Two Level Systems

1- State and Operators

- Hilbert space $\mathcal{H} = Sp\{|\uparrow\rangle, |\downarrow\rangle\}$, $|g\rangle, |e\rangle$ or $|0\rangle, |1\rangle$

  State $|\psi\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle$ with $|\alpha|^2 + |\beta|^2 = 1$

  one possible parametrization: $|\psi\rangle = \cos\theta/_2\, |\uparrow\rangle + e^{i\phi} \sin\frac{\theta}{2}\, |\downarrow\rangle$

- Hermitian operators on $\mathcal{H}$: spanned by $\hat{\mathbb{I}}, \hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z$

$\hat{\mathbb{I}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad \hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\quad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\hat{\sigma}_+ = |\uparrow\rangle\langle\downarrow| = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\quad \hat{\sigma}_- = |\downarrow\rangle\langle\uparrow| = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$$\hat{\sigma}_+ = \frac{\hat{\sigma}_x + i\hat{\sigma}_y}{2} \qquad \hat{\sigma}_- = \frac{\hat{\sigma}_x - i\hat{\sigma}_y}{2}$$

Properties of Pauli Matrices: $\mathrm{Tr}\, \hat{\sigma}_i = 0$ $\quad i = x, y, z$

commutation

$$\{\hat{\sigma}_i, \hat{\sigma}_j\} = 2\,\delta_{ij}\, \hat{\mathbb{I}} \implies \hat{\sigma}_i^2 = \hat{\mathbb{I}}$$

$[\hat{\sigma}_x, \hat{\sigma}_y] = 2i\, \hat{\sigma}_z$ + circ permutations

Notation: $\vec{\hat{\sigma}} = \begin{pmatrix} \hat{\sigma}_x \\ \hat{\sigma}_y \\ \hat{\sigma}_z \end{pmatrix}$

Representation of operators : $\hat{O} = \frac{1}{2}\left\{ \hat{\mathbb{I}}\, Tr(\hat{O}) + \sum_i Tr(\hat{O}\hat{\sigma}_i).\hat{\sigma}_i \right\}$

Think about $A = \sum_i \langle A, u_i \rangle u_i$

$$\hat{O} = \frac{1}{2}\left\{ \mathbb{I}\, Tr(\hat{O}) + \vec{n}_0\, \vec{\hat{\sigma}} \right\} \quad \text{where} \quad \vec{n}_0 = \begin{pmatrix} n_{0x} \\ n_{0y} \\ n_{0z} \end{pmatrix}$$

2. Bloch Sphere

1) Density Matrix

$\hat{\rho}$: $Tr(\hat{\rho}) = 1$ and $\hat{\rho} = \frac{1}{2}\left\{ \mathbb{I} + \vec{n}.\vec{\hat{\sigma}} \right\}$

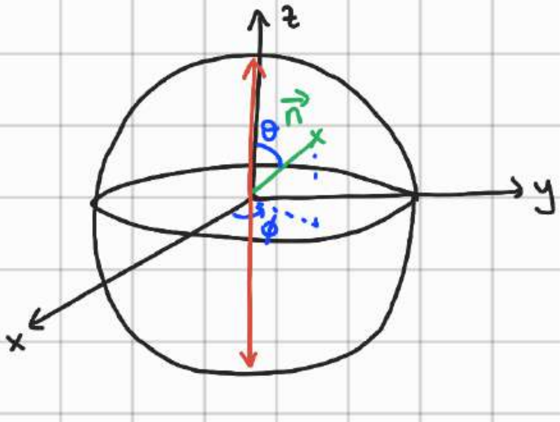
Bloch vector

Pure State

$Tr(\hat{\rho}^2) \leq 1$ with $Tr(\hat{\rho}^2) = 1 \Rightarrow \hat{\rho} = |\psi\rangle\langle\psi|$

$$Tr(\hat{\rho}^2) = Tr\left\{ \frac{1}{4} (\mathbb{I} + \vec{n}.\vec{\sigma})(\mathbb{I} + \vec{n}.\vec{\sigma}) \right\} = \frac{1 + \|\vec{n}\|^2}{2}$$

Pure States are characterized by $\|\vec{n}\|^2 = 1$



$$\vec{n} = \|\vec{n}\| \begin{pmatrix} \sin\theta \cos\phi \\ \sin\phi \sin\phi \\ \cos\theta \end{pmatrix}$$

for pure states $\|\vec{n}\| = 1$

$$|\psi\rangle = \cos\frac{\vartheta}{2}\,|\uparrow\rangle + e^{i\phi}\sin\frac{\vartheta}{2}\,|\downarrow\rangle$$

$$\hat{\rho} = |\psi\rangle\langle\psi| \qquad \mathrm{Tr}(\hat{\rho}\,\hat{\sigma}_x) = \ldots$$ → Hilbert-Schmidt Product

Example:
- $|\uparrow\rangle$: north pole for $\theta = 0$
- $|\downarrow\rangle$: South pole for $\theta = 2\pi$
- Equal weight Superposition of $|\uparrow\rangle$ and $|\downarrow\rangle$: $|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle + e^{i\phi}|\downarrow\rangle\right)$

  "equator of the BS"
- Equal weight incoherent mixture: $\hat{\rho} = \frac{1}{2}|\uparrow\rangle\langle\uparrow| + \frac{1}{2}|\downarrow\rangle\langle\downarrow|$ → center of the sphere
- General incoherent mixture : $\boxed{\hat{\rho} = p_e|\uparrow\rangle\langle\uparrow| + (1-p_e)|\downarrow\rangle\langle\downarrow|}$

  $\hat{\rho} = \hat{I} + p_e\hat{\sigma}_z$

2) Hamiltonian

$$\boxed{\hat{H} = \hat{\mathbb{I}} - \vec{h}\cdot\hat{\vec{\sigma}}}$$

Ex: spin 1/2 in a $\vec{B}$ field , $\hat{H} = -\hat{\vec{\mu}}\cdot\vec{B}$ $\qquad \hat{\vec{\mu}} = \mu\,\hat{\vec{\sigma}}$

Dynamics: $\boxed{i\hbar\frac{\partial\hat{\rho}}{\partial t} = [\hat{H}, \hat{\rho}]}$

{ Remark , don't mix this with Heisenberg picture $i\hbar\,\partial_t\hat{O} = [\hat{O}, \hat{H}]$ } → density matrix is fixed state

Look similar but different physical meaning

E_0 does not generate any dynamics

$$\hat{\rho}(t) = \frac{1}{2}\left(\mathbb{I} + \vec{a}(t)\,\hat{\vec{\sigma}}\right)$$

(

Bloch vector

$$\hat{\rho}(t) = \frac{1}{2}\left(\mathbb{I} + \vec{a}(t) \cdot \hat{\vec{\sigma}}\right)$$

$$i\hbar\dot{\hat{\rho}} = [\hat{H}, \hat{\rho}] = \left[\mathbb{I} - \vec{h}\cdot\hat{\vec{\sigma}}, \frac{1}{2}\left(\mathbb{I} + \vec{a}(t)\hat{\vec{\sigma}}\right)\right]$$

$$\hat{H}\hat{\rho} - \hat{\rho}\hat{H}$$

$$\Bigg(\cancel{\frac{1}{2}\mathbb{I}^2}^{\,\mathbb{I}} + \cancel{\frac{1}{2}\vec{a}(t)\vec{\sigma}} - \cancel{\frac{1}{2}\vec{h}\vec{\sigma}} + \frac{1}{2}\vec{h}\vec{\sigma}\,\vec{a}(t)\vec{\sigma}$$

$$-\frac{1}{2}\left(\mathbb{I} + \vec{a}(t)\hat{\vec{\sigma}}\right)\left(\mathbb{I} - \vec{h}\cdot\hat{\vec{\sigma}}\right) \qquad ?$$

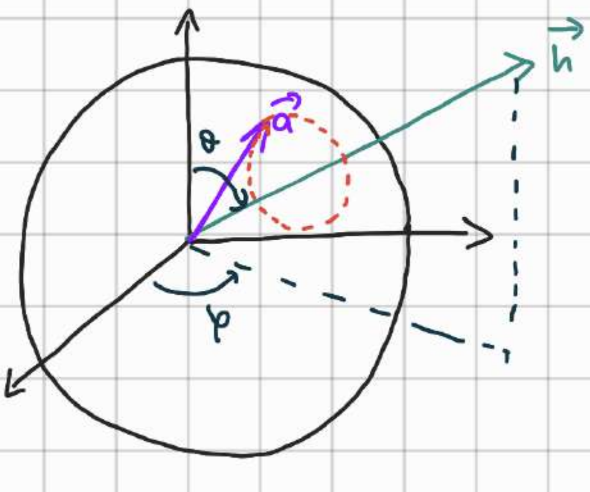
$$-\cancel{\frac{1}{2}\mathbb{I}} + \cancel{\frac{1}{2}\vec{h}\vec{\sigma}} - \cancel{\frac{1}{2}\vec{a}(t)\vec{\sigma}} + \frac{1}{2}\vec{a}(t)\vec{\sigma}\,\vec{h}\vec{\sigma} \Bigg)$$

$$i\hbar\dot{\hat{\rho}} = \dots$$

$$\boxed{\dot{\vec{a}} = \frac{2\vec{h}}{\hbar} \times \vec{a}}$$

Precession of $\vec{a}$ about the vector $\vec{h}$

Frequency: $\frac{2|\vec{h}|}{\hbar}$



Eigenstates of $\hat{H}$:

$$|\psi_+\rangle = \cos\frac{\theta}{2}|\uparrow\rangle + e^{i\varphi}\sin\frac{\theta}{2}|\downarrow\rangle$$

$$|\psi_-\rangle = \sin\frac{\theta}{2}|\uparrow\rangle - e^{i\varphi}\cos\frac{\theta}{2}|\downarrow\rangle$$

Rotations on the Bloch Sphere

$$e^{-i\frac{\hat{H}t}{\hbar}}$$ is a rotation by angle $\frac{2\|\vec{h}\|}{\hbar}\cdot t$ around the axis $\vec{h}$

Time Evolution Operator

$$\hat{R}_{\vec{n}}(\vartheta) = e^{-i\frac{\vartheta}{2}\vec{n}\cdot\vec{\sigma}}$$

Example: Rabi Oscillations $\vec{h}$ along x axis $\longrightarrow$ any $\vec{h}$ in the xy plane creates Rabi Oscillations