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MICRO-435 Hw3

$$|GHZ\rangle = \frac{1}{\sqrt{2}} (|000\rangle_A |000\rangle_B |000\rangle_C + |111\rangle_A |111\rangle_B |111\rangle_C)$$

a) Alice applies Hadamard gate to her qubit

$$\frac{1}{\sqrt{2}} (H|000\rangle_A |000\rangle_B |000\rangle_C + H|111\rangle_A |111\rangle_B |111\rangle_C)$$

$$= \frac{1}{2} \left((|00\rangle_A + |11\rangle_A) |000\rangle_B |000\rangle_C + (|00\rangle_A - |11\rangle_A) |111\rangle_B |111\rangle_C \right)$$

$$|\Psi\rangle = \frac{1}{2} \left(|000\rangle_A |000\rangle_B |000\rangle_C + |111\rangle_A |000\rangle_B |000\rangle_C + |000\rangle_A |111\rangle_B |111\rangle_C - |111\rangle_A |111\rangle_B |111\rangle_C \right)$$

b) Alice measures her qubit

Meas: $|0\rangle \longrightarrow \frac{1}{\sqrt{2}} (|000\rangle_B |000\rangle_C + |111\rangle_B |111\rangle_C)$

Meas: $|1\rangle \longrightarrow \frac{1}{\sqrt{2}} (|000\rangle_B |000\rangle_C - |111\rangle_B |111\rangle_C)$

c) Bob and Chuck share the positive even parity Bell state

$$\text{state } |\tilde{\Phi}_+\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle).$$

Protocol	m	Applied operation	
		0	1
		I	$I \otimes Z$

$$\frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \longrightarrow \frac{1}{\sqrt{2}} (|00\rangle - |11\rangle)$$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \quad \quad \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}_{4 \times 1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}_{4 \times 4} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

$I \otimes Z$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

2)

$$U = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{bmatrix} \xrightarrow{\text{Input}} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 6 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 7 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \xleftarrow{\text{input}}$$

$$|\Psi_{\text{out}}\rangle = U |\Psi_{\text{in}}\rangle$$

Start from x_2 , think zero controlled Toffoli gate

Input	Output
$x_2 x_1 x_0$	$x_2 x_1 x_0$
0 0 0	0 0 0
0 0 1	0 1 0
0 1 0	0 1 1
0 1 1	1 0 0
1 0 0	1 0 1
1 0 1	1 1 0
1 1 0	1 1 1
1 1 1	0 0 1

() TOFFOLI gate

Quantum circuit diagram illustrating the decomposition of a Toffoli gate into a CNOT gate and an X gate.

The circuit consists of three horizontal lines representing qubits x_2 , x_1 , and x_0 . The initial state is $|x_2 x_1 x_0\rangle = |000\rangle$.

Step 1: A CNOT gate is applied between x_2 and x_1 . The control x_2 is highlighted in green, and the target x_1 is highlighted in green. The result is $|000\rangle \rightarrow |001\rangle$.

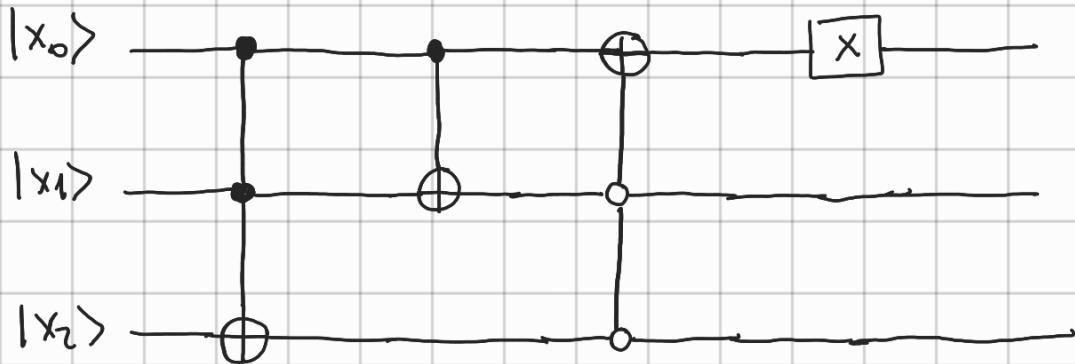
Step 2: A CNOT gate is applied between x_2 and x_0 . The control x_2 is highlighted in blue, and the target x_0 is highlighted in blue. The result is $|001\rangle \rightarrow |011\rangle$.

Step 3: An X gate is applied to x_0 . The result is $|011\rangle \rightarrow |111\rangle$.

Final state: $|x_2 x_1 x_0\rangle = |111\rangle$

Annotations:

- A green arrow points to the x_1 qubit in the first CNOT step.
- A blue arrow points to the x_0 qubit in the second CNOT step.
- A green label "CNOT gate" is placed near the first CNOT step.
- A blue label "Zero-controlled Toffoli + X gate" is placed near the final state.



$$U = (I \otimes I \otimes X) (Z_{\text{TOFFOLI}}) (I \otimes \text{CNOT}) (T_{\text{OFFOLI}}) |x_2 x_1 x_0\rangle$$

$T_{\text{OFFOLI}} =$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_2 x_1 x_0$	$x_1 x_0$	y
000	000	0
001	001	1
010	010	2
011	111	7
100	100	4
101	101	5
110	110	6
111	011	3

$$CNOT = \begin{matrix} \text{CNOT} = & \begin{array}{c} \xrightarrow{\sim} \\ \left(\begin{array}{ccccc} (0) & & (3) & (2) & (4) \\ 00 & 11 & 10 & 01 & 11 \\ 00 & 01 & 10 & 01 & 11 \end{array} \right) \end{array} \end{matrix}$$

$$I \otimes CNOT = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$(I \otimes CNOT) \otimes \text{TOFFOLI} = \begin{bmatrix} 1 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & | & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & | & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & | & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & | & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & | & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & | & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & | & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & | & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & | & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & | & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & | & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & | & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & | & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & | & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & | & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & | & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & | & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$270 \overline{FOLI} = \begin{bmatrix} 0 & 1 & 0 & 0 & | & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & | & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & | & 0 & 0 & 0 & 0 \\ - & - & - & - & | & - & - & - & - \\ 0 & 0 & 0 & 0 & | & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & | & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & | & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 1 \end{bmatrix}$$

$x_2 x_1 x_0$	$x_2 x_1 x_0$	y
0 0 0	0 0 1	1
0 0 1	0 0 0	0
0 1 0	0 1 0	2
0 1 1	0 1 1	3
1 0 0	1 0 0	4
1 0 1	1 0 1	5
1 1 0	1 1 0	6
1 1 1	1 1 1	7

$$\text{2Toffoli } (\underbrace{I \otimes \text{CNOT}}_{\text{A}}) \text{ (Toffoli)} = \begin{bmatrix} 0 & 1 & 0 & 0 & | & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & | & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & | & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & | & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & | & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & | & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & | & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & | & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & | & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & | & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & | & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & | & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & | & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & | & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & | & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & | & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & | & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & | & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & | & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{c} I \otimes I \otimes X \\ \leftarrow \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

$$= \begin{bmatrix} 01 & 00 & | & 00 & 00 \\ 10 & 00 & | & 00 & 00 \\ 00 & 01 & | & 00 & 00 \\ 00 & 10 & | & 00 & 00 \\ \hline 00 & 00 & | & 01 & 00 \\ 00 & 00 & | & 10 & 00 \\ 00 & 00 & | & 00 & 01 \\ 00 & 00 & | & 00 & 10 \end{bmatrix}$$

$$(I \otimes I \otimes X) A = \begin{bmatrix} 01 & 00 & 10 & 00 & 00 & | & 0 & 0 & 0 & 1 \\ 10 & 00 & 00 & 00 & 00 & | & 1 & 0 & 0 & 0 \\ 00 & 01 & 00 & 00 & 00 & | & 0 & 0 & 10 & 0 \\ 00 & 10 & 00 & 00 & 00 & | & 0 & 1 & 00 & 0 \\ \hline 00 & 00 & 01 & 00 & 00 & | & 0 & 0 & 00 & 1 \\ 00 & 00 & 10 & 00 & 00 & | & 0 & 0 & 00 & 0 \\ 00 & 00 & 00 & 10 & 00 & | & 0 & 0 & 01 & 0 \\ 00 & 00 & 00 & 00 & 01 & | & 0 & 0 & 00 & 0 \\ \hline 00 & 00 & 00 & 00 & 10 & | & 0 & 0 & 00 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 \\ 0 & 0 & 10 & 0 & | & 0 & 0 & 0 & 0 \\ 0 & 1 & 00 & 0 & | & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 00 & 1 & | & 0 & 0 & 0 & 0 \\ 0 & 0 & 00 & 0 & | & 1 & 0 & 0 & 0 \\ 0 & 0 & 00 & 0 & | & 0 & 0 & 0 & 0 \\ 0 & 0 & 00 & 0 & | & 0 & 0 & 10 & 0 \\ \hline 0 & 0 & 00 & 0 & | & 0 & 1 & 00 & 0 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & | & 0 & 0 & 0 & 0 \\ 0 & 0 & 10 & 0 & | & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 01 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 00 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 00 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 00 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \checkmark$$