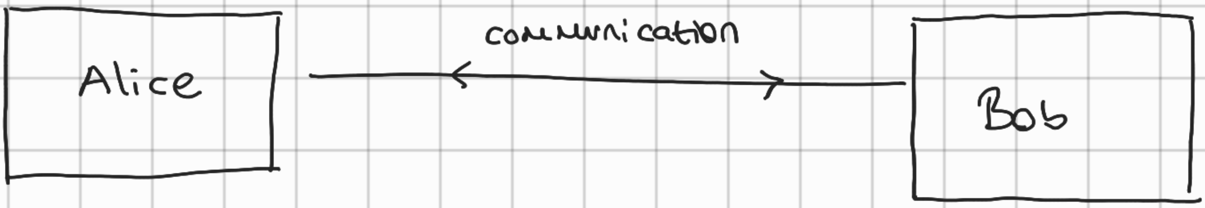
18/10/22

Week 5

Distributed Quantum Computation



Q-circuit local + local meas. device

local Q-circuit local meas. device

Data: $\vec{x} \in \mathbb{F}_2^m = \{0,1\}^m$

$\oplus$, $\times$ mod 2

addition multiplication

Data $\vec{y} \in \mathbb{F}_2^m$

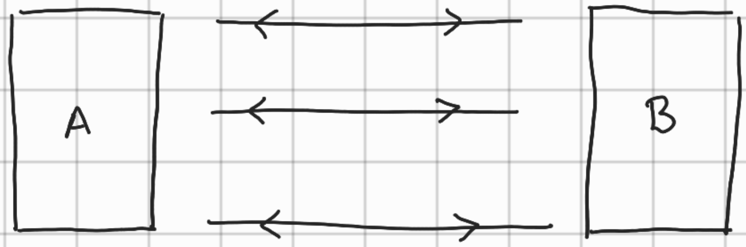
Goal $f(\vec{x},\vec{y}) = ?$

<u>Complexity of the task</u>

Communication complexity (# of bits exchanged)

Two Specific Model

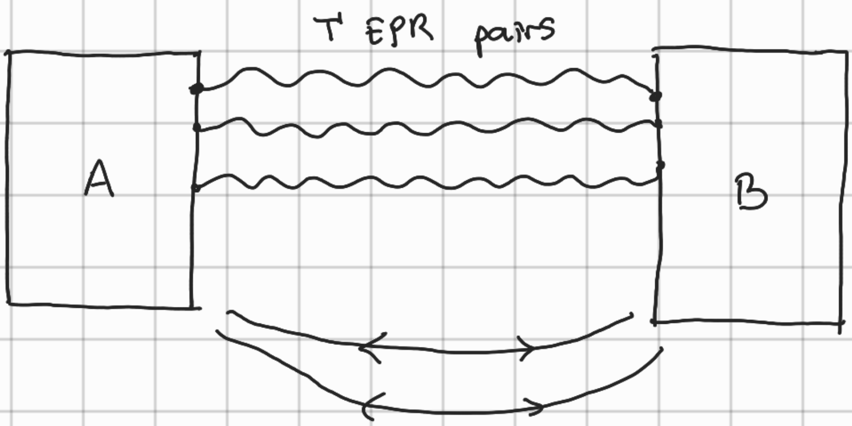
1) Yao Model



Exchange T qubits

$\boxed{\text{com. complexity} = T}$

2) Cleve-Buchman Model



Communication complexity (T, T')

Very-specific "computation"

A & B have to decide the following

$$\vec{x} = \vec{y}$$

or

$$\left(\vec{x} \neq \vec{y} \quad \text{and differ over} \quad \frac{m}{2} \ \text{binary components} \right)$$

$$\Updownarrow$$

$$\begin{cases} d_H(\vec{x}, \vec{y}) = 0 \\ \text{or} \\ d_H(\vec{x}, \vec{y}) = \frac{m}{2} \end{cases}$$

$$d_H(\vec{x}, \vec{y}) = \# \{ i \mid \text{such that } x_i \neq y_i \}$$
$$i = 0, 1, 2, \ldots, m-1$$

The function f ?

$i = 1$

$$f_i(\vec{x}, \vec{y}) = (-1)^{x_i + y_i}$$

$$\vec{f} = (f_0, f_1, \ldots, f_{m-1})$$

$$\boxed{\vec{f}(\vec{x}, \vec{y}) = \frac{1}{m} \sum_{i=0}^{m-1} f_i(\vec{x}, \vec{y})}$$

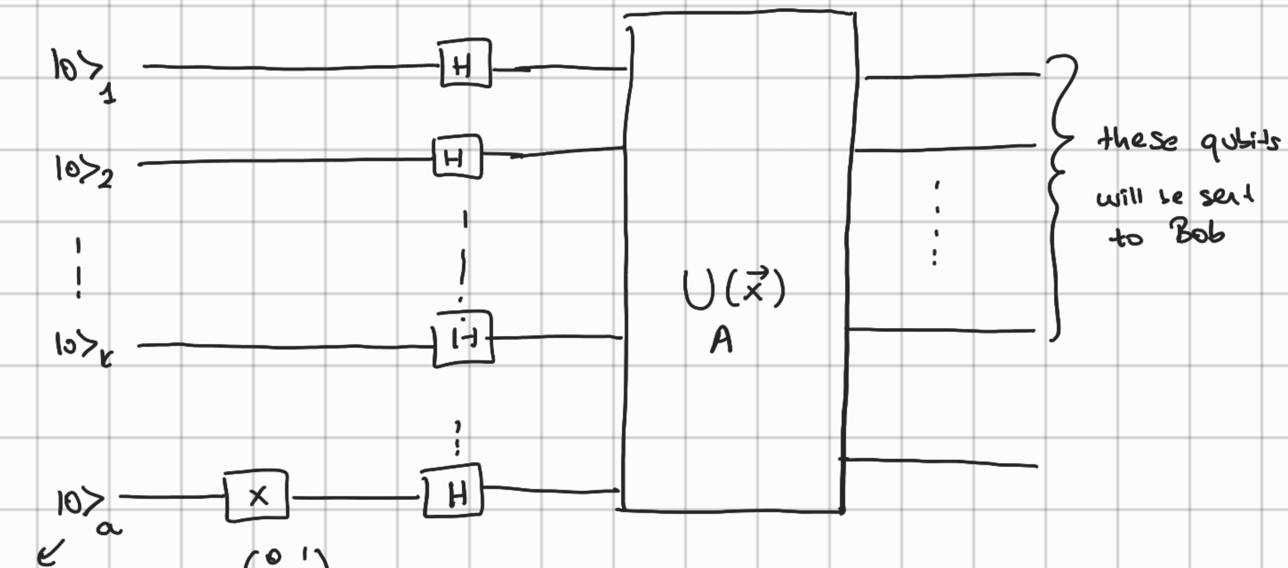
1- Analysis of Yao Model

$$f(\vec{x},\vec{y}) = \frac{1}{m}\sum_{i=0}^{m-1} (-1)^{x_i + y_i}$$

input to the circuit will be $i = 0, \dots, m-1$ where $m = 2^k$

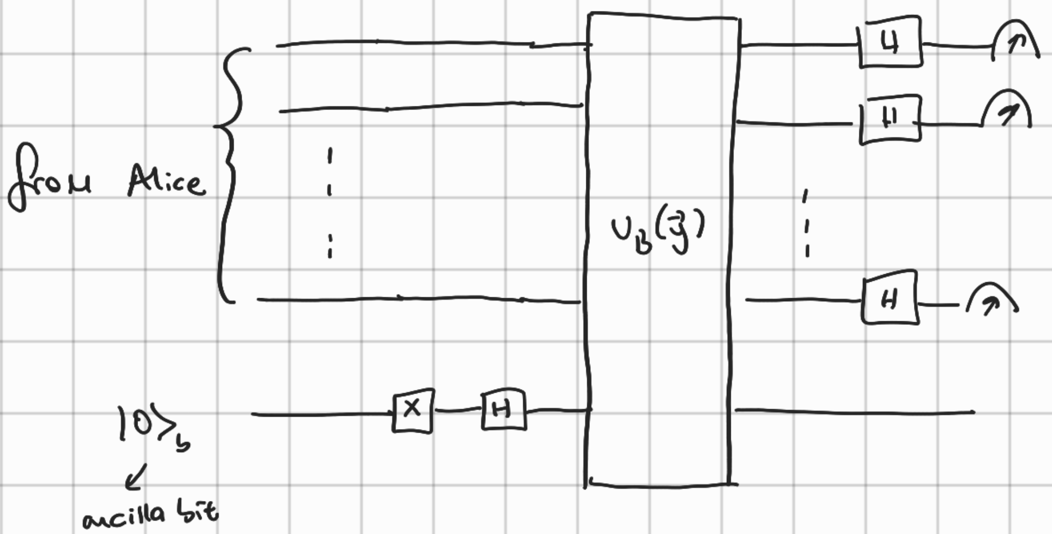
I need $k = \log_2 m$ qubits in first register

Add storage bit → ancilla bit



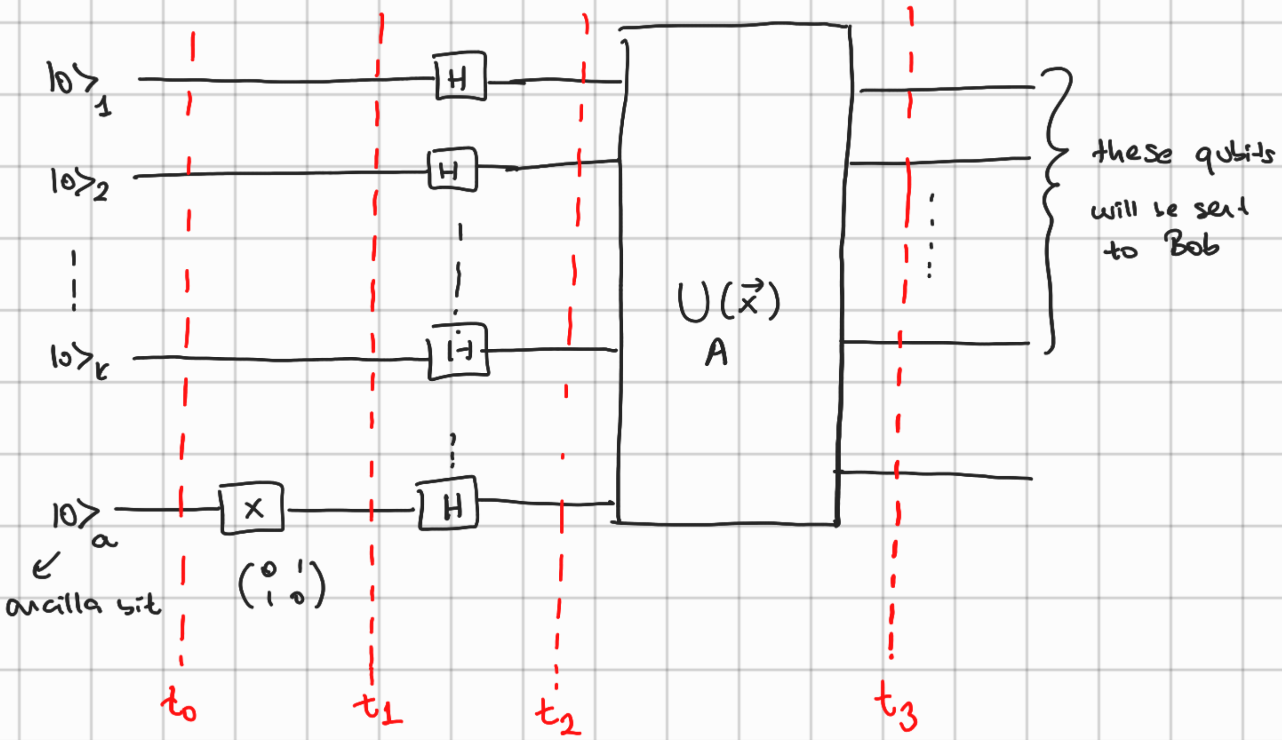
$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

On Bob's Side



$\Rightarrow$ $\vec{x} = \vec{y}$ or $d_H = \frac{m}{2}$

Analysis in A-lab



$$|\psi_0\rangle = \underbrace{|0\rangle \otimes \cdots\cdots \otimes |0\rangle}_{k \text{ qubits}} \otimes |0\rangle_a$$

→ ancilla qubit

$$|\psi_1\rangle = |0\rangle \otimes \cdots\cdots \otimes |0\rangle \otimes |1\rangle$$

$$|\psi_2\rangle = H|0\rangle \otimes \cdots\cdots H|0\rangle \otimes H|1\rangle$$

$$= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \cdots \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

$$= \frac{1}{2^{k/2}} \sum_{b_1, b_2 \in \{0,1\}^k} \underbrace{|b_1\rangle \otimes |b_2\rangle \otimes \cdots \otimes |b_k\rangle}_{|b_1 b_2 \ldots b_k\rangle = |i\rangle} \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$$

$$(b_1 \ldots\ldots b_k) = i \in \{0, \ldots, n-1\}$$

$$= \frac{1}{\sqrt{m}} \sum_{i=0}^{m-1} |i\rangle \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right)$$

To achieve $(-1)^{x_i}$

Main idea of A: $U_A(|i\rangle \otimes |z\rangle) = |i\rangle \otimes |z \oplus x_i\rangle$ for $i = 0 \ldots\ldots m-1$

$|z\rangle = |0\rangle$ or $|1\rangle$

$|i\rangle = |b_1, \ldots\ldots, b_k\rangle$

At $t = t_3$ $\quad |\psi_3\rangle = \frac{1}{\sqrt{m}} \sum_{i=0}^{m-1} \frac{|i\rangle \otimes |x_i\rangle - |i\rangle \otimes |1 \oplus x_i\rangle}{\sqrt{2}}$

$$= \frac{1}{\sqrt{m}} \sum_{i=0}^{m-1} |i\rangle \otimes \left(\frac{|x_i\rangle - |1 \oplus x_i\rangle}{\sqrt{2}} \right)$$

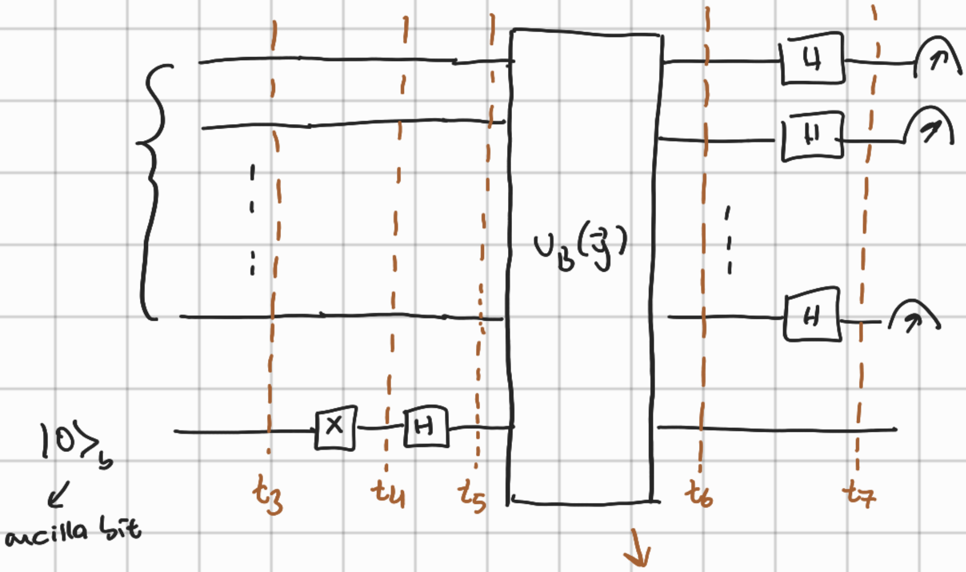
$$(-1)^{x_i} \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right)$$

magical twist :o

kickback

$$= \frac{1}{\sqrt{m}} \left\{ \sum_{i=0}^{m-1} (-1)^{x_i} |i\rangle \right\} \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right)$$

$$\frac{1}{\sqrt{m}} \sum_{i=0}^{m-1} (-1)^{x_i} |i\rangle$$



$$U_B\left(|i\rangle \otimes |z\rangle\right) = |i\rangle \otimes |z \oplus y_i\rangle$$

Time t_5

$$|\psi_5\rangle = \frac{1}{\sqrt{m}} \sum_{i=0}^{m-1} (-1)^{x_i} |i\rangle \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$$

$$\boxed{|\psi_6\rangle = \frac{1}{\sqrt{m}} \sum_{i=0}^{m-1} (-1)^{x_i} (-1)^{y_i} |i\rangle} \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$$

" Quantum Fourier Transform with last series of H's.

$$|\psi_7\rangle = \frac{1}{\sqrt{m}} \sum_{i=0}^{m-1} (-1)^{x_i + y_i} \, H^{\otimes k} |i\rangle$$

$$H^{\otimes k} |b_1 b_2 \cdots b_k\rangle = H|b_1\rangle \otimes \cdots \otimes H|b_k\rangle$$

$$= \frac{1}{2^{k/2}} \sum_{c_1 \cdots c_k \in \{0,1\}^k} (-1)^{b_1 c_1} \cdots (-1)^{b_k c_k} |c_1 c_2 \cdots c_k\rangle$$

$$= \frac{1}{2^{k/2}} \sum_{c_1 \cdots c_k} (-1)^{b_1 c_1 + \cdots + b_k c_k} |c_1 c_2 \cdots c_k\rangle$$

$$|\psi_7\rangle = \frac{1}{m} \sum_{c_1 \dots c_k \in \{0,1\}^k} \left\{ \sum_{i=0}^{m-1} (-1)^{x_i + y_i} \overset{\overset{\text{phase}}{\uparrow}}{(-1)^{\dots}} \right\} |c_1 c_2 \dots c_k\rangle =$$

$$|\psi_7\rangle = \frac{1}{m} \sum_{j=0}^{m-1} \left\{ \sum_{i=0}^{m-1} (-1)^{x_i + y_i} (-1)^{\overset{\overset{\text{inner product}}{\uparrow}}{\langle i,j\rangle}} \right\} |j\rangle$$

Local Measurement in Bob's Lab

Outcome $\rightarrow$ a basis state say $|j\rangle$

$$\text{prob}(j) \underset{\substack{\downarrow \\ \text{Born Rule}}}{=} |\langle j | \psi_7\rangle|^2 = \boxed{\frac{1}{m^2} \left| \sum_{i=0}^{m-1} (-1)^{x_i + y_i} (-1)^{\langle i,j\rangle} \right|^2}$$

$$\text{Prob}_{\text{Bob}}(j=0) = \frac{1}{m^2} \left| \sum_{i=0}^{m-1} (-1)^{x_i + y_i} \right|^2$$

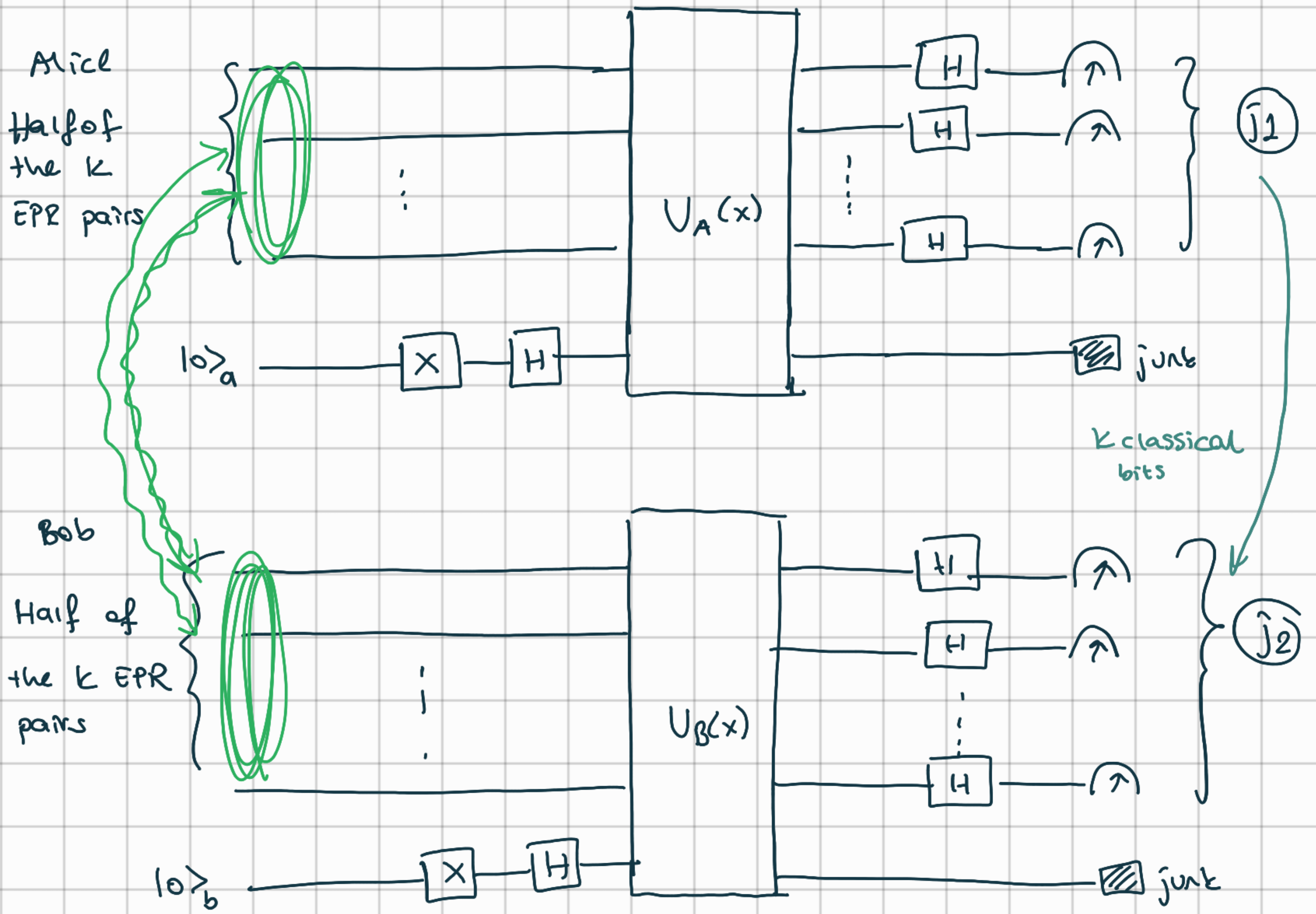
$$\downarrow$$

$= 1$ if $\vec{x} = \vec{y}$

$= 0$ if $d_H = \frac{m}{2}$

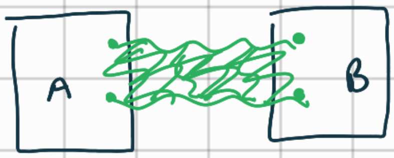
$$\underset{\swarrow \text{complexity}}{T} = \underbrace{\log_2 m}_{\text{qubits}} + \underset{\searrow \text{class. or qubit}}{1} \quad \text{(to announce the result)}$$

2- Cleve Burchman Model



1 EPR pair $\rightarrow \frac{1}{\sqrt{2}}\left(|0_A 0_B\rangle + |1_A 1_B\rangle\right)$

2 EPR pairs



$$= \frac{1}{2}\left(|0_A 0_B\rangle + |1_A 1_B\rangle\right) \otimes \left(|0_A 0_B\rangle + |1_A 1_B\rangle\right)$$

$$= \frac{1}{2}\left(|00\rangle_A \otimes |00\rangle_B + |01\rangle_A \otimes |01\rangle_B + |10\rangle_A \otimes |10\rangle_B + |11\rangle_A \otimes |11\rangle_B\right)$$

k-EPR pairs

$$\frac{1}{2^{k/2}} \sum_{b_1 \dots b_k} |b_1 \dots b_k\rangle_A \otimes |b_1 \dots b_k\rangle_B = \frac{1}{\sqrt{m}} \sum_{i=0}^{M-1} |i\rangle_A \otimes |i\rangle_B$$

$M = 2^k$

Analysis of Algorithm

At time t_0 : $|EPR\rangle_k \otimes |0\rangle_a \otimes |0\rangle_b = |\psi_0\rangle$

At time t_1 : $|EPR\rangle_k \otimes |1\rangle_a \otimes |1\rangle_b = |\psi_1\rangle$

At time t_2 : $|EPR\rangle_k \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}} = |\psi_2\rangle$

At time t_3 : $U_A(\vec{x}) \otimes U_B(\vec{y}) \left[|EPR\rangle_k \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)_a \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)_b \right] = |\psi_3\rangle$

Math

$$|\psi_3\rangle = \frac{1}{\sqrt{M}} \sum_{i=0}^{M-1} U_A(\vec{x}) \otimes U_B(\vec{y}) \left(|i\rangle_A \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}} \right) \otimes \left(|i\rangle_B \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}} \right)$$

$$|\psi_3\rangle = \frac{1}{\sqrt{M}} \sum_{i=0}^{M-1} \underbrace{U_A(\vec{x}) \left(|i\rangle_A \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right) \right)} \otimes U_B(\vec{y}) \left(|i\rangle_B \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right) \right)$$

$$= \frac{1}{\sqrt{m}} \sum_{i=0}^{M-1} (-1)^{x_i} |i\rangle_A \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right) \otimes (-1)^{y_i} |1\rangle_B \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right)$$

$$|\psi_3\rangle = \Big\{ \underbrace{\frac{1}{\sqrt{m}} \sum_{i=0}^{M-1} (-1)^{x_i+y_i} |i\rangle_A \otimes |i\rangle_B}_{\text{New entangled state}} \Big\} \otimes \left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right) \otimes \left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)$$

QFT $\longrightarrow$ $H_A^{\otimes k} \otimes H_B^{\otimes k} |\psi_3\rangle = |\psi_4\rangle$

where $H^{\otimes k}|i\rangle = \frac{1}{\sqrt{m}} \sum_{j=0}^{M-1} (-1)^{\langle i,j\rangle} |j\rangle$

$$|\psi_4\rangle = \frac{1}{M^{3/2}} \sum_{j_1=0}^{M-1} \sum_{j_2=0}^{M-1} \Big\{ \sum_{i=0}^{M-1} (-1)^{x_i+y_i} (-1)^{\langle i,j_1\rangle + \langle i,j_2\rangle} \Big\} |j_1\rangle_A \otimes |j_2\rangle_B$$

$\downarrow$ Alice (j_1) $\quad \downarrow$ Bob (j_2)

Outcome $|j_1\rangle \otimes |j_2\rangle$ with prob ☆

$$\text{Prob}(j_1, j_2) = \frac{1}{m^3} \left| \sum_{i=0}^{M-1} (-1)^{x_i+y_i} (-1)^{\langle i, j_1\rangle} (-1)^{\langle i, j_2\rangle} \right|^2$$

$$\text{Prob}(j_1 = j_2) = \sum_{j=0}^{M-1} \text{prob}(j_1 = j, j_2 = j) = \frac{M}{M^3} \left| \sum_{i=0}^{M-1} (-1)^{x_i+y_i} \right|^2$$

$$= \begin{cases} 1 & \text{if } d_H(\vec{x}, \vec{y}) = 0 \\ 0 & \text{if } d_H(\vec{x}, \vec{y}) = \frac{M}{2} \end{cases}$$

Complexity = K epr pairs + $(K+1)$ classical bits

$\swarrow$

$\log_2(m)$