2022-2023

Quantum Information and Quantum Computing, Problem set 3

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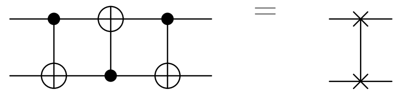
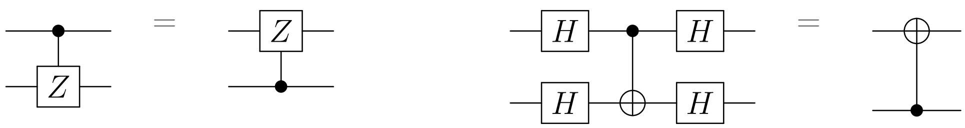
Problem 1 : Elementary quantum circuits

1. Show that this circuit is a quantum swap, i.e. it swaps the $|0\rangle$ and $|1\rangle$ states.



2. Prove the following equalities



3. For $U = V^2$, with V a single-qubit unitary, construct a $C^5(U)$ gate without using ancilla qubits. You may use multiply (up to 4 control qubits) controlled-V and $V^\dagger$ gates in addition to the universal set of gates H, S, T, CNOT.

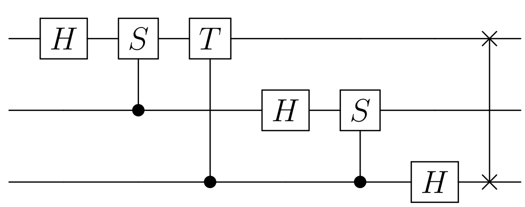
4. The discrete Fourier transform from a set of complex numbers $x_0, \dots, x_{N-1}$ to a set of complex numbers $y_0, \dots, y_{N-1}$ is defined as

$$y_k = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} x_j e^{2\pi i jk/N}$$

The quantum Fourier transform is defined in a Hilbert space of dimension N as the unitary transformation

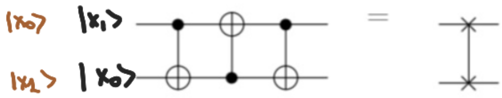
$$\sum_{j=0}^{N-1} x_j |j\rangle \rightarrow \sum_{k=0}^{N-1} y_k |k\rangle$$

where $|j\rangle$ and $|k\rangle$ are vectors of the computational basis. Show that the following circuit performs the quantum Fourier transform for $N = 2^3$. Write the corresponding unitary matrix (define $\omega = e^{2\pi i/8}$).



5. Find the quantum circuits that, starting from the state $|00\rangle$ as an input, generate the four Bell states $(|00\rangle + |11\rangle)/\sqrt{2}$, $(|00\rangle - |11\rangle)/\sqrt{2}$, $(|01\rangle + |10\rangle)/\sqrt{2}$, $(|01\rangle - |10\rangle)/\sqrt{2}$.

1. Show that this circuit is a quantum swap, i.e. it swaps the $|0\rangle$ and $|1\rangle$ states.



$x_1\ x_0$	$x_1'\ x_0'$ (1st CNOT)	$x_1''\ x_0''$ (2nd CNOT)	$x_1'''\ x_0'''$ (3rd CNOT)
0 0	0 0	0 0	0 0
0 1	0 1	1 1	1 0
1 0	1 1	0 1	0 1
1 1	1 0	1 0	1 1

From Solution:

$SWAP|x,y\rangle = |y,x\rangle$

$CNOT_{0,1}|x,y\rangle = |x, y\oplus x\rangle$

$CNOT_{1,0}|x,y\rangle = |x\oplus y, y\rangle$

$CNOT_{0,1}[CNOT_{1,0}(CNOT_{0,1}|x,y\rangle)]$

$= CNOT_{0,1}[CNOT_{1,0}|x, y\oplus x\rangle]$

$= CNOT_{0,1}[|\underbrace{x\oplus y\oplus x}_{y}, y\oplus x\rangle]$

$= |y, \underbrace{y\oplus x\oplus y}_{x}\rangle = |y,x\rangle$

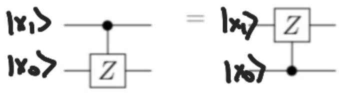
$1\oplus y\oplus 1$

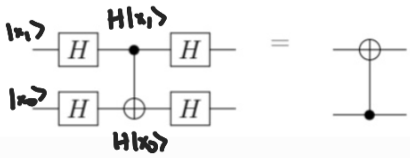
$y\oplus 1 = y$

$y=0 \to 0$

$y=1 \to 1$

2. Prove the following equalities





1)

$$\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

CZ gate

$x_1 x_0$	$x_1' x_0'$
0 0	0 0
0 1	0 1
1 0	1 0
1 1	−1 1

global phase

$x_1 x_0$	$x_1' x_0'$
0 0	0 0
0 1	0 1
1 0	1 0
1 1	−1 1

global phase

$x_1 \to$ control

$x_0 \to$ target

2)

00 01

$x_1\ x_0$	$H\|x_1\rangle H\|x_0\rangle$		CNOT
0 0	$\frac{1}{\sqrt{2}}(\|0\rangle+\|1\rangle)\otimes\frac{1}{\sqrt{2}}(\|0\rangle+\|1\rangle)$	$\frac{1}{2}(\|00\rangle+\|01\rangle+\|10\rangle+\|11\rangle)$	$\frac{1}{2}(\|00\rangle+\|01\rangle+\|11\rangle+\|10\rangle)$
0 1	$\frac{1}{\sqrt{2}}(\|0\rangle+\|1\rangle)\otimes\frac{1}{\sqrt{2}}(\|0\rangle-\|1\rangle)$	$\frac{1}{2}(\|00\rangle-\|01\rangle+\|10\rangle-\|11\rangle)$	$\frac{1}{2}(\|00\rangle-\|01\rangle+\|11\rangle-\|10\rangle)$
1 0	$\frac{1}{\sqrt{2}}(\|0\rangle-\|1\rangle)\otimes\frac{1}{\sqrt{2}}(\|0\rangle+\|1\rangle)$	$\frac{1}{2}(\|00\rangle+\|01\rangle-\|10\rangle-\|11\rangle)$	$\frac{1}{2}(\|00\rangle+\|01\rangle-\|11\rangle-\|10\rangle)$
1 1	$\frac{1}{\sqrt{2}}(\|0\rangle-\|1\rangle)\otimes\frac{1}{\sqrt{2}}(\|0\rangle-\|1\rangle)$	$\frac{1}{2}(\|00\rangle-\|01\rangle-\|10\rangle+\|11\rangle)$	$\frac{1}{2}(\|00\rangle-\|01\rangle-\|11\rangle+\|10\rangle)$

$$H|x_1\rangle \otimes H|x_0\rangle$$

$$\frac{1}{2}\Big(\tfrac{1}{2}(|00\rangle+\cancel{|01\rangle}+\cancel{|10\rangle}+\cancel{|11\rangle}) + \tfrac{1}{2}(|00\rangle-\cancel{|01\rangle}+\cancel{|10\rangle}-\cancel{|11\rangle}) + \tfrac{1}{2}(|00\rangle-\cancel{|01\rangle}-\cancel{|10\rangle}+\cancel{|11\rangle})$$
$$+\tfrac{1}{2}(|00\rangle+\cancel{|01\rangle}-\cancel{|10\rangle}-\cancel{|11\rangle})\Big) = |00\rangle$$

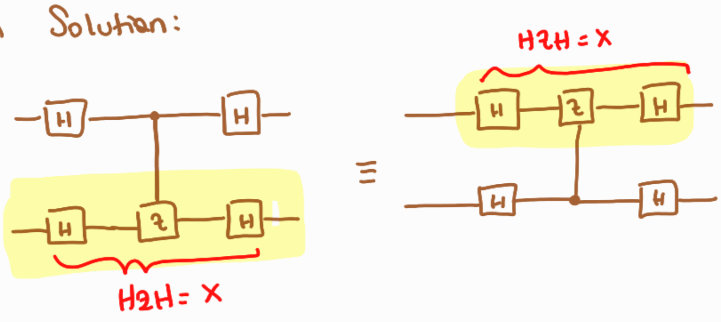
$$\frac{1}{2}\Big(\tfrac{1}{2}(\cancel{|00\rangle}+\cancel{|01\rangle}+\cancel{|10\rangle}+|11\rangle) - \tfrac{1}{2}(\cancel{|00\rangle}-\cancel{|01\rangle}+\cancel{|10\rangle}-|11\rangle) + \tfrac{1}{2}(\cancel{|00\rangle}-\cancel{|01\rangle}-\cancel{|10\rangle}+|11\rangle)$$
$$-\tfrac{1}{2}(\cancel{|00\rangle}+\cancel{|01\rangle}-\cancel{|10\rangle}-|11\rangle)\Big) = |11\rangle$$

$$\frac{1}{2}\Big(\tfrac{1}{2}(\cancel{|00\rangle}+\cancel{|01\rangle}+|10\rangle+\cancel{|11\rangle}) + \tfrac{1}{2}(\cancel{|00\rangle}-\cancel{|01\rangle}+|10\rangle-\cancel{|11\rangle}) - \tfrac{1}{2}(\cancel{|00\rangle}-\cancel{|01\rangle}-|10\rangle+\cancel{|11\rangle})$$
$$-\tfrac{1}{2}(\cancel{|00\rangle}+\cancel{|01\rangle}-|10\rangle-\cancel{|11\rangle})\Big) = |10\rangle$$

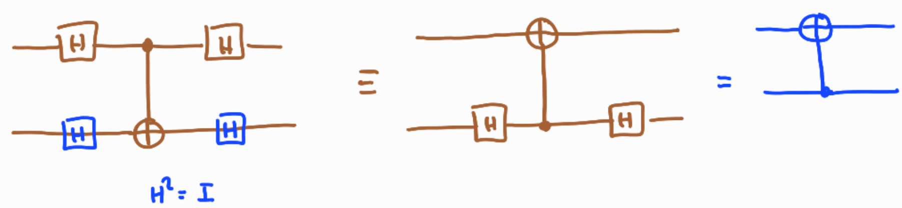
$$\frac{1}{2}\Big(\tfrac{1}{2}(\cancel{|00\rangle}+|01\rangle+\cancel{|10\rangle}+\cancel{|11\rangle}) - \tfrac{1}{2}(\cancel{|00\rangle}-|01\rangle+\cancel{|10\rangle}-\cancel{|11\rangle}) - \tfrac{1}{2}(\cancel{|00\rangle}-|01\rangle-\cancel{|10\rangle}+\cancel{|11\rangle})$$
$$+\tfrac{1}{2}(\cancel{|00\rangle}+|01\rangle-\cancel{|10\rangle}-\cancel{|11\rangle})\Big) = |01\rangle$$

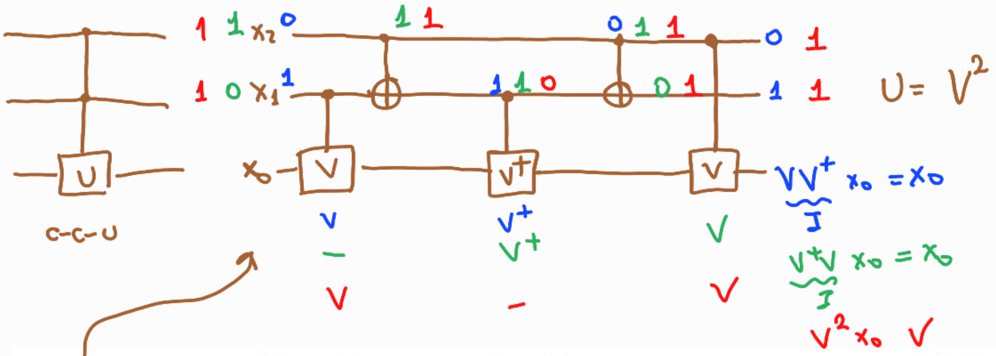
Which is CNOT → control x_0, → target x_1

From Solution:



* Notice that the relation $HZH = X$ can be applied even if the Z is a controlled operation.





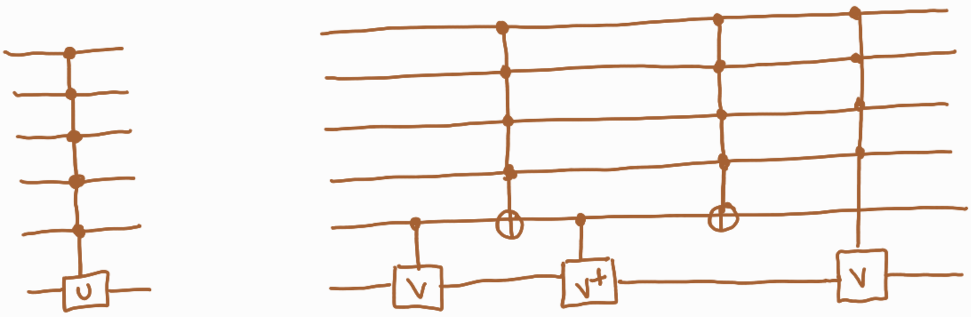
3. For $U = V^2$, with V a single-qubit unitary, construct a $C^5(U)$ gate without using ancilla qubits. You may use multiply (up to 4 control qubits) controlled-V and $V^\dagger$ gates in addition to the universal set of gates H, S, T, CNOT.

$c^2(U)$

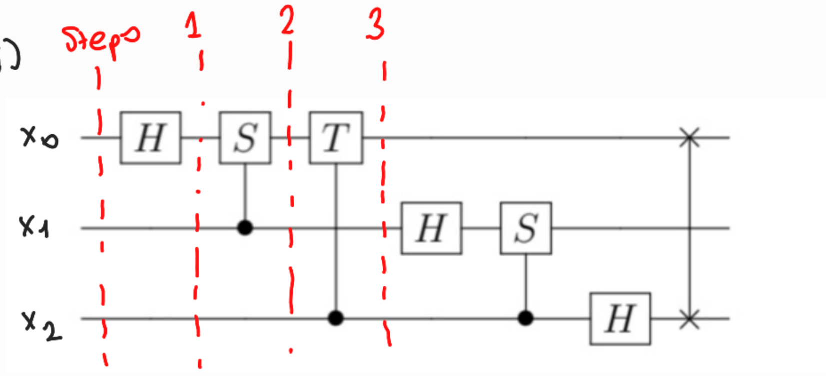
$x_2 x_1$	$x_2 x_1 x_0$
0 0	0 0 x_0
0 1	0 1 x_0
1 0	1 0 x_0
1 1	1 1 $(V^2 x_0)$ ✓

} no change

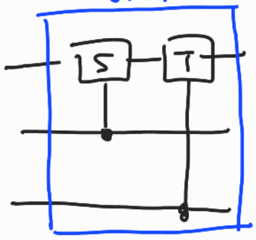
To generalize the circuit above to $c^5(U)$, we replace every controlled gate with a quatruple controlled gate.



4)



UROT



$UROT_k |x_j\rangle = e^{\frac{2\pi i}{2^k} x_j} |x_j\rangle$

$S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$ $T = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}$

Phase Gate

Control, Target

	$x_1 x_0$	$x_1' x_0'$
0	00	00
1	01	01
2	10	$\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 10$
3	11	$\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ i \end{bmatrix} = i\|11\rangle$

$cS = \begin{array}{c} x_1 x_0 \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \end{bmatrix} \end{array}$

$x_2 x_0$	$x_2' x_0'$
00	00
01	01
10	$\begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
11	$\begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ e^{i\pi/4} \end{bmatrix}$

$CT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{i\pi/4} \end{bmatrix}$

Step 0 : $|x_0 x_1 x_2\rangle$

Step 1 : $\frac{1}{\sqrt{2}}\left(|0\rangle + e^{\frac{2\pi i x_0}{2}} |1\rangle\right) \otimes |x_1 x_2\rangle = \frac{1}{\sqrt{2}}\left(|0 x_1\rangle + e^{\frac{2\pi i x_0}{2}} |1 x_1\rangle\right)$

$\underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ e^{2\pi i x_0/2} \end{pmatrix}}_{\begin{pmatrix} 1 \\ e^{\pi i x_0} \end{pmatrix}}$ $\to 0$, $\to 1$

$R_k = \begin{bmatrix} 1 & 0 \\ 0 & e^{2\pi i/2^k} \end{bmatrix}$

$R_2 = \begin{bmatrix} 1 & 0 \\ 0 & e^{\pi i/2} \end{bmatrix} = S$, $R_3 = \begin{bmatrix} 1 & 0 \\ 0 & e^{\pi i/4} \end{bmatrix} = T$

$F_Q |x\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{2\pi i k x/N} |k\rangle$

For three qubits system:

$|x\rangle \to$ computational basis

$x \in \{0, 1, 2, \ldots, 7\}$

$\omega = e^{2\pi i/8}$

$N = 8 = 2^3$

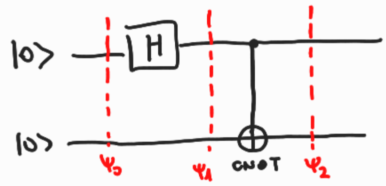
$FQ |x\rangle_3 = \frac{1}{\sqrt{8}} \sum_{k=0}^{N-1} \omega^{kx} |k\rangle_3$

Coefficients : $\frac{1}{\sqrt{8}} \omega^{xk}$ $x, k \in \{0, 1, \ldots, 7\}$

$\begin{array}{c} k \to \\ x \downarrow \begin{bmatrix} 1 & \cdots & 1 \\ & & \\ & & \end{bmatrix}_{8 \times 8} \end{array}$

5) Find the quantum circuits that, starting from the state $|00\rangle$ as an input, generate the four Bell states $(|00\rangle + |11\rangle)/\sqrt{2}$, $(|00\rangle - |11\rangle)/\sqrt{2}$, $(|01\rangle + |10\rangle)/\sqrt{2}$, $(|01\rangle - |10\rangle)/\sqrt{2}$.

$|00\rangle \rightarrow (|00\rangle + |11\rangle)/\sqrt{2}$

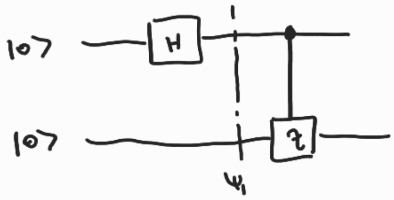


$\psi_1 = \left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right) \otimes |0\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |10\rangle\right)$

$\psi_2 = \frac{1}{2}\left(|00\rangle + |11\rangle\right)$

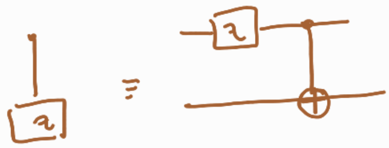
$\text{CNOT}\ H\ \psi_b\ \psi_a = \text{CNOT}\ H\ |00\rangle$

$|00\rangle \rightarrow (|00\rangle - |11\rangle)/\sqrt{2}$

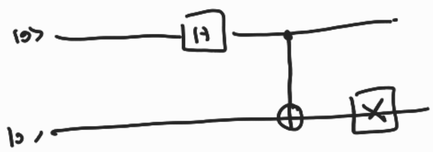


$\psi_1 = \left(\frac{|00\rangle+|10\rangle}{\sqrt{2}}\right)$

$\psi_2 = \left(\frac{|00\rangle-|11\rangle}{\sqrt{2}}\right)$



$|00\rangle \longrightarrow (|01\rangle + |10\rangle)/\sqrt{2}$



$\frac{|00\rangle+|10\rangle}{\sqrt{2}} \longrightarrow \frac{|01\rangle+|10\rangle}{\sqrt{2}}$

$|00\rangle \longrightarrow |01\rangle - |10\rangle/\sqrt{2}$