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Homework 5
Quantum Information Processing

Exercise 1 *Dynamics of Spin 1/2*

We consider a magnetic moment with spin 1/2 whose dynamics is described by a Hamiltonian of the form

$$H = \frac{\hbar\delta}{2}\sigma_z - \frac{\hbar\omega_1}{2}\sigma_x$$

where $\hbar$ is Planck's constant, δ and $\omega_1 \in \mathbb{R}$ and the Pauli matrices $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. We recall the formula :

$$\exp\left(i\frac{a}{2}\mathbf{n}\cdot\vec{\sigma}\right) = (\cos\frac{a}{2})I + i(\sin\frac{a}{2})\mathbf{n}\cdot\vec{\sigma}$$

with $a \in \mathbb{R}$ et $\mathbf{n} = (n_x, n_y, n_z)$ a unit vector, $\mathbf{n}\cdot\vec{\sigma} = n_x\sigma_x + n_y\sigma_y + n_z\sigma_z$, and $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

✓ **a)** Compute the evolution matrix (operator)

$$U(t,0) = \exp\left(-i\frac{t}{\hbar}H\right)$$

and express it in matrix form, and also in Dirac's notation. We recall the conventional coordinate representation $|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

b) Consider the case $\omega_1 << \delta$ and the initial state at $t = 0$, $\frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)$.
- Compute a good approximation of the state at time t (hint : take the limit $\omega_1 \to 0$ and δ fixed).
- Represent the trajectory on the Bloch sphere in this limit.
- Is it periodic ? If yes what is the period ?

c) Consider now the case $\delta << \omega_1$ and the initial state at $t = 0$, $|\uparrow\rangle$.
- Compute a good approximation of the final state at time t (hint : take the limit $\delta \to 0$ and ω_1 fixed).
- Represent again the trajectory on the Bloch sphere in this limit.
- Is it periodic ? If yes what is the period ?

Exercise 2 *Creation of entanglement thanks to a magnetic interaction*

We consider two spin $\frac{1}{2}$ with interaction Hamiltonian $\mathcal{H} = \hbar J \sigma_1^z \otimes \sigma_2^z$ (these can be spins of nuclei in a molecule say). The unitary evolution operator of this system is $U = \exp\left(-\frac{it}{\hbar}\mathcal{H}\right)$. Let

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle + |\downarrow\rangle\right) \otimes \frac{1}{\sqrt{2}}\left(|\uparrow\rangle - |\downarrow\rangle\right)$$

be the initial state of the two spins.

a) Show that the state after time $t = \frac{\pi}{4J}$ is

$$|\psi_t\rangle = \frac{e^{-\frac{i\pi}{4}}}{2}\left(|\uparrow\uparrow\rangle - i\,|\uparrow\downarrow\rangle + i\,|\downarrow\uparrow\rangle - |\downarrow\downarrow\rangle\right)$$

b) Show that this state is entangled, i.e., it is *impossible* to write it in the form

$$(\alpha\,|\uparrow\rangle + \beta\,|\downarrow\rangle) \otimes (\gamma\,|\uparrow\rangle + \delta\,|\downarrow\rangle)$$

c) Now we let the state obtained above still evolve for an interval of time $\frac{\pi}{4J}$. Calculate the final state and determine if it is entangled or not.

d) What happens if we let the initial state $|\Psi_0\rangle$ evolve during an interval of time $\frac{\pi}{J}$?

Exercise 3 *The no-cloning theorem*

We want to prove that *non-orthogonal* states cannot be "copied" with the *same* unitary matrix.

In other words let $|\phi_1\rangle$, $|\phi_2\rangle$ in some Hilbert space $\mathcal{H}$ and let $|O\rangle$ be a "blank" state which plays the role of a place-holder for the copy. The theorem states that there does not exist a $U : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ such that,

$$U|\phi_i\rangle \otimes |O\rangle = |\phi_i\rangle \otimes |\phi_i\rangle, \quad i = 1, 2$$

Use the following hints to find two proofs of this fact

a) The first proof only uses only that U is unitary.
Hint : assume a unitary U exists that satisfies the above two equations (for $i = 1, 2$) and find a contradiction.

b) The second proof uses the superposition principle and the linearity of U (it does not really need unitarity).
Hint : consider the action of a linear matrix U that satisfies the above equations on a state which can be writen in two equivalent ways :

$$(|\phi_1\rangle + |\phi_2\rangle) \otimes |O\rangle = |\phi_1\rangle \otimes |O\rangle + |\phi_2\rangle \otimes |O\rangle$$

and find a contradiction.

Exercise 1

a) $$U(t,0) = \exp\left(-i\frac{t}{\cancel{\hbar}} H\right) \qquad H = \frac{\cancel{\hbar}\delta}{2}\sigma_z - \frac{\cancel{\hbar}\omega_1}{2}\sigma_x$$

from $$\exp\left(i\frac{a}{2}\, n\cdot\vec{\sigma}\right) = \cos\frac{a}{2}\, I + i\sin\frac{a}{2}\, n\cdot\vec{\sigma}$$

$$\exp\left(-it\frac{\delta}{2}\sigma_z + it\frac{\omega_1}{2}\sigma_x\right) = \exp\left(-it\frac{\delta}{2}\sigma_z\right) . \exp\left(it\frac{\omega_1}{2}\sigma_x\right)$$

From Exercise 4

$$\begin{pmatrix} \cos\left(\frac{\delta t}{2}\right) - i\sin\left(\frac{\delta t}{2}\right) & 0 \\ 0 & \cos\left(\frac{\delta t}{2}\right) + i\sin\left(\frac{\delta t}{2}\right) \end{pmatrix} \begin{pmatrix} \cos\left(\frac{\omega_1 t}{2}\right) & -i\sin\left(\frac{\omega_1 t}{2}\right) \\ -i\sin\left(\frac{\omega_1 t}{2}\right) & \cos\left(\frac{\omega_1 t}{2}\right) \end{pmatrix}$$

$$= \begin{bmatrix} \overbrace{\cos\left(\frac{\omega_1 t}{2}\right)\cdot\left[\cos\left(\frac{\delta t}{2}\right) - i\sin\left(\frac{\delta t}{2}\right)\right]}^{N_{\uparrow\uparrow}(t)} & \overbrace{-i\sin\left(\frac{\omega_1 t}{2}\right)\left[\cos\left(\frac{\delta t}{2}\right) - i\sin\left(\frac{\delta t}{2}\right)\right]}^{N_{\uparrow\downarrow}(t)} \\ \underbrace{-i\sin\left(\frac{\omega_1 t}{2}\right)\left[\cos\left(\frac{\delta t}{2}\right) + i\sin\left(\frac{\delta t}{2}\right)\right]}_{N_{\downarrow\uparrow}(t)} & \underbrace{\cos\left(\frac{\omega_1 t}{2}\right)\left[\cos\left(\frac{\delta t}{2}\right) + i\sin\left(\frac{\delta t}{2}\right)\right]}_{N_{\downarrow\downarrow}(t)} \end{bmatrix}$$

Dirac Notation :

$$N_{\uparrow\uparrow}(t)\,|\uparrow\rangle\langle\uparrow| + N_{\uparrow\downarrow}(t)\,|\uparrow\rangle\langle\downarrow| + N_{\downarrow\uparrow}(t)\,|\downarrow\rangle\langle\uparrow| + N_{\downarrow\downarrow}(t)\,|\downarrow\rangle\langle\downarrow|$$

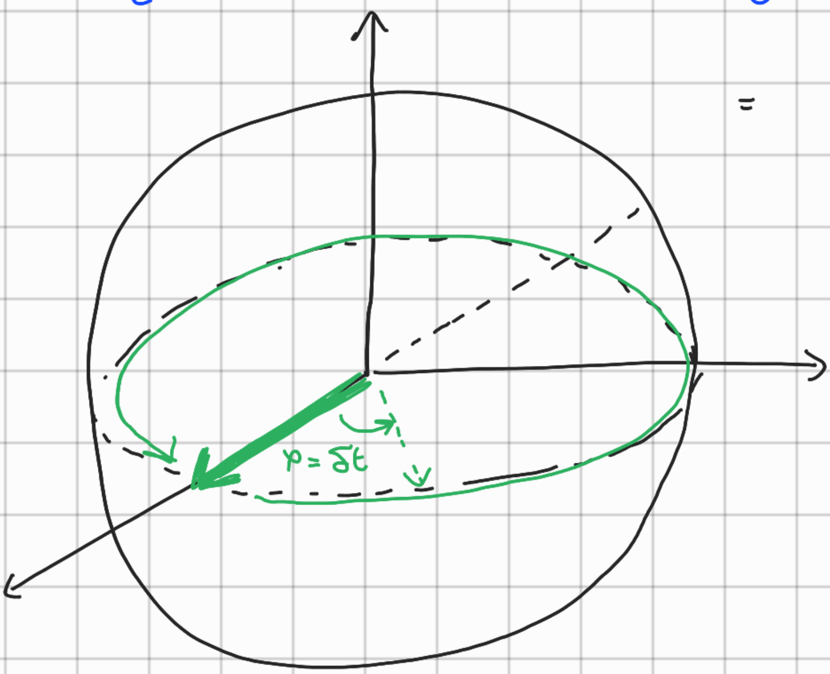
b) $\omega_1 \ll \delta$ $$|\psi\rangle_{t=0} = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle + |\downarrow\rangle\right) \qquad |\psi\rangle_t = U(t,0)\,|\psi\rangle_{t=0}$$

$$|\psi\rangle_t = N_{\uparrow\uparrow}(t)\,|\uparrow\rangle + N_{\downarrow\uparrow}(t)\,|\downarrow\rangle + N_{\uparrow\downarrow}(t)\,|\uparrow\rangle + N_{\downarrow\downarrow}(t)\,|\downarrow\rangle$$

$$= \frac{1}{\sqrt{2}}|\uparrow\rangle\left[\underbrace{\cos\left(\frac{\omega_1 t}{2}\right)}_{\to 1}\left[\cos\left(\frac{\delta t}{2}\right) - i\sin\left(\frac{\delta t}{2}\right)\right] + \cancel{-i\sin\left(\frac{\omega_1 t}{2}\right)}\left[\cos\left(\frac{\delta t}{2}\right) - i\sin\left(\frac{\delta t}{2}\right)\right]\right]$$

$$+ \frac{1}{\sqrt{2}}|\downarrow\rangle\left[\underbrace{\cancel{-i\sin\left(\frac{\omega_1 t}{2}\right)}}_{\to 0}\left[\cos\left(\frac{\delta t}{2}\right) + i\sin\left(\frac{\delta t}{2}\right)\right] + \underbrace{\cos\left(\frac{\omega_1 t}{2}\right)}_{1}\left[\cos\left(\frac{\delta t}{2}\right) + i\sin\left(\frac{\delta t}{2}\right)\right]\right]$$

$$= \frac{1}{\sqrt{2}} \left[\left(\underbrace{\cos\left(\frac{\delta t}{2}\right) - i\sin\left(\frac{\delta t}{2}\right)}_{e^{-i\frac{\delta t}{2}}} \right) |\uparrow\rangle + \left(\underbrace{\cos\left(\frac{\delta t}{2}\right) + i\sin\left(\frac{\delta t}{2}\right)}_{e^{i\frac{\delta t}{2}}} \right) |\downarrow\rangle \right]$$



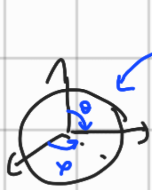
global phase

$$= e^{-i\frac{\delta t}{2}} \left[\frac{1}{\sqrt{2}} |\uparrow\rangle + \frac{1}{\sqrt{2}} e^{i\delta t} |\downarrow\rangle \right]$$

Here $\cos\frac{\theta}{2} = \frac{1}{\sqrt{2}}$

& $\sin\frac{\theta}{2} = \frac{1}{\sqrt{2}}$

Thus $\theta = \pi/2$



$\boxed{\varphi = \delta t}$

Yes it is periodic. $T = \dfrac{2\pi}{\delta}$

c) $\delta << \omega_1$ @ $t=0$ $|\psi\rangle_{t=0} = |\uparrow\rangle$

$$|\psi\rangle_t = |\uparrow\rangle \left[\cos\left(\frac{\omega_1 t}{2}\right) \cdot \left[\underbrace{\cos\left(\frac{\delta t}{2}\right)}_{1} - \underbrace{i\sin\left(\cancel{\frac{\delta t}{2}}\right)}_{0} \right] + -i\sin\left(\frac{\omega_1 t}{2}\right) \left[\underbrace{\cos\left(\frac{\delta t}{2}\right)}_{1} - \underbrace{i\sin\left(\cancel{\frac{\delta t}{2}}\right)}_{0} \right] \right]$$

$$|\psi_t\rangle = |\uparrow\rangle \, e^{-i\omega_1 t/2}$$

Global phase

No it is not periodic

Exercise 2

Initial state of two spins

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle + |\downarrow\rangle\right) \otimes \frac{1}{\sqrt{2}}\left(|\uparrow\rangle - |\downarrow\rangle\right) = \frac{1}{2}\left(|\uparrow\uparrow\rangle - |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle - |\downarrow\downarrow\rangle\right)$$

$$\mathcal{H} = \hbar J \sigma_1^z \otimes \sigma_2^z$$

$$U = \exp\left(\frac{-it}{\hbar} H\right)$$

a) $$U = \exp\left(\frac{-it}{\cancel{\hbar}} \cancel{\hbar} J \sigma_1^z \otimes \sigma_2^z\right)$$

$$U\left(t = \frac{\pi}{4J}\right) = \exp\left(-i \frac{\pi}{4\cancel{J}} \cancel{J} \sigma_1^z \otimes \sigma_2^z\right)$$

$$U\left(t = \frac{\pi}{4J}\right) = \cos\frac{\pi}{4} I - i \sin\frac{\pi}{4} \sigma_1^z \otimes \sigma_2^z$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \left(|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|\right)$$

$$|\psi_t\rangle = U\left(t = \frac{\pi}{4J}\right)|\psi_0\rangle$$

$$|\psi_t\rangle = \left[\cos\frac{\pi}{4} I - i \sin\frac{\pi}{4}\left(|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|\right) \otimes \left(|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|\right)\right]$$

$$\frac{1}{\sqrt{2}}\left(|\uparrow\rangle + |\downarrow\rangle\right) \otimes \frac{1}{\sqrt{2}}\left(|\uparrow\rangle - |\downarrow\rangle\right)$$

$$= \frac{1}{2}\left[\cos\frac{\pi}{4} I \quad \underbrace{\left(|\uparrow\rangle + |\downarrow\rangle\right) \otimes \left(|\uparrow\rangle - |\downarrow\rangle\right)}_{|\uparrow\uparrow\rangle - |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle - |\downarrow\downarrow\rangle}\right.$$

$$\left. - i \sin\frac{\pi}{4} \quad \underbrace{\left(|\uparrow\rangle - |\downarrow\rangle\right) \otimes \left(|\uparrow\rangle + |\downarrow\rangle\right)}_{|\uparrow\uparrow\rangle + |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle - |\downarrow\downarrow\rangle}\right]$$

$$= \frac{1}{2}\Big[\Big(\underbrace{\cos\frac{\pi}{4} J - i\sin\frac{\pi}{4}}_{e^{-i\pi/4}}\Big)|\uparrow\uparrow\rangle + \Big(\underbrace{-\cos\frac{\pi}{4} J - i\sin\frac{\pi}{4}}_{-e^{i\pi/4} = -e^{-i\pi/4}e^{i\pi/2}}\Big)|\uparrow\downarrow\rangle +$$

i

$$\Big(\underbrace{\cos\frac{\pi}{4} J + i\sin\frac{\pi}{4}}_{e^{i\pi/4} = e^{-i\pi/4}e^{i\pi/2}}\Big)|\downarrow\uparrow\rangle + \Big(\underbrace{-\cos\frac{\pi}{4} J + i\sin\frac{\pi}{4}}_{-e^{-i\pi/4}}\Big)|\downarrow\downarrow\rangle\Big]$$

i

$$= \frac{e^{-i\pi/4}}{2}\Big[|\uparrow\uparrow\rangle - i|\uparrow\downarrow\rangle + i|\downarrow\uparrow\rangle - |\downarrow\downarrow\rangle\Big]$$

b) $$\frac{e^{-i\pi/4}}{2}\Big[|\uparrow\rangle\big(|\uparrow\rangle - i|\downarrow\rangle\big) + |\downarrow\rangle\big(i|\uparrow\rangle - |\downarrow\rangle\big)\Big]$$

$\downarrow$

It can not be seperated, thus it's entangled

Or one can show that if det $\neq 0$ it's entangled

$$\begin{bmatrix} 1 & -i \\ i & -1 \end{bmatrix} \rightarrow \text{Det} = 1(-1) - (-i)(i) = (-1) - (\underbrace{-i^2}_{1}) = (-1) + \underbrace{i^2}_{-1} = -2 \neq 0$$

✓

c) After $\frac{\pi}{4J}$ time, At time $\frac{\pi}{2J}$

$$|\Psi_t\rangle = \frac{1}{2}\Big[\cancel{\cos\frac{\pi}{2} J}_{\,0}\quad \underbrace{(|\uparrow\rangle + |\downarrow\rangle)\otimes(|\uparrow\rangle - |\downarrow\rangle)}_{|\uparrow\uparrow\rangle - |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle - |\downarrow\downarrow\rangle}$$

$$- i\underbrace{\sin\frac{\pi}{2}}_{1}\ \underbrace{(|\uparrow\rangle - |\downarrow\rangle)\otimes(|\uparrow\rangle + |\downarrow\rangle)}_{|\uparrow\uparrow\rangle + |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle - |\downarrow\downarrow\rangle}\Big]$$

$$|\Psi_t\rangle = \frac{-i}{2}\big(|\uparrow\rangle - |\downarrow\rangle\big)\otimes\big(|\uparrow\rangle + |\downarrow\rangle\big) \rightarrow \text{It is NOT entangled}$$

d) $t = \frac{\pi}{J}$

$$|\psi_t\rangle = \frac{1}{2}\Big[\overbrace{\cos\pi}^{-1}\, I \quad \underbrace{(|\uparrow\rangle + |\downarrow\rangle) \otimes (|\uparrow\rangle - |\downarrow\rangle)}_{|\uparrow\uparrow\rangle - |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle - |\downarrow\downarrow\rangle}$$

$$- i\,\cancel{\sin\pi}^{\,0} \quad \underbrace{(|\uparrow\rangle - |\downarrow\rangle) \otimes (|\uparrow\rangle + |\downarrow\rangle)}_{|\uparrow\uparrow\rangle + |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle - |\downarrow\downarrow\rangle}\Big]$$

$$|\psi_t\rangle = \frac{-1}{2} \quad (|\uparrow\rangle + |\downarrow\rangle) \otimes (|\uparrow\rangle - |\downarrow\rangle)$$

It returns to the same initial state. Therefore $T = \frac{\pi}{J}$

Exercise 3

$$U|\phi_i\rangle \otimes |0\rangle = |\phi_i\rangle \otimes |\phi_i\rangle$$

a) Unitary $U \rightarrow U \cdot U^{+} = \mathbb{1} = U^{+}U$, Norm preserving

$\langle \phi_1 | \phi_2 \rangle \neq 0 \rightarrow$ given

$$U|\phi_1\rangle \otimes |0\rangle = |\Phi_1\rangle \otimes |\Phi_1\rangle$$

$$U|\Phi_2\rangle \otimes |0\rangle = |\Phi_2\rangle \otimes |\Phi_2\rangle$$

$$\langle 0 \Phi_2 | \underbrace{U^{+} U}_{I} | \Phi_1 0 \rangle \overset{?}{=} \langle \Phi_2 \Phi_2 | \Phi_1 \Phi_1 \rangle$$

$$\underbrace{\langle 0|0\rangle}_{1} \langle \Phi_2 | \Phi_1 \rangle \neq |\langle \Phi_2 | \Phi_1 \rangle|^2$$

b) Using Linearity

$$U(|\phi_1\rangle + |\phi_2\rangle) \otimes |0\rangle \stackrel{?}{=} U|\phi_1\rangle \otimes |0\rangle + U|\phi_2\rangle \otimes |0\rangle$$

$$(|\phi_1\rangle + |\phi_2\rangle) \otimes (|\phi_1\rangle + |\phi_2\rangle) \stackrel{?}{=} |\phi_1\rangle|\phi_1\rangle + |\phi_2\rangle|\phi_2\rangle$$

$$|\phi_1\phi_1\rangle + |\phi_1\phi_2\rangle + |\phi_2\phi_1\rangle + |\phi_2\phi_2\rangle \neq |\phi_1\phi_1\rangle + |\phi_2\phi_2\rangle$$