QUANTUM MEASUREMENT

I. PROJECTIVE MEASUREMENT

Let $\hat{M}$ be an observable $\hat{M} = \sum_m m \hat{P}_m$

where $\hat{P}_m$ projector onto the eigenspace of m

- Measurement: Upon observing outcome m, the state of the system evolves:

$$|\psi\rangle \longrightarrow \frac{\hat{P}_m|\psi\rangle}{\sqrt{p(m)}} = |\psi_m\rangle$$

The probability to observe m is $p(m) = \langle\psi|\hat{P}_m|\psi\rangle$

For a mixed state $\hat{\rho}$: $p(m) = \mathrm{Tr}(\hat{\rho}\hat{P}_m) = E[m] = \langle\hat{M}\rangle$

2 averages : - Quantum average $\langle\psi|\hat{P}_m|\psi\rangle$

- Classical prob. average

If m has been observed (as outcome of the measurement)

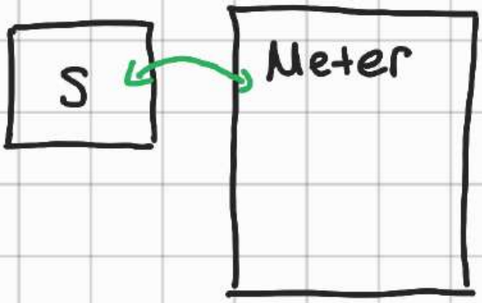
$$\hat{\rho}_m = \frac{\hat{P}_m \hat{\rho} \hat{P}_m^+}{p(m)}$$

If the measurement is performed but the outcome is not recorded

$$\hat{\rho} \longrightarrow \sum_m \hat{\rho}_m\, p(m) = \sum_m \hat{P}_m \hat{\rho} \hat{P}_m^+$$

$$\text{if } |\psi\rangle\langle\psi| \longrightarrow \sum_m p(m) \hat{P}_m |\psi\rangle\langle\psi| \hat{P}_m^+ = \sum_m p(m) |\psi_m\rangle\langle\psi_m|$$

II. System - meter formulation



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projective measurement

① Coupling between S and M

$$|\psi_{SM}\rangle \in \mathcal{H}_S \otimes \mathcal{H}_{meter}$$

$$|\Psi_{SM}\rangle = |\theta\rangle \otimes |\varphi\rangle \quad , |\theta\rangle \in \mathcal{H}_M$$

$$|\varphi\rangle \in \mathcal{H}_S$$

$$|\Psi_{SM}\rangle \longrightarrow |\Psi'\rangle = \hat{U}\,|\Psi_{SM}\rangle \neq |\theta'\rangle \otimes |\varphi'\rangle$$

② Projective Measurement of Meter:

Let r the outcome

$$\hat{P}_r = \hat{\Pi}_r \otimes \hat{\mathbb{I}}_S$$

$$|\Psi'_r\rangle = \frac{\hat{P}_r\,|\Psi'\rangle}{\sqrt{p(r)}} = \frac{(\hat{\Pi}_r \otimes \hat{\mathbb{I}}_S)|\Psi'\rangle}{\sqrt{p(r)}}$$

$$|\Psi'_r\rangle = \frac{(\hat{\Pi}_p \otimes \hat{\mathbb{I}})\,\hat{U}\,|\theta\rangle \otimes |\varphi\rangle}{\sqrt{p(r)}}$$

Suppose $\hat{\Pi}_r = |r\rangle\langle r|$

$$|\Psi'_r\rangle = \frac{|r\rangle \otimes \langle r|\hat{U}|\theta\rangle \cdot |\varphi\rangle}{\sqrt{p(r)}}$$

operator onto $\mathcal{H}_S$

let $\hat{M}_r = \langle r|\hat{U}|\theta\rangle$ operator on $\mathcal{H}_S$

So after the 2 steps:

$$|\Psi_r'\rangle = \frac{|r\rangle \otimes \hat{M}_r|\varphi\rangle}{\sqrt{p(r)}}$$

Consequence:

→ Starting from seperable state $|\theta\rangle \otimes |\varphi\rangle$
end with a seperable state $|\Psi_r'\rangle$

$$|\varphi\rangle \longrightarrow \frac{\hat{M}_r|\varphi\rangle}{\sqrt{p(r)}}$$

→ $\hat{M}_r$ is not a projector : "Measurement Operator"

→ $p(r) = \langle\Psi'|\hat{P}_r|\Psi'\rangle = \langle\varphi|M_r^\dagger M_r|\varphi\rangle$

III. Generalized Measurements

1. Definition: let $\hat{M}_r$ measurement operators,

and $\hat{E}_r = \hat{M}_r^+ \hat{M}_r$ such that

→ $\sum_r \hat{E}_r = \hat{I}$ $\hat{E}_r$ are probability operators

Hermitian, positive

→ Measurement outcomes are labelled by r

$\{\hat{E}_r, r\}$ = Positive operator-valued measurement (POVM)

Then $p(r)$ = probability of outcome r

$$p(r) = Tr(\hat{\rho}\hat{E}_r)$$

After the measurement:

→ If r has been observed $\hat{\rho} \rightarrow \hat{\rho}_r = \dfrac{\hat{M}_r \hat{\rho} \hat{M}_r^+}{p(r)}$

→ If the outcome is not recorded : $\hat{\rho} \to \sum_r \hat{\rho}_r \cdot p(r)$

$$= \sum_r \hat{M}_r \hat{\rho} \hat{M}_r^\dagger$$

Remarks:

- It is not the most general formali.
- Given $\hat{E}_r$, positive Hermition operator

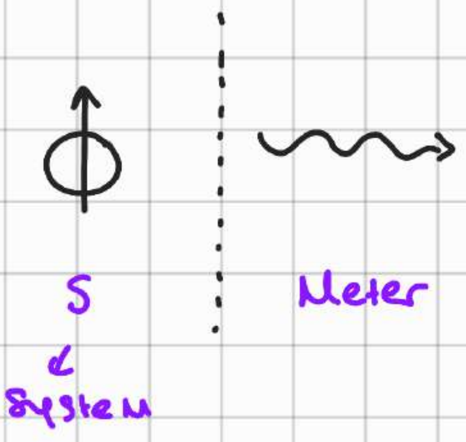
$$\hat{M}_r = \sqrt{\hat{E}_r}$$

(Cannot be repeated, photon killed)

NOT PROJECTIVE !

Example:

Photon counting + spontaneous emission



$|\Psi_{SM}\rangle$: pure state $\in \mathcal{H}_S \otimes \mathcal{H}_M$

Measurement of the # of photon :

j photon : $\hat{M}_j = \hat{I} \otimes |0\rangle\langle j|$

$\hat{I} \otimes \langle 0|j\rangle |0\rangle\langle j|$

? How

$\hat{E}_j = \hat{M}_j^{+} M_j = \hat{\mathbb{I}} \otimes |j\rangle\langle j|$, positive operator

$$\sum_j E_j = \hat{\mathbb{I}} \qquad \{\hat{E}_j, j\} \text{ POVM !}$$

After the measurement: $|\psi_{SM,j}\rangle = \dfrac{|\psi_S\rangle \otimes |0\rangle}{pr(j)}$

(pure) product state

If the result is not recorded:

$$|\psi_{SM}\rangle\langle\psi_{SM}| \longrightarrow \sum_j \hat{M}_j^{+} |\psi_{SM}\rangle\langle\psi_{SM}| \hat{M}_j^{+}$$

$$|\psi_{SM}\rangle = \sum_{kl} \alpha_{kl} |k\rangle \otimes |l\rangle$$

$$|\psi_{SM}\rangle\langle\psi_{SM}| \to \sum_j (\mathbb{I} \otimes |0\rangle\langle j|) \sum_{kl} \alpha_{kl} (|k\rangle \otimes |l\rangle)$$

$$\ldots (\langle k'| \otimes \langle l'|) \alpha^{*}_{k,l} \sum_j (\mathbb{I} \otimes |j\rangle\langle 0|)$$

$$\sum_{kk'} \left(\sum_j \alpha^*_{k'j} \hat{d}_j \, \alpha_{kj} \right) |k\rangle\langle k'| \otimes |0\rangle\langle 0|$$

$$\rightarrow \mathrm{Tr}_M \left(|\psi_{SM}\rangle\langle\psi_{SM}| \right) \otimes |0\rangle\langle 0|$$