Symmetries and The Quark Model

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Reference: Thomson Chapter 9

Symmetries & The Quark Model

1. Symmetries in quantum mechanics.

A symmetry of the universe requires that all physical predictions are invariant under the wavefunction transformation

$$\psi \rightarrow \psi' = \hat{U}\psi$$

→ A necessary requirement is that wavefunction normalisations are unchanged:

$$\langle\psi|\psi\rangle = \langle\psi'|\psi'\rangle = \langle\hat{U}\psi|\hat{U}\psi\rangle = \langle\psi|\overbrace{\hat{U}^{\dagger}\hat{U}}^{I}|\psi\rangle$$

must be unitary

→ Eigenstates of the system also must be unchanged.

$$\hat{H} \rightarrow \hat{H}' = \hat{H}$$

$$\hat{H}\psi_i = E_i\psi_i$$

$$\hat{H}'\psi_i' = \hat{H}\psi_i' = E_i\psi_i'$$

$$\hat{H}'\hat{U}\psi_i = E_i\hat{U}\psi_i = \hat{U}E_i\psi_i = \hat{U}\hat{H}\psi_i$$

$$[\hat{H},\hat{U}] = \hat{H}\hat{U} - \hat{U}\hat{H} = 0$$

★ For each symmetry of the Hamiltonian, there's a corresponding unitary operator which commutes with the Hamiltonian.

A finite continous symmetry

$$\hat{U}(\epsilon) = I + i\epsilon \hat{G}$$

infinitesimally small parameter — Generator

$$\hat{U}(\epsilon)\hat{U}^{+}(\epsilon) = (I + i\epsilon\hat{G})(I - i\epsilon\hat{G}^{+}) = I + i\epsilon(\underbrace{\hat{G}-\hat{G}^{+}}_{0}) + \cancel{O(\epsilon^2)}$$

$$\hat{G} = \hat{G}^{+}$$

★ For each symmetry of the Hamiltonian, there is corresponding unitary symmetry operation with an associated Hermitian generator $\hat{G}$. The eigenstates of a Hermition operator are real and therefore the $\hat{G}$ operator is associated with an observable quantity G.

$$[\hat{H}, \hat{U}] = 0 \rightarrow [\hat{H}, I + i\epsilon\hat{G}] = 0 \rightarrow [\hat{H}, \hat{G}] = 0$$

$$\frac{d}{dt}\langle\hat{G}\rangle = i\langle\overbrace{[\hat{H}, \hat{G}]}^{0}\rangle$$

★ For each symmetry of the Hamiltonian, there's an associated observable conserved quantity G.

Translational invariance

$$x \longrightarrow x + \epsilon$$

$$\psi(x) \longrightarrow \psi'(x) = \psi(x+\epsilon)$$

$$\psi'(x) = \psi(x+\epsilon) = \psi(x) + \frac{\partial\psi}{\partial x}\epsilon + \cancel{O(\epsilon^2)}$$ Taylor Expansion

$$\psi'(x) = \left(1 + \epsilon \frac{\partial}{\partial x}\right)\psi(x)$$

$$\hat{p}_x = -i \frac{\partial}{\partial x}$$

$$\psi'(x) = (1 + i\epsilon \underbrace{\hat{p}_x}_{\text{generator}})\psi(x)$$

★ Hence the translational invariance of Hamiltonian implies momentum conservation.

In general, $\hat{G} = \{\hat{G}_i\}$ and $\epsilon = \{\epsilon_i\}$

bold

set of generators

Ex. 3-d spatial translation $\vec{x} \to \vec{x} + \vec{\epsilon}$ $\quad \vec{p} = (\hat{p}_x, \hat{p}_y, \hat{p}_z)$

$$\hat{U}(\vec{\epsilon}) = 1 + i\vec{\epsilon}\cdot\vec{p} \equiv 1 + i\epsilon_x\hat{p}_x + i\epsilon_y\hat{p}_y + i\epsilon_z\hat{p}_z$$

1.1 Finite Transformations

Any finite symmetry transformation can be expressed as a series of infinitesimal transformations using

$$\boxed{\hat{U}(\alpha) = \lim_{n\to\infty}\left(1 + i\frac{1}{n}\alpha.\hat{G}\right)^n = \exp(i\alpha.\hat{G})}$$

Ex $\quad x \to x + x_0$

$$\hat{U}(x_0) = \exp(i x_0 \hat{p}_x) = \exp\left(x_0 \frac{\partial}{\partial x}\right)$$

$$\psi'(x) = \hat{U}\psi(x) = \exp\left(x_0 \frac{\partial}{\partial x}\right)\psi(x)$$

$$\psi'(x) = \left(1 + x_0 \frac{\partial}{\partial x} + \frac{x_0^2}{2!}\frac{\partial^2}{\partial x^2} + \cdots\right)\psi(x)$$

$$\psi'(x) = \psi(x) + x_0 \frac{\partial \psi}{\partial x} + \frac{x_0^2}{2}\frac{\partial^2 \psi}{\partial x^2} + \cdots\cdots = \psi(x + x_0)$$

2. Flavour Symmetry

Proton and neutron have very similar masses

Nuclear force is approximately charge independent. In other words, the strong force potential is the same for two protons, two neutrons or a neutron and a proton.

$$V_{pp} \approx V_{np} \approx V_{nn}$$

Heisenberg suggested if you could switch off the electric charge of the proton, there would be no way to distinguish btw a proton and a neutron.

☆ Neutron and proton could be considered as two states of a single entity (nucleon) analogous to the spin-up and spin-down states of a spin-half particle,

$$J = 1/2 \qquad\qquad J = -1/2$$

$$p = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad n = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightarrow \text{Isospin dublets}$$

Idea of isospin (physically no relation with the spins)

Ex Helium-4 with 2 neutrons and 2 protons

$$\text{Total isospin} = 2\times\left(-\tfrac{1}{2}\right) + 2\times\left(\tfrac{1}{2}\right) = 0$$

I_3 : Projection of the total isospin onto a particular axis (usually z-axis) Defined as the difference btw the number of particles that have isospin up and isospin down

$$I_3 = 2 - 2 = 0$$

2.1 Flavour Symmetry of the strong interaction

Idea of proton/neutron isospin symmetry can be extended to the quarks.

$$\hat{H} = \underbrace{\hat{H}_0}_{\text{kinetic and rest mass energy of the quarks}} + \hat{H}_{strong} + \underbrace{\hat{H}_{em}}_{\text{electro-magnetic interaction}}$$

If the effective masses of the up and down quarks are the same, and $\hat{H}_{em}$ is small compared to $\hat{H}_{strong}$, then the Hamiltonian posseses an up-down (ud) flavour symmetry.

$$u = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad d = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} u' \\ d' \end{pmatrix} = \hat{U} \begin{pmatrix} u \\ d \end{pmatrix} = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} u \\ d \end{pmatrix}$$

2x2 matrix depends on 4 complex numbers which can be described by eight real parameters. The condition $\hat{U}U^{+} = I$ imposes 4 constraints. Thus 2x2 matrix can be expressed in terms of 4 linearly independent 2x2 matrices representing the generators of the transformation

$$\hat{U} = \exp(i\alpha_i \hat{G}_i)$$

- $\hat{U} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} e^{i\phi}$ → U(1) transformation
  It is not relevant to the discussion of transformations btw different flavour states

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$ → Pauli Spin Matrices

Hermitian Generators of the SU(2)

special unitary 2 parameters

SU(2) Matrices' determinants are 1.

☆ The ud flavour symmetry corresponds to invariance under SU(2) transformations leading to three conserved quantities defined by the eigenvalues of Pauli-spin matrices.

The algebra of the ud flavour symmetry is therefore identical to that of spin for a spin-half particle.

$$\text{Isospin} \quad \boxed{\hat{T} = \frac{1}{2}\sigma}$$

Any finite transformation in the ud flavour space can be written in terms of a unitary transformation

$$\boxed{\hat{U} = e^{i\alpha.\hat{T}}}$$

such that

$$\begin{pmatrix} u' \\ d' \end{pmatrix} = e^{i\alpha.\hat{T}} \begin{pmatrix} u \\ d \end{pmatrix}$$

Rotation in flavour space

2.2 Isospin Algebra

$$[\hat{T}_1, \hat{T}_2] = i\hat{T}_3$$

$$[\hat{T}_2, \hat{T}_3] = i\hat{T}_1$$

$$[\hat{T}_3, \hat{T}_1] = i\hat{T}_2$$

Non-Abelian (non-commuting) Lie Algebra

$$\hat{T}^2 = \hat{T}_1^2 + \hat{T}_2^2 + \hat{T}_3^2$$

total isospin operator

(likewise total angular momentum)

$\hat{T}$ commutes with $\hat{T}_1, \hat{T}_2, \hat{T}_3$

Total isospin operator commutes with each of the generators, so it is Hermitian and corresponds to an observable quantity. Since the generators don't commute, they can not be known simultaneously.

✱ Thus isospinstates $\phi(I, I_3)$ are the mathematical analogues of the angular momentum states $|\ell, m\rangle$ and have the properties

$$\hat{T}^2 \, \phi(I, I_3) = I(I+1) \, \phi(I, I_3)$$
$$\hat{L}^2 \, |\ell, m\rangle = \ell(\ell+1) \, |\ell, m\rangle$$

$$\hat{T}_3 \, \phi(I, I_3) = I_3 \, \phi(I, I_3)$$
$$\hat{L}_z \, |\ell, m\rangle = m \, |\ell, m\rangle$$

Isospin 1/2 multiplet consisting of an up and down quark

$$u = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \phi\left(\tfrac{1}{2}, +\tfrac{1}{2}\right) \qquad d = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \phi\left(\tfrac{1}{2}, -\tfrac{1}{2}\right)$$

Isospin ladder operators:

$$\hat{T}_- \equiv \hat{T}_1 - i\,\hat{T}_2$$

$$\hat{T}_+ \equiv \hat{T}_1 + i\,\hat{T}_2$$

$$\hat{T}_+ \, \phi(I, I_3) = \sqrt{I(I+1) - I_3(I_3+1)} \; \phi(I, I_3+1)$$

$$\hat{T}_- \, \phi(I, I_3) = \sqrt{I(I+1) - I_3(I_3-1)} \; \phi(I, I_3-1)$$

$$\hat{T}_- \, \phi(I, -I) = 0 \qquad \hat{T}_+ \, \phi(I, +I) = 0$$

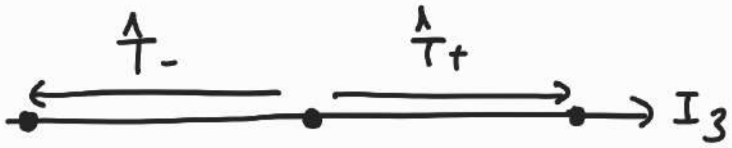
→ extreme states

$$\hat{T}_+ u = 0$$

$$\hat{T}_+ d = u$$

$$\hat{T}_- u = d$$

$$\hat{T}_- d = 0$$



3. Combining quarks into baryons

* The rules for combining isospin for a system of two quarks are identical to those for the addition of angular momentum.

$$\phi(I^a, I_3^a) \qquad \phi(I^b, I_3^b)$$

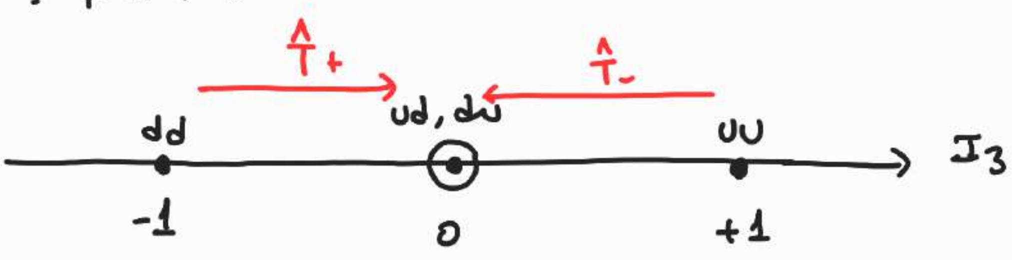
$$\boxed{I_3 = I_3^a + I_3^b \qquad |I^a - I^b| \leq J \leq |I^a + I^b|}$$

$$uu \equiv \phi\left(\tfrac{1}{2}, \tfrac{1}{2}\right)\phi\left(\tfrac{1}{2}, \tfrac{1}{2}\right) = \phi(1, +1)$$

$$dd = \phi\left(\tfrac{1}{2}, -\tfrac{1}{2}\right)\phi\left(\tfrac{1}{2}, -\tfrac{1}{2}\right) = \phi(1, -1)$$

$$ud = \phi\left(\tfrac{1}{2}, \tfrac{1}{2}\right)\phi\left(\tfrac{1}{2}, -\tfrac{1}{2}\right) = \phi(1, 0)$$

$$du : \phi(1, 0)$$



$$\hat{T}_- \phi(1,+1) = \sqrt{I(I+1) - I_3(I_3-1)}\ \phi(1,0)$$

$$\hat{T}_-\ uu = \sqrt{1.2 - 1(0)}\ (ud + du)$$

$$\hat{T}_-(uu) = \frac{(ud + du)}{\sqrt{2}}$$

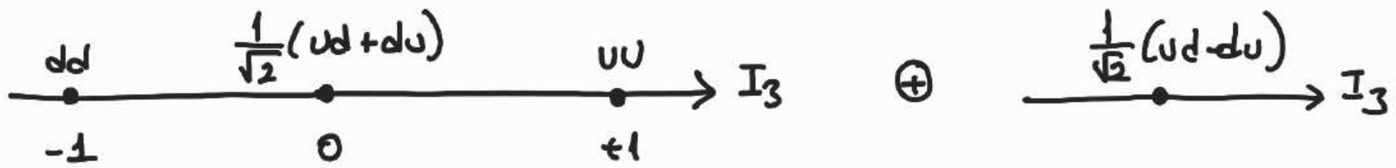
☆ The $\phi(0,0)$ state can be defined as the linear combination of ud and du that is orthogonal to $\phi(1,0)$

$$\phi(0,0) = \frac{1}{\sqrt{2}}(ud - du)$$

$I = 1 \rightarrow I_3 = -1,\ 0,\ 1$

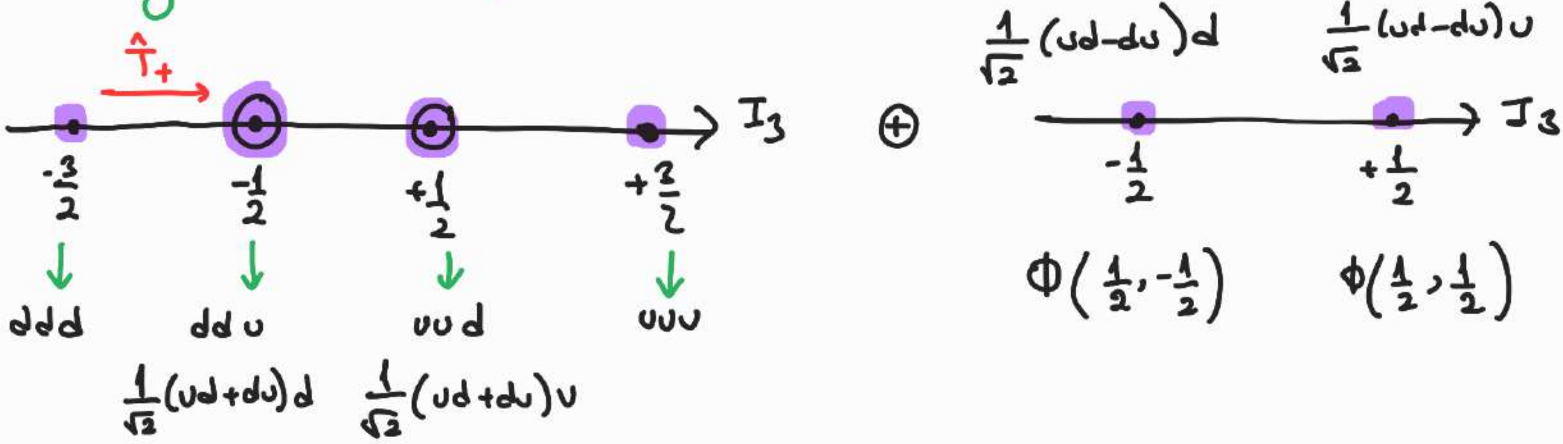
$I = 0 \rightarrow I_3 = 0$

The isospin eigenstates for the combination of two quarks:



$$2 \otimes 2 = 3 \oplus 1$$

Adding the 3rd qubit:



$\phi\left(\frac{3}{2},-\frac{1}{2}\right) \longrightarrow$ linear combination of ddu and $\frac{1}{\sqrt{2}}(ud+du)d$

can be obtained from the action of $\hat{T}_+$

$$\hat{T}_+\,\phi\left(\tfrac{3}{2},-\tfrac{3}{2}\right) = \sqrt{\underset{\frac{3}{2}}{I}\,\underbrace{(I+1)}_{\frac{5}{2}} - \underset{-\frac{3}{2}}{I_3}\,(\underset{-\frac{1}{2}}{I_3+1})}\;\phi\left(\tfrac{3}{2},-\tfrac{1}{2}\right) = \sqrt{3}\,\phi\left(\tfrac{3}{2},-\tfrac{1}{2}\right)$$

$$\frac{15}{4}-\frac{3}{4}=\frac{12}{4}=3$$

$$\boxed{\hat{T}_+(ddd) = (\underbrace{\hat{T}_+d}_{u})dd + d(\underbrace{\hat{T}_+d}_{u})d + dd[\underbrace{\hat{T}_+d}_{u}]}$$

$$\phi\left(\tfrac{3}{2},-\tfrac{1}{2}\right) = \frac{1}{\sqrt{3}}(udd+dud+ddu)$$

Four isospin-$\frac{3}{2}$ states, built from the $\underset{\text{quark quark}}{qq}$ triplet,

$$\phi\left(\tfrac{3}{2},-\tfrac{3}{2}\right) = ddd$$

$$\phi\left(\tfrac{3}{2},-\tfrac{1}{2}\right) = \frac{1}{\sqrt{3}}(udd+dud+ddu)$$

from $\hat{T}_-(uuu) \rightsquigarrow$ $\phi\left(\tfrac{3}{2},+\tfrac{1}{2}\right) = \frac{1}{\sqrt{3}}(uud+udu+duu)$

$$\phi\left(\tfrac{3}{2},\tfrac{3}{2}\right) = uuu$$

★ $\Phi\left(\frac{1}{2},-\frac{1}{2}\right)$ is orthogonal to $\Phi\left(\frac{3}{2},-\frac{1}{2}\right)$

$\Phi\left(\frac{1}{2},-\frac{1}{2}\right)$ is a linear combination of ddu and $\frac{1}{\sqrt{2}}(ud+du)d$ which is orthogonal to $\Phi\left(\frac{3}{2},-\frac{1}{2}\right)$

$$\Phi\left(\frac{1}{2},-\frac{1}{2}\right) = a\,ddu + \frac{b}{\sqrt{2}}udd + \frac{b}{\sqrt{2}}dud$$

$$\left\langle \Phi\left(\frac{1}{2},-\frac{1}{2}\right) \middle| \Phi\left(\frac{3}{2},-\frac{1}{2}\right)\right\rangle = 0$$

$$\frac{a}{\sqrt{3}} + \frac{2b}{\sqrt{6}} = 0 \Rightarrow \cancel{\sqrt{2}}a = -\overset{\sqrt{2}}{\cancel{2}}b$$

$(\sqrt{2})$

$b = x$, $a = -\sqrt{2}x$

$x^2 + 2x^2 = 1 \Rightarrow x = \frac{1}{\sqrt{3}}$

Normalization : $a^2 + \cancel{2}\frac{b^2}{\cancel{2}} = 1$

$a = -\frac{\sqrt{2}}{\sqrt{3}}$, $b = \frac{1}{\sqrt{3}}$

$$\Phi_S\left(\frac{1}{2},-\frac{1}{2}\right) = -\frac{1}{\sqrt{6}}\left(2ddu - udd - dud\right)$$

symmetric
No change when
quarks $1 \leftrightarrow 2$

Similarly,

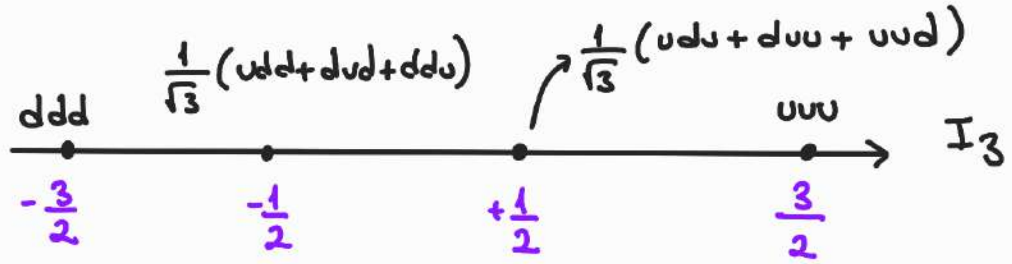
$$\Phi_S\left(\frac{1}{2},+\frac{1}{2}\right) = \frac{1}{\sqrt{6}}\left(2uud - udu - duu\right)$$

Other two states constructed from the qq isospin singlet $\Phi(0,0) = \frac{1}{\sqrt{2}}(ud-du)$

(antisymmetric)
under $1 \leftrightarrow 2$

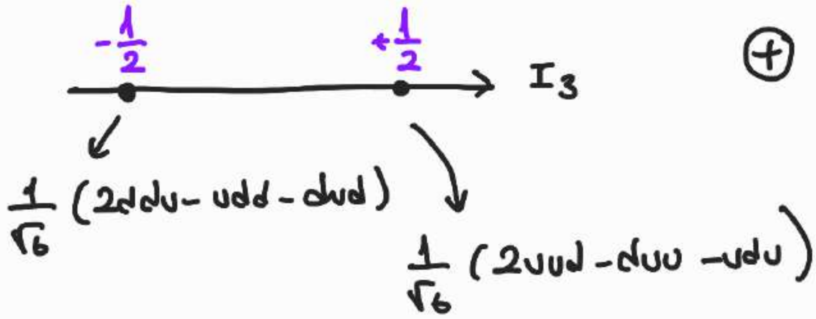
$$\Phi_A\left(\frac{1}{2},-\frac{1}{2}\right) = \frac{1}{\sqrt{2}}\left(udd - dud\right) \qquad \Phi_A\left(\frac{1}{2},\frac{1}{2}\right) = \frac{1}{\sqrt{2}}\left(udu - duu\right)$$

$J = \frac{3}{2}$

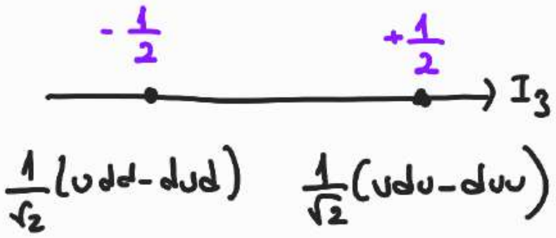


$\oplus$

$J = \frac{1}{2}$



$\oplus$



In terms of SU(2) group structure:

$$2 \otimes 2 \otimes 2 = 2 \otimes (3 \oplus 1) = (2 \otimes 3) \oplus (2 \otimes 1) = 4 \oplus 2 \oplus 2$$

3.1 Spin States of three quarks

$$\chi\left(\tfrac{3}{2}, +\tfrac{3}{2}\right) = \uparrow\uparrow\uparrow$$

$$\chi\left(\tfrac{3}{2}, +\tfrac{1}{2}\right) = \frac{1}{\sqrt{3}}\left(\uparrow\uparrow\downarrow + \uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow\right)$$

$$\chi\left(\tfrac{3}{2}, -\tfrac{1}{2}\right) = \frac{1}{\sqrt{3}}\left(\downarrow\downarrow\uparrow + \downarrow\uparrow\downarrow + \uparrow\downarrow\downarrow\right)$$

$$\chi\left(\tfrac{3}{2}, -\tfrac{3}{2}\right) = \downarrow\downarrow\downarrow$$

$$\chi_S\left(\tfrac{1}{2}, -\tfrac{1}{2}\right) = -\frac{1}{\sqrt{6}}\left(2\downarrow\downarrow\uparrow - \uparrow\downarrow\downarrow - \downarrow\uparrow\downarrow\right)$$

$$\chi_S\left(\tfrac{1}{2}, +\tfrac{1}{2}\right) = \frac{1}{\sqrt{6}}\left(2\uparrow\uparrow\downarrow - \uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow\right)$$

$$\chi_A\left(\tfrac{1}{2}, -\tfrac{1}{2}\right) = \frac{1}{\sqrt{2}}\left(\uparrow\downarrow\downarrow - \downarrow\uparrow\downarrow\right)$$

$$\chi_A\left(\tfrac{1}{2}, +\tfrac{1}{2}\right) = \frac{1}{\sqrt{2}}\left(\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow\right)$$

5. Isospin Representation of antiquarks

$$q = \begin{pmatrix} u \\ d \end{pmatrix}$$

A general SU(2) transformation of the quark doublet, $q \to q' = Uq$

$$\begin{pmatrix} u \\ d \end{pmatrix} \to \begin{pmatrix} u' \\ d' \end{pmatrix} = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \begin{pmatrix} u \\ d \end{pmatrix}$$

where $aa^* + bb^* = 1$

$$\psi' = \hat{C}\psi = i\gamma^2\psi^* \quad \text{(charge conjugation)}$$

$$\begin{pmatrix} \bar{u}' \\ \bar{d}' \end{pmatrix} = U^* \begin{pmatrix} \bar{u} \\ \bar{d} \end{pmatrix} = \begin{pmatrix} a^* & b^* \\ -b & a \end{pmatrix} \begin{pmatrix} \bar{u} \\ \bar{d} \end{pmatrix}$$

In SU(2) it is possible to place the antiquarks in a doublet that transforms in the same way as the quarks

$$\bar{q} \to \bar{q}' = U\bar{q}$$

$$\boxed{\bar{q} \equiv \begin{pmatrix} -\bar{d} \\ \bar{u} \end{pmatrix} = S \begin{pmatrix} \bar{u} \\ \bar{d} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \bar{u} \\ \bar{d} \end{pmatrix}}$$

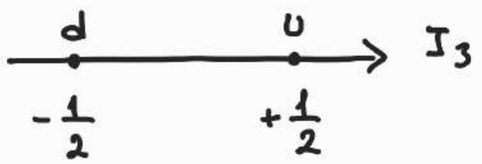
$$\begin{pmatrix} \bar{u} \\ \bar{d} \end{pmatrix} = S^{-1}\bar{q} \qquad \begin{pmatrix} \bar{u}' \\ \bar{d}' \end{pmatrix} = S^{-1}\bar{q}'$$

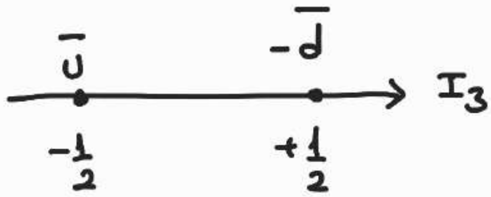
$$S^{-1}\bar{q}' = U^* S^{-1}\bar{q}$$

$$\bar{q}' = S U^* S^{-1} \bar{q}$$

$$S U^* S^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a^* & b^* \\ -b & a \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} = U$$

$$\bar{q} \to \bar{q}' = U \bar{q}$$



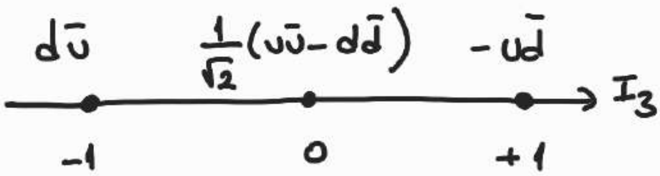


$$\hat{T}_+ \bar{u} = -\bar{d} \qquad \hat{T}_+ \bar{d} = 0$$

$$\hat{T}_- \bar{d} = -\bar{u} \qquad \hat{T}_- \bar{u} = 0$$

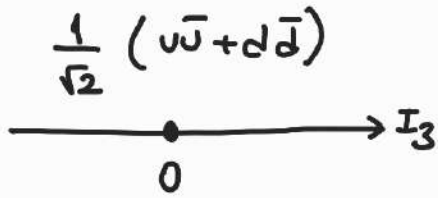
★ It is not possible to place the quarks and antiquarks in the same representation: this is a feature SU(2).
It can not applied to the SU(3) flavour symmetry.

Meson States



Triplet states

$\oplus$



Singlet State

The action of the isospin raising and lowering operators on the $\phi(0,0)$ state both give zero, confirming that it is indeed a singlet state.

6. SU(3) flavour symmetry

The strong interaction part of the Hamiltonian treats all quarks equally and therefore posseses an exact uds flavour symmetry. However since the mass of the strange quark is different from the masses of the up- and down-quarks, the overall Hamiltonian is not flavour symmetric.

Nevertheless m_s and $m_{u/d}$ difference $\sim 100\,MeV$ which is relatively small compared to binding energies of baryons $\sim 1\,GeV$.

Thus, let's assume Hamiltonian posses a uds flavour symmetry.

$$\begin{pmatrix} u' \\ d' \\ s' \end{pmatrix} = \hat{U} \begin{pmatrix} u \\ d \\ s \end{pmatrix} = \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{pmatrix} \begin{pmatrix} u \\ d \\ s \end{pmatrix}$$

In general 3×3 matrix $\longrightarrow$ 9 complex numbers $\equiv$ 18 real parameters

From $\hat{U}^\dagger \hat{U} = I \rightarrow 9$ constraints

★ Thus, $\hat{U}$ can be expressed in terms 9 linearly independent 3×3 matrices.

As before in the case of SU(2), one of these matrices is the identity matrix multiplied by a complex phase, not relevent to transformations btw different flavour states

The remaining 8 matrices form an SU(3) group and can be expressed in terms of the eight independent Hermitian generators $\hat{T}_i$ such that the general SU(3) flavour transformation can be expressed:

$$\boxed{\hat{U} = e^{i\alpha . \hat{T}}}$$

$$\hat{T} = \frac{1}{2}\lambda \quad \text{(eight } \lambda\text{-matrices)}$$

$$u = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad d = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad s = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

* SU(3) uds flavour symmetry contains the subgroup of SU(2) u↔d flavour symmetry.

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\hat{T}_3 = \frac{1}{2}\lambda_3$$

such that $\hat{T}_3 u = +\frac{1}{2}u$ $\quad \hat{T}_3 d = -\frac{1}{2}d$ $\quad$ and $\hat{T}_3 s = 0$

SU(3) uds flavour symmetry also contains the subgroups of SU(2) u↔s and SU(2) d↔s flavour symmetries, both of which can also be expressed in terms of the Pauli spin-matrices.

λ- matrixes for the $u \leftrightarrow s$ symmetry

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \qquad \lambda_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

For the $d \leftrightarrow s$ symmetry they are

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad \lambda_y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

★ Here λ_3, λ_x, λ_y can be expressed in terms of other two

Because the $u \leftrightarrow d$ symmetry is nearly exact, retain λ_3 as one of the eight generators of SU(3) flavour symmetry.

The final generator is chosen as the linear combination of λ_x and λ_y that treats u and d quarks symmetrically

WHY?

$$\lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} + \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

6.1 SU(3) flavour states

$$\hat{T}^2 = \sum_{i=1}^{8} \hat{T}_i^2 = \frac{1}{4}\sum_{i=1}^{8} \lambda_i^2 = \frac{4}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

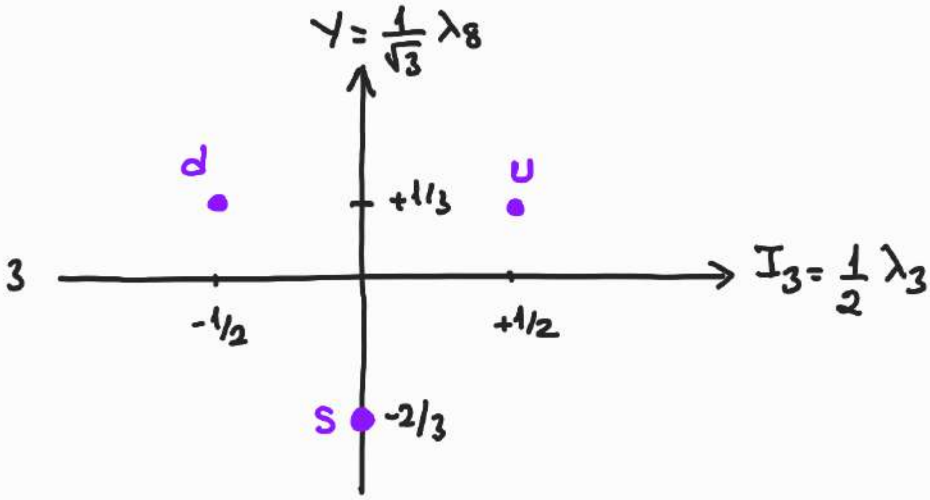
total isospin

★ Of the eight SU(3) generators, only $T_3 = \frac{1}{2}\lambda_3$ and $T_8 = \frac{1}{2}\lambda_8$ commute and thus describe compatible observable quantities

In addition to the analogue of the total isospin, SU(3) states are described in terms of the eigenstates of the λ_3 and λ_8 matrices.

$$\hat{T}_3 = \frac{1}{2}\lambda_3 \quad \text{and} \quad \hat{Y} = \frac{1}{\sqrt{3}}\lambda_8$$

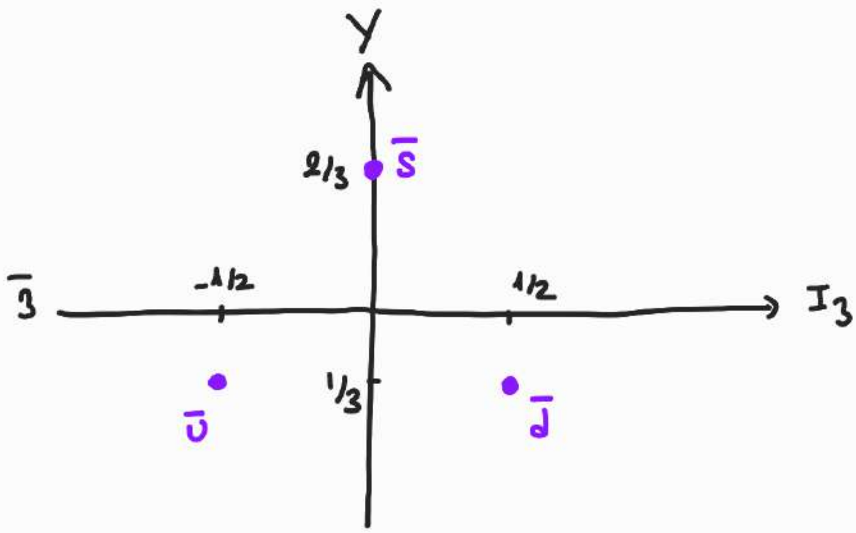
↳ flavour hypercharge



$\hat{T}_3 u = +\frac{1}{2}u \qquad \hat{Y}u = +\frac{1}{3}u$

$\hat{T}_3 d = -\frac{1}{2}d \qquad \hat{Y}d = +\frac{1}{3}d$

$\hat{T}_3 s = 0 \qquad \hat{Y}s = -\frac{2}{3}s$



$$I_3 = n_u - n_d$$

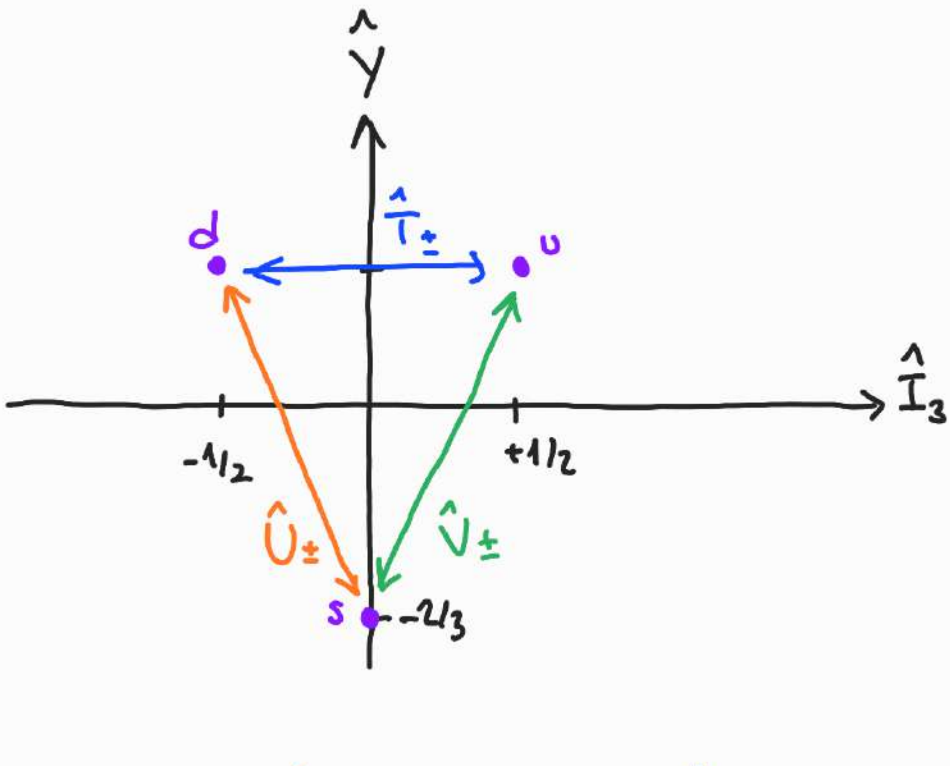
$$Y = \frac{1}{3}(n_u + n_d - 2n_s)$$

While the Gell-Mann λ_3 and λ_8 matrices label the SU(3) states, remaining six λ-matrices can be used to define ladder operators.

$$\hat{T}_\pm = \frac{1}{2}(\lambda_1 \pm i\lambda_2) \qquad d \leftrightarrow u$$

$$\hat{V}_\pm = \frac{1}{2}(\lambda_4 \pm i\lambda_5) \qquad s \leftrightarrow u$$

$$\hat{U}_\pm = \frac{1}{2}(\lambda_6 \pm i\lambda_7) \qquad d \leftrightarrow s$$



$\hat{V}_+ s = +u \qquad \hat{U}_+ s = +d \qquad \hat{T}_+ d = +u$

$\hat{V}_- u = +s \qquad \hat{U}_- d = +s \qquad \hat{T}_- u = +d$

All other combinations giving zero.

6.2 The light mesons

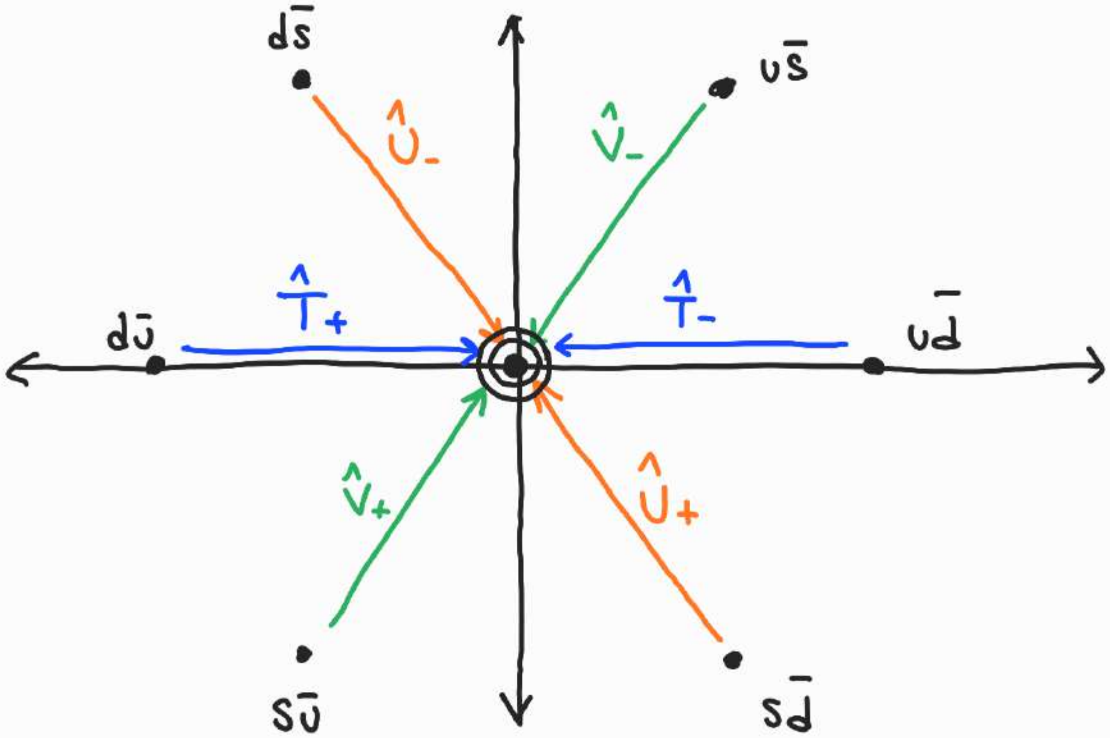
In the discussion of SU(2) flavour symmetry, the third component of isospin is an additive quantum number, in analogy with angular momentum.

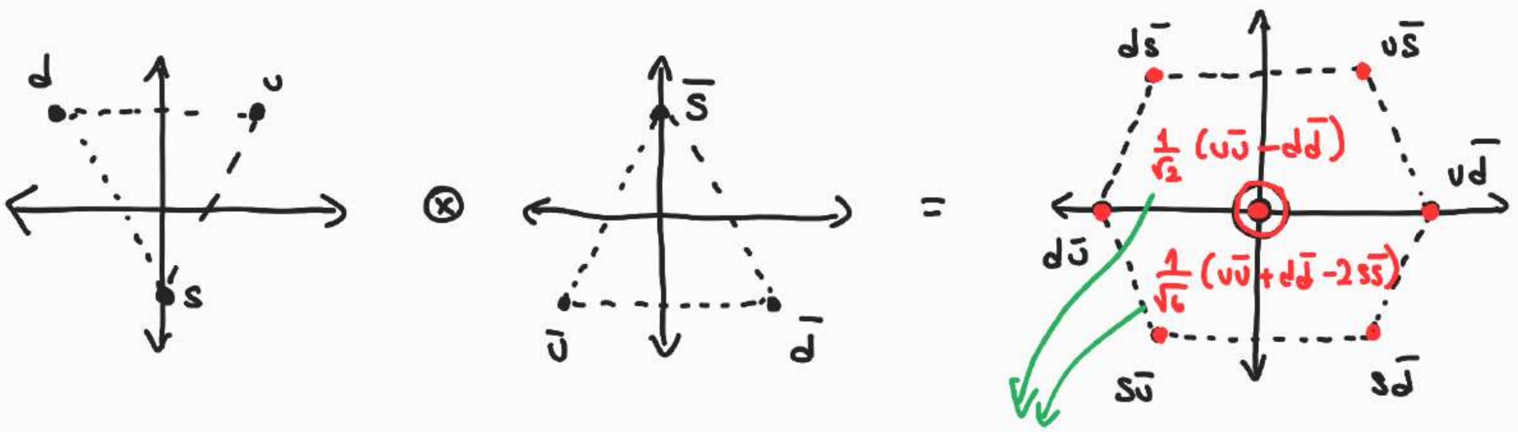
In SU(3) flavour symmetry, both I_3 and Y are additive quantum numbers, which together specify the flavour content of a state.

Light meson ($q\bar{q}$) states, formed from combinations of u, d and s quarks / antiquarks, can be constructed using this additive property to identify the extreme states within an SU(3) multiplet.

↓

Having identified the extreme states, the ladder operators can be used to obtain the full multiplet structure.

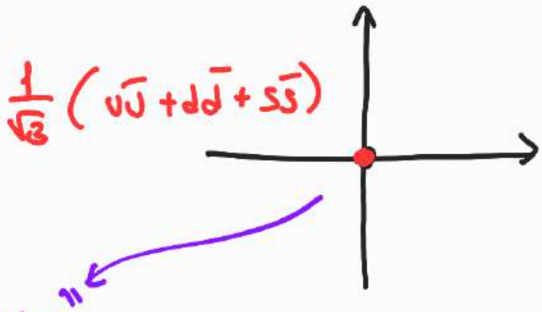




orthogonal to the singlet state

$$3 \otimes \bar{3} = 8 \oplus 1$$

$\oplus$



"flavourless state"

carrying no information about the flavours of its constituents

6.3 The L=0 Mesons

$$\Psi(\text{meson}) = \phi_{\text{flavour}}\, \chi_{\text{spin}}\, \xi_{\text{colour}}\, \eta_{\text{space}}$$

There are two possible spin states S=0 and S=1

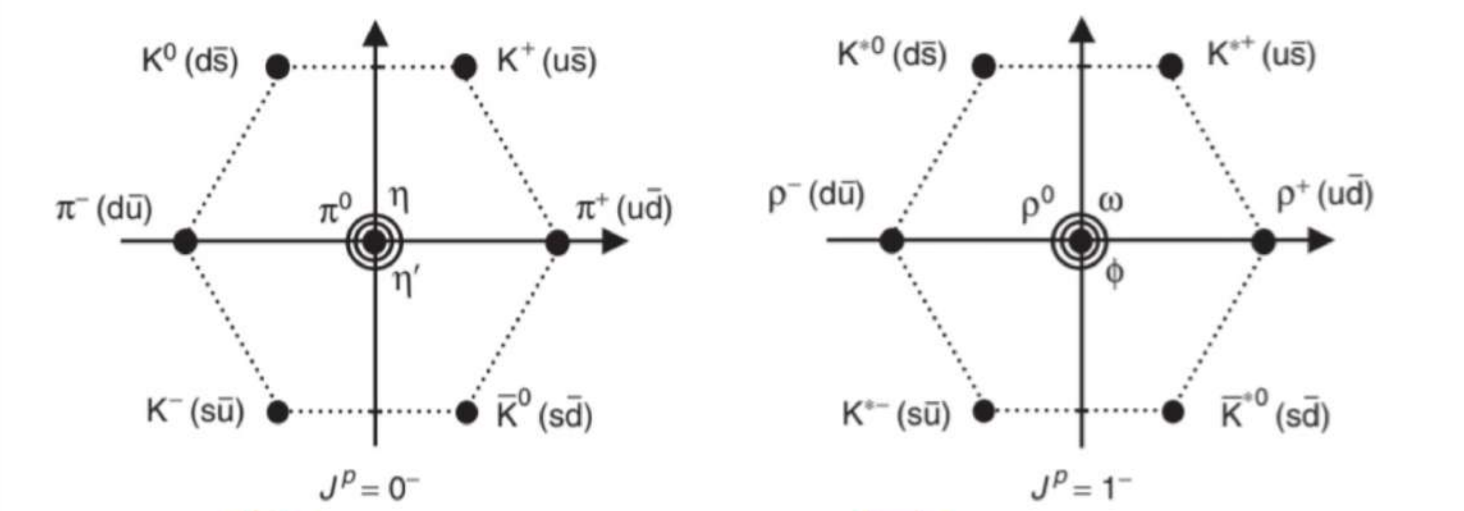
For the lightest mesons, which have zero orbital angular momentum ($l=0$), the total angular momentum J is determined by spin state alone.

J=0 (pseudoscalar mesons) J=1 (the vector mesons)

$$P(q\bar{q}) = P(q)P(\bar{q}) \times (-1)^{\ell} = (+1)(-1)(-1)^{\ell}$$

parity

symmetry of the orbital wavefunction



he nine $\ell = 0$, $s = 0$ pseudoscalar mesons and nine $\ell = 0$, $s = 1$ vector mesons formed from the light
uarks, plotted in terms of I_3 and Y.

If the SU(3) flavour symmetry were exact, all the states in pseudoscalar meson octet would have the same mass. The observed mass differences can be ascribed to the fact that the strange quark is more massive than the up- and down quarks

Table 9.1 The $L = 0$ pseudoscalar and vector meson masses.

Pseudoscalar mesons		Vector mesons	
π^0	135 MeV	ρ^0	775 MeV
$\pi^\pm$	140 MeV	$\rho^\pm$	775 MeV
$K^\pm$	494 MeV	$K^{*\pm}$	892 MeV
$K^0, \overline{K}^0$	498 MeV	$K^{*0}/\overline{K}^{*0}$	896 MeV
η	548 MeV	ω	783 MeV
η'	958 MeV	ϕ	1020 MeV

However, only the mass of strange quark does not explain why the vector mesons are more massive than their pseudoscalar counterparts.

For instance the π and ρ states are the same, but their masses very different (140 MeV, 770 MeV)

The only difference is the spin wavefunction.

In QED, the potential energy between two magnetic dipoles contains a term proportional to scalar product of the two dipole moments $\mu_i \cdot \mu_j$.

fine structure constant

$$U \propto \frac{e}{m_i} S_i \cdot \frac{e}{m_j} S_j \propto \frac{\alpha}{m_i m_j} S_i \cdot S_j$$

This QED interaction term, which contributes to the hyperfine splitting of the energy levels of the hydrogen atom, relatively small

arises from the coupling of the electron spin and nuclear spin

QCD vertex has the same form as that of QED. Thus, there will be a corresponding QCD chromomagnetic spin-spin interaction giving:

$(\alpha_s \sim 1) >> (\alpha \sim 1/137)$

$$U \propto \frac{\alpha_s}{m_i m_j} S_i \cdot S_j$$

$$m(q_1 q_2) = m_1 + m_2 + \frac{\overset{\text{from experiment}}{A}}{m_1 m_2} \underbrace{\langle S_1 \cdot S_2 \rangle}_{}$$

expectation value

total spin

$$\vec{S} = \vec{S}_1 + \vec{S}_2$$

$$S^2 = S_1^2 + 2 S_1 . S_2 + S_2^2$$

$$S_1 . S_2 = \frac{1}{2} \left[S^2 - S_1^2 - S_2^2 \right]$$

$$\langle S_1 . S_2 \rangle = \frac{1}{2} \left[\langle S^2 \rangle - \langle S_1^2 \rangle - \langle S_2^2 \rangle \right]$$

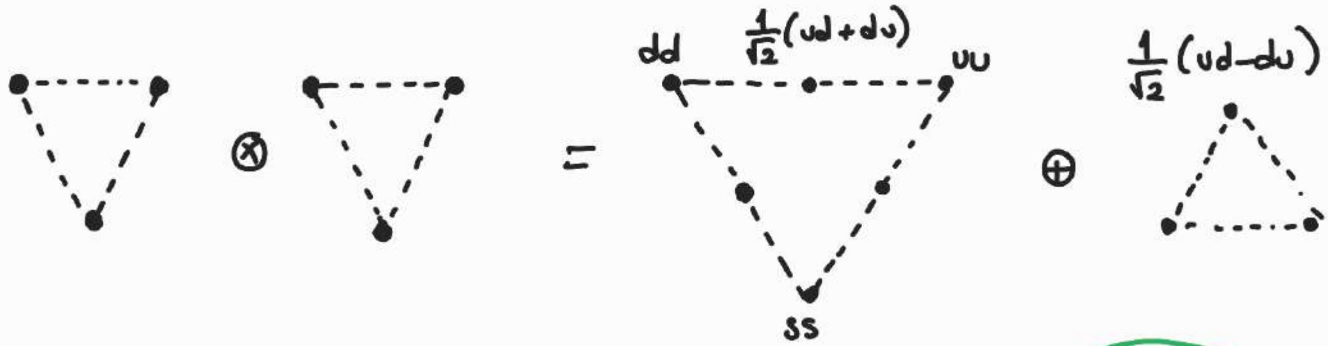
$$= \frac{1}{2} \left[s(s+1) - s_1(s_1+1) - s_2(s_2+1) \right]$$

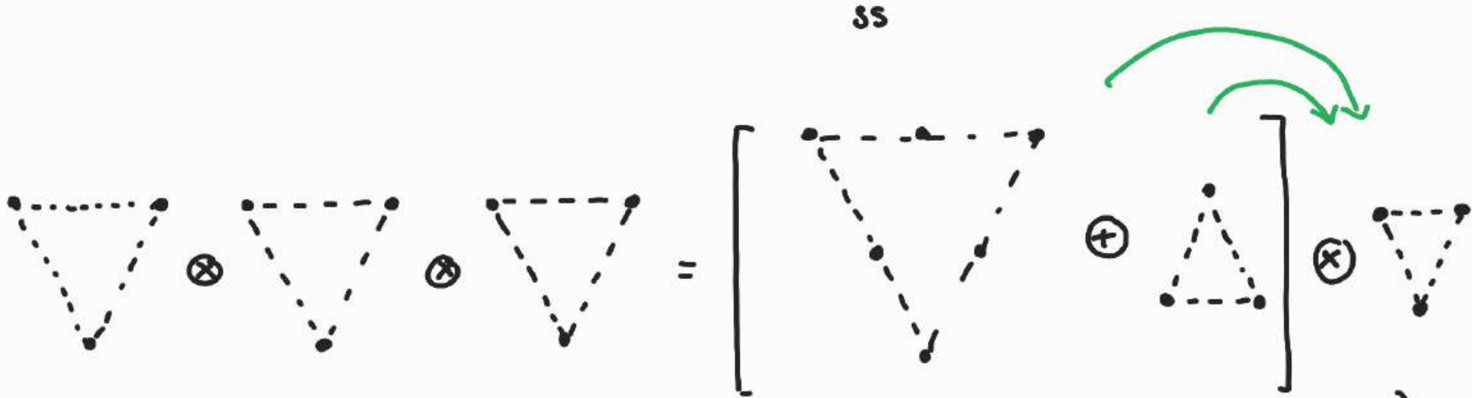
$s_1 = s_2 = \frac{1}{2}$ and s is the total spin of the $q\bar{q}$ system

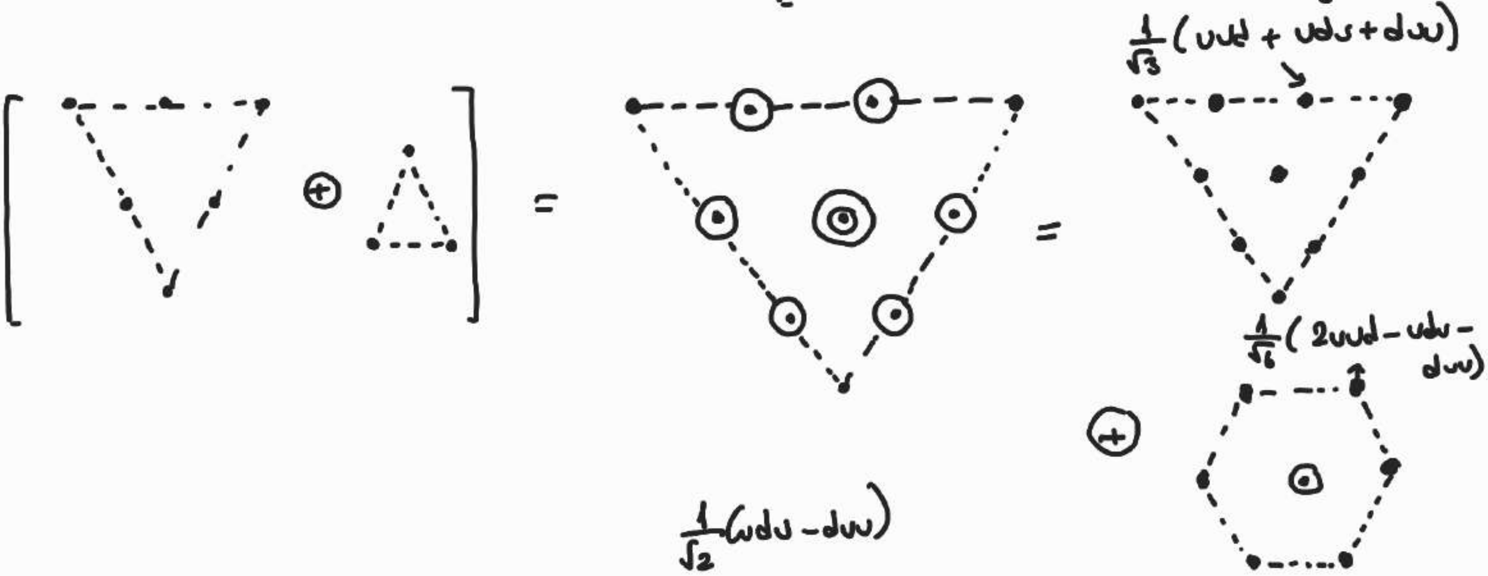
Pseudoscalar mesons ($s=0$): $\quad m_P = m_1 + m_2 - \dfrac{3A}{4 m_1 m_2}$

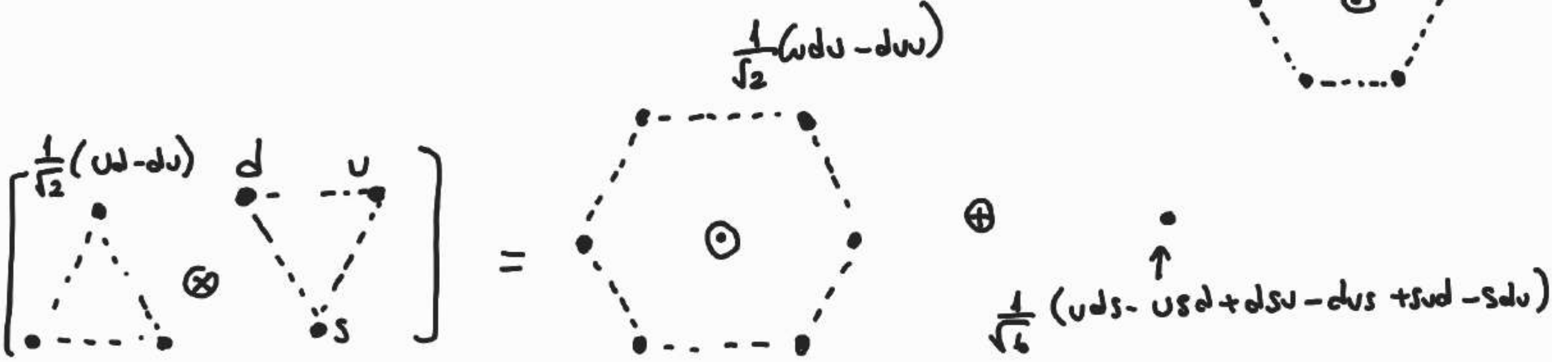
Vector mesons ($s=1$): $\quad m_V = m_1 + m_2 + \dfrac{A}{4 m_1 m_2}$

6.4 The L=0 uds baryons









$$3 \otimes 3 \otimes 3 = 3 \otimes (6 \oplus \bar{3}) = 10 \oplus 8 \oplus 8 \oplus 1$$

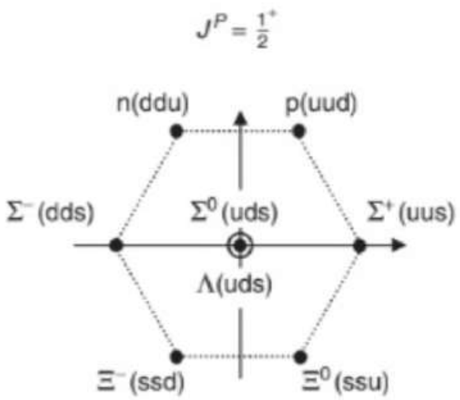
Symmetric Decuplet

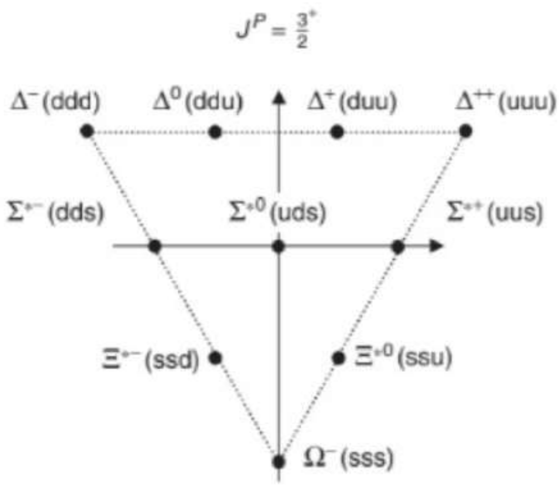
Mixed Symmetry Octet

Singlet Anti-Symmetric

Table 9.2 Measured masses and number of strange quarks for the $L = 0$ light baryons.

s quarks	Octet		Decuplet	
0	p, n	940 MeV	Δ	1230 MeV
1	Σ	1190 MeV	Σ^*	1385 MeV
1	Λ	1120 MeV		
2	Ξ	1320 MeV	Ξ^*	1533 MeV
3			Ω	1670 MeV





★★ In this chapter a number of important concepts were introduced. Symmetries of the Hamiltonian were associated with unitary transformations expressed in terms of Hermition generators:

$$\hat{U}(\alpha) = \exp(i\alpha . \hat{G})$$