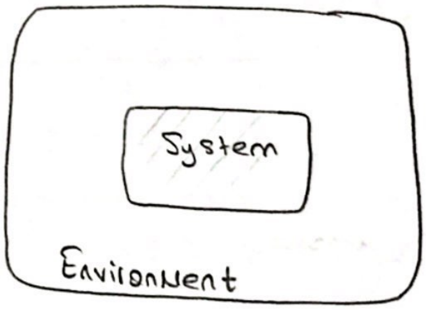
Density Matrix

23/11/22

Need → Generalize notion of quantum state

Till now $|\psi\rangle \in \mathcal{H}$ for isolated system

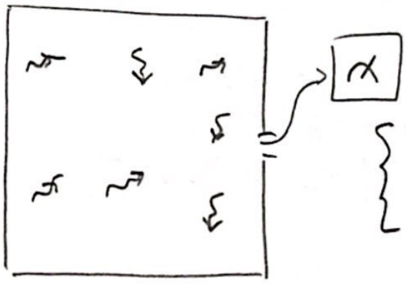
New situation systems are not isolated.



· Another new situation :

Statistical mixture of state

gas of photons



N qubits possible states:

$$\begin{cases} |\psi_1\rangle & |\psi_2\rangle & \cdots & |\psi_K\rangle \\ p_1 & p_2 & & p_K \end{cases}$$

$$0 \leq p_i \leq 1 \quad , \quad \sum_{i=1}^{K} p_i = 1$$

① Description of Statistical Mixtures

Def: Stat. Mixture is an ensemble of states $|\varphi_1\rangle, |\varphi_2\rangle, \ldots, |\varphi_k\rangle \in \mathcal{H}$

with each state drawn with probabilities $p_1, p_2, p_3, \ldots, p_k$.

$$\left(0 \le p_i \le 1 \;,\; \sum_{i=1}^{k} p_i = 1\right)$$

The Density Matrix of the statistical Mixture is $\boxed{\rho \overset{def}{=} \sum_{i=1}^{K} p_i \underbrace{|\varphi_i\rangle\langle\varphi_i|}_{\text{Projector Matrix onto } |\varphi_i\rangle}}$

Dimension of ρ is $[\dim \mathcal{H} \times \dim \mathcal{H}]$

Properties: 1) $\rho \ge 0$ positive semi-definite $\begin{cases} \text{All eigenvalues are } \ge 0 \\ \langle\psi|\rho|\psi\rangle \ge 0 \end{cases}$

2) $\rho^{+} = \rho$ hermitian → eigen values are real

3) $\underset{d}{\mathrm{Tr}}\, \rho = 1$ $\qquad \mathrm{Tr}\rho = \sum_{i=1}^{K} p_i \overbrace{\mathrm{Tr}\,|\varphi_i\rangle\langle\varphi_i|}^{1} = 1$ $\qquad \underbrace{\mathrm{Tr}(\langle\varphi_i|\varphi_i\rangle)}_{\langle\varphi_i|\varphi_i\rangle = 1}$

sum of the diagonal elements (eigen values ?)

Cyclicity of Trace: $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$

$$\mathrm{Tr}(|\varphi_i\rangle\langle\varphi_i|) = \mathrm{Tr}(\langle\varphi_i|\varphi_i\rangle) = 1$$

<u>Lemma:</u> Any square matrix ρ with the projection $\rho^{+} = \rho$, $\rho \geq 0$, $\mathrm{Tr}\,\rho = 1$ is a density matrix for some statistical mixture,

$$\boxed{\rho = \sum_{i=1}^{k} p_i |\varphi_i\rangle\langle\varphi_i|}$$

for some $(|\varphi_1\rangle, |\varphi_2\rangle, \ldots, |\varphi_k\rangle \,.\, \underbrace{p_1, \ldots, p_k}_{\text{probabilities}})$

<u>Proof:</u> Since $\rho^{+} = \rho$ the spectral Thm says that

Real eigenvalues & eigenvectors form an orthonormal basis.

$$\rho|\chi_\alpha\rangle = \lambda_\alpha|\chi_\alpha\rangle \quad , \lambda_\alpha \in \mathbb{R} \quad \langle\chi_\alpha|\chi_\beta\rangle = \delta_{\alpha\beta}$$

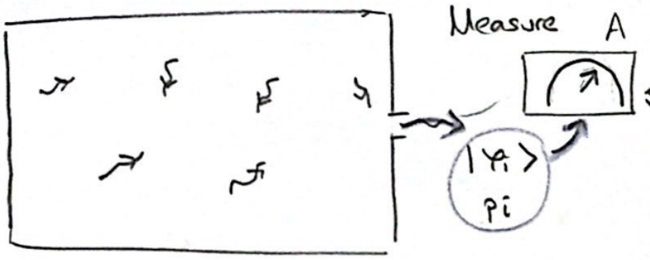
Thus $\boxed{\rho = \sum_{\alpha=1}^{D} \lambda_\alpha |\chi_\alpha\rangle\langle\chi_\alpha|}$ since $\rho \geq 0$: $0 \leq \lambda_\alpha$ $\mathrm{Tr}\,\rho = 1 = \sum_\alpha \lambda_\alpha \Rightarrow \lambda_\alpha \leq 1$

$$\rho|\chi_\alpha\rangle = \lambda_\alpha|\chi_\alpha\rangle$$

$$\rho|\chi_\alpha\rangle\langle\chi_\alpha| = \lambda_\alpha|\chi_\alpha\rangle\langle\chi_\alpha|$$

$$\rho\underbrace{\boxed{\sum_\alpha|\chi_\alpha\rangle\langle\chi_\alpha|}}_{\mathbb{I}} = \sum_\alpha \lambda_\alpha|\chi_\alpha\rangle\langle\chi_\alpha|$$

Measure A observable



$|\chi_\alpha\rangle$ with prob. given by Born Rule $|\langle\chi_\alpha|\varphi_i\rangle|^2$

and I record the Meas. result μ_α

$$\underbrace{\text{Average of } A \text{ over measurement}}_{\langle A\rangle \;\; Av(A)} = \boxed{Tr(A\rho)}$$

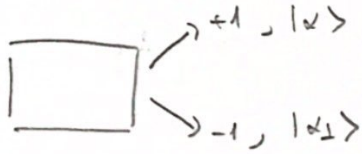
$$\boxed{E(f(x)) = \int dx\, f(x)\, p(x)} \rightarrow \text{Analogy}$$

$A^{\dagger} = A$ has spectral decomposition $A|X_0\rangle = \mu_\alpha |X_0\rangle$

$$A = \sum_\alpha \mu_\alpha |X_\alpha\rangle\langle X_\alpha|$$

According to meas. postulate, I modeled the meas. apparatus by the orthonormal basis $\{X_\alpha\}_{\alpha=1,\dots,\dim\mathcal{H}}$

(Previously)
Polarization Obs. $A = (+1)|\alpha\rangle\langle\alpha| + (-1)|\alpha_\perp\rangle\langle\alpha_\perp|$



$$Av(A) = \sum_i p_i \sum_\alpha \mu_\alpha |\langle \chi_\alpha|\varphi_i\rangle|^2$$

(Conditional on state being measured is $|\varphi_i\rangle$, the avg. is $\sum_\alpha \mu_\alpha |\langle X_\alpha|\varphi_i\rangle|^2$)

$$Av(A) = \sum_\alpha \mu_\alpha \sum_i p_i \langle X_\alpha|\varphi_i\rangle\langle\varphi_i|X_\alpha\rangle = \sum_\alpha \mu_\alpha \langle X_\alpha| \underbrace{\left(\sum_i p_i |\varphi_i\rangle\langle\varphi_i|\right)}_{\rho} |X_\alpha\rangle$$

$$= \sum_\alpha \mu_\alpha \langle x_\alpha | \rho | x_\alpha \rangle = \mathrm{Tr}\Big(\rho \underbrace{\sum_\alpha \mu_\alpha |x_\alpha\rangle\langle x_\alpha|}_{A}\Big)$$

$$\mathrm{Tr}\left(\langle x_\alpha | \rho | x_\alpha \rangle\right) = \mathrm{Tr}\left(\rho | x_\alpha \rangle \langle x_\alpha |\right)$$

$$\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$$

$$\boxed{Av(A) = \mathrm{Tr}(\rho A) = \mathrm{Tr}(A\rho)}$$

Variance of A : $(\Delta A)^2 = Av(A^2) - (Av(A))^2$

$$= \mathrm{Tr}(\rho A^2) - (\mathrm{Tr}\,\rho A)^2$$

$$Var(x) = \int dx\, x^2 p(x) - \left(\int dx\, x\, p(x)\right)^2$$ → Analogy

Scanned with CamScanner

Measurements & Density Matrix

System (statistical mix) described by ρ

Measurement with appratus with basis $\{|X_\alpha\rangle\} \Leftrightarrow \{ \overset{\text{projector}}{P_\alpha} = |X_\alpha\rangle\langle X_\alpha| \}$

If initially state is $|\varphi_i\rangle \longrightarrow$ after meas. $\dfrac{P_\alpha|\varphi_i\rangle}{\| P_\alpha|\varphi_i\rangle\|} = |X_\alpha\rangle$

Initially $\rho = \sum_{i=1}^{k} p_i |\varphi_i\rangle\langle\varphi_i|$ what is the DM after the meas?

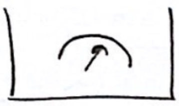
$$\rho^{after} = \sum_{i=1}^{k} p_i \left(\frac{P_\alpha|\varphi_i\rangle\langle\varphi_i| P_\alpha}{\|P_\alpha|\varphi_i\rangle\| \, \|P_\alpha|\varphi_i\rangle\|} \right)$$

$\langle\varphi_i|P_\alpha P_\alpha|\varphi_i\rangle$

$\langle\varphi_i|P_\alpha|\varphi_i\rangle = \langle\varphi_i|X_\alpha\rangle\langle X_\alpha|\varphi_i\rangle$

$$\rho^{after} = \sum_{i=1}^{k} p_i \frac{\overset{|X_\alpha\rangle\langle X_\alpha|}{P_\alpha}|\varphi_i\rangle\langle\varphi_i|\overset{|X_\alpha\rangle\langle X_\alpha|}{P_\alpha}}{\langle\varphi_i|X_\alpha\rangle\langle X_\alpha|\varphi_i\rangle} = \sum_{i=1}^{k} p_i \underbrace{|X_\alpha\rangle\langle X_\alpha|}_{P_\alpha}$$

$$\boxed{\rho^{after} = \Big(\overbrace{\sum_i p_i}^{1}\Big) |X_\alpha\rangle\langle X_\alpha| = P_\alpha}$$

$\rho \longrightarrow$  $\rightarrow \rho^{after} = \dfrac{P_\alpha \rho P_\alpha}{Tr(P_\alpha \rho P_\alpha)} = \dfrac{|X_\alpha\rangle\langle X_\alpha|\rho|X_\alpha\rangle\langle X_\alpha|}{Tr(|X_\alpha\rangle\langle X_\alpha|\rho|X_\alpha\rangle\langle X_\alpha|)}$

$$\underbrace{\langle X_\alpha|X_\alpha\rangle}_{1}\underbrace{\langle X_\alpha|\rho|X_\alpha\rangle}_{\text{some number}}$$

$$\rho^{after} = \frac{|X_\alpha\rangle\cancel{\langle X_\alpha|\rho|X_\alpha\rangle}\langle X_\alpha|}{\cancel{\langle X_\alpha|\rho|X_\alpha\rangle}} = |X_\alpha\rangle\langle X_\alpha|$$

Prob. to get $|X_\alpha\rangle\langle X_\alpha| = Tr(P_\alpha \rho)$

$$Tr(P_\alpha \rho P_\alpha) = Tr(P_\alpha^2 \rho) = Tr(P_\alpha \rho) = Tr(|X_\alpha\rangle\langle X_\alpha|\rho)$$

$$= \boxed{\langle X_\alpha|\rho|X_\alpha\rangle}$$

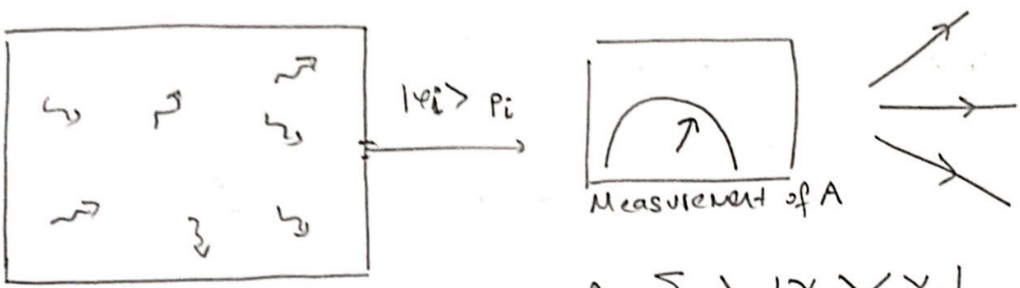
$$\rho = \sum_i p_i |\varphi_i\rangle\langle\varphi_i|$$

Average $\leftarrow \langle X_\alpha|\rho|X_\alpha\rangle = \sum_i p_i \underbrace{\langle X_\alpha|\varphi_i\rangle\langle\varphi_i|X_\alpha\rangle} = \sum_i p_i |\langle X_\alpha|\varphi_i\rangle|^2$

24/11/22

Density Matrix Review

System statistical mixture of states



$|\chi_\alpha\rangle$, λ_α with probability $|\langle \chi_\alpha | \varphi_i \rangle|^2$

$$A = \sum_\alpha \lambda_\alpha |\chi_\alpha\rangle\langle\chi_\alpha|$$

$|\varphi_1\rangle, |\varphi_2\rangle, \ldots, |\varphi_k\rangle$
$p_1 \quad p_2 \quad \ldots \quad p_k$

$$Av(A) = \sum_{i=1}^{k} p_i \sum_\alpha \lambda_\alpha |\chi_\alpha\rangle\langle\chi_\alpha| \Rightarrow Av(A) = Tr(A\rho) \text{ with } \rho = \sum_{i=1}^{k} p_i |\varphi_i\rangle\langle\varphi_i|$$

Convex combination of states in the mixture.

(ρ = Density Matrix (DM) = Mixed states)

Terminology: Pure States $\rightarrow$ usual vectors of Hilbert space $|\psi\rangle \in \mathcal{H}$

$$\Leftrightarrow \quad |\psi\rangle\langle\psi| = \rho_\psi = \{|\psi\rangle, p=1\}$$

Mixed States $\rightarrow$ Density matrices of stat. mixture

$$\{|\varphi_1\rangle, \ldots, |\varphi_k\rangle, p_1, \ldots, p_k\}$$

$$\rho = \sum_{i=1}^{k} p_i |\varphi_i\rangle\langle\varphi_i|$$

Remark $\quad \rho_\psi^2 = \rho_\psi \rightarrow$ Pure State

but in general $\rho^2 \neq \rho$

Measurement of a Mixed state

$\rho \rightarrow$ 

$|v_\alpha\rangle\langle v_\alpha|$ with

$$\text{prob}(\alpha) = \mathrm{Tr}(\rho P_\alpha) = \langle v_\alpha|\rho|v_\alpha\rangle$$

$\{|v_\alpha\rangle\langle v_\alpha| = P_\alpha$

$\alpha = 1 \ldots D\}$

orthonormal basis

$\rho_\psi = |\psi\rangle\langle\psi|$ $\rightarrow$ $\text{prob}\ \alpha = \langle v_\alpha|\psi\rangle\langle\psi|v_\alpha\rangle = |\langle v_\alpha|\psi\rangle|^2$ $\rightarrow$ Born Rule

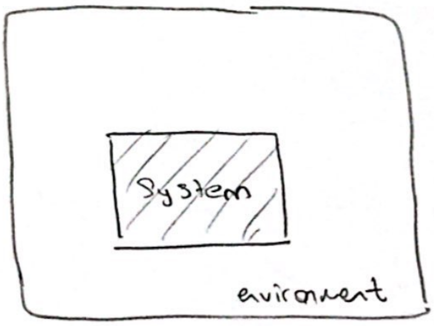
Pure states

$$|v_\alpha\rangle\langle v_\alpha| = \frac{P_\alpha \rho P_\alpha}{\mathrm{Tr}(\underbrace{P_\alpha \rho P_\alpha}_{\rho P_\alpha^2 = \rho P_\alpha})} = \frac{P_\alpha \rho P_\alpha}{\mathrm{Tr}(\rho P_\alpha)}$$

$\rightarrow$ Projected from both sides

(for states (vectors) only one projection is enough)

Density Matrix after the Measurement



$\rightarrow$ description of S only involves ρ_S

Eliminate degrees of freedom of environment

$|\psi\rangle\langle\psi|$

Density Matrix of one qubit

State vector of 1 qubit

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2} e^{i\psi}|1\rangle$$

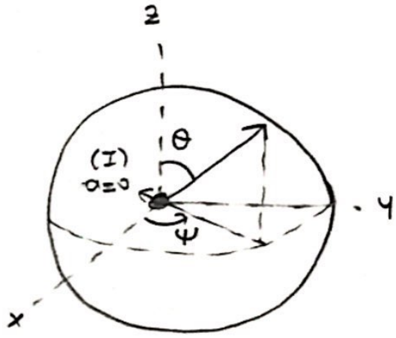
$\mathcal{H} = \mathbb{C}^2$

$$\rho = \frac{a_0}{2} I + \frac{a_1}{2}\sigma_x + \frac{a_2}{2}\sigma_y + \frac{a_3}{2}\sigma_z$$

$\underbrace{I, \sigma_x, \sigma_y, \sigma_z \quad I, X, Y, Z}_{}$

Pauli Matrices

$$\boxed{\begin{array}{l} \cdot\ \rho^{+} = \rho \Rightarrow a_0, a_1, a_2, a_3 \Rightarrow \text{real} \\ \cdot\ \mathrm{Tr}\,\rho = 1 \Rightarrow a_0 = 1 \\ \cdot\ \rho \geqslant 0 \Rightarrow \|\vec{a}\| \leq 1 \end{array}}$$



$$\boxed{\rho = \frac{1}{2}\left(I + \vec{a}.\vec{\sigma}\right)}$$

$$\vec{a} = r\left(\sin\theta\cos\psi,\ \sin\theta\sin\psi,\ \cos\theta\right)$$

$0 \leq r \leq 1$

If $r = 1$ then $\rho^2 = \rho \Rightarrow \rho = |\psi\rangle\langle\psi|$ (pure state) on the surface of the Bloch ball

Proof: $\rho \geq 0 \Rightarrow \|\vec{a}\| = 1$

$Tr\ \rho = 1 \Rightarrow$ I have 2 eigenvalues (?)

$$\begin{cases} \lambda_0 + \lambda_1 = 1 \\ \lambda_0 \cdot \lambda_1 \geq 0 \end{cases}$$ (semi definite)

$$\boxed{\lambda_0 \lambda_1 = \det \rho \geq 0}$$

$$\det \left[\frac{I}{2} + \frac{\vec{a}}{2} \cdot \vec{\sigma} \right] = \det \underbrace{\begin{bmatrix} \frac{1+a_1}{2} & \frac{a_1 + i a_2}{2} \\ \frac{a_1 - i a_2}{2} & \frac{1 - a_3}{2} \end{bmatrix}}_{\downarrow \text{ yields}} \geq 0$$

$\|a\| \geq 1$