EVOLUTION OF DENSITY MATRICES

I. Super - operators

Preskill Notes

1) Evolution Paths for $\hat{\rho}$

→ Unitary evolutions (closed system)

$$i\hbar \partial_t \hat{\rho} = [\hat{H}, \hat{\rho}]$$

$$\hat{\rho}(t) = \hat{U}(t)\, \hat{\rho}\, \hat{U}^{+}(t)$$

→ Measurements

- with results recorded:

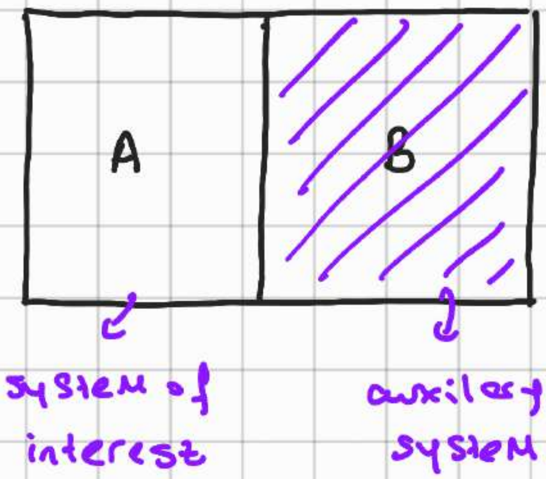
$$\hat{\rho} \longrightarrow \hat{\rho}' = \frac{1}{\sqrt{p(r)}} \hat{M}_r \hat{\rho} \hat{M}_r^{+}$$

with $\hat{M}_r$ measurement operators $\sum_r \hat{M}_r^{+} \hat{M}_r = \mathbb{I}$

- without recording the result:

$$\hat{\rho}' = \sum_r \hat{M}_r \hat{\rho} \hat{M}_r^{+}$$

→ Unitary evolution on extended space



$$\hat{\rho} = \hat{\rho}_A \otimes \underbrace{|0\rangle\langle 0|}_{B}$$

unitary evolution:

$$\hat{\rho}' : \hat{U}\left(\hat{\rho}_A \otimes |0\rangle\langle 0|\right)\hat{U}^{+}$$

operator on $\mathcal{H}_A \otimes \mathcal{H}_B \neq \hat{U}_A \otimes \hat{U}_B$

Partial-trace over B:

$$\hat{\rho}_A' = \mathrm{Tr}_B\, \hat{\rho}'$$

$$= \sum_{\mu} \langle \mu_B | \hat{U}\left(\hat{\rho}_A \otimes |0\rangle\langle 0|\right)\hat{U}^{+} | \mu_B \rangle$$

$$= \sum_{\mu} \underbrace{\langle \mu_B | \hat{U} | 0 \rangle}_{\substack{\text{operator on } \mathcal{H}_A \\ \hat{M}_\mu}} \cdot \hat{\rho}_A \cdot \underbrace{\langle 0 | \hat{U}^{+} | \mu_B \rangle}_{\hat{M}_\mu^{+}}$$

Common feature: $\hat{\rho}' = \sum_{\mu} \hat{M}_\mu \hat{\rho} \hat{M}_\mu^{+}$ → operator sum representation

2) Completely Positive Maps

$$S: \hat{\rho} \longrightarrow \hat{\rho}'$$

$\hat{\rho}$: operator on $\mathcal{H}_A$

Requirements:
- preserves Hermiticity
- trace-preserving
- positive $\hat{\rho} \geq 0 \rightarrow \hat{\rho}' \geq 0$
- $\forall \mathcal{H}_B$ Hilbert space of system B

  $S \otimes \mathbb{I}$ is a positive operator

complete positivity

Completely positive map: Super-Operator $\hat{S} = \hat{\rho} \rightarrow \hat{\rho}'$

Krauss Theorem: Any completely positive map has an operator sum representation.

$$\hat{S}(\hat{\rho}) = \sum_{\nu} \hat{M}_{\nu} \hat{\rho} \hat{M}_{\nu}^{+} \quad \text{for a set of } \{\hat{M}_{\nu}\}$$

$$\text{such that } \sum_{\nu} \hat{M}_{\nu}^{+} \hat{M}_{\nu} = \mathbb{I}$$

Remarks :

- if $\dim(\mathcal{H}_A) = N$ then there are at most N^2 operators

- the representation is not unique

unitary matrix

$$\hat{N}_\nu = \sum_\mu U_{\nu\mu} \hat{M}_\mu$$

- Any super-operator can be interpreted as a POVM

Decoherence

- In general, a super operator IS NOT invertible

the only case where S.O IS invertible is unitary evolution.