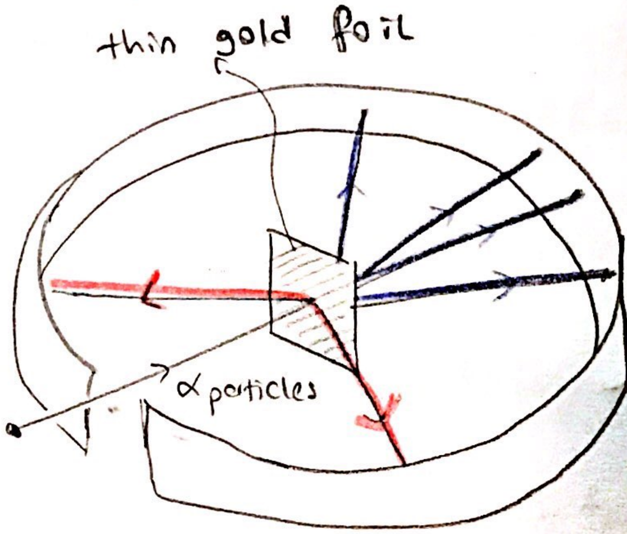
Rutherford Scaterring



positive charge is concentrated, the nucleus.

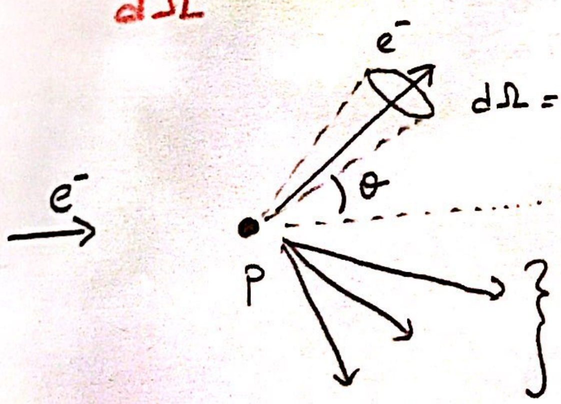
$$\sigma = \frac{\boxed{\#\text{ of interactions}}\text{ per unit time per target}}{\text{incident flux}}$$

σ → Cross Section

flux → # of incident particles/unit area/unit time

☆ σ is a measure of the probability with which an interaction occurs.

$$\underbrace{\frac{d\sigma}{d\Omega}}_{\text{differential cross section}} = \frac{\#\text{ of interactions per unit time per target into } d\Omega}{\text{incident flux}}$$



$d\Omega = d(\cos\theta)\,d\phi$

$$\boxed{\sigma = \int \frac{d\sigma}{d\Omega}\,d\Omega}$$

Relativistic kinematics

4-component vectors

$$X = (x_0, x_1, x_2, x_3) = (ct, \vec{x}) = (t, x)$$

$$P = (p_0, p_1, p_2, p_3) = (E/c, \vec{p}) = (E, p)$$

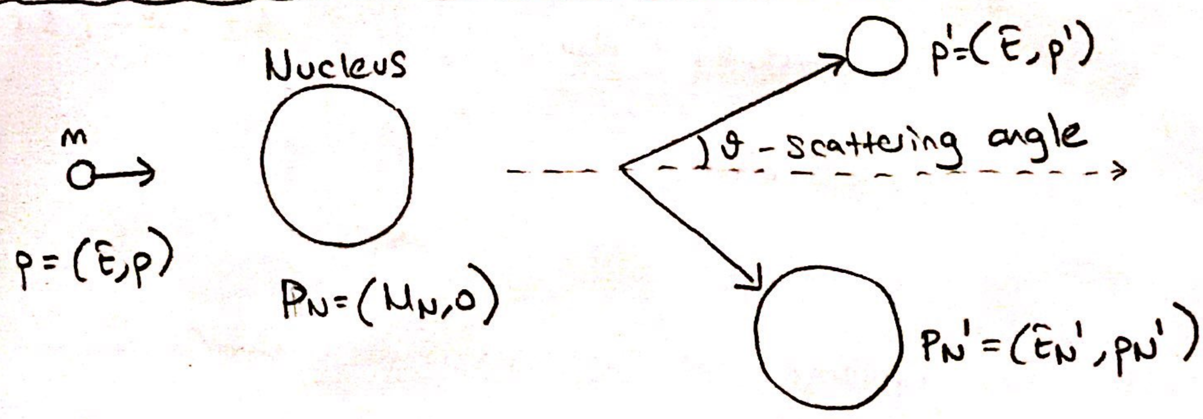
* Scaler product of two 4-vectors is invariant, i.e. independent of a reference frame.

$$a \cdot b = ab = a_0 b_0 - a_1 b_1 - a_2 b_2 - a_3 b_3 = a_0 b_0 - ab$$

For a 4-momentum $\rightarrow$ $P^2 = E^2 - p^2 = m^2$ m ↳ mass of particle at rest

$$\sum_i P_i = \text{constant} \Rightarrow m^2 = \left(\sum_i P_i\right)^2 = \text{constant}$$

Kinematics of an Elastic Scattering



* From momentum-energy conservation

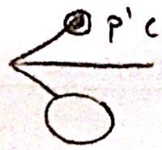
$$(p + P_N)^2 = (p' + P_N')^2$$

$$\cancel{p^2} + 2pP_N + \cancel{P_N^2} = \cancel{p'^2} + 2p'P_N' + \cancel{P_N'^2}$$

$$p^2 = p'^2 = m^2 \quad \& \quad P_N^2 = P_N'^2 = M_N^2$$

$$\boxed{pP_N = p'P_N'}$$

for a scattered particle momentum p' : $p'(E', p')$



$p P_N = p' P_N'$

$p P_N = p'(p + P_N - p') = p'p + p' P_N - p'^2$ $\quad \mu^2$

$$\boxed{p P_N = p'p + p' P_N - \mu^2}$$

$E M_N = (E'E - p'p) + E' M_N - m^2 \quad (2)$

If $\mu \ll E$, then $|p| \simeq E$, $|p'| \approx E'$, and μ^2 can be neglected

$E^2 = p^2 + \mu^2 \qquad E'^2 = p'^2 + \mu^2$

$E M_N = E'E(1 - \cos\theta) + E' M_N$

$E M_N = E'(E(1-\cos\theta) + M_N)$

$$E' = \frac{E M_N}{E(1-\cos\theta) + M_N} = \frac{E}{1 + \frac{E}{M_N}(1-\cos\theta)}$$

scatter particle's energy

If M_N is large $\rightarrow$ small energy transfer

If M_N is small $\rightarrow$ large energy transfer at large θ

Rutherford Cross Section

The probability $\sigma(\theta)$ of particles to go into the solid angle $d\Omega$ ($d\Omega = \sin\theta \, d\theta \, d\phi$)

$$dN = N \, \sigma(\theta) \sin\theta \, d\theta \, d\phi$$

differential Rutherford cross section

$$\sigma(\theta) = \left(\frac{z Z e^2}{4 E_{kin}} \right)^2 \frac{1}{\sin^4(\theta/2)}$$

kinetic energy

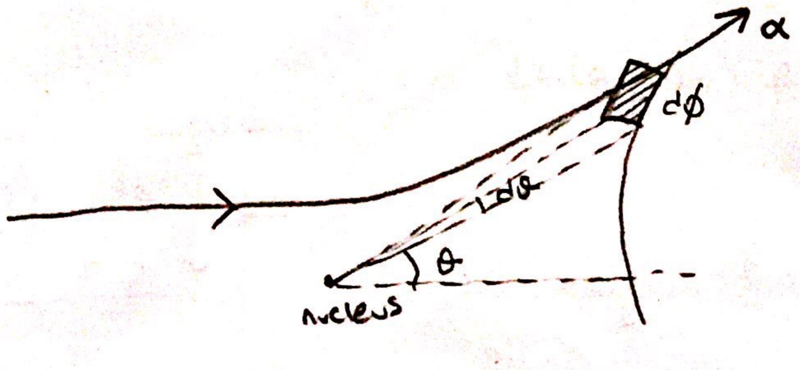
scattering of a particle with charge ze and E_{kin}

target nucleus charge Ze

Relativistic Rutherford Cross Section

$$\sigma(\theta) = (z Z \alpha)^2 \frac{E'^2}{|q|^4}$$

where $\alpha = e^2$, $q = p' - p$



1. Full Rutherford Scattering Cross Section

for $\theta > \theta_0$

$$\sigma_{full} = \int_{\theta_0}^{\pi} \sigma(\theta)\, 2\pi \sin\theta \, d\theta$$

$\theta \to d(\sin\frac{\theta}{2}) = \cos\left(\frac{\theta}{2}\right) \cdot \frac{1}{2} \, d\theta$

$d\theta = \frac{2}{\cos(\frac{\theta}{2})} \, d\theta$

$\downarrow$ Substitude θ with $\sin\frac{\theta}{2}$

$$= \int_{\theta_0}^{\pi} \sigma(\theta) \; 2\pi \cancel{\sin\theta}^{\,2.\sin\frac{\theta}{2}.\cancel{\cos\frac{\theta}{2}}} \cdot \frac{2}{\cancel{\cos(\frac{\theta}{2})}} \; d\left(\sin\frac{\theta}{2}\right)$$

$$= \int_{\theta_0}^{\pi} \left(\frac{2 Z e^2}{4 E_{kin}}\right)^2 \frac{1}{\sin^4(\theta/2)} \; 8\pi \overbrace{\sin(\theta/2)}^{x} \; d \overbrace{\sin(\theta/2)}^{x}$$

x^4

$\int \underbrace{\frac{1}{x^3}}_{x^{-3}} dx = \frac{x^{-2}}{-2}$

$$\left(\frac{2 Z e^2}{4 E_{kin}}\right)^2 \frac{-4\pi}{\sin^2(\theta/2)} \Bigg|_{\theta_0}^{\pi}$$

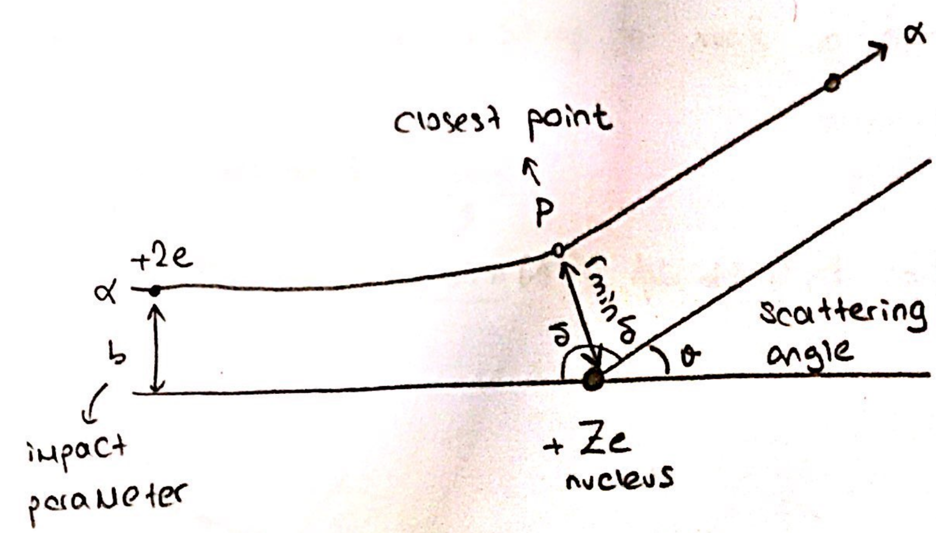
$$\left(1 - \frac{1}{\sin^2\frac{\theta}{2}} = \cot^2\frac{\theta}{2}\right)$$

$$\boxed{\sigma_{full} = +4\pi \left(\frac{2 Z e^2}{4 E_{kin}}\right)^2 \cot^2(\theta_0/2)}$$

$\searrow$ ★ Formula can not be used for very small angles.

Since $\theta = 0 \longrightarrow \sigma_{full} \to \infty$

Derivation of Rutherford Cross Section with classical mechanics



Angular momentum conservation:

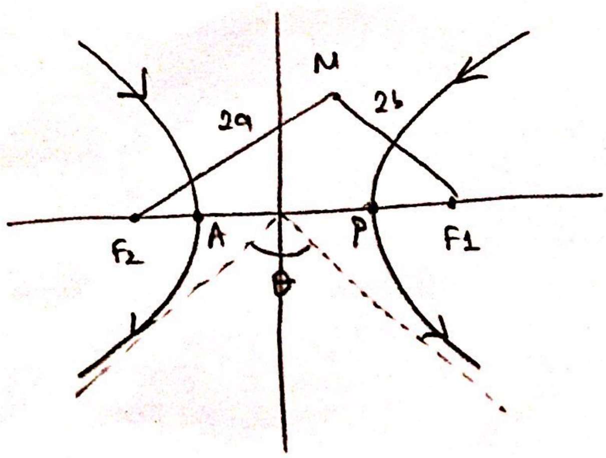
$$b v_\infty = r_{min} v_{min}$$

At point P :

$$\frac{m v_{min}^2}{2} + \frac{z Z e^2}{r_{min}} = E_{tot} = \frac{m v_\infty^2}{2}$$

$$r_{min}^2 - \frac{2 z Z e^2}{m v_\infty^2} r_{min} - b^2 = 0$$

$\to E_{kin}$



Hyperbola trajectory for any point P

$$\big|\, |PF_1| - |PF_2| \,\big| = 2a$$

$$r_1^{min} + r_2^{min} = \frac{z Z e^2}{E_{kin}}$$

$$= F_2P - F_2A = AP = 2a$$

$$\cot\frac{\theta}{2} = \frac{2b}{2a} = \frac{b E_{kin}}{z Z e^2}$$

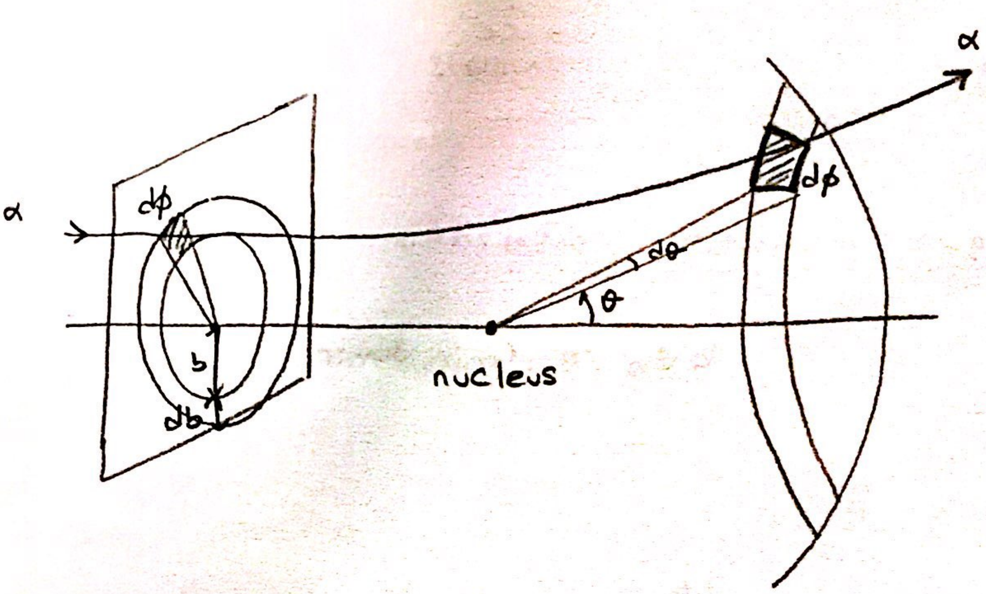
?

Experimentally we cannot measure one particle deflection

Instead we have a flux a particle N per unit of area and per unit of time.

prob. σ of particles to go into solid angle

$$dN = N \underline{b\,db}\, d\phi = N \sigma(\theta) \underline{\sin\theta\, d\theta}\, d\phi \qquad d\Omega = \sin\theta\, d\theta\, d\phi$$



$$\sigma(\theta) = \frac{b}{\sin\theta} \frac{db}{d\theta} = \left(\frac{zZe^2}{4E_{kin}} \right)^2 \frac{1}{\sin^4(\theta/2)}$$