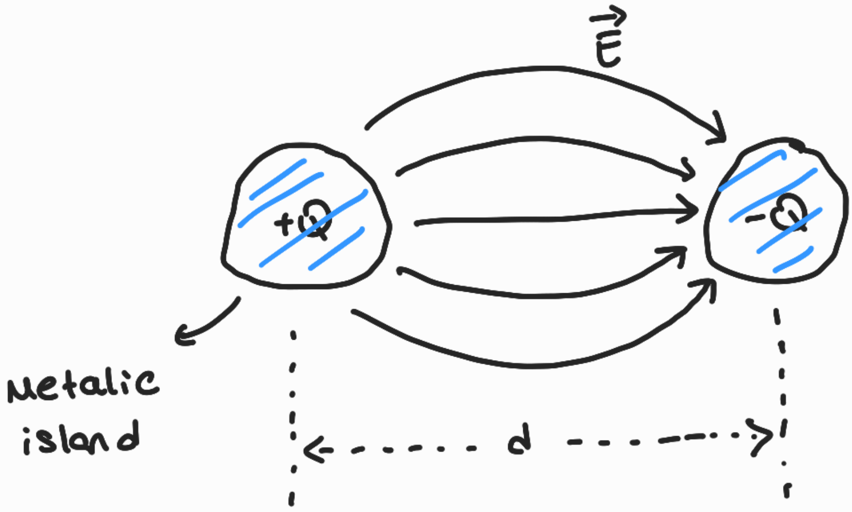
Quantization of electrical circuits

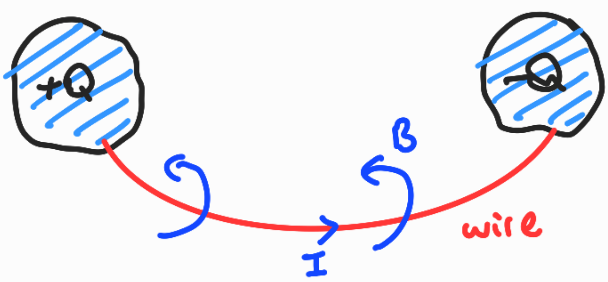


The energy required to move one unit charge from one to the other island is path-independent and given by the voltage V.

$$V = \frac{Q}{C}$$

Total energy stored in the electric field:

$$E_{el} = \int_0^Q dQ' \, V(Q') = \int_0^Q dQ' \, \frac{Q'}{C} = \frac{Q^2}{2C}$$



$$I = -\frac{dQ}{dt} = -\dot{Q}$$

$$\Phi_B = L \, I$$

Magnetic flux

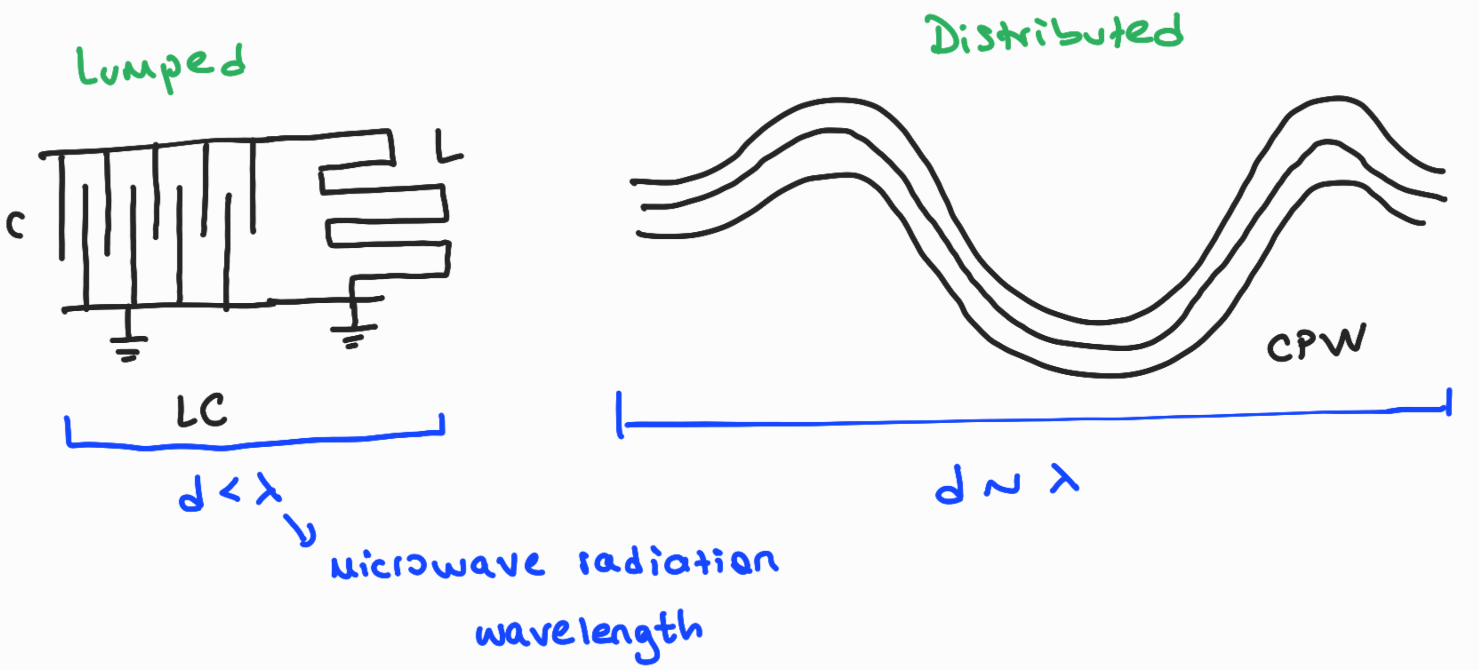
inductance (geometry dependent)

induced voltage

$$\Phi_B(t) = \int_0^t v(t') \, dt' \quad (?)$$

Energy stored in magnetic field $\vec{B}$:

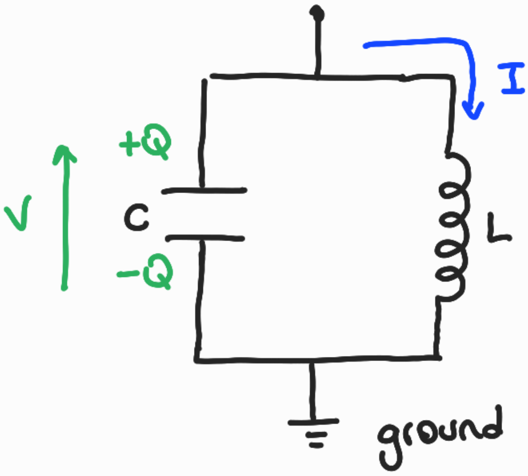
$$E_{mag} = \int_0^I dI' \, \Phi(I') = \frac{1}{2} L I^2 = \frac{\Phi^2}{2L}$$



For GHz photons $\longrightarrow$ $\lambda \sim$ cm

For the low frequency dynamics of this system are fully captured by two effective parameters L, C.

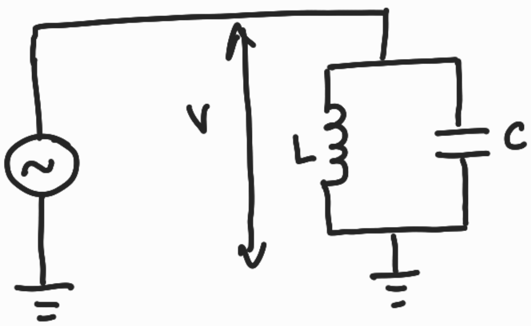
Quantization of the LC Resonator



Faraday's law of induction $\dot{\Phi} = V$

$$\ddot{\Phi} = \dot{V} = \frac{\dot{Q}}{C} = -\frac{I}{C} = -\frac{\Phi}{LC} = -\omega_0^2 \Phi \quad , \quad \omega_0 = \frac{1}{\sqrt{LC}}$$

Electronic Harmonic Oscillator



$Q = CV$

$\Phi = LI$

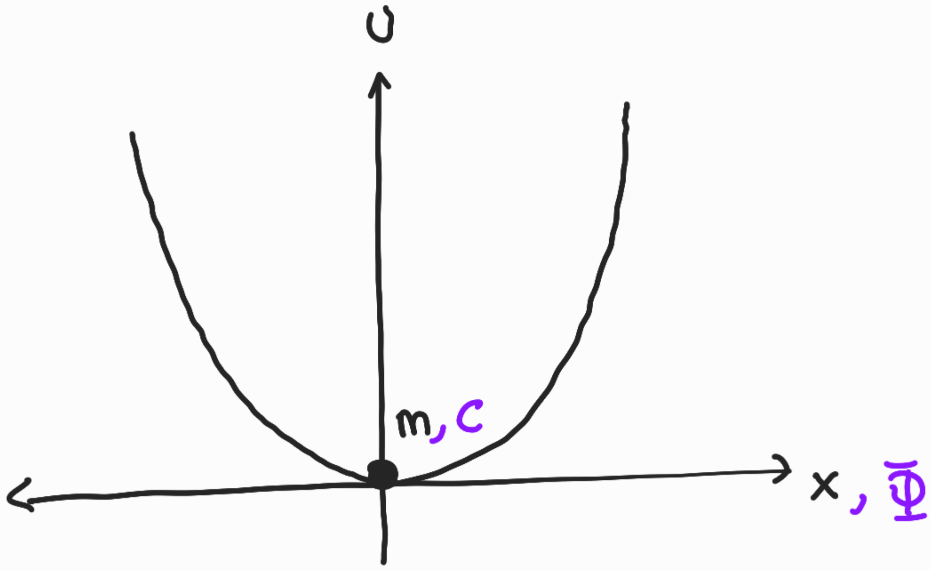
$$V = \frac{Q}{C} = -L\dot{I} = -\dot{\Phi}$$

Hamiltonian :
$$\frac{CV^2}{2} + \frac{LI^2}{2} = \frac{Q^2}{2C} + \frac{\Phi^2}{2L}$$

Mechanical Oscillator $\leftarrow$ $H = \frac{p^2}{2m} + \frac{kx^2}{2}$

kinetic, K

potential, U



Characteristic Quantities

Mechanical

Position x

Momentum p

Mass m

Spring constant k

resonance freq $\omega = \sqrt{\frac{k}{m}}$

Electronic

Flux Φ

Charge Q

Capacitance C

inverse inductance $1/L$

$$\omega = \sqrt{\frac{1}{LC}}$$

$$\hat{x} = x$$

$$\hat{p} = -i\hbar \frac{\partial}{\partial x}$$

$$\hat{\Phi} = \Phi$$

$$\hat{Q} = -i\hbar \frac{\partial}{\partial \Phi}$$

$$[\hat{x},\hat{p}] = (\hat{x}\hat{p} - \hat{p}\hat{x})\psi$$

$$= x(-i\hbar)\frac{\partial}{\partial x}\psi - (-i\hbar)\frac{\partial}{\partial x}(x\psi)$$

$$= \cancel{-i\hbar x\frac{\partial\psi}{\partial x}} + i\hbar\psi + \cancel{i\hbar\frac{\partial\psi}{\partial x}x} = i\hbar$$

$$[\hat{\Phi},\hat{Q}] = i\hbar$$

HAMILTONIAN OPERATOR

Using conjugate variables Q, Φ

$$\hat{H} = \frac{\hat{Q}^2}{2C} + \frac{\Phi^2}{2L} = -\frac{\hbar^2}{2C}\frac{\partial^2}{\partial\Phi^2} + \frac{1}{2L}\Phi^2$$

m = number operators

$$\hat{H} = \hbar\omega\left[\hat{a}^+\hat{a} + \frac{1}{2}\right]$$

$$\hat{a}^+ = \frac{1}{\sqrt{2\hbar Z_c}}\left(Z_c\hat{Q}^+ - i\hat{\Phi}^+\right) \rightarrow \text{creation}$$

$$\hat{a} = \frac{1}{\sqrt{2\hbar Z_c}}\left(Z_c\hat{Q} + i\hat{\Phi}\right) \rightarrow \text{annihilation}$$

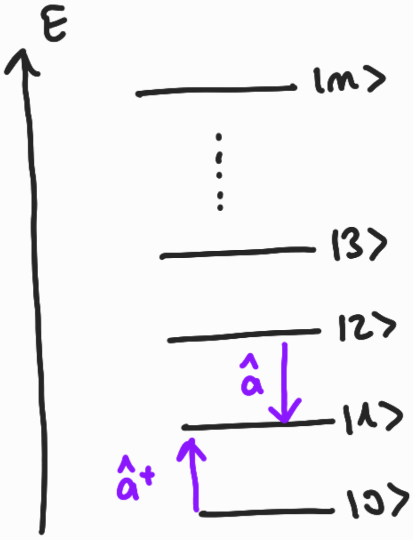
with $Z_c = \sqrt{\frac{L}{C}}$ impedance of oscillator

Properties of $\hat{a}^+$ and $\hat{a}$

$$\hat{a}^+ |n\rangle = \sqrt{n+1}\, |n+1\rangle$$

$$\hat{a}\, |n\rangle = \sqrt{n}\, |n-1\rangle$$

$$\hat{a}^+ \hat{a}\, |n\rangle = n\, |n\rangle$$



with $|n\rangle$ number (Fock) state of harmonic oscillator

$$\hat{Q} = \sqrt{\frac{\hbar}{2Z_c}}\left(\hat{a}^+ + \hat{a}\right)$$

$$\hat{\Phi} = i\sqrt{\frac{\hbar Z_c}{2}}\left(\hat{a}^+ - \hat{a}\right)$$

$$\hat{V} = \sqrt{\frac{\hbar\omega}{2C}}\left(\hat{a}^+ + \hat{a}\right)$$

$$\hat{I} = i\sqrt{\frac{\hbar\omega}{2L}}\left(\hat{a}^+ - \hat{a}\right)$$

where $\omega = \frac{1}{\sqrt{C}}$ $V = \frac{Q}{C}$

$I = \frac{\Phi}{L}$

Lagrangian Formalism

$$\mathcal{L}(x, \dot{x})$$

$$\mathcal{L} = E_k - E_p$$

Hamiltonian $\leftarrow \mathcal{H} = \mathcal{H}(x, p)$

Conjugate variable $\quad p = \dfrac{\partial \mathcal{L}}{\partial \dot{x}}$

(generalized momentum)

Connection between Lagrangian and Hamiltonian via Legendre form:

$$\boxed{\mathcal{H}(x, p) = \dot{x}p - \mathcal{L}(x, \dot{x})}$$

$\downarrow$

for complicated circuits (?)