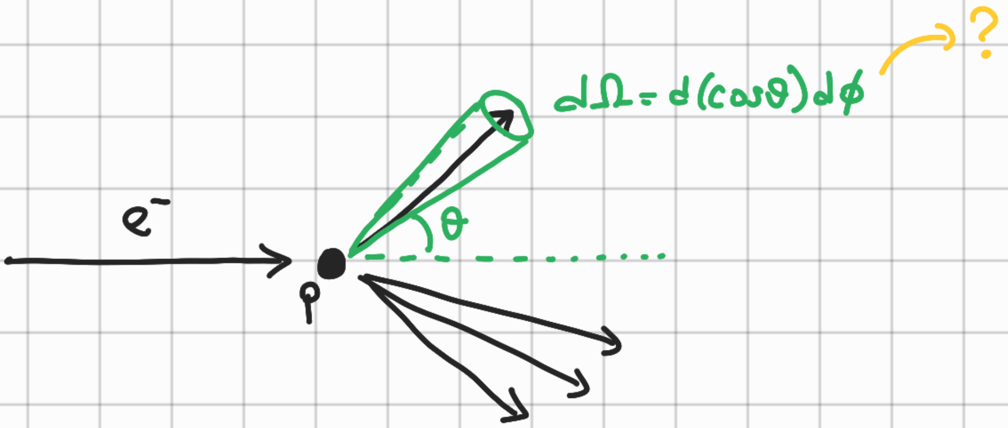
Week 5 Particle Scattering

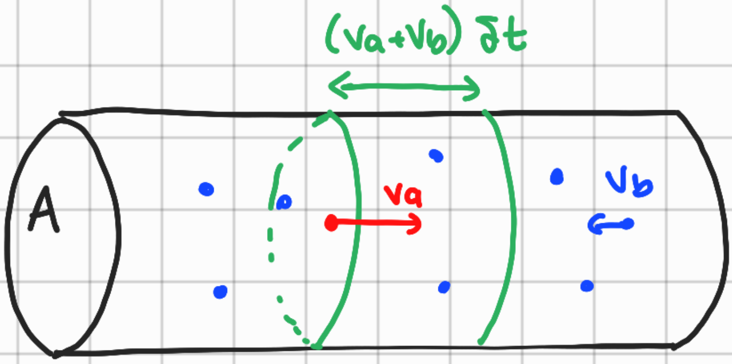
Cross Sections

$$\sigma = \frac{\text{\# of interactions per unit time per target}}{\text{incident flux}}$$

$$\frac{d\sigma}{d\Omega} = \frac{\text{\# of interactions per unit time per target into } d\Omega}{\text{incident flux}}$$

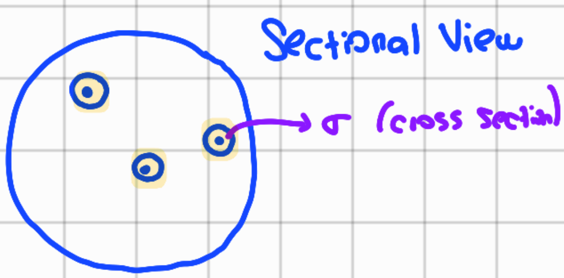


$$\rightarrow \sigma = \int \frac{d\sigma}{d\Omega}\, d\Omega$$



in time δt a single particle a traverses region

containing $n_b (v_a + v_b) A \delta t$ particles of type b



$$\text{Interaction probability} = \frac{n_b \overbrace{(v_a + v_b)}^{v} \cancel{A}\, \delta t\, \sigma}{\cancel{A}}$$

$$= \boxed{n_b\, v\, \delta t\, \sigma}$$

Rate for a single particle a = $n_b v \sigma$ $\quad \frac{dP}{dt} = r_a$ (prob. / rate)

Consider volume V, total reaction rate:

$$\text{Total Rate} = \underbrace{(n_b v \sigma)}_{\text{rate } a} \cdot (\underbrace{n_a}_{\#\text{density}} V) = (n_b V)(n_a v)\sigma$$

$$= N_b\, \phi_a\, \sigma$$

Total Rate = $(\#\text{ of targets})_b \cdot (\text{Flux})_a \cdot (\text{Cross Section})$

Cross Section Calculations

"Scattering Process"

$$1 + 2 \rightarrow 3 + 4$$

why only p_3 & p_4?

Fermi's Golden Rule:

$$\Gamma_{fi} = (2\pi)^4 \int |T_{fi}|^2\, \delta(E_1 + E_2 - E_3 - E_4)\, \delta^3(\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - \vec{p}_4)\, \frac{d^3\vec{p}_3}{(2\pi)^3} \frac{d^3\vec{p}_4}{(2\pi)^3}$$

$$\frac{\text{Rate}}{\text{Volume}} = (\#\,\text{Density of } 2)\cdot(\text{Flux of } 1)\times\sigma = n_2\cdot n_1(v_1+v_2)\cdot\sigma$$

For 1 target particle per unit volume

$$\text{Rate} = (v_1+v_2)\,\sigma \Longrightarrow \sigma = \frac{\Gamma_{fi}}{v_1+v_2}$$

$$\sigma = \frac{(2\pi)^4}{v_1+v_2}\int |T_{fi}|^2\,\delta(E_1+E_2-E_3-E_4)\,\delta^3(p_1+p_2-p_3-p_4)\,\frac{d^3p_3}{(2\pi)^3}\,\frac{d^3p_4}{(2\pi)^3}$$

Lorentz Invariant Parts

To obtain a Lorentz invariant form use wave-functions normalised to $2E$ particles per unit volume $\psi' = (2E)^{1/2}\,\psi$

Again, define $M_{fi} = (2E_1\,2E_2\,2E_3\,2E_4)^{1/2}\,T_{fi}$

$$\sigma = \frac{(2\pi)^{-2}}{2E_1\cdot 2E_2\,(v_1+v_2)}\int |M_{fi}|^2\,\delta(E_1+E_2-E_3-E_4)\,\delta^3(p_1+p_2-p_3-p_4)\,\frac{d^3p_3}{2E_3}\,\frac{d^3p_4}{2E_4}$$

Lorentz-Invariant form

$$F = 2E_1\,2E_2\,(v_1+v_2) \xrightarrow[\text{of 4-vector scalar product}]{\text{in the form}} F = 4\left[(p_1^{\mu}p_{2\mu})^2 - m_1^2 m_2^2\right]^{1/2}$$

Two Special cases of Lorentz Invariant Flux

1. CoM frame: $F = 4E_1E_2(v_1+v_2)$

$$= 4E_1E_2\left(\frac{|\vec{p}^*|}{E_1} + \frac{|\vec{p}^*|}{E_2}\right)$$

$$= 4|\vec{p}^*|(E_1+E_2) = 4|\vec{p}^*|\sqrt{s}$$

Mandelstam Variable "s" $(p_1+p_2)^2$

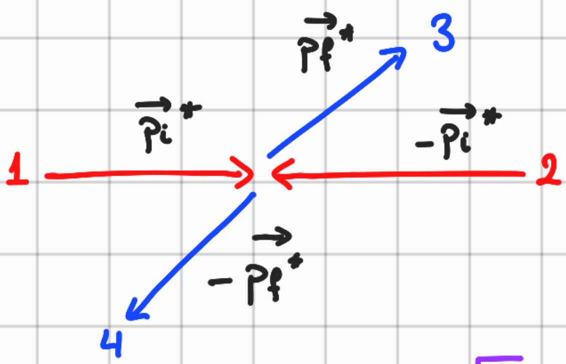
2. Target particle (particle 2) at rest $\rightarrow E_2 = m_2$, $v_2 = 0$

$$F = 4E_1E_2(v_1+v_2)$$

$$= 4E_1 m_2 v_1$$

$$= 4\cancel{E_1} m_2 \frac{|p_1|}{\cancel{E_1}} = 4 m_2 |\vec{p_1}|$$

2 → 2 Body Scattering in CoM frame



$$\sigma = \frac{(2\pi)^{-2}}{\underbrace{(2E_1)(2E_2)(v_1+v_2)}_{4|\vec{p_i}^*|\sqrt{s}}} \int |M_{fi}|^2 \, \delta(\overbrace{E_1+E_2}^{\sqrt{s}}-E_3-E_4) \, \delta^3(\overbrace{\vec{p_1}+\vec{p_2}}^{0}-\vec{p_3}-\vec{p_4}) \frac{d^3p_3}{2E_3}\frac{d^3p_4}{2E_4}$$

$$\vec{p_1}+\vec{p_2} = 0 \longrightarrow E_1+E_2 = \sqrt{s}$$

$$\sigma = \frac{(2\pi)^{-2}}{4\,|\vec{p}_i^{\,*}|\sqrt{s}} \underbrace{\int |M_{fi}|^2\, \delta(\sqrt{s} - E_3 - E_4)\, \delta^3(\vec{p}_3 + \vec{p}_4)\, \frac{d^3p_3}{2E_3}\, \frac{d^3p_4}{2E_4}}$$

similar to case in particle decay calculation

$$\sigma = \frac{(2\pi)^{-2}}{4\,|\vec{p}_i^{\,*}|\sqrt{s}} \overbrace{\frac{|\vec{p}_f^{\,*}|}{4\,m_i} \int |M_{fi}|^2\, d\Omega^*}$$

$\uparrow \sqrt{s}$

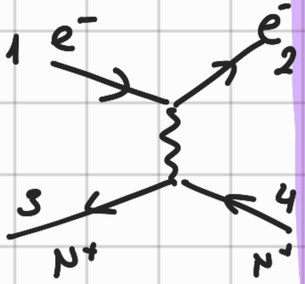
$$\sigma = \frac{1}{64\pi^2 s} \frac{|\vec{p}_f^{\,*}|}{|\vec{p}_i^{\,*}|} \int |M_{fi}|^2\, d\Omega^*$$

For an elastic scattering $|\vec{p}_i^{\,*}| = |\vec{p}_f^{\,*}|$

$$\sigma_{\text{elastic}} = \frac{1}{64\pi^2 s} \int |M_{fi}|^2\, d\Omega^*$$

$\swarrow$

total cross section



$$d\sigma = \frac{1}{64\pi^2 s} \frac{|\vec{p}_f^{\,*}|}{|\vec{p}_i^{\,*}|} |M_{fi}|^2\, d\Omega^*$$

$\downarrow$

$d(\cos\theta^*)\, d\phi^*$

in the C.o.M frame

! let's find a L.I expression for $d\sigma$

Mandelstam "t"

$$t = q^2 = (p_1 - p_3)^2 = p_1^2 + p_3^2 - 2p_1 \cdot p_3 = m_1^2 + m_3^2 - 2p_1 \cdot p_3$$

$$p_1^{*\mu} = \left(E_1^*, 0, 0, |\vec{p}_1^{\,*}| \right)$$

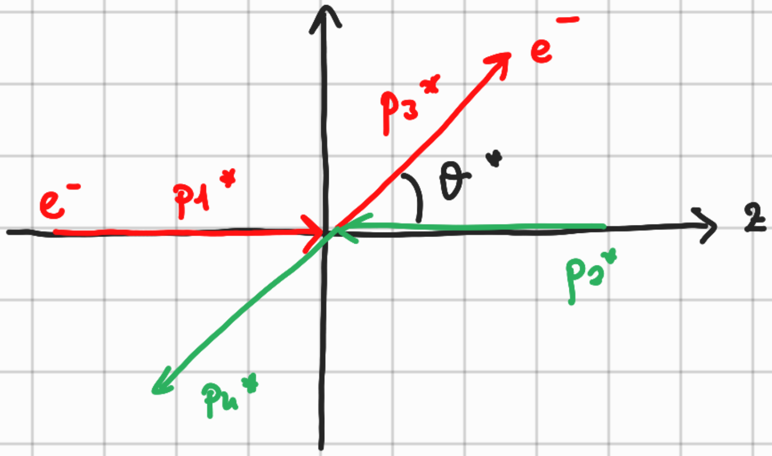
$$p_3^{*\mu} = \left(E_3^*, |\vec{p}_3^{\,*}| \sin\theta^*, 0, |\vec{p}_3^{\,*}| \cos\theta^* \right)$$

$$p_1^{\mu} p_{3\mu} = E_1^* E_3^* - |\vec{p}_1^{\,*}| |\vec{p}_3^{\,*}| \cos\theta^*$$

only free parameter

$$t = m_1^2 + m_3^2 - 2 E_1^* E_3^* + 2 p_1^* p_3^* \cos\theta^*$$

$$dt = 2 p_1^* p_3^* \, d(\cos\theta^*)$$



$$d\Omega^* = d(\cos\theta^*) \, d\phi^*$$

$$d\Omega^* = \frac{dt \, d\phi^*}{2 |\vec{p}_1^{\,*}| |\vec{p}_3^{\,*}|}$$

Mandelstam "t"

$$d\sigma = \frac{1}{64\pi^2 s} \frac{|\vec{p}_f^{\,*}|}{|\vec{p}_i^{\,*}|} |M_{fi}|^2 \, d\Omega^* = \frac{1}{2 \cdot 64\pi^2 s \cdot |\vec{p}_i^{\,*}|^2} |M_{fi}|^2 \, d\phi^* \, dt$$

Assume no ϕ^* dependence of $|M_{fi}|^2$

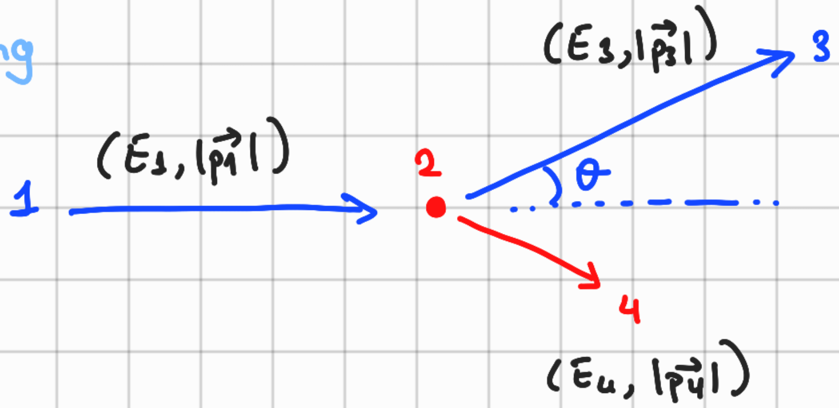
$\downarrow$ Integrate over $d\phi^*$, introduces a factor of 2π

$$\frac{d\sigma}{dt} = \frac{1}{64\pi s \, p_i^{*2}} \left| M_{fi} \right|^2$$

Lorentz Invariant

$$p_i^{*2} = \frac{1}{4s}\left[s - (M_1+M_2)^2\right]\left[s - (M_1-M_2)^2\right]$$

Scattering



$m_1 = m_3 = 0$

$M_2 = m_4 = M$

$$d\Omega = 2\pi \, d(\cos\theta)$$

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma}{dt}\frac{dt}{d\Omega} = \frac{1}{2\pi}\frac{t}{d(\cos\theta)}\frac{d\sigma}{dt}$$

$$p_1 = (E_1, 0, 0, E_1)$$

$$p_2 = (M, 0, 0, 0)$$

$$p_3 = (E_3, E_3\sin\theta, 0, E_3\cos\theta)$$

$$p_4 = (E_4, \vec{p_4})$$

$$t = (p_1 - p_3)^2 = -2 p_1 \cdot p_3 = -2E_1E_3(1-\cos\theta)$$

from $(E, \vec{p})$ conversion $p_1 + p_2 = p_3 + p_4$

equate these

$$t = (p_2 - p_4)^2 = 2M^2 - 2p_2 p_4 = 2M^2 - 2ME_4$$

$$t = 2M^2 - 2M(\underbrace{E_1 + M - E_3}_{E_2}) = -2M(E_1 - E_3)$$

E_1 is a constant (the 'E_1' energy of the incoming particle)

$$\frac{dt}{d(\cos\theta)} = 2M \frac{dE_3}{d(\cos\theta)}$$

$$E_3 = \frac{E_1 M}{M + E_1 - E_1 \cos\theta}$$

$$\frac{dE_3}{d\cos\theta} = \frac{E_1^2 M}{(M + E_1 - E_1\cos\theta)^2} = E_1^2 M \left(\frac{E_3}{E_1 M}\right)^2 = \frac{E_3^2}{M}$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{2\pi} \frac{dt}{d(\cos\theta)} \frac{d\sigma}{dt} = \frac{1}{2\pi} 2M \frac{E_3^2}{M} \frac{d\sigma}{dt}$$

$$= \frac{E_3^2}{\pi} \frac{d\sigma}{dt} = \frac{E_3^2}{\pi} \frac{1}{16\pi (s - M^2)^2} |M_{fi}|^2$$

?

since $p_2^2 = 0$

$$s = (p_1 + p_2)^2 = M^2 + 2 p_1 p_2 = M^2 + 2ME_1$$

$$(s - M^2) = 2ME_1$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\left(\frac{E_3}{ME_1}\right)^2 |M_{fi}|^2$$

in the limit of $m_1 \to 0$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\left(\frac{1}{M + E_1 - E_1\cos\theta}\right)^2 |M_{fi}|^2$$

Summary

Particle decay: $$\Gamma = \frac{|\vec{p}^*|}{32\pi^2 m_i^2} \int |M_{fi}|^2 \, d\Omega$$

where $$p^* = \frac{1}{2m_i}\sqrt{\left[m_i^2 - (m_1+m_2)^2\right]\left[m_i^2 - (m_1-m_2)^2\right]}$$

$$\sigma = \frac{1}{64\pi^2 s} \frac{|\vec{p_f}^*|}{|\vec{p_i}^*|} \int |M_{fi}|^2 d\Omega^*$$

Lab frame $(m_1 = 0)$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\left(\frac{E_3}{M E_1}\right)^2 |M_{fi}|^2 \iff \frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\left(\frac{1}{M + E_1 - E_1\cos\theta}\right)^2 |M_{fi}|^2$$

Lab frame $(m_1 \neq 0)$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \cdot \frac{1}{|\vec{p_1}| m_1} \cdot \frac{|\vec{p_3}|^2}{|\vec{p_3}|(E_1 + m_2) - E_3 |\vec{p_1}| \cos\theta} \cdot |M_{fi}|^2$$

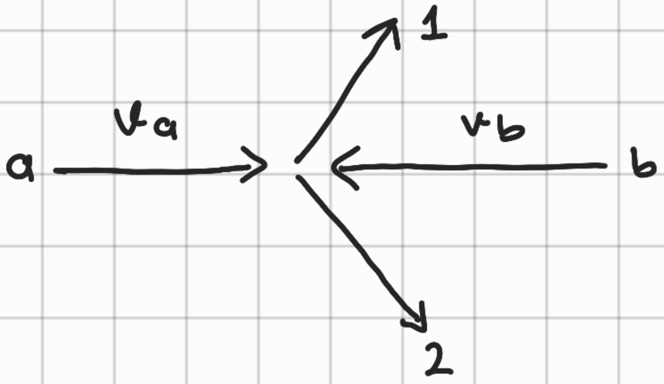
$$E_1 + m_2 = \sqrt{|\vec{p_3}|^2 + m_3^2} + \sqrt{|\vec{p_1}|^2 + |\vec{p_3}|^2 - 2|p_1||p_3|\cos\theta + m_4^2}$$

?

NOTES FROM THOMSON BOOK

Lorentz - invariant flux

$a + b \longrightarrow 1 + 2$



$$\text{Total Rate} = \phi_a \, (n_b \overset{\text{volume}}{V}) \, \sigma$$

$$\Gamma_{fi} = (v_a + v_b) \, n_a (n_b V) \sigma$$

Normalising the wavefunctions to 1 particle in a volume V

$\downarrow$ yields

$n_a = n_b = 1/V$

$$\Gamma_{fi} = (v_a + v_b) \frac{1}{V} \cdot \frac{1}{\cancel{V}} \cancel{V} \cdot \sigma$$

$$\Gamma_{fi} = \frac{(v_a + v_b)}{V} \sigma$$

again using normalisation V won't appear in final result

$$\sigma = \frac{\Gamma_{fi}}{v_a + v_b}$$

$$\sigma = \frac{(2\pi)^{-2}}{(v_a + v_b) \, 4 E_a E_b} \int |M_{fi}|^2 \, \delta(E_a + E_b - E_1 - E_2) \, \delta^3(\vec{p_a} + \vec{p_b} - \vec{p_1} - \vec{p_2}) \, \frac{d^3 p_1}{2E_1} \, \frac{d^3 p_2}{2E_2}$$

$\downarrow$

Lorentz - invariant flux factor F

$$F = 4E_aE_b(v_a+v_b) = 4E_aE_b\left(\frac{p_a}{E_a} + \frac{p_b}{E_b}\right)$$

$$F = 4(E_ap_b + E_bp_a)$$

$p_a = \gamma m_a v_a = E_a v_a$

$E_a = \gamma m_a c^2 \; (c=1)$

$$F^2 = 16\left(E_a^2 p_b^2 + E_b p_a^2 \; \underbrace{+2E_aE_bp_ap_b}\right)$$

4-vector

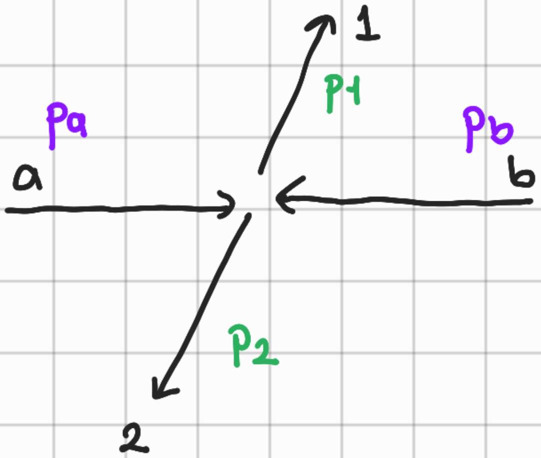
$$(p_a \cdot p_b)^2 = (E_aE_b + p_ap_b)^2 = E_a^2E_b^2 + p_a^2p_b^2 + 2E_aE_bp_ap_b$$

$$2E_aE_bp_ap_b = (p_a \cdot p_b)^2 - E_a^2E_b^2 + p_a^2p_b^2$$

$$F^2 = 16\left[(p_ap_b)^2 - \overbrace{(E_a^2 - p_a^2)}^{m_a^2}\overbrace{(E_b^2 - p_b^2)}^{m_b^2}\right]$$

$$F = 4\left[(p_a \cdot p_b)^2 - m_a^2m_b^2\right]^{1/2} \rightarrow \text{Lorentz Invariant}$$

Scattering in the Centre-of-Mass Frame



$$p_a = -p_b = p_i^*$$

$$p_1 = -p_2 = p_f^*$$

$$\sqrt{s} = (E_a^* + E_b^*)$$

$$F = 4E_a^* E_b^* (v_a^* + v_b^*) = 4E_a^* E_b^* \left(\underset{(E_b^*)}{\frac{p_i^*}{E_a^*}} + \underset{(E_a^*)}{\frac{p_i^*}{E_b^*}} \right)$$

$$= 4 p_i^* \underbrace{(E_a^* + E_b^*)}$$

$$= 4 p_i^* \sqrt{s}$$

$$\sigma = \frac{(2\pi)^{-2}}{\underbrace{(v_a + v_b)\, 4E_a^* E_b^*}_{F}} \int |M_{fi}|^2 \, \delta(\underbrace{E_a^* + E_b^*}_{\sqrt{s}} - E_1 - E_2)\, \delta^3(\cancel{\vec{p}_a} + \cancel{\vec{p}_b} - \vec{p}_1 - \vec{p}_2) \frac{d^3p_1}{2E_1} \frac{d^3p_2}{2E_2}$$

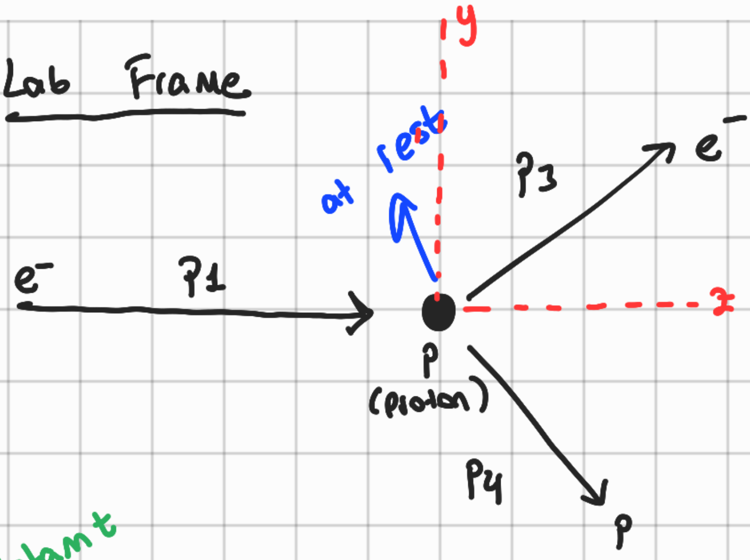
$$\sigma = \frac{(2\pi)^{-2}}{4p_i^* \sqrt{s}} \int |M_{fi}|^2 \, \delta(\sqrt{s} - E_1 - E_2)\, \delta^3(p_1 + p_2) \frac{d^3p_1}{2E_1} \frac{d^3p_2}{2E_2}$$

$$\sigma = \frac{1}{16\pi^2 p_i^* \sqrt{s}} \cdot \frac{p_f^*}{4\sqrt{s}} \int |M_{fi}|^2 \, d\Omega^*$$

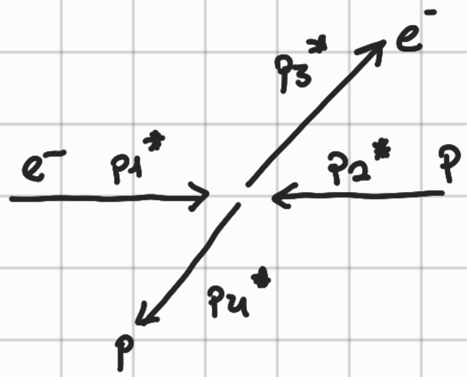
Cross-section of any 2→2 body process

$$\boxed{\sigma = \frac{1}{64\pi^2 s} \frac{p_f^*}{p_i^*} \int |M_{fi}|^2 \, d\Omega^*}$$

Lab Frame



CoM Frame



Mandelstam t

4-vector

$$t = (p_1^* - p_3^*)^2 = p_1^{*2} + p_3^{*2} - 2p_1^* p_3^*$$

$$= m_1^2 + m_3^2 - 2(E_1^* E_3^* - \vec{p}_1^* \cdot \vec{p}_3^*)$$

$$t = \underbrace{m_1^2 + m_3^2 - 2E_1^* E_3^*}_{\text{fixed}} + 2p_1^* p_3^* \cos\theta$$

magnitude of 3 vector

$$dt = 2p_1^* p_3^* \, d(\cos\theta^*)$$

$$d\Omega^* = d(\cos\theta^*) \, d\phi^* = \frac{dt \, d\phi^*}{2p_1^* p_3^*}$$

$p_1^* = p_i^*$

$p_3^* = p_f^*$

$$d\sigma = \frac{1}{64\pi^2 s} \frac{p_f^*}{p_i^*} |M_{fi}|^2 d\Omega^*$$

$$d\sigma = \frac{1}{64\pi^2 s} \frac{\cancel{p_f^*}}{p_i^*} |M_{fi}|^2 \frac{dt \, d\phi^*}{2p_i^* \cancel{p_f^*}}$$

$$\downarrow$$

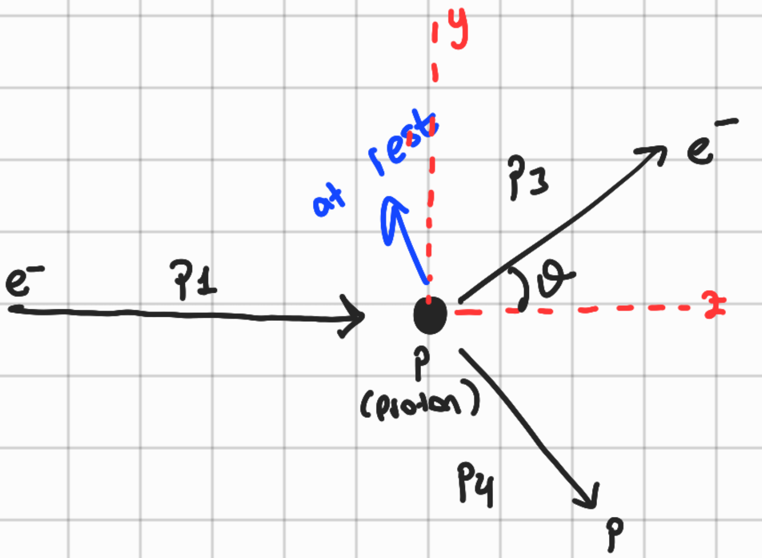
$$d\sigma = \frac{1}{128\pi^2 s\, p_i^{*2}} |M_{fi}|^2 d\phi^* dt$$

Integration over $d\phi^*$ yields 2π

$$\frac{d\sigma}{dt} = \frac{1}{64\pi s\, p_i^{*2}} |M_{fi}|^2$$

$$p_i^{*2} = \frac{1}{4s}\left[s - (M_1+M_2)^2\right]\left[s-(M_1-M_2)^2\right]$$

Laboratory Frame Differential Cross Section



$$p_1 = \overbrace{\gamma m_1}^{E_1} \overset{\approx 1}{\beta} \overset{(1)}{c}$$

$$E_1 = \gamma m_1$$

$$p_1 \simeq (E_1, 0, 0, E_1)$$ assuming $E_1 >> m_1$

$$p_2 \simeq (m_p, 0, 0, 0)$$

$$p_3 \simeq (E_3, 0, E_3 \sin\theta, E_3 \cos\theta)$$

$$p_4 \simeq (E_4, \vec{p_4})$$

Using p_i^{*2} equation: $\quad p_i^{*2} \approx \dfrac{(s-m_p^2)^2}{4s}$ since $m_e << m_p$

$$s = (p_1+p_2)^2 = p_1^2 + p_2^2 + 2p_1p_2$$

$$\simeq \cancel{m_1^2} + m_2^2 + 2p_1p_2$$

$$s \simeq m_p^2 + 2E_1 m_p$$

Thus $$p_i^{*2} = \frac{E_1^2 m_p^2}{s}$$

To obtain laboratory frame scattering angle:

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma}{dt}\left|\frac{dt}{d\Omega}\right| = \underbrace{\frac{1}{2\pi}}_{\text{from } d\phi}\frac{dt}{d(\cos\theta)}\frac{d\sigma}{dt}$$

$$t = (p_1-p_3)^2 = \underbrace{\cancel{m_1^2}}_{m_e} + \underbrace{\cancel{m_3^2}}_{m_e} - 2E_1E_3 + 2\overset{E_1}{p_1}\,\overset{E_3}{p_3}\cos\theta$$

$$t = (p_1-p_3)^2 \approx -2E_1E_3(1-\cos\theta)$$

From Conservation of energy and momentum

$$t = (p_1-p_3)^2 = (p_2-p_4)^2 = p_2^2 + p_4^2 - 2p_2p_4$$

$$E_1 + \overset{M_p}{E_2} = E_3 + E_4$$

$$E_4 = E_1 + M_p - E_3$$

$$= 2M_p^2 - 2m_p E_4$$

$$= \cancel{2M_p^2} - \cancel{2m_p^2} - 2M_pE_1 + 2M_pE_3$$

$$t = -2m_p(E_1 - E_3)$$

$$\boxed{-2E_1E_3(1-\cos\theta) = -2M_p(E_1-E_3)}$$

$$\downarrow$$

$$-\cancel{2}E_1E_3 + \cancel{2}E_1E_3\cos\theta = -\cancel{2}M_pE_1 + \cancel{2}M_pE_3$$

$$E_3(-E_1 + E_1\cos\theta - M_p) = -M_pE_1$$

$$E_3 = \frac{M_pE_1}{E_1 + M_p - E_1\cos\theta}$$

$$t = -2m_p\,(\overset{\text{fixed}}{E_1} - E_3)$$

$$\frac{dt}{d(\cos\theta)} = +2m_p\,\frac{dE_3}{d\cos\theta}$$

$$\frac{dE_3}{d\cos\theta} = \frac{m_pE_1^2}{(M_p + E_1 - E_1\cos\theta)^2} = (m_pE_1^2)\cdot\left(\frac{E_3}{M_pE_1}\right)^2 = \frac{E_3^2}{m_p}$$

$$\frac{dt}{d(\cos\theta)} = 2M_p\cdot\frac{E_3^2}{m_p} = 2E_3^2$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{2\pi}\cdot(2E_3^2)\,\frac{d\sigma}{dt} = \frac{E_3^2}{64\pi^2\cancel{s}\,p_i^{*2}}|M_{fi}|^2$$

$$p_i^{*2} \to \frac{E_1^2M_p^2}{\cancel{s}}$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\left(\frac{E_3}{M_pE_1}\right)^2|M_{fi}|^2$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\left(\frac{1}{M_p + E_1 - E_1\cos\theta}\right)^2|M_{fi}|^2$$