Single and Two Qubit Gates

A pulse in the z-direction:

$B_z(t) \rightarrow$ time-dependent

$$i\hbar \frac{d}{dt}|\Psi(t)\rangle = \hat{H}|\Psi(t)\rangle$$

$$\hat{H} = -\frac{\hbar\gamma}{2} B_z(t)\,\sigma_z \quad , \quad |\Psi(t)\rangle = c_0(t)|0\rangle + c_1(t)|1\rangle$$

$$i\hbar \begin{bmatrix} \dot{c}_0(t) \\ \dot{c}_1(t) \end{bmatrix} = -\frac{\hbar\gamma}{2} \begin{bmatrix} B_z(t) & 0 \\ 0 & B_z(t) \end{bmatrix} \begin{bmatrix} c_0(t) \\ c_1(t) \end{bmatrix}$$

$$\begin{bmatrix} c_0(t) \\ c_1(t) \end{bmatrix} = \underbrace{\begin{bmatrix} e^{i\delta/2} & 0 \\ 0 & e^{-i\delta/2} \end{bmatrix}}_{R_z(-\delta)} \begin{bmatrix} c_0(0) \\ c_1(0) \end{bmatrix} \quad \text{where} \quad \delta = \gamma \int_0^t dt' B_z(t')$$

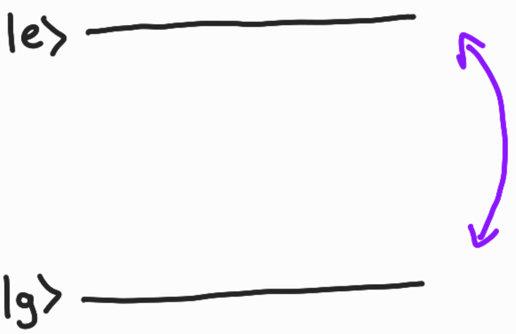
$$R_z(-\delta)|\Psi(\vartheta,\phi)\rangle = |\Psi(\vartheta, \phi-\delta)\rangle$$

* Note: One can reach any point on the Bloch-sphere with two consecutive rotations along two not-parallel axis.

Single Qubit Gates

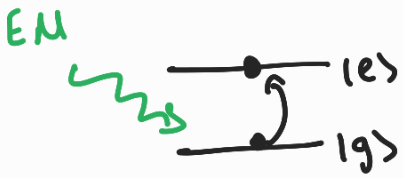
Use small amplitude harmonic perturbation of some qubit parameters.

Rabi - NMR (nuclear magnetic resonance)



$\Delta E = E_e - E_g = \hbar\omega_0$

artificial atom



If EM radiation $\omega \sim \omega_0$ ⤳ resonant freq. , photon can be absorbed and the state can change.

$\hat{H}_{int} = -\vec{d} \cdot \vec{E}(t)$ ⤳ Field - Atom Interaction Hamiltonian

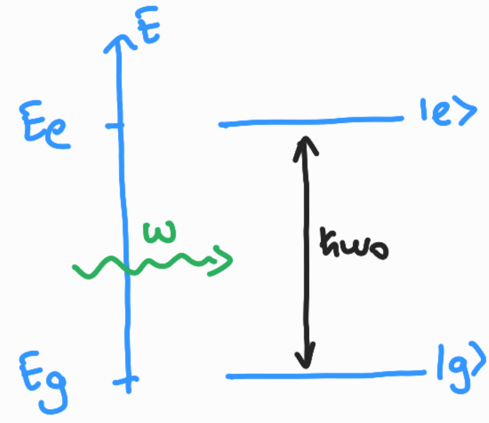
$\vec{d}$ ↙ dipole moment operator

Assume harmonic radiation $\vec{E}(t) = \vec{E}_0 \cos(\omega t + \phi)$ with frequency close to that of the atom resonance

$|\omega_0 - \omega| \ll \omega_0$

In Exercise 2

$$\hat{H} = -\frac{\hbar\omega_0}{2}\sigma_z - A\cos(\omega t+\phi)\,\sigma_x$$



where $\underset{?}{A} = \langle e|\vec{d}\cdot\vec{E}|g\rangle = -\hbar\Omega_1$

Ansatz for the state:

$$|\psi(t)\rangle = c_g(t)\,e^{-iE_g t/\hbar}|g\rangle + c_e(t)\,e^{-iE_e t/\hbar}|e\rangle$$

with $E_g = -\frac{\hbar\omega_0}{2}$, $E_e = \frac{\hbar\omega_0}{2}$

From the Schrödinger Eqn: $i\hbar\frac{d}{dt}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle$

$$\dot{c}_g(t) = \frac{i}{\hbar}A\cos(\omega t+\phi)\,e^{-i\omega_0 t}\,c_e$$

$$\dot{c}_e(t) = \frac{i}{\hbar}A\cos(\omega t+\phi)\,e^{+i\omega_0 t}\,c_g$$

$$\cos(\omega t+\phi) = \frac{\left[e^{i(\omega t+\phi)} + e^{-i(\omega t+\phi)}\right]}{2}$$

One can find slowly rotating forms $e^{\pm i(\omega-\omega_0)t}$ and fast oscillating terms $\cancel{e^{\pm i(\omega+\omega_0)t}}$

omit this term (RWA)

* since time evolution induced by the field is much slower than ω_0

Leaving:

$$\dot{c}_g(t) = \frac{i}{2\hbar} A e^{i\phi} e^{i(\omega-\omega_0)t} c_e \quad (1)$$

$$\dot{c}_e(t) = \frac{i}{2\hbar} A e^{-i\phi} e^{-i(\omega-\omega_0)t} c_g \quad (2)$$

From Eq (1) and (2):

$$\ddot{C}_e + i \overbrace{(\omega-\omega_0)}^{\Delta \text{ detuning}} \dot{C}_e + \frac{1}{4}\frac{A^2}{\hbar^2} C_e = 0$$

From the trial solution $c_e(t) = e^{i\lambda t}$

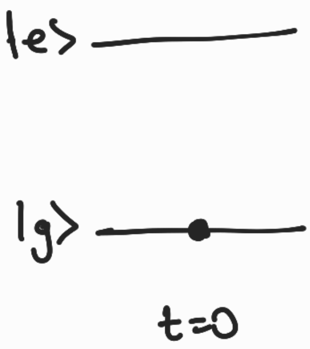
$$\lambda_\pm \; \frac{1}{2}\left(\Delta \pm \underbrace{\sqrt{\Delta^2 + \frac{A^2}{\hbar^2}}}_{\Omega_R \,\to\, \text{Rabi frequency}}\right) = \frac{1}{2}\left(\Delta \pm \Omega_R\right)$$

$$c_e(t) = C_+ e^{i\lambda_+ t} + C_- e^{i\lambda_- t}$$

$$c_g(t) = \frac{2\hbar}{iA} e^{i\phi} e^{-i\Delta t}\left[i\lambda_+ C_+ e^{i\lambda_+ t} + i\lambda_- C_- e^{i\lambda_- t}\right]$$

Example: At time $t=0$ in $|g\rangle$

$$c_g(0)=1 \quad , \quad c_e(0)=0$$



$$c_e(t)=\frac{iA}{\Omega\hbar}\, e^{i\Delta t/2} \sin\left(\frac{\Omega_e t}{2}\right)$$

$$c_g(t)= e^{i\phi} . e^{-i\Delta t/2}\left[\cos\left(\frac{\Omega_e t}{2}\right)+ i\frac{\Delta}{\Omega_e} \sin\left(\frac{\Omega_e t}{2}\right)\right]$$

$$P_e(t) = |\langle e|\psi(t)\rangle|^2 = |c_e(t)|^2 = \frac{A^2}{\Omega_e^2\hbar^2} \sin\left(\frac{\Omega_e t}{2}\right)^2$$

Let's move to

Rotating Frame

@ freq ω_0

$$H_{rot} = i\hbar\,\dot{U}U^\dagger + UHU^\dagger$$

Derivation: $|\tilde{\Psi}(t)\rangle = U(t)\,|\Psi(t)\rangle$

$$i\hbar\frac{d}{dt}|\tilde{\Psi}(t)\rangle = i\hbar\frac{\partial}{\partial t}U(t)\,|\overset{U^\dagger\tilde{\Psi}(t)}{\Psi(t)}\rangle + U(t)\,\overset{\substack{U^\dagger\tilde{\Psi} \\ \uparrow \\ H|\Psi\rangle}}{i\hbar\frac{d}{dt}|\Psi(t)\rangle}$$

$$i\hbar\frac{\partial}{\partial t}|\tilde{\Psi}(t)\rangle = i\hbar\Big(\overbrace{\frac{\partial}{\partial t}U(t)}^{\dot{U}}\Big)U^\dagger\tilde{\Psi}(t) + U(t)\,H\,U^\dagger(t)\,\tilde{\Psi}(t)$$

$$\boxed{H_{rot} = i\hbar\,\dot{U}U^\dagger + UHU^\dagger}$$

For this case: $U(t) = e^{-i\frac{\omega_0 t}{2}\hat{\sigma}_z}$

...

↑

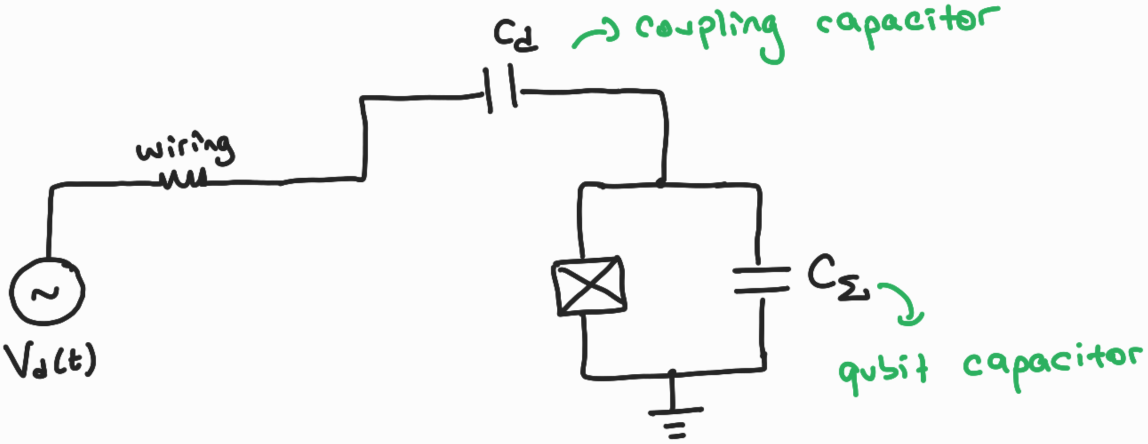
Look at Exc2

$$H_{rot} = -\frac{A}{2}\left[\begin{pmatrix} 0 & e^{i\phi}e^{i\overset{\Delta=0}{(\omega-\omega_0)}t} \\ e^{i\phi}e^{i\underset{RWA}{\cancel{(\omega+\omega_0)}}t} & 0 \end{pmatrix} + \begin{pmatrix} 0 & e^{-i\phi}e^{-i\overset{RWA}{\cancel{(\omega+\omega_0)}}t} \\ e^{-i\phi}e^{-i\underset{\Delta=0}{(\omega-\omega_0)}t} & 0 \end{pmatrix}\right.$$

For $\Delta=0$

$$-\frac{A}{2}\begin{pmatrix} 0 & e^{+i\phi} \\ e^{-i\phi} & 0 \end{pmatrix} = -\frac{A}{2}\left[\cos\phi\,\sigma_x - \sin\phi\,\sigma_y\right]$$

Example: Capacitive coupling of MW to a superconducting circuit can be used to derive single-qubit gates



@ Room temperature

Capacitive Coupling: X, Y control

$$H = -\frac{\omega_q}{2}\sigma_z + \Omega V_d(t)\sigma_y = H_0 + H_{int}$$

$$\Omega = \left(\frac{C_d}{C_\Sigma}\right) Q_{zpf} \quad (?) \qquad \omega_q = \frac{(E_1 - E_0)}{\hbar}$$

Move to Rotating Frame: (with frequency ω_q)

$$U_{rf} = e^{iH_0 t} = U_{H_0}^{+}$$

$$|\Psi_{rf}(t)\rangle = U_{rf}|\Psi_0\rangle$$

$$i\frac{\partial}{\partial t}|\Psi_{rf}(t)\rangle = i\left(\partial_t U_{rf}\right)|\Psi_0\rangle + i\,U_{rf}\left(\partial_t|\Psi_0\rangle\right)$$

$$= i\,\dot{U}_{rf}U_{rf}^{+}|\Psi_{rf}\rangle + U_{rf}H_0|\Psi_0\rangle$$

$$= \left(\underbrace{i\,\dot{U}_{rf}U_{rf}^{+} + U_{rf}H_0U_{rf}^{+}}_{\tilde{H}_0}\right)|\Psi_{rf}\rangle$$

$$\frac{d}{dt} e^{iH_0 t} = iH_0 e^{iH_0 t}$$

$$\tilde{H}_0 = i\,(iH_0 e^{iH_0 t})e^{-iH_0 t} + e^{iH_0 t} H_0 e^{-iH_0 t}$$

$$\tilde{H}_0 = -H_0 + H_0 = 0$$

driving Hamiltonian

One can find $\tilde{H}_d = \Omega V_d(t)\left[\cos(\omega_q t)\sigma_y - \sin(\omega_q t)\sigma_x\right]$

Assume $V_d(t) = V_0 v(t)$

$$v(t) = s(t)\sin(\omega_d t + \phi)$$

$$= s(t)\left[\cos(\phi)\sin(\omega_d t) + \sin(\phi)\cos(\omega_d t)\right]$$

dimensionless envelope function } $s(t)$ sets the shape and amplitude $[V_0 s(t)]$ of the drive

Microwave Pulses $\begin{cases} I = \cos\phi & \rightarrow \text{"In phase component"} \\ Q = \sin\phi & \rightarrow \text{"Out phase component"} \end{cases}$

Rewrite the driving Hamiltonian

$$\tilde{H}_d = \Omega V_0 s(t)\left[I\sin(\omega_d t) - Q\cos(\omega_d t)\right]\left[\cos(\omega_q t)\sigma_y - \sin(\omega_q t)\sigma_x\right]$$

$\delta\omega = \omega_q - \omega_d$

$$\tilde{H}_d = \frac{1}{2}\Omega V_0 s(t)\Big[\Big((-I\cos(\delta\omega t) + Q\sin(\delta\omega t)\Big)\sigma_x + \Big((I\sin(\delta\omega t) - Q\cos(\delta\omega t)\Big)\sigma_y\Big]$$

$(\omega + \omega_d)$ terms were dropped with RWA.

$$\boxed{\tilde{H}_d = \frac{-\Omega}{2} V_0 s(t) \begin{bmatrix} 0 & e^{i(\delta\omega t+\phi)} \\ e^{-i(\delta\omega t+\phi)} & 0 \end{bmatrix}}$$

Driving Hamiltonian in the rotating frame is a powerful tool for understanding single qubit gates in Super-conducting qubits

. Let's apply a pulse at the qubit frequency ω_q

$$\omega_d = \omega_q \rightarrow \delta\omega = \omega_d - \omega_q = 0$$

$$\tilde{H}_d = -\frac{\Omega}{2} V_0 s(t) \left(\underbrace{I}_{\cos\phi} \sigma_x + \underbrace{Q}_{\sin\phi} \sigma_y \right)$$

→ In-phase pulse ($\phi = 0$ or I quadrative)

yields rotation around the X-axis

→ Out-phase pulse ($\phi = \frac{\pi}{2}$, or Q quadrative)

yields rotation around the Y-axis

Concrete example of an in-phase pulse:

$$\underbrace{U_d^{\phi=0}(t)}_{\text{Rabi Driving}} = \exp\left[\left(i\frac{\Omega}{2}V_0\int_0^t s(t')dt'\right)\sigma_x\right]$$

V_0 and $s(t)$ are controlled using the Arbitrary Waveform Generator (AWG)

One can define $\theta(t) = -\Omega V_0 \int_0^t s(t')dt'$

To implement a π-pulse around the X-axis,

one should solve $\theta(t) = \pi$

In this framework, a sequence of pulses $\theta_k, \theta_{k-1}, \ldots, \theta_0$

is converted to a sequence of gates operating on a qubit

$$U_k \ldots\ldots U_1 U_0 = \prod_{m=0}^{k} \exp\left[-\frac{i}{2}\theta_m(t)\left(I_m\sigma_x + Q_m\sigma_y\right)\right]$$

I-Q (Microwave Pulses) $\Longrightarrow$ Rotation Around X, Y, Z $\Rightarrow$ Set of Unitary Gates

Physical Implementation of CNOT Gate:

$$H_{tot} = H_{qubit1} + H_{qubit2} + H_{INT}$$

$$H_{qubit1} = -\frac{E_C^1(t)}{2}\left(\sigma_1^z \otimes I_2\right) - \frac{E_J^1(t)}{2}\left(\sigma_1^x \otimes I_2\right)$$

? what happened them

$$H_{qubit2} = -\frac{E_C^2(t)}{2}\left(I_1 \otimes \sigma_2^z\right) - \frac{E_J^2(t)}{2}\left(I_1 \otimes \sigma_2^x\right)$$

$$H_{INT} = \frac{E_{INT}(t)}{2}\left(\sigma_1^z \otimes \sigma_2^z\right)$$

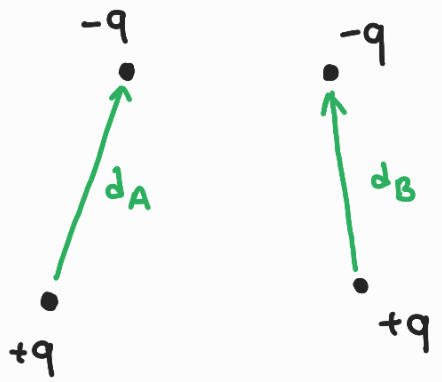
$\downarrow$ One can switch on this interaction term for a time giving the following transformation in the computational basis

$$\begin{bmatrix} c_{00}^{final} \\ c_{01}^{f} \\ c_{10}^{f} \\ c_{11}^{f} \end{bmatrix} = \begin{bmatrix} e^{-i\delta/2} & & & \\ & e^{+i\delta/2} & & \\ & & e^{+i\delta/2} & \\ & & & e^{-i\delta/2} \end{bmatrix} \begin{bmatrix} c_{00}^{initial} \\ c_{01}^{i} \\ c_{10}^{i} \\ c_{11}^{i} \end{bmatrix}$$

$$= e^{-i\delta/2} \begin{bmatrix} 1 & & & \\ & e^{i\delta} & & \\ & & e^{i\delta} & \\ & & & 1 \end{bmatrix} \begin{bmatrix} c_{00}^{i} \\ c_{01}^{i} \\ c_{10}^{i} \\ c_{11}^{i} \end{bmatrix} \quad \text{with } \delta = \frac{1}{\hbar}\int dt' E_{int}(t')$$

Accumulate $\delta = \frac{3}{2}\pi \rightarrow e^{-i3/4\pi} \begin{bmatrix} 1 & & & \\ & -i & & \\ & & -i & \\ & & & 1 \end{bmatrix}$ $\rightsquigarrow$ Apply other 1-qubit gates to retrieve CNOT

Two Qubit Gate via Dipolar Interaction



Atom A: $\{E_{0A}, E_{1A}\}$; $\{|0\rangle_A, |1\rangle_B\}$

Atom B: $\{E_{0B}, E_{1B}\}$; $\{|0\rangle_B, |1\rangle_B\}$

$$H_A = E_{0A}\, |0\rangle_A \langle 0|_A + E_{1A} |1\rangle_A \langle 1|_A$$

$$H_A = \begin{bmatrix} E_{0A} & 0 \\ 0 & E_{1A} \end{bmatrix} = \begin{bmatrix} \frac{E_{0A}-E_{1A}}{2} & 0 \\ 0 & \frac{E_{1A}-E_{0A}}{2} \end{bmatrix} + \cancel{\begin{bmatrix} \frac{E_{0A}+E_{1A}}{2} & 0 \\ 0 & \frac{E_{0A}+E_{1A}}{2} \end{bmatrix}}$$

why?

$$H_A = -\frac{1}{2}\underbrace{(E_{1A} - E_{0A})}_{\hbar\omega_A}\sigma_z^A$$

Similarly $H_B = -\frac{\hbar}{2}\omega_B \sigma_z^B$

$$H_{int} = -J\, \vec{d}_A \vec{d}_B$$ "Dipole-dipole interaction"

$$H_{int} = -J\left[\sum_{i,j} |i\rangle_A\langle i|_A\, \vec{d}_A\, |j\rangle_A\langle j|_A\right] \otimes \left[\sum_{i,j} |i\rangle_B\langle i|_B\, \vec{d}_B\, |j\rangle_B\langle j|_B\right]$$

$$\Downarrow$$

$$\langle 0|\vec{d}|0\rangle = \langle 1|\vec{d}|1\rangle = 0 \quad \text{due to symmetry}$$

In fact, $\vec{d} = \vec{r}q \Rightarrow q\langle 0|\vec{r}|0\rangle = \int d^3r \, \underbrace{e^{-r^2/a^2}}_{\text{even}} \, \underbrace{\vec{r}}_{\text{odd}} \, e^{-r^2/a^2} = 0$

So $J_A \, {}_A\langle 0|\vec{d}_A|1\rangle_A \overset{?}{=} J_B \, {}_B\langle 0|\vec{d}_B|1\rangle_B \overset{\text{def}}{\equiv} \hbar g$

$$H_{int} = -\hbar g \Big[\underbrace{|0\rangle_A {}_A\langle 1| + |1\rangle_A {}_A\langle 0|}_{\sigma_x^A} \Big] \otimes \Big[\underbrace{|0\rangle_B {}_B\langle 1| + |1\rangle_B {}_B\langle 0|}_{\sigma_x^B} \Big]$$

$$\boxed{H_{int} = -\hbar g \, \sigma_x^A \otimes \sigma_x^B}$$

$$H_{tot} = -\frac{1}{2}\hbar\omega_A\sigma_z^A - \frac{1}{2}\hbar\omega_B\sigma_z^B - \hbar g\sigma_x^A\sigma_x^B$$

Let's move to Rotating frame

$$U = e^{-i\frac{\omega_A t}{2}\sigma_z^A} \otimes e^{-i\frac{\omega_B t}{2}\sigma_z^B}$$

....

$$\tilde{H} = -\frac{1}{4}\hbar g\left[e^{-i(\omega_A-\omega_B)}\sigma_+^A\sigma_-^B + e^{+i(\omega_A-\omega_B)}\sigma_-^A\sigma_+^B\right]$$

$$\sigma_+ = \sigma_x + i\sigma_y = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$$

$$\sigma_- = \sigma_x - i\sigma_y = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$$

for $\omega_A = \omega_B$ (resonance)

$$\tilde{H} = -\frac{1}{2}\hbar g\left[\sigma_x^A\sigma_x^B + \sigma_y^A\sigma_y^B\right]$$

XY Coupling is very common in superconducting qubits.

What kind of 2-qubit gate this Hint generates?

$$U = \exp\left[-i\frac{\tilde{H}_{int}}{\hbar}t\right] = \exp\left[igt\frac{\sigma_{xA}}{2}\right]\otimes\exp\left[igt\frac{\sigma_{xB}}{2}\right]\cdot$$

$$\cdot\exp\left[igt\frac{\sigma_{yA}}{2}\right]\otimes\exp\left[igt\frac{\sigma_{yB}}{2}\right]$$

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(gt) & i\sin(gt) & 0 \\ 0 & i\sin(gt) & \cos(gt) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[t=\frac{\pi}{2g}]{\text{if}} U = \begin{bmatrix} 1 & & & \\ & 0 & i & \\ & i & 0 & \\ & & & 1 \end{bmatrix}$$

i SWAP gate