Density Operator (Matrix)

1. Ensembles of quantum states

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

↳ pure state

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \xrightarrow{U} \sum_i p_i U|\psi_i\rangle\langle\psi_i|U^\dagger = U\rho U^\dagger$$

$$p(m|i) = \langle\psi_i| M_m^\dagger M_m |\psi_i\rangle = \mathrm{Tr}\left(M_m^\dagger M_m |\psi_i\rangle\langle\psi_i|\right)$$

↓ probability

$$p(m) = \sum_i p(m|i)\,p(i)$$

↙ probability

$$= \sum_i p(i)\,\mathrm{Tr}\left(M_M^\dagger M_M |\psi_i\rangle\langle\psi_i|\right)$$

$$p(m) = \mathrm{Tr}\left(M_m^\dagger M_m\, \rho\right)$$

Density operator after obtaining the measurement result m?

$$|\psi_i^m\rangle = \frac{M_m |\psi_i\rangle}{\sqrt{\langle\psi_i| M_m^\dagger M_m |\psi_i\rangle}}$$

$$\rho_m = \sum_i p(i|m)\, |\psi_i^m\rangle\langle\psi_i^m| = \sum_i p(i|m) \frac{M_m|\psi_i\rangle\langle\psi_i|M_m^\dagger}{\langle\psi_i|M_m^\dagger M_m|\psi_i\rangle}$$

$$p(i|m)\cdot p(m) = p(i,m) = p(m|i)\,p(i)$$

$$p(i|m) = \frac{p(m|i)\,p(i)}{p(m)}$$

$$\rho_m = \sum_i p(m|i) \frac{p(i)}{p(m)} \frac{M_m|\psi_i\rangle\langle\psi_i|M_m^\dagger}{\langle\psi_i|M_m^\dagger M_m|\psi_i\rangle} = \frac{M_m \rho M_m^\dagger}{Tr(M_m^\dagger M_m \rho)}$$

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