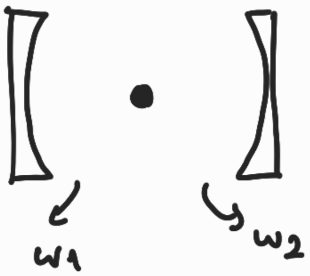
# Atom In Two Modes Cavity




$$H_{at} = \frac{1}{2}\omega_A \sigma_z^A$$

$$H_{cav} = \omega_1 \hat{a}_1^\dagger \hat{a}_1 + \omega_2 \hat{a}_2^\dagger \hat{a}_2$$

$$H_{int} = g_1(\sigma_-^A \hat{a}_1^\dagger + \sigma_+^A \hat{a}_1) + g_2(\sigma_-^A \hat{a}_2^\dagger + \sigma_+^A \hat{a}_2)$$

$$N_{exc} = \hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2 + \hat{\sigma}_+ \sigma_-$$

# of excitations

$n_1, n_2$

$n_1 \quad n_2+1$

$n_1+1 \quad n_2$

$(n_1+1) \quad (n_2+1)$

| Eigenstates | Eigenvalues |
|---|---|
| ~~$\lvert e, n_1, n_2\rangle$~~<br>~~$\lvert g, n_1, n_2+1\rangle$~~<br>~~$\lvert g, n_1+1, n_2\rangle$~~ | ~~$n_1+n_2$~~ |
| $\lvert g, n_1+\cancel{1}, n_2+\cancel{1}\rangle$<br>$\lvert e, n_1+\cancel{1}, n_2^{-1}\rangle$<br>$\lvert e, n_1^{-1}, n_2+\cancel{1}\rangle$ | $n_1+n_2$ |

WRONG METHOD!

$[H, N_{exc}] = 0$ ⇝ Thus same eigenvectors.

| Eigenstates | Eigenvalue |
|---|---|
| $\|g, n_1, n_2\rangle$ | |
| $\|e, n_1-1, n_2\rangle$ | $n_1+n_2$ |
| $\|e, n_1, n_2-1\rangle$ | |

$$\hat{H} = \begin{array}{c} \\ \\ \end{array}\left[\begin{array}{cccc} |g,0,0\rangle & & & \\ & \ddots & & \\ \cdots & & \begin{array}{ccc} |g,n_1,n_2\rangle & |e,n_1-1,n_2\rangle & |e,n_1,n_2-1\rangle \\ \boxed{\begin{matrix} A & X & Y \\ X & B & 0 \\ Y & 0 & C \end{matrix}} & & \end{array} & \\ & & & \ddots \end{array}\right]$$

$$H = \frac{1}{2}\omega_A \sigma_z^A + \omega_1 \hat{a}_1^\dagger \hat{a}_1 + \omega_2 \hat{a}_2^\dagger \hat{a}_2 + g_1(\sigma_-^A \hat{a}_1^\dagger + \sigma_+^A \hat{a}_1) + g_2(\sigma_-^A \hat{a}_2^\dagger + \sigma_+^A \hat{a}_2)$$

$$\hat{H}^{(n_1+n_2)} = \begin{bmatrix} A & X & Y \\ X & B & 0 \\ Y & 0 & X \end{bmatrix} \rightsquigarrow \det \begin{bmatrix} A-\lambda & X & Y \\ X & B-\lambda & 0 \\ Y & 0 & C-\lambda \end{bmatrix} = 0$$

$$+ \quad - \quad +$$

$$Y \begin{vmatrix} X & Y \\ B-\lambda & 0 \end{vmatrix} + (C-\lambda) \begin{vmatrix} A-\lambda & X \\ X & B-\lambda \end{vmatrix} = 0$$

$$-Y^2 \cdot (B-\lambda) + (C-\lambda)\left[(A-\lambda)(B-\lambda) - X^2\right] = 0$$

$$-Y^2(B-\lambda) + (A-\lambda)(B-\lambda)(C-\lambda) - (C-\lambda)X^2 = 0$$

$$(B-\lambda)\Big[\underbrace{-Y^2 + (A-\lambda)(C-\lambda)}_{-Y^2 + AC - \lambda(A+C) + \lambda^2}\Big] - (C-\lambda)X^2 = 0$$

$$\begin{array}{l} -BY^2 + BAC - \lambda B(A+C) + B\lambda^2 \\ +\lambda Y^2 \quad -\lambda AC \quad + \lambda^2(A+C) \quad - \lambda^3 \\ +\;\; -CX^2 \quad +\lambda X^2 \\ \hline \end{array}$$

$$+\lambda^3 - \lambda^2(A+B+C) + \lambda(AB+BC+AC-Y^2-X^2) + (BY^2+CX^2-ABC) = 0$$

Solution for the 3rd degree polynomial

$\lambda_1$

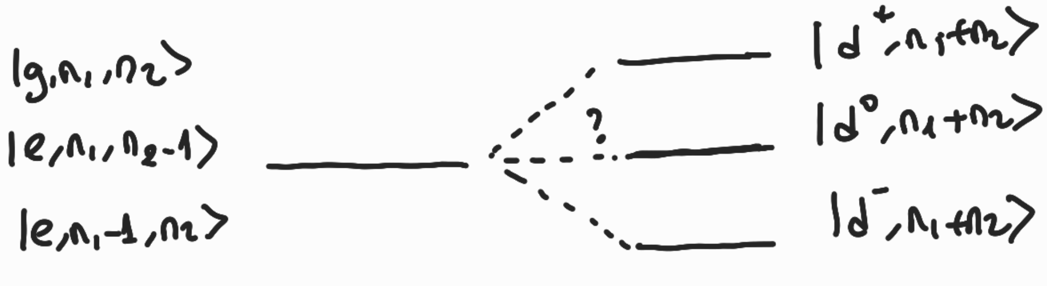
$\lambda_2 = \lambda_1\left(-\frac{1}{2}\right) + i\,\lambda_1\left(\frac{\sqrt{3}}{2}\right)$

$\lambda_3 = \lambda_1\left(-\frac{1}{2}\right) - i\,\lambda_1\left(\frac{\sqrt{3}}{2}\right)$

$|d^+, n_1+n_2\rangle$

$|d^0, n_1+n_2\rangle$

$|d^-, n_1+n_2\rangle$



$$
\begin{array}{cccccc}
0\,0\,1 & 0\,1\,0 & 0\,1\,1 & 1\,0\,0 & 1\,0\,1 & 1\,1\,0 \\
|g,n_1,n_2+1\rangle & |g,n_1+1,n_2\rangle & |g,n_1+1,n_2+1\rangle & |e,n_1,n_2\rangle & |e,n_1,n_2+1\rangle & |e,n_1+1,n_2\rangle
\end{array}
$$

$$
\begin{bmatrix}
A & 0 & 0 & X & 0 & 0 \\
0 & B & 0 & Y & 0 & 0 \\
0 & 0 & C & 0 & Y & X \\
X & Y & 0 & D & 0 & 0 \\
0 & 0 & Y & 0 & E & 0 \\
0 & 0 & X & 0 & 0 & F
\end{bmatrix}
$$

$$
H = \frac{1}{2}\omega_A \sigma_z^A + \omega_1 \hat{a}_1^{\dagger}\hat{a}_1 + \omega_2 \hat{a}_2^{\dagger}\hat{a}_2 + g_1\left(\sigma_-^A \hat{a}_1^{\dagger} + \sigma_+^A \hat{a}_1\right) + g_2\left(\sigma_-^A \hat{a}_2^{\dagger} + \sigma_+^A \hat{a}_2\right)
$$

$$
\left[\hat{\sigma}_z |g\rangle = -\frac{1}{2}\omega_A \,, \quad \hat{\sigma}_z |e\rangle = +\frac{1}{2}\omega_A\right]
$$

$$A = -\frac{\omega_A}{2} + \omega_1(n_1) + \omega_2(n_2+1)$$

$$B = -\frac{\omega_A}{2} + \omega_1(n_1+1) + \omega_2(n_2)$$

$$C = -\frac{\omega_A}{2} + \omega_1(n_1+1) + \omega_2(n_2+1)$$

$$D = +\frac{\omega_A}{2} + \omega_1(n_1) + \omega_2(n_2)$$

$$E = +\frac{\omega_A}{2} + \omega_1(n_1) + \omega_2(n_2+1)$$

$$F = +\frac{\omega_A}{2} + \omega_1(n_1+1) + \omega_2(n_2)$$

$X = \langle g, n_1, n_2+1 | \hat{H} | e, n_1, n_2 \rangle = g_2\sqrt{n_2+1}$

Only the Hint survives: ( $\sigma_- \hat{a}^+$ term only)

$$\hat{H}_{int} = (\cancel{g_1 \sigma_-^A \hat{a}_1^+} + g_2 \sigma_-^A \hat{a}_2^+)$$

$Y = \langle g, n_1+1, n_2 | \hat{H} | e, n_1, n_2 \rangle = g_1\sqrt{n_1+1}$

$$g_1 \sigma_-^A \hat{a}_1^+ | e, n_1, n_2 \rangle = g_1\sqrt{n_1+1}\, | g, n_1+1, n_2 \rangle$$

$Z = \langle g, n_1+1, n_2+1 | \hat{H} | e, n_1, n_2+1 \rangle = g_1\sqrt{n_1+1} = Y$

$$g_1 \sigma_-^A \hat{a}_1^+ | e, n_1, n_2+1 \rangle = g_1\sqrt{n_1+1}$$

$W = \langle g, n_1+1, n_2+1 | \hat{H} | e, n_1+1, n_2 \rangle = g_2\sqrt{n_2+1} = X$

$$g_2 \sigma_-^A \hat{a}_2^+$$

# Block Matrix, Eigenvalues, Eigenvectors

$$
\begin{array}{c}
+\\ -\\ +\\ -\\ +\\ -
\end{array}
\left[\begin{array}{ccc|ccc}
A & 0 & 0 & X & 0 & 0 \\
0 & B & 0 & Y & 0 & 0 \\
0 & 0 & C & 0 & Y & X \\
\hline
X & Y & 0 & D & 0 & 0 \\
0 & 0 & Y & 0 & E & 0 \\
0 & 0 & X & 0 & 0 & F
\end{array}\right]
$$

$$
\overbrace{(A-\lambda)\begin{vmatrix}
B-\lambda & 0 & Y & 0 & 0 \\
0 & C-\lambda & 0 & Y & X \\
Y & 0 & D-\lambda & 0 & 0 \\
0 & Y & 0 & E-\lambda & 0 \\
0 & X & 0 & 0 & F-\lambda
\end{vmatrix}}^{(1)}
\; - X \;
\overbrace{\begin{array}{c} + \\ - \\ \\ \\ \\ \end{array}\begin{vmatrix}
0 & 0 & X & 0 & 0 \\
B-\lambda & 0 & Y & 0 & 0 \\
0 & C-\lambda & 0 & Y & X \\
0 & Y & 0 & E-\lambda & 0 \\
0 & X & 0 & 0 & F-\lambda
\end{vmatrix}}^{(2)}
= 0
$$

(1)

$$
(A-\lambda)\left[(B-\lambda)\underbrace{\begin{vmatrix}
C-\lambda & 0 & Y & X \\
0 & D-\lambda & 0 & 0 \\
Y & 0 & E-\lambda & 0 \\
X & 0 & 0 & F-\lambda
\end{vmatrix}} + Y \underbrace{\begin{array}{cccc} + & - & + & - \\ \end{array}\begin{vmatrix}
0 & Y & 0 & 0 \\
C-\lambda & 0 & Y & X \\
Y & 0 & E-\lambda & 0 \\
X & 0 & 0 & F-\lambda
\end{vmatrix}}\right]
$$

$$
-(D-\lambda)\begin{vmatrix}
C-\lambda & Y & X \\
Y & E-\lambda & 0 \\
X & 0 & F-\lambda
\end{vmatrix}
\begin{array}{c} + \\ - \\ + \end{array}
\qquad
-Y\begin{vmatrix}
C-\lambda & Y & X \\
Y & E-\lambda & 0 \\
X & 0 & F-\lambda
\end{vmatrix}
$$

det → z $\qquad$ det → z

(1)

$$
(A-\lambda)\left[-(B-\lambda)(D-\lambda)z - Y^2 z\right] = \boxed{-(A-\lambda)z\left[(B-\lambda)(D-\lambda)+Y^2\right]}
$$

$$
z = -X^2(E-\lambda) + (F-\lambda)\left[(C-\lambda)(E-\lambda) - Y^2\right]
$$

$$
z = -X^2(E-\lambda) + (F-\lambda)(C-\lambda)(E-\lambda) - (F-\lambda)Y^2
$$

(2)

$$+X(B-\lambda)\begin{array}{cccc} + & - & & \\ \end{array}$$
$$\begin{vmatrix} \cancel{0} & X & \cancel{0} & \cancel{0} \\ C-\lambda & \cancel{0} & Y & X \\ Y & \cancel{0} & E-\lambda & 0 \\ X & \cancel{0} & 0 & F-\lambda \end{vmatrix}$$

$$-X^2(B-\lambda)\underbrace{\begin{vmatrix} C-\lambda & Y & X \\ Y & E-\lambda & 0 \\ X & 0 & F-\lambda \end{vmatrix}}_{z}$$

$$(2) = \boxed{-X^2(B-\lambda)z}$$

$$(1) + (2) = (0)$$

$$\boxed{+(A-\lambda)z\left[(B-\lambda)(D-\lambda)+Y^2\right]} \quad \boxed{+X^2(B-\lambda)z} = 0$$

$$\underset{z=0}{z}\left[\underbrace{+(A-\lambda)(B-\lambda)(D-\lambda)+(A-\lambda)Y^2+X^2(B-\lambda)}_{W=0}\right]=0$$

$$z = -X^2(E-\lambda) + (F-\lambda)(C-\lambda)(E-\lambda) - (F-\lambda)Y^2 = 0$$

Let's examine W

$$W = +(A-\lambda)(B-\lambda)(D-\lambda) + (A-\lambda)Y^2 + X^2(B-\lambda) = 0$$

$$(A-\lambda)\left[BD - \lambda(B+D) + \lambda^2 + Y^2\right] + X^2(B-\lambda) = 0$$

$$ABD - \lambda A(B+D) + A\lambda^2 + AY^2$$
$$-\lambda BD + \lambda^2(B+D) - \lambda^3 - \lambda Y^2$$
$$+X^2B - X^2\lambda$$
$$+$$

$$\boxed{+\lambda^3 - \lambda^2(A+B+D) + \lambda(AB+AD+BD+Y^2+X^2) - (ABD+AY^2+BX^2) = 0}$$

$\downarrow$

$\lambda_1, \lambda_2, \lambda_3$

$v_1, v_2, v_3$

Let's examine Z:

$$Z = -X^2(E-\lambda) + (F-\lambda)(C-\lambda)(E-\lambda) - (F-\lambda)Y^2 = 0$$

$$(F-\lambda)\left[CE - \lambda(C+E) + \lambda^2 - Y^2\right] - X^2(E-\lambda) = 0$$

$$FCE - \lambda F(C+E) + F\lambda^2 - FY^2$$
$$-\lambda CE + \lambda^2(C+E) - \lambda^3 + \lambda Y^2$$
$$-X^2E + X^2\lambda$$
$$+$$

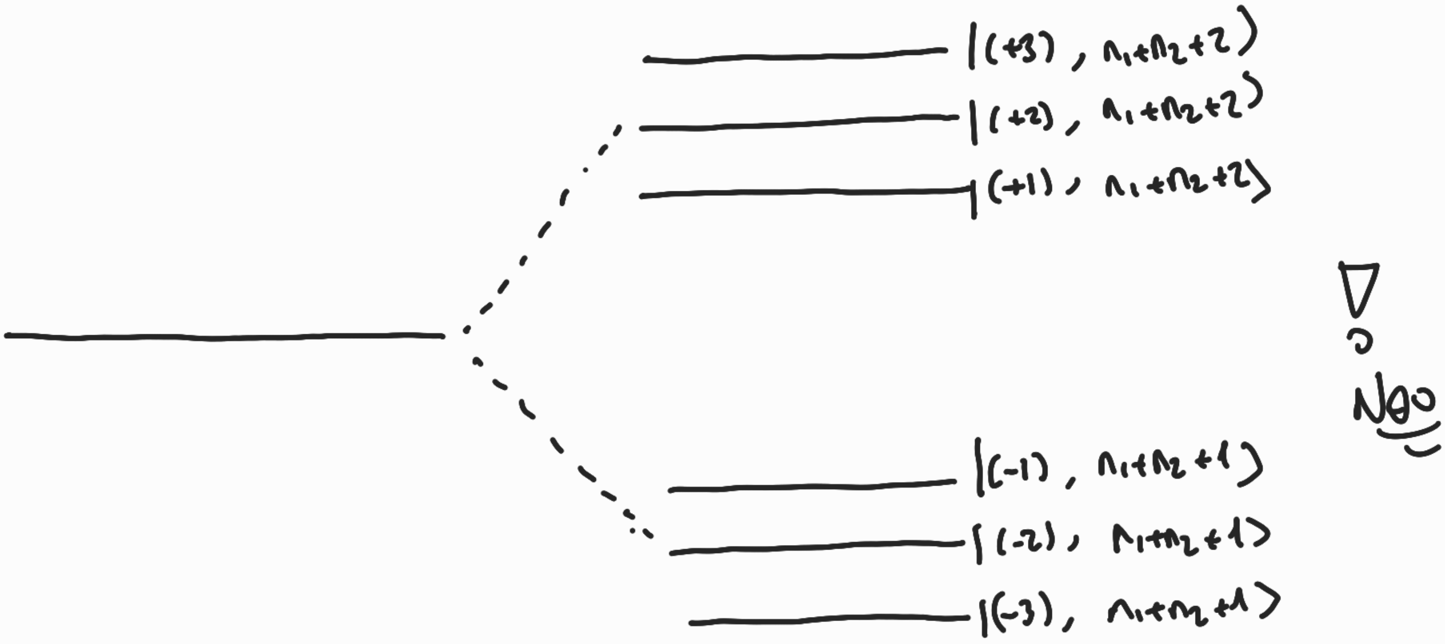
no need

$$\boxed{+\lambda^3 - \lambda^2(F+C+E) + \lambda(FC+FE+CE-Y^2-X^2) - (FCE-FY^2-EX^2) = 0}$$

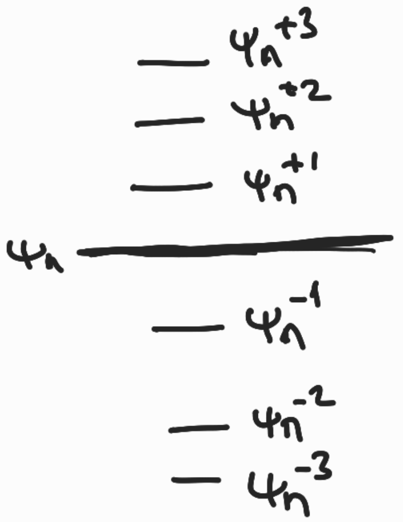
$\downarrow$

$\lambda_4, \lambda_5, \lambda_6$

$v_4, v_5, v_6$




Look AT How JC Model Eigenstates are found?




} degenerate states

HOW TO FIND EIGENVECTORS

or do we need?

$\lambda_1$

$\lambda_2 = \lambda_1\left(-\frac{1}{2}\right) + i\,\lambda_1\left(\frac{\sqrt{3}}{2}\right)$

$\lambda_3 = \lambda_1\left(-\frac{1}{2}\right) - i\,\lambda_1\left(\frac{\sqrt{3}}{2}\right)$

$\lambda_4$

$\lambda_5 = \lambda_4\left(-\frac{1}{2}\right) + i\,\lambda_1\left(\frac{\sqrt{3}}{2}\right)$

$\lambda_6 = \lambda_5\left(-\frac{1}{2}\right) - i\,\lambda_2\left(\frac{\sqrt{3}}{2}\right)$

$|\xi\rangle \otimes |q_a, q_b\rangle \rightsquigarrow$ 2 eigenvectors

(How to find coefficients) $\leftarrow$

$|e,0\rangle = a_1 \underset{\to\lambda_1}{v_1} + a_2 \underset{\to\lambda_2}{v_2} + a_3 \underset{\to\lambda_3}{v_3} + a_4 \underset{\to\lambda_4}{v_4} + a_5 \underset{\to\lambda_5}{v_5} + a_6 \underset{\to\lambda_6}{v_6}$

$|g,1\rangle = b_1 v_1 + \cdots + b_6 v_6$

Look at 2-photon reference $\to \lambda = K + iQ$ (?) $\boxed{\cos(\theta)\,\sin(\theta) = ?}$