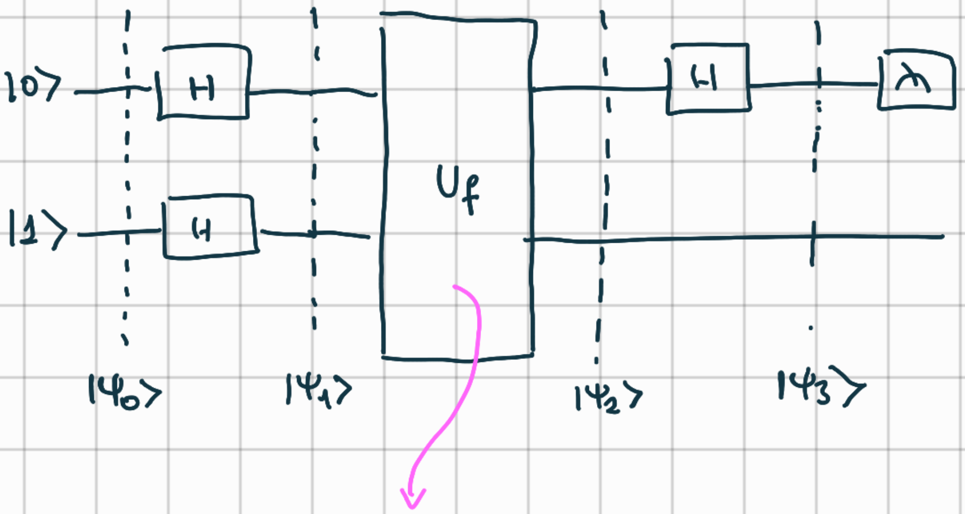
27/10/22

Deutsch's Algorithm

Task $\quad f(x) : \{0,1\} \rightarrow \{0,1\}$

can be degenerate or not ?

$$f(0) = f(1) = ?$$



$U_f |x\rangle |y\rangle = |x\rangle |y \oplus f(x)\rangle$

$$|\psi_3\rangle = \frac{1}{2\sqrt{2}} \left[\left[(-1)^{f(0)} + (-1)^{f(1)} \right] |0\rangle + \left[(-1)^{f(0)} - (-1)^{f(1)} \right] |1\rangle \right] \otimes \left(|0\rangle - |1\rangle \right)$$

phase kickback

Deutsch - Jozsa Algorithm

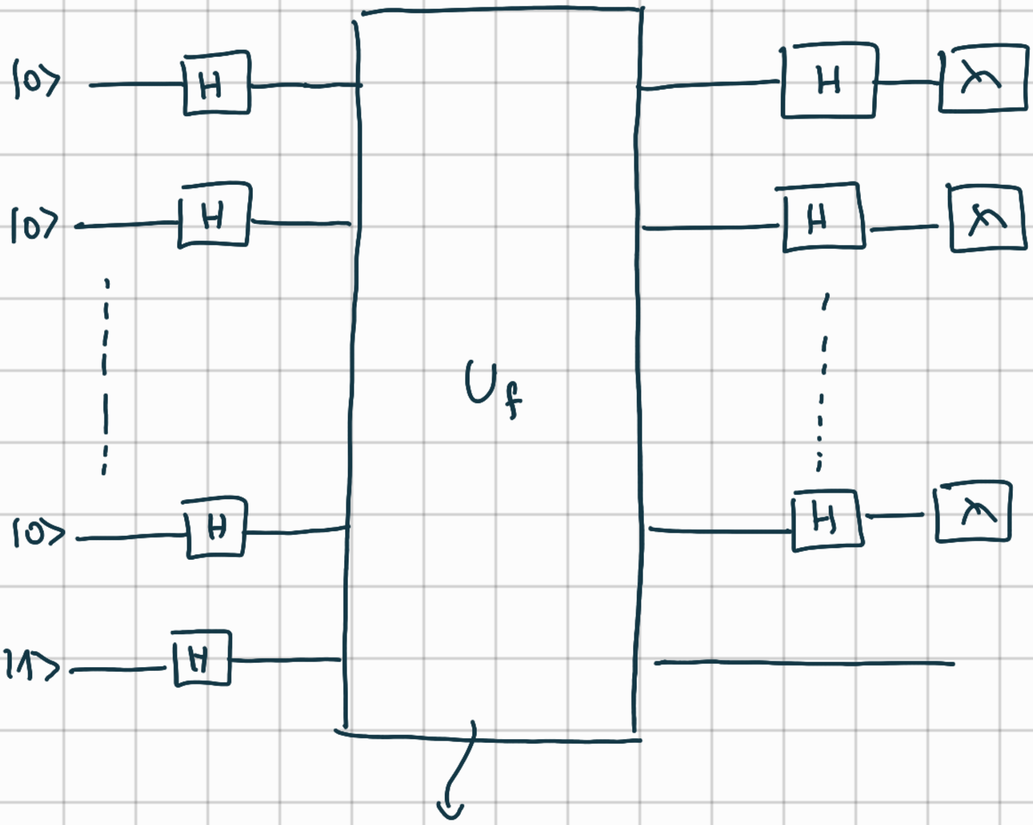
Task: $f(x) : \{0,1\}^n \longrightarrow \{0,1\}$

We know that $f(x)$ is either constant or balanced. Which one?

constant: $f(x) = f(x') \quad \forall x, x'$

balanced: half as many x $f(x)=0$ / $f(x)=1$

Classical Algorithm: 2^{n-1} computations of $f(x)$ in the worst case.



$$U_f |x\rangle |y\rangle = |x\rangle |y \oplus f(x)\rangle$$

If f is constant : $|0\rangle$ with 100 % probability

" " balanced : $|0\rangle$ with 0 probability

What if we ask:

Is f balanced with prob $p = 1 - \varepsilon$?

Suppose f balanced and we get $f(x) = 1$ or 0 $1k$ times.

$$p = \frac{1}{2^{k-1}}$$

Prob. that by measuring constant k times, f is balanced

$$p' = \frac{1}{2^{k-1} + 1}$$

Total prob. of error after k comp.

$$P_{err} = 1 - P = \frac{1}{2^{k-1}(2^{k-1}+1)} \simeq 2^{-k} = \varepsilon$$

probability of success

$$k \sim \frac{1}{2} \log\left(\frac{1}{\varepsilon}\right)$$

Deutsch Jozsa - 1992 poly(n) adv over worst case

Bernstein - Vazirani - 1995 poly(n) adv over all cases

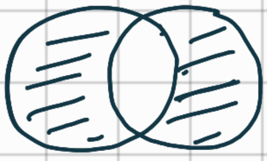
Simon - 1995 exp(n) adv. over all case

Probabilistic answer accurate with ε

→ Seperation between BQP and BPP

BPP bounded-error probabilistic polynom

BQP " " quantum polynom



relative to an oracle

Quantum Fourier Transform

Discrete FT over $N \sim 2^n$ values

DFT $O(N^2)$

FFT $O(nN)$

QFT $O(n^2)$

DFT input $(x_0, x_1, \dots, x_{n-1})$

output $(y_0, y_1, \dots, y_{n-1})$

$$y_k = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} x_j e^{2\pi i jk/N}$$

QFT $\{ |0\rangle, |1\rangle, |2\rangle, |3\rangle \dots |N-1\rangle \}$

QFT $|j\rangle \rightarrow \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{2\pi i jk/N} |k\rangle$

$$\sum_{j=0}^{N-1} x_j |j\rangle \longrightarrow \sum_{k=0}^{N-1} y_k |k\rangle$$

$x_j \in \mathbb{C}$

Def: $\quad j \in \{0, 1, \ldots, 2^n - 1\}$

$$\frac{j}{N} = \frac{j}{2^n} < 1 \rightarrow 0{,}713$$

$$\frac{j}{N} = \frac{j_1}{2} + \frac{j_2}{4} + \cdots + \frac{j_n}{2^n}$$ → binary decimal expansion

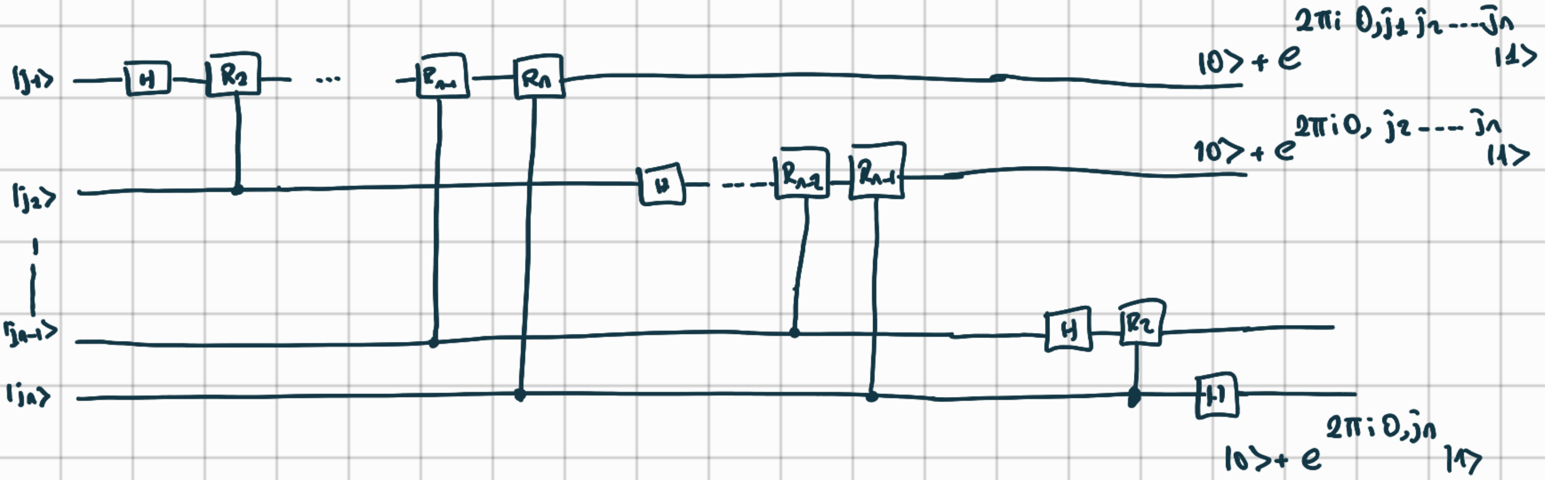
$$\frac{j}{N} = 0.j_1 j_2 \ldots j_n$$ (binary decimal)

"Theorem"

$$\frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{2\pi i jk/N} |k\rangle =$$

$$= \frac{1}{\sqrt{2^n}} \left(|0\rangle + e^{2\pi i\, 0.j_1} |1\rangle \right) \otimes \left(|0\rangle + e^{2\pi i\, 0.j_{n-1} j_n} |1\rangle \right) \otimes \ldots$$

$$\ldots \otimes \left(|0\rangle + e^{2\pi i\, 0.j_1 j_2 \ldots j_n} |1\rangle \right)$$



$$R_k = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i/2^k} \end{pmatrix}$$

Quantum Phase Estimation (QPE)

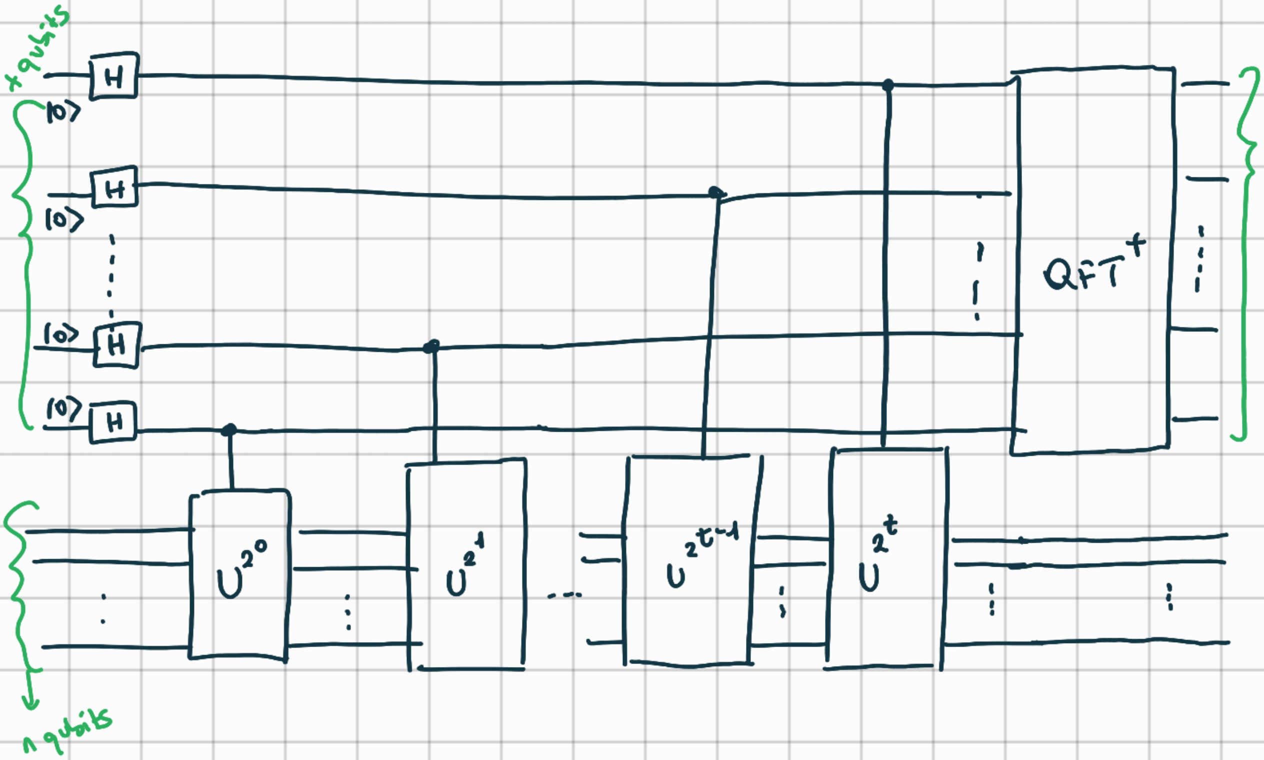
Task: Given $\hat{U}$ unitary on n qubits

given $|u\rangle$ eigenstates of $\hat{U}$ $\rightarrow$ $\hat{U}|u\rangle = e^{2\pi i \varphi}|u\rangle$

length must be "1" since U is unitary

Find (estimate) φ !

QPE requires $C\text{-}U^j$



$|\varphi\rangle = |\varphi_1 \dots \varphi_t\rangle$

$\varphi_j = \frac{\varphi}{2^j}$

$2^t\varphi$ is a t-bit integer