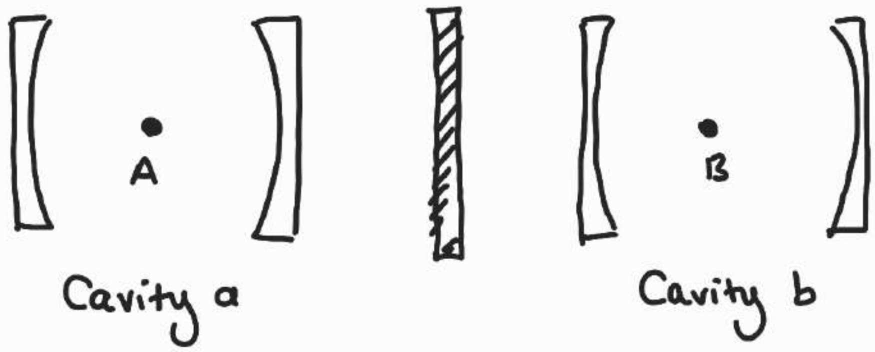
More Photon Cases:



$$H_{tot} = \frac{\omega_0}{2}\sigma_z^A + g\left(a^\dagger \sigma_-^A + \sigma_+^A a\right) + \omega a^\dagger a +$$
$$+ \frac{\omega_0}{2}\sigma_z^B + g\left(b^\dagger \sigma_-^B + \sigma_+^B b\right) + \omega b^\dagger b \quad \text{where } \hbar = 1$$

ω_0: frequency of atoms

ω: frequency of cavities

$$H_{JC}|\Psi_n^\pm\rangle = \lambda_n^\pm |\Psi_n^\pm\rangle$$

$$\lambda_n^\pm = n\omega + \frac{1}{2}\left(\Delta \pm \sqrt{\Delta^2 + G_n^2}\right)$$

where $\Delta = \omega - \omega_0$ & $G_n = 2g\sqrt{n}$

$$|\Psi_n\rangle = |g,n\rangle$$

$$|\Psi_n^+\rangle = c_n|e,n-1\rangle + s_n|g,n\rangle$$

$$|\Psi_n^-\rangle = -s_n|e,n-1\rangle + c_n|g,n\rangle \qquad \sin(\theta_n) = \frac{G_n}{\sqrt{\Delta^2 + G_n^2}}$$

$$c_n = \cos\left(\frac{\theta_n}{2}\right), \quad s_n = \sin\left(\frac{\theta_n}{2}\right), \quad \cos(\theta_n) = \frac{\Delta}{\sqrt{\Delta^2 + G_n^2}}$$

A. Partially entangled Bell states $|\Phi_{AB}\rangle$

$$|\Phi_{AB}\rangle = \cos\alpha\,|e_A, e_B\rangle + \sin\alpha\,|g_A, g_B\rangle$$

$$|\Phi(0)\rangle = |\Phi_{AB}\rangle \otimes |n_a, n_b\rangle$$

$$|\Phi(0)\rangle = \cos\alpha\,|\underset{A}{\uparrow}\underset{B}{\uparrow}\underset{a}{n_a}\underset{b}{n_b}\rangle + \sin\alpha\,|\downarrow\downarrow n_a n_b\rangle$$

Dressed eigenstates:

$$|e_A, (n_a-1)\rangle = c\,|\Psi_{n_a}^{+}\rangle - s\,|\Psi_{n_a}^{-}\rangle$$

$$|g_A, n_a\rangle = s\,|\Psi_{n_a}^{+}\rangle + c\,|\Psi_{n_a}^{-}\rangle$$

$$|g_A, (n_a-1)\rangle = |\Psi_{(n_a-1)}\rangle$$

$$|\Phi(0)\rangle = \cos\alpha\,|e_A, n_a-1\rangle \otimes |e_B, n_b-1\rangle + \sin\alpha\,|g_A, (n_a-1)\rangle \otimes |g_B, (n_b-1)\rangle$$

$$|\Phi(0)\rangle = \cos\alpha\left(c\,|\Psi_{n_a}^{+}\rangle_A - s\,|\Psi_{n_a}^{-}\rangle_A\right) \otimes \left(c\,|\Psi_{n_b}^{+}\rangle_B - s\,|\Psi_{n_b}^{-}\rangle_B\right)$$

$$+ \sin\alpha\,|\Psi_{(n_a-1)}\rangle \otimes |\Psi_{(n_b-1)}\rangle$$

$$|\psi^{\pm}(t)\rangle = e^{-i\lambda^{\pm}t}|\psi^{\pm}(0)\rangle$$

$$|\Phi(t)\rangle = \cos\alpha\left(c\,e^{-i\lambda^+t}|\psi_{n_a}^+\rangle_A - s\,e^{-i\lambda^-t}|\psi_{n_a}^-\rangle_A\right)\otimes\left(c\,e^{-i\lambda^+t}|\psi_{n_b}^+\rangle_B - s\,e^{-i\lambda^-t}|\psi_{n_b}^-\rangle_B\right)$$

$$+\sin\alpha\,|\psi_{(n_a-1)}\rangle\otimes|\psi_{(n_b-1)}\rangle$$

$\Big\}$ back to bare bases e, g

$$|\Phi(t)\rangle = \cos\alpha\Big[c_n e^{-i\lambda^+t}\underbrace{\left(c_n|e_A, n_A-1\rangle + s_n|g_A, n_A\rangle\right)}_{|\psi_{n_a}^+\rangle_A} -$$

$$-s_n e^{-i\lambda^-t}\underbrace{\left(-s_n|e_A, n_A-1\rangle + c_n|g_A, n_A\rangle\right)}_{|\psi_{n_a}^-\rangle_A}\Big]\otimes$$

$$\Big[c_n e^{-i\lambda^+t}\underbrace{\left(c_n|e_B, n_B-1\rangle + s_n|g_B, n_B\rangle\right)}_{|\psi_{n_b}^+\rangle_B} -$$

$$-s_n e^{-i\lambda^-t}\underbrace{\left(-s_n|e_B, n_B-1\rangle + c_n|g_B, n_B\rangle\right)}_{|\psi_{n_b}^-\rangle_B}\Big]$$

$$+\sin\alpha\,|g_A, n_a-1\rangle\otimes|g_B, n_b-1\rangle$$

$$|\Phi(t)\rangle = \cos\alpha\Big[\left(c_n^2 e^{-i\lambda^+t} + s_n^2 e^{-i\lambda^-t}\right)|e_A, n_a-1\rangle$$

$$+\left(c_n s_n e^{-i\lambda^+t} - c_n s_n e^{-i\lambda^-t}\right)|g_A, n_a\rangle\Big]\otimes$$

$$\Big[\left(c_n^2 e^{-i\lambda^+t} + s_n^2 e^{-i\lambda^-t}\right)|e_B, n_b-1\rangle + \left(c_n s_n e^{-i\lambda^+t} - c_n s_n e^{-i\lambda^-t}\right)|g_B, n_b\rangle\Big]$$

$$+\sin\alpha\,|g_A, n_a-1\rangle\otimes|g_B, n_b-1\rangle$$

$$\boxed{\begin{aligned}|\Phi(t)\rangle = {} & x_1 |\uparrow\uparrow (n_a-1)(n_b-1)\rangle + x_2 |\downarrow\downarrow n_a n_b\rangle + x_3 |\uparrow\downarrow (n_a-1) n_b\rangle \\ & + x_4 |\downarrow\uparrow n_a (n_b-1)\rangle + x_5 |\downarrow\downarrow (n_a-1)(n_b-1)\rangle\end{aligned}}$$

$$x_1 = \big(\overbrace{c_n^2}^{L} e^{-i\lambda^+ t} + \overbrace{s_n^2}^{M} e^{-i\lambda^- t}\big)^2$$

$$x_2 = \big(\underbrace{c_n s_n}_{\sqrt{LM}} e^{-i\lambda^+ t} - \underbrace{c_n s_n}_{\sqrt{LM}} e^{-i\lambda^- t}\big)^2$$

$$x_3 = \big(\underbrace{c_n^2}_{L} e^{-i\lambda^+ t} + \underbrace{s_n^2}_{M} e^{-i\lambda^- t}\big)\big(\underbrace{c_n s_n}_{\sqrt{LM}} e^{-i\lambda^+ t} - \underbrace{c_n s_n}_{\sqrt{LM}} e^{-i\lambda^- t}\big)$$

$$x_4 = \big(\underbrace{c_n s_n}_{N=\sqrt{LM}} e^{-i\lambda^+ t} - \underbrace{c_n s_n}_{\sqrt{LM}} e^{-i\lambda^- t}\big)\big(\underbrace{c_n^2}_{L} e^{-i\lambda^+ t} + \underbrace{s_n^2}_{M} e^{-i\lambda^- t}\big)$$

$$x_5 = \sin\alpha$$

A.1 $C_{AB}(t)$

Similar to one photon case $C^{AB}(t) = 2|x_1||x_5| - 2|x_3||x_4|$

$$|x_1| = \left|\big(L e^{-i\lambda_+ t} + M e^{-i\lambda_- t}\big)^2 \cos\alpha\right| \qquad |x_5| = |\sin\alpha|$$

$$|x_3| = \left|\big(L e^{-i\lambda_+ t} + M e^{-i\lambda_- t}\big)\big(e^{-i\lambda_+ t} N - e^{-i\lambda_- t} N\big)\cos\alpha\right|$$

$$|x_3| = |x_4|$$

For resonance case: $(\Delta = 0)$

$$L = M = N = \frac{1}{2}$$

$$\lambda^{\pm} = \vartheta + \cancel{\frac{\Delta}{2}} \pm \frac{\sqrt{\cancel{\Delta^2} + G_n^2}}{2} = \vartheta \pm \frac{G_n}{2}$$

$$|x_1| = \left| \left(\frac{1}{2} e^{-i\lambda^+ t} + \frac{1}{2} e^{-i\lambda_- t} \right)^2 . \cos\alpha \right|$$

$$= \left| \frac{1}{4} \left(e^{-i\lambda^+ t} + e^{-i\lambda^- t} \right)^2 \cos\alpha \right| = \frac{1}{4} |\cos\alpha| \left| \left(e^{-i\lambda^+ t} + e^{-i\lambda^- t} \right) \right|^2$$

$$= \frac{1}{4} |\cos\alpha| \left| \left(\cancel{e^{-i\vartheta t}} . e^{-i\frac{G_n t}{2}} + \cancel{e^{-i\vartheta t}} e^{+i\frac{G_n t}{2}} \right)^2 \right|$$

$$= \frac{1}{4} |\cos\alpha| \left| \left(2\cos\left(\frac{G_n}{2} t\right) \right)^2 \right| = |\cos\alpha| \cos^2\left(\frac{G_n t}{2}\right)$$

$$\rightarrow |x_1||x_5| = |\cos\alpha| |\sin\alpha| \cos^2\left(\frac{G_n t}{2}\right)$$

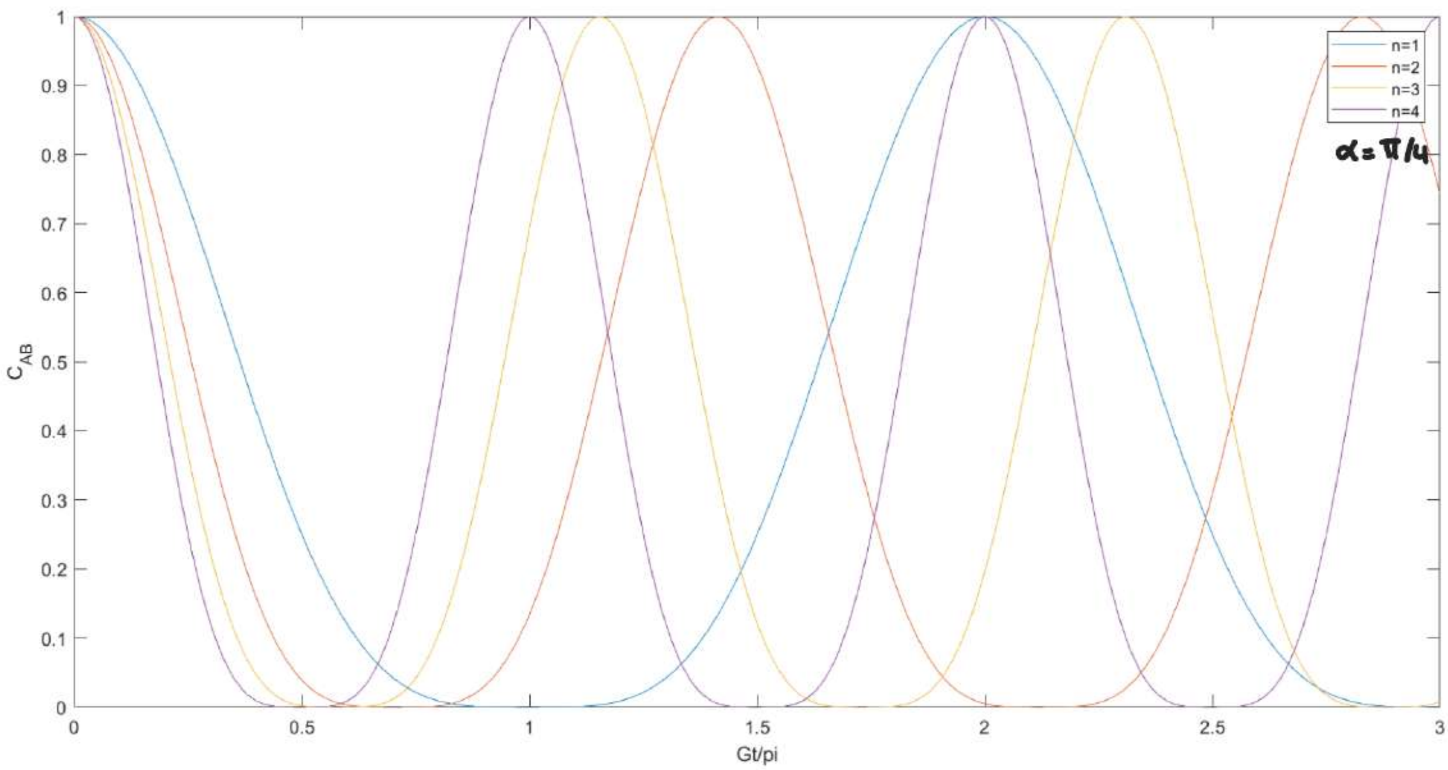
$$|x_3| = \left| \cos\left(\frac{G_n t}{2}\right) \right| \left| \sin\left(\frac{G_n t}{2}\right) \right| |\cos\alpha| = |x_4|$$

$$\rightarrow |x_3| |x_4| = \cos^2\left(\frac{G_n t}{2}\right) \sin^2\left(\frac{G_n t}{2}\right) \cos^2\alpha$$

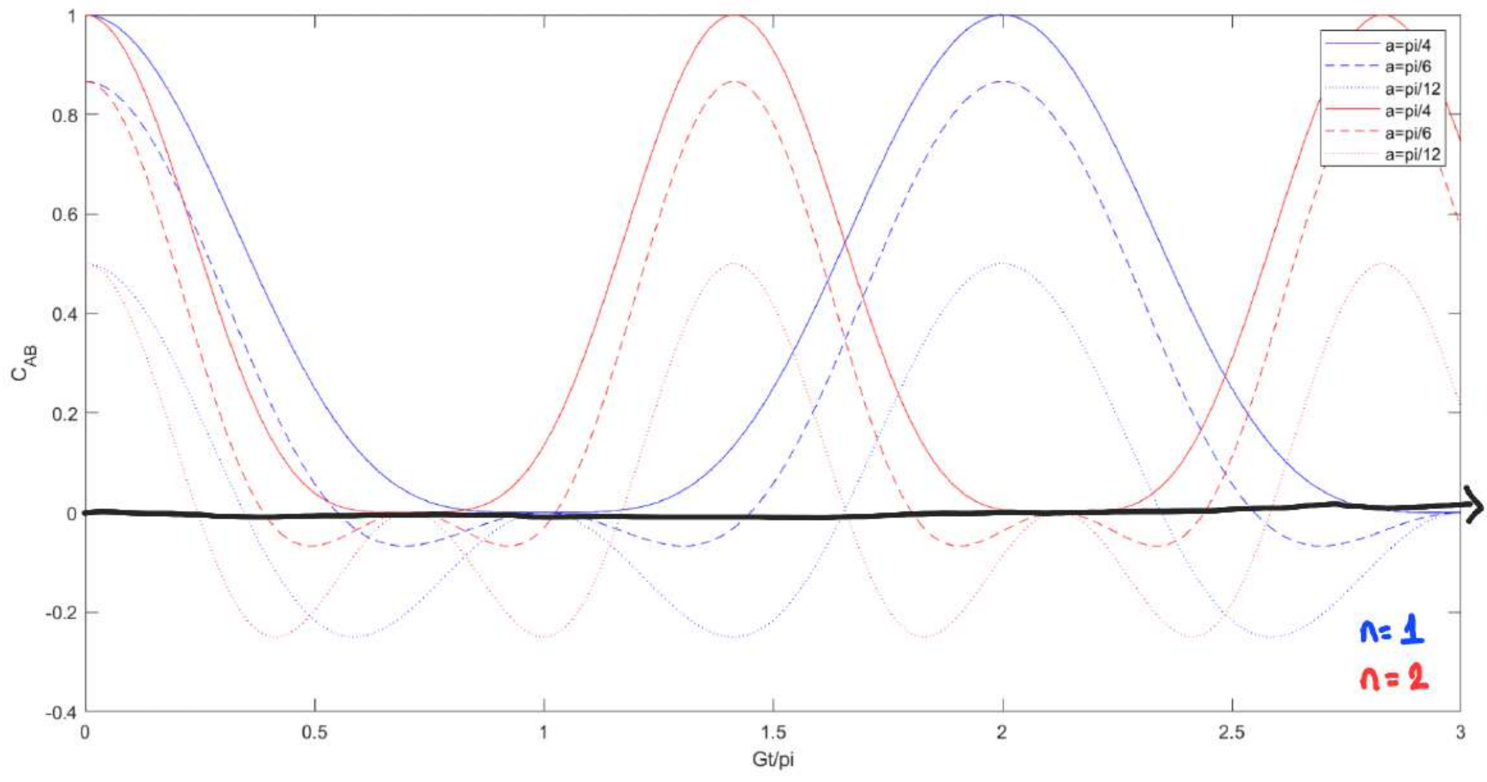
$$Q(t) = 2 \left[|x_1||x_5| - |x_3||x_4| \right]$$

$$= \cos^2\left(\frac{G_n t}{2}\right) \left[|\sin 2\alpha| - 2\sin^2\left(\frac{G_n t}{2}\right) \cos^2\alpha \right]$$

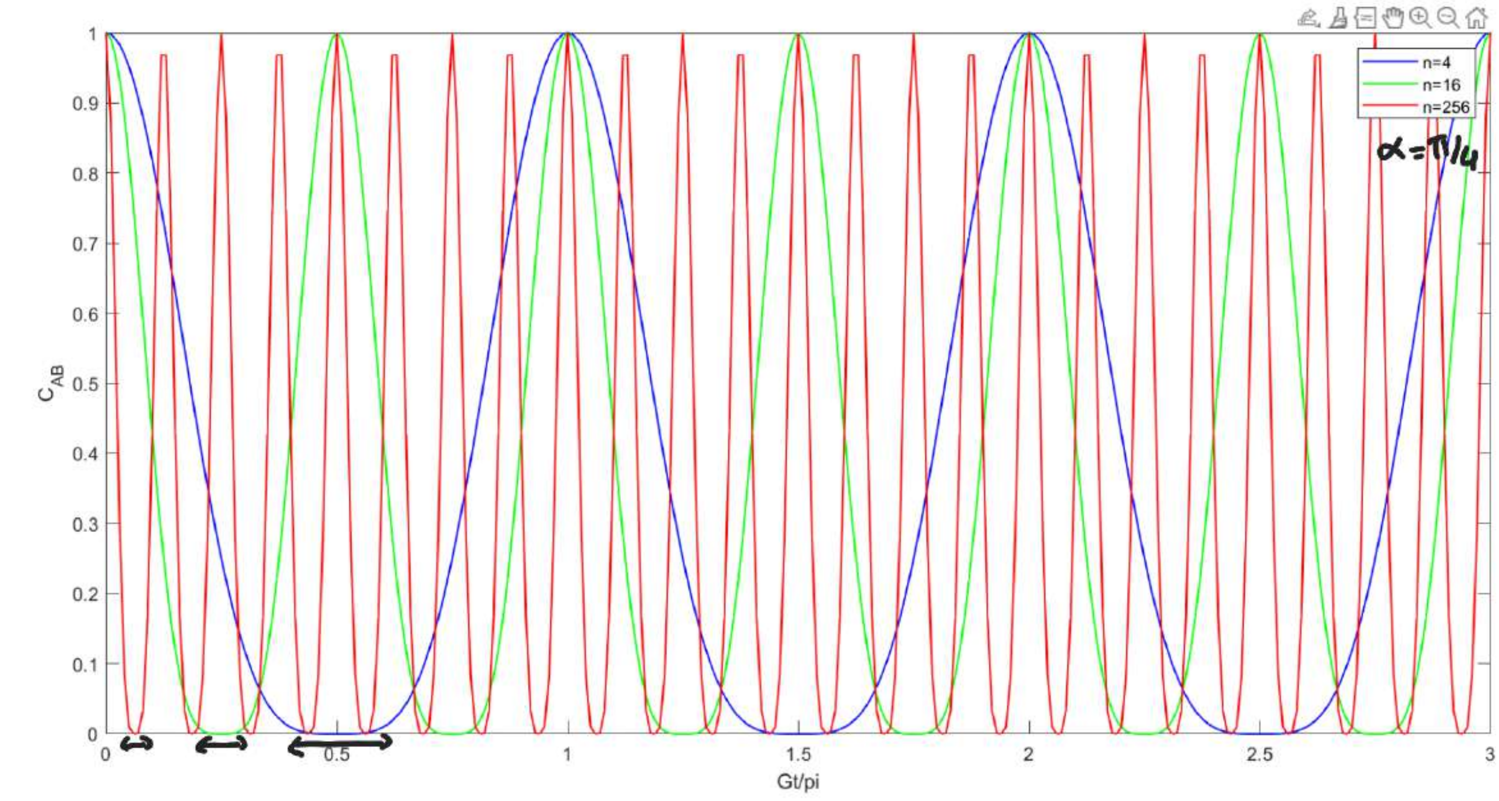
where $G_n = 2g\sqrt{n}$



n=1
n=2
n=3
n=4
α=π/4
C_AB
Gt/pi



a=pi/4
a=pi/6
a=pi/12
a=pi/4
a=pi/6
a=pi/12
n=1
n=2
C_AB
Gt/pi

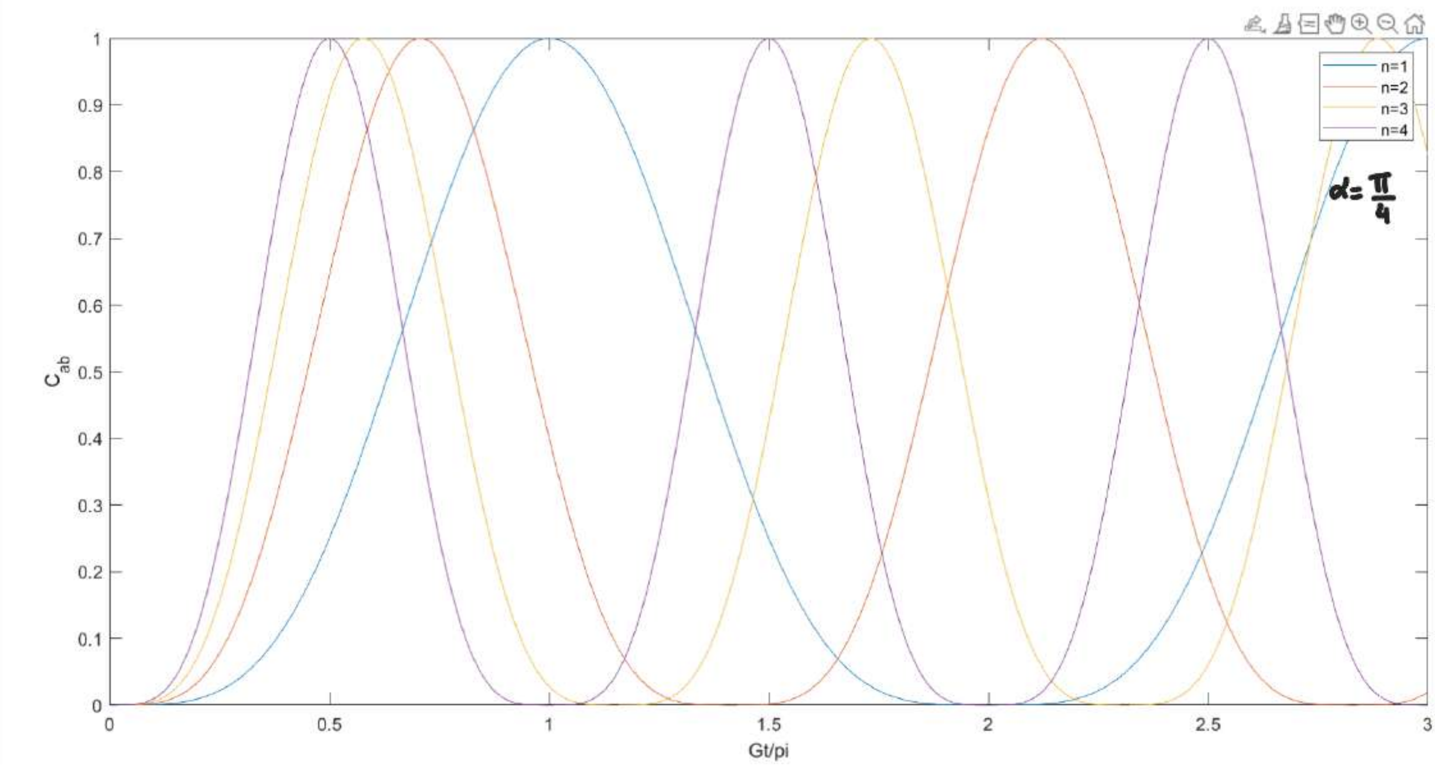


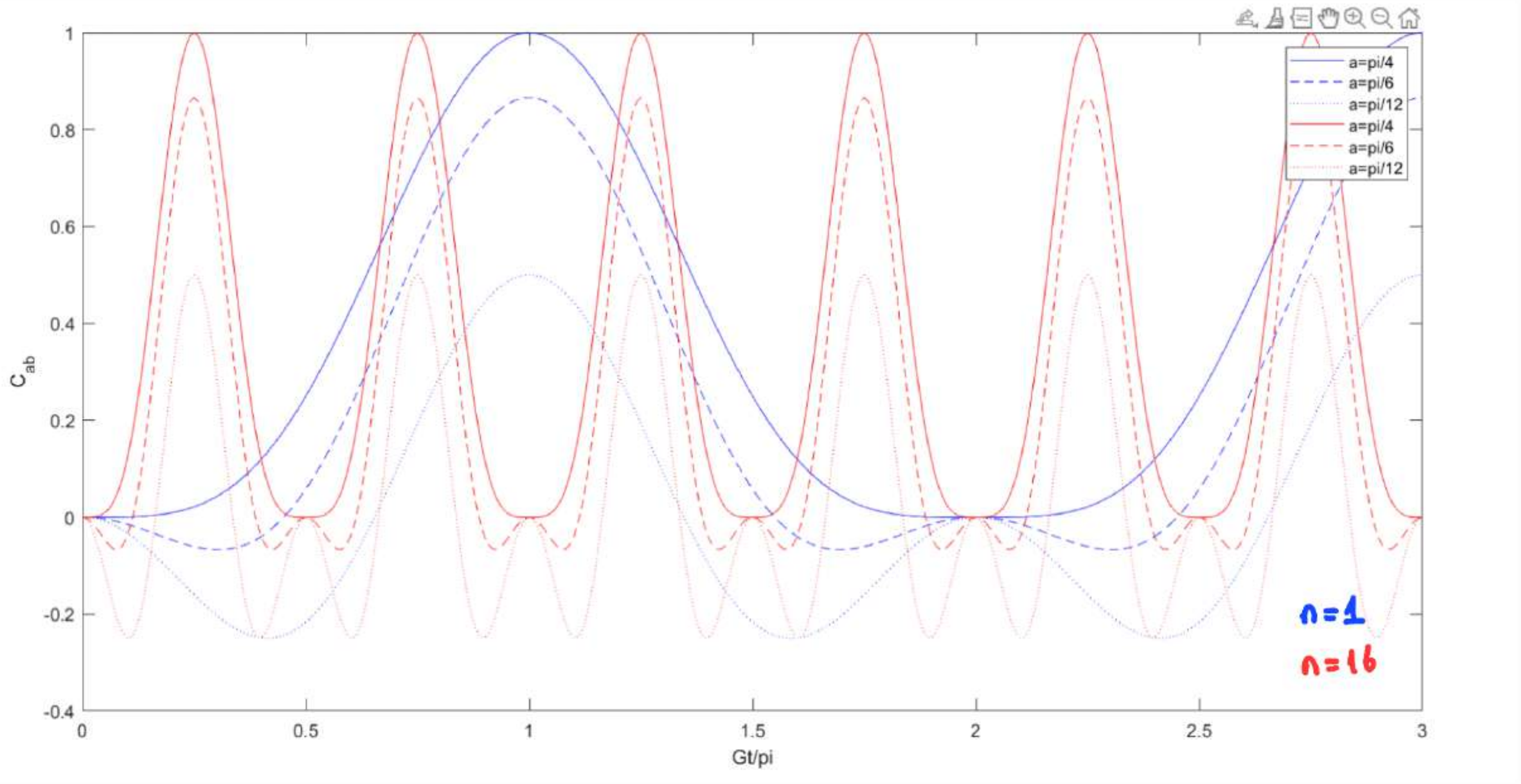
As photon number increases, ESD time decreases

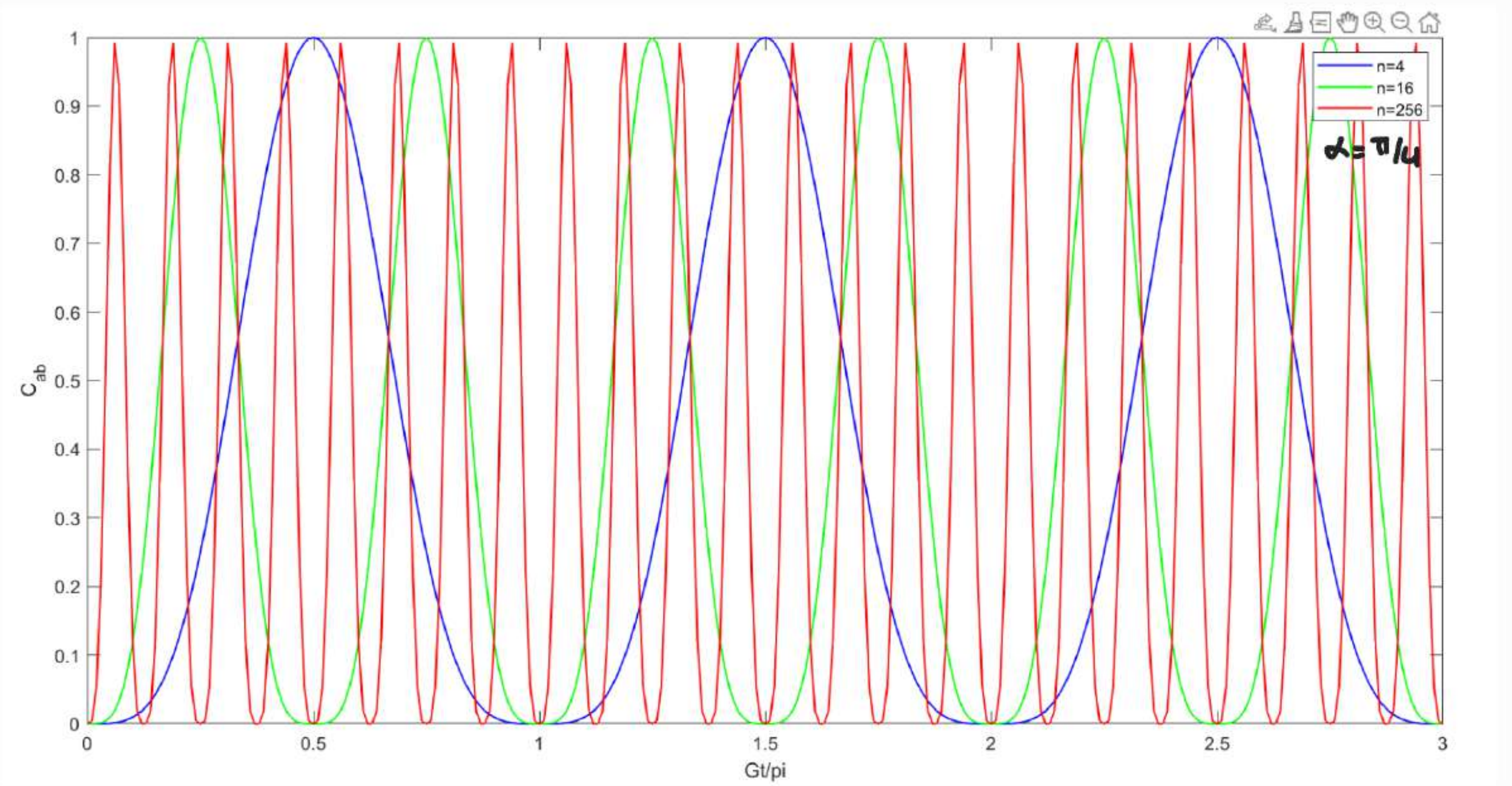
A.2 $C_{ab}(t)$

Using the previous results belonging to 1 photon case.

$$C_{ab}(t) = \sin^2\left(\frac{G_n t}{2}\right)\left[\,|\sin 2\alpha| - 2\cos^2\left(\frac{G_n t}{2}\right)\cos^2\alpha\right]$$

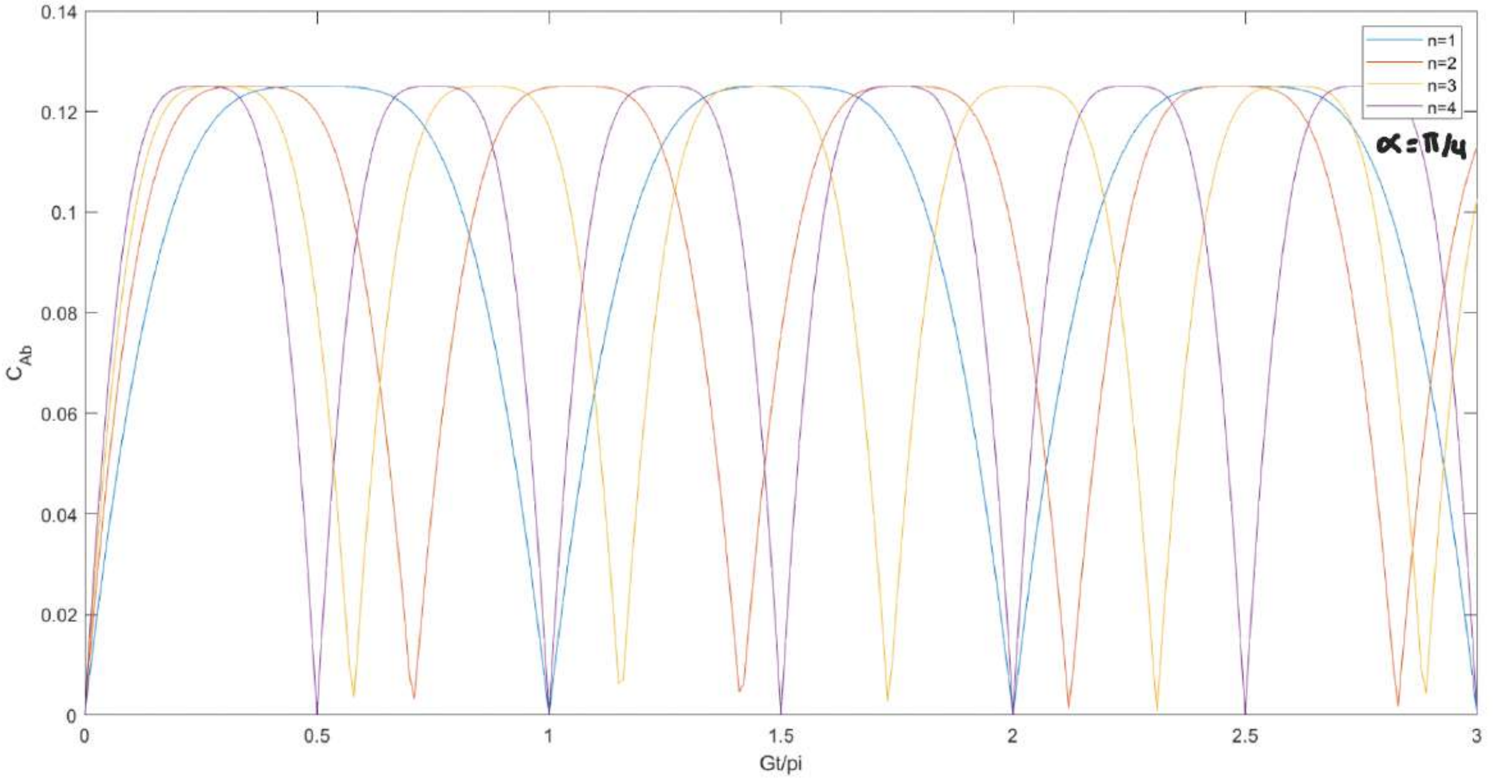




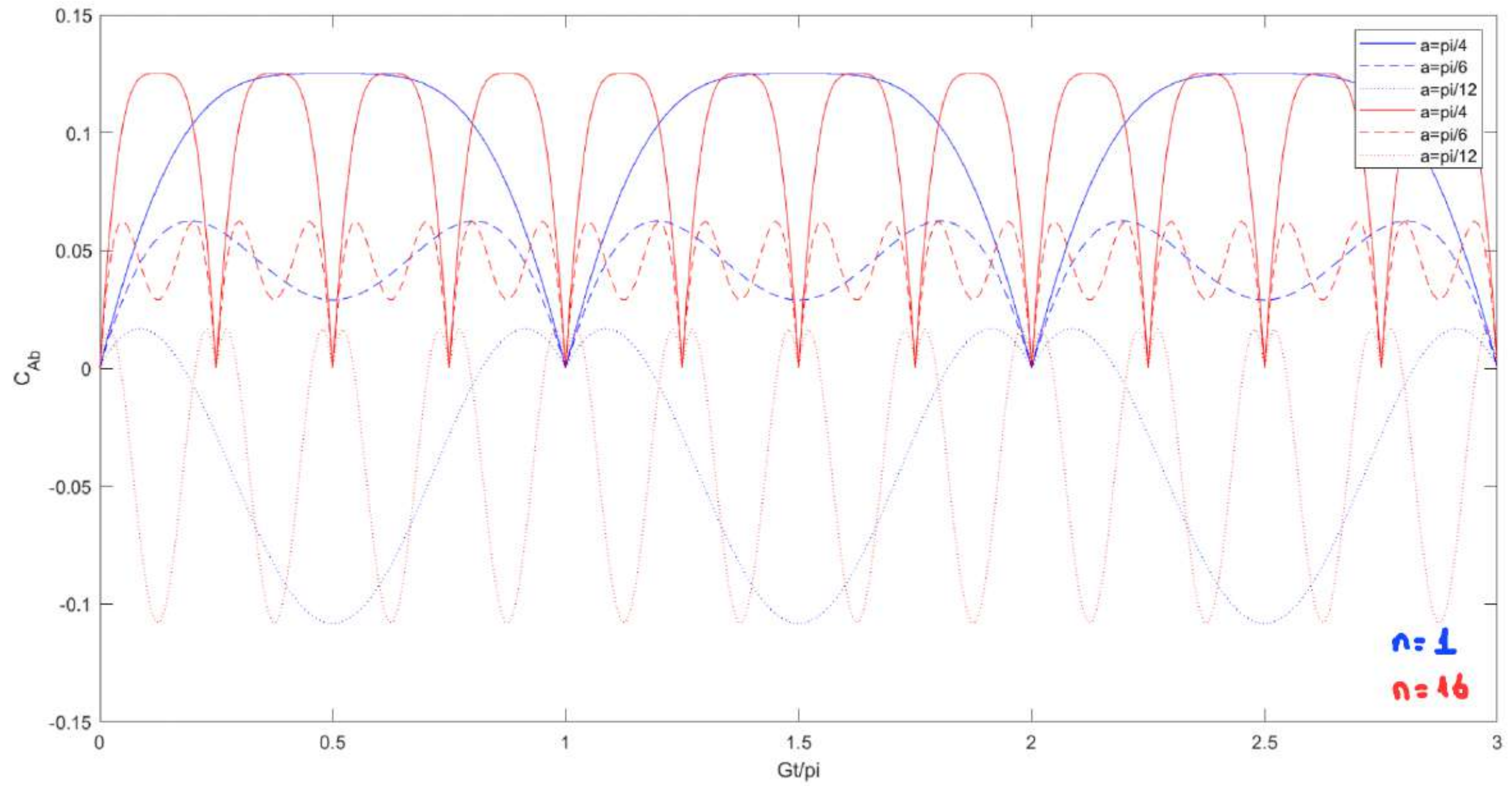


A.3 $C_{Ab}(t)$

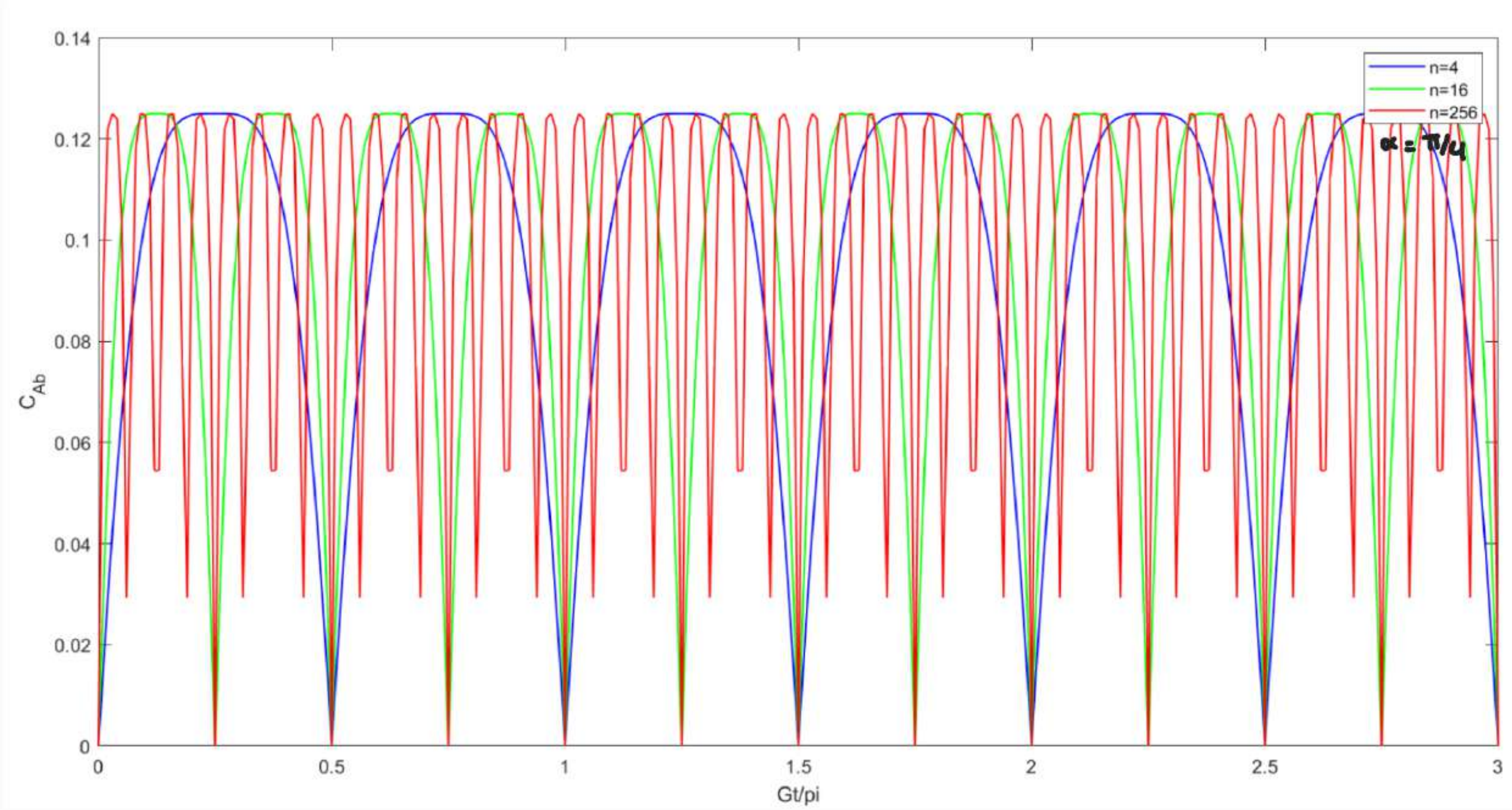
$$C_{Ab}(t) = \frac{1}{4}\cos^2\alpha \left|\sin(G_n t)\right| \left(2\left|\tan\alpha\right| - \left|\sin(G_n t)\right| \right)$$



n=1
n=2
n=3
n=4
α=π/4
C_{Ab}
Gt/pi



a=pi/4
a=pi/6
a=pi/12
a=pi/4
a=pi/6
a=pi/12
n=1
n=16
C_{Ab}
Gt/pi

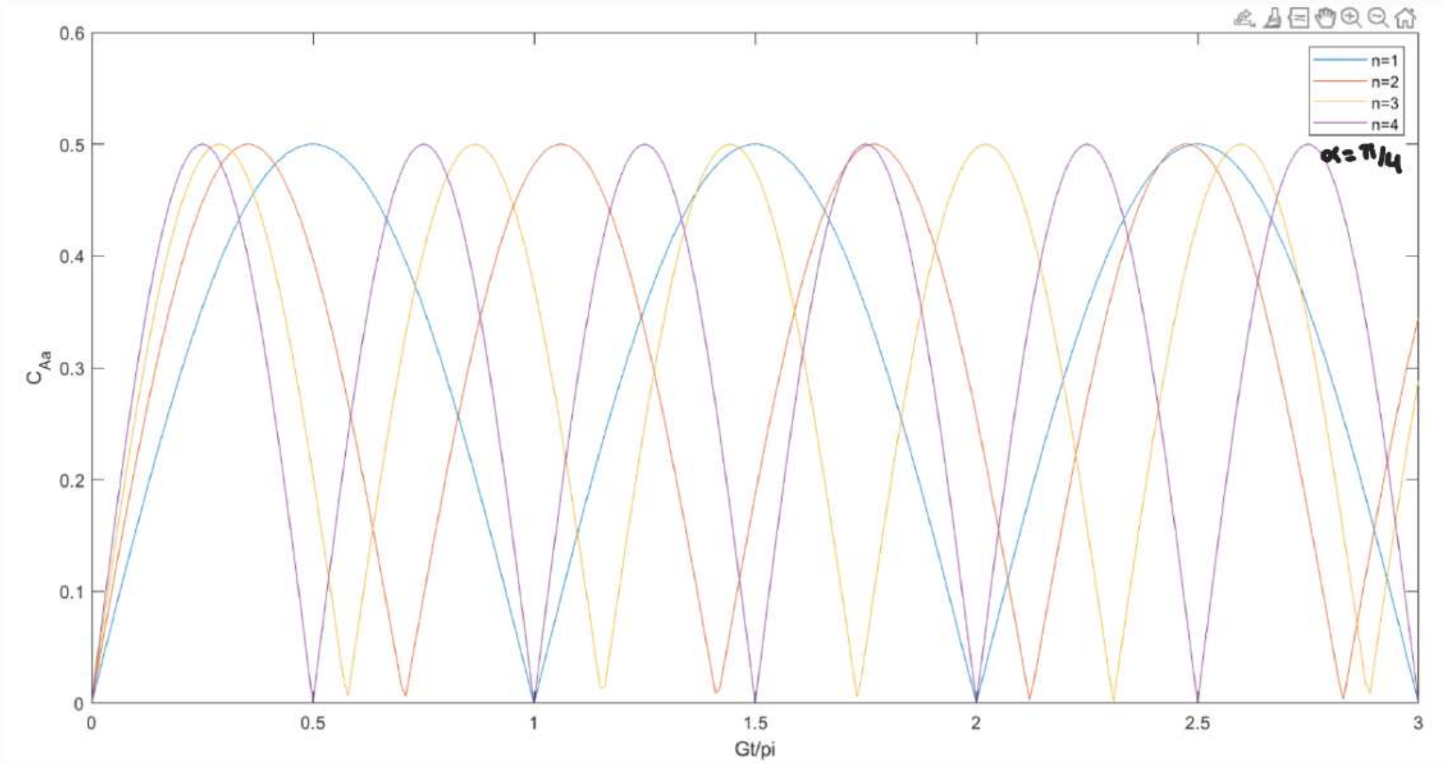


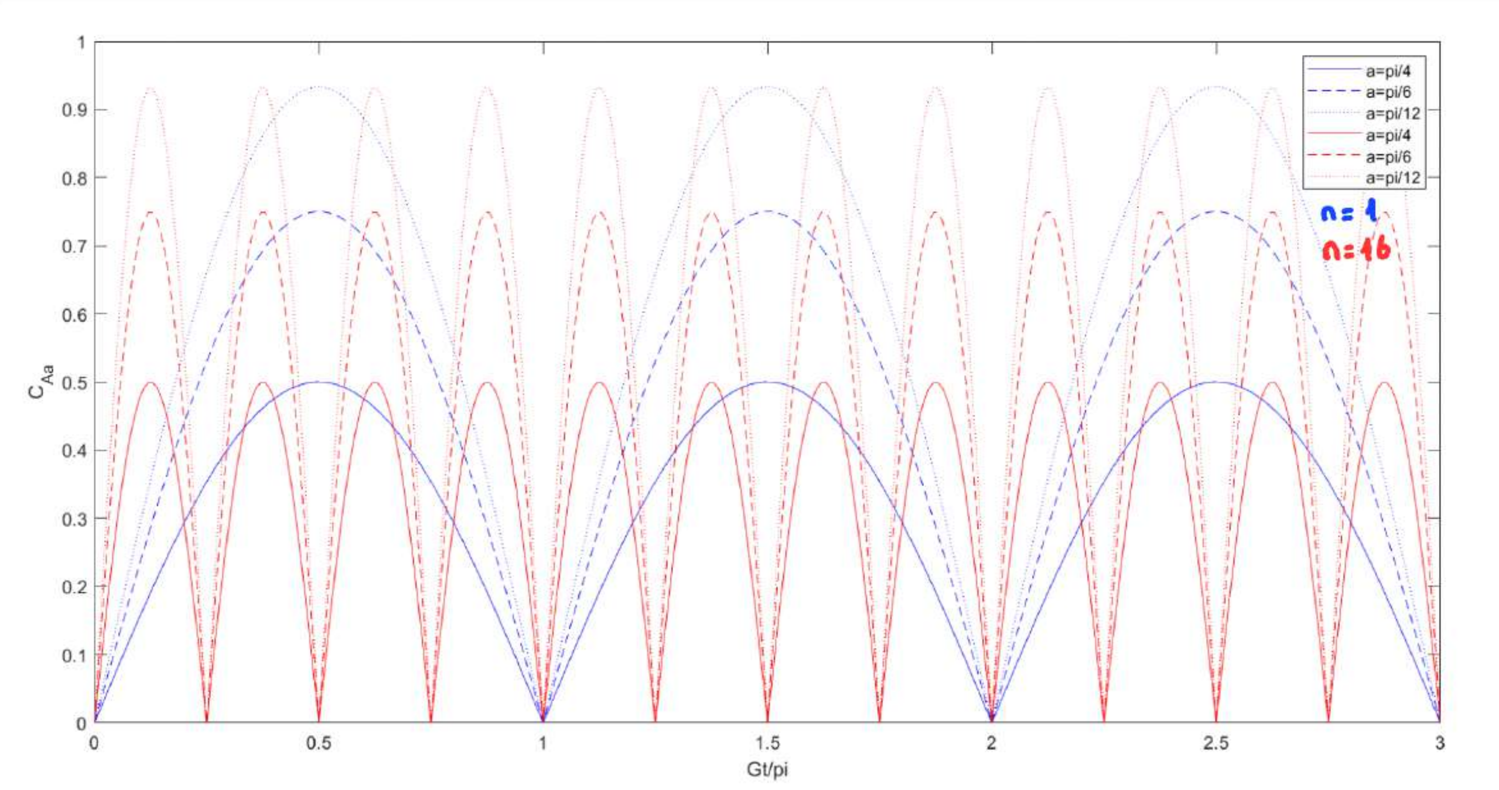
A.4 $C_{aB}(t)$

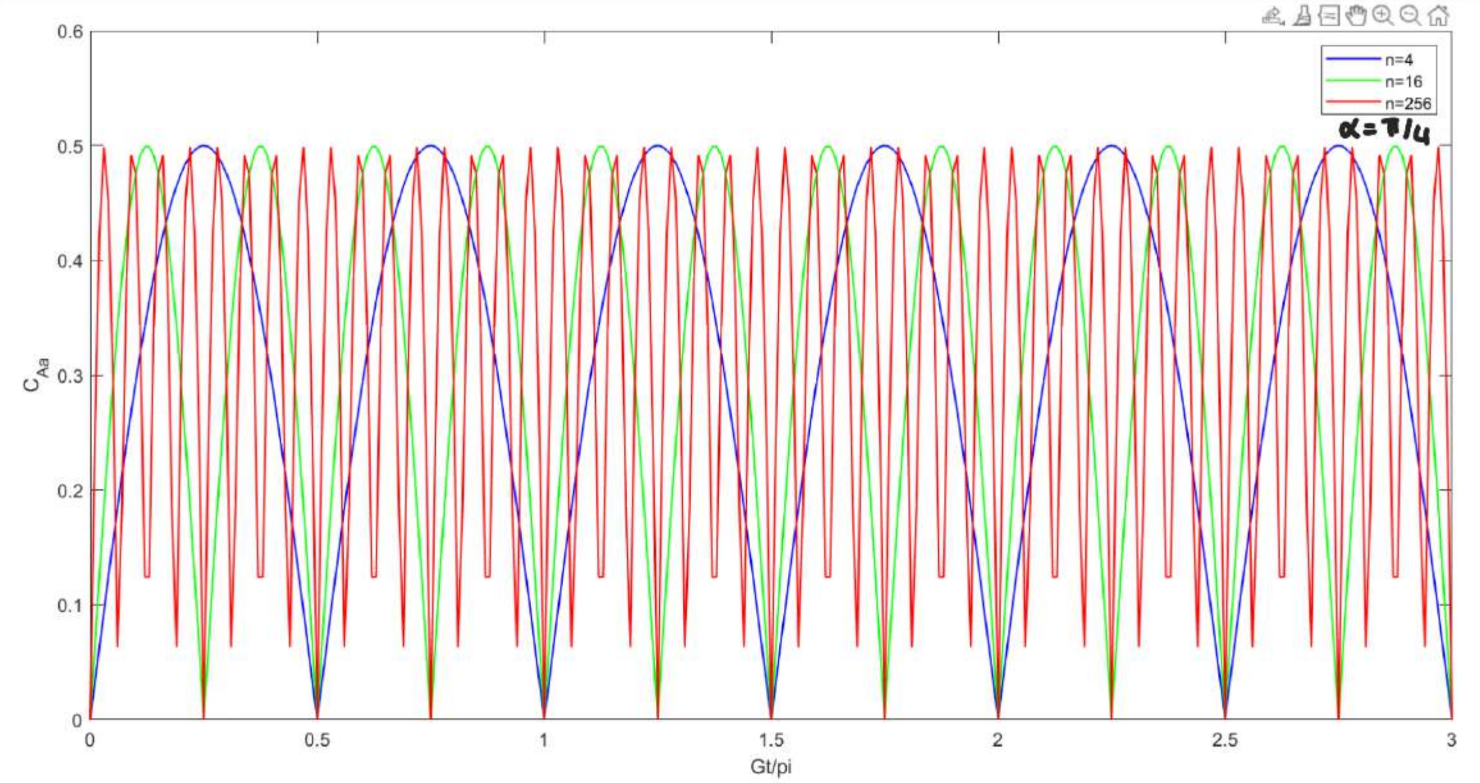
From the symmetry $C_{Ab}(t) = C_{aB}(t)$

A.5 $C_{Aa}(t)$

$$C_{Aa}(t) = |\sin(G_n t)| \cos^2\alpha$$







A.6 $C_{Bb}(t)$

From the symmetry, $C_{Aa}(t) = C_{Bb}(t)$

B. Partially Entangled Bell States $|\Psi_{AB}\rangle$

$$|\Psi(0)\rangle = |\Psi_{AB}\rangle \otimes |n_a, n_b\rangle$$

$$|\Psi(0)\rangle = \left(\cos\alpha\, |e_A, g_B\rangle + \sin\alpha\, |g_A, e_B\rangle\right) \otimes |n_a, n_b\rangle$$

Using previous relations:

$$\boxed{\begin{aligned} |\Psi(t)\rangle = \; & x_1 \,|\uparrow\downarrow (n_a-1)(n_b-1)\rangle + x_2\, |\downarrow\uparrow (n_a-1)(n_b-1)\rangle \\ & + x_3\, |\downarrow\downarrow n_a (n_b-1)\rangle + x_4\, |\downarrow\downarrow (n_a-1)\, n_b\rangle \end{aligned}}$$

$$x_1 = \cos\alpha \left(L e^{-i\lambda^+ t} + M e^{-i\lambda^- t} \right) \qquad L = \frac{1}{2}\left(1 + \frac{\Delta}{\sqrt{\Delta^2+G^2}}\right) = \cos^2\left(\frac{\theta}{2}\right)$$

$$x_2 = \sin\alpha \left(L e^{-i\lambda^+ t} + M e^{-i\lambda^- t} \right)$$

$$M = \frac{1}{2}\left(1 - \frac{\Delta}{\sqrt{\Delta^2+G^2}}\right) = \sin^2\left(\frac{\theta}{2}\right)$$

$$x_3 = \cos\alpha \left(N e^{-i\lambda^+ t} - N e^{-i\lambda^- t} \right)$$

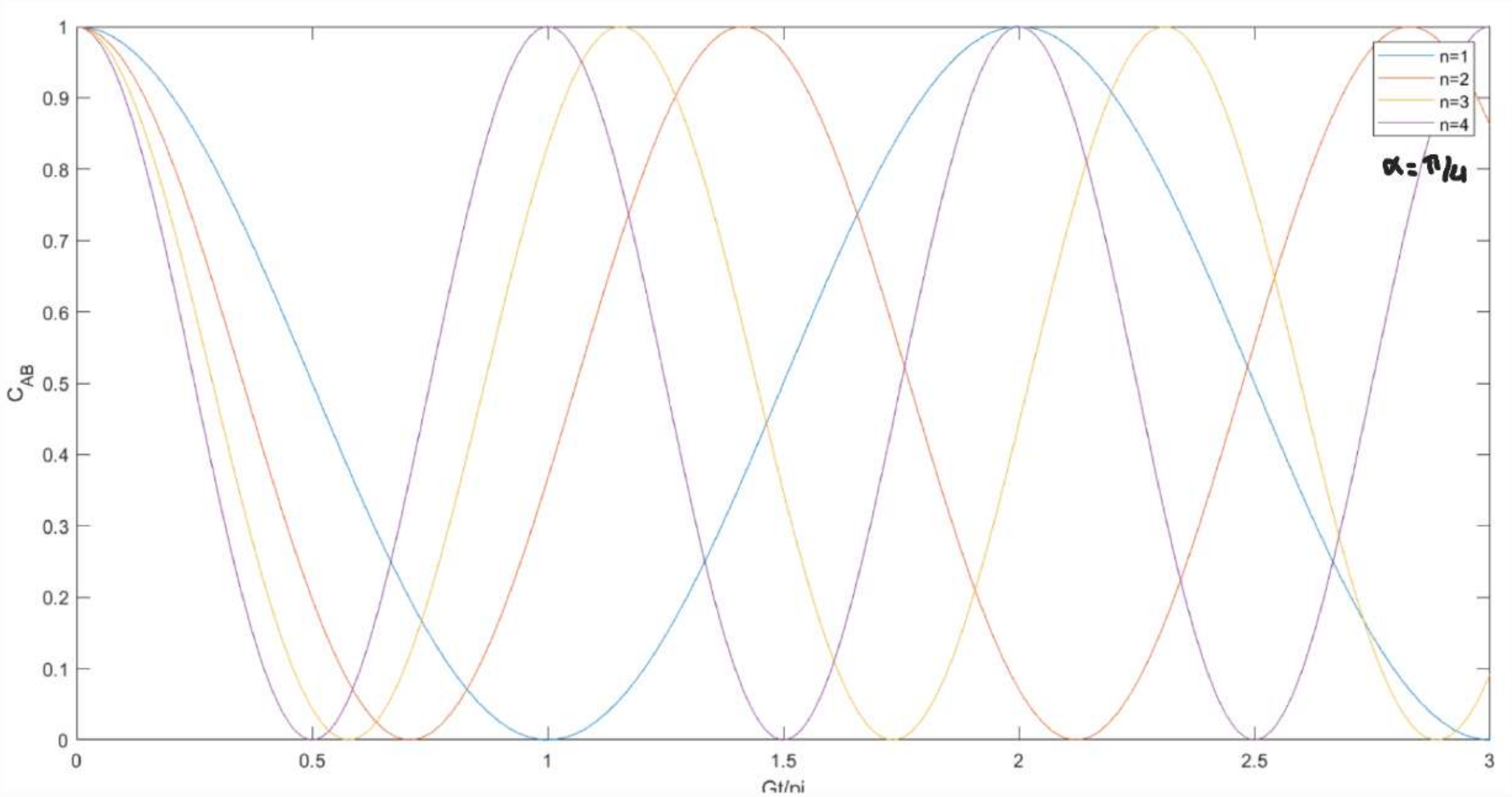
$$N = \frac{G}{2\sqrt{\Delta^2+G^2}} = \frac{1}{2}\sin(\theta)$$

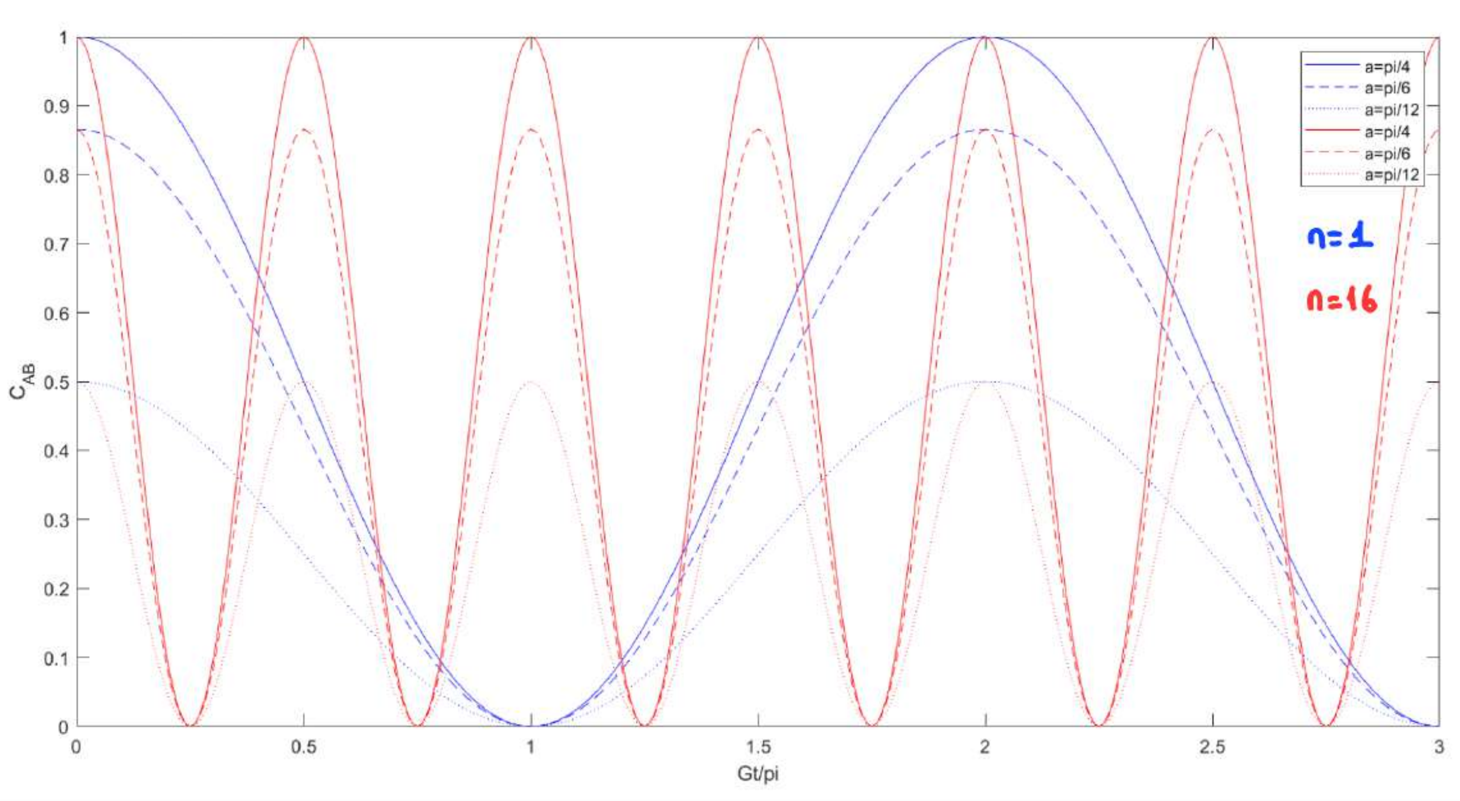
$$x_4 = \sin\alpha \left(N e^{-i\lambda^+ t} - N e^{-i\lambda^- t} \right)$$

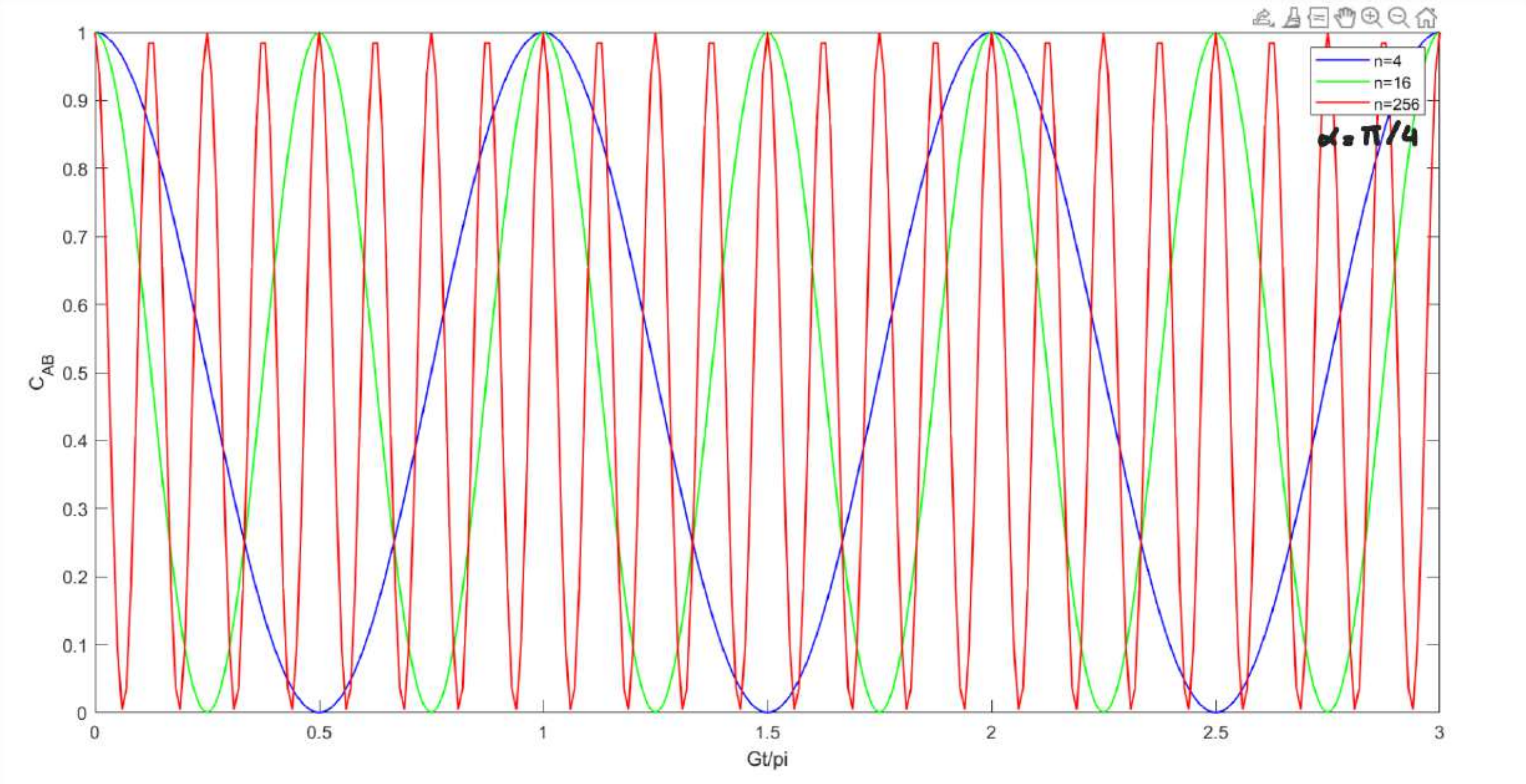
B.1 $C_{AB}(t)$

From the one photon case, one can nowplate the concurrence:

$$C_{AB} = |\sin(2\alpha)| \cos^2\left(\frac{G n t}{2}\right)$$

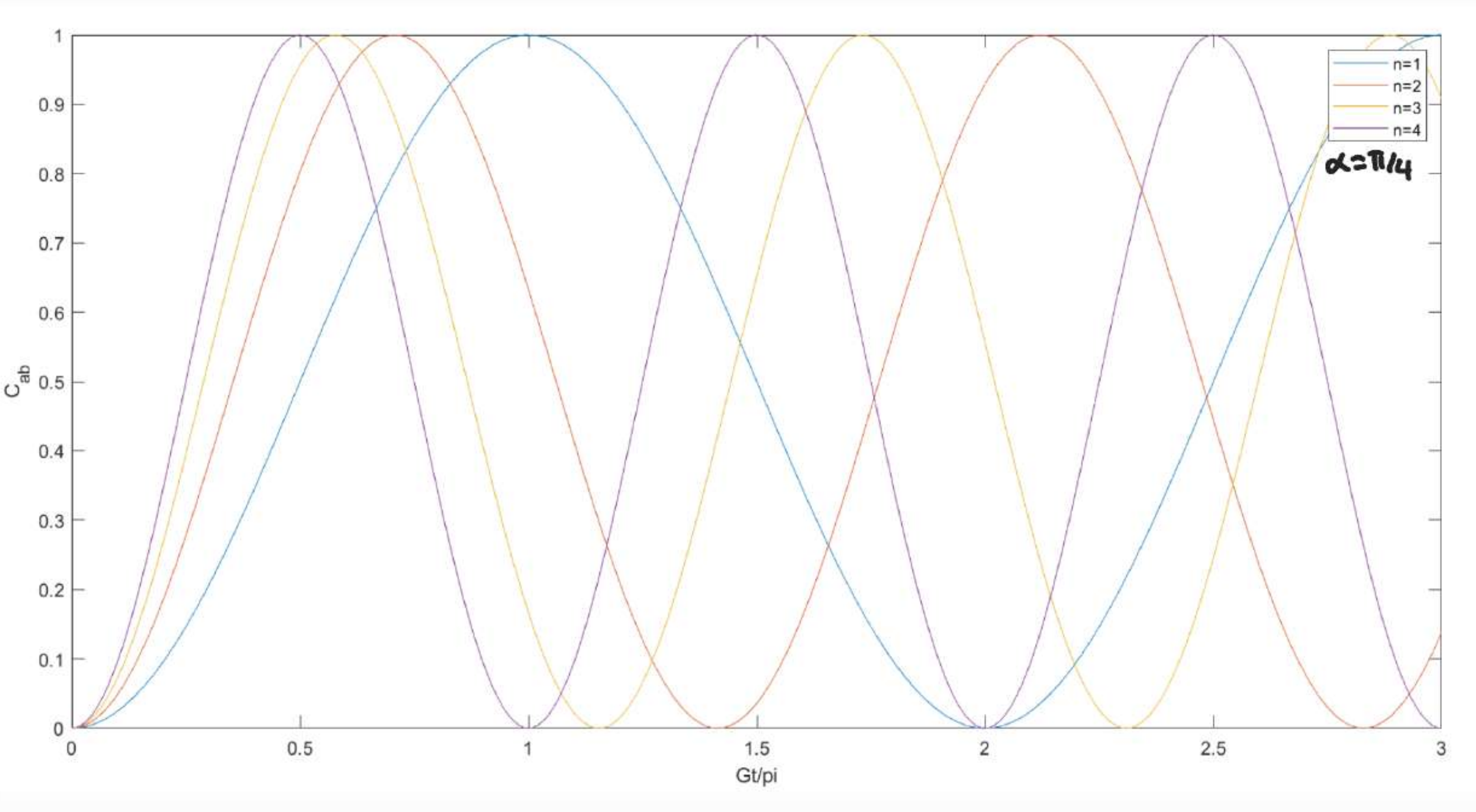


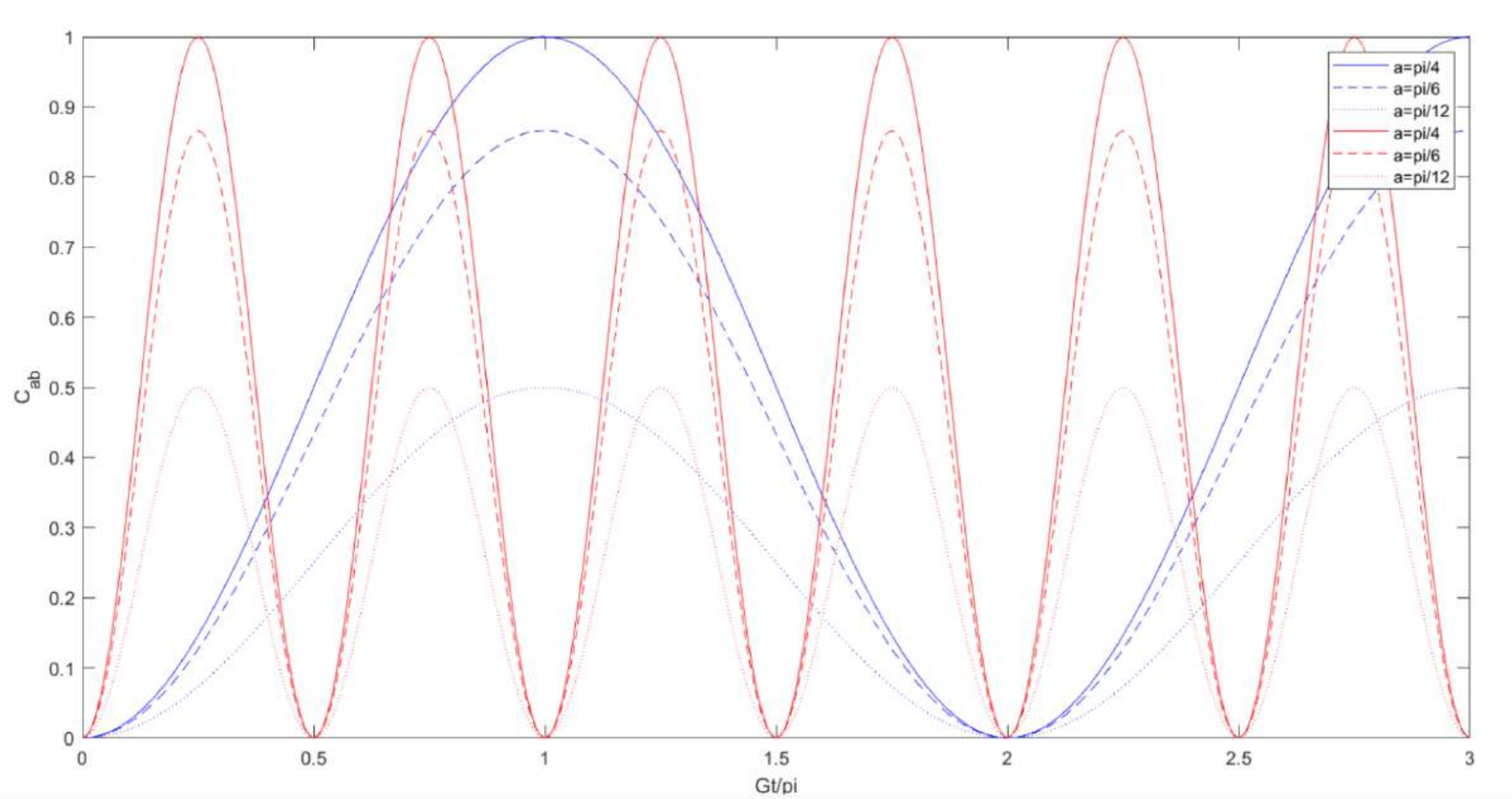


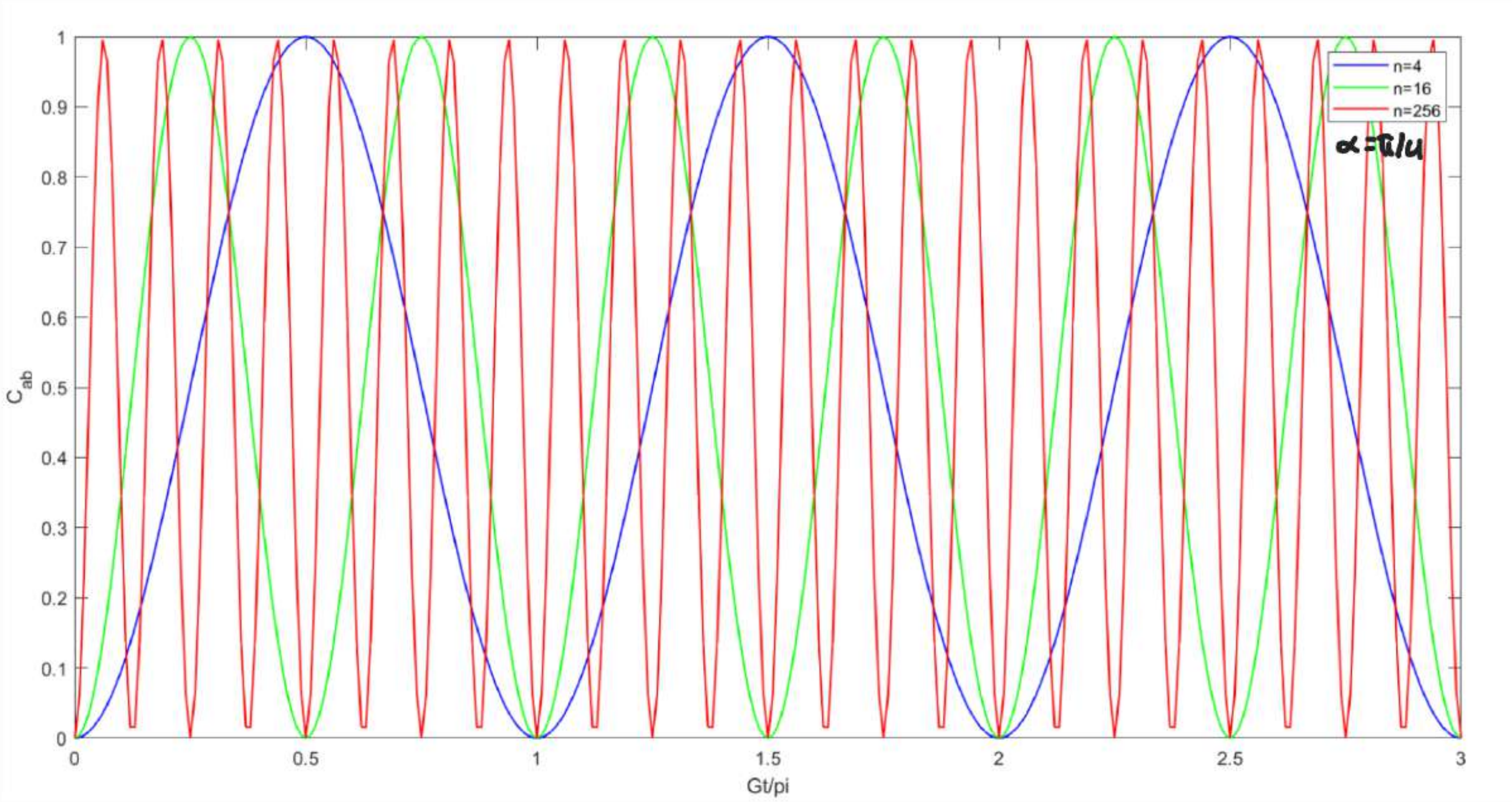


B.2 $C_{ab}(t)$

$$C_{ab}(t) = |\sin(2\alpha)| \sin^2\left(\frac{G_n t}{2}\right)$$

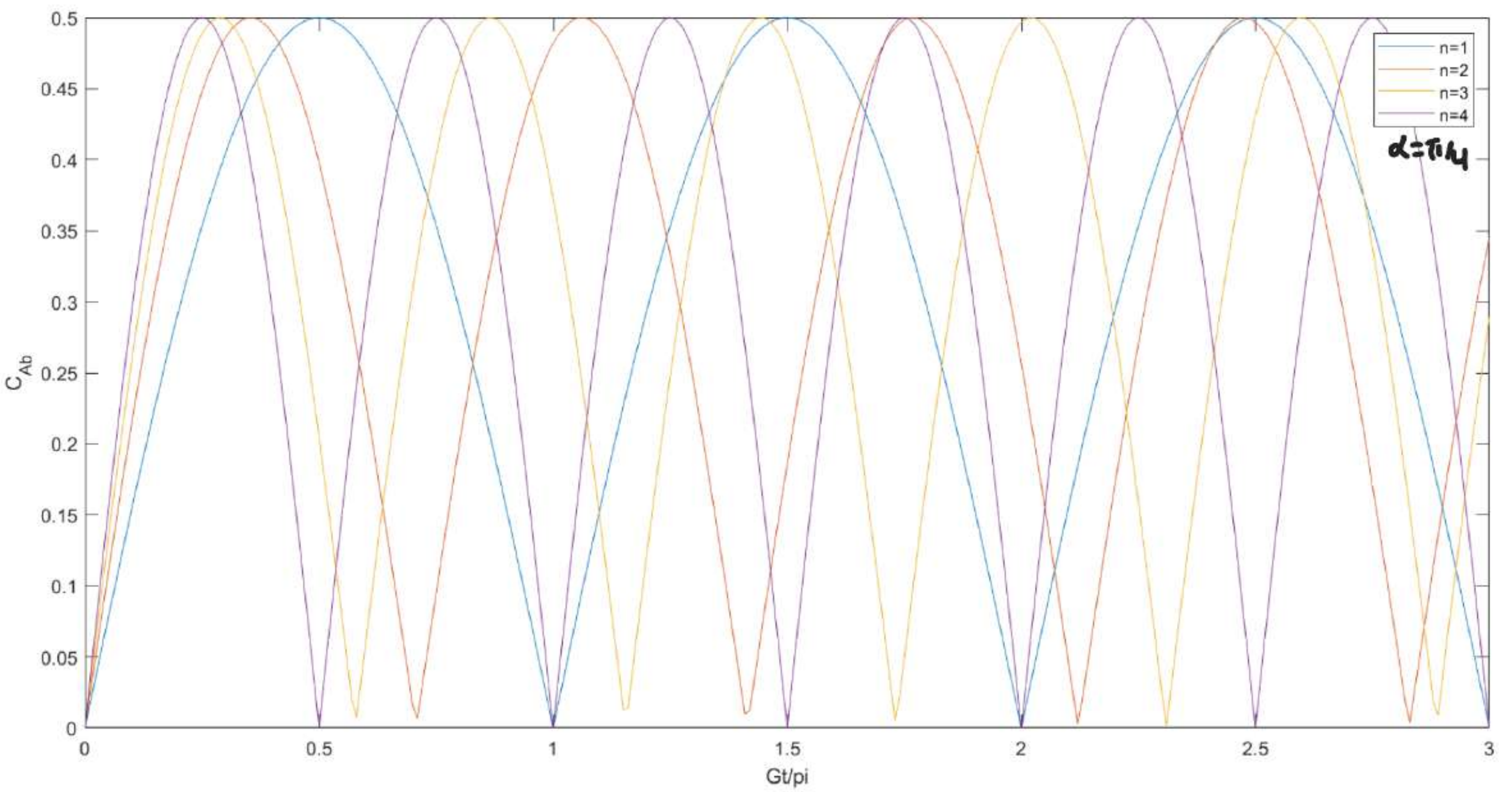




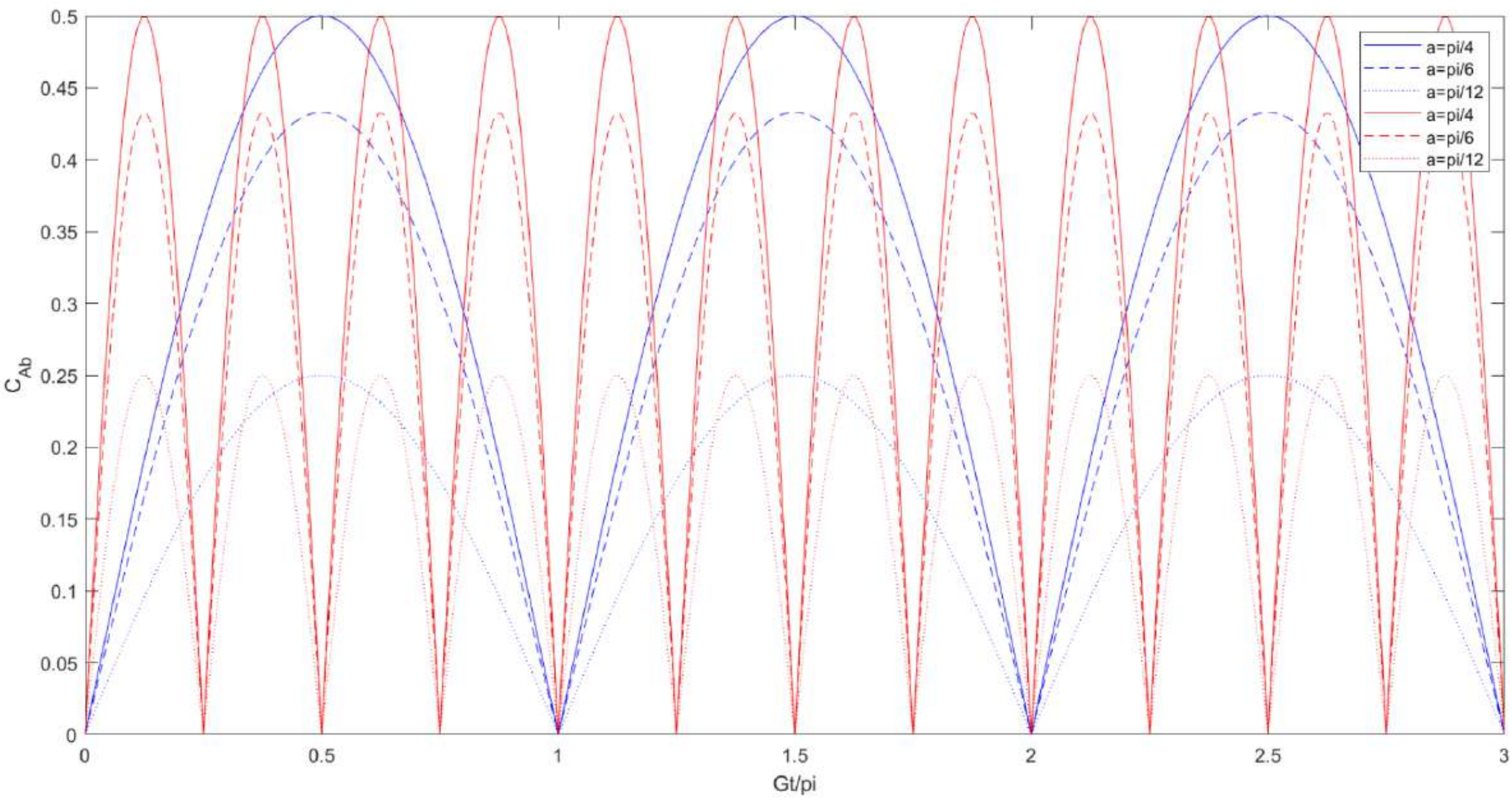


B.3 $C_{Ab}(t)$

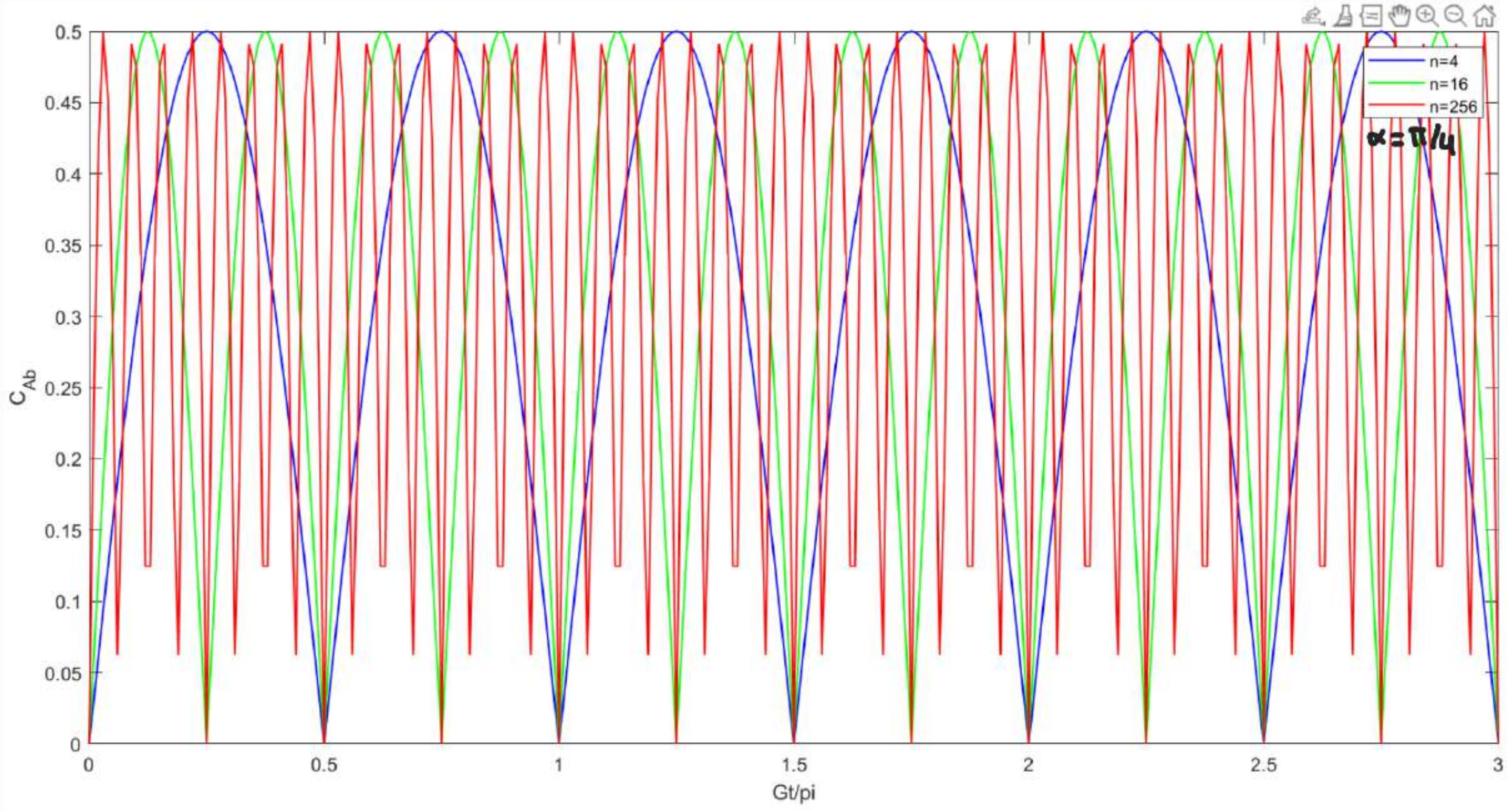
$$C_{Ab}(t) = \frac{1}{2}\left|\sin(2\alpha)\right|\left|\sin(G_n t)\right|$$



n=1
n=2
n=3
n=4
α=π/4
C_Ab
Gt/pi



a=pi/4
a=pi/6
a=pi/12
a=pi/4
a=pi/6
a=pi/12
C_Ab
Gt/pi

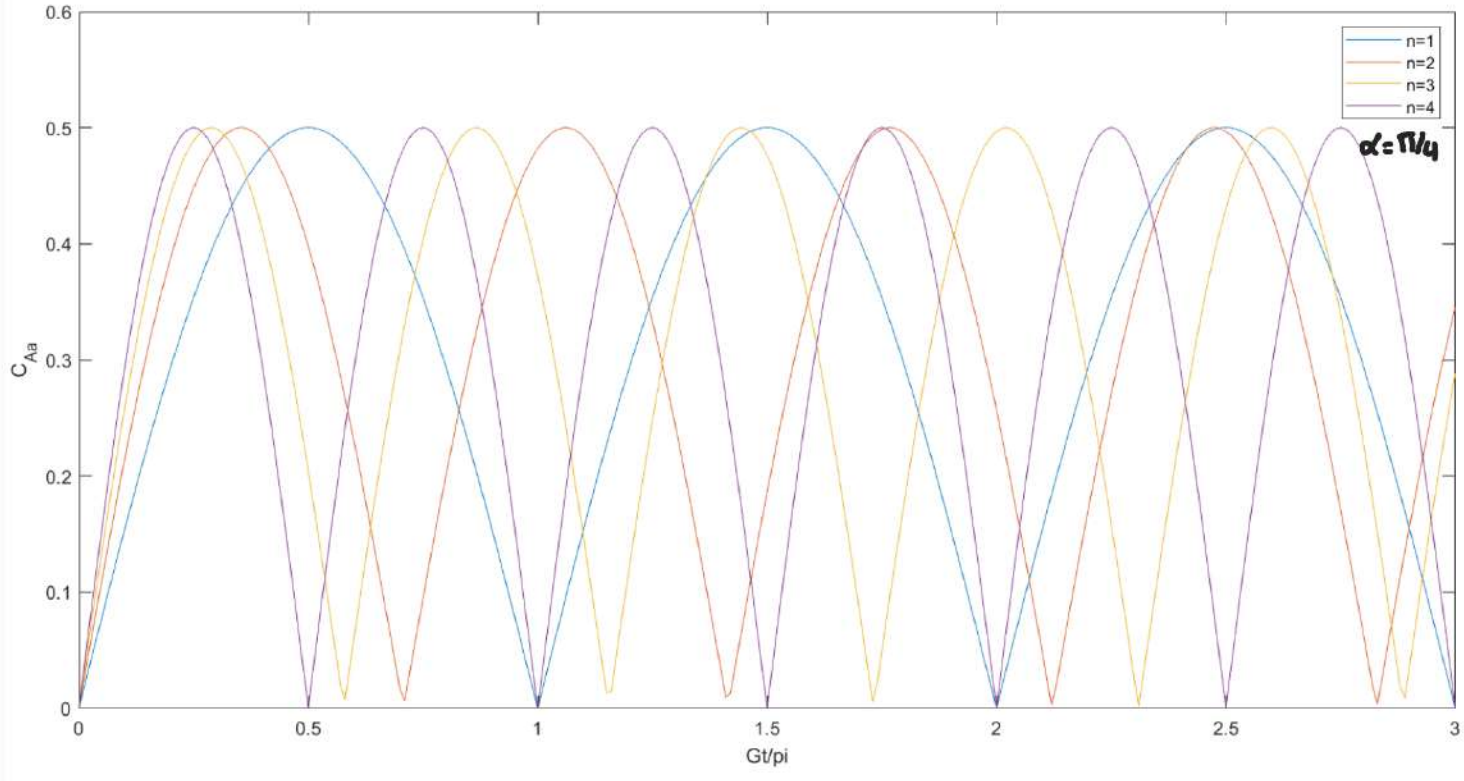


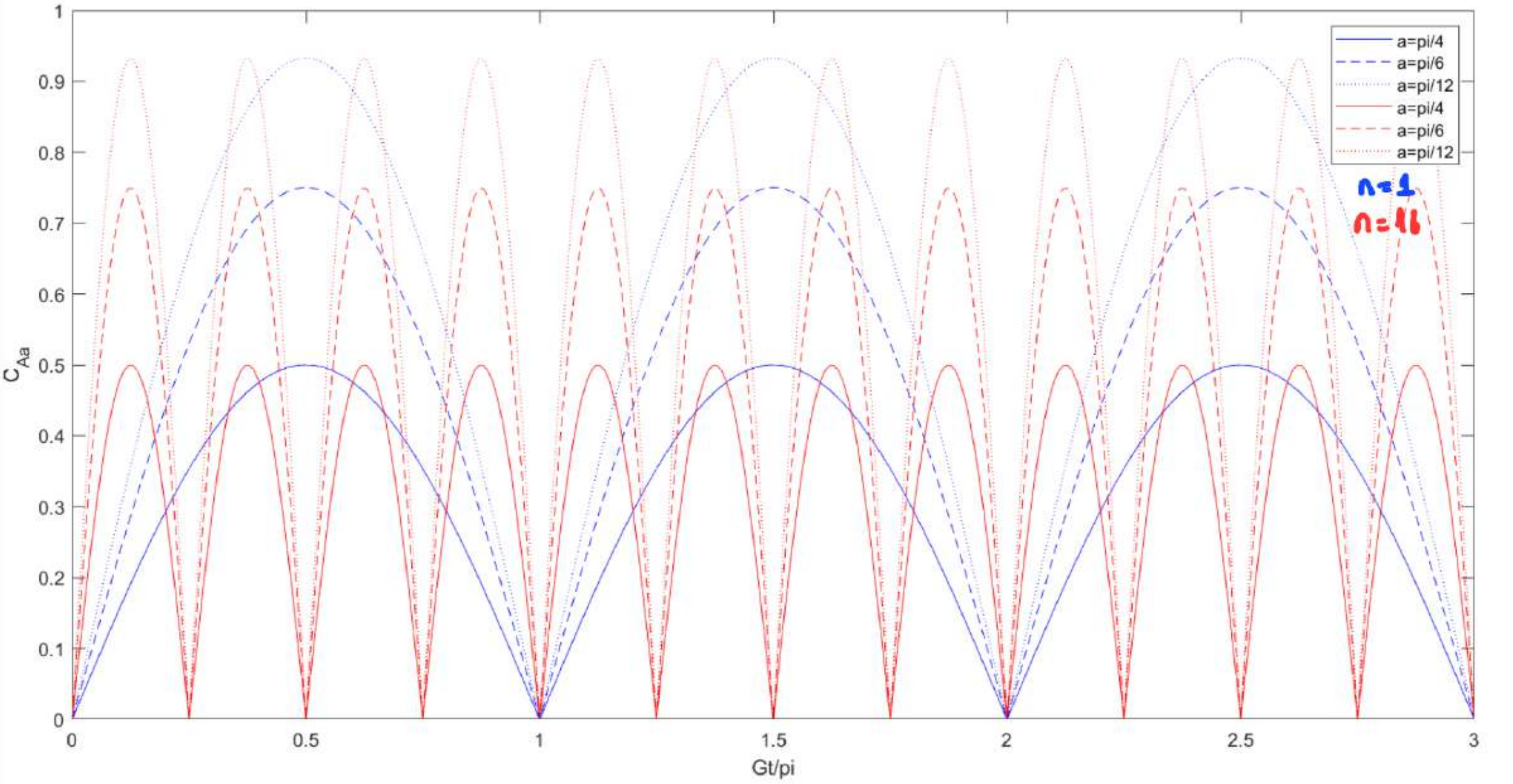
B.4 $C_{Ba}(t)$

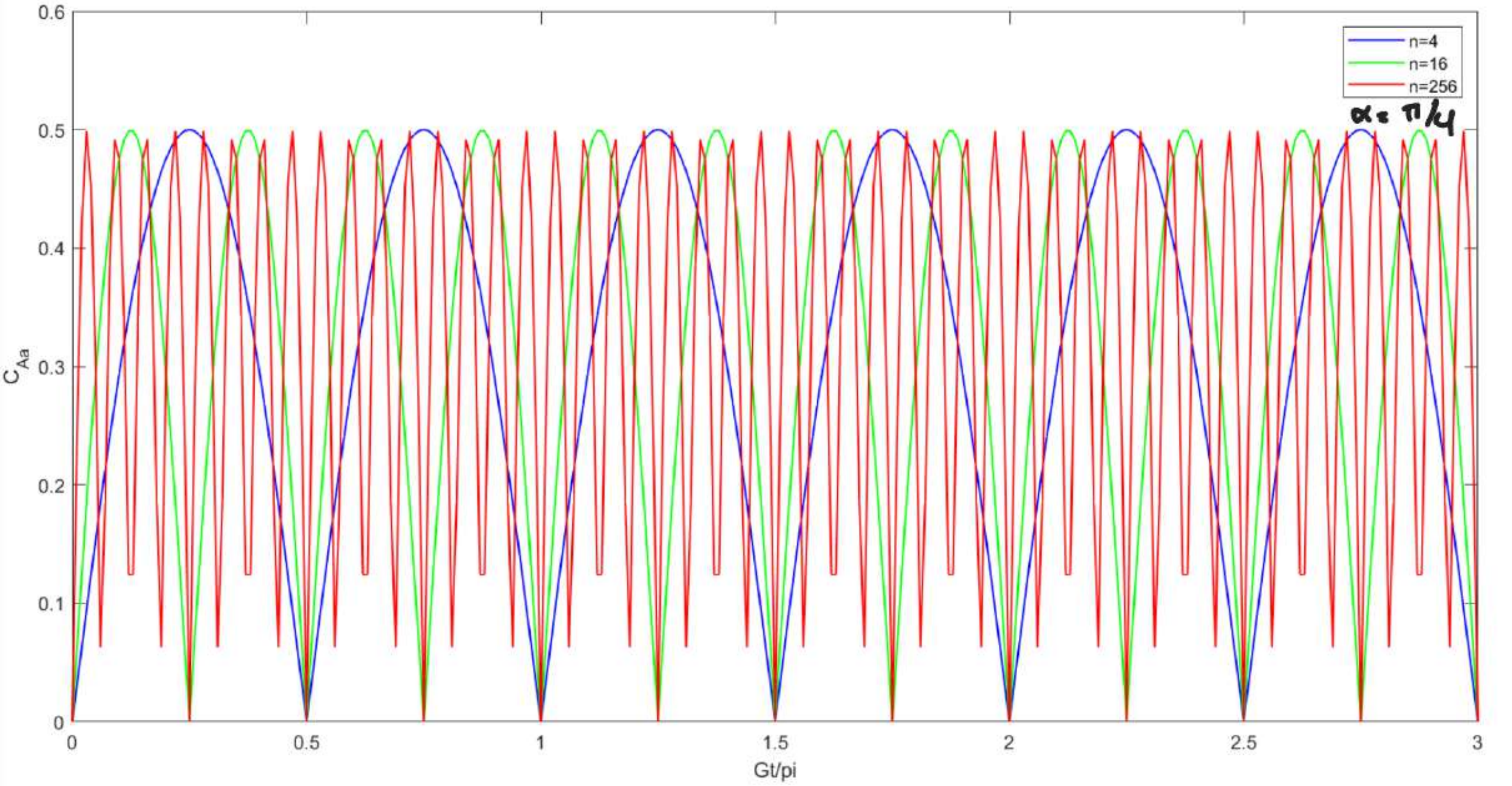
From the symmetry $C_{Ab}(t) = C_{Ba}(t)$

B.5 $C_{Aa}(t)$

$$C_{Aa}(t) = |\sin(G_n t)| \cos^2\alpha$$

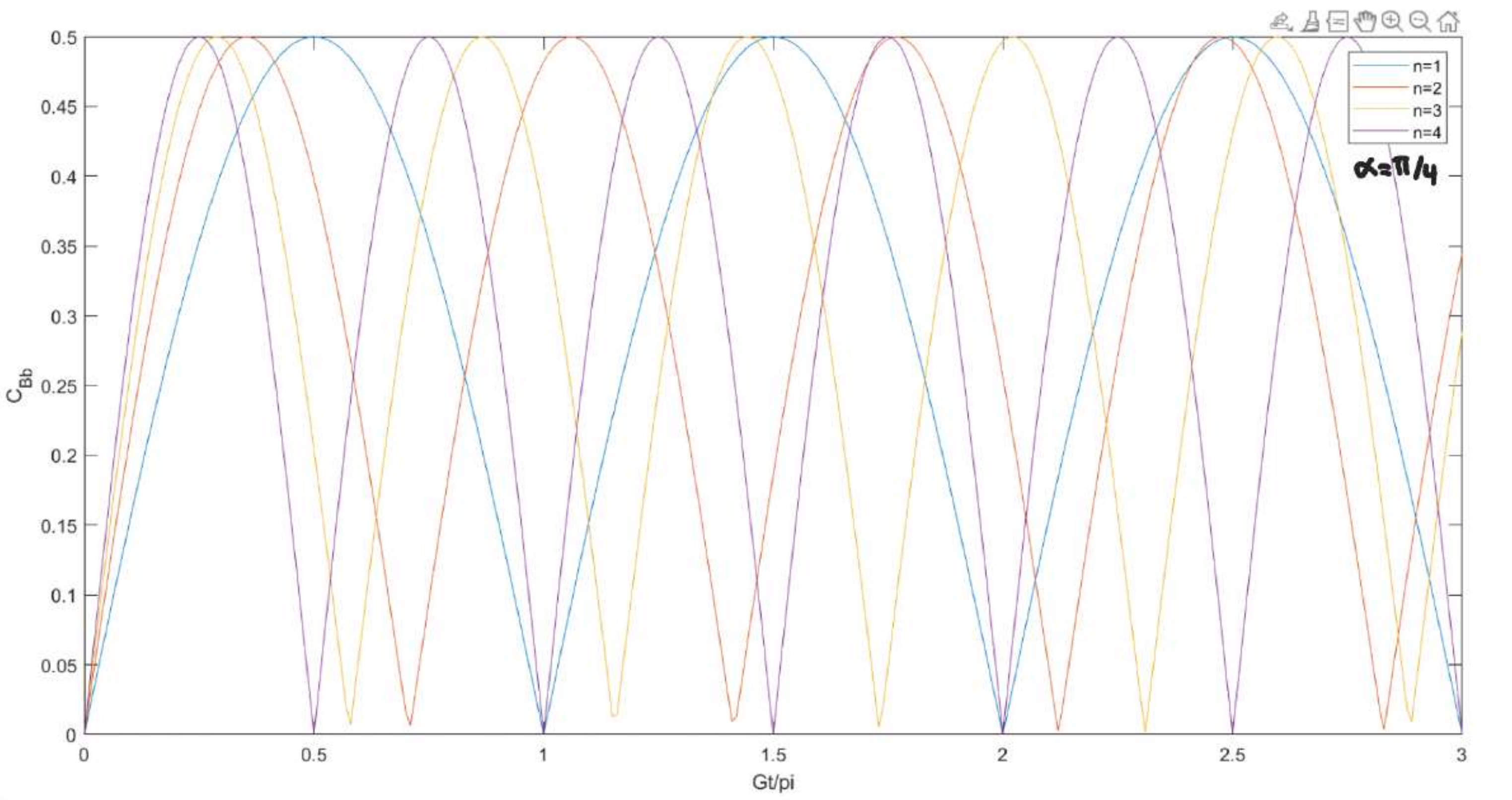


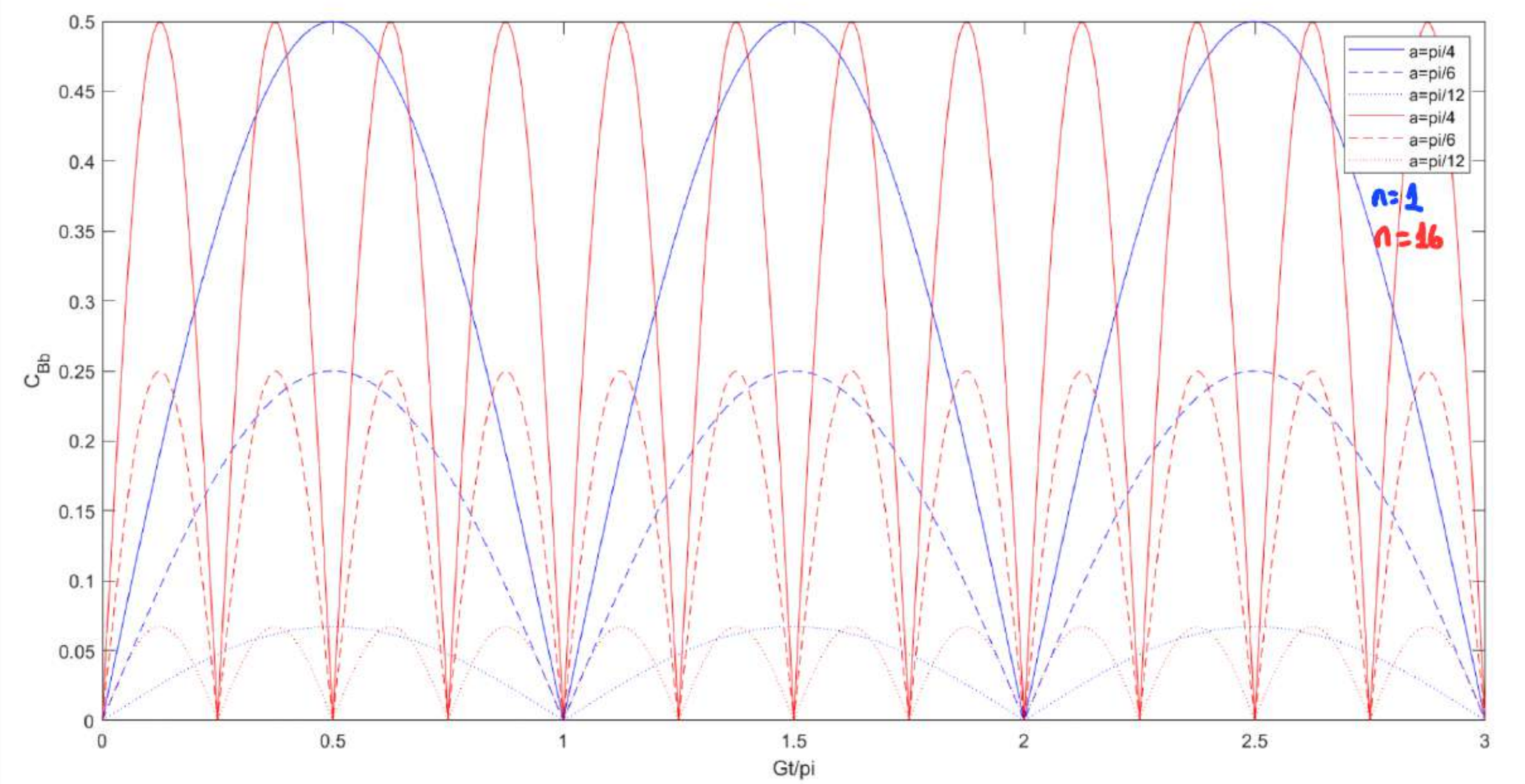

a=pi/4
a=pi/6
a=pi/12
a=pi/4
a=pi/6
a=pi/12
n=1
n=16
C_Aa
Gt/pi

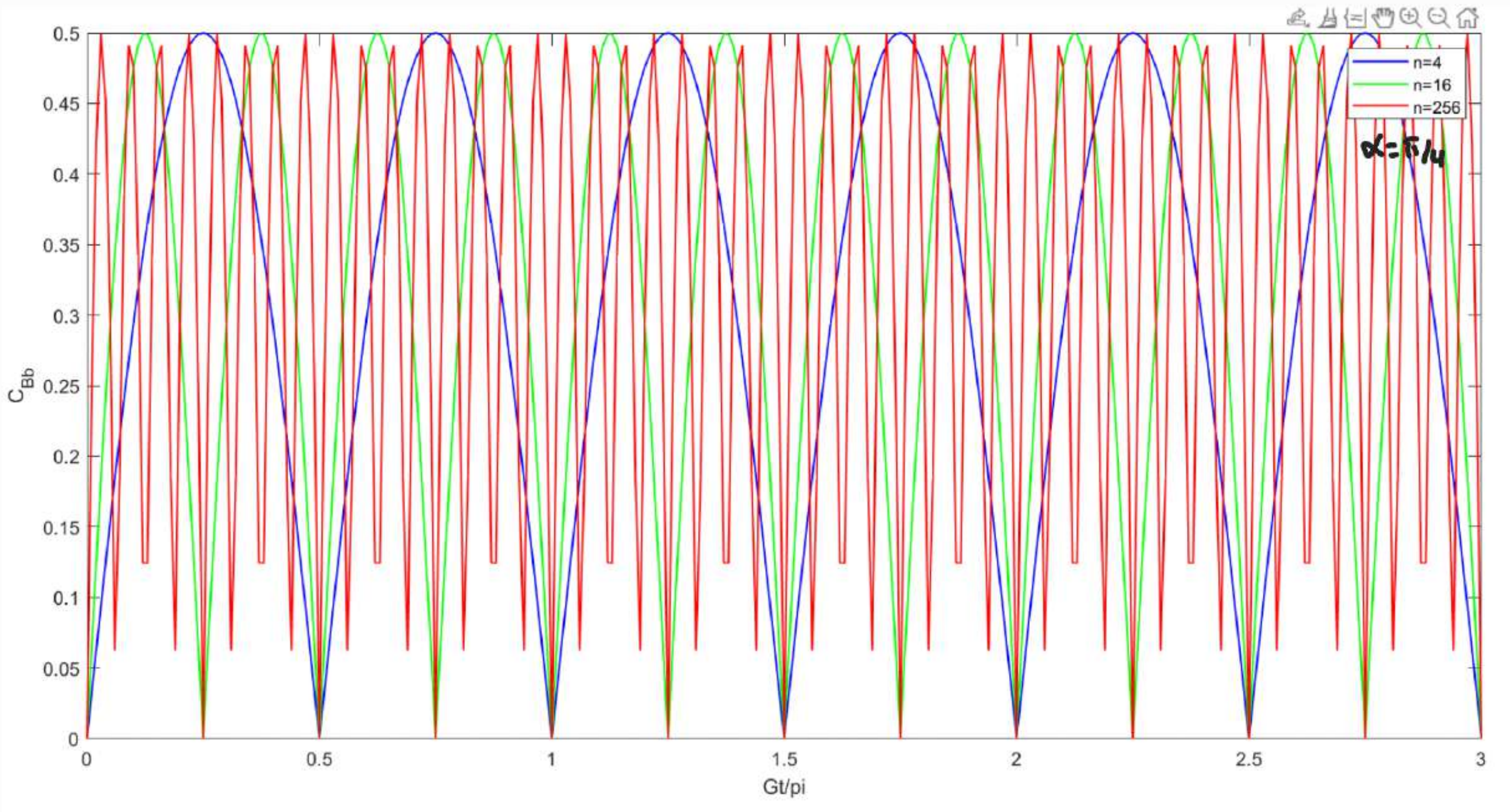

n=4
n=16
n=256
α = π/4
C_Aa
Gt/pi

B.6 $C_{Bb}(t)$

$$C_{Bb}(t) = \sin^2\alpha \, |\sin(G_n t)|$$







n=4
n=16
n=256
α=π/4
C_{Bb}
Gt/pi