2} \end{pmatrix} \begin{pmatrix} \omega_0 e^{-i\omega t/2} & \omega_1 e^{-i\omega t/2} \\ \omega_1 e^{i\omega t/2} & -\omega_0 e^{i\omega t/2} \end{pmatrix}$$

$$\tilde{H} = \frac{\hbar}{2}\begin{pmatrix} \omega_0 & \omega_1 \\ \omega_1 & -\omega_0 \end{pmatrix} - i\hbar \begin{pmatrix} -i\omega/2 & 0 \\ 0 & i\omega/2 \end{pmatrix}$$

$$= \frac{\hbar}{2}\begin{pmatrix} \Delta\omega & \omega_1 \\ \omega_1 & -\Delta\omega \end{pmatrix} \quad \text{where} \quad \Delta\omega = \omega - \omega_0$$

$$i\hbar \frac{d}{dt}\begin{pmatrix} b_0(t) \\ b_1(t) \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} \Delta\omega & \omega_1 \\ \omega_1 & -\Delta\omega \end{pmatrix}\begin{pmatrix} b_1(t) \\ b_0(t) \end{pmatrix}$$

2. Calculate the probability of being in the excited state $|1\rangle$ under the assumption that at time zero $t = 0$ the system is in the ground state $|\psi(0)\rangle = 0$

$$P_1(t) = |\langle 1|\psi(t)\rangle|^2 \qquad (3)$$

Hint: Use the normalization condition of the eigenvectors.

$$P_1(t) = |\langle 1|\psi(t)\rangle|^2 = |\langle 1|\tilde{\psi}(t)\rangle|^2$$

$$|\tilde{\psi}(t)\rangle = e^{i\frac{E_0}{\hbar}t}|v_0\rangle\langle v_0|\tilde{\psi}(0)\rangle + e^{i\frac{E_1}{\hbar}t}|v_1\rangle\langle v_1|\tilde{\psi}(0)\rangle$$

$E_0, E_1 \rightarrow$ eigenenergies

$v_0, v_1 \rightarrow$ eigen vectors

$$\boxed{\begin{aligned} E_0 &= \frac{\hbar}{2}\sqrt{\Delta\omega^2 + \omega_1^2} \\ E_1 &= -\frac{\hbar}{2}\sqrt{\Delta\omega^2 + \omega_1^2} \end{aligned}}$$

$$\det(\lambda I - A) = 0$$

$$A = \lambda A$$

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} + \begin{pmatrix} -\Delta\omega & -\omega_1 \\ -\omega_1 & +\Delta\omega \end{pmatrix}$$

$$\begin{matrix} \lambda - \Delta\omega & -\omega_1 \\ -\omega_1 & \lambda + \Delta\omega \end{matrix}$$

$$\lambda^2 - \Delta\omega^2 - (\omega_1)^2 = 0$$

$$\lambda^2 = (\omega_1)^2 + (\Delta\omega)^2$$

$$\lambda_{0,1} = \pm\frac{\hbar}{2}\sqrt{\Delta\omega^2 + \omega_1^2}$$

$|v_0\rangle = \begin{pmatrix} a \\ b \end{pmatrix} = ?$

$$\tilde{H} \begin{pmatrix} a \\ b \end{pmatrix} = E_0 \begin{pmatrix} a \\ b \end{pmatrix}$$

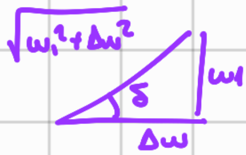
$$\frac{\hbar}{2} \begin{pmatrix} \Delta\omega & \omega_1 \\ \omega_1 & -\Delta\omega \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = E_0 \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\frac{\hbar}{2} \Delta\omega a + \frac{\hbar}{2} b \omega_1 = a E_0$$

$$b = - \frac{a\left(\Delta\omega - \frac{2}{\hbar} E_0\right)}{\omega_1}$$

$|a|^2 + |b|^2 = 1 \longrightarrow$ Due to Normalization

$$|a|^2 + |a|^2 \left(\frac{\Delta\omega - \sqrt{\Delta\omega^2 + \omega_1^2}}{\omega_1} \right)^2 = 1$$



Let's introduce angle δ such that $\tan\delta = \frac{\omega_1}{\Delta\omega}$

$$\sin\delta = \frac{\omega_1}{\sqrt{\Delta\omega^2 + \omega_1^2}}$$

$$\cos\delta = \frac{\Delta\omega}{\sqrt{\Delta\omega^2 + \omega_1^2}}$$

$$|a|^2 + |a|^2 \Big(\underbrace{\frac{1}{\tan\delta} - \frac{1}{\sin\delta}}_{\tan(\delta/2)} \Big)^2 = 1$$

$$|v_0\rangle = \begin{pmatrix} \cos\delta/2 \\ e^{i\phi} \sin\delta/2 \end{pmatrix}$$

$$|a|^2 + |a|^2 \tan\left(\frac{\delta}{2}\right)^2 = 1 \longrightarrow a = \cos(\delta/2)$$

Don't forget phase $\quad b = e^{i\phi} \sin(\delta/2)$

Similar to steps in $|v_0\rangle$ finding ...

$$|v_1\rangle = \begin{pmatrix} \sin\delta/2 \\ -e^{i\phi}\cos\delta/2 \end{pmatrix}$$

$$|\tilde{\Psi}(t)\rangle = e^{+i\frac{\sqrt{\Delta w^2 + w_1^2}\,t}{2}\,B} |v_0\rangle \cos\delta/2 + e^{-i\frac{\sqrt{\Delta w^2 + w_1^2}\,t}{2}\,B} |v_1\rangle \sin\delta/2$$

$(0\ 1) \leftarrow \langle 1|v_0\rangle \binom{a}{b} = b \qquad \binom{1}{0} \langle 0|v_1\rangle \ (a\ b) = a$

$$P_1(t) = |\langle 1|\tilde{\Psi}(t)\rangle|^2 = \left| e^{i\phi}\sin\delta/2\cos\delta/2\, e^{+iB} - e^{i\phi}\cos\delta/2\sin\delta/2\, e^{-iB}\right|^2$$

$$= \left| \underset{1}{e^{i\phi}} \cos\frac{\delta}{2}\sin\frac{\delta}{2}\left(\underbrace{e^{iB} - e^{-iB}}_{\substack{c+is\ -(c-is) \\ 2is}}\right)\right|^2$$

$$= \left|\cos\frac{\delta}{2}\sin\frac{\delta}{2}\, 2i\sin(B)\right|^2$$

$$= \underbrace{4\cos^2\frac{\delta}{2}\sin^2\frac{\delta}{2}}_{\sin(\delta)^2} \sin^2\left(\frac{\sqrt{\Delta w^2 + w_1^2}}{2}\,t\right)$$

$$P_1(t) = \frac{w_1^2}{\Delta w^2 + w_1^2}\sin^2\left(\frac{\Omega_R}{2}t\right)$$

$$\Omega_R = \sqrt{\Delta w^2 + w_1^2} \rightarrow \text{Rabi Frequency}$$

3. Calculate the time average of the probability of being in the excited state

$$\bar{P}_1 = \lim_{t\to\infty} \frac{1}{T} \int_0^T P_1(t)dt \tag{4}$$

$E[\cos(x)] = 0$

$$\lim_{t\to\infty} \frac{1}{T} \int_0^T \sin^2(x) \rightarrow \frac{1}{2} - \frac{1}{2}\cos(x)\Big|^T \rightarrow \frac{1}{2}$$

$$\overline{P_1} = \frac{1}{2} \frac{\omega_1^2}{\Delta\omega^2 + \omega_1^2}$$

Exercise 2 : Ramsey Fringes

Exercise 2 : Ramsey fringes

Consider the following Hamiltonian:

$$\mathcal{H} = -\frac{\hbar}{2}(\Delta\sigma_z + \Omega\sigma_x).$$

In the following, you will calculate a pulse sequence that is often used in characterizing qubits or even in using an atomic clock. It is called Ramsey sequence and starts with the qubit in its ground state, then applies a $\pi/2$ pulse, followed by free evolution for time $t = T$ and another $\pi/2$ pulse.

1. Start with the qubit in its ground state $|g\rangle$ at time $t = 0$. Apply a $\pi/2$ pulse and calculate the state after this pulse. Remember: the pulse is applied in the resonant case and for a time duration $T_{\pi/2} = \frac{\pi}{2}\frac{1}{\Omega}$

$\hookrightarrow H = -\frac{\hbar}{2}\Omega\sigma_x$

$$U\left(t = \frac{\pi}{2}\right) = e^{-i\frac{H}{\hbar}t} = e^{\frac{-i\left(-\frac{\hbar}{2}\Omega\sigma_x\right)\cdot\frac{\pi}{2}\frac{1}{\Omega}}{\hbar}}$$

$$= e^{i\frac{1}{2}\frac{\pi}{2}\sigma_x} = e^{i\frac{\pi}{4}\sigma_x} = \cos\frac{\pi}{4}\,\mathbb{I} + i\sin\frac{\pi}{4}\sigma_x$$

$$= \frac{1}{\sqrt{2}}(\mathbb{I} + i\sigma_x)$$

ground state

$$U\left(\frac{\pi}{2}\right)|0\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + i|1\rangle\right)$$

2. Now, let the state evolve freely ($\Omega = 0$). What is the state after time $t = T$? Calculate the population of the excited state after this time. $\rightarrow$ $P_1 = |\langle 1|\psi'\rangle|^2$

$$U(t) = e^{-i\frac{H}{\hbar}t} = e^{i\frac{\Delta t \sigma_z}{2}}$$

$$H = -\frac{\hbar}{2}\Delta\sigma_z \qquad = \cos\left(\frac{\Delta t}{2}\right) I + i\sin\left(\frac{\Delta t}{2}\right)\sigma_z$$

$$\Psi = \frac{1}{\sqrt{2}}\left(|0\rangle + i|1\rangle\right)$$

$$|\psi'\rangle = U(T)|\psi\rangle = \frac{1}{\sqrt{2}}\left(e^{i\frac{\Delta T}{2}}|0\rangle + ie^{-i\frac{\Delta T}{2}}|1\rangle\right)$$

$$P_1 = \left|\frac{1}{\sqrt{2}}\, ie^{-i\frac{\Delta T}{2}}\right|^2 = \frac{1}{2}_{\,//}$$

3. Apply now a second $\pi/2$ pulse. What is the state after this pulse? Calculate the population of the excited state after this time. $\rightarrow$ $P_1 = |\langle 1|\psi_2\rangle|^2$

$$|\psi_2\rangle = \underbrace{U\left(\frac{\pi}{2}\right)}_{\frac{1}{\sqrt{2}}(\mathbb{I}+i\sigma_x)}|\psi'\rangle \rightarrow \frac{1}{\sqrt{2}}\left(e^{i\frac{\Delta T}{2}}|0\rangle + ie^{-i\frac{\Delta T}{2}}|1\rangle\right)$$

$$= \frac{1}{2}\left(e^{i\frac{\Delta T}{2}}|0\rangle + ie^{-i\frac{\Delta T}{2}}|1\rangle + ie^{i\frac{\Delta T}{2}}|1\rangle - e^{-i\frac{\Delta T}{2}}|0\rangle\right)$$

$$= \frac{1}{2}\left(|0\rangle\left(\underbrace{e^{i\frac{\Delta T}{2}} - e^{-i\frac{\Delta T}{2}}}_{C+iS-(C-iS)}\right) + i|1\rangle\left(\underbrace{e^{i\frac{\Delta T}{2}} + e^{-i\frac{\Delta T}{2}}}\right)\right)$$

$$= \cancel{\frac{1}{2}}\left(\cancel{2i}\sin\left(\frac{\Delta T}{2}\right)|0\rangle + \cancel{i}\,\cancel{2}\cos\left(\frac{\Delta T}{2}\right)|1\rangle\right)$$

i = global phase

$$P_1 = |\langle 1|\psi_2\rangle|^2 = \left| \cos\left(\frac{\Delta \tau}{2}\right) \right|^2 = \cos^2\left(\frac{\Delta \tau}{2}\right)$$