WHAT'S LEARNED

- Density Matrix
- Quantum Coherence
- Superposition and Mixed State
- Reduced Density Matrix
- Dephasing Rates (Local and Mixed)
- Wootter's Concurrence
- Entanglement Decay

  A. Under Two-Qubit Dephasing Channel $\mathcal{E}_{AB}$

  B. Under One-Qubit Dephasing Channel $\mathcal{E}_A$ or $\mathcal{E}_B$

-
- Conclusions

1. Density Matrix

$$\rho = \sum_j p_j \, |\underbrace{\psi_j\rangle\langle\psi_j}_{\text{pure state}}|$$

Probabilistic Mixture $\neq$ Superposition

$|\psi_1\rangle \rightarrow \frac{1}{2}$

$|\psi_2\rangle \rightarrow \frac{1}{2}$

$$\rho = \frac{1}{2}|\psi_1\rangle\langle\psi_1| + \frac{1}{2}|\psi_2\rangle\langle\psi_2|$$

Assume $|\psi_1\rangle$ & $|\psi_2\rangle$ are orthogonal

$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$|\psi_1\rangle\langle\psi_1|$

$\begin{pmatrix} 1 \\ 0 \end{pmatrix}(1 \; 0)$ $\begin{pmatrix} 0 \\ 1 \end{pmatrix}(0 \; 1)$

$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$$\rho = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Superposition $\rightarrow$ $|\psi\rangle = \frac{1}{\sqrt{2}}(|\psi_1\rangle + |\psi_2\rangle)$

$$|\psi\rangle\langle\psi| = \frac{1}{2}\left(|\psi_1\rangle + |\psi_2\rangle\right)\left(\langle\psi_1| + \langle\psi_2|\right)$$

$$= \underbrace{\frac{1}{2}|\psi_1\rangle\langle\psi_1|}_{\begin{pmatrix}1&0\\0&0\end{pmatrix}} + \underbrace{\frac{1}{2}|\psi_1\rangle\langle\psi_2|}_{\begin{pmatrix}1\\0\end{pmatrix}(0\,1) = \begin{pmatrix}0&1\\0&0\end{pmatrix}}$$

$$+ \underbrace{\frac{1}{2}|\psi_2\rangle\langle\psi_1|}_{\begin{pmatrix}0\\1\end{pmatrix}(1\,0) \;\; \begin{pmatrix}0&0\\1&0\end{pmatrix}} + \underbrace{\frac{1}{2}|\psi_2\rangle\langle\psi_2|}_{\begin{pmatrix}0&0\\0&1\end{pmatrix}}$$

$$\rho = \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} \neq \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

2. Quantum Coherence, Decoherence

Representation of coherence with density matrices.

Off-diagonal elements of density matrix are "coherence"

When a system interacts with its environment the off-diagonal elements decay and the final density matrix is the diagonal one.

off-diagonal elements $\rightsquigarrow$ Related to Coherence

Pure State (Equal Superposition)

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$\rho_+ = |+\rangle\langle+| = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\rho_+ = \frac{1}{2}\left(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|\right)$$

Completely Mixed State

$$\{|0\rangle, |1\rangle\}$$

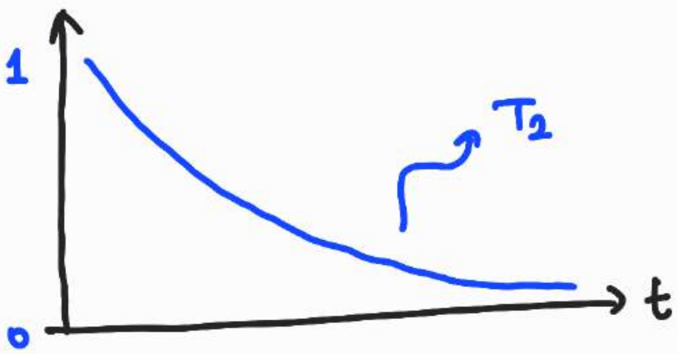
$$\left\{\frac{1}{2}, \frac{1}{2}\right\}$$

$$\rho_c = \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right)$$

$$\rho_c = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\rho_{pure} \xrightarrow{\text{Decoherence}} \rho_{mixed}$$

$$\rho_+ = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \xrightarrow{\text{Dephasing}} \rho_c = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$



2.1 Copenhagen Interpretation

Niels Bohr and Werner Heisenberg

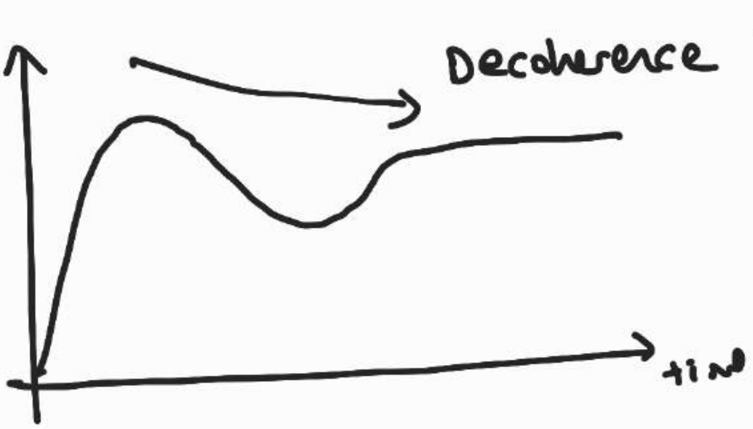
observations and measurement processes — Look at later on :)

↳ Born Rule

! Decoherence can be viewed as the loss of info. from a system into the environment

Phase Spaces and Hilbert Spaces → Look at later

Thermal Bath → Look at later



At first atom can be both in two states at a time

But in the end, the atom can be only in one state at a time, randomly chosen.

2.1 Classical prob. – Quantum Prob.

Transition from quantum to classical probabilities

$$\mathrm{Prob}_{before}(\Psi \to \phi) = |\langle \Psi | \phi \rangle|^2 = \left| \sum_i \Psi_i^* \phi_i \right|^2$$

$$\Psi_i = \langle i | \Psi \rangle \qquad \Psi_i^* = \langle \Psi | i \rangle \qquad \phi_i = \langle i | \phi \rangle$$

$$\left| \sum_i \Psi_i^* \phi_i \right|^2 = \sum_i |\Psi_i^* \phi_i|^2 + \underbrace{\sum_{\substack{i,j \\ (i \neq j)}} \Psi_i^* \Psi_j \phi_j^* \phi_i}_{\text{interference}}$$

$$\mathrm{Prob}_{after}(\Psi \to \phi) = \sum_j |\langle after | \phi, \epsilon_j \rangle|^2$$

$$= \sum_j \left| \sum_i \Psi_i^* \langle i, \epsilon_i | \phi, \epsilon_j \rangle \right|^2 = \sum_j \left| \sum_i \Psi_i^* \phi_i \underbrace{\langle \epsilon_i | \epsilon_j \rangle}_{\delta_{ij}} \right|^2$$

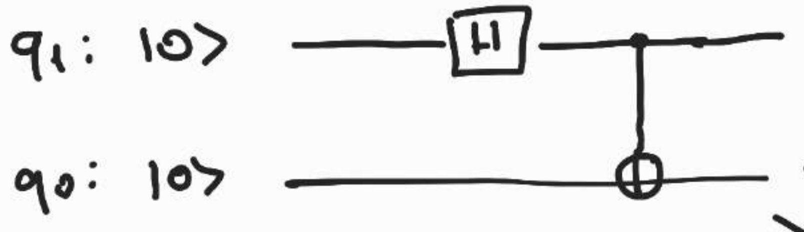
$$\mathrm{Prob}_{after}(\Psi \to \phi) = \sum_i |\Psi_i^* \phi_i|^2$$

* Here environment introduced decoherence

Quantum interference terms vanished!

The decoherence has irreversibly converted quantum behaviour (additive probability amplitudes) to classical behaviour (additive probabilities!)

2.2 Reduced Density Matrixes



$|\psi\rangle = \frac{1}{\sqrt{2}}(|\underset{A}{0}\underset{B}{0}\rangle + |\underset{A}{1}\underset{B}{1}\rangle)$

$|\psi\rangle \neq |a\rangle \otimes |b\rangle$

Alice $\boxed{\nearrow}$ = 0, 1, 0, 0, 1, ...

Bob $\boxed{\nearrow}$ =

0, 1, 0, 0, 1, ...

with equal prob

~~= 0, 0, 0..~~

but it's not the case

Assume Bob does not know Alice exists

Guess

$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |+\rangle$

$|+\rangle \xrightarrow{H} |0\rangle$

(for measurement)

Observe

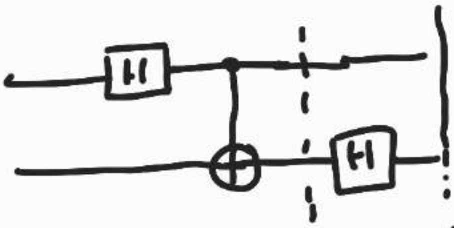
$\{|0\rangle, |1\rangle\}$

$\{\frac{1}{2}, \frac{1}{2}\}$ prob

(Mixed State)

$\rho_B = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$

Reduced Density Matrix



$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

$\frac{1}{\sqrt{2}}(|0+\rangle + |1-\rangle)$

"

$\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle - |11\rangle)$

$\rho_B = Tr_A(\rho_{AB})$

$\rho_A = Tr_B(\rho_{AB})$

$\downarrow$

The Portial Trace

Subsystem A: $\{|\phi_v\rangle\}_{v=0}^{N}$ $N \rightarrow$ # of qubits

Subsystem B: $\{|X_v\rangle\}_{v=0}^{M}$ $M \rightarrow$ # of qubits

$$\rho_A = Tr_B(\rho_{AB}) = \sum_{v=0}^{M} (I_A \otimes \langle X_v|)\, \rho_{AB}\, (I_A \otimes |X_v\rangle)$$

Assumptions (for now) $\rightarrow$ it can be applied to entangle states

1) ρ_{AB} is an pure state $\rho_{AB} = |\psi\rangle\langle\psi|$

2) $|\psi\rangle$ is seperable : $|\psi\rangle = |a\rangle \otimes |b\rangle$

Prob. Amplitude of measuring $|X_v\rangle$

$$\rho_A = Tr_B(\rho_{AB}) = \sum_{v=0}^{M} \underbrace{(I_A \otimes \langle X_v|)|\psi\rangle}\underbrace{\langle\psi|(I_A \otimes |X_v\rangle)}$$

$(I_A \otimes \langle X_v|)(|a\rangle \otimes |b\rangle) = |a\rangle \underbrace{(\langle X_v|b\rangle)}_{\alpha_v}$

eigenstate arbitrary

$(\langle a| \otimes \langle b|)(I_A \otimes |X_v\rangle) = \langle a| \underbrace{(\langle b|X_v\rangle)}$

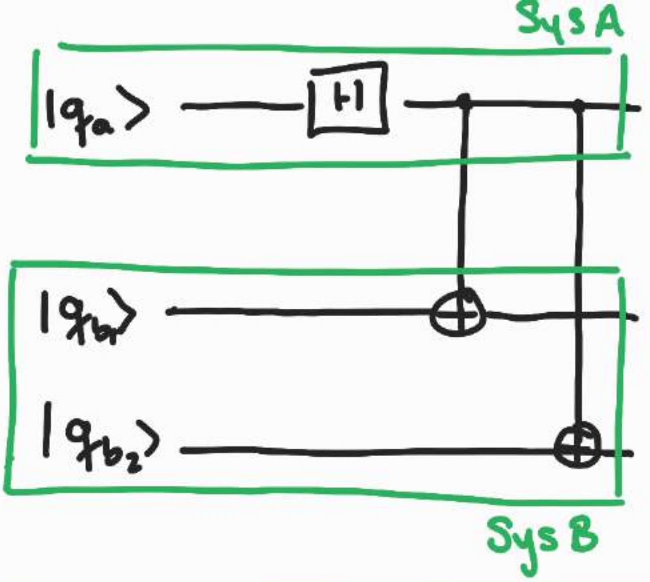
$\underbrace{(\langle X_v|b\rangle)^*}_{\alpha_v^*}$

number

$$\rho_A = \sum_{v=0}^{M} (|a\rangle \cdot \alpha_v)(\langle a| \alpha_v^*)$$

$$\rho_A = \sum_{v=0}^{M} \underbrace{|\alpha_v|^2}\, |a\rangle\langle a| = |a\rangle\langle a|$$

$$\rho_B = Tr_A(\rho_{AB}) = \sum_{v=0}^{N} (\langle\phi_v| \otimes I_B)\, \rho_{AB}\, (|\phi_v\rangle \otimes I_B)$$



$\rightarrow \{|0\rangle, |1\rangle\}$

$\rightarrow \{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$

$$|\xi\rangle = \frac{1}{\sqrt{2}}\left(\overbrace{|0\rangle_A \otimes |00\rangle_B}^{|000\rangle} + \underbrace{|1\rangle_A \otimes |11\rangle_B}_{|111\rangle}\right)$$

$$\boxed{|\xi\rangle = \sum_{u=0}^{N}\sum_{v=0}^{M} \alpha_{uv} |\phi_u\rangle \otimes |\chi_v\rangle}$$

$$= \alpha_{00}|0\rangle|00\rangle + \alpha_{01}|0\rangle|01\rangle + \alpha_{02}|0\rangle|10\rangle + \alpha_{03}|0\rangle|11\rangle$$

$$+ \alpha_{10}|1\rangle|00\rangle + \alpha_{11}|1\rangle|01\rangle + \alpha_{12}|1\rangle|10\rangle + \alpha_{13}|1\rangle|11\rangle$$

$\alpha_{00} = \frac{1}{\sqrt{2}}$, $\alpha_{13} = \frac{1}{\sqrt{2}}$, all other $\alpha_{uv} = 0$

$$\Omega = |\xi\rangle\langle\xi| = \left(\sum_{u=0}^{N}\sum_{v=0}^{M} \alpha_{uv} |\phi_u\rangle \otimes |\chi_v\rangle\right)\left(\sum_{w=0}^{N}\sum_{k=0}^{M} \alpha_{wk}^{*} \langle\phi_w| \otimes \langle\chi_k|\right)$$

↳ density matrix representation for any arbitrary state

$$\Omega = \sum_{u,w=0}^{N}\sum_{v,k=0}^{M} \gamma_{uvwk} \underbrace{|\phi_u\rangle\langle\phi_w|}_{\substack{|0\rangle\langle 0| \\ |1\rangle\langle 1|}} \otimes \underbrace{|\chi_v\rangle\langle\chi_k|}_{\substack{|00\rangle\langle 00| \\ |01\rangle\langle 01| \\ |10\rangle\langle 10| \\ |11\rangle\langle 11|}}$$ ☆☆

$\gamma_{uvwk} \downarrow \alpha_{uv}\alpha_{wk}^{*}$

$$\sum_{u=0}^{N}\sum_{w=0}^{N}$$

$$|\xi\rangle \xrightarrow{U} U|\xi\rangle$$

$$\Omega \xrightarrow{U} \Omega' = U\Omega U^{\dagger}$$

$$U = U_A \otimes U_B$$

$$\Omega' = (U_A \otimes U_B) \sum_{u,w=0}^{N} \sum_{v,k=0}^{M} \gamma_{uvwk} \overbrace{|\phi_u\rangle\langle\phi_w|}^{\text{Sys A}} \otimes \overbrace{|\chi_v\rangle\langle\chi_k|}^{\text{Sys B}} \overbrace{(U_A \otimes U_B)^{\dagger}}^{U_A^{\dagger} \otimes U_B^{\dagger}}$$

$$\Omega' = \sum_{u,w=0}^{N} \sum_{v,k=0}^{M} \gamma_{uvwk}\, U_A |\phi_u\rangle\langle\phi_w| U_A^{\dagger} \otimes U_B |\chi_v\rangle\langle\chi_k| U_B^{\dagger}$$

$$\boxed{\rho_A = Tr_B(\rho_{AB}) = \sum_{t=0}^{M} \big(\underbrace{\overset{A}{I_A} \otimes \overset{B}{\langle\chi_t|}}_{\Gamma}\big) \rho_{AB} \big(\underbrace{\overset{A}{I_A} \otimes \overset{B}{|\chi_t\rangle}}_{\Gamma^{\dagger}}\big)}$$

$$\rho_A = \sum_{t=0}^{M} (I_A \otimes \langle\chi_t|) \left[\sum \sum \gamma_{uvwk} |\phi_u\rangle\langle\phi_w| \otimes |\chi_v\rangle\langle\chi_k| \right] (I_A \otimes |\chi_t\rangle)$$

$$\rho_A = \sum_{t=0}^{M} \sum_{u,w=0}^{N} \sum_{v,k=0}^{M} \gamma_{uvwk}\, I_A |\phi_u\rangle\langle\phi_w| I_A \otimes \underbrace{\langle\chi_t|\chi_v\rangle}_{t=v} \underbrace{\langle\chi_k|\chi_t\rangle}_{t=k}$$

$$v = k$$

$$Tr(|\chi_v\rangle\langle\chi_k|)$$

$$\rho_A = \cancel{\sum_{t=0}^{M}} \sum_{u,w=0}^{N} \sum_{v,k=0}^{M} \gamma_{uvwk} |\phi_u\rangle\langle\phi_w| \otimes Tr(|\chi_v\rangle\langle\chi_k|)$$

$$\rho_A = \sum_{u,w=0}^{N} \sum_{v,k=0}^{M} \gamma_{uvwk} \, |\phi_u\rangle\langle\phi_w| \otimes \underbrace{\mathrm{Tr}\left(|\chi_v\rangle\langle\chi_k|\right)}_{\langle\chi_k|\chi_v\rangle = \delta_{kv}}$$

$$\rho_B = \sum_{u,w=0}^{N} \sum_{v,k=0}^{M} \gamma_{uvwk} \, \underbrace{\mathrm{Tr}\left(|\phi_u\rangle\langle\phi_w|\right)}_{\substack{\langle\phi_w|\phi_u\rangle \\ \delta_{wu}}} \otimes |\chi_v\rangle\langle\chi_k|$$

$$S^A = \mathrm{Tr}_B\{\rho\}$$

To study coherence decay of a single qubit under the two qubit dephasing channel $\mathcal{E}_{AB}$ (27)

$$\rho = \begin{array}{c} \\ 00 \\ 01 \\ 10 \\ 11 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} AB\;00 & 01 & 10 & 11 \end{array} \\ \begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} \\ \rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} \\ \rho_{41} & \rho_{42} & \rho_{43} & \rho_{44} \end{pmatrix} \end{array}$$

$$S^A = \mathrm{Tr}_B\{\rho\} = \begin{array}{c} \\ 0 \\ 1 \end{array}\!\!\begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{pmatrix} \rho_{11}+\rho_{22} & \rho_{13}+\rho_{24} \\ \rho_{31}+\rho_{42} & \rho_{33}+\rho_{44} \end{pmatrix} \end{array}$$

$$S^B = \mathrm{Tr}_A\{\rho\} = \begin{pmatrix} \rho_{11}+\rho_{33} & \rho_{12}+\rho_{34} \\ \rho_{21}+\rho_{43} & \rho_{22}+\rho_{44} \end{pmatrix}$$

Remark: Array formalism

$$M = \begin{array}{c} \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \\ \begin{pmatrix} M_{0000} & M_{0001} & M_{0010} & M_{0011} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \end{array}\!\!\begin{array}{c} AB \\ 00 \\ 01 \\ 10 \\ 11 \end{array}$$

$\mathbb{C}^2 \otimes \mathbb{C}^2$

C

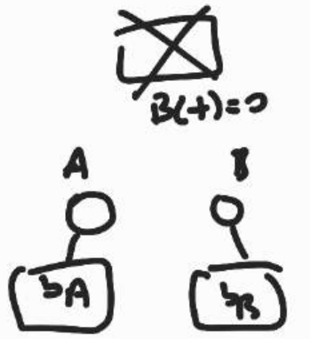
D

$$\mathrm{Tr}_B M = \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix}$$

$$\mathrm{Tr}_A M = \left(\boxed{\begin{matrix} \cdot & \cdot \\ \cdot & \cdot \end{matrix}} \right)$$

C+D

$$S_{12}^{A}(t) = \rho_{13}(t) + \rho_{24}(t) = \gamma_A S_{12}^{A}(0)$$



Two-qubit local dephasing channel

where $\rho(t) = \mathcal{E}_{AB}(\rho(0))$

$$\rho(t) = \begin{bmatrix} \rho_{11} & \gamma_B \rho_{12} & \gamma_A \rho_{13} & \gamma_A\gamma_B \rho_{14} \\ \gamma_B \rho_{21} & \rho_{22} & \gamma_A\gamma_B \rho_{23} & \gamma_A \rho_{24} \\ \gamma_A \rho_{31} & \gamma_A\gamma_B \rho_{32} & \rho_{33} & \gamma_B \rho_{34} \\ \gamma_A\gamma_B \rho_{41} & \gamma_A \rho_{42} & \gamma_B \rho_{43} & \rho_{44} \end{bmatrix}$$

$$\gamma_A(\underbrace{\rho_{13} + \rho_{24}}_{S_{12}^{A}(0)})$$

$$S_{12}^{B}(t) = \gamma_B \rho_{12} + \gamma_B \rho_{34} = \gamma_B S_{12}^{B}(0)$$

$$\gamma_A(t) = e^{-t/2T_2^A} \quad \text{where } T_2^A = 1/\Gamma_A$$

$$\gamma_B(t) = e^{-t/2T_2^B} \quad \text{where } T_2^B = 1/\Gamma_B$$

$$\rho_{ij}(t) = e^{-\Gamma_{ij} t} \rho_{ij}(0) \rightarrow T_{dec} = \frac{1}{\Gamma_{ij}}$$

$$\Gamma_{ij}^{A} = \frac{1}{2T_2^A} \qquad \tau_A = \frac{1}{\Gamma_{ij}} = 2T_2^A = \frac{2}{\Gamma_A}$$

$$\Gamma_{ij}^{B} = \frac{1}{2T_2^B} \qquad \tau_B = \frac{1}{\Gamma_{ij}} = 2T_2^B = \frac{2}{\Gamma_B}$$

Mixed Dephasing Rate (τ)

$$\rho(t) = \begin{bmatrix} \rho_{11} & \gamma_B \rho_{12} & \gamma_A \rho_{13} & \gamma_A\gamma_B \rho_{14} \\ \gamma_B \rho_{21} & \rho_{22} & \gamma_A\gamma_B \rho_{23} & \gamma_A \rho_{24} \\ \gamma_A \rho_{31} & \gamma_A\gamma_B \rho_{32} & \rho_{33} & \gamma_B \rho_{34} \\ \gamma_A\gamma_B \rho_{41} & \gamma_A \rho_{42} & \gamma_B \rho_{43} & \rho_{44} \end{bmatrix}$$

Decoherence Rate $1/\tau$ = Mixed Dephasing Rate

$1/\tau$ is determined by the slower decaying elements.

$\Rightarrow$ $1/\tau$ is not shorter than the local dephasing rates ?

$$\tau \geqslant \tau_A, \tau_B \rightarrow \frac{1}{\tau} \leq \frac{1}{\tau_A}, \frac{1}{\tau_B}$$

$$\tau_A \rightarrow \frac{2}{\Gamma_A} \qquad \tau_B \rightarrow \frac{2}{\Gamma_B}$$

$$\gamma_A\gamma_B = e^{-t/\tau_2^A} \cdot e^{-t/\tau_2^B} \qquad \Big| \qquad \rho_{ij}(t) = e^{-\Gamma_{ij} t} \rho_{ij}(0)$$

$$= e^{-t\left(\frac{1}{\tau_{2A}} + \frac{1}{\tau_{2B}}\right)} \qquad \qquad \tau = \frac{1}{\Gamma_{ij}}$$

$$\tau = \frac{1}{\frac{1}{\tau_{2A}} + \frac{1}{\tau_{2B}}} = \frac{1}{\Gamma_A + \Gamma_B}$$

$$\tau_{2A} \rightarrow \frac{1}{\Gamma_A} \qquad \tau_{2B} \rightarrow \frac{1}{\Gamma_B}$$

However disentanglement rate can be faster than $\underbrace{1/\tau_A, 1/\tau_B}_{\text{local rates}}$

2.3 Relation btw dephasing and decoherence

Decoherence

Dephasing

" Z- Noise"

Reduce the off-diagonal matrices

Relaxation

$|1\rangle \longrightarrow |0\rangle$

Relax back to state 0

T_2: "Phase Relaxation Time"

3. Entanglement Decay

$$\tilde{\rho} = \rho (\sigma_y^A \otimes \sigma_y^B) \rho^* (\sigma_y^A \otimes \sigma_y^B)$$

$$\sigma_y^{A,B} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$-i.(-i) = i^2 = -1$

$$\sigma_y^A \otimes \sigma_y^B = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

Time Reversal Operation for a spin-1/2 $\rightarrow \underbrace{(-\sigma_y \otimes \sigma_y)|\psi\rangle^*}_{|\tilde{\psi}\rangle}$

$$|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$$

$$|\tilde{\psi}\rangle = (\sigma_y \otimes \sigma_y)|\psi^*\rangle$$

$$|\tilde{\psi}\rangle \underbrace{\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & +1 & 0 \\ 0 & +1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}}_{4\times4} \underbrace{\begin{pmatrix} a^* \\ b^* \\ c^* \\ d^* \end{pmatrix}}_{4\times1} = \begin{pmatrix} -d^* \\ +c^* \\ +b^* \\ -a^* \end{pmatrix}$$

$$|\tilde{\psi}\rangle = -d^*|00\rangle + c^*|01\rangle + b^*|10\rangle - a^*|11\rangle$$

$$C(|\psi\rangle) = |\langle\psi| \underbrace{\sigma_y^A \otimes \sigma_y^B |\psi^*\rangle}_{\tilde{\psi}}| = 2|ad-bc| \;?$$

$$\begin{pmatrix} a & b & c & d \end{pmatrix} \begin{pmatrix} -d^* \\ c^* \\ b^* \\ -a^* \end{pmatrix} = |-ad^* + bc^* + cb^* - a^*d|$$

$$|\psi\rangle = a_1|1\rangle + a_2|2\rangle + a_3|3\rangle + a_4|4\rangle$$

$$C(|\psi\rangle) = 2|a_1a_4 - a_2a_3| \;?$$

Wootter's Concurrence

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

↳ Pure States

(for entangled states ✓)

How can we directly say this?

$$E(\psi) = -\mathrm{Tr}(\rho_A \log_2 \rho_A) = -\mathrm{Tr}(\rho_B \log_2 \rho_B)$$

↙ entanglement

↓

Entropy $S(\rho) = -\mathrm{Tr}(\rho \log \rho)$

quantifies degree of uncertainty

$$E(\rho) = \min \sum_i p_i E(\psi_i)$$

Spin-flip transformation:

$\rightarrow$ For a spin 1/2 particle it's the standard time reversal operation

$$|\tilde{\psi}\rangle = \sigma_y |\psi^*\rangle$$

For a general state ρ of two qubits:

$$\tilde{\rho} = (\sigma_y^A \otimes \sigma_y^B)\, \rho^* \,(\sigma_y^A \otimes \sigma_y^B)$$

$$E(\psi) = \mathcal{E}(C(\psi))$$

$$C(\psi) = |\langle \psi | \tilde{\psi} \rangle|$$

convex function

function $\mathcal{E}$ $\rightsquigarrow$ $\mathcal{E}(C) = h\left(\dfrac{1+\sqrt{1-C^2}}{2}\right)$

goes from 0 to 1

where $h(x) = -x\log_2 x - (1-x)\log_2(1-x)$

Don't understand why?

$$C(\rho) = \max\left(0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\right)$$

where λ_is are the eigenvalues of matrix $\rho\tilde{\rho}$

Non-Hermitian

For a pure state $|\psi\rangle$

$$C(|\psi\rangle) = \left| \langle\psi|\, \sigma_y^A \otimes \sigma_y^B \,|\psi^*\rangle \right|$$

A. Entanglement decay under two-qubit dephasing channel

$$|\psi\rangle = a_1 \overset{++}{|1\rangle_{AB}} + a_2 \overset{+-}{|2\rangle_{AB}} + a_3 \overset{-+}{|3\rangle_{AB}} + a_4 \overset{--}{|4\rangle_{AB}}$$

$$\sigma_y^A \otimes \sigma_y^B = \begin{pmatrix} & & & -1 \\ & & 1 & \\ & 1 & & \\ -1 & & & \end{pmatrix} \qquad |\psi\rangle = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}$$

$$\langle\psi|\, \underbrace{\sigma_y^A \otimes \sigma_y^B}\, |\psi^*\rangle = (a_1\ a_2\ a_3\ a_4) \underbrace{\begin{pmatrix} & & & -1 \\ & & 1 & \\ & 1 & & \\ -1 & & & \end{pmatrix} \begin{pmatrix} a_1^* \\ a_2^* \\ a_3^* \\ a_4^* \end{pmatrix}}$$

$$= (a_1\ a_2\ a_3\ a_4) \begin{pmatrix} -a_4^* \\ a_3^* \\ a_2^* \\ -a_1^* \end{pmatrix}$$

$$C(|\psi\rangle) = \left| -a_1 a_4^* + a_2 a_3^* + a_3 a_2^* - a_1^* a_4 \right|$$

If a_i's are real $\rightsquigarrow$ $2\left| -a_1 a_4 + a_2 a_3 \right|$

$$= 2\left| a_1 a_4 - a_2 a_3 \right|$$

$|\Psi\rangle\langle\Psi| = ?$

$$(a_1|1\rangle + a_2|2\rangle + a_3|3\rangle + a_4|4\rangle)(a_1^*\langle 1| + a_2^*\langle 2| + a_3^*\langle 3| + a_4^*\langle 4|)$$

$$\rho(t) = \mathcal{E}_{AB}(\rho) = \begin{bmatrix} |a_1|^2 & \gamma_B a_1 a_2^* & \gamma_A a_1 a_3^* & \gamma_A\gamma_B a_1 a_4^* \\ \gamma_B a_2 a_1^* & |a_2|^2 & \gamma_A\gamma_B a_2 a_3^* & \gamma_A a_2 a_4^* \\ \gamma_A a_3 a_1^* & \gamma_A\gamma_B a_3 a_2^* & |a_3|^2 & \gamma_B a_3 a_4^* \\ \gamma_A\gamma_B a_4 a_1^* & \gamma_A a_4 a_2^* & \gamma_B a_4 a_3^* & |a_4|^2 \end{bmatrix}$$

Concurrence is always decay to zero on a time scale determined by the dephasing time τ.

Show that entangled states decay rate is faster than the dephasing rate of individual qubits.

. $C(\rho)$ is a convex function $\sum_N p_N = 1$

$$\boxed{C\left(\sum_{N=1}^{n} p_N \rho_N\right) \leq \sum_{N=1}^{n} p_N C(\rho_N)}$$

Jensen's Inequality

Using $\rho(t) = \mathcal{E}_{AB}(0) = \sum_{N=1}^{4} M_N^{\dagger} \rho(0) M_N$

$$C(\rho(t)) \leq \sum_{\mu=1}^{4} C(\underbrace{M_\mu^\dagger \rho(0) M_\mu}_{\rho_{out}})$$

For diagonal matrices $D_1 A D_2 = D_2 A D_1$

Here M_μ's are diagonal

$$K = \rho_{out} (\sigma_y^A \otimes \sigma_y^B)\, \rho_{out}^* (\sigma_y^A \otimes \sigma_y^B)$$

$$= M_\mu^\dagger \rho(0) M_\mu (\sigma_y^A \otimes \sigma_y^B) M_\mu^T \rho^* M_\mu^* (\sigma_y^A \otimes \sigma_y^B)$$

K and K' have same eigenvalues, How?

$$K' = \rho \underbrace{M_\mu (\sigma_y^A \otimes \sigma_y^B) M_\mu^T} \rho^* \underbrace{M_\mu^* (\sigma_y^A \otimes \sigma_y^B) M_\mu^\dagger}$$

$$(M_\mu^\dagger = M_\mu = M_\mu^*)$$

$$M_\mu (\sigma_y^A \otimes \sigma_y^B) M_\mu = \gamma_A \gamma_B (\sigma_y^A \otimes \sigma_y^B)$$

for $\mu = 1$ $\quad M_1 = E_1 F_1$

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & \gamma_A \end{pmatrix} \otimes I = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \gamma_A & \\ & & & \gamma_A \end{pmatrix}$$

$$F_1 = I \otimes \begin{pmatrix} 1 & 0 \\ 0 & \gamma_B \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & \gamma_B & & \\ & & 1 & \\ & & & \gamma_B \end{pmatrix}$$

$$= \begin{pmatrix} & & & -\gamma_A\gamma_B \\ & & \gamma_A\gamma_B & \\ & \gamma_A\gamma_B & & \\ -\gamma_A\gamma_B & & & \end{pmatrix}$$

$$\begin{pmatrix} 1 & & & \\ & \gamma_B & & \\ & & \gamma_A & \\ & & & \gamma_A\gamma_B \end{pmatrix} \begin{pmatrix} & & & -1 \\ & & +1 & \\ & +1 & & \\ -1 & & & \end{pmatrix} \begin{pmatrix} 1 & & & \\ & \gamma_B & & \\ & & \gamma_A & \\ & & & \gamma_A\gamma_B \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & \gamma_B & & \\ & & \gamma_A & \\ & & & \gamma_A\gamma_B \end{pmatrix} \begin{pmatrix} & & & -\gamma_A\gamma_B \\ & & \gamma_A & \\ & \gamma_B & & \\ -1 & & & \end{pmatrix}$$

for $N=2$ $\quad M_2 = E_1 F_2$

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & \gamma_A \end{pmatrix} \otimes I = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \gamma_A & \\ & & & \gamma_A \end{pmatrix}$$

$$F_2 = I \otimes \begin{pmatrix} 0 & 0 \\ 0 & \omega_B \end{pmatrix} = \begin{pmatrix} 0 & & & \\ & \omega_B & & \\ & & 0 & \\ & & & \omega_B \end{pmatrix}$$

$$\begin{pmatrix} 0 & & & \\ & \omega_B & & \\ & & 0 & \\ & & & \gamma_A\omega_B \end{pmatrix} \begin{pmatrix} & & & -1 \\ & & +1 & \\ & +1 & & \\ -1 & & & \end{pmatrix} \begin{pmatrix} 0 & & & \\ & \omega_B & & \\ & & 0 & \\ & & & \gamma_A\omega_B \end{pmatrix} = \overbrace{\begin{pmatrix} 0 & & & \\ & \omega_B & & \\ & & 0 & \\ & & & \gamma_A\omega_B \end{pmatrix} \begin{pmatrix} & & & -\gamma_A\omega_B \\ & & 0 & \\ & \omega_B & & \\ 0 & & & \end{pmatrix}}^{\begin{pmatrix} & & & 0 \\ & & 0 & \\ & 0 & & \\ 0 & & & \end{pmatrix}}$$

So,

$$M_N \left(\sigma_y^A \otimes \sigma_y^B \right) M_N = \begin{cases} \gamma_A \gamma_B \left(\sigma_y^A \otimes \sigma_y^B \right) & \text{if } N=1 \\ 0 & \text{if } N \neq 1 \end{cases}$$

we know

$$K' = \rho \underbrace{M_N \left(\sigma_y^A \otimes \sigma_y^B \right) M_N}_{\gamma_A\gamma_B (\sigma_y^A \otimes \sigma_y^B)} \rho^* \underbrace{M_N \left(\sigma_y^A \otimes \sigma_y^B \right) M_N}_{\gamma_A\gamma_B (\sigma_y^A \otimes \sigma_y^B)}$$

Finally

$$K' = \begin{cases} \gamma_A^2 \gamma_B^2 \, \rho \left(\sigma_y^A \otimes \sigma_y^B \right) \rho^* \left(\sigma_y^A \otimes \sigma_y^B \right) & \text{if } N=1 \\ 0 & \text{otherwise} \end{cases}$$

From definition : $C(\rho) = \max\left(0, \overbrace{\sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}}^{\sqrt{\gamma_A \gamma_B}\, C(\rho(0))}\right)$

$$C(k') = \max\left(0, \gamma_A \gamma_B\, C(\rho(0))\right) = \gamma_A \gamma_B\, C(\rho(0))$$

Thus, $C(\rho(t)) \leq \sum_{\mu=1}^{4} C(\rho_{out})$

$$\Downarrow$$

$$C(\rho(t)) \leq \gamma_A \gamma_B\, C(\rho(0))$$

$$\left.\begin{array}{l} \gamma_A(t) = e^{-t/2T_2^A} \\ \gamma_B(t) = e^{-t/2T_2^B} \end{array}\right\} e^{-t\left(\frac{1}{2T_2^A} + \frac{1}{2T_2^B}\right)}$$

$$* \left\{\begin{array}{l} \rho_{ij}(t) = e^{-\Gamma_{ij} t} \rho_{ij}(0) \\ \\ C(\rho(t)) = e^{-t\left(\frac{1}{2T_2^A} + \frac{1}{2T_2^B}\right)} C(\rho(0)) \end{array}\right.$$

$$\Gamma_{ij} = \frac{1}{2\underbrace{T_2^A}_{\frac{1}{\Gamma_A}}} + \frac{1}{2T_2^B} = \underbrace{\frac{\Gamma_A}{2}}_{\frac{1}{\tau_A}} + \underbrace{\frac{\Gamma_B}{2}}_{\frac{1}{\tau_B}} = \underbrace{\Gamma_e}_{\frac{1}{\tau_e}}$$

* Entanglement Decay Rate is shorter than mixed dephasing time.

! $\underbrace{\tau_e < \tau_A, \tau_B}_{\text{local dephasing time}} \ll \underbrace{\tau}_{\text{mixed dephasing time}}$

If $\quad \tau_A = \tau_B \rightarrow \tau_e = \frac{\tau_A}{2} = \frac{\tau_B}{2} = \frac{\tau}{2}$

? How is it related to the phase coherence relaxation rate T_2 and the diagonal element decay rate T_1 in open quantum systems (Ref. 37)

$$|\Phi_1\rangle = a_1|1\rangle_{AB} + a_2|2\rangle_{AB} + a_4|4\rangle_{AB}$$

$$|\Phi_2\rangle = a_1|1\rangle_{AB} + a_3|3\rangle_{AB} + a_4|3\rangle_{AB}$$

$$|\Psi_1\rangle = a_1|1\rangle_{AB} + a_2|2\rangle_{AB} + a_3|3\rangle_{AB}$$

$$|\Psi_2\rangle = a_2|2\rangle_{AB} + a_3|3\rangle_{AB} + a_4|4\rangle_{AB}$$

Ex1.

$$C(|\Phi_1\rangle) = \left|\langle \Phi_1|\, \sigma_4^A \otimes \sigma_4^B \,|\Phi_1^*\rangle\right|$$

$$|\Phi_1\rangle = \begin{pmatrix} a_1 \\ a_2 \\ 0 \\ a_4 \end{pmatrix}$$

$$C(|\Phi_1\rangle) = \begin{pmatrix} a_1 & a_2 & 0 & a_4 \end{pmatrix} \begin{pmatrix} & & & -1 \\ & & 1 & \\ & 1 & & \\ -1 & & & \end{pmatrix} \begin{pmatrix} a_1^* \\ a_2^* \\ 0 \\ a_4^* \end{pmatrix}$$

$$= \begin{pmatrix} a_1 & a_2 & 0 & a_4 \end{pmatrix} \begin{pmatrix} -a_4^* \\ 0 \\ a_2^* \\ -a_1^* \end{pmatrix} = \left|\left(-a_1 a_4^* - a_1^* a_4\right)\right|$$

if $a_1, a_4 \in \mathbb{R}$ $\quad C(|\Phi_1\rangle) = 2|a_1 a_4|$

$$\rho(t) = \mathcal{E}_A(\underbrace{\rho}_{|\phi_1\rangle\langle\phi_1|}) = \begin{bmatrix} |a_1|^2 & \gamma_B a_1 a_2^* & 0 & \gamma_A\gamma_B a_1 a_4^* \\ \gamma_B a_2 a_1^* & |a_2|^2 & 0 & \gamma_A a_2 a_4^* \\ 0 & 0 & 0 & 0 \\ \gamma_A\gamma_B a_4 a_1^* & \gamma_A a_4 a_2^* & 0 & |a_4|^2 \end{bmatrix}$$

$$\begin{pmatrix} a_1 \\ a_2 \\ 0 \\ a_4 \end{pmatrix} (a_1^* \; a_2^* \; 0 \; a_4^*)$$

$$k = \underbrace{\rho(\sigma_y^A \otimes \sigma_y^B)}_{B}\ \underbrace{\rho^*(\sigma_y^A \otimes \sigma_y^B)}_{A}$$

$$\begin{bmatrix} |a_1|^2 & \gamma_B a_1^* a_2 & 0 & \gamma_A\gamma_B a_1^* a_4 \\ \gamma_B a_2^* a_1 & |a_2|^2 & 0 & \gamma_A a_2^* a_4 \\ 0 & 0 & 0 & 0 \\ \gamma_A\gamma_B a_4^* a_1 & \gamma_A a_4^* a_2 & 0 & |a_4|^2 \end{bmatrix} \begin{bmatrix} & & & -1 \\ & & 1 & \\ & 1 & & \\ -1 & & & \end{bmatrix}$$

$$A = \begin{bmatrix} -\gamma_A\gamma_B a_1^* a_4 & 0 & \gamma_B a_1^* a_2 & -|a_1|^2 \\ -\gamma_A a_2^* a_4 & 0 & |a_2|^2 & -\gamma_B a_2^* a_1 \\ 0 & 0 & 0 & 0 \\ -|a_4|^2 & 0 & \gamma_A a_4^* a_2 & -\gamma_A\gamma_B a_4^* a_1 \end{bmatrix}$$

$$B = \begin{bmatrix} -\gamma_A\gamma_B a_1 a_4^* & 0 & \gamma_B a_1 a_2^* & -|a_1|^2 \\ -\gamma_A a_2 a_4^* & 0 & |a_2|^2 & -\gamma_B a_2 a_1^* \\ 0 & 0 & 0 & 0 \\ -|a_4|^2 & 0 & \gamma_A a_4 a_2^* & -\gamma_A\gamma_B a_4 a_1^* \end{bmatrix}$$

$$
B.A = \begin{array}{c} + \\ - \\ + \\ - \end{array}
\begin{bmatrix}
((\gamma_A\gamma_B)^2+1)|a_1a_4|^2-\lambda & 0 & -\gamma_A(1+\gamma_B^2)|a_1|^2 a_2^* a_4 & 2\gamma_A\gamma_B|a_1|^2 a_1 a_4^* \\
\gamma_B(1+\gamma_A^2)\, a_1^* a_2 |a_4|^2 & 0-\lambda & -2\gamma_A\gamma_B\, a_1^* a_2^2 a_4^* & \gamma_A(1+\gamma_B)^2|a_4|^2 a_2 a_4^* \\
0 & 0 & 0-\lambda & 0 \\
2\gamma_A\gamma_B|a_4|^2 a_1^* a_4 & 0 & -\gamma_B(1+\gamma_A^2) a_1^* a_2 |a_4|^2 & ((\gamma_A\gamma_B)^2+1)|a_1a_4|^2-\lambda
\end{bmatrix}
$$

$\Big\} \downarrow$ Eigenvalues of K

det

$$
|K - \lambda I| = 0
$$

$$
\Big[\underbrace{((\gamma_A\gamma_B)^2+1)|a_1a_4|^2}_{X}-\lambda\Big]
\begin{vmatrix}
-\lambda & \cdots & \cdots \\
0 & -\lambda & \cdots \\
0 & \cdots & \cdots
\end{vmatrix}
$$

$$
= \Big[\underbrace{((\gamma_A\gamma_B)^2+1)|a_1a_4|^2}_{X}-\lambda\Big](-\lambda)
\begin{vmatrix}
-\lambda & 0 \\
\cdots & \underbrace{((\gamma_A\gamma_B)^2+1)|a_1a_4|^2}_{X}-\lambda
\end{vmatrix}
$$

$$
= (-\lambda)(X-\lambda)(X-\lambda)(-\lambda) = \boxed{\lambda^2(X-\lambda)^2}
$$

$$
= \lambda^2 \cdot (X^2 - 2X\lambda + \lambda^2)
$$

$$\gamma_B(1+\gamma_A^2)a_1^* a_2 |a_4|^2 \begin{vmatrix} 0 & \sim & \sim \\ 0 & - & - \\ 0 & \underbrace{-}_{0} & - \end{vmatrix} = 0$$

c)

$$-2\gamma_A\gamma_B|a_4|^2 a_1^* a_4 \begin{array}{c} + \\ - \\ + \end{array}\begin{vmatrix} 0 & \sim & \sim \\ -\lambda & \sim & \sim \\ 0 & -\lambda & 0 \end{vmatrix}$$

$$(-2\gamma_A\gamma_B|a_4|^2 a_1^* a_4)(+\lambda) \begin{vmatrix} \sim & 2\gamma_A\gamma_B|a_1|^2 a_1 a_4^* \\ -\lambda & 0 \end{vmatrix}$$

$$-2\gamma_A\gamma_B|a_4|^2 a_1^* a_4 \lambda \left(\lambda\, 2\gamma_A\gamma_B|a_1|^2 a_1 a_4^*\right)$$

$$= -\lambda^2 \underbrace{4(\gamma_A\gamma_B)^2 |a_1 a_4|^4}_{Y^2} \longrightarrow Y = 2\gamma_A\gamma_B|a_1 a_4|^2$$

Characteristic Equation:

$$\lambda^2(X^2-2X\lambda+\lambda^2)+\lambda^2 Y^2 = 0$$

$$\lambda^2\left[\underbrace{X^2-2X\lambda+\lambda^2-Y^2}_{0}\right] = 0$$

$\lambda_{1,2} \neq 0$

$$X = \left[(\gamma_A\gamma_B)^2 + 1\right] |a_1 a_4|^2$$

$$Y = 2\gamma_A\gamma_B\, |a_1 a_4|^2$$

• Put $\lambda_1 = (1+\gamma_A\gamma_B)^2 |a_1 a_4|^2$ ✓

$$\left((\gamma_A\gamma_B)^2+1\right)^2 \cancel{|a_1a_4|^4} - 2\left[(\gamma_A\gamma_B)^2+1\right](1+\gamma_A\gamma_B)^2 \cancel{|a_1a_4|^2}$$

$$+ (1+\gamma_A\gamma_B)^4 \cancel{|a_1a_4|^4} - \cancel{4(\gamma_A\gamma_B)^2} \cancel{|a_1a_4|^4} = 0$$

$$\left[1+(\gamma_A\gamma_B)^2\right]^2 = \cancel{1} + \boxed{\cancel{2(\gamma_A\gamma_B)^2}} + \boxed{\cancel{(\gamma_A\gamma_B)^4}}$$

$$-2\left[(\gamma_A\gamma_B)^2+1\right]\left(1+2\gamma_A\gamma_B+\gamma_A^2\gamma_B^2\right)$$

$$= \left[\cancel{-2\gamma_A^2\gamma_B^2}\ \cancel{-2}\ \boxed{\cancel{-4\gamma_A^3\gamma_B^3}}\ \boxed{\cancel{-4\gamma_A\gamma_B}}\ \boxed{\cancel{-2\gamma_A^4\gamma_B^4}}\ \cancel{-2\gamma_A^2\gamma_B^2}\right]$$

$$(1+\gamma_A\gamma_B)^4 = \binom{4}{0}1^4 + \binom{4}{1}1^3.(\gamma_A\gamma_B) + \binom{4}{2}1^2.(\gamma_A\gamma_B)^2 + \binom{4}{3}1(\gamma_A\gamma_B)^3$$

$$+\binom{4}{4}1^0.(\gamma_A\gamma_B)^4$$

$$= \cancel{1} + \boxed{\cancel{4\gamma_A\gamma_B}} + \boxed{\cancel{6\gamma_A^2\gamma_B^2}} + \boxed{\cancel{4\gamma_A^3\gamma_B^3}} + \boxed{\cancel{\gamma_A^4\gamma_B^4}}$$

$$\lambda_2 = (1 - \gamma_A \gamma_B)^2 \, |a_1 a_4|^2$$

$$\lambda_1 = (1 + \gamma_A \gamma_B)^2 \, |a_1 a_4|^2$$

$$C(\rho(t)) = \max \{ 0, \sqrt{\lambda_1} - \sqrt{\lambda_2} \}$$

$$\sqrt{\lambda_1} - \sqrt{\lambda_2} = (1 + \gamma_A \gamma_B) |a_1 a_4| - (1 - \gamma_A \gamma_B) |a_1 a_4|$$

$$\boxed{C(\rho(t)) = 2 \gamma_A \gamma_B \, |a_1 a_4|}$$

Ex 2

$$|\psi_1\rangle = a_1 |1\rangle + a_2 |2\rangle + a_3 |3\rangle$$

$$|\psi_1\rangle\langle\psi_1| = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ 0 \end{pmatrix} \begin{pmatrix} a_1^* & a_2^* & a_3^* & 0 \end{pmatrix} = \begin{pmatrix} |a_1|^2 & a_1 a_2^* & a_1 a_3^* & 0 \\ a_2 a_1^* & |a_2|^2 & a_2 a_3^* & 0 \\ a_3 a_1^* & a_3 a_2^* & |a_3|^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\rho(t) = \mathcal{E}_{AB}(\rho) = \begin{pmatrix} |a_1|^2 & \gamma_B \, a_1 a_2^* & \gamma_A \, a_1 a_3^* & 0 \\ \gamma_B \, a_2 a_1^* & |a_2|^2 & \gamma_A \gamma_B \, a_2 a_3^* & 0 \\ \gamma_A \, a_3 a_1^* & \gamma_A \gamma_B \, a_3 a_2^* & |a_3|^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$K = \underbrace{\rho\,(\sigma_y{}^A \otimes \sigma_y{}^B)}_{B}\ \underbrace{\rho^*\,(\sigma_y{}^A \otimes \sigma_y{}^B)}_{A}$$

$$A = \begin{pmatrix} |a_1|^2 & \gamma_B\, a_1^* a_2 & \gamma_A\, a_1^* a_3 & 0 \\ \gamma_B\, a_2^* a_1 & |a_2|^2 & \gamma_A\gamma_B\, a_2^* a_3 & 0 \\ \gamma_A\, a_3^* a_1 & \gamma_A\gamma_B\, a_3^* a_2 & |a_3|^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} & & & -1 \\ & & 1 & \\ & 1 & & \\ -1 & & & \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & \gamma_A\, a_1^* a_3 & \gamma_B\, a_1^* a_2 & -|a_1|^2 \\ 0 & \gamma_A\gamma_B\, a_2^* a_3 & |a_2|^2 & -\gamma_B\, a_2^* a_1 \\ 0 & |a_3|^2 & \gamma_A\gamma_B\, a_3^* a_2 & -\gamma_A\, a_3^* a_1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & \gamma_A a_1 a_3^* & \gamma_B a_1 a_2^* & -|a_1|^2 \\ 0 & \gamma_A\gamma_B\, a_2 a_3^* & |a_2|^2 & -\gamma_B a_2 a_1^* \\ 0 & |a_3|^2 & \gamma_A\gamma_B a_3 a_2^* & -\gamma_A\, a_3 a_1^* \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$K = \begin{bmatrix} 0-\lambda & \gamma_B(1+\gamma_A^2)\, a_1 a_2^* |a_3|^2 & \gamma_A(1+\gamma_B^2)\, a_2 a_3^* |a_2|^2 & -\gamma_A(1+\gamma_B^2)\, a_2^* a_3^* |a_1|^2 \\ 0 & +(1+(\gamma_A\gamma_B)^2)\, |a_2 a_3|^2 - \lambda & 2\gamma_A\gamma_B\, |a_2|^2 a_2 a_3^* & -\gamma_A(1+\gamma_B)^2 a_1 a_3^* |a_2|^2 \\ 0 & -\,2\gamma_A\gamma_B\, a_2^* a_3 |a_3|^2 & (1+(\gamma_A\gamma_B)^2)\, |a_2 a_3|^2 - \lambda & -\gamma_B(1+\gamma_A)^2 a_1 a_2^* |a_3|^2 \\ 0 & +\quad 0 & 0 & 0-\lambda \end{bmatrix}$$

$$\det(K - I\lambda) = 0$$

$$(-\lambda)\begin{vmatrix} +\, x-\lambda & y & z \\ -\, w & x-\lambda & T \\ 0 & 0 & -\lambda \end{vmatrix}$$

$$(-\lambda)\left[(x-\lambda)\begin{vmatrix} x-\lambda & T \\ 0 & -\lambda \end{vmatrix} - w\begin{vmatrix} y & z \\ 0 & -\lambda \end{vmatrix}\right]$$

$$(-\lambda)\cdot\left[(x-\lambda)(-\lambda)(x-\lambda) - w\,(-\lambda)\,y\right] = 0$$

$$(\lambda)^2\left[\underbrace{(x-\lambda)^2 - wy}_{0}\right] = 0$$

$$w = 2\gamma_A\gamma_B\, a_2^* a_3 |a_3|^2$$

$$y = 2\gamma_A\gamma_B\, a_2 a_3^* |a_2|^2$$

$$\boxed{x^2 - 2x\lambda + \lambda^2 - wy = 0}$$

$$\lambda_1 = (1+\gamma_A\gamma_B)^2 |a_2 a_3|^2$$

$$\lambda_2 = (1-\gamma_A\gamma_B)^2 |a_2 a_3|^2$$

* Conclusion:

Concurrence is determined by the fast-decaying off diagonal elements ρ_{13}, ρ_{23}

Ex 1 Ex 2

$$C(\rho(t)) = \max\left\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2}\right\} = \boxed{2\gamma_A\gamma_B |a_2 a_3|}$$

B. Entanglement decay under one-qubit dephasing channel

Show that $$C(\mathcal{E}_A(\rho(0))) \leq \gamma_A C(\rho(0))$$

$$C(\mathcal{E}_B(\rho(0))) \leq \gamma_B C(\rho(0))$$

$$C\left(\sum_{\mu=1}^{n} p_\mu \rho_\mu\right) \leq \sum_{\mu=1}^{n} p_\mu C(\rho_\mu)$$ → "Jensen's" Inequality

$$C(\rho(t)) = \sum_{\mu=1}^{2} C(\underbrace{E_\mu^\dagger \rho(0) E_\mu}_{\rho_{out}})$$

$$K = \rho_{out} (\sigma_y^A \otimes \sigma_y^B) \rho_{out}^* (\sigma_y^A \otimes \sigma_y^B)$$

$$K = E_\mu^\dagger \rho \underbrace{E_\mu (\sigma_y^A \otimes \sigma_y^B)} E_\mu^T \rho^* E_\mu^* (\sigma_y^A \otimes \sigma_y^B)$$

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & \gamma_A \end{pmatrix} \otimes I = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \gamma_A & \\ & & & \gamma_A \end{pmatrix}$$

$$E_2 = \begin{pmatrix} 0 & 0 \\ 0 & \omega_A \end{pmatrix} \otimes I = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & \omega_A & \\ & & & \omega_A \end{pmatrix}$$

$\Big\}$ $E_\mu^\dagger = E_\mu^* = E_\mu^T$

$$K' = \rho \, E_\mu (\sigma_y^A \otimes \sigma_y^B) E_\mu \, \rho^* \, E_\mu (\sigma_y^A \otimes \sigma_y^B) E_\mu$$

$$E_\nu \left(\sigma_y^A \otimes \sigma_y^B\right) E_\nu = A$$

for $\nu = 1$ $\quad E_1 = \begin{pmatrix} 1 &&& \\ & 1 && \\ && \gamma_A & \\ &&& \gamma_A \end{pmatrix}$

$$A = \begin{pmatrix} 1 &&& \\ & 1 && \\ && \gamma_A & \\ &&& \gamma_A \end{pmatrix} \begin{pmatrix} &&& -1 \\ && 1 & \\ & 1 && \\ -1 &&& \end{pmatrix} \begin{pmatrix} 1 &&& \\ & 1 && \\ && \gamma_A & \\ &&& \gamma_A \end{pmatrix}$$

$$A = \begin{pmatrix} 1 &&& \\ & 1 && \\ && \gamma_A & \\ &&& \gamma_A \end{pmatrix} \begin{pmatrix} &&& -\gamma_A \\ && \gamma_A & \\ & 1 && \\ -1 &&& \end{pmatrix}$$

$$= \begin{pmatrix} &&& -\gamma_A \\ && \gamma_A & \\ & \gamma_A && \\ -\gamma_A &&& \end{pmatrix} = \gamma_A \left(\sigma_y^A \otimes \sigma_y^B\right)$$

$$K': \ \rho A \rho^* A = \gamma_A^2 \underbrace{\rho \left(\sigma_y^A \otimes \sigma_y^B\right) \rho^* \left(\sigma_y^A \otimes \sigma_y^B\right)}_{C(\rho|0)}$$

for $\nu = 2$ $\quad E_2 = \begin{pmatrix} 0 &&& \\ & 0 && \\ && w_A & \\ &&& w_A \end{pmatrix}$

$$\begin{pmatrix} 0 &&& \\ & 0 && \\ && w_A & \\ &&& w_A \end{pmatrix} \begin{pmatrix} &&& -1 \\ && 1 & \\ & 1 && \\ -1 &&& \end{pmatrix} \begin{pmatrix} 0 &&& \\ & 0 && \\ && w_A & \\ &&& w_A \end{pmatrix}$$

$$\begin{pmatrix} 0 &&& \\ & 0 && \\ && w_A & \\ &&& w_A \end{pmatrix} \begin{pmatrix} &&& -w_A \\ && w_A & \\ & 0 && \\ 0 &&& \end{pmatrix} = \begin{pmatrix} &&& 0 \\ && 0 & \\ & 0 && \\ 0 &&& \end{pmatrix}$$

$$K' = \begin{cases} \gamma_A^2\, C(\rho(0)) & \text{if } N = 1 \\ 0 & N \neq 1 \end{cases}$$

$$C(K') = \max\Big(0,\ \underset{\substack{\Downarrow \\ \gamma_A C(\rho(0))}}{\sqrt{\lambda_1}} \ldots \Big)$$

Thus, $\boxed{C(\rho(t)) \leq \gamma_A\, C(\rho(0))}$

One-qubit local dephasing channels can completely destroy the quantum entanglement after the dephasing times τ_A or τ_B.

<u>Ex</u> $$|\phi\rangle = \frac{1}{\sqrt{3}}\left(|1\rangle_{AB} + |3\rangle_{AB} + |4\rangle_{AB}\right)$$

$$|\phi\rangle\langle\phi| = \frac{1}{3}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 & 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 0 & \overset{\gamma_A}{1} & \overset{\gamma_A}{1} \\ 0 & 0 & 0 & 0 \\ {}^{\gamma_A}1 & 0 & 1 & 1 \\ {}^{\gamma_A}1 & 0 & 1 & 1 \end{pmatrix}$$

$$\rho(t) = \mathcal{E}_A(\rho) = \begin{pmatrix} \frac{1}{3} & 0 & \frac{\gamma_A}{3} & \frac{\gamma_A}{3} \\ 0 & 0 & 0 & 0 \\ \frac{\gamma_A}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{\gamma_A}{3} & 0 & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

$$K = \underbrace{\rho\,(\sigma_y^A \otimes \sigma_y^B)}_{A}\ \underbrace{\rho^*\,(\sigma_y^A \otimes \sigma_y^B)}_{A} \qquad \text{where } \rho^* = \rho$$

$$A = \begin{pmatrix} \frac{1}{3} & 0 & \frac{\gamma_A}{3} & \frac{\gamma_A}{3} \\ 0 & 0 & 0 & 0 \\ \frac{\gamma_A}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{\gamma_A}{3} & 0 & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} & & & -1 \\ & & 1 & \\ & 1 & & \\ -1 & & & \end{pmatrix}$$

$$A = \begin{bmatrix} -\frac{\gamma_A}{3} & \frac{\gamma_A}{3} & 0 & -\frac{1}{3} \\ 0 & 0 & 0 & 0 \\ -\frac{1}{3} & \frac{1}{3} & 0 & -\frac{\gamma_A}{3} \\ -\frac{1}{3} & \frac{1}{3} & 0 & -\frac{\gamma_A}{3} \end{bmatrix}$$

$-\frac{\gamma_A}{9}$

$-\frac{\gamma_A}{9}$

$$K = A^2 = \begin{bmatrix} -\frac{\gamma_A}{3} & \frac{\gamma_A}{3} & 0 & -\frac{1}{3} \\ 0 & 0 & 0 & 0 \\ -\frac{1}{3} & \frac{1}{3} & 0 & -\frac{\gamma_A}{3} \\ -\frac{1}{3} & \frac{1}{3} & 0 & -\frac{\gamma_A}{3} \end{bmatrix} \begin{bmatrix} -\frac{\gamma_A}{3} & \frac{\gamma_A}{3} & 0 & -\frac{1}{3} \\ 0 & 0 & 0 & 0 \\ -\frac{1}{3} & \frac{1}{3} & 0 & -\frac{\gamma_A}{3} \\ -\frac{1}{3} & \frac{1}{3} & 0 & -\frac{\gamma_A}{3} \end{bmatrix}$$

$$= \begin{array}{c} + \\ - \\ + \\ - \end{array} \begin{bmatrix} \frac{\gamma_A^2+1}{9} - \lambda & -\frac{(\gamma_A^2+1)}{9} & 0 & \frac{2\gamma_A}{9} \\ 0 & 0 - \lambda & 0 & 0 \\ \frac{2\gamma_A}{9} & -\frac{2\gamma_A}{9} & 0 - \lambda & \frac{\gamma_A^2+1}{9} \\ \frac{2\gamma_A}{9} & -\frac{2\gamma_A}{9} & 0 & \frac{\gamma_A^2+1}{9} - \lambda \end{bmatrix}$$

a)

$$\left(\frac{\gamma_A^2+1}{g}-\lambda\right)\begin{array}{c} + \\ - \\ + \end{array}\begin{vmatrix} -\lambda & 0 & 0 \\ -\frac{2\gamma_A}{g} & -\lambda & \frac{\gamma_A^2+1}{g} \\ -\frac{2\gamma_A}{g} & 0 & \frac{\gamma_A^2+1}{g}-\lambda \end{vmatrix}$$

$$\left(\frac{\gamma_A^2+1}{g}-\lambda\right)\left[(-\lambda)\begin{vmatrix} -\lambda & \\ 0 & \frac{\gamma_A^2+1}{g}-\lambda \end{vmatrix} + \frac{2\gamma_A}{g}\begin{vmatrix} 0 & 0 \\ 0 & - \end{vmatrix} - \frac{2\gamma_A}{g}\begin{vmatrix} & 0 \\ & \cdots \end{vmatrix}\right]$$

$$\left(\frac{\gamma_A^2+1}{g}-\lambda\right)(-\lambda)(-\lambda)\left(\frac{\gamma_A^2+1}{g}-\lambda\right) = \lambda^2\left(\frac{\gamma_A^2+1}{g}-\lambda\right)^2$$

b)

$$\frac{2\gamma_A}{g}\begin{vmatrix} -\frac{(\gamma_A^2+1)}{g} & 0 & \frac{2\gamma_A}{g} \\ -\lambda & 0 & 0 \\ -\frac{2\gamma_A}{g} & 0 & \frac{\gamma_A^2+1}{g} \end{vmatrix} = 0$$

c)

$$-\frac{2\gamma_A}{g}\begin{vmatrix} -\frac{(\gamma_A^2+1)}{g} & 0 & \frac{2\gamma_A}{g} \\ -\lambda & 0 & 0 \\ -\frac{2\gamma_A}{g} & -\lambda & \frac{\gamma_A^2+1}{g} \end{vmatrix}$$

$$\lambda^2\left(\frac{2\gamma_A}{g}\right)^2$$

$$-\frac{2\gamma_A}{g}(-\lambda)\begin{vmatrix} 0 & \frac{2\gamma_A}{g} \\ -\lambda & \sim \end{vmatrix} = \frac{2\gamma_A}{g}\lambda\left(\lambda\frac{2\gamma_A}{g}\right)$$

$$\det(K - I\lambda) = 0$$

Characteristic Equation:

$$\lambda^2 \left[\underbrace{\left(\frac{\gamma_A^2+1}{9} - \lambda\right)^2 + \left(\frac{2\gamma_A}{9}\right)^2}_{0} \right] = 0$$

$$\lambda^2 - 2\lambda\left(\frac{\gamma_A^2+1}{9}\right) + \left(\frac{\gamma_A^2+1}{9}\right)^2 + \left(\frac{2\gamma_A}{9}\right)^2 = 0$$

$$a\lambda^2 + b\lambda + c = 0$$

where $a = 1$

$$b = -2\left(\frac{\gamma_A^2+1}{9}\right)$$

$$c = \left(\frac{\gamma_A^2+1}{9}\right)^2 + \left(\frac{2\gamma_A}{9}\right)^2$$

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac} \rightarrow \Delta}{2a}$$

$$\Delta = b^2 - 4ac = \cancel{4\left(\frac{\gamma_A^2+1}{9}\right)^2} - 4\left[\cancel{\left(\frac{\gamma_A^2+1}{9}\right)^2} + \left(\frac{2\gamma_A}{9}\right)^2\right]$$

$$= -4\left(\frac{2\gamma_A}{9}\right)^2$$

$$\sqrt{\Delta} = \left(\frac{4\gamma_A}{9}\right) i$$

$$\lambda_1 = \frac{-2\left(\frac{\gamma_A^2+1}{9}\right) + \left(\frac{4\gamma_A}{9}\right)i}{2} = -\frac{\gamma_A^2}{9} - \frac{1}{9} + \frac{2\gamma_A}{9} i$$

$$\lambda_1 = -\frac{1}{9}\left(\underbrace{\gamma_A^2 - 2\gamma_A i + 1}_{(\gamma_A - i)^2 + 2}\right)$$

$$\sqrt{\lambda_1} = \frac{i}{3}\sqrt{(\gamma_A - i)^2 + 2}$$

$$\sqrt{\lambda_2} = \frac{i}{3}\sqrt{(\gamma_A + i)^2 + 2}$$

$$\lambda_2 = \frac{-2\left(\frac{\gamma_A^2+1}{9}\right) - \left(\frac{4\gamma_A}{9}\right)i}{2}$$

$$= -\frac{1}{9}\left(\underbrace{\gamma_A^2 + 2\gamma_A i + 1}_{(\gamma_A + i)^2 + 2}\right)$$

✱ Decoherence process for the composite 2-qubit system becomes frozen after the local dephasing times τ_A or τ_B. $\rightarrow$ UNDER 1 QUBIT CHANNEL $\mathcal{E}_A$ or $\mathcal{E}_B$

Conclusion / Results

Still Questions

- What can be modeled, how death and rebirth regime can be seen from this paper?