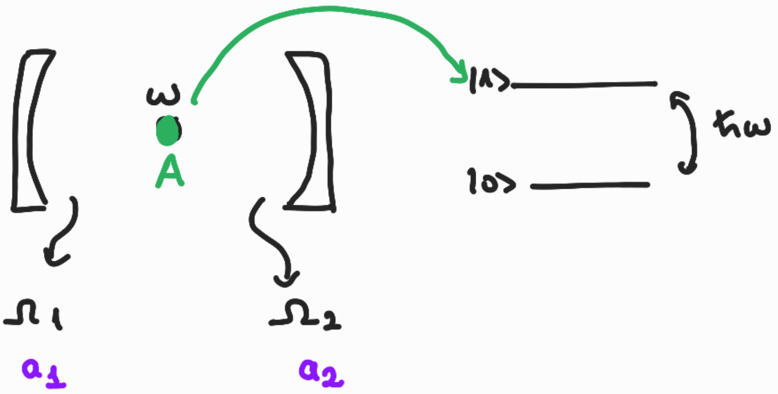
Two-Level Atom Interacting with 2-modes of Light in a Cavity



$\Omega_1 + \Omega_2 \simeq \omega$

$\Omega = \dfrac{\Omega_1 + \Omega_2}{2}$ carrier freq.

$\epsilon = \dfrac{\Omega_1 - \Omega_2}{2}$ modulation freq.

$\Delta = \omega - 2\Omega$ detuning

$$\hat{H} = \hat{H}_0 + \hat{H}_i$$

$$\hat{H}_0 = \frac{\omega}{2}\hat{\sigma}_z + (\overbrace{\Omega + \epsilon}^{\Omega_1})\hat{a}_1^{+}\hat{a}_1 + (\overbrace{\Omega - \epsilon}^{\Omega_2})\hat{a}_2^{+}\hat{a}_2$$

$$\hat{H}_i = g\left(\hat{a}_1^{+}\hat{a}_2^{+}\hat{\sigma}_- + \hat{\sigma}_+\hat{a}_1\hat{a}_2\right)$$

Note: In the Eberly's "Finite Time Disentanglement via Spontaneous Emission"

$$\hat{H}_i = \sum_{k=1}^{2}\left(g_k^{*}\hat{\sigma}_-\hat{a}_k^{+} + g_k\hat{\sigma}_+\hat{a}_k\right)$$

$$\hat{H} = \begin{array}{cc} |g, n_1+1, n_2+1\rangle & |e, n_1, n_2\rangle \\ \end{array}$$

$$\hat{H} = \begin{bmatrix} A(n_1, n_2) & g\sqrt{(n+1)(n+2)} \\ \underbrace{g\sqrt{(n_1+1)(n_2+1)}}_{\langle e, n_1, n_2 | H | g, n_1+1, n_2+1\rangle} & A(n_1, n_2) - \Delta \end{bmatrix}$$

$$\langle e, n_1, n_2 | \hat{\sigma}_+ \hat{a}_1 \hat{a}_2 | g, n_1+1, n_2+1\rangle = \sqrt{(n+1)(n+2)}$$

$a|n\rangle = \sqrt{n}\,|n-1\rangle$ $\quad\hat{a}_1|n_1+1\rangle = \sqrt{n_1+1}\,|n_1\rangle$ $\quad\hat{a}_2|n_2+1\rangle = \sqrt{n_2+1}\,|n_2\rangle$

$\hat{a}^{+}|n\rangle = \sqrt{n+1}\,|n+1\rangle$

Here, $A(n_1, n_2) = \frac{\omega}{2}(n_1 + n_2 + 1) - \frac{\Delta}{2}(n_1 + n_2) + \epsilon(n_1 - n_2)$

$$= \frac{\omega}{2} - \Omega_1 n_2 - \Omega_2 n_1$$

Eigenvalues $\lambda^{\pm} = K(n_1, n_2) \pm Q(n_1, n_2)$

$$K(n_1, n_2) = \frac{\omega - \Delta}{2}(n_1 + n_2 + 1) + \epsilon(n_1 - n_2)$$

$$Q(n_1, n_2) = \sqrt{\left(\frac{\Delta^2}{4} + \frac{4}{4} g^2 (n_1 + 1)(n_2 + 1)\right)}$$

$$= \frac{1}{2}\sqrt{(\Delta^2 + G_n^2)} \quad \text{where } G_n = 2g(n_1+1)(n_2+1)$$

$$\cos\theta_{n_1,n_2} = \frac{\Delta}{\sqrt{\Delta^2 + G_n^2}} \qquad \sin\theta_{n_1,n_2} = \frac{G_n}{\sqrt{\Delta^2 + G_n^2}}$$

Eigenstates

$$|\Psi_0\rangle = |g, 0, 0\rangle$$

$$|\Psi_{n_1,n_2}^{+}\rangle = c_{n_1,n_2}|e, n_1, n_2\rangle + s_{n_1,n_2}|g, n_1+1, n_2+1\rangle$$

$$|\Psi_{n_1,n_2}^{-}\rangle = -s_{n_1,n_2}|e, n_1, n_2\rangle + c_{n_1,n_2}|g, n_1+1, n_2+1\rangle$$

$$c_{n_1,n_2} = \cos\left(\frac{\theta_{n_1,n_2}}{2}\right) \qquad s_{n_1,n_2} = \sin\left(\frac{\theta_{n_1,n_2}}{2}\right)$$

A 4-Qubit Model with 2-modes cavity field

A. Partially Entangled Bell States $|\Phi_{AB}\rangle$

what happens if not started with "0" ?

$$|\Phi(0)\rangle = |\Phi_{AB}\rangle \otimes |0_{a_1}, 0_{a_2}, 0_{b_1}, 0_{b_2}\rangle$$

$$|\Phi(0)\rangle = (\cos\alpha\, |e_A, e_B\rangle + \sin\alpha\, |g_A, g_B\rangle) \otimes |0_{a_1}, 0_{a_2}, 0_{b_1}, 0_{b_2}\rangle$$

$$|\Phi(0)\rangle = \cos\alpha\, |e_A, 0_{a_1}, 0_{a_2}\rangle \otimes |e_B, 0_{b_1}, 0_{b_2}\rangle + \sin\alpha\, |g_A, 0_{a_1}, 0_{a_2}\rangle \otimes |g_B, 0_{b_1}, 0_{b_2}\rangle$$

Eigenstates:

$$|\Psi_0\rangle = |g, 0, 0\rangle$$

$$|\Psi_{0,0}^+\rangle = c_{0,0}\, |e, 0, 0\rangle + s_{0,0}\, |g, 1, 1\rangle$$

$$|\Psi_{0,0}^-\rangle = -s_{0,0}\, |e, 0, 0\rangle + c_{0,0}\, |g, 1, 1\rangle$$

$$|e, 0, 0\rangle = c_{0,0}\, |\Psi_{0,0}^+\rangle - s_{0,0}\, |\Psi_{0,0}^-\rangle$$

$$\boxed{\begin{aligned}|\Phi(0)\rangle = {} & \cos\alpha\, (c\, |\Psi_0^+\rangle_A - s\, |\Psi_0^-\rangle_A) \otimes (c\, |\Psi_0^-\rangle_B - s\, |\Psi_0^-\rangle_B) \\ & + \sin\alpha\, |\Psi_0\rangle_A \otimes |\Psi_0\rangle_B\end{aligned}}$$

$$|\Phi(t)\rangle = e^{-i\lambda^{\pm} t} |\Phi(0)\rangle$$

let's apply the same procedure with the single-mode cavity field

$$
\begin{aligned}
|\Phi(t)\rangle = \; & x_1 \, |e_A, e_B, 0_{a_1}, 0_{a_2}, 0_{b_1}, 0_{b_2}\rangle \\
& + x_2 \, |g_A, g_B, 1_{a_1}, 1_{a_2}, 1_{b_1}, 1_{b_2}\rangle \\
& + x_3 \, |e_A, g_B, 0_{a_1}, 0_{a_2}, 1_{b_1}, 1_{b_2}\rangle \\
& + x_4 \, |g_A, e_B, 1_{a_1}, 1_{a_2}, 0_{b_1}, 0_{b_2}\rangle \\
& + x_5 \, |g_A, g_B, 0_{a_1}, 0_{a_2}, 0_{b_1}, 0_{b_2}\rangle
\end{aligned}
$$

$$x_1 = \left(\overbrace{c^2}^{L} e^{-i\lambda^+ t} + \overbrace{s^2}^{M} e^{-i\lambda^- t}\right)^2 \cos\alpha$$

$$x_2 = \left(\overbrace{cs}^{\sqrt{LM}}\, e^{-i\lambda^+ t} - cs\, e^{-i\lambda^- t}\right)^2 \cos\alpha$$

$$x_3 = \left(c^2 e^{-i\lambda^+ t} + s^2 e^{-i\lambda^- t}\right)\left(s^2 e^{-i\lambda^+ t} - c^2 e^{-i\lambda^- t}\right) \cos\alpha$$

$$x_4 = x_3$$

$$x_5 = \sin\alpha$$

A.1 $C_{AB}(t)$

$$\rho^{AB} = \mathrm{Tr}_{a_1 a_2, b_1 b_2}\left[\,|\Phi(t)\rangle\langle\Phi(t)|\,\right]$$

$$\mathrm{Tr}_{0_{a_1},0_{a_2},0_{b_1},0_{b_2}} = |x_1|^2\,|e_A e_B\rangle\langle e_A e_B| + |x_5|^2\,|g_A g_B\rangle\langle g_A g_B| + x_1 x_5^*\,|e_A e_B\rangle\langle g_A g_B| + x_1^* x_5\,|g_A g_B\rangle\langle e_A e_B|$$

$$\mathrm{Tr}_{0_{a_1},0_{a_2},1_{b_1},1_{b_2}} = |x_3|^2\,|e_A g_B\rangle\langle e_A g_B|$$

$$\mathrm{Tr}_{1_{a_1},1_{a_2},0_{b_1},0_{b_2}} = |x_4|^2\,|g_A e_B\rangle\langle g_A e_B|$$

$$\mathrm{T}_{1_{a_1},1_{a_2},1_{b_1},1_{b_2}} = |x_2|^2\,|g_A g_B\rangle\langle g_A g_B|$$

$$\rho^{AB} = \begin{array}{c} \uparrow\uparrow \\ \uparrow\downarrow \\ \downarrow\uparrow \\ \downarrow\downarrow \end{array} \begin{bmatrix} \underbrace{|x_1|^2}_{a} & 0 & 0 & \underbrace{x_1 x_5^*}_{w} \\ 0 & \underbrace{|x_3|^2}_{b} & 0\ z & 0 \\ 0 & 0\ z^* & \underbrace{|x_4|^2}_{c} & 0 \\ \underbrace{x_1^* x_5}_{w^*} & 0 & 0 & \underbrace{|x_2|^2+|x_5|^2}_{d} \end{bmatrix}$$

$$C(\rho) = 2\max\left[0,\ |z| - \sqrt{ad},\ |w| - \sqrt{bc}\right]$$

$$Q(t) = 2\,|x_1|\,|x_5| - 2\,|x_3|\,|x_4|$$

<u>For $\Delta = 0$:</u>

$$L = c^2 = \cos^2\left(\frac{\theta}{2}\right) = \frac{1}{2}\left(1 + \frac{\cancelto{0}{\Delta}}{\sqrt{\Delta^2 + G_n^2}}\right) = \frac{1}{2}$$

$$M = s^2 = \sin^2\left(\frac{\theta}{2}\right) = \frac{1}{2}\left(1 - \frac{\cancelto{0}{\Delta}}{\sqrt{\Delta^2 + G^2}}\right) = \frac{1}{2}$$

$$\lambda^{\pm} = K(n_1,n_2) \pm Q(n_1,n_2)$$

$\hookrightarrow$ becomes global phase

$$Q(n_1,n_2) = \frac{1}{2}\sqrt{\cancel{\Delta^2} + G_n^2} = \frac{G_n}{2}$$

$$|x_1| = \left|\left(\frac{1}{2}e^{-i\lambda^+ t} + \frac{1}{2}e^{-i\lambda^- t}\right)^2 \cos\alpha\right|$$

$$= \frac{1}{4}|\cos\alpha| \left|\left(e^{-i\lambda^+ t} + e^{-i\lambda^- t}\right)^2\right|$$

$$= \frac{1}{4}|\cos\alpha| \left|\left(\underbrace{\cancel{e^{-iKt}}}_{\text{global phase}} e^{-iQt} + \cancel{e^{-iKt}} e^{+iQt}\right)^2\right| \qquad Q \to \frac{G}{2}$$

$$= \frac{1}{4}|\cos\alpha| \left(2\cos\left(\frac{Gt}{2}\right)\right)^2 = |\cos\alpha| \cos^2\left(\frac{Gt}{2}\right)$$

$$|x_3| = \cos^2\left(\frac{Gt}{2}\right)\sin^2\left(\frac{Gt}{2}\right)\cos^2\alpha$$

$$Q(t) = 2\left[|x_1||x_5| - |x_4||x_3|\right]$$

$$= \cos^2\left(\frac{Gt}{2}\right)\left[|\sin 2\alpha| - 2\sin^2\left(\frac{Gt}{2}\right)\cos^2\alpha\right]$$

★ Same result with the single mode case since

$$G = G_{n_1,n_2} = G_{0,0} = 2g\sqrt{(0+1)(0+1)} = 2g$$

* If the initial state of the cavity will be changed, $Q_{AB}(t)$ will change. Since G_n will change.

Thus changing Hamiltonian could be helpful:

$$H_{at} = \frac{1}{2}\omega_A \sigma_z^A + \frac{1}{2}\omega_B \sigma_z^B$$

$$H_{cav} = \omega_1 \hat{a}_1^+ a_1 + \omega_2 \hat{a}_2^+ a_2 + \nu_1 \hat{b}_1^+ \hat{b}_1 + \nu_2 \hat{b}_2^+ b_2$$

$$H_{int} = g_1\left(\sigma_-^A \hat{a}_1^+ + \sigma_+^A \hat{a}_1\right) + g_2\left(\sigma_-^A \hat{a}_2^+ + \sigma_+^A \hat{a}_2\right)$$

$$+ f_1\left(\sigma_-^B \hat{b}_1^+ + \sigma_+^B \hat{b}_1\right) + f_2\left(\sigma_-^B \hat{b}_2^+ + \sigma_-^B \hat{b}_2^+\right)$$

First only investigate the Atom A and cavity a_1, a_2:

Eigenstates : $|e, n_1, n_2\rangle$ & $|g, n_1+1, n_2+1\rangle$

$$\hat{H} = \begin{array}{cc} |g, n_1+1, n_2+1\rangle & |e, n_1, n_2\rangle \\ \left[\begin{array}{c} B(n_1,n_2) \\ \langle e,n_1,n_2|\hat{H}|g,\dots\rangle \end{array}\right. & \left.\begin{array}{c} \langle g,\dots|\hat{H}|e,n_1,n_2\rangle \\ B(n_1,n_2)-\Delta \end{array}\right] \end{array}$$

Interaction

Interaction

$$B(n_1,n_2) = \frac{\omega_A}{2} + \omega_1(n_1+1) + \omega_2(n_2+1)$$

$$B(n_1,n_2) - \Delta = -\frac{\omega_A}{2} + \omega_1 n_1 + \omega_2 n_2 \quad \text{where } \Delta = \omega_A - \underbrace{(\omega_1+\omega_2)}_{2\Omega \to \frac{\omega_1+\omega_2}{2}}$$

$$\langle e, n_1, n_2 | \hat{H} | g, n_1+1, n_2+1 \rangle$$

$$= \left(g_1 \hat{\sigma}_+^A \hat{a}_1 + g_2 \hat{\sigma}_+^A \hat{a}_2 \right) | g, n_1+1, n_2+1 \rangle$$

$$\langle e, n_1, n_2 | g_1 \hat{\sigma}_+^A \hat{a}_1 | g, n_1+1, n_2+1 \rangle = 0$$

So, we need to change the eigen states

$n_1 \; n_2$
$n_1 \; n_2+1$
$(n_1+1) \; n_2$
$(n_1+1)(n_2+1)$

$$N_{exc} = \hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2 + \hat{\sigma}_+ \hat{\sigma}_-$$

$$N_{exc} | g, n_1+1, n_2+1 \rangle = (n_1+1+n_2+1) | g, n_1+1, n_2+1 \rangle$$

$$N_{exc} | e, n_1, n_2 \rangle = (1+n_1+n_2) | e, n_1, n_2 \rangle$$

$$N_{exc} | e, n_1+1, n_2 \rangle = (1+n_1+1+n_2) | e, n_1+1, n_2 \rangle$$

$| e, n_1, n_2+1 \rangle$

$$N_{exc} | g, n_1, n_2+1 \rangle = (n_1+n_2+1) | g, n_1, n_2+1 \rangle$$

$| g, n_1+1, n_2 \rangle$

$$\det \begin{array}{cccc} |g,n_1+1,n_2\rangle & |e,n_1+1,n_2\rangle & & \\ |g,n_1,n_2+1\rangle & |e,n_1,n_2+1\rangle & |g,n_1+1,n_2+1\rangle & |e,n_1,n_2\rangle \end{array}$$

$$\det \begin{bmatrix} A-\lambda & & & X \\ & B-\lambda & Y & \\ & Y & C-\lambda & \\ X & & & D-\lambda \end{bmatrix} = 0$$

$$X = \langle g, n_1+1, n_2 | \hat{H} | e, n_1, n_2 \rangle + \langle g, n_1, n_2+1 | \hat{H} | e, n_1, n_2 \rangle$$

$$\hat{H} | e, n_1, n_2 \rangle = \left(g_1 \sigma_-^A \hat{a}_1^+ + g_2 \sigma_-^A \hat{a}_2^+ \right) | e, n_1, n_2 \rangle$$

$$g_1 \sqrt{n_1+1}\, | g, n_1+1, n_2 \rangle + g_2 \sqrt{n_2+1}\, | g, n_1, n_2+1 \rangle$$

$$X = g_1 \sqrt{n_1+1} + g_2 \sqrt{n_2+1}$$

$$Y = \langle e, n_1, n_2+1 | \underbrace{\hat{H} | g, n_1+1, n_2+1 \rangle} + \langle e, n_1+1, n_2 | \hat{H} | g, n_1+1, n_2+1 \rangle$$

$$g_1 \sigma_+^A \hat{a}_1 + g_2 \sigma_+^A \hat{a}_2 \, | g, n_1+1, n_2+1 \rangle$$

$$g_1 \sqrt{n_1+1}\, | e, n_1, n_2+1 \rangle + g_2 \sqrt{n_2+1}\, | e, n_1+1, n_2 \rangle$$

$$Y = g_1 \sqrt{n_1+1} + g_2 \sqrt{n_2+1}$$

$$X = Y$$

Let's find λ:

$$\det \begin{array}{cccc} |g,n_1+1,n_2\rangle & |e,n_1+1,n_2\rangle & & \\ |g,n_1,n_2+1\rangle & |e,n_1,n_2+1\rangle & |g,n_1+1,n_2+1\rangle & |e,n_1,n_2\rangle \end{array}$$

$$\det \begin{bmatrix} A-\lambda & & & X \\ & B-\lambda & X & \\ & X & C-\lambda & \\ X & & & D-\lambda \end{bmatrix} = 0$$

$$A-\lambda \begin{vmatrix} B-\lambda & X & \\ X & C-\lambda & \\ & & D-\lambda \end{vmatrix} - X \begin{vmatrix} & & X \\ B-\lambda & X & \\ X & C-\lambda & \end{vmatrix}$$

$$(A-\lambda)(D-\lambda) \begin{vmatrix} B-\lambda & X \\ X & C-\lambda \end{vmatrix} - (X)(X) \begin{vmatrix} B-\lambda & X \\ X & C-\lambda \end{vmatrix}$$

$$(A-\lambda)(D-\lambda)\left[(B-\lambda)(C-\lambda) - X^2\right] - X^2\left[(B-\lambda)(C-\lambda) - X^2\right] = 0$$

$$\left[\underbrace{(B-\lambda)(C-\lambda)}_{x^2} - X^2\right]\left[\underbrace{(A-\lambda)(D-\lambda)}_{x^2} - X^2\right] = 0$$

let's find A, B, C, D $\longrightarrow$ $A=C$ & $B=D$

$$-\frac{\omega_A}{2} + \omega_1(n_1+1) + \omega_2(n_2+1)$$

$$\hat{\sigma}_z |g\rangle = -\frac{\omega_A}{2}$$

$$\hat{\sigma}_z |e\rangle = +\frac{\omega_A}{2}$$

$$+\frac{\omega_A}{2} + \omega_1(n_1+1) + \omega_2(n_2+1)$$

$$(A-\lambda)(D-\lambda) - x^2 = 0$$

$$AD - (A+D)\lambda + \lambda^2 - x^2 = 0$$

$$\Delta = \omega_A - (\omega_1 + \omega_2)$$

$$\lambda^2 - (A+D)\lambda + (AD - x^2) = 0$$

$$\lambda^{\pm} = \frac{(A+D) \pm \sqrt{(A+D)^2 - 4(AD - x^2)}}{2}$$

$$\underbrace{A^2 + 2AD + D^2 - 4AD}_{(A-D)^2} + x^2$$

Δ ? Check it

$$\lambda^{\pm} = K(n_1, n_2) \pm Q(n_1, n_2)$$

$$Q(n_1, n_2) = \frac{1}{2}\sqrt{\Delta^2 + \underbrace{4\cdot(g_1\sqrt{n_1+1} + g_2(\sqrt{n_2+1})^2}_{G_n}}$$

$$\star \quad G_n = 2\left(g_1\sqrt{n_1+1} + g_2\sqrt{n_2+1}\right)$$