Exercise 1 Corrupted dense coding

$$|\psi\rangle = (1+\delta^2)^{-1/2} \left\{ |B_{00}\rangle + \overset{\text{corruption}}{\delta e^{i\gamma} |01\rangle_{AB}} \right\}$$

$$|B_{00}\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle_{AB} + |11\rangle_{AB} \right)$$

a) Alice sends message (00).

No gate is applied. Thus $|\psi\rangle$ does not change

$$|\psi\rangle = (1+\delta^2)^{-1/2} \left\{ \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle + \delta e^{i\gamma}|01\rangle \right\}$$

Bob applies CNOT, then H

$$CNOT|\psi\rangle = (1+\delta^2)^{-1/2} \left\{ \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|10\rangle + \delta e^{i\gamma}|01\rangle \right\}$$

$$H_A CNOT|\psi\rangle = (1+\delta^2)^{-1/2} \left\{ \frac{1}{2}\left(|00\rangle + |10\rangle\right) + \frac{1}{2}\left(|00\rangle - |10\rangle\right) + \frac{\delta e^{i\gamma}}{\sqrt{2}}\left(|01\rangle + |11\rangle\right) \right\}$$

$$|\tilde{\psi}\rangle = (1+\delta^2)^{-1/2} \left\{ |00\rangle + |01\rangle\frac{\delta e^{i\gamma}}{\sqrt{2}} + |10\rangle\underbrace{\left(\frac{1}{2}-\frac{1}{2}\right)}_{0} + |11\rangle\frac{\delta e^{i\gamma}}{\sqrt{2}} \right\}$$

$$P(00) = \frac{1}{(1+\delta^2)} \qquad P(10) = 0$$

$$P(01) = \frac{\delta^2}{2(1+\delta^2)}$$

$$P(11) = \frac{\delta^2}{2(1+\delta^2)}$$

$$P(00) + P(01) + P(11) + P(10) = 1 \ \checkmark$$

b) Alice sends message (10)

Alice applies Z gate to her qubit, Bob receives $|\psi'\rangle$

$$|\psi'\rangle = Z_A \otimes I_B |\psi\rangle = \frac{1}{(1+\delta^2)^{1/2}} \left\{ \frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{2}}|11\rangle + \delta e^{i\gamma}|01\rangle \right\}$$

Bob applies $HCNOT|\psi'\rangle$

$$CNOT|\psi'\rangle = \frac{1}{(1+\delta^2)^{1/2}} \left\{ \frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{2}}|10\rangle + \delta e^{i\gamma}|01\rangle \right\}$$

$$HCNOT|\psi'\rangle = \frac{1}{(1+\delta^2)^{1/2}} \left\{ \frac{1}{2}(|00\rangle + |10\rangle) - \frac{1}{2}(|00\rangle - |10\rangle) + \frac{\delta e^{i\gamma}}{\sqrt{2}}(|01\rangle + |11\rangle) \right\}$$

$$|\psi_f\rangle = \frac{1}{(1+\delta^2)^{1/2}} \left\{ |00\rangle\left(\cancel{\tfrac{1}{2}-\tfrac{1}{2}}_{\,0}\right) + |01\rangle\frac{\delta e^{i\gamma}}{\sqrt{2}} + |10\rangle\left(\frac{1}{2}+\frac{1}{2}\right) + |11\rangle\frac{\delta e^{i\gamma}}{\sqrt{2}} \right\}$$

$$P(00) = 0$$

$$P(01) = \frac{\delta^2}{2(1+\delta^2)}$$

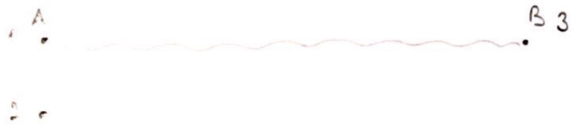
$$P(10) = \frac{1}{(1+\delta^2)}$$

$$P(11) = \frac{\delta^2}{2(1+\delta^2)}$$

$P(00) + P(01) + P(10) + P(11) = 1$ ✓

Exercise 2

$$|\psi'\rangle = (1+\delta^2)^{-1/2}\left\{|B_{00}\rangle + \delta e^{i\gamma}|01\rangle\right\}$$



$|\varphi\rangle = \alpha|0\rangle + \beta|1\rangle$

a) Alice does a measurement in the perfect Bell basis in her lab.

$$\Big(\underbrace{|B_{ij}\rangle_{12}}_{\text{ket with Alice}}\langle B_{ij}|_{12} \otimes I_3\Big)|\psi\rangle \otimes |\varphi\rangle_2$$

for $i,j = 00$ $= P_{00}|\psi\rangle$

$$\frac{1}{\sqrt2}\left(\langle 00|_{12} + \langle 11|_{12}\right) \quad (1+\delta^2)^{-1/2}\left\{\frac{1}{\sqrt2}|00\rangle_{13} + \frac{1}{\sqrt2}|11\rangle_{13} + \delta e^{i\gamma}|01\rangle_{13}\right\} \otimes (\alpha|0\rangle_2 + \beta|1\rangle_2)$$

$$= \frac{1}{\sqrt2}(1+\delta^2)^{-1/2}\left(\frac{\alpha}{\sqrt2}|0\rangle_3 + \alpha\delta e^{i\gamma}|1\rangle_3 + \frac{\beta}{\sqrt2}|1\rangle_3\right) = \frac{1}{\sqrt2}(1+\delta^2)^{-1/2}\left(\frac{\alpha}{\sqrt2}|0\rangle_3 + \left(\alpha\delta e^{i\gamma} + \frac{\beta}{\sqrt2}\right)|1\rangle_3\right)$$

$$\text{Prob} = \| P_{00}|\psi\rangle\|^2 = \frac{1}{2}(1+\delta^2)^{-1}\left(\frac{\alpha}{\sqrt2}\langle 0|_3 + \left(\alpha\delta e^{-i\gamma} + \frac{\beta}{\sqrt2}\right)\langle 1|_3\right)\left(\frac{\alpha}{\sqrt2}|0\rangle_3 + \left(\alpha\delta e^{i\gamma} + \frac{\beta}{\sqrt2}\right)|1\rangle_3\right)$$

$$= \frac{1}{2}(1+\delta^2)^{-1}\left(\frac{\alpha^2}{2} + \alpha^2\delta^2 + \underbrace{\delta e^{-i\gamma}\frac{\alpha\beta}{\sqrt2} + \delta e^{+i\gamma}\frac{\alpha\beta}{\sqrt2}}_{\frac{\delta\beta\alpha}{\sqrt2}(\underbrace{e^{-i\gamma} + e^{+i\gamma}}_{2\cos(\gamma)})} + \frac{\beta^2}{2}\right)$$

$$= \frac{1}{2}\,\frac{1}{(1+\delta^2)}\left(\frac{1}{2} + \alpha^2\delta^2 + \frac{\delta\beta\alpha}{\sqrt2}.2\cos(\gamma)\right)$$

for $i,j = 01$ $= P_{01}|\psi\rangle$

$$\frac{1}{\sqrt2}\left(\langle 00|_{12} - \langle 11|_{12}\right)(1+\delta^2)^{-1/2}\left\{\frac{1}{\sqrt2}|00\rangle_{13} + \frac{1}{\sqrt2}|11\rangle_{13} + \delta e^{i\gamma}|01\rangle_{13}\right\} \otimes (\alpha|0\rangle_2 + \beta|1\rangle_2)$$

$$= \frac{1}{\sqrt2}(1+\delta^2)^{-1/2}\left(\frac{\alpha}{\sqrt2}|0\rangle_3 + \alpha\delta e^{i\gamma}|1\rangle_3 - \frac{\beta}{\sqrt2}|1\rangle_3\right) = \frac{1}{\sqrt2}(1+\delta^2)^{-1/2}\left(\frac{\alpha}{\sqrt2}|0\rangle_3 + \left(\alpha\delta e^{i\gamma} - \frac{\beta}{\sqrt2}\right)|1\rangle_3\right)$$

$$\left(\delta e^{-i\gamma} - \frac{\beta}{6}\right)$$

$$\text{Prob} = \|P_{01}|\psi\rangle\|^2 = \frac{1}{2}(1+\delta^2)^{-1}\left(\frac{\alpha^2}{2} + \alpha^2\delta^2 - \delta e^{+i\gamma}\frac{\beta}{\sqrt2} - \delta e^{-i\gamma}\frac{\beta}{\sqrt2} + \frac{\beta^2}{2}\right)$$

$$= \frac{1}{2}\,\frac{1}{(1+\delta^2)}\left(\frac{1}{2} + \alpha^2\delta^2 - \frac{\delta\beta\alpha}{\sqrt2}2.\cos(\gamma)\right)$$

for $i,j = 10$ $= P_{10}|\Psi\rangle$

$$\frac{1}{\sqrt{2}}\left(\langle 01|_{12} + \langle 10|_{12}\right)(1+\delta^2)^{-1/2}\left\{\frac{1}{\sqrt{2}}|00\rangle_{13} + \frac{1}{\sqrt{2}}|11\rangle_{13} + \delta e^{i\gamma}|01\rangle_{13}\right\}\otimes\left(\alpha|0\rangle_2 + \beta|1\rangle_2\right)$$

$$= \frac{1}{\sqrt{2}}(1+\delta^2)^{-1/2}\left(\frac{\beta}{\sqrt{2}}|0\rangle_3 + \underbrace{\beta\delta e^{i\gamma}|1\rangle_3 + \frac{\alpha}{\sqrt{2}}|1\rangle_3}\right)$$

$$\left(\beta\delta e^{i\gamma} + \frac{\alpha}{\sqrt{2}}\right)\left(\beta\delta e^{-i\gamma} + \frac{\alpha}{\sqrt{2}}\right)$$

$$\text{Prob} = \frac{1}{2}\frac{1}{(1+\delta^2)}\left(\frac{\beta^2}{2} + \beta^2\delta^2 + \frac{\beta\delta\alpha}{\sqrt{2}}e^{i\gamma} + \frac{\beta\delta\alpha}{\sqrt{2}}e^{-i\gamma} + \frac{\alpha^2}{2}\right)$$

$$= \frac{1}{2}\frac{1}{(1+\delta^2)}\left(\frac{1}{2} + \beta^2\delta^2 + \frac{\beta\delta\alpha}{\sqrt{2}}\cdot 2\cos(\gamma)\right)$$

for $i,j = 11$ $= P_{11}|\Psi\rangle$

$$\frac{1}{\sqrt{2}}\left(\langle 01|_{12} - \langle 10|_{12}\right)(1+\delta^2)^{-1/2}\left\{\frac{1}{\sqrt{2}}|00\rangle_{13} + \frac{1}{\sqrt{2}}|11\rangle_{13} + \delta e^{i\gamma}|01\rangle_{13}\right\}\otimes\left(\alpha|0\rangle_2 + \beta|1\rangle_2\right)$$

$$= \frac{1}{\sqrt{2}}(1+\delta^2)^{-1/2}\left(\frac{\beta}{\sqrt{2}}|0\rangle_3 + \beta\delta e^{i\gamma}|1\rangle_3 - \frac{\alpha}{\sqrt{2}}|1\rangle_3\right) \qquad \left(\beta\delta e^{i\gamma} - \frac{\alpha}{\sqrt{2}}\right)\left(\beta\delta e^{-i\gamma} - \frac{\alpha}{\sqrt{2}}\right)$$

$$\text{Prob} = \| P_{11}|\Psi\rangle\|^2 = \frac{1}{2}(1+\delta^2)^{-1}\left(\frac{\beta^2}{2} + \beta^2\delta^2 \underbrace{- \frac{\beta\alpha\delta e^{i\gamma}}{\sqrt{2}} - \frac{\beta\alpha\delta e^{-i\gamma}}{\sqrt{2}}}_{-\frac{\beta\alpha\delta}{\sqrt{2}}(2\cos\gamma)} + \frac{\alpha^2}{2}\right)$$

$$= \frac{1}{2}\frac{1}{(1+\delta^2)}\left(\frac{1}{2} + \beta^2\delta^2 - \frac{\beta\delta\alpha}{\sqrt{2}}\cdot 2\cos(\gamma)\right)$$

Check $P(00) + P(01) + P(10) + P(11) \overset{?}{=} 1$ ✓

Prob

$$= \frac{1}{2}\cdot\frac{1}{(1+\delta^2)}\left(\frac{1}{2} + \alpha^2\delta^2 + \cancel{\ldots} + \frac{1}{2} + \alpha^2\delta^2 - \cancel{\ldots} + \frac{1}{2} + \beta^2\delta^2 + \cancel{\ldots} + \frac{1}{2} + \beta^2\delta^2 - \cancel{\ldots}\right)$$

$$= \frac{1}{2}\cdot\frac{1}{(1+\delta^2)}\left(2 + 2\alpha^2\delta^2 + 2\beta^2\delta^2\right)$$

$$= \frac{1}{\cancel{2}}\frac{1}{\cancel{(1+\delta^2)}}\cancel{2}\left(1 + \cancel{\underbrace{(\alpha^2+\beta^2)}_{1}\delta^2}\right) = 1$$

Scanned with CamScanner

Alice sends classical 2 bits:

00 $\longrightarrow$ $|\psi\rangle_{tel} = \frac{1}{\sqrt{2}}(1+\delta^2)^{-1/2}\left(\underbrace{\frac{\alpha}{\sqrt{2}}|0\rangle_3 + \left(\alpha\delta e^{i\gamma}+\frac{\beta}{\sqrt{2}}\right)|1\rangle_3}_{\frac{1}{\sqrt{2}}\underbrace{(\alpha|0\rangle_3+\beta|1\rangle_3)}_{|\varphi\rangle} + \alpha\delta e^{i\gamma}|1\rangle_3}\right)$

01 $\longrightarrow$ $|\psi\rangle_{tel} = \frac{1}{\sqrt{2}}(1+\delta^2)^{-1/2}\left(\frac{\alpha}{\sqrt{2}}|0\rangle_3 + \left(\alpha\delta e^{i\gamma}-\frac{\beta}{\sqrt{2}}\right)|1\rangle_3\right)$

Bob applies Z_3

$\downarrow$

$|\psi\rangle' = \frac{1}{\sqrt{2}}(1+\delta^2)^{-1/2}\left(\underbrace{\frac{\alpha}{\sqrt{2}}|0\rangle_3 - \left(\alpha\delta e^{i\gamma}-\frac{\beta}{\sqrt{2}}\right)|1\rangle_3}_{\frac{1}{\sqrt{2}}\underbrace{(\alpha|0\rangle_3+\beta|1\rangle_3)}_{|\varphi\rangle} - \alpha\delta e^{i\gamma}|1\rangle_3}\right)$

10 $\longrightarrow$ $|\psi\rangle_{tel} = \frac{1}{\sqrt{2}}(1+\delta^2)^{-1/2}\left(\frac{\beta}{\sqrt{2}}|0\rangle_3 + \left(\beta\delta e^{i\gamma}+\frac{\alpha}{\sqrt{2}}\right)|1\rangle_3\right)$

Bob applies X_3

$\downarrow$

$|\psi\rangle' = \frac{1}{\sqrt{2}}(1+\delta^2)^{-1/2}\left(\underbrace{\frac{\beta}{\sqrt{2}}|1\rangle_3 + \left(\beta\delta e^{i\gamma}+\frac{\alpha}{\sqrt{2}}\right)|0\rangle_3}_{\frac{1}{\sqrt{2}}|\varphi\rangle + \beta\delta e^{i\gamma}|0\rangle_3}\right)$

11 $\longrightarrow$ $|\psi\rangle_{tel} = \frac{1}{\sqrt{2}}(1+\delta^2)^{-1/2}\left(\frac{\beta}{\sqrt{2}}|0\rangle_3 + \left(\beta\delta e^{i\gamma}-\frac{\alpha}{\sqrt{2}}\right)|1\rangle_3\right)$

Bob applies Z_3X_3

$\downarrow$

$|\psi\rangle' = \frac{1}{\sqrt{2}}(1+\delta^2)^{-1/2}\left(\underbrace{-\frac{\beta}{\sqrt{2}}|1\rangle_3 + \left(\beta\delta e^{i\gamma}-\frac{\alpha}{\sqrt{2}}\right)|0\rangle_3}_{-\frac{1}{\sqrt{2}}|\varphi\rangle + \beta\delta e^{i\gamma}|0\rangle_3}\right)$

Exercise 3 An entanglement criterion for 2qubits

$$|\psi\rangle = a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|10\rangle + a_{11}|11\rangle$$

a) Show that $|\psi\rangle$ is a product state if and only if

det $A = 0$ where $A = \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix}$ $\rightarrow \det A = a_{00}a_{11} - a_{10}a_{01} = 0$ ✓

$$\left.\begin{aligned} |\psi_1\rangle &= \alpha_1|0\rangle + \beta_1|1\rangle \\ |\psi_2\rangle &= \alpha_2|0\rangle + \beta_2|1\rangle \end{aligned}\right\} \quad |\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$$

$$|\psi\rangle = \alpha_1\alpha_2|00\rangle + \alpha_1\beta_2|01\rangle + \alpha_2\beta_1|10\rangle + \beta_1\beta_2|11\rangle$$

$$\underbrace{(\alpha_1\alpha_2)}_{a_{00}}\underbrace{(\beta_1\beta_2)}_{a_{11}} = \underbrace{(\alpha_1\beta_2)}_{a_{01}}.\underbrace{(\alpha_2\beta_1)}_{a_{10}} \rightarrow \det A = 0 ✓$$

b) Use this criterion to determine when the state is entangled

$$|\psi_1\rangle = \frac{1}{\sqrt{1+\delta^2+\epsilon^2}}\left(\underbrace{\frac{1}{\sqrt{2}}}_{a_{00}}|00\rangle + \underbrace{\frac{1}{\sqrt{2}}}_{a_{11}}|11\rangle + \underbrace{\delta e^{i\gamma}}_{a_{10}}|10\rangle + \underbrace{\epsilon}_{a_{01}}|01\rangle\right)$$

$$(a_{00}).(a_{11}) = (a_{10})(a_{01})$$

$$\left(\frac{1}{\sqrt{2}}\right)\cdot\left(\frac{1}{\sqrt{2}}\right) \overset{?}{\neq} (\delta e^{i\gamma})(\epsilon) \rightarrow \text{It is entangled.}$$

Exercise 4 $|W\rangle$ state

$$|W\rangle = \frac{1}{\sqrt{3}}\left(|100\rangle + |010\rangle + |001\rangle\right)$$

a)
$$\left.\begin{aligned} |\psi_1\rangle &= \alpha_1|0\rangle + \beta_1|1\rangle \\ |\psi_2\rangle &= \alpha_2|0\rangle + \beta_2|1\rangle \\ |\psi_3\rangle &= \alpha_3|0\rangle + \beta_3|1\rangle \end{aligned}\right\} \quad |W\rangle \overset{?}{=} |\psi_1\rangle \otimes |\psi_2\rangle \otimes |\psi_3\rangle$$

$$|W\rangle \neq \overbrace{\alpha_1\alpha_2\alpha_3}^{0}|000\rangle + \overbrace{\alpha_1\alpha_2\beta_3}^{1/\sqrt{3}}|001\rangle + \overbrace{\alpha_1\beta_2\alpha_3}^{1/\sqrt{3}}|010\rangle + \overbrace{\alpha_1\beta_2\beta_3}^{0}|011\rangle$$
$$+ \underbrace{\beta_1\alpha_2\alpha_3}_{1/\sqrt{3}}|100\rangle + \underbrace{\beta_1\alpha_2\beta_3}_{0}|101\rangle + \underbrace{\beta_1\beta_2\alpha_3}_{0}|110\rangle + \underbrace{\beta_1\beta_2\beta_3}_{0}|111\rangle$$

It can not be possible!

Since $\alpha_1\alpha_2\beta_3 = 1/\sqrt{3}$, $\alpha_1\beta_2\alpha_3 = 1/\sqrt{3}$, $\beta_1\alpha_2\alpha_3 = 1/\sqrt{3}$

None of them can be zero.

Thus $\alpha_1\alpha_2\alpha_3 \neq 0$ similar case is accessible for others.

b) $|W\rangle \overset{?}{\neq} |\psi_1\rangle \otimes |\psi_{23}\rangle$

not same

No. $|W\rangle = \frac{1}{\sqrt{3}}\left(|1\rangle\underline{|00\rangle} + |0\rangle\underline{(|10\rangle + |01\rangle)}\right)$. Thus

it can not be separated with this way as well.

Exercise 5 a) b)

$$|\Psi\rangle = |B_{00}\rangle_{12} \otimes |B_{00}\rangle_{34} \otimes |B_{00}\rangle_{56}$$

A local measurement on earth which project states 1, 3, 5 on the state $|GHZ\rangle_{135} = \frac{1}{\sqrt{2}}\left(|000\rangle_{135} + |111\rangle_{135}\right)$

$$|GHZ\rangle_{135}\langle GHZ|_{135} \otimes I_2 \otimes I_4 \otimes I_6 \; |\Psi\rangle$$

$$\frac{1}{\sqrt{2}}\left(\langle 000|_{135} + \langle 111|_{135}\right) \frac{1}{\sqrt{2}}\left(|00\rangle_{12} + |11\rangle_{12}\right) \otimes \frac{1}{\sqrt{2}}\left(|00\rangle_{34} + |11\rangle_{34}\right) \otimes \frac{1}{\sqrt{2}}\left(|00\rangle_{56} + |11\rangle_{56}\right)$$

$$= \frac{1}{\sqrt{2}}\left(|000\rangle_{246} + |111\rangle_{246}\right) \rightsquigarrow \text{resulting State}$$

Global state: $\frac{1}{\sqrt{2}}\left(|000\rangle_{135} + |111\rangle_{135}\right) \otimes \frac{1}{\sqrt{2}}\left(|000\rangle_{246} + |111\rangle_{246}\right)$

where $P = |GHZ\rangle_{135}\langle GHZ|_{135} \otimes I_2 \otimes I_4 \otimes I_6$