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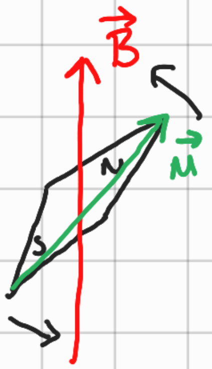
SPIN 1/2 & Magnetic Moments

Magnetic needle



Energy function

$$\boxed{E = -\vec{M} \cdot \vec{B}}$$

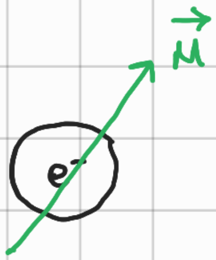
$$\vec{M} = (M_x, M_y, M_z)$$

$$\vec{B} = (B_x, B_y, B_z)$$

Many origins for Magn. Movement

* electric current loops can create $\vec{M}$

* Intrinsic magnetic moments of electrons



Quantum description of intrinsic magn. moment of type spin 1/2

Spin 1/2 $\boxed{\vec{\mu} = \gamma \frac{\hbar}{2} \vec{S}}$

Vector with matrix components

$$\vec{S} = (\sigma_x, \sigma_y, \sigma_z) = (x, y, z)$$

Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\hbar = \frac{h}{2\pi} = \frac{\text{Planck constant}}{2\pi} \quad [J \cdot s]$$

$$\gamma = g \frac{q}{2m}$$

q → charge

m → mass

depends on particle

$$\begin{cases} e^- & g \simeq 2 \\ \text{proton} & g \simeq 5 \end{cases}$$

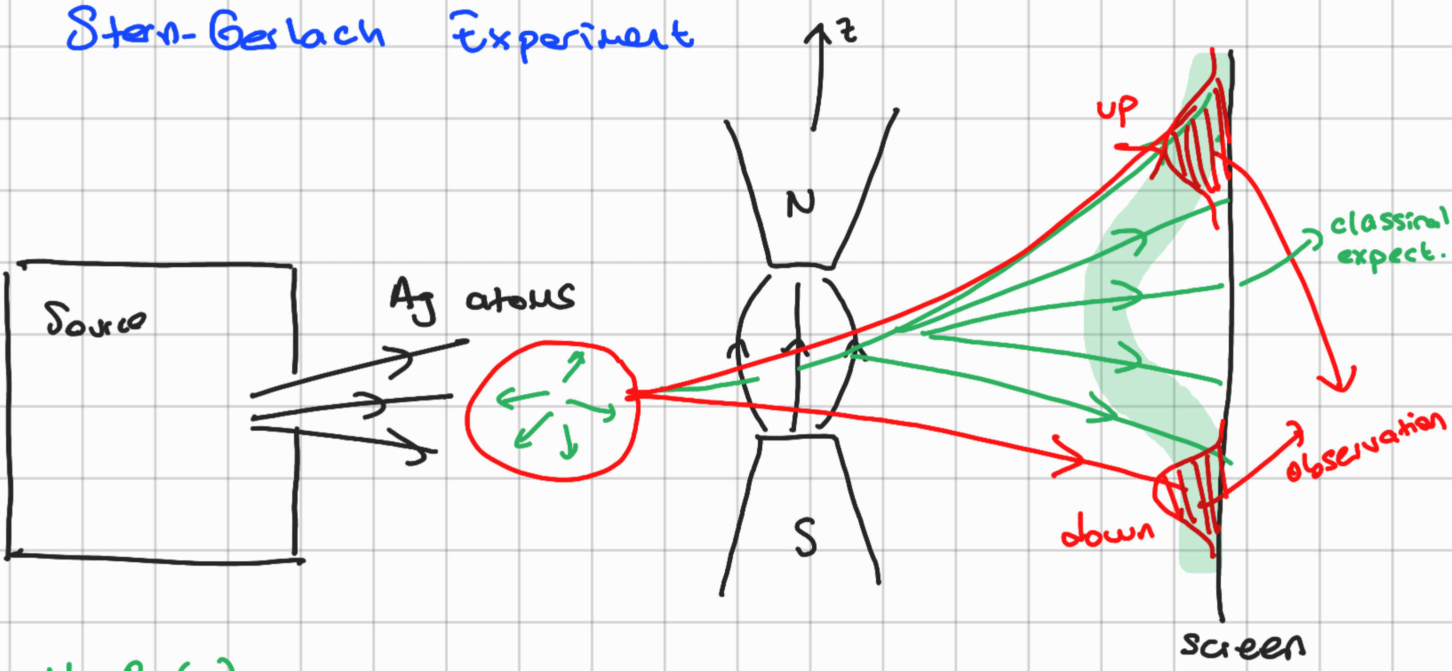
Hamiltonian describe dynamics

$$\boxed{H = -\gamma \frac{\hbar}{2} \vec{S} \cdot \vec{B}} = -\gamma \frac{\hbar}{2} \{ \sigma_x B_x + \sigma_y B_y + \sigma_z B_z \}$$

H → observable energy

$$= \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix}$$

Stern-Gerlach Experiment

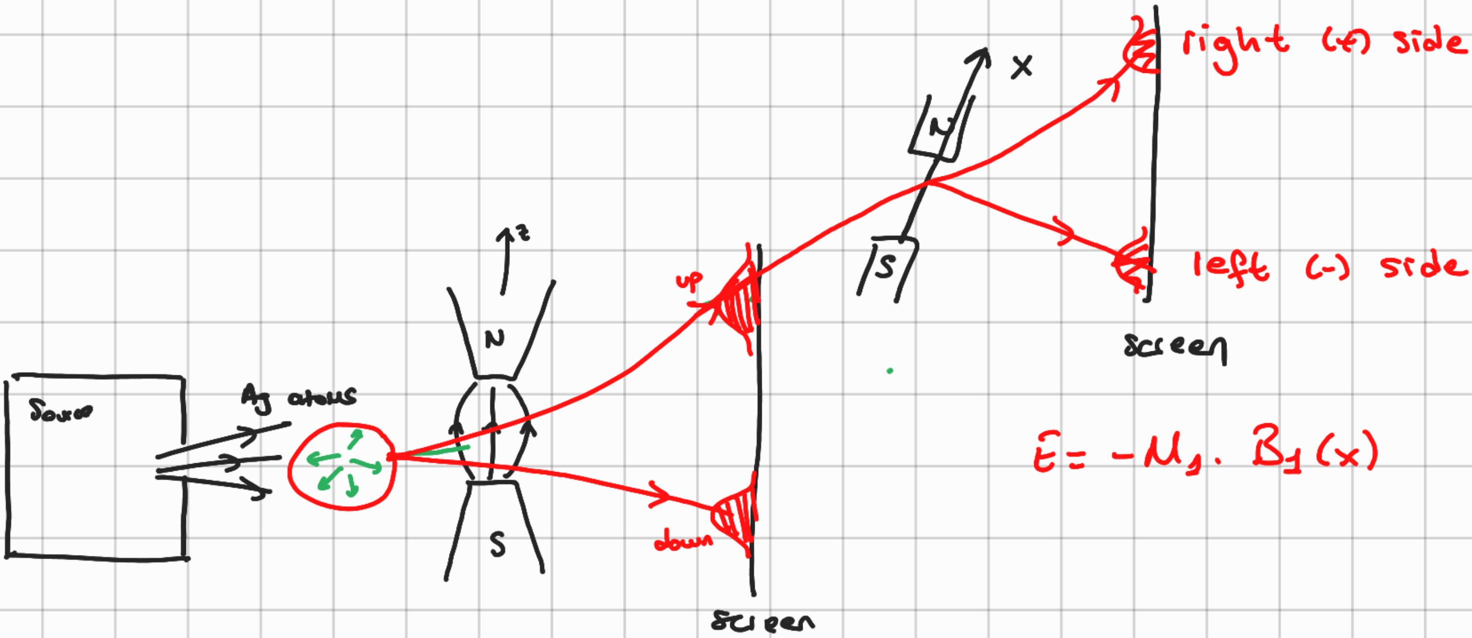


$$E = -M_3 \, B_3(z)$$

$$B = (0, 0, B_3(z))$$

$$-\frac{dE}{dz} = +M_3 \frac{dB_3(z)}{dz}$$

$$\frac{d\,B_3(z)}{dz} > 0$$



$$E = -M_1 \cdot B_1(x)$$

Explanation:

Ag → $47 e^-$ each with intrinsic magnetic moment

Remains one magnetic moment not compensated.

$$M_3 = (+1)\,|\uparrow\rangle\langle\uparrow| + (-1)\,|\downarrow\rangle\langle\downarrow|$$

First setting $\Rightarrow$ states $|up\rangle = |\uparrow\rangle$

after the measurement $|down\rangle = |\downarrow\rangle$

General state $\alpha\,|\uparrow\rangle + \beta\,|\downarrow\rangle = \begin{pmatrix}\alpha\\ \beta\end{pmatrix} \in \mathbb{C}^2$

Record $+1$ for $|\uparrow\rangle$

-1 for $|\downarrow\rangle$

$\vec{M} = \gamma\frac{\hbar}{2}\,\vec{S}$ without units

Construct observable summarize meas results

$$S_3 = (+1)\,|\uparrow\rangle\langle\uparrow| + (-1)\,|\downarrow\rangle\langle\downarrow|$$

$$= \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}$$

Pauli z or σ_z matrix

State before meas $|\psi\rangle$ = $Av(S_3) = \langle\psi|\,S_3\,|\psi\rangle$

Example: $|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle + |\downarrow\rangle\right)$

$$prob(+1) = |\langle\uparrow|\psi\rangle|^2 = \frac{1}{2}$$

$$prob(-1) = |\langle\downarrow|\psi\rangle|^2 = \frac{1}{2}$$

Magnets along x-axis

State before here $|\uparrow\rangle$

State after on the screen $|+\rangle, \ |-\rangle$

$$\frac{1}{\sqrt{2}}\left(|\uparrow\rangle + |\downarrow\rangle\right) \qquad \frac{1}{\sqrt{2}}\left(|\uparrow\rangle - |\downarrow\rangle\right)$$

Look at Feynman Lectures about this experiment

Observable that records meas. results

$$S_1 = (+1)\cdot |+\rangle\langle +| + (-1)\,|-\rangle\langle -|$$

$$\mathrm{Prob}(+1) = \frac{1}{2} = \left|\langle +|\uparrow\rangle\right|^2$$

$$\mathrm{Prob}(-1) = \frac{1}{2} = \left|\langle -|\uparrow\rangle\right|^2$$

$$S_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_x \quad X$$

Along y-direction $\quad M_2 = \gamma \frac{\hbar}{2} S_2$

$|\circlearrowleft\rangle, \quad |\circlearrowright\rangle$

two states $|\circlearrowleft\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle - i|\downarrow\rangle\right)$

$$|\circlearrowright\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle + i|\downarrow\rangle\right)$$

$$S_2 = (+1)\,|\circlearrowleft\rangle\langle\circlearrowleft| + (-1)\,|\circlearrowright\rangle\langle\circlearrowright|$$

Properties of Pauli Matrices

$$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\sigma_x \sigma_y = - \sigma_y \sigma_x$$

$$\sigma_y \sigma_z = - \sigma_z \sigma_y$$

$$\sigma_x \sigma_z = - \sigma_z \sigma_x$$

$$\left.\begin{array}{l} \sigma_x \sigma_y - \sigma_y \sigma_x = 2i\, \sigma_z \\ \\ + \text{ cyclic permutation of } x, y, z \end{array}\right\}$$

Called commutation $[A,B] \equiv AB - BA$

$$[\sigma_x , \sigma_y] = 2i\, \sigma_z$$

$$\sigma_x^{+} = \sigma_x \quad , \quad \sigma_y^{+} = \sigma_y \quad , \quad \sigma_z^{+} = \sigma_z$$

Hamiltonian of spin 1/2 in $\vec{B} = (B_x, B_y, B_z)$

$$H = -\gamma \frac{\hbar}{2} \vec{S}\cdot\vec{B} = -\gamma \frac{\hbar}{2}\left\{ \sigma_x B_x + \sigma_y B_y + \sigma_z B_z \right\}$$

$$= -\gamma\frac{\hbar}{2}\left(\begin{array}{c|c} B_z & B_x - iB_y \\ \hline B_x + iB_y & B_z \end{array}\right)$$

$H = H^{\dagger}$ is an observable that records meas. of energy of the magnetic moment or $\vec{B}$?

Physically you can $\vec{B} // e$ $\quad \vec{B} = (0, 0, B)$

$\downarrow$

$$\vec{H} = -\frac{\gamma\hbar}{2} B \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

unit of energy

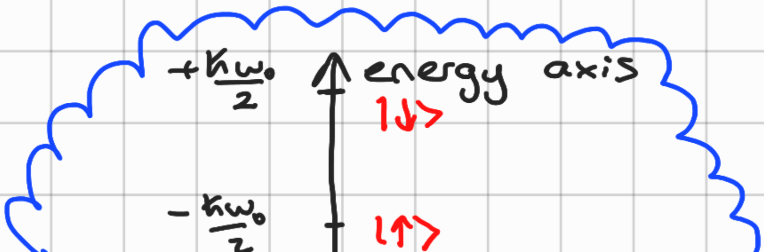
$$\hbar\frac{\gamma B}{2} = \hbar\omega_0$$ → Larmor frequency

$$[\zeta] = [J.s]\left[\frac{1}{s}\right]$$

$$H = -\frac{\hbar\omega_0}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Dirac Not.

$$= -\frac{\hbar\omega_0}{2}\left(|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow| \right)$$



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Hamiltonian of spin 1/2 in magnetic field

$$\vec{B} = (0, 0, B)$$

$$H = -\gamma \frac{\hbar}{2} \sigma_z B = -\frac{\gamma \hbar B}{2} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{z = \sigma_z}$$

$\omega_0 \left[\frac{1}{s}\right]$ Larmor frequency $\frac{\hbar\omega_0}{2}$

Eigenvalues & Eigenvectors of H

$\frac{\hbar\omega_0}{2}$ $\qquad H|\downarrow\rangle = +\frac{\hbar\omega_0}{2}|\downarrow\rangle \qquad |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$-\frac{\hbar\omega_0}{2}$ $\qquad H|\uparrow\rangle = -\frac{\hbar\omega_0}{2}|\uparrow\rangle \qquad |\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

"Two Level System" = qubit

$$H = \left(-\frac{\hbar\omega_0}{2}\right)|\uparrow\rangle\langle\uparrow| + \left(+\frac{\hbar\omega_0}{2}\right)|\downarrow\rangle\langle\downarrow|$$

two possible values $\nearrow -\frac{\hbar\omega_0}{2}$, $\searrow +\frac{\hbar\omega_0}{2}$

Geometrical representation of qubit states

Bloch Sphere

state vector in $\mathbb{C}^2$

$$\alpha|\uparrow\rangle + \beta|\downarrow\rangle = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$

$$\alpha\alpha^* + \beta\beta^* = 1$$

↳ Fix α to be real

First principle:

$|\psi\rangle$ equivalent to $e^{i\gamma}|\psi\rangle$, $\gamma \in [0, 2\pi]$

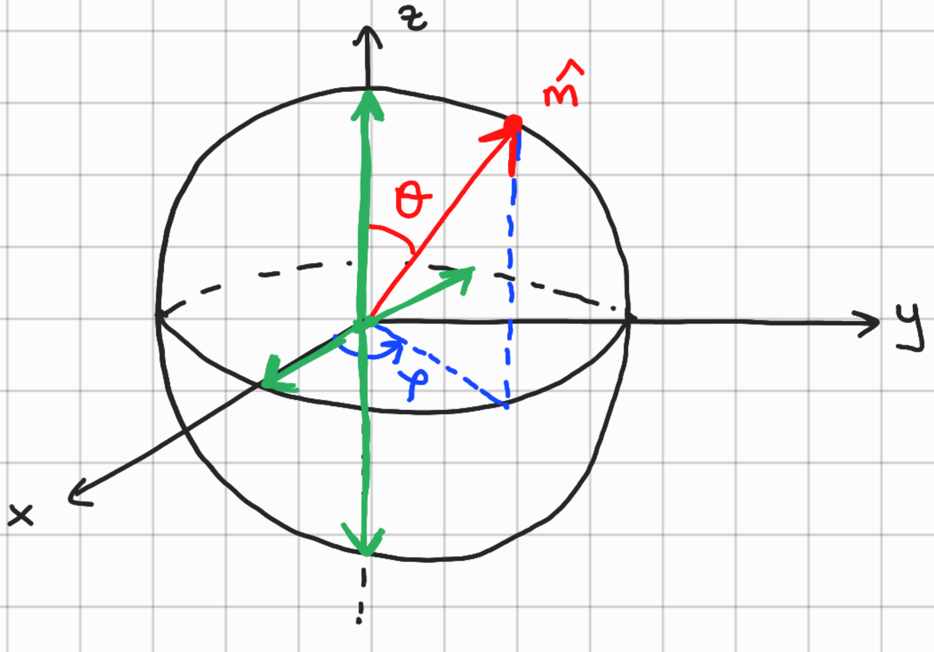
state vector

? "Global phase in bloch sphere"

In fact qubits described by 2 real parameters.

$$\alpha = \cos\left(\frac{\theta}{2}\right) \qquad \beta = e^{i\varphi}\sin\left(\frac{\theta}{2}\right)$$

$$0 \leq \theta \leq \pi \qquad 0 \leq \varphi \leq 2\pi$$



$$|\psi\rangle = \cos\frac{\theta}{2}\,|\uparrow\rangle + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)|\downarrow\rangle$$

$$\hat{m} = (\sin\theta\cos\alpha\ ,\ \sin\theta\sin\alpha,\ \cos\theta)$$

they're orthogonal even though it does seem parallel in Bloch sphere

$$|\uparrow\rangle \iff \theta = 0 \quad , \quad \varphi \in [0, 2\pi]$$

$$|\downarrow\rangle \iff \theta = \pi \quad , \quad \varphi \in [0, 2\pi]$$

$$|+\rangle = \frac{1}{\sqrt{2}}|\uparrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\rangle \qquad \theta = \frac{\pi}{2} \quad , \quad \varphi = 0$$

$$|-\rangle = \frac{1}{\sqrt{2}}|\uparrow\rangle - \frac{1}{\sqrt{2}}|\downarrow\rangle \qquad \theta = \frac{\pi}{2} \quad , \quad \varphi = \pi$$

<u>Dynamics</u> $\qquad |\psi_t\rangle = U_t\,|\psi_0\rangle$

Equation to compute U_t

↓

Schrödinger Equation (1926)

↓

$$\boxed{i\hbar\frac{d}{dt}U_t = H\,U_t} \quad ; \quad \boxed{U_{t=0} = \mathbb{1}}$$

Photon Polarization State $\quad \overbrace{e^{-i\omega t}}^{U_t}|\alpha\rangle$

$$i\hbar\frac{d}{dt}\left(e^{\underbrace{-i\omega t}_{U_t}}\right) = \hbar\omega\, \underbrace{e^{-i\omega t}}_{U_t}$$

Solution of Schr eqn. for $\vec{B} = (0,0,B)$

Remark: if H is independent of time

$$U_t = \exp\left(-\frac{it}{\hbar} H\right)$$

$$H = -\frac{\hbar\omega_0}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -\frac{\hbar\omega_0}{2} & 0 \\ 0 & +\frac{\hbar\omega_0}{2} \end{pmatrix}$$

$$e^{\frac{-it}{\hbar}H} = \begin{pmatrix} e^{i\frac{\omega_0}{2}t} & 0 \\ 0 & e^{-i\frac{\omega_0}{2}t} \end{pmatrix}$$

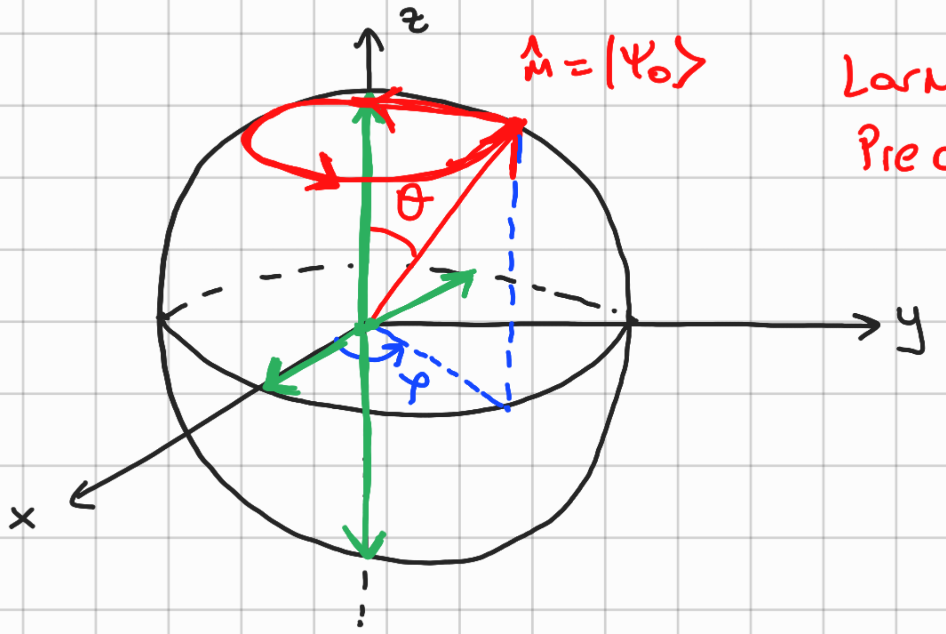
$$|\psi_t\rangle = \begin{pmatrix} e^{i\omega_0 t/2} & 0 \\ 0 & e^{-i\omega_0 t/2} \end{pmatrix} |\psi_0\rangle$$

$$|\psi_0\rangle = \cos\frac{\theta}{2}|\uparrow\rangle + e^{i\varphi}\sin\frac{\theta}{2}|\downarrow\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ \left(\sin\frac{\theta}{2}\right)e^{i\varphi} \end{pmatrix}$$

$$|\psi_t\rangle = \begin{pmatrix} e^{i\omega_0 t/2}\cos\theta/2 \\ e^{-i\omega_0 t/2}\left(\sin\frac{\theta}{2}\right)e^{i\varphi} \end{pmatrix}$$

$$T = \frac{2\pi}{\omega_0}$$

$$|\psi_t\rangle = \underbrace{\cancel{e^{i\omega_0 t/2}}}_{\text{global phase}} \begin{pmatrix} \cos\theta/2 \\ \sin\frac{\theta}{2}\, e^{i(\alpha-\omega_0 t)} \end{pmatrix}$$



Larmor Precession