# Circuit Quantum Electrodynamics

## Superconducting platform

Covering: basic concepts, measurement techniques, implementations, qubit approaches, current trends

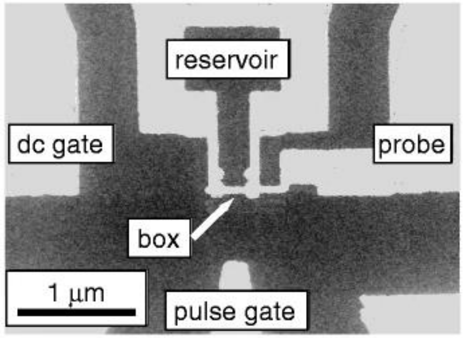
With figures and slides borrowed from

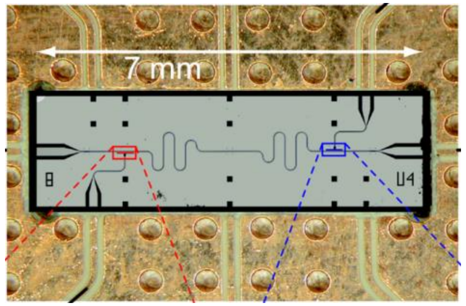
A. Wallraff (ETH-Zurich), A. Fedorov (University of Queensland), C. Eichler (FAU Erlangen).

# Goals of this part of the course

- Superconducting
  - Understand basic superconducting theory
  - What is a Josephson junction?
- Quantum
  - How do we make quantum circuits with superconductor?
- Bits
  - Superconductor as information carriers
  - How to design a qubit
  - How to operate a qubit

# History of superconducting qubits




1980: Leggett – Macroscopic quantum tunnelling

1985: Devoret, Martinis: Measurements of MQT in Josephson junction

1999: Nakamura: First superconducting qubit

2003: Hans Mooij: First flux qubit

2004: Wallraff, Blais et al: Circuit QED

2007: Koch et al: Transmon qubit

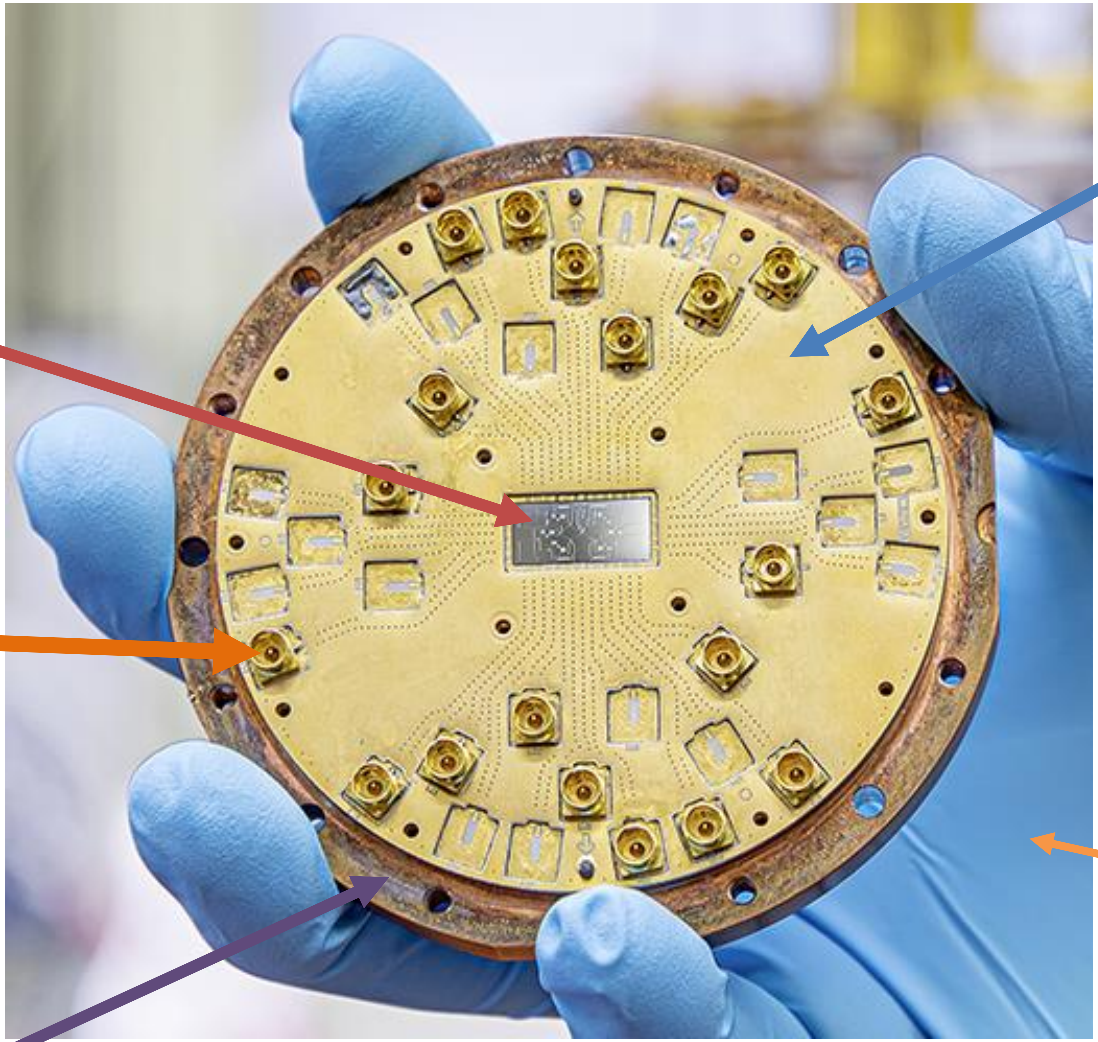
2009: Dicarlo et al: Demonstration of two-qubit algorithms

2011: Paik et al: High coherence transmon with 3D

2016: IBM: 5 qubit device on the cloud

2019: Google: Quantum computation beyond classical with 53 qubits

# A superconducting quantum processor



Quantum device
(7 qubits)

Microwave
connector

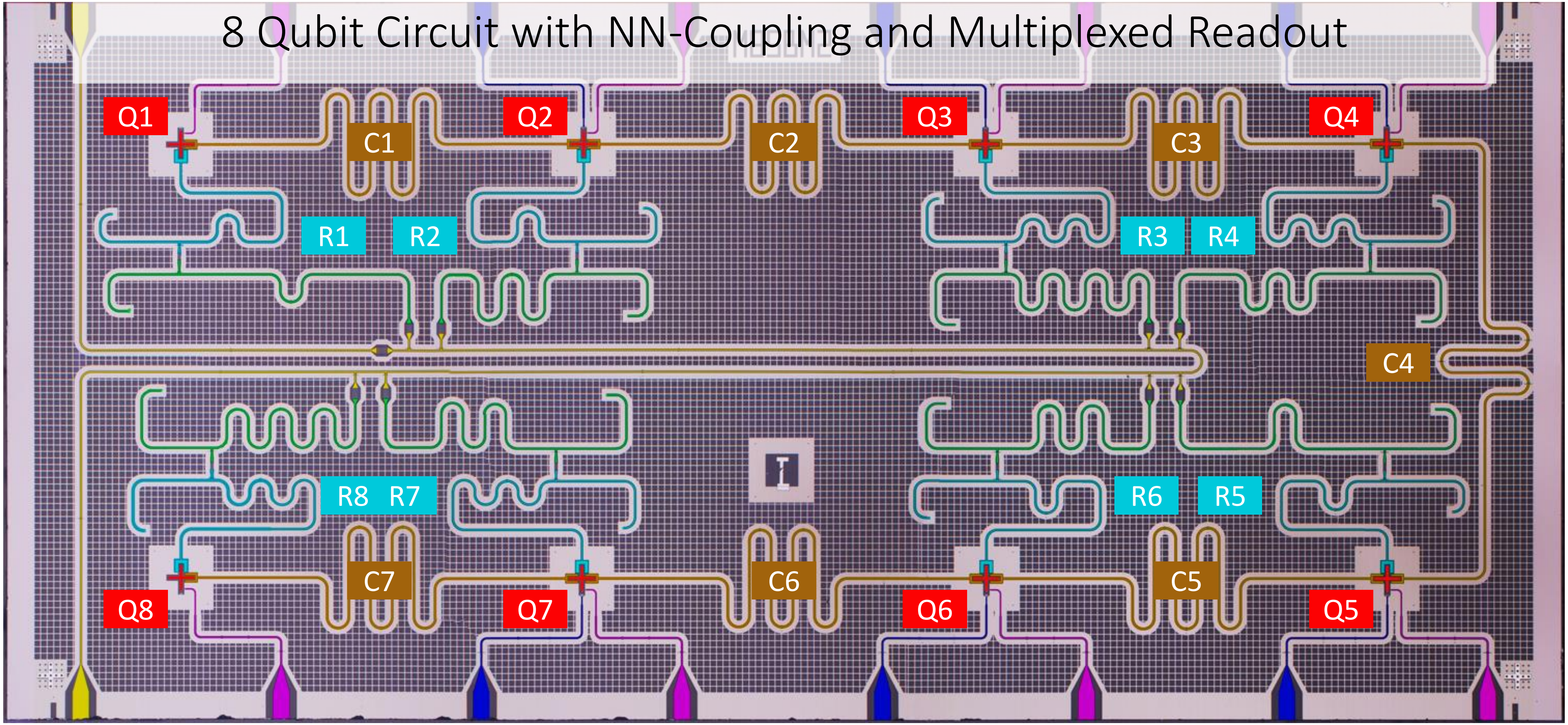
Cu holder

Printed circuit board

human hand

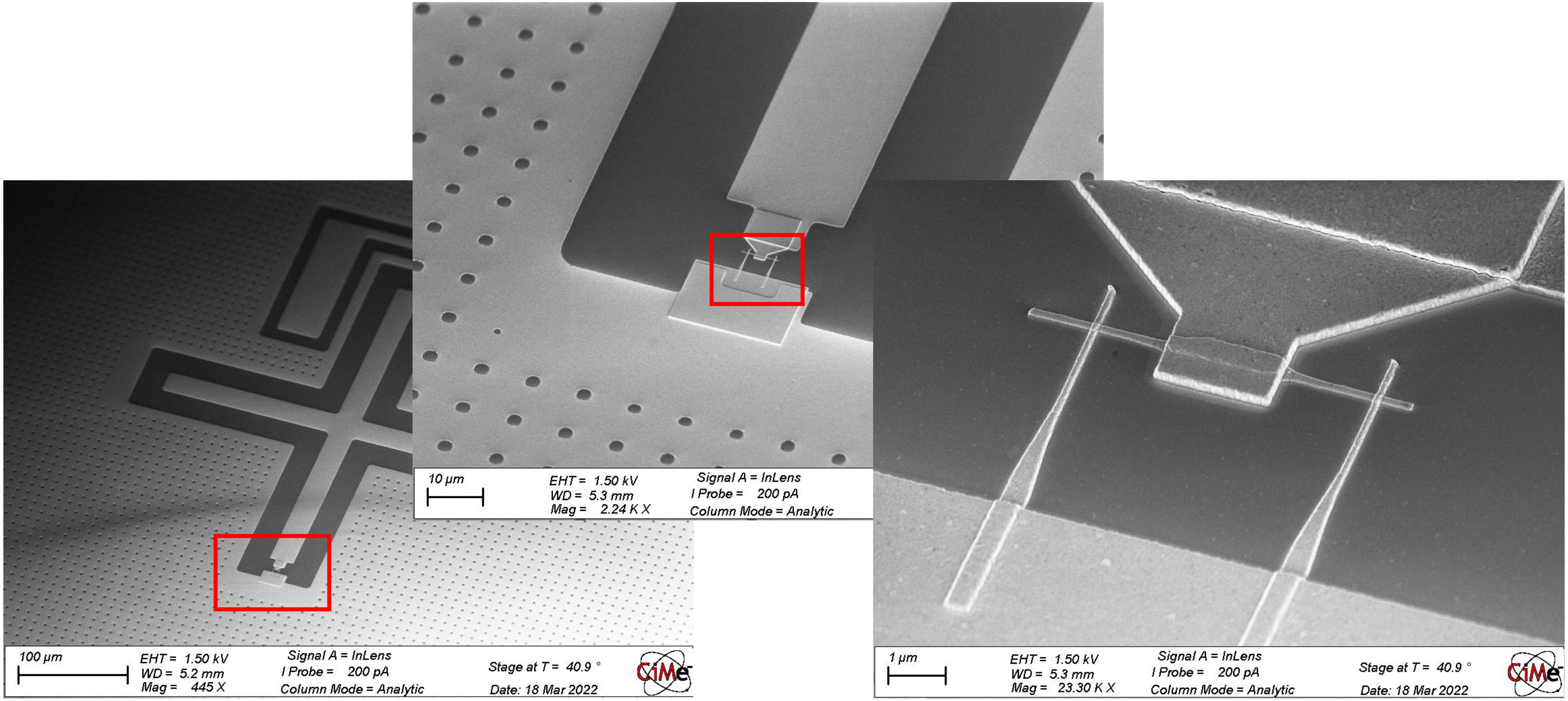
Wallraff group (ETH-Zurich)

# 8 Qubit Circuit with NN-Coupling and Multiplexed Readout




Qubits
Readout resonators
Purcell filters
Coupling Bus resonators
Charge lines
Flux lines
Feed line

# Transmons in EPFL




In collaboration with AQUA and LPQM labs in EPFL (pictures credit: Simone Frasca)

# Outline

- Brief  introduction and motivation. Artificial vs natural quantum systems

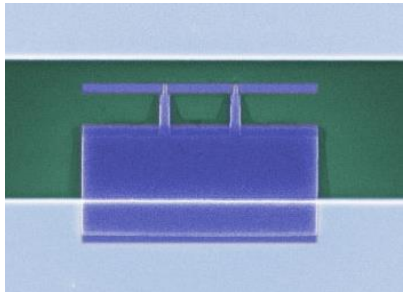
- Superconductivity

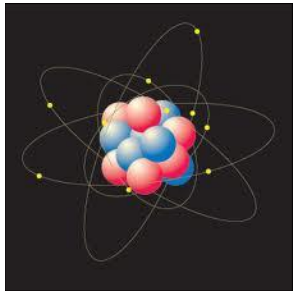
- Quantization of an LC oscillator. Classical and quantum state of the oscillator

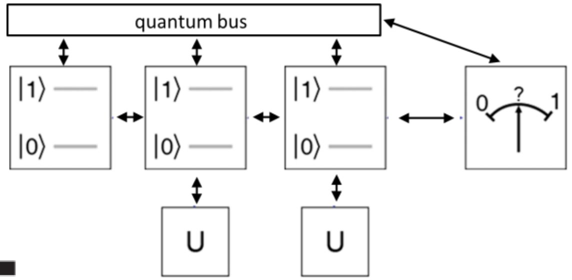
- Dissipation less nonlinearity: Josephson junction

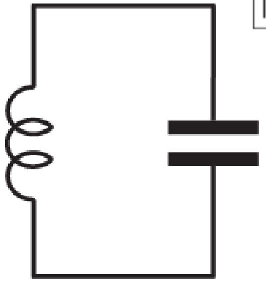
- Cooper Pair Box and Transmon qubits

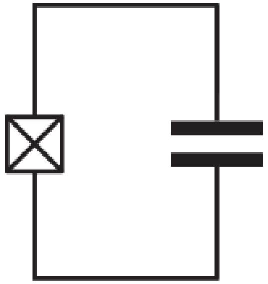
- Circuit Quantum Electrodynamics architecture

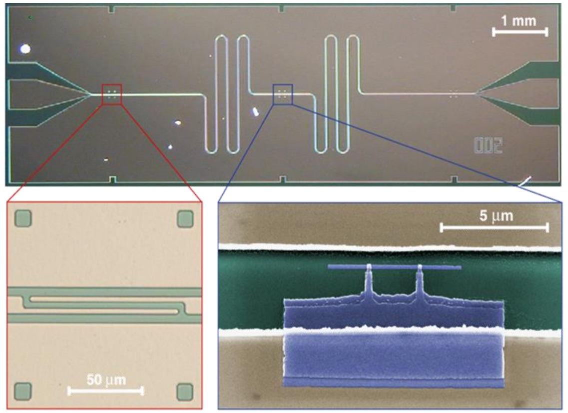






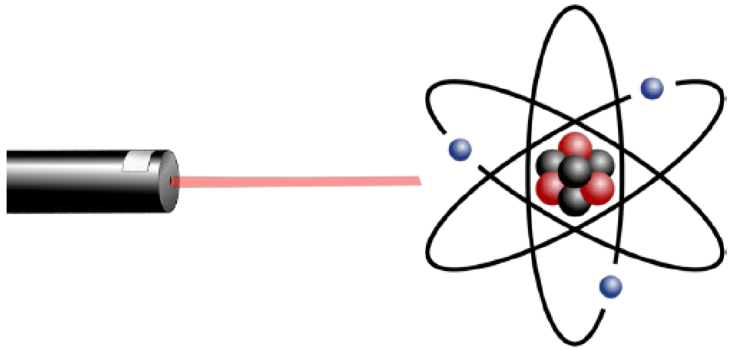



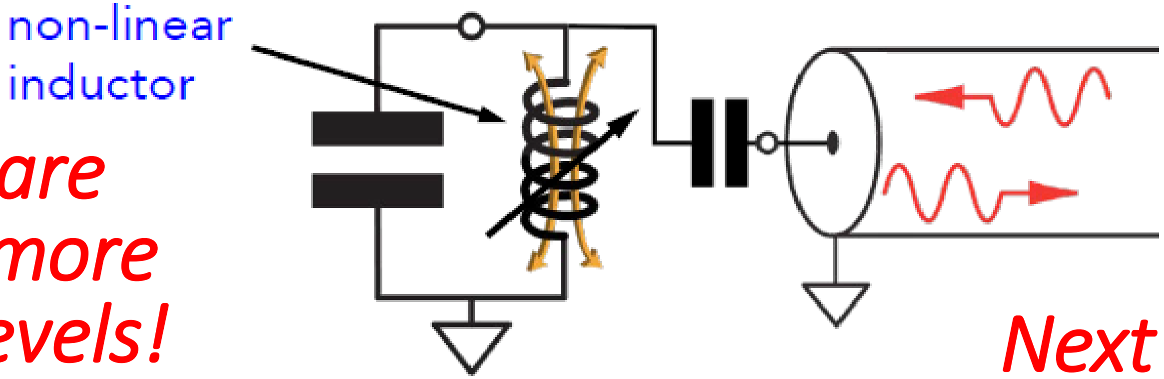




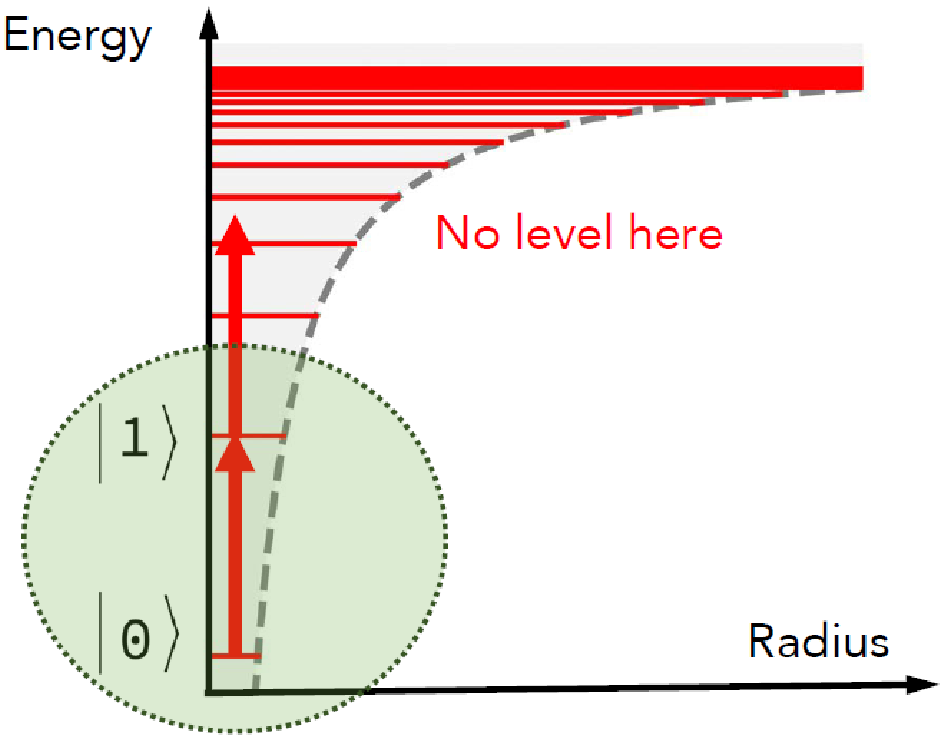
## Real Atoms

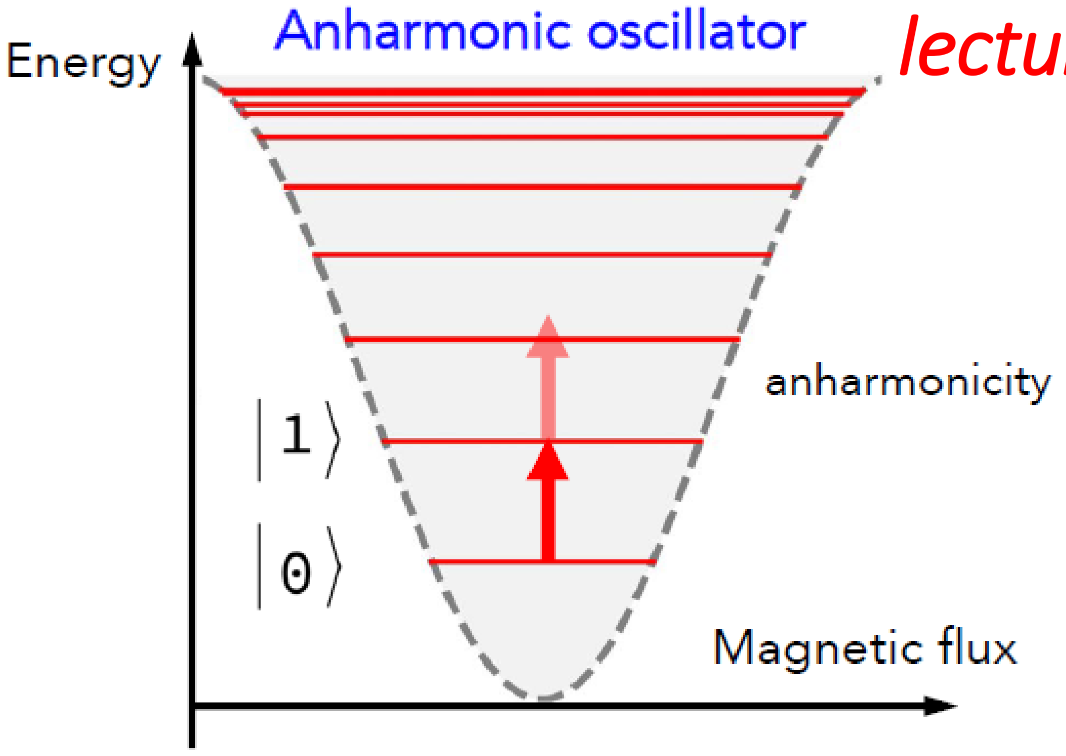
## Artificial Systems






*There are always more than 2 levels!*

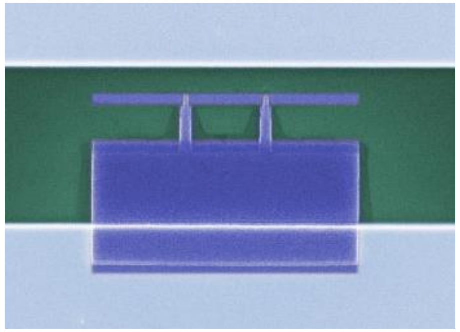


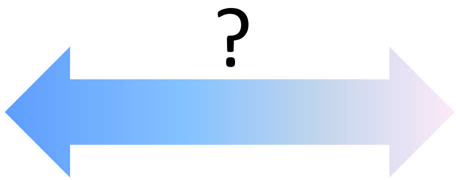




*Next lecture*

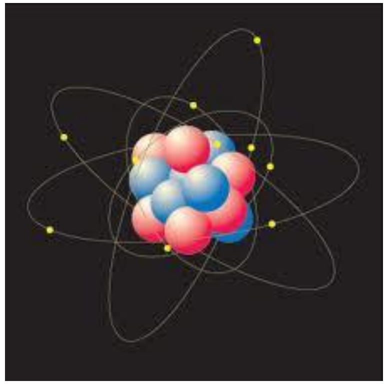
# Artificial and “natural”

Engineered quantum systems






“Natural” quantum particles



**Pros**

- Parameters can be designed for specific purpose
- Many parameters can be also controlled *in situ*

Cons

- Typically, solid-state fabricated structure hence relatively “dirty”

**Pros**

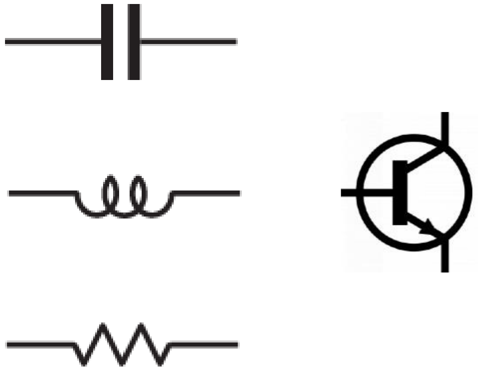
- “Clean” quantum systems: long quantum coherence

Cons

- Many parameters are fixed and cannot be tuned
- Hard to couple with each other

# Conventional electronic circuits

Conventional circuit elements:

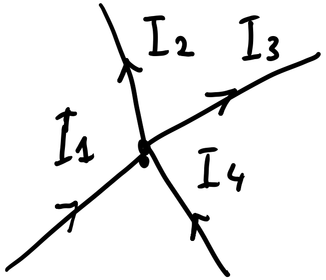


Described by classical variables (real numbers):

- $V$ (voltages) and $q$ (charges)
- $\Phi$ (fluxes) and $I$ (currents)

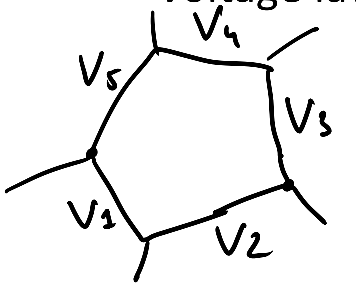
**Kirghoff's circuit laws:**

Current law




$$\sum_n I_n = 0$$

Voltage law

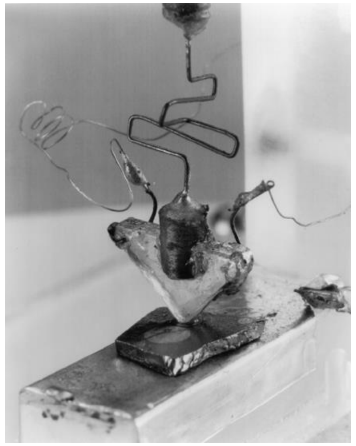



$$\sum_n V_n = 0$$

- classical physics
- no quantum mechanics
- no superposition principle
- no quantization of fields

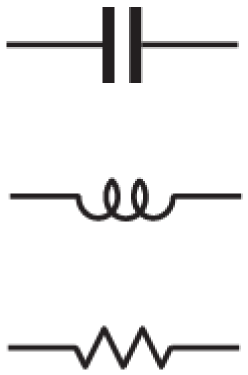
Very complex electronic circuits as the base for modern electronic technology:


first transistor at Bell Labs (1947)



# Quantum Electronic Circuits

basic circuit elements:



properties :

- classical physics
- no quantum mechanics
- no superposition principle
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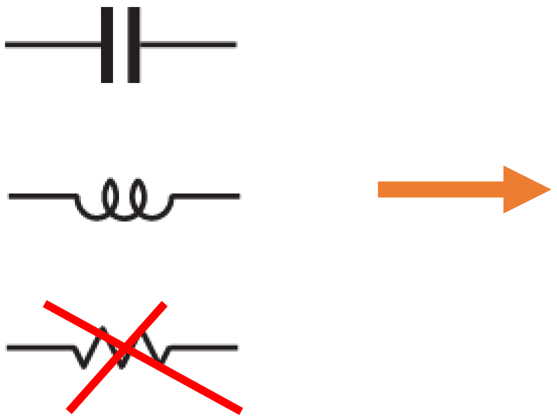
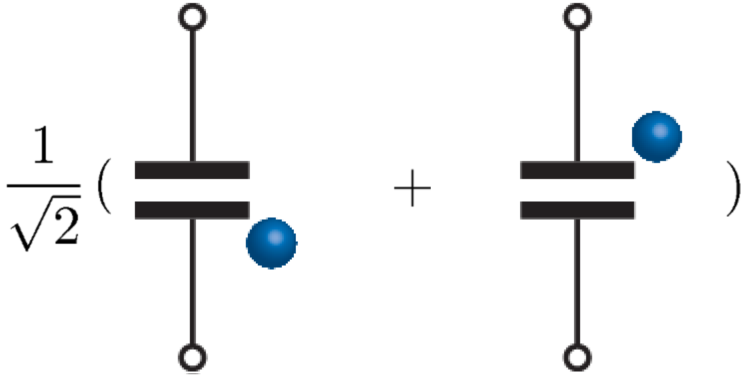
# Quantum Electronic Circuits

basic circuit elements:



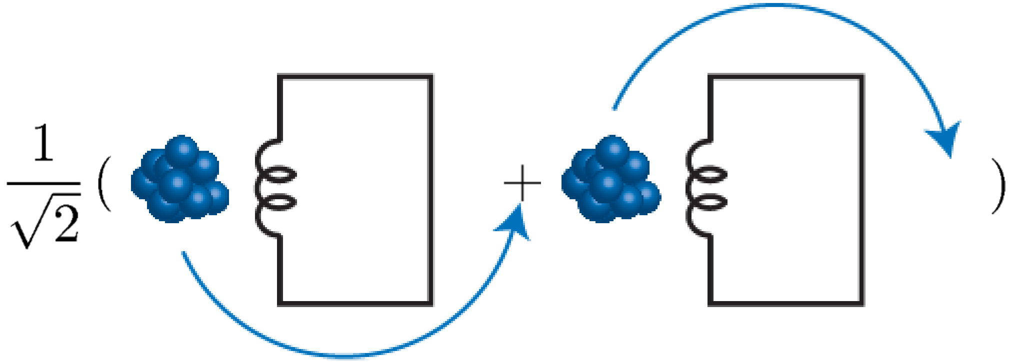
charge on a capacitor:



quantum superposition states of:

- charge $Q$
- flux $\phi$

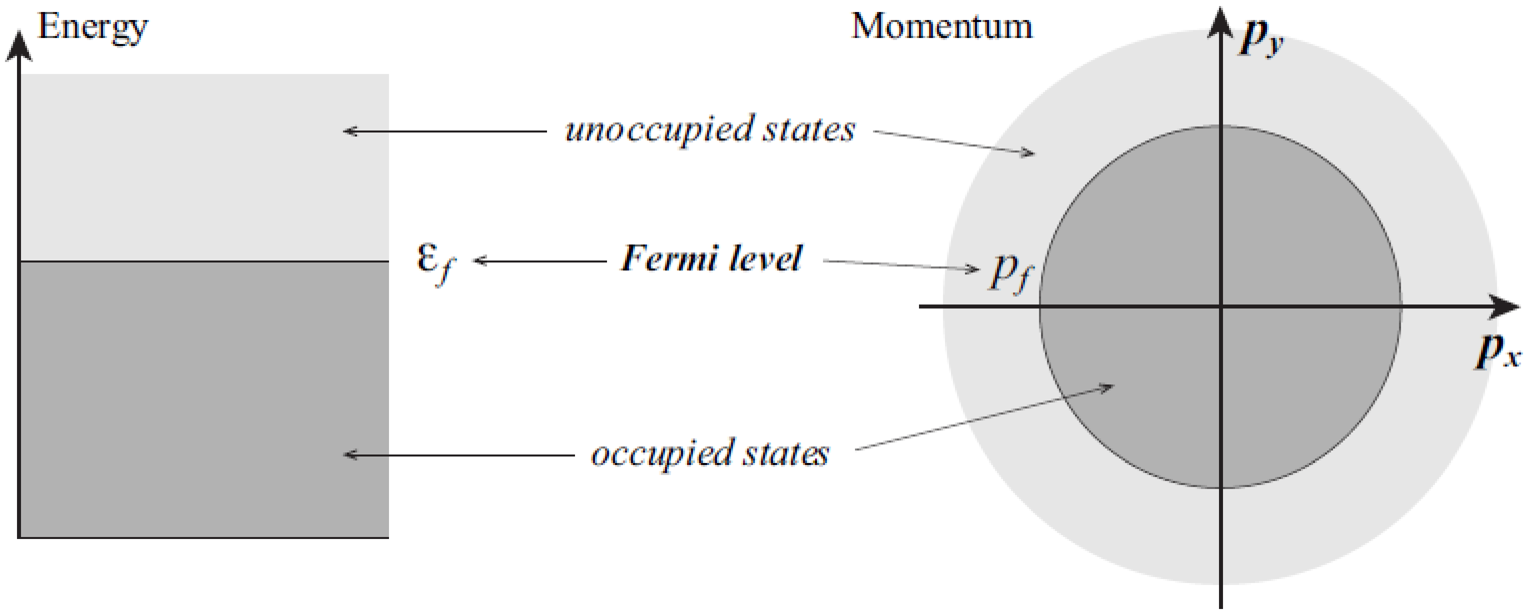
current or magnetic flux in an inductor:



$Q$, $\phi$ are conjugate variables

quantum uncertainty relation $\Delta\phi\Delta Q > h$

# Review: Normal Metal and Superconductors

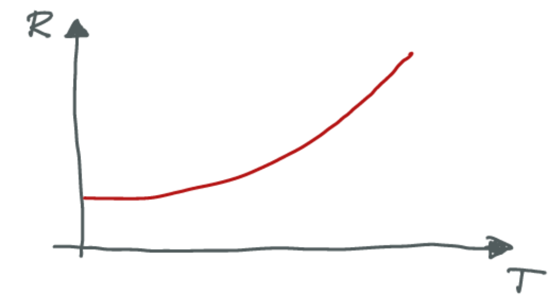


# Superconductivity

FIRST MEASUREMENT:

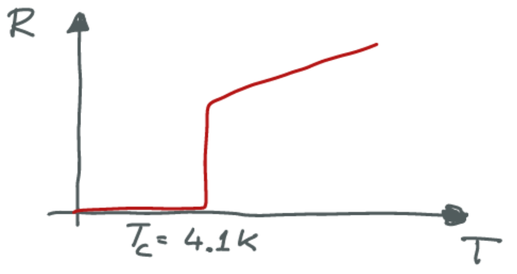
- 1911 KAMERLINGH ONNES (LEIDEN)

MEASURING RESISTANCE OF METALS IN LIQUID $^4He$




GOLD, PLATINUM

(PHONONS FREEZE OUT
RESIDUAL RESISTANCE)
LINEAR DEPENDENCE



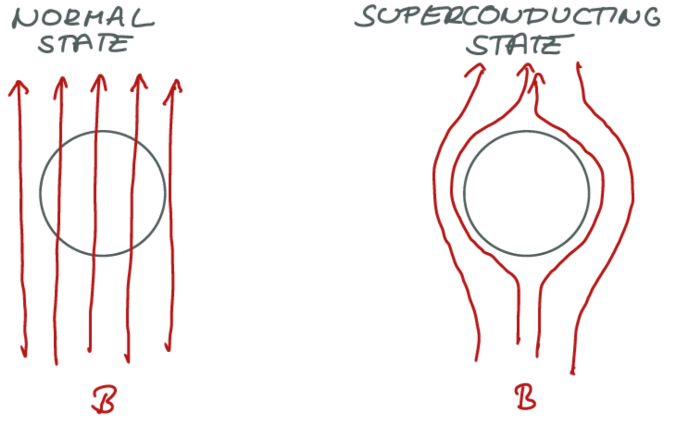

MERCURY

FOR EXAMPLE, Al ~1.2K
Nb ~9K
Ta ~4K

**1. ZERO ELECTRICAL RESISTANCE BELOW $T_c$**

"SECOND" MEASUREMENT

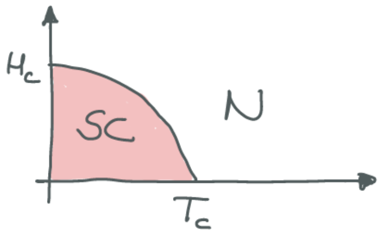
- 1933 MEISSNER & OCHSENFELD



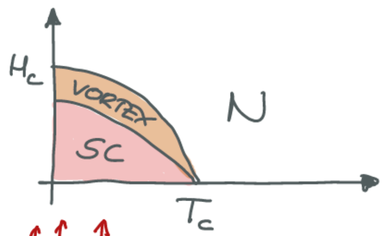

**2. THE MAGNETIC FIELD IS EXPELLED FROM SC.**

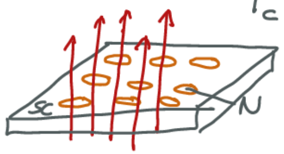
→ IT BREAKS DOWN ABOVE $H_c(T)$

PHASE DIAGRAM OF TYPE I SC



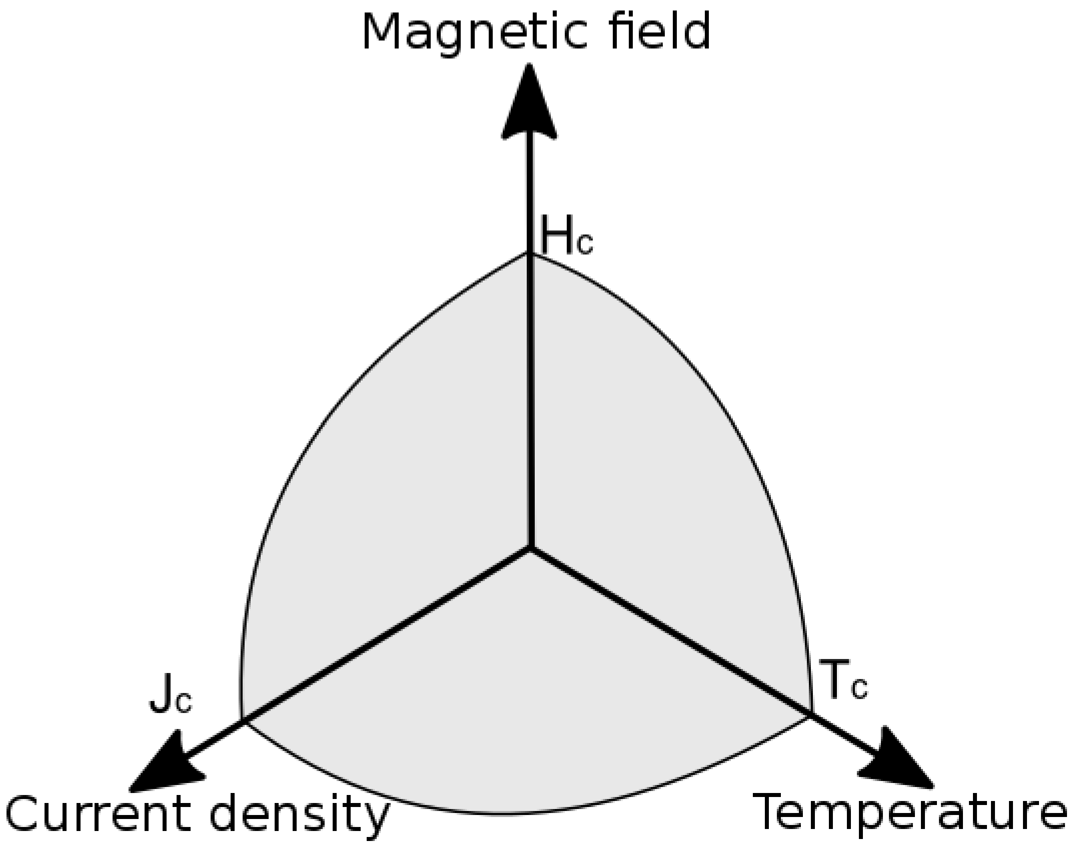

PHASE DIAGRAM OF TYPE II SC




EACH VORTEX CORE CARRIES $\Phi_0 = h/2e$ FLUX.

# How to destroy superconductivity?

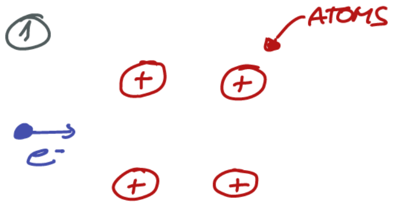


# BCS THEORY (1956) BARDEEN, COOPER, SCHRIEFFER

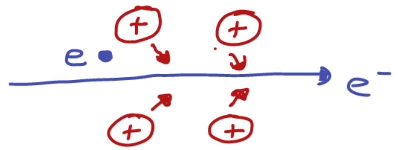
MICROSCOPIC THEORY OF SUPERCONDUCTIVITY

- HOW CAN ELECTRONS ATTRACT EACH OTHER ??

① 

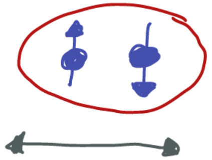



② 




③ 

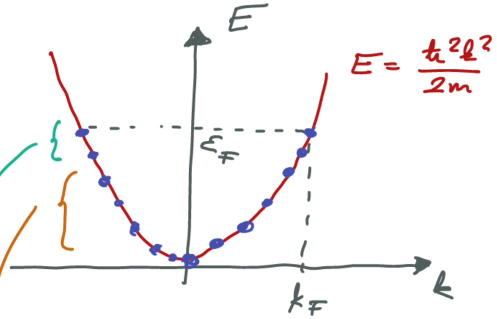

LATTICE - MEDIATED ELECTRON - ELECTRON INTERACTION

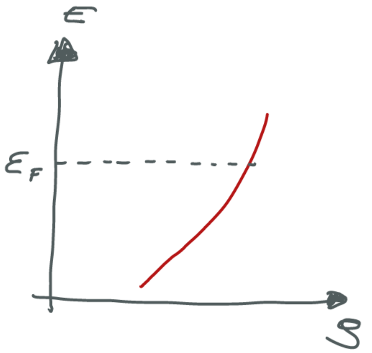


COOPER PAIRS (COMPOUND BOSONS)

$\xi_0$ COHERENCE LENGTH.

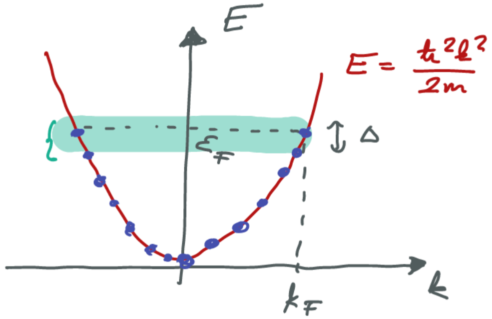
WHICH ELECTRONS FORM COOPER PAIRS IN A METAL ?

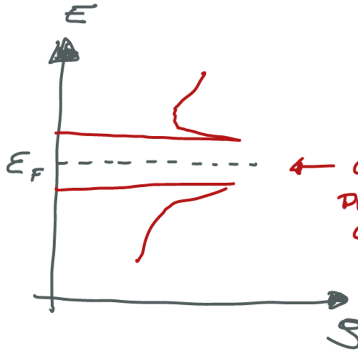



ELECTRONS ARE FROZEN HERE

ELECTRONS ARE MORE FREE HERE → THEY MOVE MORE EASILY




← COOPER PAIRS AT 0 ENERGY

COOPER PAIRS ARE FORMED AROUND THE FERMI SURFACE ($\Delta$), $\Delta = 1.76\, k_B T_C$

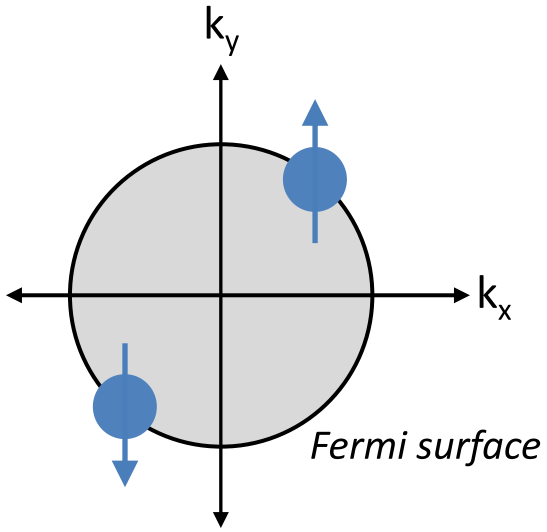
$(k,\uparrow) \leftrightarrow (-k,\downarrow)$

GAP IS FORMED → CAN NOT GIVE ENERGY DURING SCATTERING.
→ MAKE A QUBIT
→ HOW CAN WE PUT ENERGY LEVELS INTO IT? QUANTUM CIRCUITS.

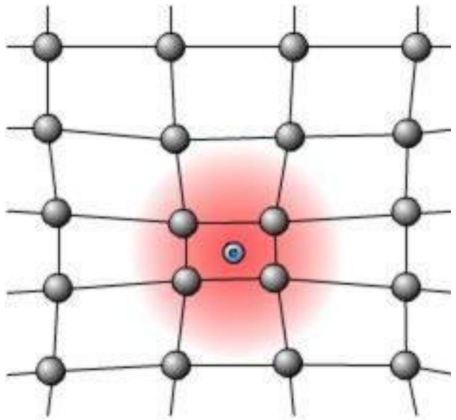
# BCS theory

- Bardeen, Cooper, Schrieffer (1957)
- Microscopic theory of superconductivity

[1] Electrons with opposite momentum and opposite spin form bound states: **Cooper pairs**



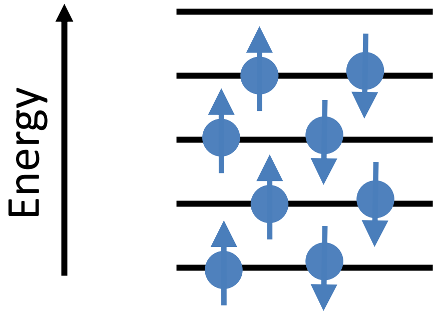

[2] The attraction between the electrons is provided by **electron-phonon interaction** (i.e., interaction with lattice vibrations)



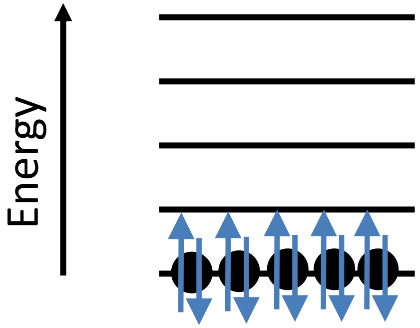
# BCS theory

[3] Cooper pairs are (composite) ***bosons***, they condense to a single ground state

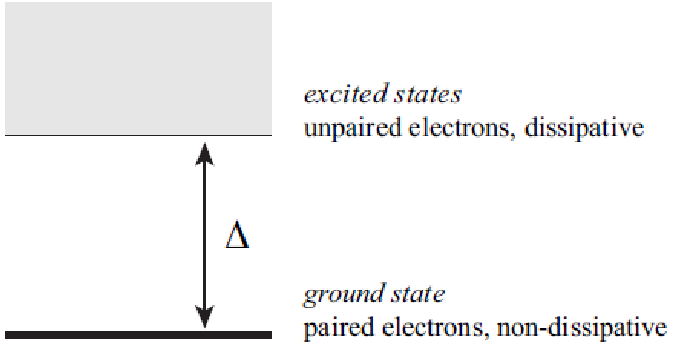
*Fermions*



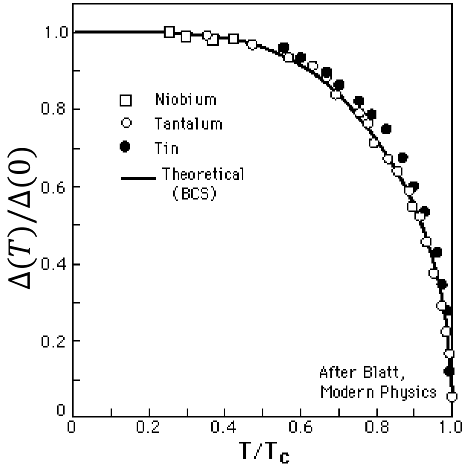

*Bosons (Cooper pairs)*



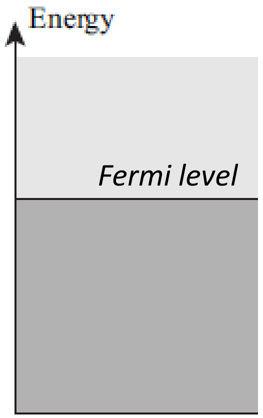

[4] Pairing results in an ***energy gap*** in the




*Superconductor*




Dimensionless energy gap Δ(T )/Δ0 in niobium, tantalum, and tin.




*Normal metal*

# BCS theory

[5] The superconducting state is described by a ***single macroscopic wave function***

$$\psi = |\psi| e^{i\phi}$$

The superconducting state can be described by an order parameter $\psi$

$|\psi|^2 = n_s$ (density of Cooper pairs), $\phi$: phase of the condensate

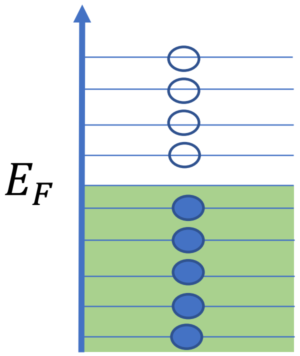
**Wave function is the same everywhere in the superconductor**
*(*provided there are no magnetic fields)*

In the superconducting state, the wavefunction acquires a certain rigidity in that there is a global phase $\phi$.
This has important consequences (including the Meissner effect and flux quantization).

# Superconductors vs normal conductors

## Normal metal

Charge carrier: electrons, fermions (S=1/2)

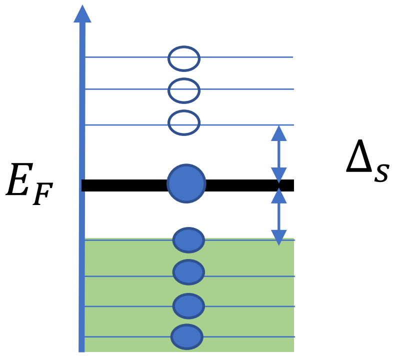



$T = 0$

Scattering with impurities can take some energy -> create resistance

## Superconductor

Charge carrier: Cooper pairs, pairs of electrons, bosons (S=0)





Excitations are gapped ->

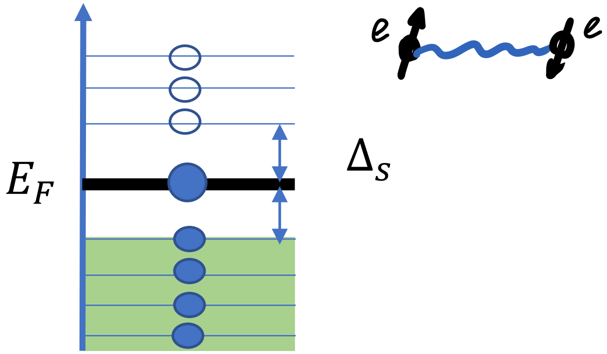
No electrical resistance for $T < T_c = \frac{\Delta_s}{k_B}$

Typical superconducting materials:

- Niobium (Nb): $T_c$ = 9.2 K, $\frac{2\Delta_s}{h} = 725\ \mathrm{GHz}$
- Aluminium (Al): $T_c$ = 1.2 K, $\frac{2\Delta_s}{h} = 100\ \mathrm{GHz}$

# Superconducting phase

Charge carrier: Cooper pairs, pairs of electrons, bosons (S=0)




Chunk of a superconductor


No magnetic field:

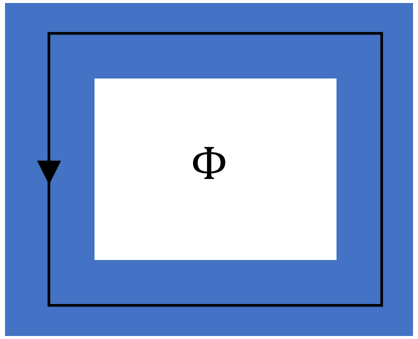
All Cooper pairs condensed in one macroscopic wave function

$$\psi = \sqrt{n}\,\exp(i\delta)$$

$n$ is Cooper pair density and $\delta$ is global phase.

With magnetic field:

$\Delta\delta = \frac{2\pi\Phi}{\Phi_0}$, where $\Phi_0 = \frac{h}{2e} = 2.067 \times 10^{-15}$ Wb is called the flux quantum

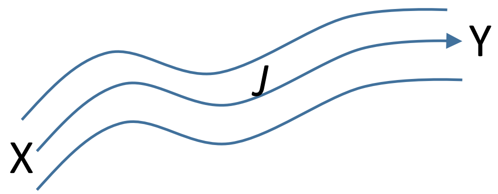



As $\Delta\delta = 2\pi n$ -> flux quantization:

$$\Phi = n\,\Phi_0$$

# Flux quantization

Consider the change in phase when current flows through a superconductor from X -> Y



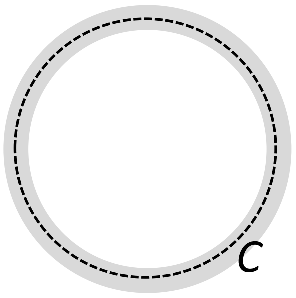

$$\Delta\phi_{XY} = \frac{1}{\hbar}\int_X^Y p(\boldsymbol{r}).\, d\boldsymbol{r}$$

In the presence of a magnetic field, ***B***:

p= $2m\boldsymbol{v} + 2e\boldsymbol{A}$ $\quad \boldsymbol{\nabla} \times \boldsymbol{A} = \boldsymbol{B}$

$$\longrightarrow \quad \Delta\phi_{XY} = \frac{m}{en_s\hbar}\int_X^Y \boldsymbol{J}(\boldsymbol{r}) \cdot d\boldsymbol{r} + \frac{2e}{\hbar}\int_X^Y \boldsymbol{A}(\boldsymbol{r}) \cdot d\boldsymbol{r}$$

Consider supercurrent flowing around a ring: ***flux quantization***




$$\Delta\phi_C = n2\pi$$

phase of the superconducting condensate must be single-valued

$$\Phi' \equiv \frac{m}{2e^2n}\oint_C \boldsymbol{J}(\boldsymbol{r}) \cdot d\boldsymbol{r} + \Phi_A = n\Phi_0$$

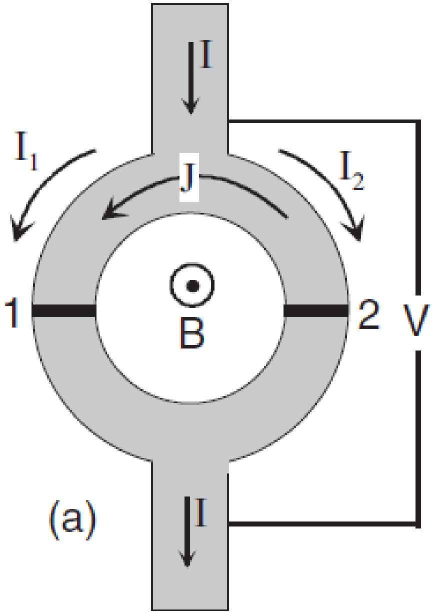
Total flux in SC ring quantized in units of SC flux quantum, $\Phi_0 = h/2e$

In general, phase difference in a SC circuit can be expressed as: $\phi = 2\pi\Phi_A/\Phi_0$

# SQUID loop (following class)

- Superconducting quantum ***interference*** device
- Results in a flux-tunable critical current
- Used very often in SC qubits to tune $E_J$

$$H = -E_J \cos \phi_1 - E_J \cos \phi_2$$
$$\phi_1 - \phi_2 - \frac{2\pi\Phi}{\Phi_0} = 2\pi n$$
$$\frac{\phi_1 + \phi_2}{2} = \phi$$




Small exercise: show that the Hamiltonian can be written as

$$H = -E_J(\Phi) \cos \phi$$

# Brief recap of Electromagnetism

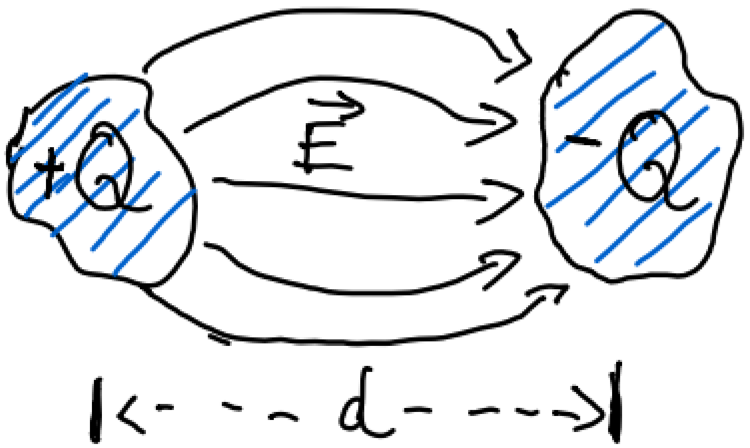
# Quantization of Electrical Circuits

GOAL: given an electrical circuit composed of inductors and capacitors, find the system Hamiltonian.

N.B. (for the moment) there are no Josephson junctions and no resistors.

## Lumped Element representation:

Consider two metallic islands charged with charge + and - Q




$\vec{E}$ electric field

$d$ characteristic lenght of the circuit

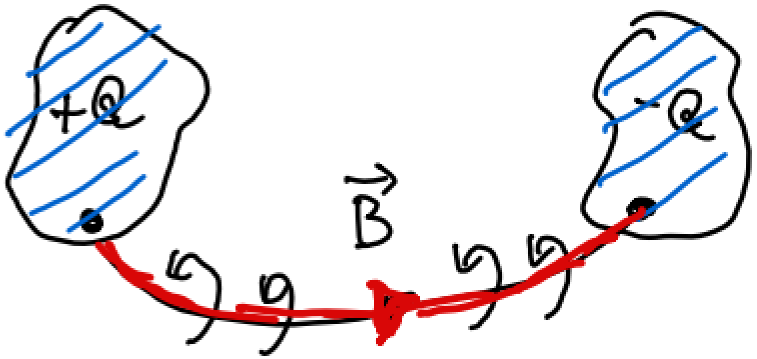
- In a static configuration, charges rearrange on the surface of the metals and the electric field vanishes inside the metal.
- The time to reach a static charge configuration is $\boldsymbol{\tau \sim d/c}$, with $d$ the typical conductor dimention and $c$ the velocity of light.
- If $Q(t)$ changes slowly compared to $\tau$, the electric field follows quasi-instantaneously.

The energy required to move one unit of charge from one to the other island it path-independent and given by the voltage $V = Q/C$, with $C$ the capacitance of the conductor configuration (which depends on geometry and dielectric medium.

Total energy stored in the electric field $\Rightarrow$

$$E_{el} = \int_0^Q dQ' \, V(Q') = \int_0^Q dQ' \, \frac{Q'}{C} = \frac{Q^2}{2C}$$

Now consider an additional metallic wire connecting the two islands:




$I = -\dot{Q}$

$\Phi = L I \rightarrow$ current

$\Phi$: magnetic flux

$L$: Inductance (geometry-dependent)

Total energy stored in a magnetic field: $\Rightarrow$

$$E_{mag} = \int_0^I dI' \, \Phi(I') = \frac{1}{2} L I^2 = \Phi^2/2L$$

# There are two types of on-chip electronic components:

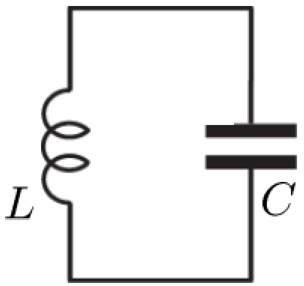
**Lumped elements** and **Distributed elements**.

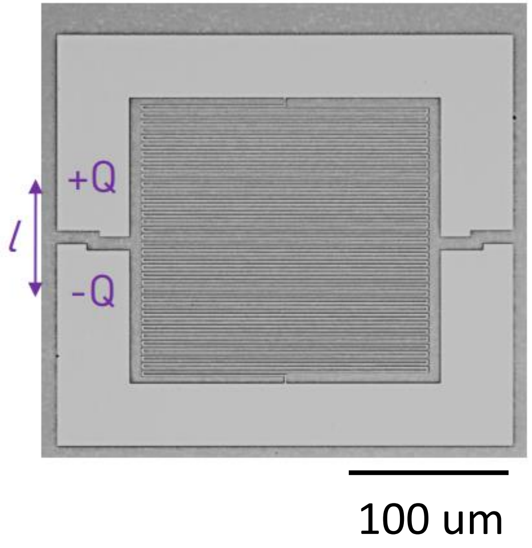
We can distinguish the two types by their typical dimention ‘d’ compared to the wavelength ‘λ’ of the microwave radiation at the relevant frequency.

- For lumped elements, the size *‘d’* of the components is much smaller than the wavelength ‘λ’: $d << \lambda$
- For distributed elements, the size of the components *‘d’* is roughly of the same size as ‘λ’ : $d \sim \lambda$

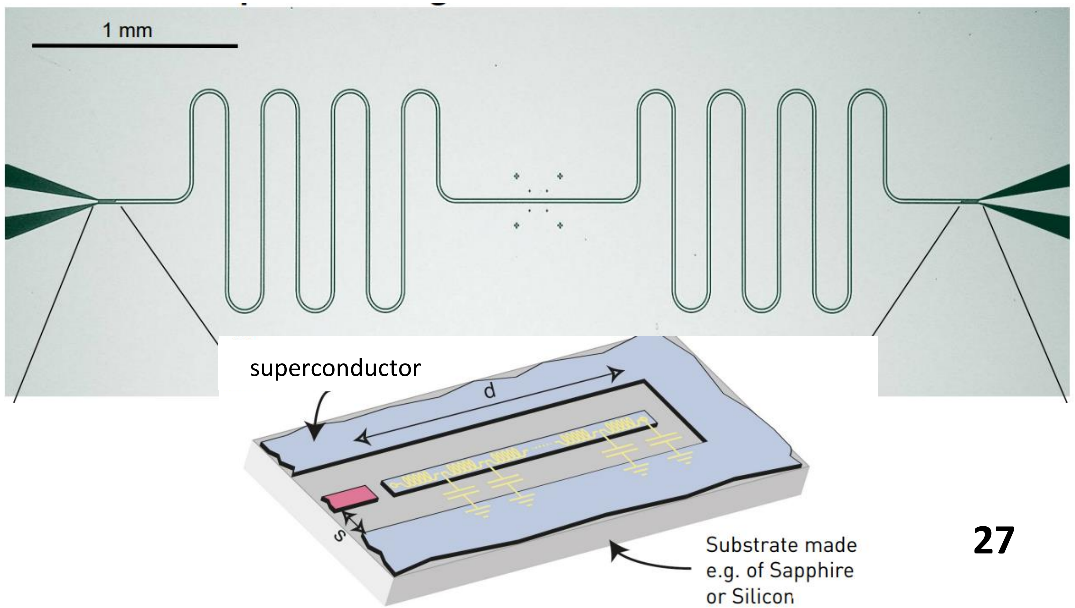
Resonance frequencies typically between $4 - 8\ \mathrm{GHz}$

**Lumped elements resonators**



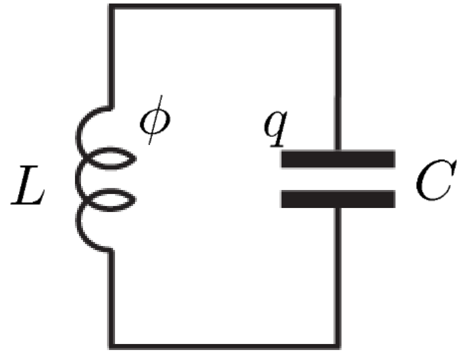




**Distributed resonators:**



# Classical dynamics of harmonic oscillator

harmonic LC oscillator:




$$\omega = \frac{1}{\sqrt{LC}}$$

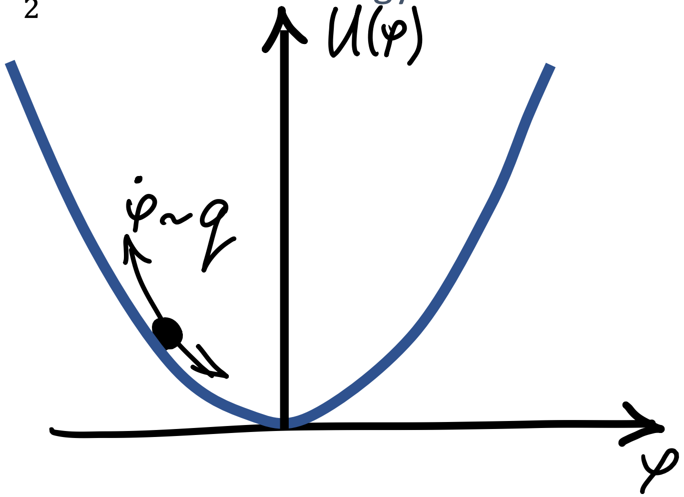
From Kirchhoff's rules and classical electrodynamics, the variable are related to each other:

$$q = CV_C$$
$$\Phi = LI$$
$$V_L = -L\dot{I} = -\dot{\Phi} = -V_C$$

Take $\Phi$ as a new coordinate of a fictitious particle and identify :

Inductive energy $U(\phi) = \frac{\Phi^2}{2L}$ -> as potential energy

Electrostatic energy $K(\dot{\phi}) = \frac{CV_C^2}{2} = \frac{C\dot{\Phi}^2}{2}$ -> as kinetic energy




General procedure to find quantum Hamiltonian

1) Set up the Lagrangian function:

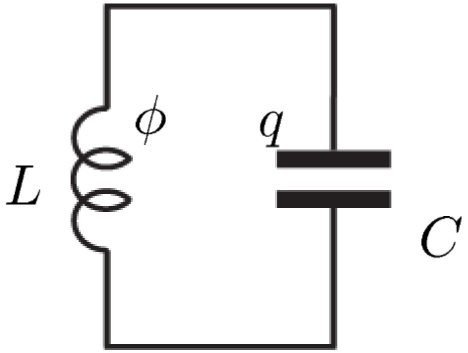
$$\mathcal{L}(\dot{\Phi}, \Phi) = K - U = \frac{C\dot{\Phi}^2}{2} - \frac{\Phi^2}{2L}$$

-> generalized momentum $p = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}} = C\dot{\Phi} = q$

Particle with mass $C$ momentum $q$ and coordinate $\Phi$ in a harmonic trap

# Quantization of harmonic oscillator

harmonic LC oscillator:




$$\omega = \frac{1}{\sqrt{LC}}$$

2) Legendre Transformation:

$$q = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}} = C\dot{\Phi}$$

3) The Hamiltonian function:

$$H(q, \Phi) = \dot{\Phi} q - \mathcal{L} = \frac{q^2}{C} - \frac{q^2}{2C} - (-\frac{\Phi^2}{2L}) = \frac{q^2}{2C} + \frac{\Phi^2}{2L}$$

where we have chosen to write the Hamiltonian in terms of the charge and magnetic flux.

Now we are stuck since we do not know the quantum operators for Q and Φ.

For inspiration, let us turn to the mechanical harmonic oscillator. Recall the Hamiltonian for the classic mechanical harmonic oscillator, for example a mass on a spring, is:

$$H_{\mathrm{cl,mech}} = \frac{p^2}{2m} + \frac{kx^2}{2} \qquad H_{\mathrm{cl,LC}} = \frac{Q^2}{2C} + \frac{\Phi^2}{2L}$$

where *p* is the momentum of the mass *m, k* the spring constant and *x* is the displacement of the mass from equilibrium.

Comparing these two Hamiltonians gives us the idea to think of *Q* as *p* and *x* as *Φ*.
As further motivation we recall that *p* and *x* satisfy the canonical equations of motion

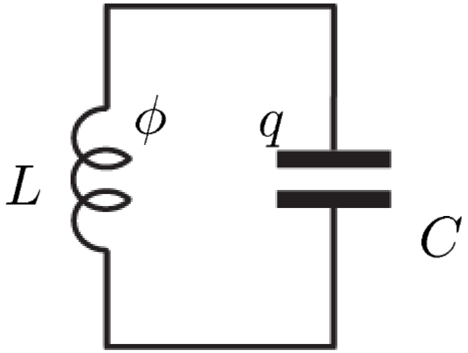
$$\frac{\delta H}{\delta p} = \dot{x} \qquad \frac{\delta H}{\delta x} = -\dot{p}$$

| | Classic Mechanical | Classic Electronic | Quantum Mechanical | Quantum Electronic |
|---|---|---|---|---|
| Displacement | $x$ | $\Phi$ | $\hat{x}$ | $\hat{\Phi}$ |
| Flow | $p$ | $Q$ | $\hat{p} = -i\hbar\frac{d}{dx}$ | $\hat{Q} = -i\hbar\frac{d}{d\Phi}$ |
| Force Proportionality | $m$ | $C$ | $m$ | $C$ |
| Restoring Proportionality | $k$ | $\frac{1}{L}$ | $k$ | $\frac{1}{L}$ |
| Resonant Frequency | $\omega = \sqrt{\frac{k}{m}}$ | $\omega = \frac{1}{\sqrt{LC}}$ | $\omega = \sqrt{\frac{k}{m}}$ | $\omega = \frac{1}{\sqrt{LC}}$ |
| Commutation Relations | - | - | $[\hat{x}, \hat{p}] = i\hbar$ | $\left[\hat{\Phi}, \hat{Q}\right] = i\hbar$ |

Relationship of variables between the mechanical and electronic harmonic oscillator as well as classically and quantum mechanically.

# Quantization of harmonic oscillator

harmonic LC oscillator:




$$\omega = \frac{1}{\sqrt{LC}}$$

2) Legendre Transformation:

$$q = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}} = C\dot{\Phi}$$

3) The Hamiltonian function:

$$H(q, \Phi) = \dot{\Phi} q - \mathcal{L} = \frac{q^2}{C} - \frac{q^2}{2C} - (-\frac{\Phi^2}{2L}) = \frac{q^2}{2C} + \frac{\Phi^2}{2L}$$

4) Quantization:

Flux and charge operator (similar to coordinate and momentum) with commutation relation $[\hat{\Phi}, \hat{q}] = i\hbar$

$$\hat{\Phi} = \Phi$$

$$\hat{q} = -i\hbar \frac{\partial}{\partial \Phi}$$

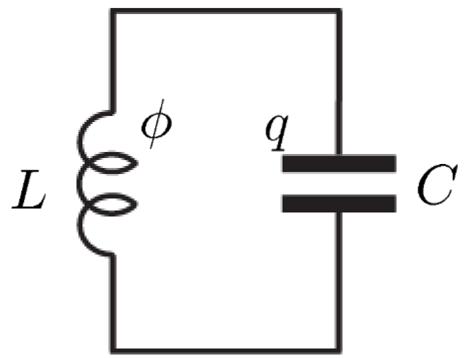
The quantum Hamiltonian

$$\hat{H}(q, \Phi) = \frac{\hat{q}^2}{2C} + \frac{\hat{\Phi}^2}{2L}$$

Diagonalizing the Hamiltonian -> find the allowed energies of the system

# Creation and annihilation operators

harmonic LC oscillator:




$$\omega = \frac{1}{\sqrt{LC}}$$

5) Express H in terms of annihilation and creation operators

$$\hat{H}(q,\Phi) = \frac{\hat{q}^2}{2C} + \frac{\hat{\Phi}^2}{2L}$$

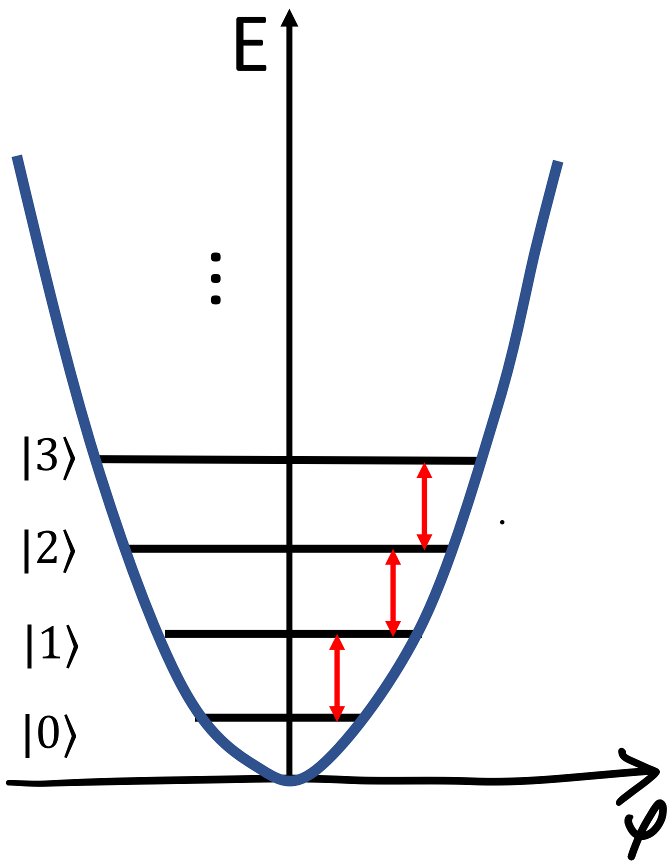
Introduce creation and annihilation operators as

$$\hat{a}^\dagger = \sqrt{\frac{\omega C}{2\hbar}}\left(\hat{\Phi} - i\frac{\hat{q}}{C\omega}\right)$$

$$\hat{a} = \sqrt{\frac{\omega C}{2\hbar}}\left(\hat{\Phi} + i\frac{\hat{q}}{C\omega}\right)$$

Solution of the Hamiltonian are number states $|n\rangle$:

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$$
$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$
$$\hat{a}^\dagger\hat{a}|n\rangle = n|n\rangle$$




Can we define a qubit with an harmonic system?

# Find the system Hamiltonian for a general circuit network

# Lagrangian formalism

- Formalism of mechanics that deals with energies rather than forces (Newtonian mechanics)

- Lagrangian of a system is expressed with terms involving the generalized coordinates only and generalized velocities only.

$$\mathcal{L} = \mathcal{L}(X, \dot{X})$$

- Often represents the difference between the kinetic energy (velocity dependent) and potential energy (position dependent)

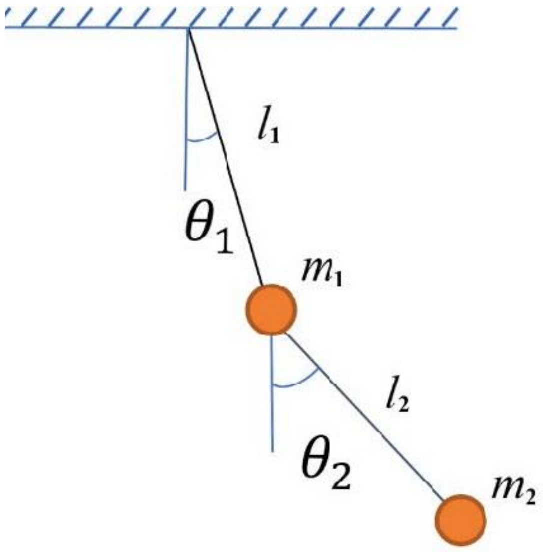
$$\mathcal{L} = E_k - E_p$$

- Hamiltonian of the system is expressed with terms involving generalized coordinates only and generalized momenta only

$$\mathcal{H} = \mathcal{H}(X, P)$$

- Conjugate variable (generalized momentum) can be expressed as: $P = \frac{\partial \mathcal{L}}{\partial \dot{X}}$
- Connection between Lagrangian and Hamiltonian via the Legendre transformation:

$$\mathcal{H}(X, P) = \dot{X}P - \mathcal{L}(X, \dot{X})$$



# Quantize the LC circuit – for reals!

- Step 1: Write down Lagrangian

$$\mathcal{L} = E_{cap} - E_{ind} = \frac{1}{2} C \dot{\Phi}^2 - \frac{1}{2L} \Phi^2$$

- Step 2: Find conjugate variable

$$Q = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}} = C \dot{\Phi}$$

- Step 3: Calculate classical Hamiltonian

$$\mathcal{H}(\Phi, P) = C \dot{\Phi}^2 - \mathcal{L}\left(\Phi, \dot{\Phi}\right) = \frac{1}{2C} Q^2 + \frac{1}{2L} \Phi^2$$

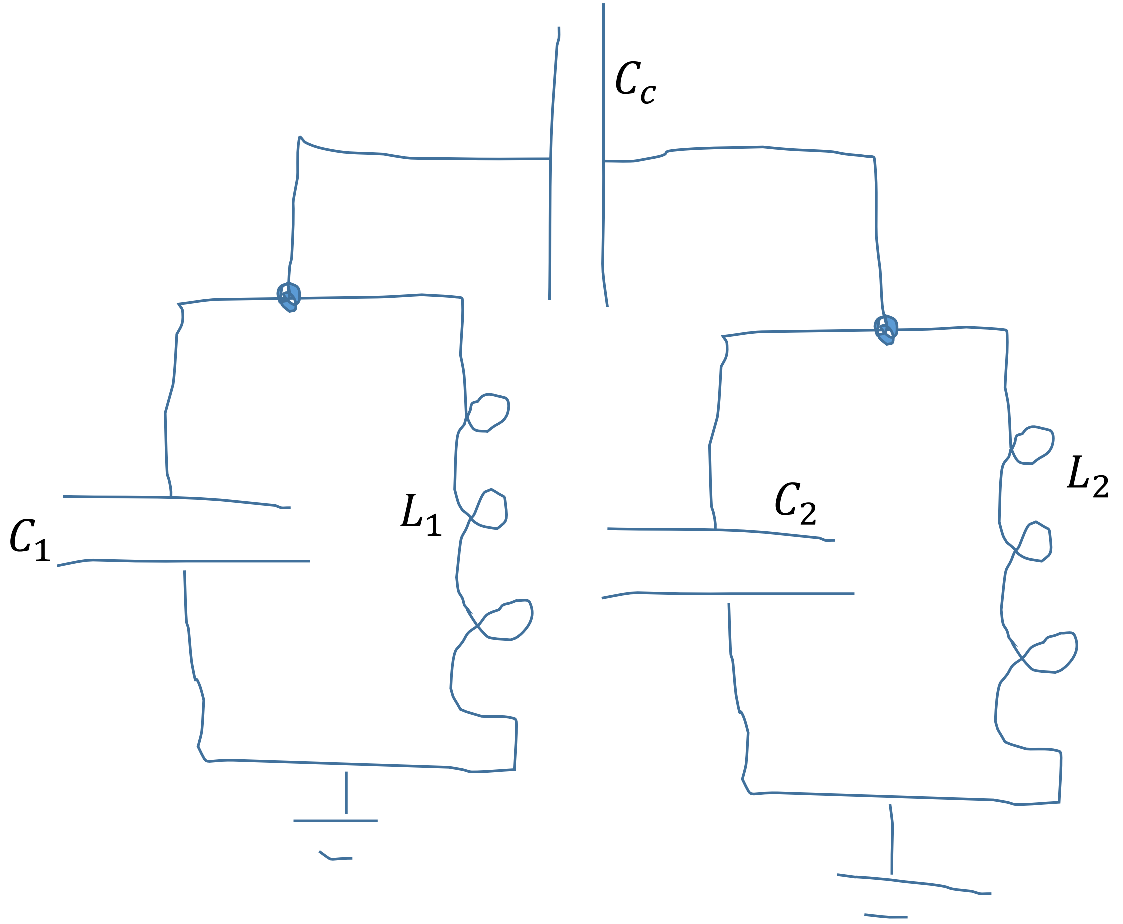
- Step 4: Quantize the Hamiltonian

$$\left[\hat{\Phi}, \hat{Q}\right] = i\hbar \qquad H = \frac{1}{2L} \hat{\Phi}^2 + \frac{1}{2C} \hat{Q}^2$$

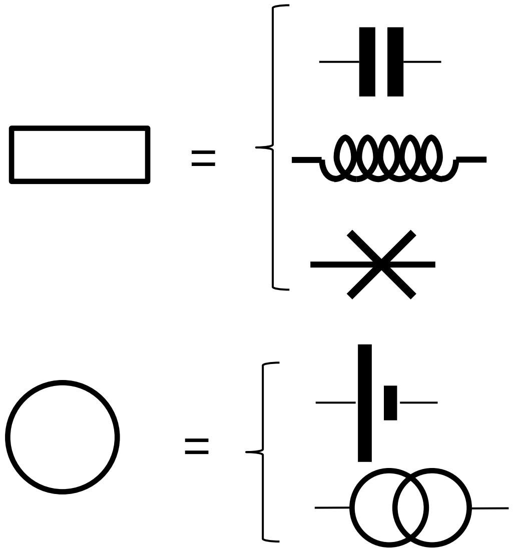
# Exercise 1

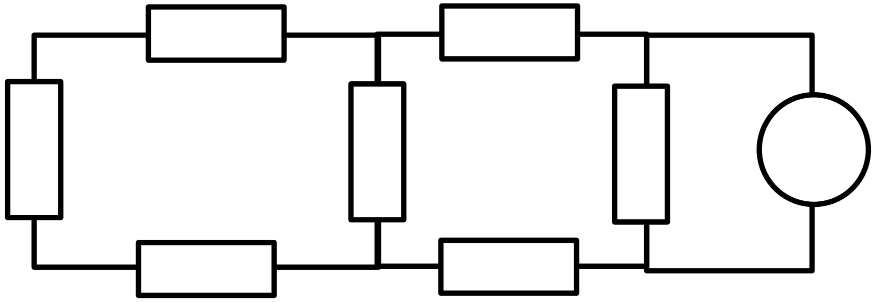
- Consider the circuit of two capacitively coupled LC circuit
  - Write the Lagrangian
  - Calculate the quantum Hamiltonian

- Hints:
  - consider each node to have a given flux $\phi_i$
  - Energy of capacitor connecting nodes $\phi_i$ and $\phi_j$: $\frac{1}{2}C\left(\dot{\phi}_i - \dot{\phi}_j\right)^2$
  - Energy of inductor connecting nodes $\phi_i$ and $\phi_j$: $\frac{1}{2L}\left(\phi_i - \phi_j\right)^2$
  - Node flux of ground is 0



# Hamiltonian of an arbitrary circuit





HAMILTONIAN ???

Correct procedure described in :

M. H. Devoret, p. 351 in *Quantum fluctuations* (Les Houches 1995)

Girvin, S. M. (2011), Circuit QED: superconducting qubits coupled to microwave photons.

G. Wendin and V. Shumeiko, cond-mat/0508729

U. Vool, M.H. Devoret, Introduction to quantum electromagnetic circuits, https://doi.org/10.1002/cta.2359

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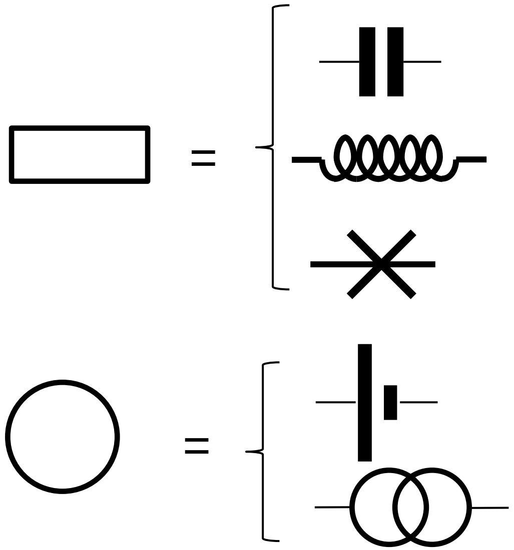
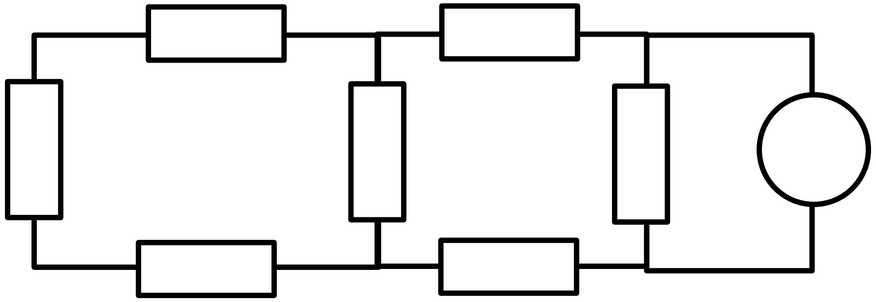
# Hamiltonian of an arbitrary circuit





HAMILTONIAN ???

1) Identify the relevant independent circuit variables

2) Write the circuit Lagrangian

3) Determine the canonical conjugate variables and the Hamiltonian

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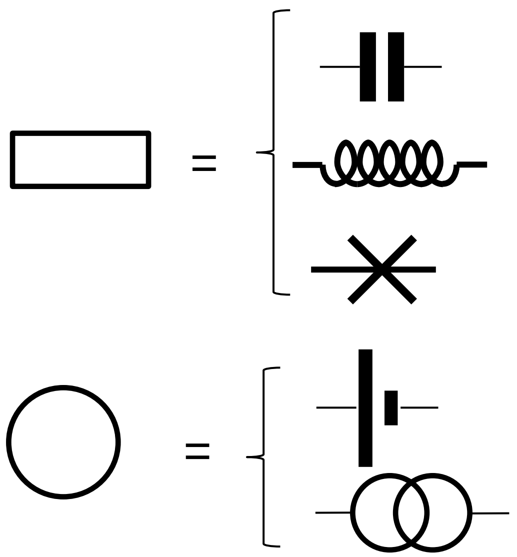
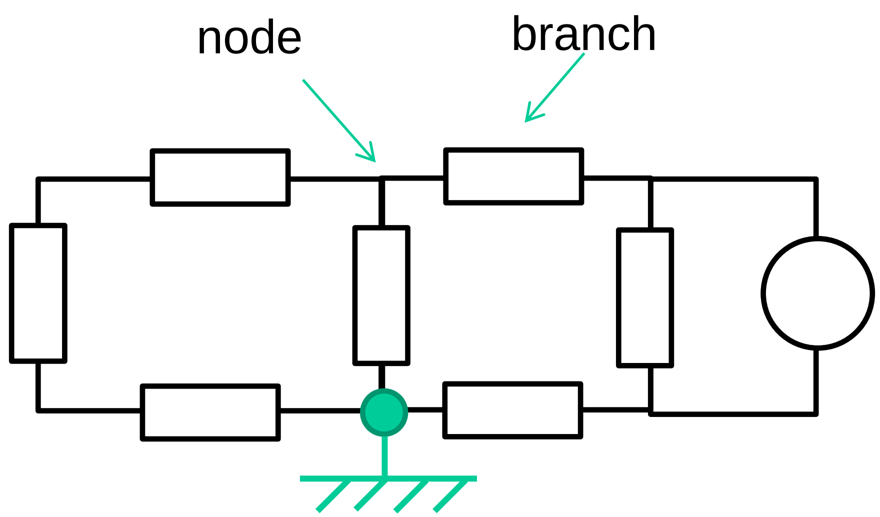
# Hamiltonian of an arbitrary circuit






Identifying the relevant independent circuit variables

1) Choose reference node (ground)

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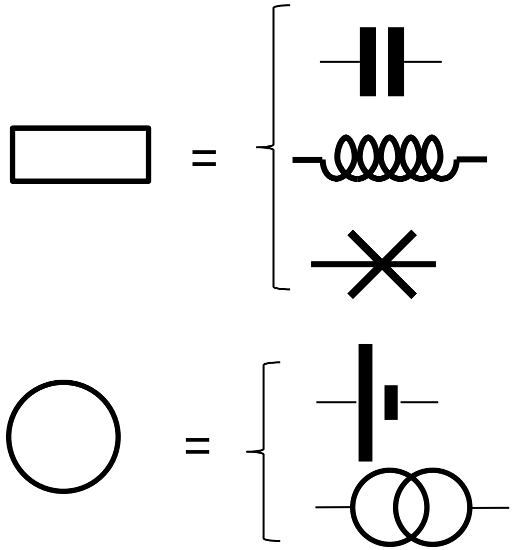
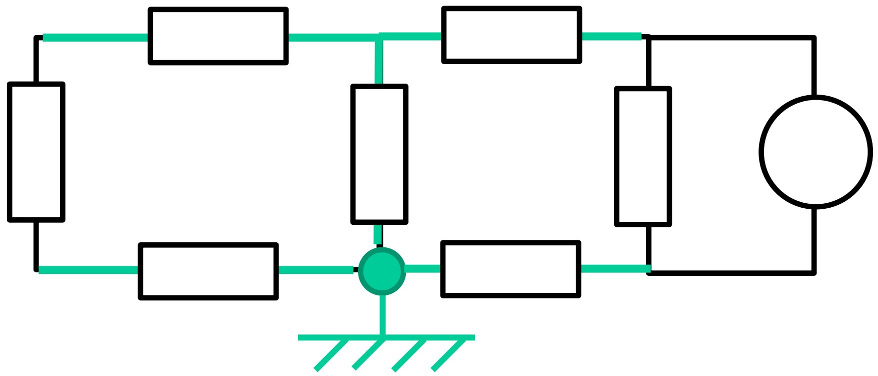
# Hamiltonian of an arbitrary circuit



Identifying the relevant independent circuit variables

1) Choose reference node (ground)

2) Choose « spanning tree » (no loop)

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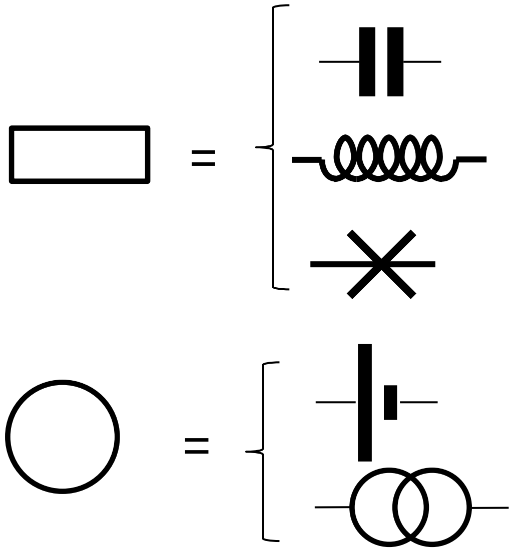
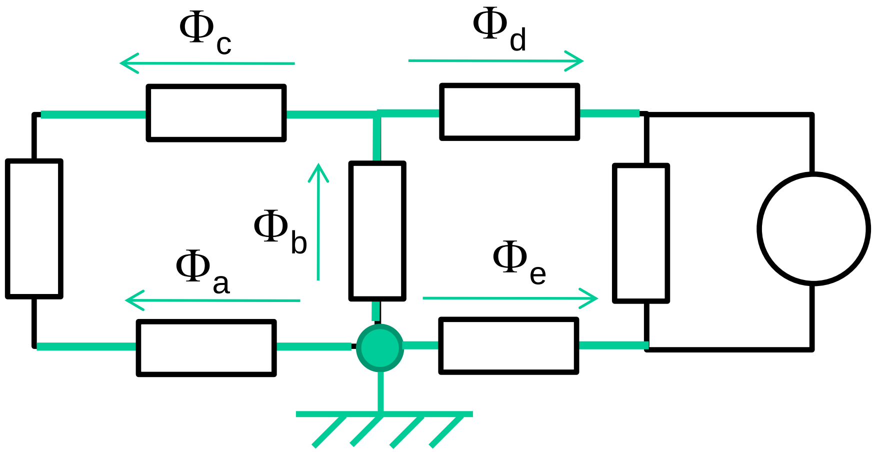
# Hamiltonian of an arbitrary circuit






Identifying the relevant independent circuit variables

1) Choose reference node (ground)

2) Choose « spanning tree » (no loop)

3) Define « tree branch fluxes » $\Phi_i(t) = \int_{-\infty}^{t} V(t')dt'$

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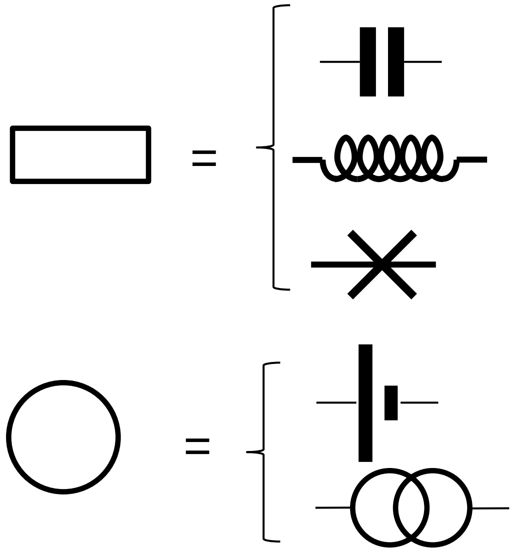
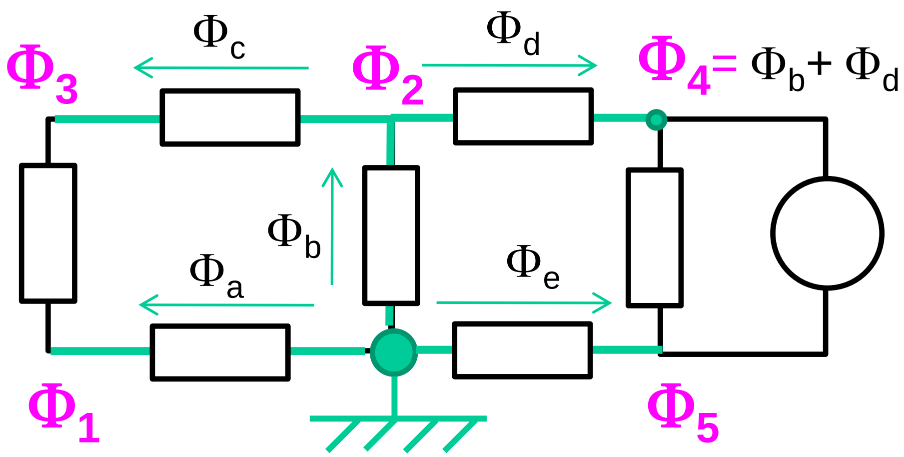
# Hamiltonian of an arbitrary circuit




Identifying the relevant independent circuit variables

1) Choose reference node (ground)

2) Choose « spanning tree » (no loop)

3) Define « tree branch fluxes » $\Phi_i(t) = \int_{-\infty}^{t} V(t')dt'$

4) Define **node fluxes** = sum of branch fluxes from ground

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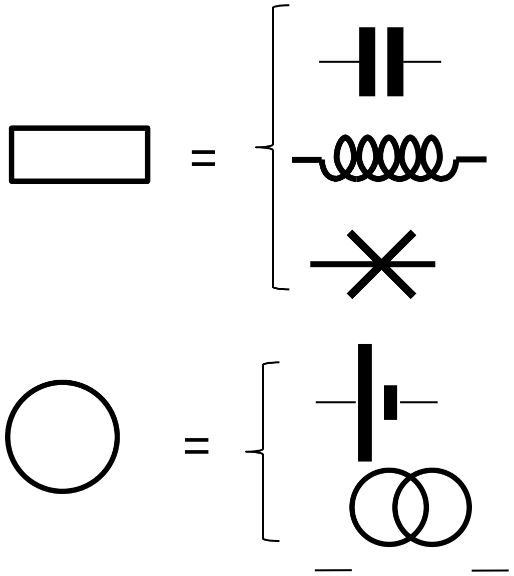
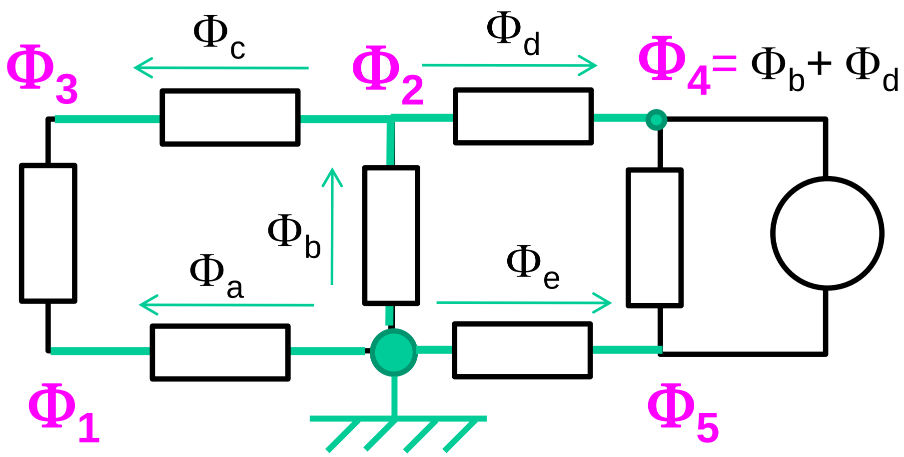
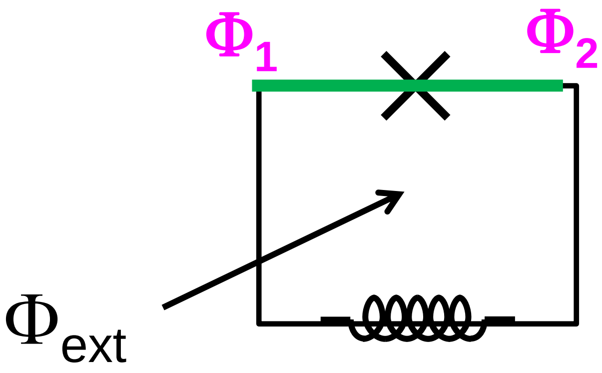
# Hamiltonian of an arbitrary circuit






Write Lagrangian $$L(\Phi_i, \dot{\Phi}_i) = L_{el}(\dot{\Phi}_i) - L_{pot}(\Phi_i)$$

taking into account constraints imposed by external biases (fluxes or charges)




$$L_{pot} = E_J \cos\left(\Phi_2 - \Phi_1\right) - \frac{\left(\Phi_1 - \Phi_2 - \Phi_{ext}\right)^2}{2L}$$

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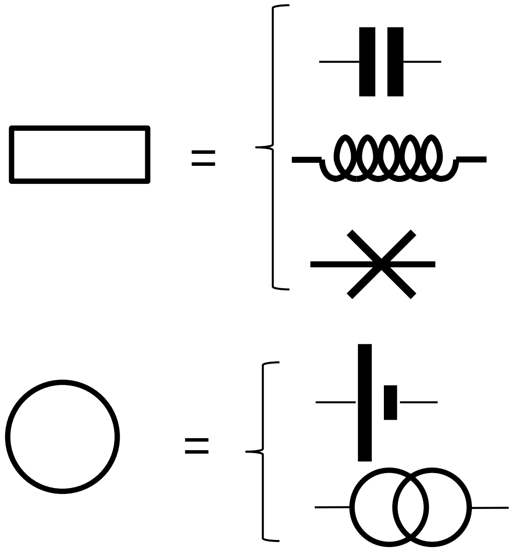
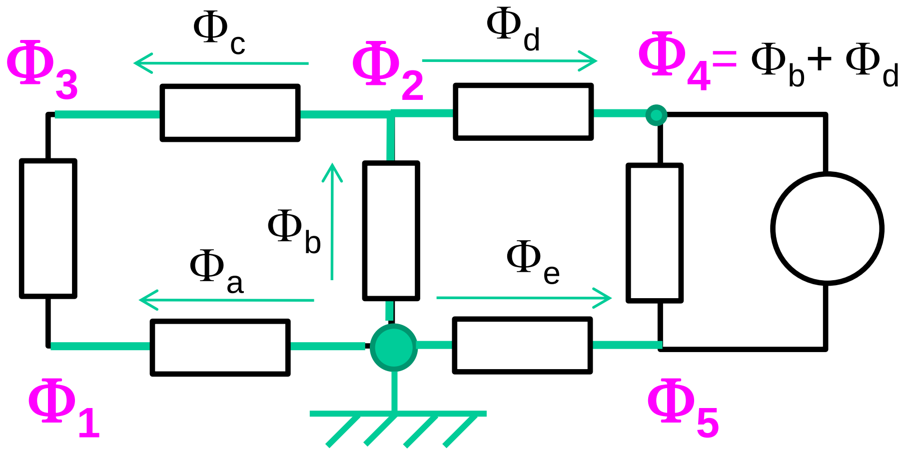
# Hamiltonian of an arbitrary circuit




Conjugate variables : $Q_i = \dfrac{\partial L}{\partial \dot{\Phi}_i}$

Classical Hamiltonian $H(\Phi_i, Q_i) = \sum Q_i \dot{\Phi}_i - L$

Quantum Hamiltonian $H(\hat{\Phi}_i, \hat{Q}_i)$ With $\boxed{\left[\hat{\Phi}_i, \hat{Q}_i\right] = i\hbar}$

or $H(\hat{\theta}_i, \hat{n}_i)$ with $\hat{n}_i = \hat{Q}_i / 2e$, $\hat{\theta}_i = \hat{\Phi}_i (2e/\hbar)$ $\left[\hat{\theta}_i, \hat{n}_i\right] = i$

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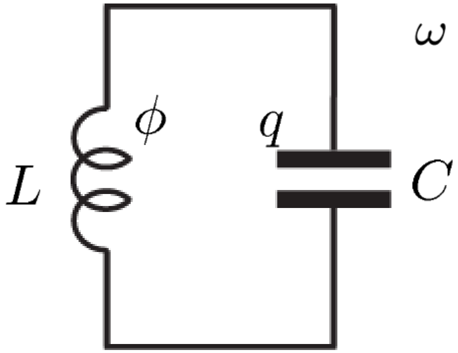
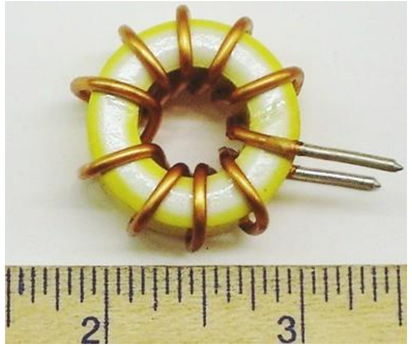
# Classical and quantum states of the harmonic oscillator

harmonic LC oscillator:



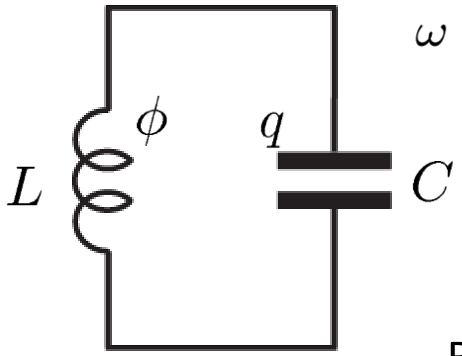

$$\omega = \frac{1}{\sqrt{LC}}$$

Question: take 1 nF capacitor and 1 nH inductor and solder an LC oscillator. Why we don't see quantum effects with it?

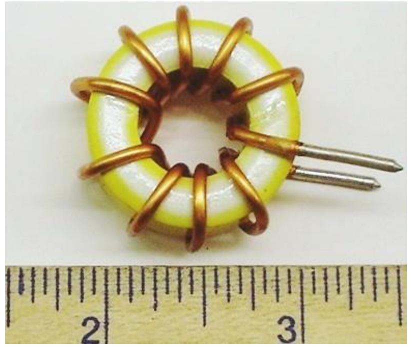


# Classical and quantum states of the harmonic oscillator

harmonic LC oscillator:




$$\omega = \frac{1}{\sqrt{LC}}$$



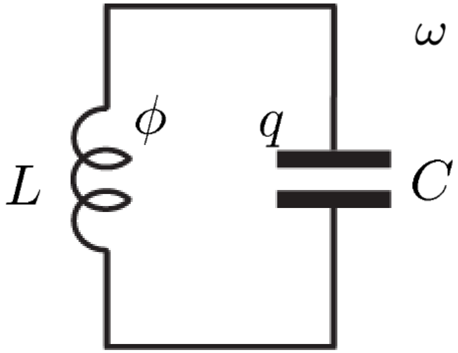

Resonance frequency of this resonator is $f \simeq 150\ \mathrm{MHz}$

If compare to thermal energy:

$$\frac{k_B 300K}{\hbar\omega} \simeq 40\ 000$$

# Classical and quantum states of the harmonic oscillator

harmonic LC oscillator:




$$\omega = \frac{1}{\sqrt{LC}}$$

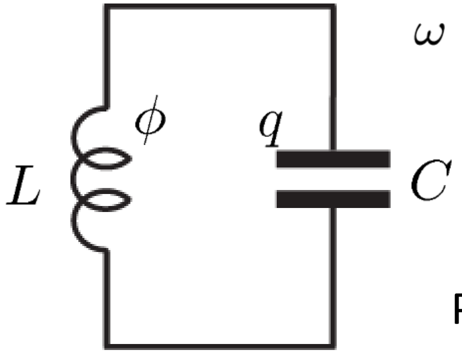
Question: what state will the oscillator attain if put in contact with environment with temperature $T$?

Note that due to the Boltzmann distribution: The probability to find the state of the system with energy $E_i$ is given by

$$p_i = \frac{e^{-\frac{E_i}{k_B T}}}{\sum_i e^{-\frac{E_i}{k_B T}}}$$

# Classical and quantum states of the harmonic oscillator

harmonic LC oscillator:




$$\omega = \frac{1}{\sqrt{LC}}$$

Probability to find $n$ photons is $p_n = \dfrac{e^{-\frac{\hbar\omega\left(n+\frac{1}{2}\right)}{k_B T}}}{\sum_n e^{-\frac{\hbar\omega\left(n+\frac{1}{2}\right)}{k_B T}}}$

Let's calculate the sum using the geometric progression formula:

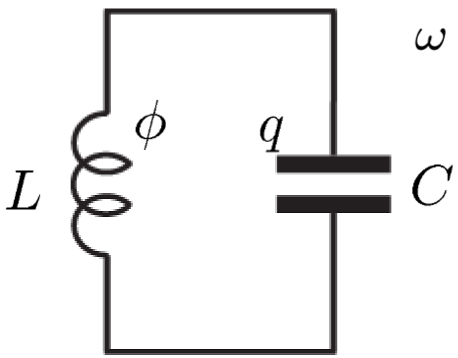
$$\sum_n e^{-\frac{\hbar\omega n}{k_B T}} = \frac{1}{1 - e^{-\frac{\hbar\omega}{k_B T}}}$$

The result is:

$$p_n = e^{-\frac{\hbar\omega n}{k_B T}}\left(1 - e^{-\frac{\hbar\omega}{k_B T}}\right)$$

# Classical and quantum states of the harmonic oscillator

harmonic LC oscillator:

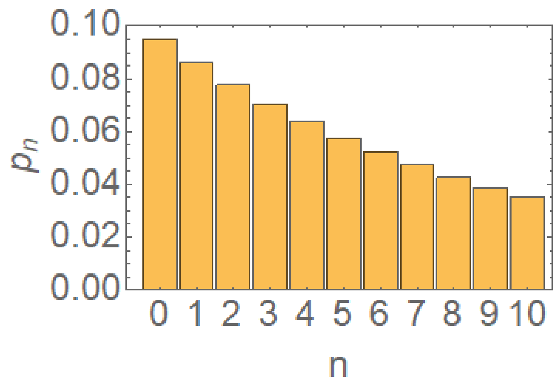



$$\omega = \frac{1}{\sqrt{LC}}$$

The HO oscillator will be described by the density operator:

$$\rho_{th} = \begin{pmatrix} p_0 & 0 & \dots \\ 0 & p_1 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} \text{ with } p_n = e^{-\frac{\hbar\omega n}{k_B T}}\left(1 - e^{-\frac{\hbar\omega}{k_B T}}\right)$$

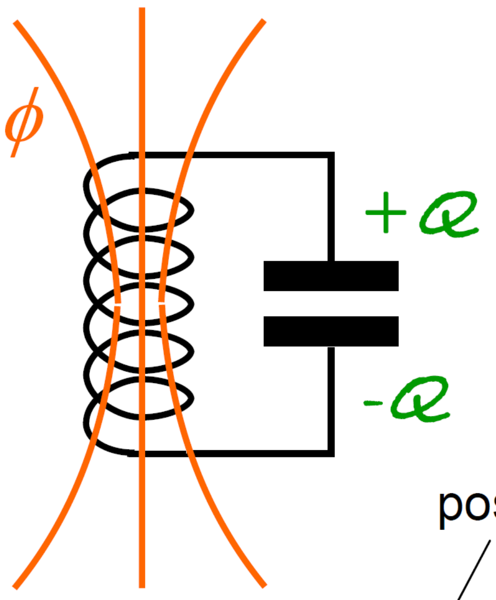
For $\frac{k_B T}{\hbar\omega} = 10$ (average number of photons):




Statistical mixture not a pure state!
-> no quantum effects

There is only one thermal state which is a pure state:
Ground state at T = 0 K

# Classical and quantum states of the harmonic oscillator



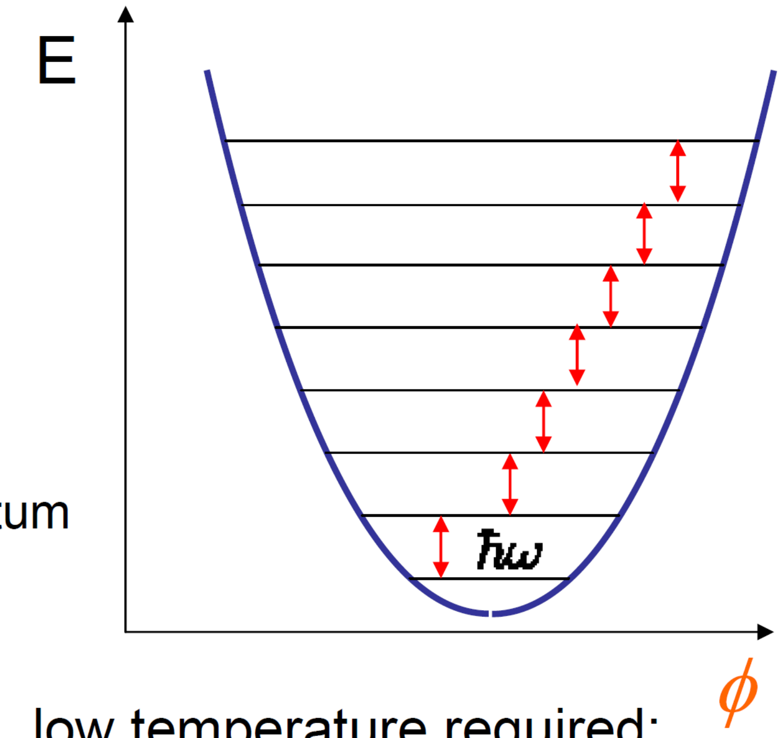

$$[\phi, q] = i\hbar$$

position momentum

Hamiltonian $$H = \frac{\phi^2}{2L} + \frac{q^2}{2C}$$

$$\omega = 1/\sqrt{LC}$$

$$H = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)$$




low temperature required:

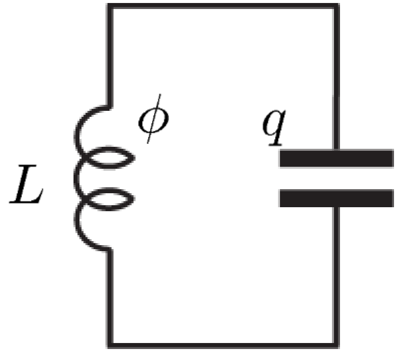
$$\hbar\omega \gg k_B T$$

10 GHz ~ 500 mK 20 mK

$$\langle n_{\text{th}} \rangle = \frac{1}{\exp\left(h\nu/k_B T\right) - 1} \sim 10^{-11}$$

# Different Flavors of Superconducting Resonators

harmonic LC oscillator:




$$\omega = \frac{1}{\sqrt{LC}}$$

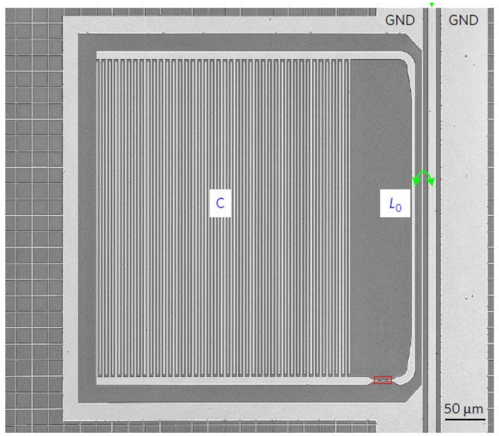
$$\frac{\omega}{2\pi} \sim 5 \text{ GHz}$$

To observe quantum effects we need to initialise an oscillator in a pure state.

Use dilution cryostat can be cooled down to its ground state at

$T = 20 \text{ mK} \left(\frac{k_B T}{\hbar\omega} = 0.1\right)$

**lumped element resonator:**




Use nanofabrication to shrink the dimensions and increase the frequency

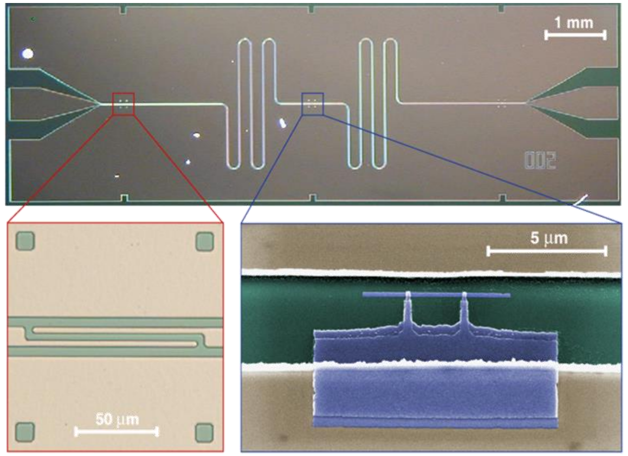
F. Yoshihara *et al., Nature Physics* (2017)

**3D cavity:**

Use 3D cavities as oscillators. Their dimension defines the frequency

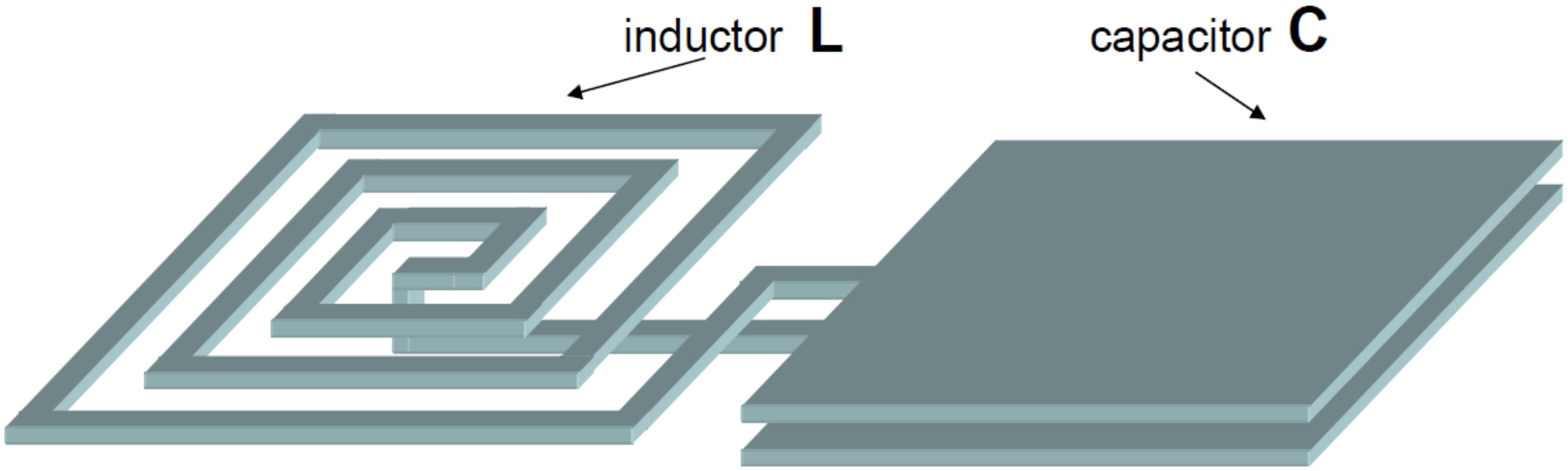
M. Jerger *et al.,* Nature Comm. 7, 12930 (2016)

**planar transmission line:**

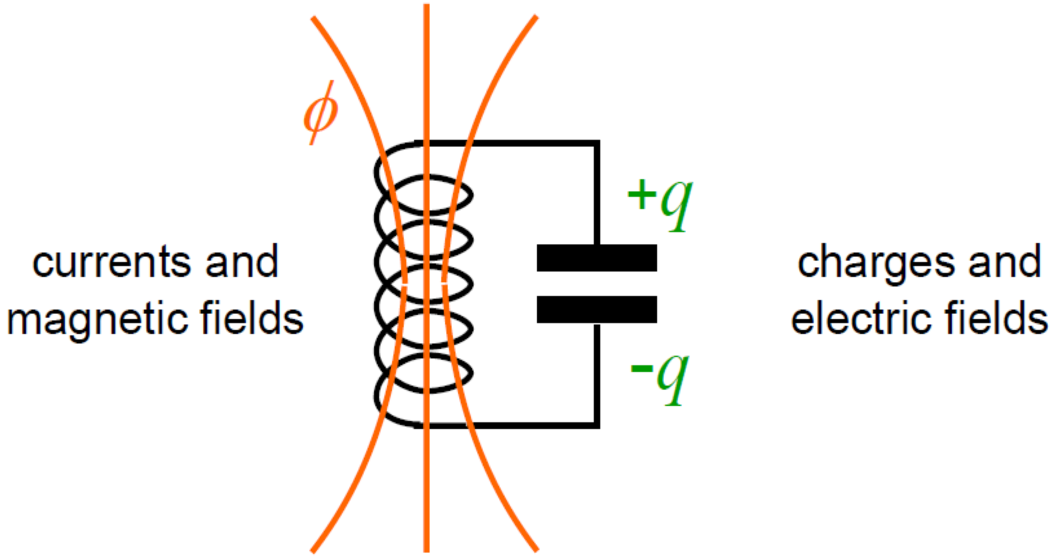



Wallraff *et al.*, *Nature* **431**, 162 (2004)

# Lumped element resonator



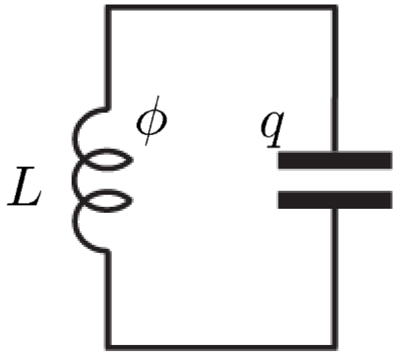

a harmonic oscillator



# Lumped element superconducting resonators

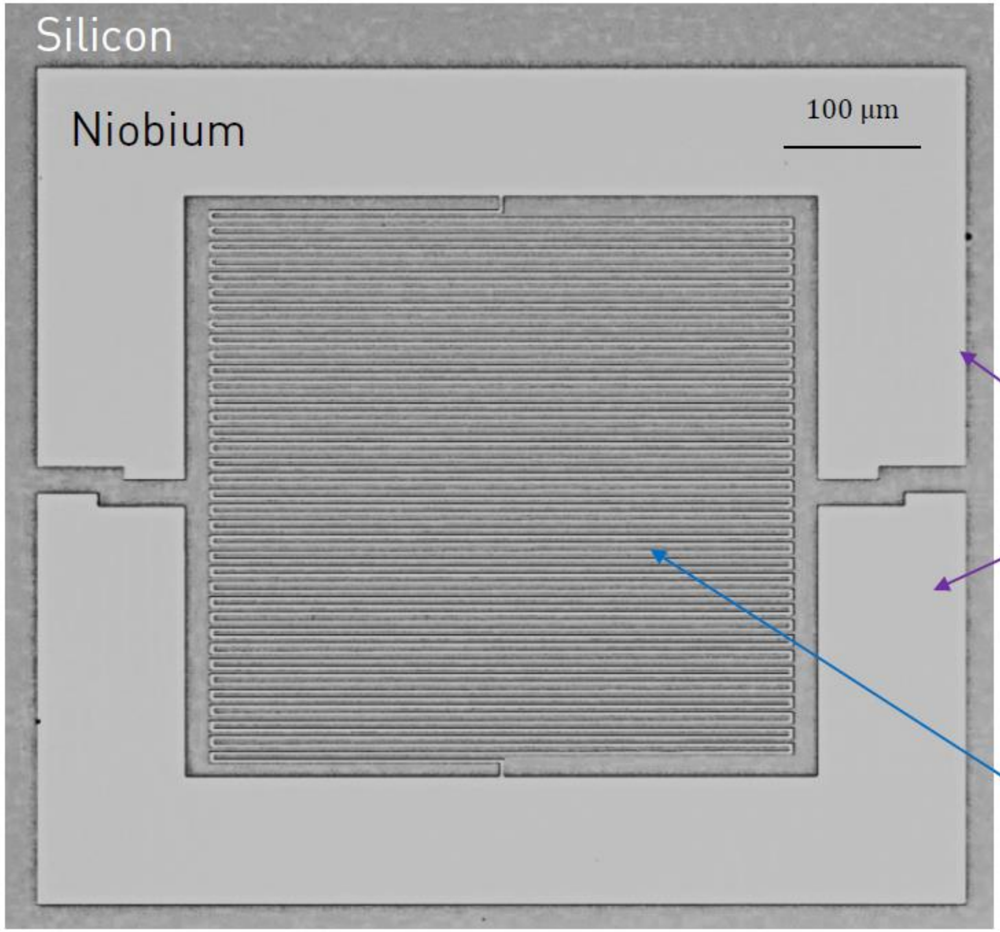
- Resonators fabricated from a 150 nm thin Niobium film on a Silicon substrate.
- Inductance and capacitance controlled by designed geometry.
- Characteristic size much smaller than characteristic wavelength.
- Accurate prediction of resonance frequency needs to take stray capacitance of inductor into account.
- Measured in a metallic cavity with a sender (IN) and receiver (OUT) antenna.

harmonic LC oscillator:




$$\omega = \frac{1}{\sqrt{LC}}$$

$$\frac{\omega}{2\pi} \sim 5\ \mathrm{GHz}$$



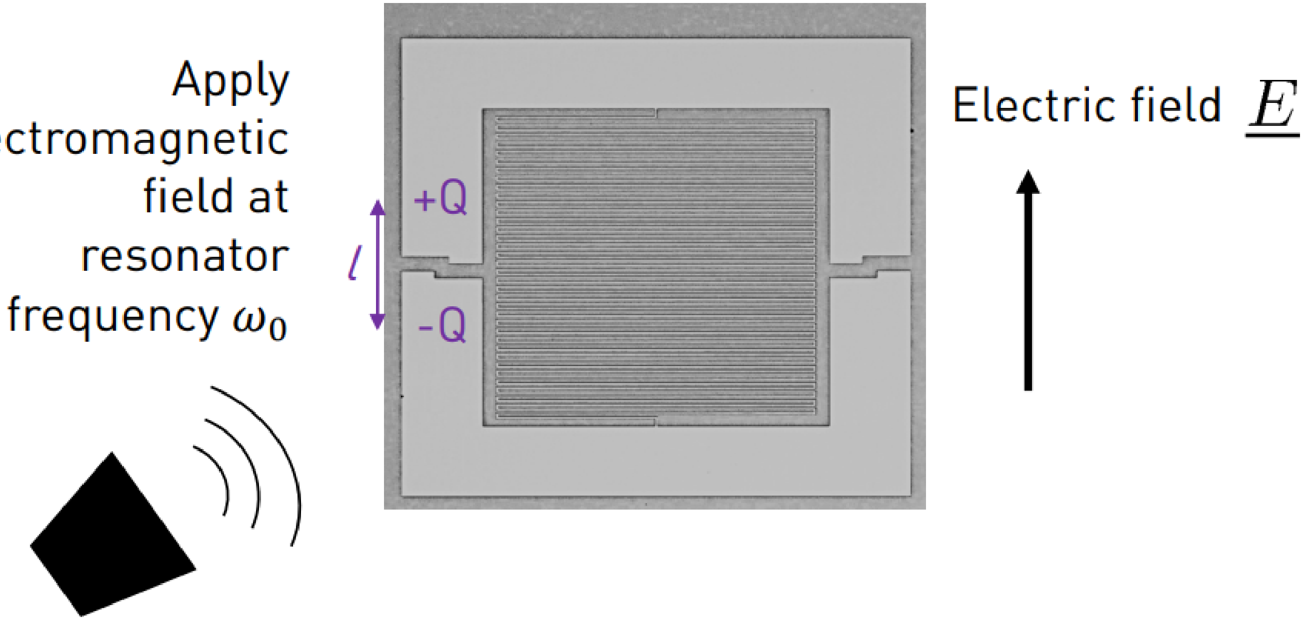

Resonance frequencies typically between $4 - 8$ GHz

Two plates forming a capacitor. Typical values: $C \sim 50 - 500$ fF

Meandering wire forming an inductor. Typical values $L \sim 2 - 20$ nH

# Coupling Planar Lumped Element Resonators to an E-field

Apply Electromagnetic field at resonator frequency $\omega_0$




- Coupling mediated by dipole-field interaction

$$H_\mathrm{d} = \underline{d} \cdot \underline{E}$$

- Dipole moment

$$\underline{\hat{d}} = \int \mathrm{d}^3 r \, \underline{r} \, \rho(\underline{r}) \equiv \hat{Q} \, \underline{l}$$

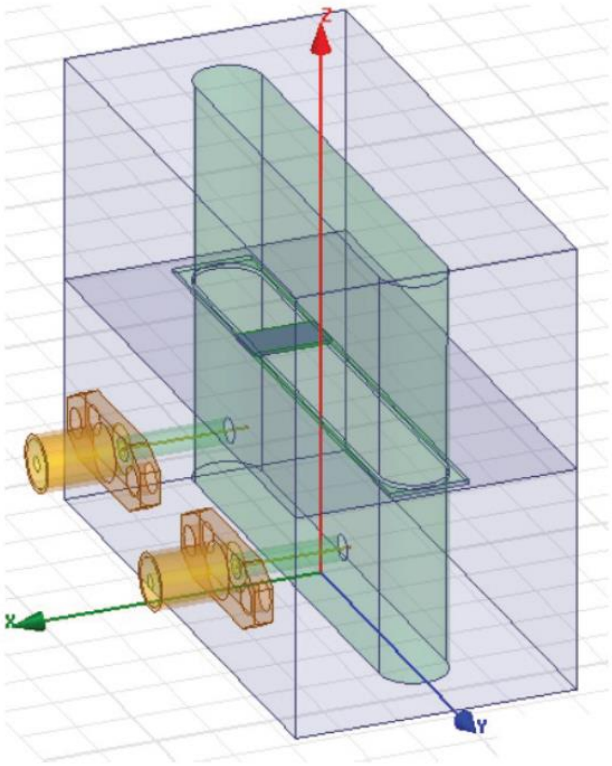
- With average charge separation $l$.
- For an LC resonator we thus obtain

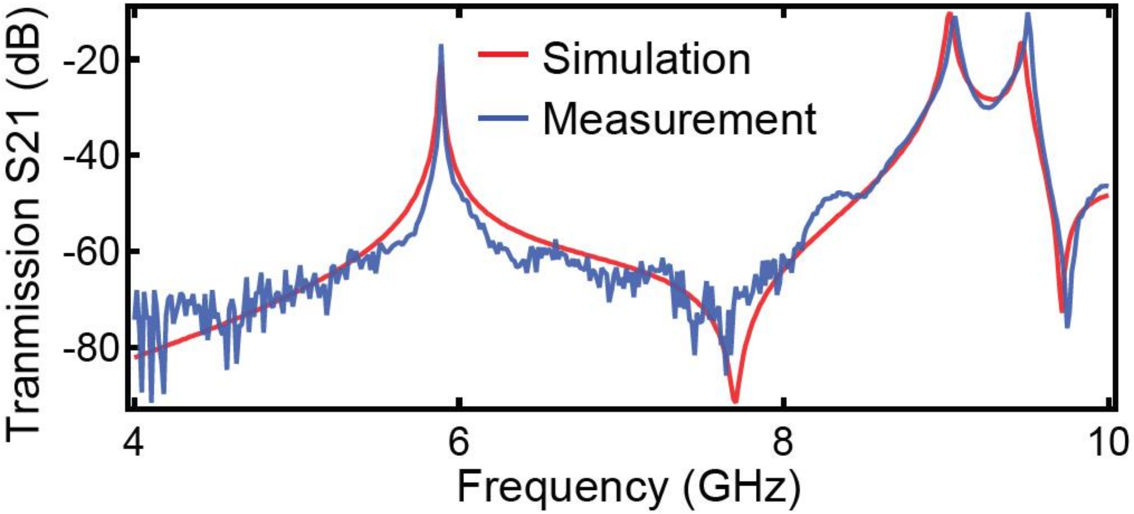
$$H_\mathrm{d} = \underline{l} \cdot \underline{E} \, Q_\mathrm{zpf}(a + a^\dagger)$$

Note: Applying $H_\mathrm{d}$ for short duration and with $\underline{E}(t) = \underline{E}_0 \cos(\omega_0 t)$ generates a coherent state:

$$|0\rangle \rightarrow |\alpha\rangle$$

# Aluminium 3D cavity



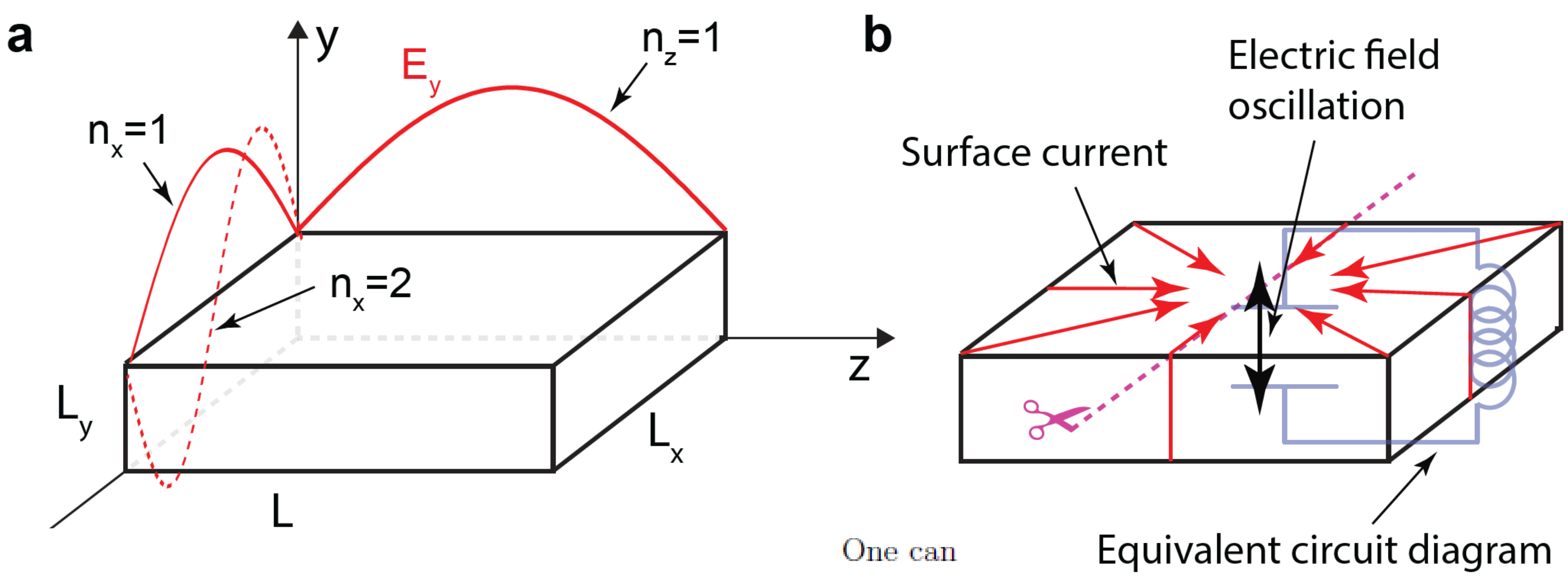





**HFSS simulation for cavity transmission:** The cavity transmission is simulated by HFSS (red curve) which is in agreement with actual measurement (blue curve).

Mahdi Naghiloo, Introduction to Experimental Quantum Measurement with Superconducting Qubits, arXiv:1904.09291

# Electromagnetic Modes of a 3D cavity




One can show that for a 3D cavity described by $L_x, L_y, L_z$ dimensions and depicted in Figure 3.1, the Equation 2.1.2 simply generalized to,

$$E(\vec{r},t) = \mathcal{E}\, q(t) \sin(\vec{k}\cdot\vec{r}) \quad (3.1.1a)$$

$$B(\vec{r},t) = \mathcal{E}\, \frac{\mu_0\varepsilon_0}{k}\dot{q}(t) \cos(\vec{k}\cdot\vec{r}), \quad (3.1.1b)$$

where $\vec{k} = (n_x\pi/L_x, n_z\pi/L_z, n_z\pi/L_z)$ and $\vec{r} = (x, y, z)$ and the corresponding resonance frequency of modes are,

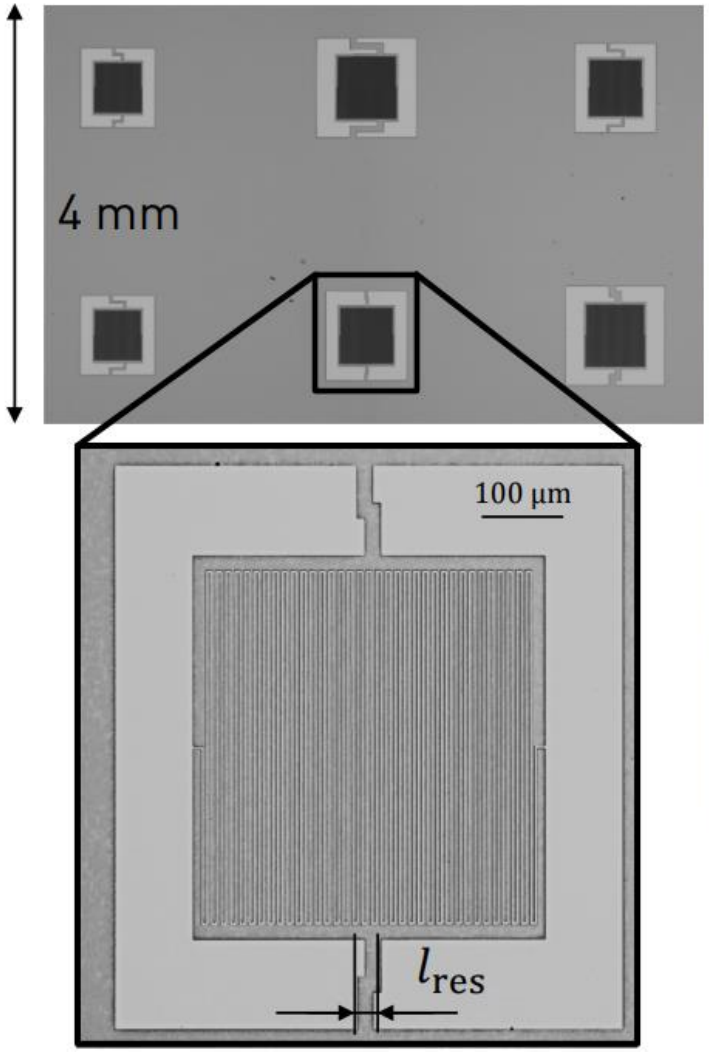
$$f = \omega_c/2\pi = \frac{c}{2}\sqrt{(\frac{n_x}{L_x})^2 + (\frac{n_y}{L_y})^2 + (\frac{n_z\pi}{L_z})^2}, \quad (3.1.2)$$

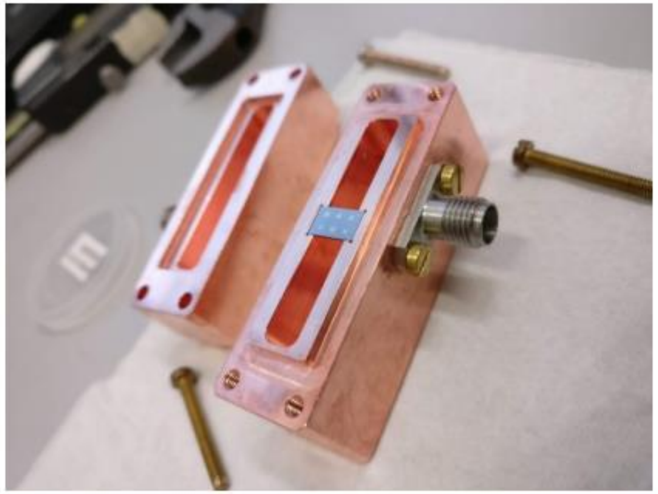
where $c$ is the speed of light inside the cavity. Each mode is described by a set of integers $n_x, n_y, n_z$,

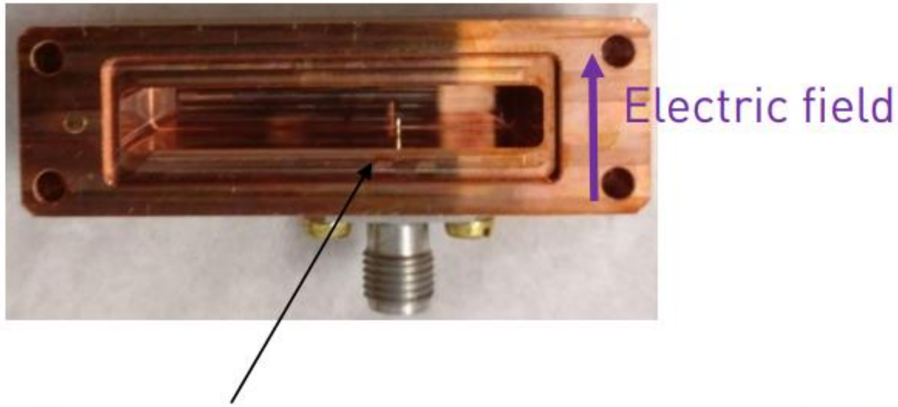
typical cavity geometry is shown. Red lines
$TE_{101}$ mode. The electric field oscillations
ɔscillations (red arrows) have been depicted
e center (opposite charges at the top and
ı for a section of the cavity. The cut-line
ace current is tangential.

Mahdi Naghiloo, Introduction to Experimental Quantum Measurement with Superconducting Qubits, arXiv:1904.09291

# Lumped Element Resonators in 3D Cavities



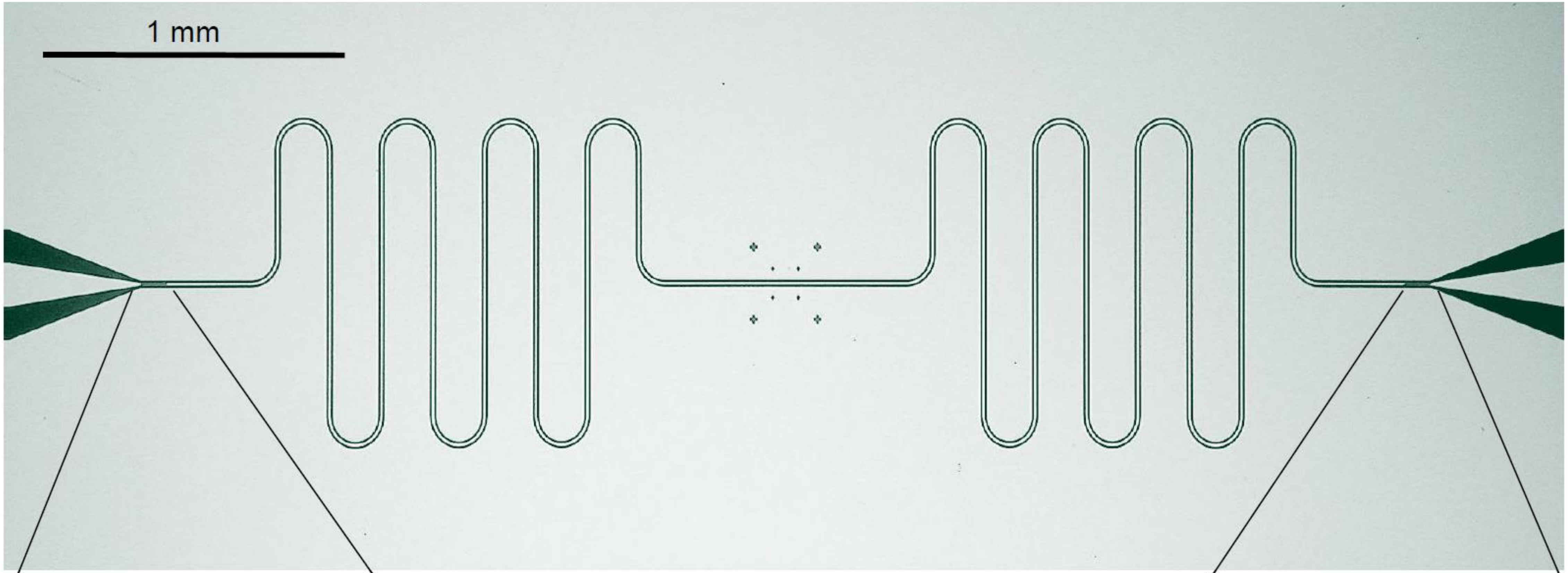


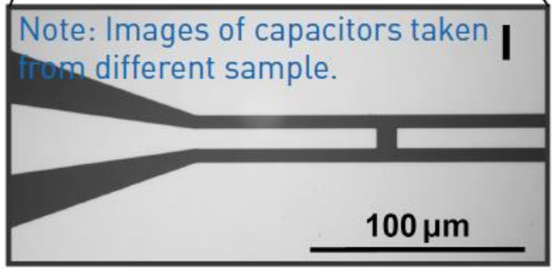

"Antenna" to apply and receive EM field

Properties of the system:

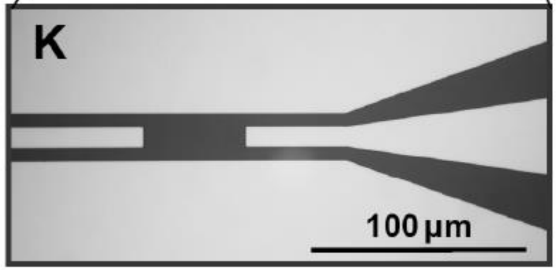
- multiple lumped element with different frequencies on a chip
- 3-Dimenesional cavity, typically made of copper or aluminium
- TE101 mode of the cavity couples to the dipole moment of the resonators
- Inner conductor of the connector acts as an antenna to send and receive electromagnetic fields to/from the resonators
- Typical use case: Establish well-controlled environment to test material-related loss mechanisms.

# Distributed coplanar waveguide resonators



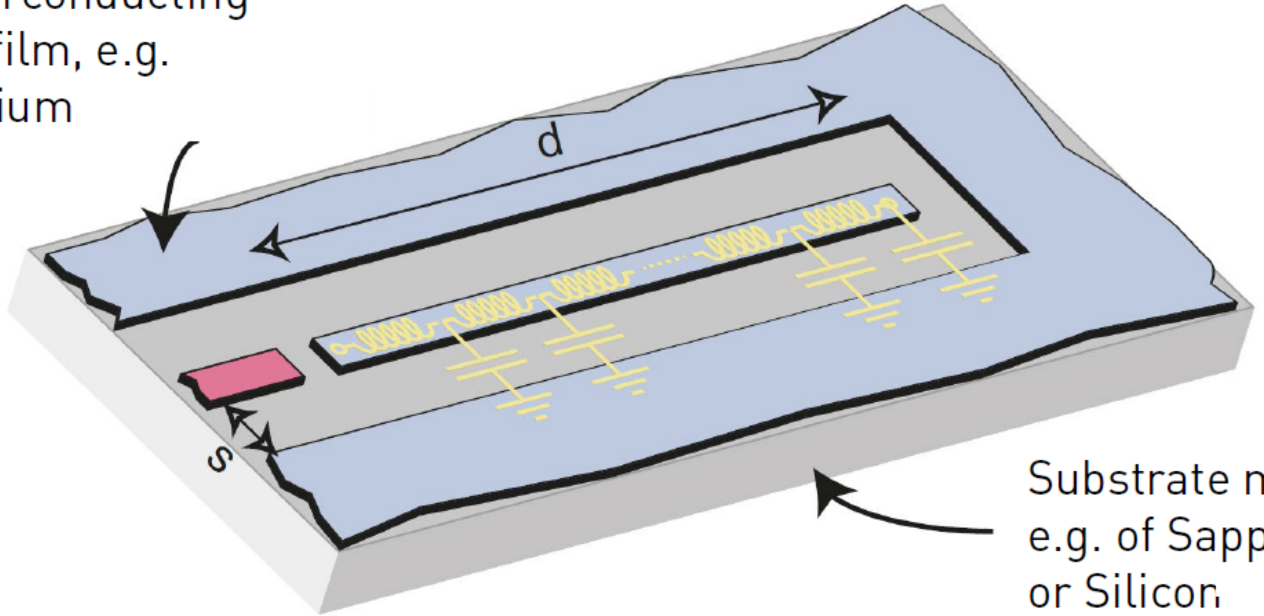

- Open ends to support standing waves with wavelengths $\lambda_n/2 = d/n$. Here, $n$ takes integer values and $d$ is the length of transmission line resonator.
- Resonance frequency $f_n = v/\lambda_n$ with phase velocity $v$.



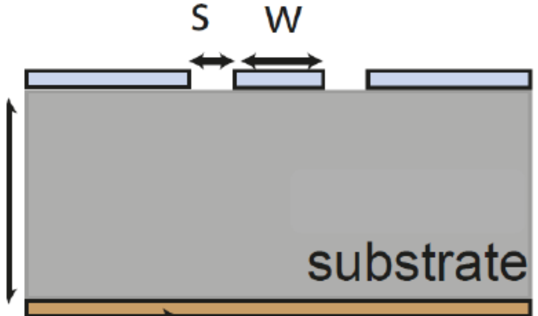

M. Goeppl et al., J. Appl. Phys. 104, 113904 (2008)

# Distributed coplanar waveguide resonators

Superconducting thin film, e.g. Niobium




Substrate made e.g. of Sapphire or Silicon




Typical dimensions:

$s = 4.5\ \mu m$

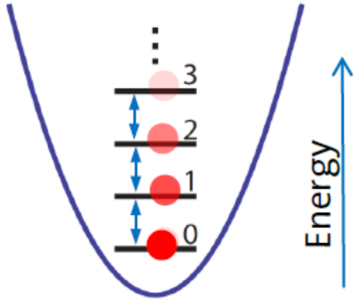
$w = 10\ \mu m$

$d \sim 5\ mm$

Radiation field stored inside:

$$H = \hbar\omega \ a^\dagger a$$

harmonic oscillators

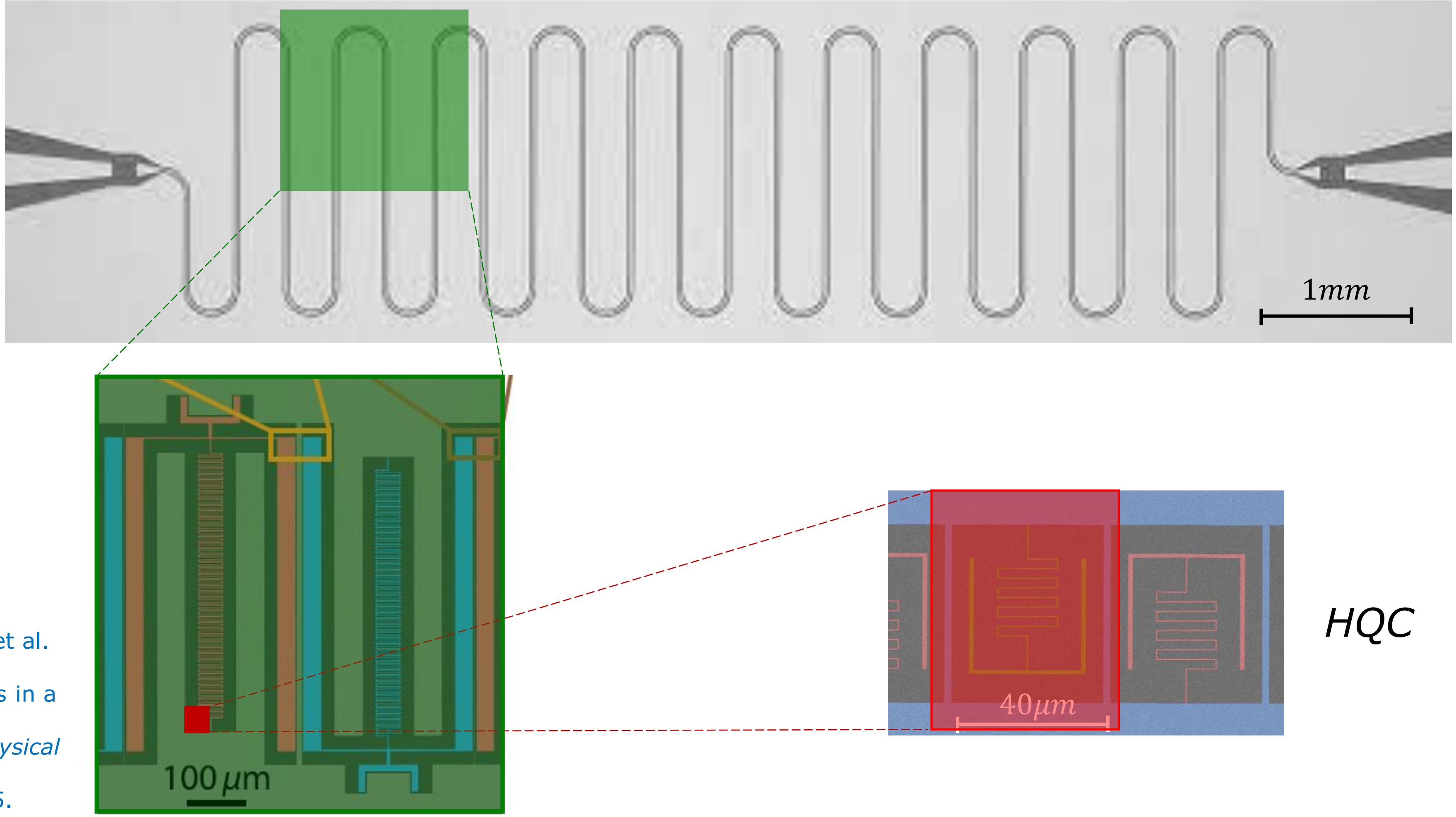



classical coherent state

Need nonlinear circuit element to build qubit.

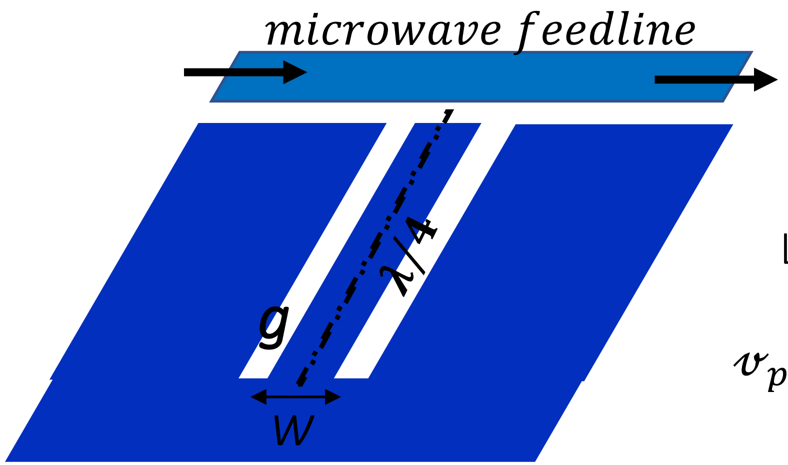
# Comparison to other kind of microwave resonators

Göppl, Martin, et al. "Coplanar waveguide resonators for circuit quantum electrodynamics." *Journal of Applied Physics* 104.11 (2008): 113904.




Kim, Eunjong, et al. "Quantum electrodynamics in a topological waveguide." *Physical Review X* 11.1 (2021): 011015.

# Compact High Impedance resonators (distributed)




$\ell_r$= *resonator length*

$\mathcal{L} = inductance\ per\ unit\ length$

$\mathcal{C} = capacitance\ per\ unit\ length$

Let's consider a $\lambda/4$ resonator:

$v_{ph} = 1/\sqrt{\mathcal{L}\mathcal{C}}$ is the phase velocity

$$\ell_r \propto v_{ph}/4f_r$$

$Z_r = \sqrt{\mathcal{L}/\mathcal{C}}$

$\mathcal{C} = \mathcal{C}_g$

$\mathcal{L} = \mathcal{L}_g + \mathcal{L}_k$

$kinetic\ inductance$

$geometric\ inductance$

For a typical CPW 50 Ω res. with *w*=10 um and gap *g*=6um we have:

$\mathcal{L}_g$ ~ 420 nH/m and $\mathcal{C}_g$ ~ 0.166 fF/m so $v_{pf}$~120 *10^6 m/s.
A $f_r$ ~ 6 GHz $\lambda/4$ resonator will be $\ell_r$~5 mm long

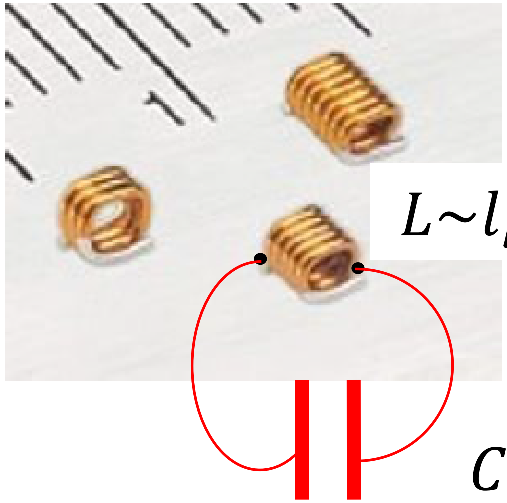
*The kinetic inductance of disordered superconducting thin film nanowire resonator (with w ~200 nm) can be $\mathcal{L}_k$~1000 $\mathcal{L}_g$ so the length od the resonator can be ~ 33 times shorter*

# Need for High Impedance:

## ordinary inductors fail before their impedance exceeds $R_Q$

$$\mathbf{g} \propto V_0^{\mathrm{rms}} \propto \sqrt{Z_0}$$

Coilcraft



$L \sim l\mu_0$

$C \sim l\varepsilon_0$

$\omega_0 = 1/\sqrt{LC}$ *Self resonance frequency*

$Z_0 \equiv \omega_0 L = \sqrt{L/C}$ *Impedance*

$R_Q = h/(2e)^2 \cong 6.5\ k\Omega$

$\alpha \sim 1/137$

*fine structure constant*

*Vacuum Impedance*

$$\sqrt{L/C} \approx \sqrt{\mu_0/\varepsilon_0} = 8\,\alpha\,R_Q \equiv Z_{vac} = 377\ \Omega$$

$$Z_0 \approx Z_{vac}$$

***How can we engineer an environment with α >> 1/137?***

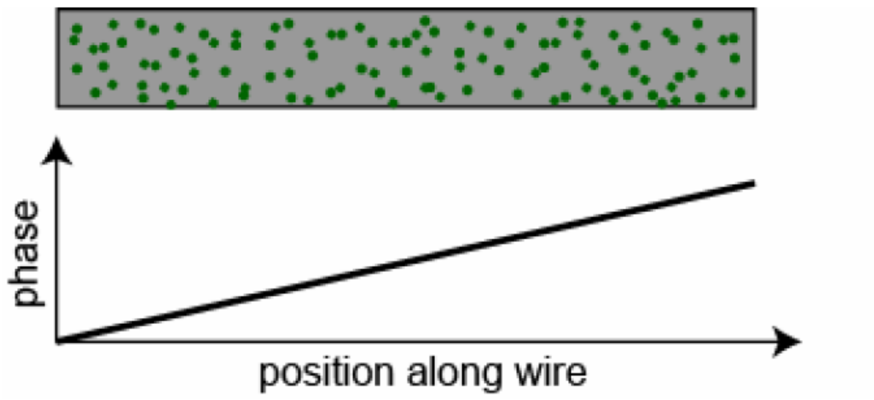
N. A. Masluk et al., *PRL* **109**, 137002 (2012)

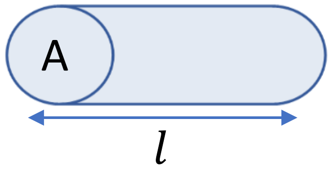
V. Manucharyan's PhD thesis

# Different ways to achieve a high kinetic inductance

Credits
Vl. E. Manucharyan
PhD thesis

disordered superconducting thin films




$$I = (2e)nAv$$

$$E_K = \frac{1}{2}(2m_e)nAdv^2 \equiv \frac{1}{2}L_K I^2$$

$$L_K = \frac{m_e}{2en}\frac{l}{A} = \mu_0\lambda^2\frac{l}{A}$$

narrow wire
etched in thin film

disordered superconductor

disordered superconducting thin films can be made compatible with in-plane magnetic field

TiN

M.R. Vissers, et al., *Appl. Phys. Lett.* **97**, 232509 (2010).
H.G. Leduc, et al., *Appl. Phys. Lett.* **97**, 102509 (2010).
L.J. Swenson, er al., *J. Appl. Phys.* **113**, 104501 (2013).
A. Shearrow et al., *Appl. Phys. Lett.* **113**, 212601 (2018)

NbTiN

R. Barends, et al., *Appl. Phys. Lett.* **97**, 033507 (2010).
N. Samkharadze, et al., *Phys. Rev. Appl.* **5**, 044004 (2016).
T.M. Hazard, et al., *Phys. Rev. Lett.* **122**, 010504 (2019).

Granular aluminum

L. Grünhaupt, et al., *Phys. Rev. Lett*. **121**, 117001
N. Maleeva et al., *Nat. Commun.* **9**, 3889 (2018)

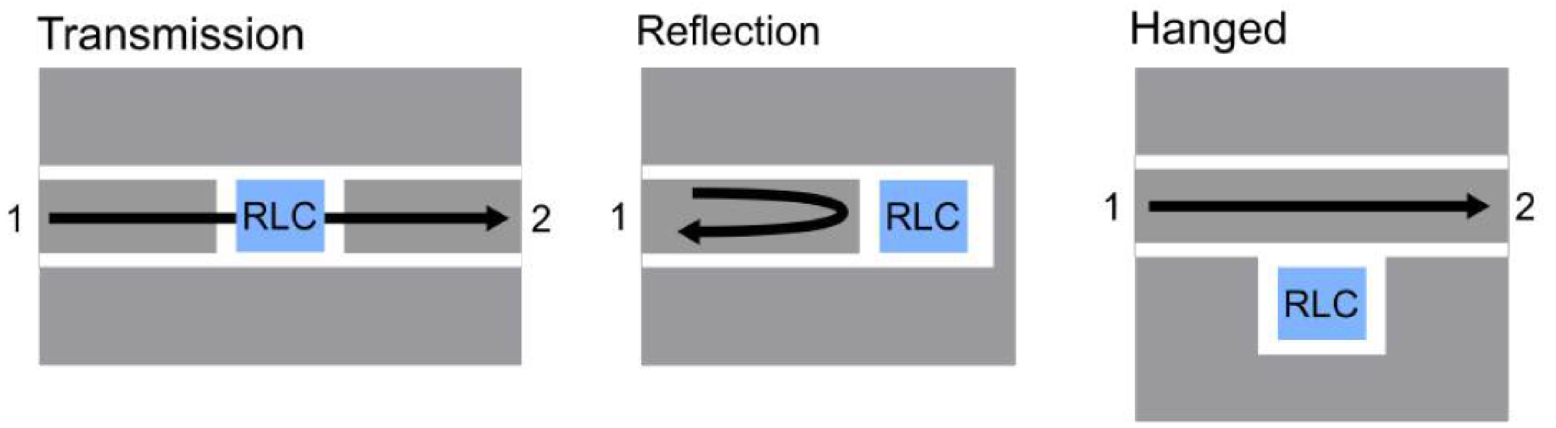
NbN

G.J. Grabovskij, et al., *Appl. Phys. Lett.* **93**, 134102 (2008).
J. Luomahaara, et al., *Nat. Commun.* **5**, 4872 (2014).
D. Niepce, et al., *Phys. Rev. Applied* **11**, 044014 (2019).

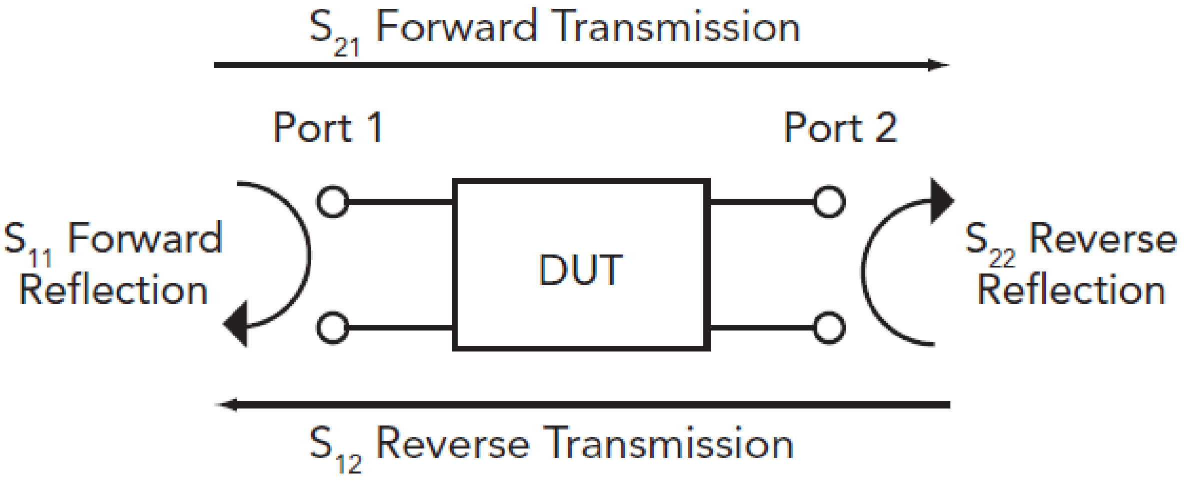
InOx

O. Dupre’, et al., *Supercond. Sci. Technol.* **30**, 045007 (2017).
S. E. de Graaf, et al., *Nat. Phys.* **14**, 590–594 (2018).

# How to measure the scattering parameters of microwave resonators

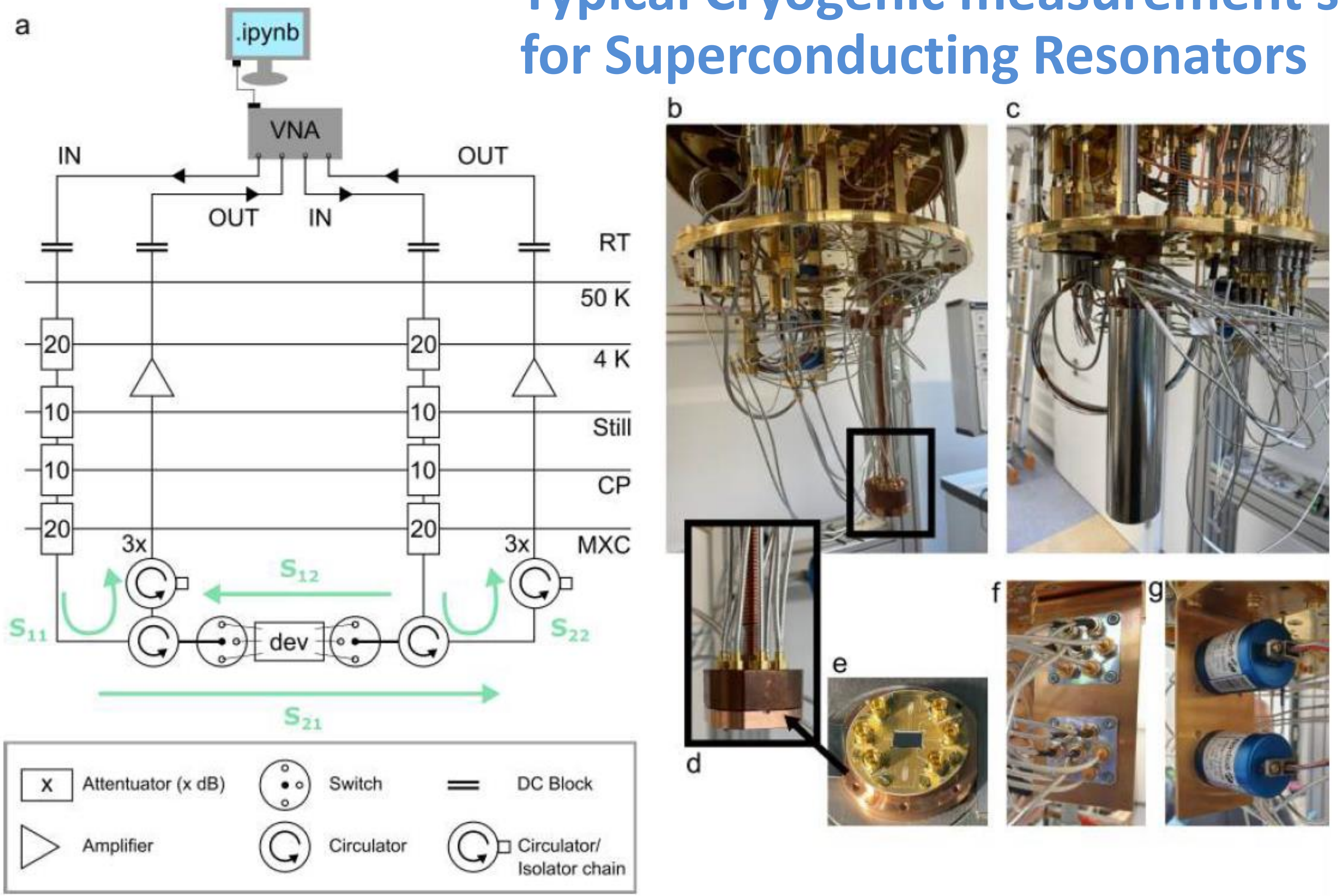



**Three Types of Resonator Measurement.** Transmission: The resonator is coupled to an input and an output feedline on each side. Reflection: The resonator is probed by coupling it to one end of the feedline. Hanged: The resonator couples to a feedline whose transmission is measured.




*The four scattering parameters are shown in this figure.*

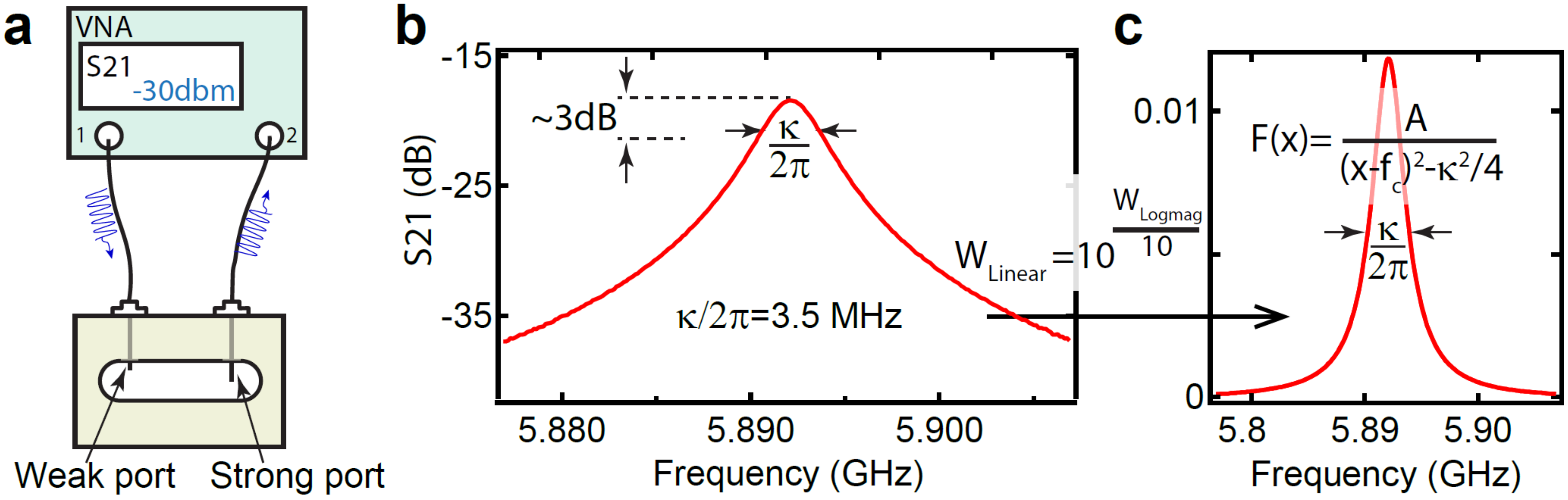
# Typical Cryogenic measurement setup for Superconducting Resonators




HQC

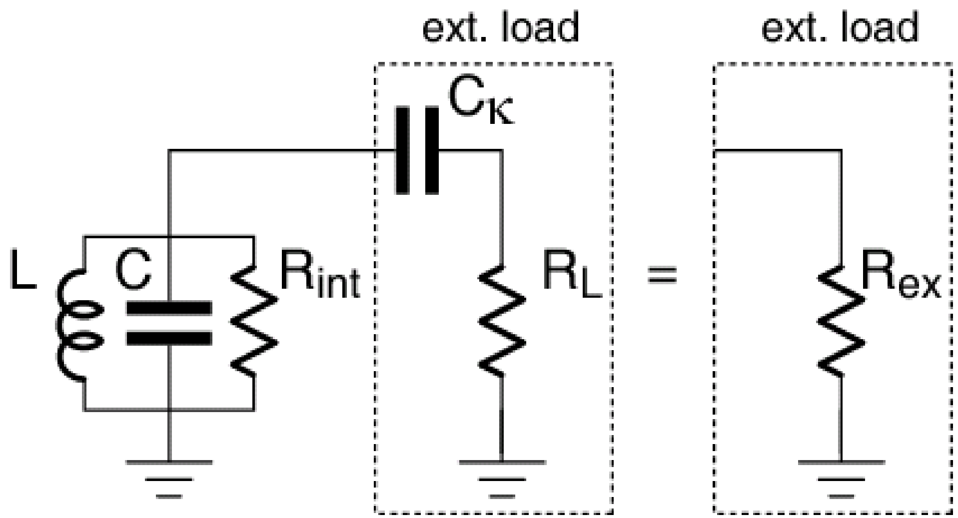
EPFL

# How to measure the scattering parameters of microwave resonators




**The cavity linewidth characterization:** The cavity linewidth $\kappa$ can be quantified by measuring the scattering parameters of the cavity. **a**,**b** In the transmission measurement S21, the cavity linewidth can be estimated by the bandwidth that the transmission signal drops by 3 dB. **c**, More carefully, one can scale the transmission in the linear form and fit to the Lorentzian function. The FWHM of the Lorentzian function would be the cavity linewidth.

# Internal and External Dissipation in an LC Oscillator




internal losses: conductor, dielectric $R_{\text{int}}$

external losses: radiation, coupling $R_{\text{ext}}$

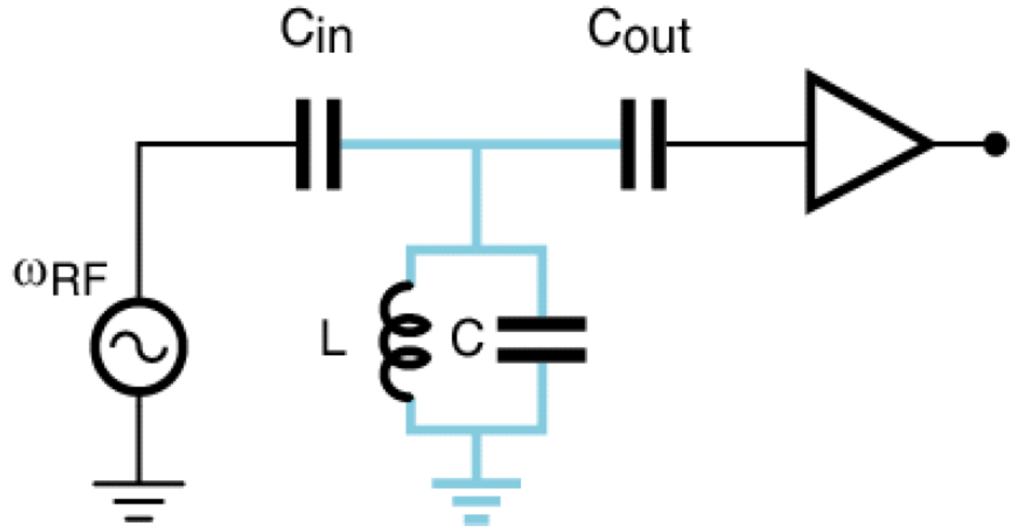
total losses $$\frac{1}{R} = \frac{1}{R_{\text{int}}} + \frac{1}{R_{\text{ext}}}$$

quality factor $$Q = \frac{R}{Z} = \omega_0 RC$$ with impedance $$Z = \sqrt{\frac{L}{C}}$$

excited state decay rate $$\Gamma_1 = \frac{\omega_0}{Q} = \frac{1}{RC}$$

for $Q = 10^6$ and $\omega_0 \sim 2\pi$ 1.5 GHz
decay rate $\Gamma_1$ = 10 kHz corresponding to lifetime $T_1$ = 100 $\mu$s

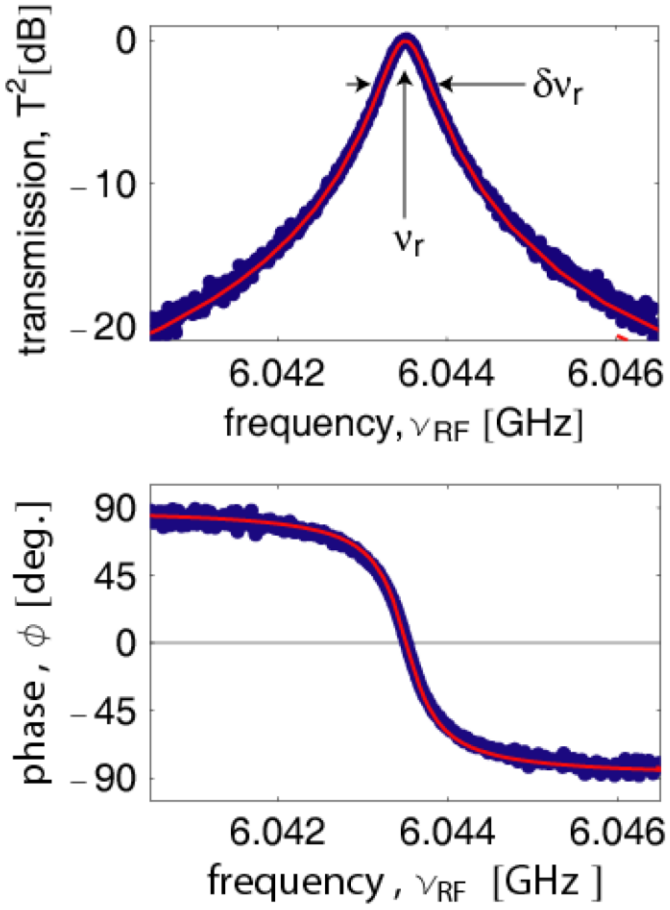
# Measure Transmission through a resonator




$$\frac{1}{Q} = \frac{1}{Q_L} = \frac{1}{Q_{int}} + \frac{1}{Q_{ext}}$$

$Q_{ext} \propto 1/\kappa_{ext}$ represents the effective controlled coupling of the resonator to the MW feedline

$Q_{int} \propto 1/\kappa_{int}$ represents the effective uncontrolled coupling of the resonator to the impurities or defects which can absorbe the resonator energy.




resonance frequency:

$$\nu_r = 6.04\,\mathrm{GHz}$$

quality factor:

$$Q = \frac{\nu_r}{\delta\nu_r} \approx 10^4$$

photon decay rate:

$$\frac{\kappa}{2\pi} = \frac{\nu_r}{Q} \approx 0.8\,\mathrm{MHz}$$

photon lifetime:

$$T_\kappa = 1/\kappa \approx 200\,\mathrm{ns}$$

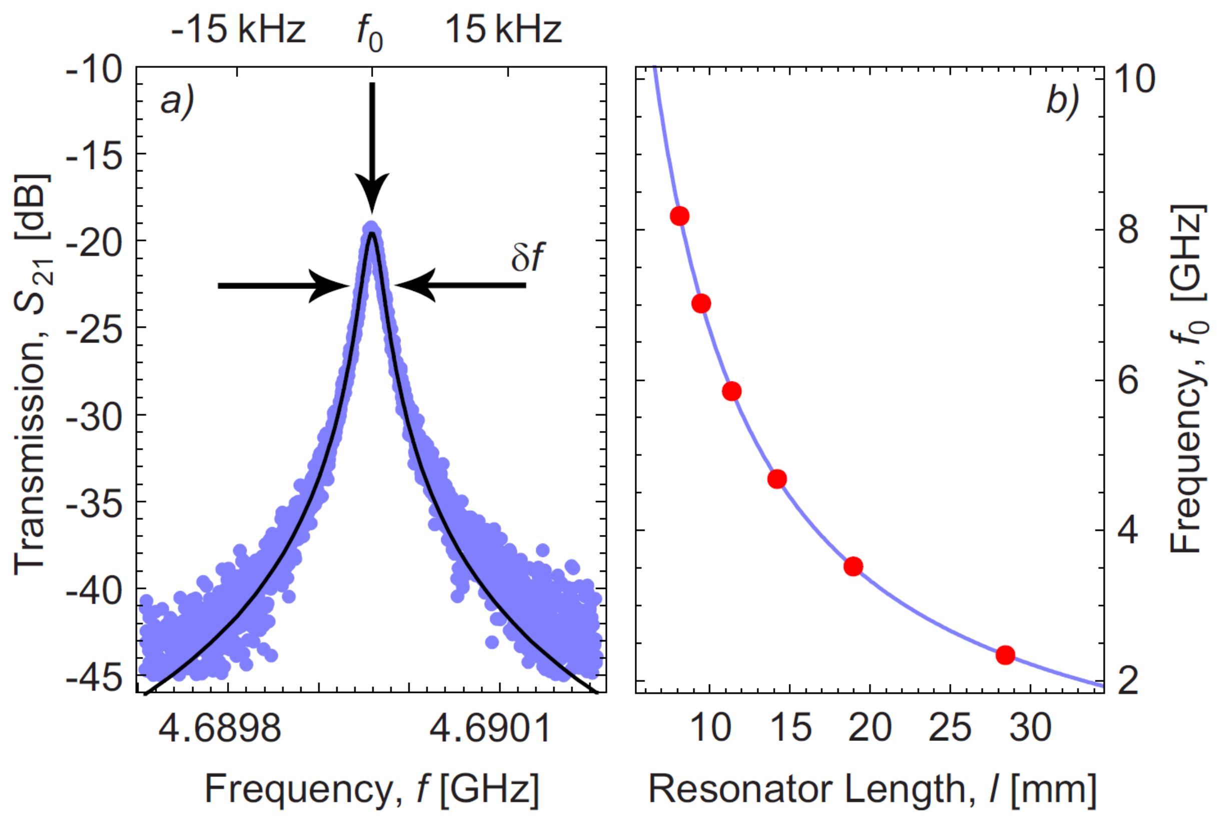
for Q = $10^6$ and $\omega_0$ ~ $2\pi$ 1.5 GHz
decay rate $\kappa$ = 10 kHz corresponding to lifetime $T_1$ = 100 µs

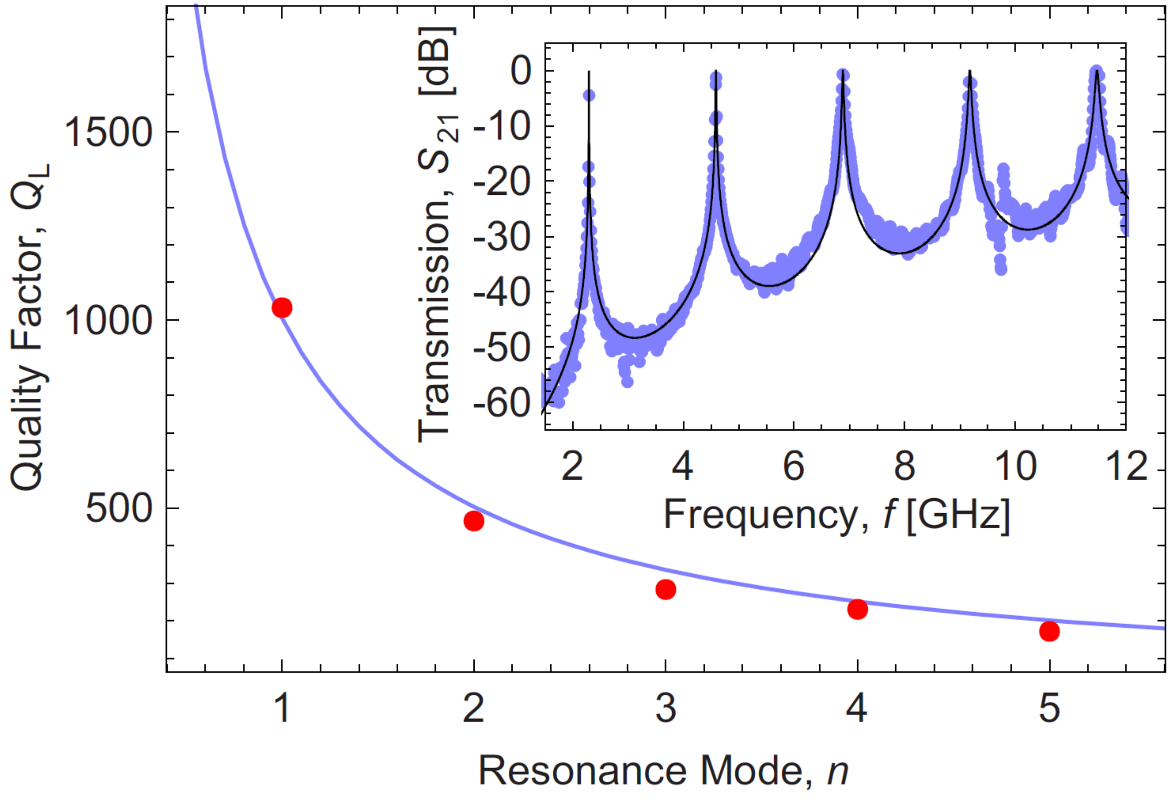
# Measure Transmission through a resonator

$$f_0 = \frac{c}{\sqrt{\epsilon_{\mathrm{eff}}}} \frac{1}{2l}.$$

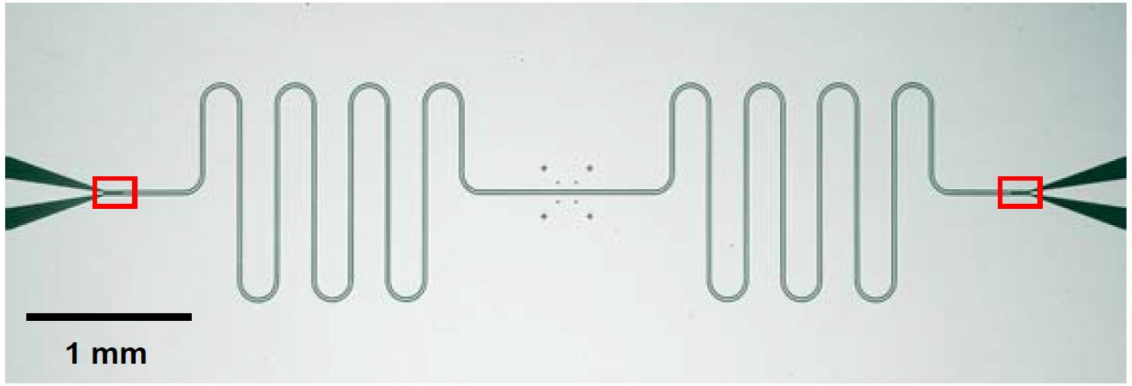
$$c/\sqrt{\epsilon_{\mathrm{eff}}} = v_{\mathrm{ph}}$$

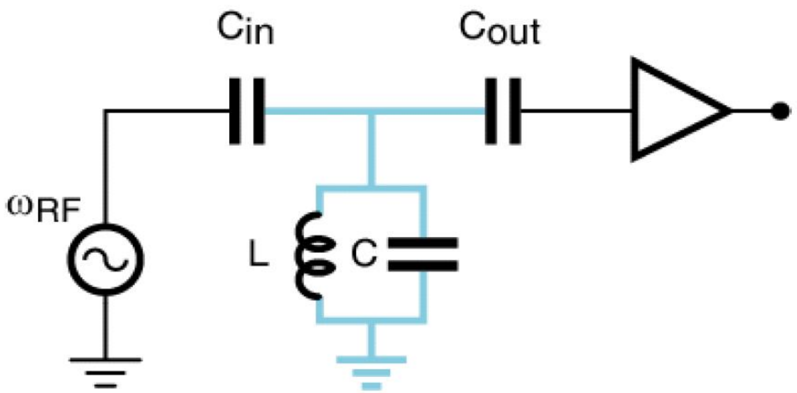
$$v_{\mathrm{ph}} = 1/\sqrt{L_\ell C_\ell}$$



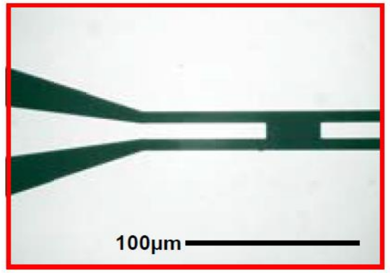



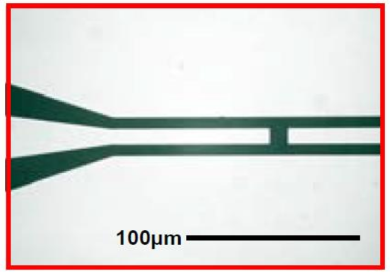
# Controlling the Photon Lifetime $\propto 1/Q_{ext}$

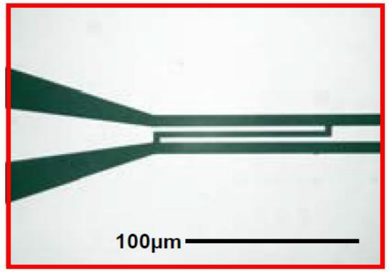


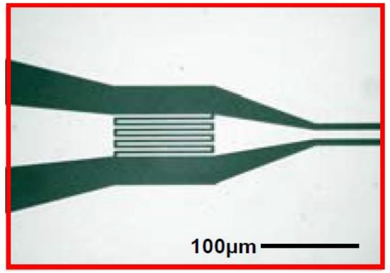




photon lifetime (quality factor)
controlled by coupling capacitor $C_{in/out}$



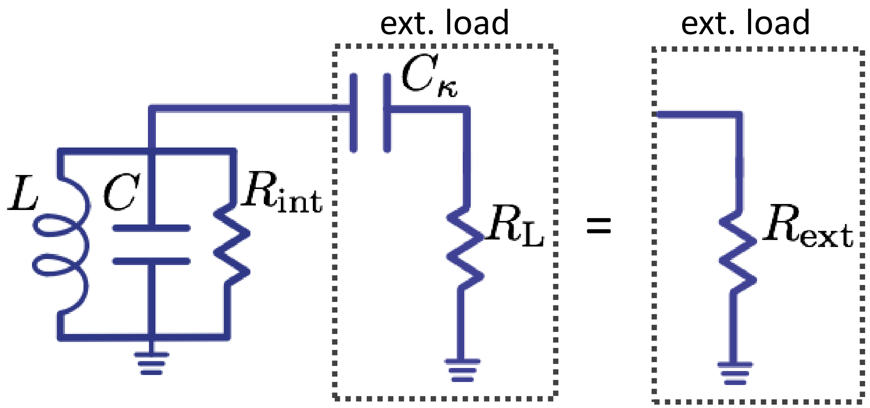



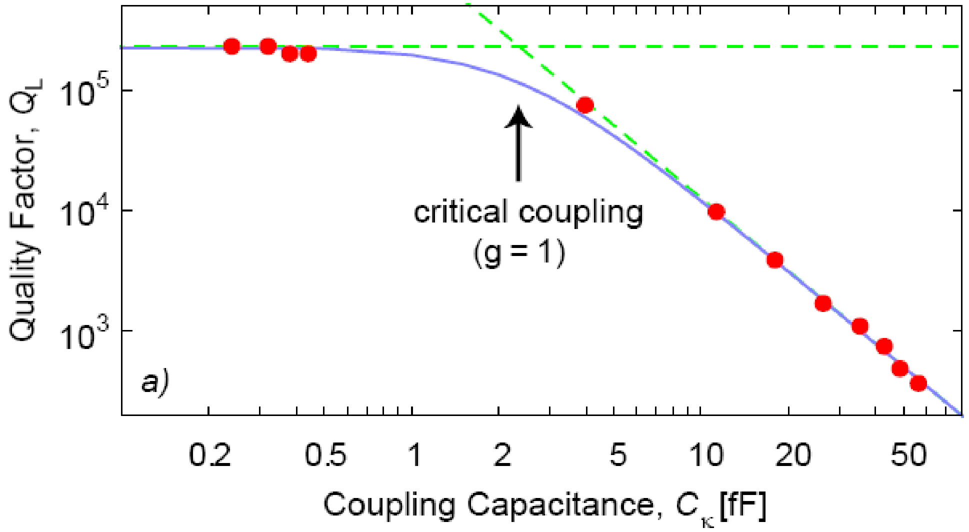



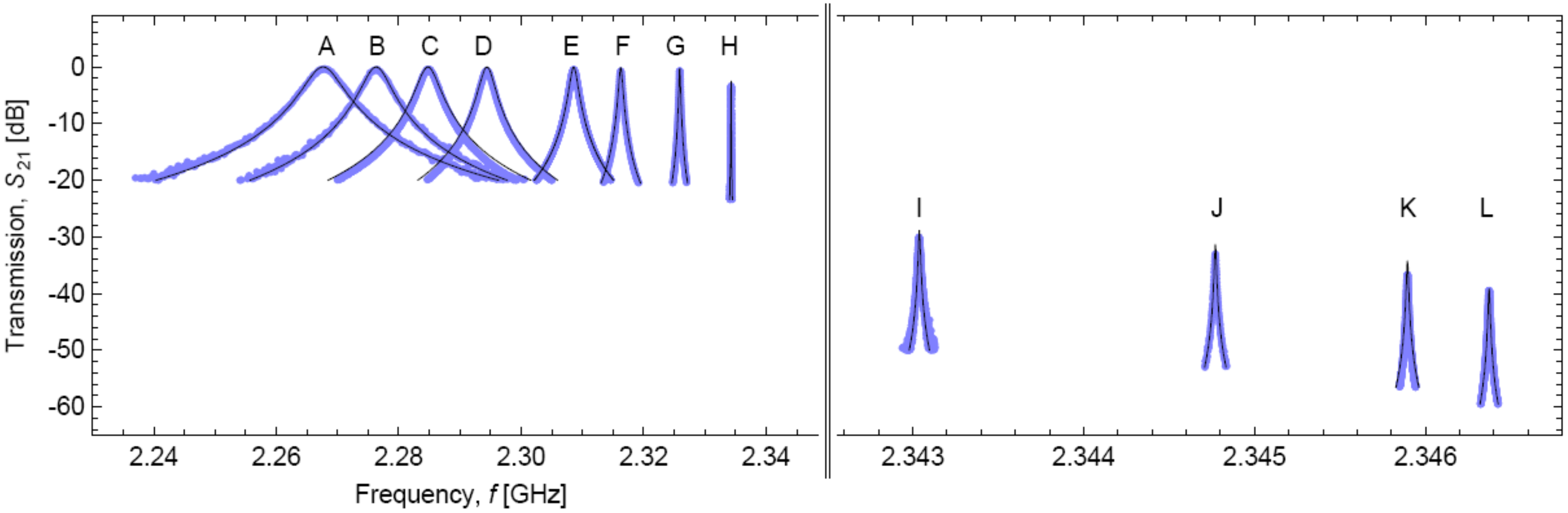



# Internal and External Dissipation in an LC Oscillator

M. Goeppl et al., *J. Appl. Phys.* **104**, 113904 (2008)



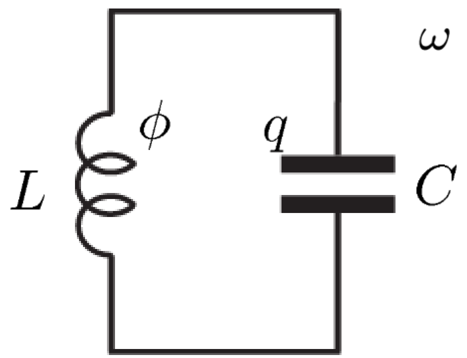




for $Q = 10^6$ and $\omega_0 \sim 2\pi$ 1.5 GHz

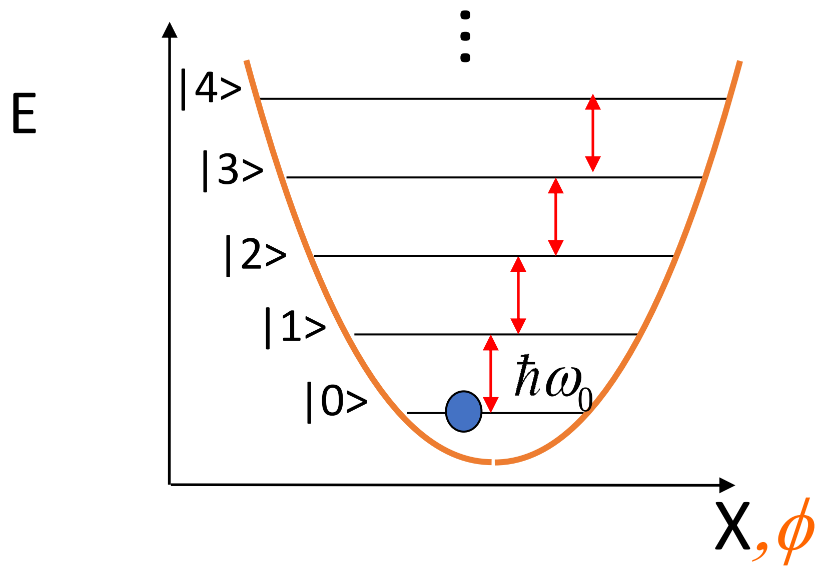
decay rate $\Gamma_1$ = 10 kHz corresponding to lifetime $T_1$ = 100 μs

# Harmonic oscillator as a qubit?

harmonic LC oscillator:




$$\omega = \frac{1}{\sqrt{LC}}$$



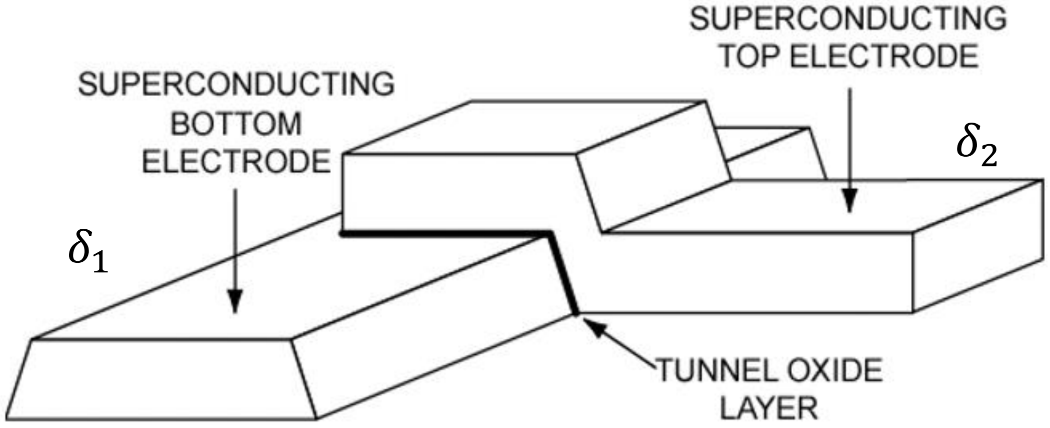

Even if we prepare the oscillator in its ground state we do not have an easy way neither to control its quantum states nor to measure.

The system is harmonic -> cannot define a qubit

Need to some nonlinear and non-dissipative quantum circuits for that!

# Next lecture

# The Josephson Junction




$\phi = \delta_2 - \delta_1$

$\phi_0 = \frac{h}{2e}$ is the flux quantum

**Nobel Prize in Physics 1973**

*"for his theoretical predictions of the properties of a supercurrent through a tunnel barrier, in particular those phenomena which are generally known as the Josephson effects".*

Non-linear current-phase relation
The potential energy is not harmonic
No dissipation

Josephson relations:

$$I = I_c \sin\phi$$

$$V = \frac{\Phi_0}{2\pi}\dot{\phi}$$

Josephson inductance

Specific Josephson inductance

$$V = \frac{\Phi_0}{2\pi I_0}\frac{1}{\cos\phi}\dot{I} = L_J\dot{I}$$

*Josephson energy*

Specific Josephson energy

$$E_J = \int VI\,dt = \frac{I_0\Phi_0}{2\pi}(1-\cos\phi)$$

$I_0$ = 100 nA corresponds to $L_{J0}$ ~ 3 nH

$I_0$ = 100 nA corresponds to $E_{J0}/h$ ~ 50 GHz

Inductive energy $U(\phi) = \frac{\phi^2}{2L}$ *for an harmonic oscillator*