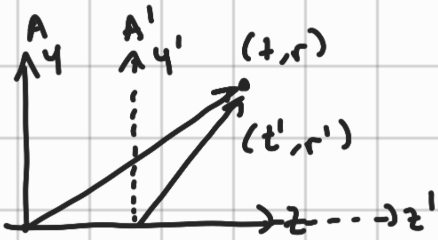
Four-vectors and Lorentz Invariance

$$\beta = \frac{v}{c} \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}$$



$$t' = \gamma(t - \beta z)$$
$$x' = x$$
$$y' = y$$
$$z' = \gamma(z - \beta t)$$

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \quad (1) \quad (2.4)$$

$$t = \gamma(t' + \beta z')$$
$$x = x'$$
$$y = y'$$
$$z = \gamma(z' + \beta t')$$

$$\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & +\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ +\gamma\beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} \quad (2) \quad (2.6)$$

Contravariant space-time four-vector $\rightarrow x^{\mu} = (t, x, y, z)$

where $\mu = 0,1,2,3$ $\quad x^0 = t \quad x^1 = x, \; x^2 = y, \; x^3 = z$

Covariant space-time four vector $\rightarrow x_{\mu} = (t, -x, -y, z)$

Einstein's summation convention

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} \, x^{\nu} \quad \leftarrow \text{Eq from (1)}$$

$$x^{\mu} x_{\mu} = x^0 x_0 + x^1 x_1 + x^2 x_2 + x^3 x_3 = t^2 - x^2 - y^2 - z^2$$

Lorentz invariance of the spacetime interval

Relation between covariant and contravariant four-vectors

$$x_\mu = g_{\mu\nu} x^\nu$$

$$\downarrow$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Diagonal metric tensor

★ Any expression that can be written in terms of 4-vector scalar product is guaranteed to be Lorentz invariant.

Four Momentum

$$E = \gamma m c^2 \xrightarrow{c=1} E = \gamma m$$

$$p = \gamma m v \xrightarrow{v = \beta c^{(1)}} p = \gamma m \beta$$

$$p^\mu = (E, p_x, p_y, p_z)$$

$$p_\mu = (E, -p_x, -p_y, -p_z)$$

$$p^\mu p_\mu = E^2 - \vec{p}^{\,2}$$ ⇝ Lorentz-invariant quantity

$$E^2 - (p_x^2 + p_y^2 + p_z^2)$$

For a particle at rest : $p^\mu = (m, 0, 0, 0) = p_\mu$

$$p^\mu p_\mu = E^2 - p^2 = m^2 \rightsquigarrow E^2 = p^2 + m^2$$ "Einstein energy-momentum relationship"

For a system of particles :

$$p^\mu = \sum_{i=1}^{N} p_i^\mu$$ is also 4-vector

$$p^\mu p_\mu = \left(\sum_{i=1}^{n} E_i\right)^2 - \left(\sum_{i=1}^{n} p_i\right)^2 = M^2$$

invariant mass of the system

$$a \longrightarrow 1 + 2$$

$$(p_1+p_2)^\mu (p_1+p_2)_\mu = p_a^\mu p_{a\mu} = M_a^2$$

Four- derivative

$$\frac{\partial}{\partial z'} = \underbrace{\left(\frac{\partial z}{\partial z'}\right)}_{\gamma}\frac{\partial}{\partial z} + \underbrace{\left(\frac{\partial t}{\partial z'}\right)}_{\gamma\beta}\frac{\partial}{\partial t}$$

$$t = \gamma(t' + \beta z')$$

$$z = \gamma(z' + \beta t')$$

$$\frac{\partial}{\partial t'} = \underbrace{\left(\frac{\partial z}{\partial t'}\right)}_{\gamma\beta}\frac{\partial}{\partial z} + \underbrace{\left(\frac{\partial t}{\partial t'}\right)}_{\gamma}\frac{\partial}{\partial t}$$

$$\frac{\partial}{\partial z'} = \gamma\frac{\partial}{\partial z} + \gamma\beta\frac{\partial}{\partial t} \qquad \frac{\partial}{\partial t'} = \gamma\beta\frac{\partial}{\partial z} + \gamma\frac{\partial}{\partial t}$$

$$\begin{pmatrix} \partial/\partial t' \\ \partial/\partial x' \\ \partial/\partial y' \\ \partial/\partial z' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & \gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & \gamma \end{pmatrix}\begin{pmatrix} \partial/\partial t \\ \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \begin{matrix} \to \partial_0 \\ \to \partial_1 \\ \to \partial_2 \\ \to \partial_3 \end{matrix}$$

Covariant 4-vector

$$\partial_\mu = \frac{\partial}{\partial x^\mu}$$

Corresponding Contravariant 4-derivative

$$\partial^\mu = \left(\frac{\partial}{\partial t}, -\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z}\right)$$

$$\partial^\mu \partial_\mu = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} = \Box$$

↳ d'Alembertian

Vector & 4-vector notation in Thomson

$\mathbf{p}_1 \cdot \mathbf{p}_2$ ⇝ 3-vector scalar product

↳ in bold

$$a \cdot b = a^\mu b_\mu = g_{\mu\nu} a^\mu b^\nu = a^0 b^0 - a^1 b^1 - a^2 b^2 - a^3 b^3$$

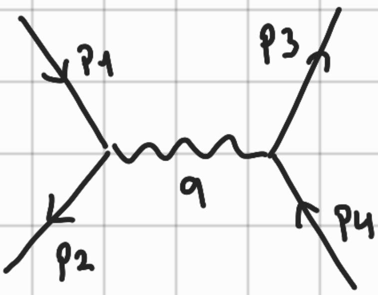
$$p^2 = p \cdot p = E^2 - \mathbf{p}^2$$

↳ italic ↳ bold

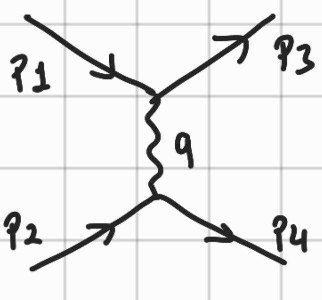
$$p = |\mathbf{p}|$$

↳ magnitude of a 3-vector

Mandelstam variables

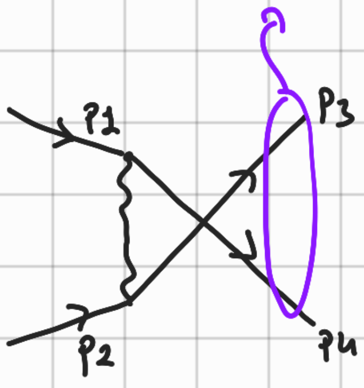


s-channel
(annihilation process)



t-channel
(scattering)

if the final part. are same



u-channel
(scattering)

$$s = (p_1+p_2)^2 = (p_3+p_4)^2$$
$$t = (p_1-p_3)^2 = (p_2-p_4)^2$$
$$u = (p_1-p_4)^2 = (p_2-p_3)^2$$

$q^2 = q \cdot q$ — exchanged boson's 4-momentum squared

In the CM frame: → No net Momentum

colliding particles:
$$p_1 = (E_1^*, p^*)$$
$$p_2 = (E_2^*, -p^*)$$

$$p_1 p_2 = (E_1^* + E_2^*)^2 - (\cancel{p^* - p^*})^2_{\to 0} = (E_1^* + E_2^*)^2$$

(?) $s = (p_1+p_2)^2 = p_1 p_2 = (E_1^* + E_2^*)^2$

Wave Mechanics and Schrödinger Equation

$$\Psi(x,t) \propto e^{i(\vec{k}\cdot\vec{x} - \omega t)}$$

$$\vec{k} = \frac{\vec{p}}{\hbar} \xrightarrow{(\hbar=1)} \vec{k} = \vec{p}$$
$$E = \hbar\omega \xrightarrow{\hbar=1} E = \omega$$
$$\Psi(x,t) = N\, e^{i(\vec{p}\cdot\vec{x} - Et)}$$

$$\hat{A}\Psi = a\Psi$$

$\hat{A}$: operator correspond. physical observable

a: Eigenvalue

$(\hat{A} = \hat{A}^{\dagger})$ Hermitian

$$\hat{p} = -i\nabla$$

$$\hat{p}\Psi = -i\nabla\Psi = p\Psi$$

$$\hat{E} = i\frac{\partial}{\partial t} = E\Psi$$

For a non-relativistic particle: $E = \underset{\text{Hamiltonian}}{H} = \underset{\text{kin.}}{T} + \underset{\text{Pot.}}{V} = \frac{p^2}{2m} + V$

$$i\frac{\partial\Psi(x,t)}{\partial t} = -\frac{1}{2m}\frac{\partial^2\Psi(x,t)}{\partial x^2} + \hat{V}\Psi(x,t)$$

Probability density and probability current

$$\rho(x,t) = \Psi^*(x,t)\Psi(x,t)$$ → Probability Density

$$\underbrace{\frac{\partial}{\partial t}\int_V \rho \, dV}_{\text{Total Prob.}} = -\int_S \vec{j}\cdot d\vec{S}$$

$\downarrow$ Divergence Theorem

$$\frac{\partial}{\partial t}\int_V \rho \, dV = -\int_V \nabla\cdot\vec{j} \, dV$$

$\downarrow$ Since it holds for any arbitrary volume

$$\boxed{\nabla \cdot \vec{j} + \frac{\partial \rho}{\partial t} = 0}$$

The continuity equation for the conservation of quantum mechanical probability

For a free particle $(V=0)$

$$i \frac{\partial \psi}{\partial t} = -\frac{1}{2m} \nabla^2 \psi \qquad (1)$$

$$-i \frac{\partial \psi^*}{\partial t} = -\frac{1}{2m} \nabla^2 \psi^* \qquad (2)$$

$\psi^* (1) - \psi (2)$

$$-\frac{1}{2m}\left(\psi^* \nabla^2 \psi - \psi \nabla^2 \psi^*\right) = i\left(\psi^* \frac{\partial \psi}{\partial t} + \psi \frac{\partial \psi^*}{\partial t}\right)$$

$$= -\frac{1}{2m} \nabla \cdot \left(\psi^* \nabla \psi - \psi \nabla \psi^*\right) = i \frac{\partial}{\partial t}(\psi^* \psi) = i \frac{\partial \rho}{\partial t}$$

$$\boxed{j = \frac{1}{2im}\left(\psi^* \nabla \psi - \psi \nabla \psi^*\right)}$$

Also $\boxed{j = |N|^2 \frac{p}{m} = n v}$

velocity

density

Time Dependence and conserved quantities

$$\hat{H}\Psi_i(x,t) = E_i\Psi_i(x,t)$$

$$i\frac{\partial\Psi_i(x,t)}{\partial t} = E_i\Psi_i(x,t)$$

$$\Psi_i(x,t) = \phi_i(x)e^{-iE_it}$$

$$\langle\hat{A}\rangle = \langle\Psi|\hat{A}|\Psi\rangle = \int \psi^\dagger\hat{A}\psi\, d^3x$$

$$\frac{d\langle\hat{A}\rangle}{dt} = \int\left[\frac{\partial\psi^\dagger}{\partial t}\hat{A}\psi + \psi^\dagger\hat{A}\frac{\partial\psi}{\partial t}\right]d^3x$$

$$\frac{d\langle\hat{A}\rangle}{dt} = \int\left[\underbrace{\left\{\frac{1}{i}\hat{H}\psi\right\}^\dagger}_{\substack{(i)\\ \{-i\hat{H}\psi\}^\dagger \\ +i\,\psi^\dagger\hat{H}^\dagger}}\hat{A}\psi + \psi^\dagger\hat{A}\underbrace{\left\{\frac{1}{i}\hat{H}\psi\right\}}_{\{-i\hat{H}\psi\}}\right]d^3x$$

$$= i\int\left[\psi^\dagger\hat{H}^\dagger\hat{A}\psi - \psi^\dagger\hat{A}\hat{H}\psi\right]d^3x$$

Hermition

$$= i\int\psi^\dagger(\underbrace{\hat{H}\hat{A} - \hat{A}\hat{H}}_{\text{commutator}})\psi\, d^3x$$

$$\boxed{\frac{d\langle\hat{A}\rangle}{dt} = i\langle[\hat{H},\hat{A}]\rangle}$$

If $\hat{A}$ commutes with Hamiltonian (H) $\xrightarrow{\text{corresponds}}$ Conserved Quantity

★ For an eigenstates of the Hamiltonian, the expectation
★ value of any operator is constant

$$\frac{d\langle \hat{A} \rangle}{dt} = \int \left[[i E_i \psi_i^+] \hat{A} \psi_i + \psi_i^+ \hat{A} [-i E_i \psi_i] \right] d^3x = 0$$

↳ Eigenstates are known as "stationary" states

$$|\psi\rangle = \sum_i c_i |\psi_i\rangle$$

$$|\psi(x,t)\rangle = \sum_i c_i \underbrace{|\phi_i(x)\rangle e^{-iE_i t}}$$

→ time evolution of the stationary states

Commutation Relations and Compatible Observables

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A} = 0$$

$$\hat{A}|\phi\rangle = a|\phi\rangle$$

$$\hat{A}\hat{B}|\phi\rangle = \hat{B}\hat{A}|\phi\rangle = a\hat{B}|\phi\rangle$$

$\hat{B}|\phi\rangle$ is also an eigenstate of $\hat{A}$

⇓

$$\hat{B}|\phi\rangle = b|\phi\rangle$$

$\{a, b\}$ → compatible observables ⇝ quantum numbers

$|\phi\rangle$ is the simultaneous eigenstate of both $\hat{A}$ and $\hat{B}$

$$\Delta A \Delta B \geqslant \frac{1}{2}\left|\langle i[\hat{A},\hat{B}]\rangle\right| \rightarrow$$ Generalized Uncertainty Principle

Position-Momentum uncertainty relation

$$\hat{x}\psi = x\psi$$

$$\hat{p}_x\psi = -i\frac{\partial}{\partial x}\psi$$

$$[\hat{x},\hat{p}_x]\psi = -ix\frac{\partial}{\partial x}\psi + i\frac{\partial}{\partial x}(x\psi)$$

$$[\hat{x}\hat{p}_x - \hat{p}_x\hat{x}]\psi$$

$$= \cancel{-ix\frac{\partial\psi}{\partial x}} + i\psi + \cancel{ix\frac{\partial\psi}{\partial x}} = i\psi$$

$$[\hat{x},\hat{p}_x] = +i$$

$\downarrow$ Heisenberg Uncertainty

$\hbar$ (Re insert hidden factor $\hbar$)

$$\Delta x \Delta p_x \geqslant \frac{1}{2}\left|\langle i\underbrace{[\hat{x},\hat{p}_x]}_{+i}\rangle\right|$$

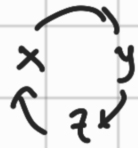
$$\Delta x \Delta p_x \geqslant \frac{\hbar}{2}$$

Angular momentum in quantum mechanics

classical

$$\vec{L} = \vec{r} \times \vec{p} = (y p_z - z p_y \, , \, z p_x - x p_z \, , \, x p_y - y p_x)$$

$$\hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y \qquad [\hat{L}_x, \hat{L}_y] = i\hat{L}_z$$



$$\hat{L}_y = \hat{z}\hat{p}_x - \hat{x}\hat{p}_z \qquad [\hat{L}_y, \hat{L}_z] = i\hat{L}_x$$

$$\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x \qquad [\hat{L}_z, \hat{L}_x] = i\hat{L}_y$$

$$\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$$

$$\downarrow$$

$$[\hat{L}^2, \hat{L}_x] = [\hat{L}^2, \hat{L}_y] = [\hat{L}^2, \hat{L}_z] = 0$$

Raising Operator: $\hat{L}_+ = \hat{L}_x + i\hat{L}_y$

Lowering Operator: $\hat{L}_- = \hat{L}_x - i\hat{L}_y$

$$\left.\begin{matrix} \hat{L}_+^\dagger = \hat{L}_- \\ \hat{L}_-^\dagger = \hat{L}_+ \end{matrix}\right\}$$

$$\boxed{[\hat{L}_z, \hat{L}_\pm] = \pm \hat{L}_\pm}$$

$$[\hat{L}^2, \hat{L}_\pm] = 0$$

$$\boxed{\hat{L}^2 = \hat{L}_-\hat{L}_+ + \hat{L}_z + \hat{L}_z^2}$$

$\downarrow$ Simultaneous eigenstates of $\hat{L}^2$, $\hat{L}_z$

$\hat{L}_z|\lambda,m\rangle = m|\lambda,m\rangle$ $\qquad$ $\hat{L}^2|\lambda,m\rangle = \lambda|\lambda,m\rangle$

Consider the state $\psi = \hat{L}_+|\lambda,m\rangle$

Since $\hat{L}^2$ commutes with $\hat{L}_+$

$$\hat{L}^2\psi = \hat{L}^2\hat{L}_+|\lambda,m\rangle = \hat{L}_+\underbrace{\hat{L}^2|\lambda,m\rangle}_{\lambda|\lambda,m\rangle} = \lambda\underbrace{\hat{L}_+|\lambda,m\rangle}_{\psi} = \lambda\psi$$

Also $\hat{L}_z\hat{L}_+ = \hat{L}_+\hat{L}_z + \hat{L}_+$. Thus

$$\hat{L}_z\psi = \hat{L}^2[\hat{L}_+|\lambda,m\rangle] = (\hat{L}_+\hat{L}_z + \hat{L}_+)|\lambda,m\rangle$$

$$= m\hat{L}_+|\lambda,m\rangle + \hat{L}_+|\lambda,m\rangle$$

$$= (m+1)[\underbrace{\hat{L}_+|\lambda,m\rangle}_{\psi}] = (m+1)\psi$$

Thus $|\psi\rangle = \hat{L}_+|\lambda,m\rangle$ is also a simultaneous eigenstate of $\hat{L}^2$ and $\hat{L}_z$, with respective eigenvalues of λ and $m+1$

$$\langle\hat{L}_z^2\rangle \leq \langle\hat{L}^2\rangle$$

$\downarrow$

This implies that for a particular λ, there must be max and min. values of m

Assume $m = \ell$ (extreme case)

$$\hat{L}_+ \, |\lambda, \overset{m}{\ddot{\ell}}\rangle = 0$$

$$\hat{L}^2 |\lambda,\ell\rangle = (\hat{L}_-\hat{L}_+ + \hat{L}_z + \hat{L}_z^2) |\lambda,\ell\rangle$$

$$\lambda |\lambda,\ell\rangle = (0 + \ell + \ell^2) |\lambda,\ell\rangle$$

$$\lambda = \ell(\ell+1)$$

$$m = -\ell, \; -\ell+1, \; \ldots\ldots, +\ell-1, \; +\ell$$

ℓ is quantised $\rightarrow$ can take only integer or half-integer values

$$\hat{L}_z |\ell,m\rangle = m|\ell,m\rangle \qquad \hat{L}^2 |\ell,m\rangle = \ell(\ell+1)|\ell,m\rangle$$

?

$$\hat{L}_+ |\ell,m\rangle = \alpha_{\ell,m} |\ell, m+1\rangle$$

$$\hat{L}_+^\dagger = \hat{L}_-$$

$$[\hat{L}_+|\ell,m\rangle]^\dagger = \langle \ell,m|\hat{L}_- = \alpha_{\ell m}^* \langle \ell,m+1|$$

$$\langle \ell,m|\hat{L}_-\hat{L}_+|\ell,m\rangle = |\alpha_{\ell m}|^2 \underbrace{\langle \ell,m+1|\ell,m+1\rangle}_{1}$$

$$|\alpha_{e,m}|^2 = \langle l,m|\hat{L}_-\hat{L}_+|l,m\rangle$$

$$= \langle l,m|\underbrace{\hat{L}^2 - \hat{L}_z - \hat{L}_z^2}|l,m\rangle$$

$$|\alpha_{e,m}|^2 = \left(l(l+1) - m - m^2\right) \underbrace{\langle l,m|l,m\rangle}_{1}$$

$$\hat{L}_+|l,m\rangle = \sqrt{l(l+1) - m(m+1)}\;|l,m+1\rangle$$

$$\hat{L}_-|l,m\rangle = \sqrt{l(l+1) - m(m-1)}\;|l,m-1\rangle$$

Fermi's Golden Rule

$$\hat{H}_0\,\phi_k = E_k\,\phi_k$$

$\hat{H}_0$ = unperturbed time-independent

$$\langle\phi_j|\phi_k\rangle = \delta_{jk}$$

presence of an interaction Hamiltonian

$$i\frac{d\Psi}{dt} = \left[\hat{H}_0 + \hat{H}'(x,t)\right]\Psi$$

$$\Psi(x,t) = \sum_k \underbrace{c_k(t)}\,\phi_k(x)\,e^{-iE_kt}$$

↳ allow for transition btw states

$$i \sum_k \left[\frac{dc_k}{dt} \phi_k e^{-iE_k t} - \cancel{i E_k c_k \phi_k e^{-iE_k t}} \right] = \cancel{\sum_k c_k H_0 \phi_k e^{-iE_k t}} + \sum_k H' c_k \phi_k e^{-iE_k t}$$

$$i \sum_k \frac{dc_k}{dt} \phi_k e^{-iE_k t} = \sum_k H' c_k \phi_k e^{-iE_k t}$$

At time $t=0$ $\quad |i\rangle = \phi_i \quad$ & $\quad c_k(0) = \delta_{ik}$

If the perturbing Hamiltonian → constant for $t>0$

→ sufficiently small that all times $c_i(t) \simeq 1$ & $c_{k\neq i}(t) \simeq 0$

$$i \sum_k \frac{dc_k}{dt} \phi_k e^{-iE_k t} \simeq \hat{H}' \phi_i e^{-iE_i t}$$

→ final state

$|f\rangle = \phi_f$

$$i \frac{dc_f}{dt} \underbrace{\phi_f}_{|f\rangle} e^{-iE_f t} = H' \phi_i e^{-iE_i t}$$

$$i \langle f| \frac{dc_f}{dt} |f\rangle = \langle f| H' |i\rangle e^{\overset{-i(E_i - E_f)t}{\downarrow} \, i(E_f - E_i)t}$$

$$\frac{dc_f}{dt} = -i \underbrace{\langle f| H' |i\rangle}_{} e^{i(E_f - E_i)t}$$

$$\langle f | H' | i \rangle = \int_V \phi_f^*(x)\, H'\, \phi_i(x)\, d^3x$$

T_{fi} ⇝ Transition Matrix Element

$$\frac{dc_f}{dt} = -i \langle f | \hat{H}' | i \rangle\, e^{i(E_f - E_i)t}$$

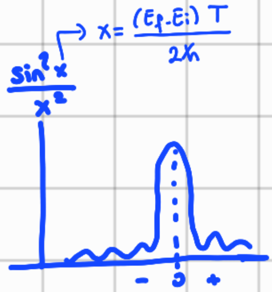
$$\rightsquigarrow \quad \frac{dc_f}{dt} = -i\, T_{fi}\, e^{i(E_f - E_i)t}$$

$$c_f(T) = -i \int_0^T T_{fi}\, e^{i(E_f - E_i)t}\, dt$$

$t = T$

if H's time-independent

$$\rightsquigarrow \quad c_f(T) = -i\, T_{fi} \int_0^T e^{i(E_f - E_i)t}\, dt$$



$$P_{fi} = c_f(T)\, c_f^*(T) = |T_{fi}|^2 \int_0^T \int_0^T e^{i(E_f - E_i)t}\, e^{-i(E_f - E_i)t'}\, dt\, dt'$$

$$d\Gamma_{fi} = \frac{P_{fi}}{T} = \frac{1}{T} |T_{fi}|^2 \int_{-T/2}^{+T/2} \int_{-T/2}^{+T/2} e^{i(E_f - E_i)t}\, e^{-i(E_f - E_i)t'}\, dt\, dt'$$

$d\Gamma_{fi}$: Transition Rate

$$d\Gamma_{fi} = |T_{fi}|^2 \lim_{T\to\infty} \left\{ \frac{1}{T} \int_{-T/2}^{T/2} \int_{-T/2}^{T/2} e^{i(E_f-E_i)t} e^{-i(E_f-E_i)t'} dt\,dt' \right\}$$

Use Dirac delta-function

$$dt' = 2\pi\,\delta(E_f - E_i)$$

$$d\Gamma_{fi} = 2\pi\,|T_{fi}|^2 \lim_{T\to\infty} \left\{ \frac{1}{T} \int_{-T/2}^{T/2} e^{i(E_f-E_i)t}\,\delta(E_f-E_i)\,dt \right\}$$

If there are dn accessible final states for $E_f \to E_f + dE_f$

$$\Gamma_{fi} = 2\pi \int |T_{fi}|^2 \frac{dn}{dE_f} \lim_{T\to\infty} \left\{ \frac{1}{T} \int_{-T/2}^{+T/2} e^{i(E_f-E_i)t}\,\delta(E_f-E_i)\,dt \right\} dE_f$$

Total transition rate

$E_f = E_i$

?

$$\Gamma_{fi} = 2\pi \int |T_{fi}|^2 \frac{dn}{dE_f}\,\delta(E_f-E_i) \lim_{T\to\infty} \left\{ \frac{1}{T} \int_{-T/2}^{T/2} dt \right\} dE_f$$

$$= 2\pi \int |T_{fi}|^2 \frac{dn}{dE_f}\,\delta(E_f-E_i)\,dE_f$$

$$\Gamma_{fi} = 2\pi\,|T_{fi}|^2 \left|\frac{dn}{dE_f}\right|_{E_i} \longrightarrow \text{Density of states} = \rho(E_i)$$

Fermi's Golden Rule:

$$\Gamma_{fi} = 2\pi \, |T_{fi}|^2 \rho(E_i)$$

$$\frac{dc_f}{dt} \approx -i \langle f|\hat{H}'|i\rangle e^{i(E_f - E_i)t} +$$

constant for $t>0$

$$(-i)^2 \sum_{k \neq i} \langle f|\hat{H}'|k\rangle e^{i(E_f - E_k)t} \underbrace{\int_0^t \langle k|\hat{H}'|i\rangle e^{i(E_k - E_i)t'} dt'}_{\langle k|\hat{H}'|i\rangle \frac{e^{i(E_k - E_i)t}}{i(E_k - E_i)}}$$

$$\frac{dc_f}{dt} = -i \underbrace{\left[\langle f|\hat{H}'|i\rangle + \sum_{k \neq i} \frac{\langle f|\hat{H}'|k\rangle\langle k|\hat{H}'|i\rangle}{E_i - E_k} \right]}_{T_{fi}} e^{i(E_f - E_i)t}$$

H', H?

$$T_{fi} = \langle f|\hat{H}'|i\rangle + \underbrace{\sum_{k \neq i} \frac{\langle f|\hat{H}'|k\rangle\langle k|\hat{H}'|i\rangle}{E_i - E_k}}$$

2nd order term

corresponds to the transition occuring via some intermediate state