The Postulates of Quantum Mechanics

1. State Space

Complex vector space with inner product (Hilbert Space)

$$|\psi\rangle = a|0\rangle + b|1\rangle$$

$$\langle\psi|\psi\rangle = 1 \rightarrow |a|^2 + |b|^2 = 1$$

2. Evolution

$$|\psi'\rangle = U|\psi\rangle$$

@ time t_2

depends only on t_1 and t_2

@ time t_1

Describes how the quantum states of a closed quantum system at <u>two different times</u> are related.

2'. Evolution of a quantum system in continous time

$$i\hbar \frac{d|\psi\rangle}{dt} = H|\psi\rangle$$ "Schrödinger Equation"

$$H = \sum_E E\,|E\rangle\langle E|$$

stationary states

eigenvectors

Hamiltonian is Hermitian

eigenvalues

it has a spectral decomposition

States $|E\rangle$ are known as "Stationary States"

$$|E\rangle \longrightarrow \underbrace{\exp(-iEt/\hbar)}_{\text{Numerical factor}}\, |E\rangle$$

Connection btw Hamiltonian and the unitary operator :

time- independent

$$|\Psi(t_2)\rangle = \underbrace{\exp\left[\frac{-iH(t_2-t_1)}{\hbar}\right]}_{U(t_1,t_2)} |\Psi(t_1)\rangle = U(t_1,t_2)\,|\Psi(t_1)\rangle$$

$$\Psi(t) = U(t)\,\Psi(0) \longrightarrow U(t) = e^{-iKt/\hbar}$$

time independent Hermitian operator

$$-K = K^{+}$$

- real eigenvalues
- orthogonal eigenvectors
- observable physical quantities

Let's substitude into Schrödinger

$$i\hbar\frac{\partial}{\partial t}\left(e^{-iKt/\hbar}\,\Psi(0)\right) = H\,e^{-iKt/\hbar}\,\Psi(0)$$

$$\downarrow$$

$$(i\hbar)\frac{(-iK)}{\hbar}\,\underbrace{e^{-iKt/\hbar}\,\Psi(0)}_{\Psi(t)} + (i\hbar)\,e^{-iKt/\hbar}\,\frac{\partial\Psi(0)}{\partial t}$$

assume 0 time-independent

$$i\hbar\frac{\partial}{\partial t}\Psi(t) = K\Psi(t) + i\hbar\,e^{-iKt/\hbar}\,\cancel{\frac{\partial\Psi(0)}{\partial t}}$$

$$\downarrow$$

$$i\hbar\frac{\partial}{\partial t}\Psi(t) = K\Psi(t) = H\Psi(t)$$

$K = H$

$$U(t) = e^{-iHt/\hbar}$$

3. Measurement

Closed quantum systems evolve acc. to unitary evolution.

Observation ⟶ no longer closed system thus not necessarily subject to unitary evolution.

Collection $\{M_m\}$ of measurement operators

↳ m: measurement outcomes

↳ Generally non-orthogonal

$$p(m) = \langle \psi | M_m^{+} M_m | \psi \rangle$$

probability that m occurs after measurement

System after meas. : $$\frac{M_m|\psi\rangle}{\sqrt{\langle \psi | M_m^{+} M_m | \psi \rangle}}$$

$$\sum_m M_m^{+} M_m = I$$

↓

$$\sum_m p(m) = \sum_m \langle \psi | M_m^{+} M_m | \psi \rangle = 1$$

3.1. Distinguishing quantum states

Non-orthogonal states can't be reliably distinguished!

3.2 Projective Measurements → Repeatible

$$\underbrace{M}_{\text{observable}} = \sum_m \underbrace{m}_{\text{eigenvalues}} P_m \longrightarrow \text{projecter onto the eigenspaces of } M$$

orthogonal ↑ (projecter)

$$\sum_m p(m) = 1 \leftarrow p(m) = \langle \psi | P_m | \psi \rangle$$

p(m): probability

$$\text{The state immediately after the measurement} = \frac{P_m |\psi\rangle}{\sqrt{p(m)}}$$

$$* \quad E(M) = \sum_m m\, p(m) = \sum_m m \langle \psi | P_m | \psi \rangle$$

$$= \sum_m \langle \psi | m P_m | \psi \rangle = \langle \psi | \underbrace{\left(\sum_m m P_m \right)}_{M} | \psi \rangle$$

$$= \langle \psi | M | \psi \rangle$$

$$\langle M \rangle = \langle \psi | M | \psi \rangle$$

3.3 POVM measurements (Positive Operator-Valued Measure)

From postulate 3:

$M_m \rightarrow$ measurement operators

$$p(m) = \langle \psi | M_m^{\dagger} M_m | \psi \rangle$$

Define $E_m = M_m^{\dagger} M_m$

positive operator st $\sum_m E_m = I$

4. Composite Systems

$$|\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots\cdots \otimes |\psi_n\rangle$$