Zeynepnur Sahinel

COM-309 Hw2

Exercise 1

1.1) $\{ |a\rangle, |a_\perp\rangle \}$ and $\{ |R\rangle, |L\rangle \}$ are orthonormal basis?

$\langle a_\perp | a \rangle \overset{?}{=} 0 \; ✓$

$\langle a_\perp | = -\sin\alpha \langle x| + \cos\alpha \langle y|$

$\langle a_\perp | a \rangle = (-\sin\alpha \langle x| + \cos\alpha \langle y|)(\cos\alpha |x\rangle + \sin\alpha |y\rangle)$

$= -\sin\alpha . \cos\alpha \underbrace{\langle x | x \rangle}_{1} + \cos\alpha . \sin\alpha \underbrace{\langle y|y\rangle}_{1}$

$= -\sin\alpha . \cos\alpha + \cos\alpha \sin\alpha = 0 \,//$

$\langle L | R \rangle \overset{?}{=} 0 \; ✓$

$\langle L | = \frac{1}{\sqrt{2}} (\langle x| + i \langle y|)$

$\langle L | R \rangle = \frac{1}{\sqrt{2}} (\langle x| + i\langle y|) \frac{1}{\sqrt{2}} (|x\rangle + i |y\rangle)$

$= \frac{1}{2} \left(\underbrace{\langle x | x \rangle}_{1} + \underbrace{i^2}_{-1} \underbrace{\langle y | y \rangle}_{1} \right) = \frac{1}{2} (1 - 1) = 0 \,//$

1.2) $|\psi\rangle \longrightarrow \boxed{M_1 \; \{|x\rangle, |y\rangle\}}$

Outcome		Probability
$\|x\rangle$	$\longrightarrow$	$\| \langle x \| \psi \rangle \|^2 = \| \cos\theta \|^2 = \cos^2\theta$
$\|y\rangle$	$\longrightarrow$	$\| \langle y \| \psi \rangle \|^2 = \| e^{i\varphi} \sin\theta \|^2$

$= \underbrace{|e^{i\varphi}|^2}_{1} . |\sin^2\theta|$

$= \sin^2\theta$

$|\psi\rangle = \cos\theta \, |x\rangle + (\sin\theta) e^{i\varphi} |y\rangle$

$\langle x | \psi \rangle = \cos\theta$

$\langle y | \psi \rangle = \sin\theta \, e^{i\varphi}$

check: $\cos^2\theta + \sin^2\theta = 1 \; ✓$

$$
\begin{tikzcd}
 & & \text{Outcome} & \text{Probability} \\
 & & {|R\rangle} \arrow[r] & {\left|\langle R|\psi\rangle\right|^2} \\
{|\psi\rangle} \arrow[r] & \boxed{\begin{matrix} M_2 \\ \{|R\rangle, |L\rangle\} \end{matrix}} \arrow[ur] \arrow[dr] & & \\
 & & {|L\rangle} \arrow[r] & {\left|\langle L|\psi\rangle\right|^2}
\end{tikzcd}
$$

$$|\psi\rangle = \cos\theta|x\rangle + (\sin\theta)e^{i\varphi}|y\rangle$$

$$\langle R|\psi\rangle = \cos\theta\,\langle R|x\rangle + \sin\theta\, e^{i\varphi}\,\langle R|y\rangle = \cos\theta \cdot \frac{1}{\sqrt{2}} + \sin\theta\, e^{i\varphi}\left(\frac{-i}{\sqrt{2}}\right)$$

$$\left.\begin{aligned}
\langle R|x\rangle &= \frac{1}{\sqrt{2}}\left(\langle x| - i\langle y|\right)|x\rangle = \frac{1}{\sqrt{2}}\underbrace{\langle x|x\rangle}_{1} = \frac{1}{\sqrt{2}} \\
\langle R|y\rangle &= \frac{1}{\sqrt{2}}\left(\langle x| - i\langle y|\right)|y\rangle = -\frac{i}{\sqrt{2}}\underbrace{\langle y|y\rangle}_{1} = -\frac{i}{\sqrt{2}}
\end{aligned}\right\}$$

$$\left|\langle R|\psi\rangle\right|^2 = \frac{1}{2}\left|\cos\theta + (-i)\sin\theta\, \underbrace{e^{i\varphi}}_{(\cos\alpha + i\sin\alpha)}\right|^2$$

$$= \frac{1}{2}\left|\cos\theta - i\sin\theta\,(\cos\alpha + i\sin\alpha)\right|^2$$

$$= \frac{1}{2}\left|\cos\theta - i\sin\theta\cos\alpha \underbrace{-i^2}_{+1}\sin\theta\sin\alpha\right|^2$$

$$= \frac{1}{2}\left(\cos\theta + \sin\theta\sin\alpha - i\sin\theta\cos\alpha\right)\cdot\left(\cos\theta + \sin\theta\sin\alpha + i\sin\theta\cos\alpha\right)$$

$$= \frac{1}{2}\left((\cos\theta + \sin\theta\sin\alpha)^2 + (\sin\theta\cos\alpha)^2\right)$$

$$= \frac{1}{2}\left(\left(\cos^2\theta + 2\sin\theta\cos\theta\sin\alpha + \sin^2\theta\sin^2\alpha\right) + \sin^2\theta\cos^2\alpha\right)$$

$$= \frac{1}{2}\left(\cos^2\theta + \sin 2\theta\sin\alpha + \sin^2\theta\,\underbrace{(\sin^2\alpha + \cos^2\alpha)}_{1}\right)$$

$$= \frac{1}{2}\left(1 + \sin 2\theta\sin\alpha\right)$$

$$\langle L|\psi\rangle = \frac{1}{\sqrt{2}}\left(\langle x| + i\langle y|\right)\left(\cos\theta|x\rangle + (\sin\theta)e^{i\varphi}|y\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\cos\theta + \frac{i}{\sqrt{2}}(\sin\theta)e^{i\varphi}$$

$$\| \langle L|\psi\rangle |^2 = \left\| \frac{1}{\sqrt{2}}\cos\theta + \frac{i}{\sqrt{2}}\sin\theta\,(\cos\alpha + i\sin\alpha) \right\|^2$$

$$= \frac{1}{2}\left|\left(\cos\theta + i\sin\theta\,(\cos\alpha + i\sin\alpha)\right|^2$$

$$= \frac{1}{2}\left| \cos\theta + i\sin\theta\cos\alpha - \sin\theta\sin\alpha \right|^2$$

$$= \frac{1}{2}\Big(\left(\cos\theta - \sin\theta\sin\alpha\right) + i\sin\theta\cos\alpha\Big)\Big(\left(\cos\theta - \sin\theta\sin\alpha\right) - i\sin\theta\cos\alpha\Big)$$

$$= \frac{1}{2}\left(\left(\cos\theta - \sin\theta\sin\alpha\right)^2 + \sin^2\theta\cos^2\alpha\right)$$

$$= \frac{1}{2}\left(\cos^2\theta - 2\sin\theta\cos\theta\sin\alpha + \sin^2\theta\sin^2\alpha + \sin^2\theta\cos^2\alpha\right)$$

$$= \frac{1}{2}\left(\cos^2\theta - \sin 2\theta\sin\alpha + \sin^2\theta\,(\underbrace{\sin^2\alpha + \cos^2\alpha}_{1})\right)$$

$$= \frac{1}{2}\left(1 - \sin 2\theta\sin\alpha\right)$$

Check: $\underbrace{\frac{1}{2}\left(1 + \sin 2\theta\sin\alpha\right)}_{P(|R\rangle)} + \underbrace{\frac{1}{2}\left(1 - \sin 2\theta\sin\alpha\right)}_{P(|L\rangle)} = 1$ //

$$\begin{tikzcd}
 & & \text{Outcome} & \text{Prob.} \\
 & & |\alpha\rangle \arrow[r] & \|\langle\alpha|\psi\rangle\|^2 \\
|\psi\rangle \arrow[r] & \boxed{\begin{matrix} M_3 \\ \{|\alpha\rangle, |\alpha_\perp\rangle\} \end{matrix}} \arrow[ur] \arrow[dr] & & \\
 & & |\alpha_\perp\rangle \arrow[r] & \|\langle\alpha_\perp|\psi\rangle\|^2
\end{tikzcd}$$

$$\langle\alpha|\psi\rangle = \left(\cos\alpha\langle x| + \sin\alpha\langle y|\right)\left(\cos\theta|x\rangle + (\sin\theta)e^{i\varphi}|y\rangle\right)$$

$$|\langle\alpha|\psi\rangle|^2 = \left|\cos\alpha\cos\theta + \sin\alpha(\sin\theta)e^{i\varphi}\right|^2$$

$$|\langle\alpha|\psi\rangle|^2 = \left| \cos\alpha\cos\theta + \sin\alpha\sin\theta\,(\cos\varphi + i\sin\varphi) \right|^2$$

$$= \left| \underbrace{\cos\alpha\cos\theta + \sin\alpha\sin\theta\cos\varphi}_{a} + \underbrace{i\sin\alpha\sin\theta\sin\varphi}_{ib} \right|^2$$

$$= (\cos\alpha\cos\theta + \sin\alpha\sin\theta\cos\varphi)^2 + (\sin\alpha\sin\theta\sin\varphi)^2$$

$$= \Big(\cos^2\alpha\cos^2\theta + 2\cos\alpha\cos\theta\sin\alpha\sin\theta\cos\varphi + \sin^2\alpha\sin^2\theta\cos^2\varphi + \sin^2\alpha\sin^2\theta\sin^2\varphi\Big)$$

$\sin^2\alpha\sin^2\theta$

$(1-\sin^2\alpha)(1-\sin^2\theta)$

$$= \left(\cos^2\alpha\cos^2\theta + \frac{1}{2}\sin(2\alpha)\cdot\sin(2\theta)\cos\alpha + \sin^2\alpha\sin^2\theta\right)$$

$$\langle\alpha_\perp|\psi\rangle = \left(-\sin\alpha\langle x| + \cos\alpha\langle y|\right)\left(\cos\theta|x\rangle + (\sin\theta)e^{i\varphi}|y\rangle\right)$$

$$|\langle\alpha_\perp|\psi\rangle|^2 = \left|-\sin\alpha\cos\theta + \cos\alpha(\sin\theta)e^{i\varphi}\right|^2$$

$$= \left|-\sin\alpha\cos\theta + \cos\alpha(\sin\theta)(\cos\varphi + i\sin\varphi)\right|^2$$

$$= \left|-\sin\alpha\cos\theta + \cos\alpha\sin\theta\cos\varphi + i\cos\alpha\sin\theta\sin\varphi\right|^2$$

$$= \left(-\sin\alpha\cos\theta + \cos\alpha\sin\theta\cos\varphi\right)^2 + \cos^2\alpha\sin^2\theta\sin^2\varphi\,)$$

$$= \Big(\sin^2\alpha\cos^2\theta - 2\sin\alpha\cos\theta\cos\alpha\sin\theta\cos\alpha + \underbrace{\cos^2\alpha\sin^2\theta\cos^2\varphi + \cos^2\alpha\sin^2\theta\sin^2\varphi}_{\cos^2\alpha\sin^2\theta}\Big)$$

$$= \left(\sin^2\alpha\cos^2\theta + \cos^2\alpha\sin^2\theta - \frac{1}{2}\sin(2\alpha)\sin(2\theta)\cos\alpha\right)$$

Check: $\left(\cos^2\alpha\cos^2\theta + \sin^2\alpha\sin^2\theta + \frac{1}{2}\sin(2\alpha)\sin(2\theta)\cos\alpha\right) + \left(\sin^2\alpha\cos^2\theta + \cos^2\alpha\sin^2\theta - \frac{1}{2}\sin(2\alpha)\sin(2\theta)\cos\alpha\right)$

$$= \cos^2\alpha\cos^2\theta + \sin^2\alpha\cos^2\theta + \sin^2\alpha\sin^2\theta + \cos^2\alpha\sin^2\theta$$

$$= \cos^2\theta\,(1) + \sin^2\theta\,(1) = 1 \quad \checkmark$$

Exercise 2

2.1) Compute $S|H\rangle$, $S|V\rangle$, $R|H\rangle$, $R|V\rangle$

$$S = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \qquad R = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \qquad |H\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |V\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$S|H\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$S|V\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \end{pmatrix}$$

$$R|H\rangle = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ i \end{pmatrix}$$

$$R|V\rangle = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} i \\ 0 \end{pmatrix}$$

2.2) $SRS|H\rangle$

$$RS = \underbrace{\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}}_{R}\ \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}}_{S} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & i \\ i & -1 \end{pmatrix}$$

$$SRS = \frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\begin{pmatrix} -1 & i \\ i & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$SRS|H\rangle = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

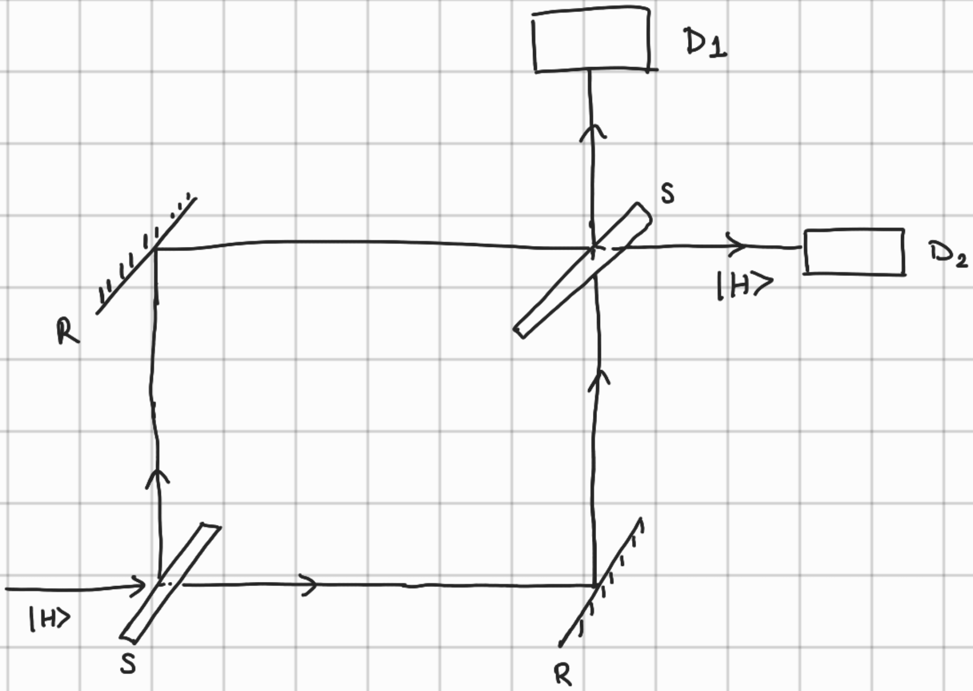
$$\langle H|SRS|H\rangle = \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} -1 \\ 0 \end{pmatrix} = -1$$

$$|\langle H|SRS|H\rangle|^2 = 1$$

$$\langle V|SRS|H\rangle = \begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 0 \end{pmatrix} = 0$$

$$|\langle V|SRS|H\rangle|^2 = 0$$

Make a picture of the experimental set-up.



2.3) $D = \begin{pmatrix} e^{i\varphi_1} & 0 \\ 0 & e^{i\varphi_2} \end{pmatrix}$

$D^\dagger = \begin{pmatrix} e^{-i\varphi_1} & 0 \\ 0 & e^{-i\varphi_2} \end{pmatrix}$ $\quad D^\dagger D = \begin{pmatrix} e^{-i\varphi_1} & 0 \\ 0 & e^{-i\varphi_2} \end{pmatrix} \begin{pmatrix} e^{i\varphi_1} & 0 \\ 0 & e^{i\varphi_2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$ ✓

Dephaser is unitary.

$$D|H\rangle = \begin{pmatrix} e^{i\varphi_1} & 0 \\ 0 & e^{i\varphi_2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} e^{i\varphi_1} \\ 0 \end{pmatrix} = e^{i\varphi_1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = e^{i\varphi_1} |H\rangle$$

$$D|V\rangle = \begin{pmatrix} e^{i\varphi_1} & 0 \\ 0 & e^{i\varphi_2} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ e^{i\varphi_2} \end{pmatrix} = e^{i\varphi_2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = e^{i\varphi_2} |V\rangle$$

First apply S to $|H\rangle$

$$S|H\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

Then apply D

$$DS|H\rangle = \begin{pmatrix} e^{i\varphi_1} & 0 \\ 0 & e^{i\varphi_2} \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\varphi_1} \\ i e^{i\varphi_2} \end{pmatrix}$$

Apply R

$$RDS|H\rangle = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\varphi_1} \\ i e^{i\varphi_2} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -e^{i\varphi_2} \\ i e^{i\varphi_1} \end{pmatrix}$$

Apply S

$$SRDS|H\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} -e^{i\varphi_2} \\ i e^{i\varphi_1} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -e^{i\varphi_2} - e^{i\varphi_1} \\ -i e^{i\varphi_2} + i e^{i\varphi_1} \end{pmatrix}$$

$$\left| \langle H | SRDS | H \rangle \right|^2 = \left| \frac{1}{2} (1 \ 0) \begin{pmatrix} -e^{i\varphi_2} - e^{i\varphi_1} \\ -i e^{i\varphi_2} + i e^{i\varphi_1} \end{pmatrix} \right|^2 = \frac{1}{4} \cdot \left| -e^{i\varphi_2} - e^{i\varphi_1} \right|^2$$

$$= \frac{1}{4} \left| -e^{i\varphi_2} \left(1 + e^{i(\varphi_1 - \varphi_2)} \right) \right|^2 = \frac{1}{4} \left| 1 + e^{i(\varphi_1 - \varphi_2)} \right|^2$$

$$= \frac{1}{4} \left| 1 + 2 e^{i(\varphi_1 - \varphi_2)} + e^{2i(\varphi_1 - \varphi_2)} \right|$$

$$= \frac{1}{4} \left| e^{i(\varphi_1 - \varphi_2)} \left(e^{-i(\varphi_1 - \varphi_2)} + 2 + e^{i(\varphi_1 - \varphi_2)} \right) \right|$$

$$= \frac{1}{4} \left| \underbrace{\cos(\varphi_1 - \varphi_2) - \cancel{i \sin(\varphi_1 - \varphi_2)} + 2 + \cos(\varphi_1 - \varphi_2) + \cancel{i \sin(\varphi_1 - \varphi_2)}}_{>0} \right|$$

$$= \frac{1}{4} \left(2 + 2\cos(\varphi_1 - \varphi_2) \right) = \frac{1}{2} \left(1 + \cos(\varphi_1 - \varphi_2) \right)$$

$$\left| \langle V | SRDS | H \rangle \right|^2 = \left| \frac{1}{2} (0 \ 1) \begin{pmatrix} -e^{i\varphi_2} - e^{i\varphi_1} \\ -i e^{i\varphi_2} + i e^{i\varphi_1} \end{pmatrix} \right|^2 = \frac{1}{4} \left| -i e^{i\varphi_2} + i e^{i\varphi_1} \right|^2$$

$$= \frac{1}{4} \left| -i e^{i\varphi_2} \left(+1 - e^{i(\varphi_1 - \varphi_2)} \right) \right|^2$$

$$= \frac{1}{4} \left| \left(1 - e^{i(\varphi_1-\varphi_2)}\right) \right|^2$$

$$= \frac{1}{4} \left| 1 - 2e^{i(\varphi_1-\varphi_2)} + e^{2i(\varphi_1-\varphi_2)} \right|$$

$$= \frac{1}{4} \left| e^{i(\varphi_1-\varphi_2)} \left(e^{-i(\varphi_1-\varphi_2)} - 2 + e^{i(\varphi_1-\varphi_2)} \right) \right|$$

$$= \frac{1}{4} \left| 2\cos(\varphi_1-\varphi_2) - 2 \right| = \frac{1}{2} \left| \overbrace{\cos(\varphi_1-\varphi_2) - 1}^{<0} \right|$$

$$= \frac{1}{2} \left(1 - \cos(\varphi_1-\varphi_2) \right)$$

Prob Check $= \left| \langle H | SDRS | H \rangle \right|^2 + \left| \langle V | SDRS | H \rangle \right|^2 \overset{?}{=} 1$

$$= \frac{1}{2}\left(1 + \cos(\varphi_1-\varphi_2)\right) + \frac{1}{2}\left(1 - \cos(\varphi_1-\varphi_2)\right)$$

$$= \frac{1}{2} + \frac{1}{2} = 1 \quad \checkmark$$

SDRS is unitary ? ✓

We know that S, R, D are unitary

$$(SRDS)^\dagger = (S^\dagger D^\dagger R^\dagger S^\dagger)$$

$$(SRDS)^\dagger (SRDS) \overset{?}{=} \mathbb{I}$$

$$\underbrace{(S^\dagger \underbrace{D^\dagger \underbrace{R^\dagger \underbrace{S^\dagger)(S}_{I} R}_{I} D}_{I} S)}_{S^\dagger \underbrace{I S}_{S} = S^\dagger S = \mathbb{I}} \overset{\checkmark}{=} \mathbb{I}$$

Experimental situation: