WEEK 11 PART2 01/12/22

Density Matrix

→ Statistical mixtures $\{ |\varphi_1\rangle, p_1 ; |\varphi_2\rangle, p_2 ; \ldots\ldots |\varphi_k\rangle, p_k \}$

$$\rho = \sum_{i=1}^{k} p_i \, |\varphi_i\rangle\langle\varphi_i|$$

↳ these projections are not necessarily orthonormal

why

(no need to create a basis)

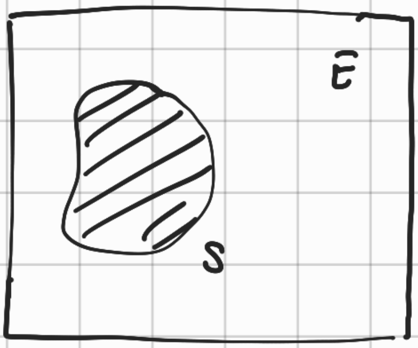
Mixed States

→ System S in contact with Environment

Total state in $\mathcal{H}_S \otimes \mathcal{H}_E$ $\begin{cases} |\Psi\rangle_{SE} \langle\Psi|_{SE} \\ \text{or} \\ \rho_{SE} \end{cases}$

↳ pure states

↳ density Matrix (Mixed states)



Reduced Density Matrix of S (RDM)

$\Downarrow$

$$\rho_S = Tr_E \, \rho_{SE}$$

acts on $\mathcal{H}_S$

Properties of RDM

→ Local Meas. of observable $A_S \otimes \mathbb{1}_E$

$$\mathrm{Av}(A_S \otimes \mathbb{1}_E) = \mathrm{Tr}_{\mathcal{H}_S} A_S \rho_S$$

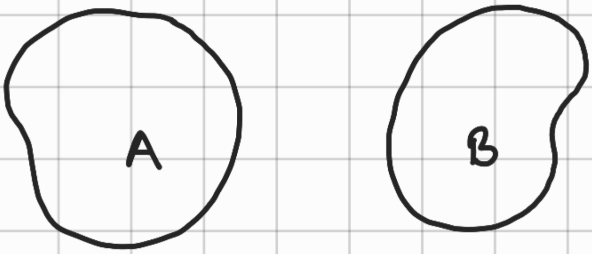
→ Unitary U_t on a system described by ρ_0

At time t: $\rho_t = U_t \rho_0 U_t^{\dagger}$

Generalization of $|\psi_t\rangle = U_t |\psi_0\rangle$

→ For a local unitary operation acting on S (not on E)

$U_S(t) \otimes \mathbb{1}_E$ you have $\rho_S(t) = U_S(t)\, \rho_S(0)\, U_S^{\dagger}(t)$



U_A U_B

Total operation $U_{AB}^{(t)} = U_A^{(t)} \otimes U_B^{(t)}$

$$\rho_{AB}(t) = U_A(t) \otimes U_B(t)\, \rho_{AB}(0)\, U_A^{\dagger}(t) \otimes U_B^{\dagger}(t)$$

$$\rho_A(t) = \mathrm{Tr}_{\mathcal{H}_B}\left(U_A(t) \otimes U_B(t)\, \rho_{AB}(0)\, U_A^{\dagger}(t) \otimes U_B^{\dagger}(t) \right)$$

$\mathbb{1}$

$$\rho_A(t) = \mathrm{Tr}_{\mathcal{H}_B} U_A(t)\, \rho_{AB}(0)\, U_A^{\dagger}(t)$$

$$\boxed{\rho_A(t) = U_A(t)\, \rho_A(0)\, U_A^{\dagger}(t)}$$

$$\boxed{\rho_B(t) = U_B(t)\,\rho_B(0)\,U_B^{+}(t)}$$

Notion of Von Neuman Entropy

Recap of classical Shanon Entropy:

X random variable $\quad X \in \{a_1, a_2, \dots, a_i\}$

$p_1, \dots\dots, p_i$

$p(x = a_i) = p_i$

Measure of how much uncertainity you have

$$H(X) = -\sum_{i=1}^{K} p_i \log_2 p_i$$

$$p_1 = p_2 = \dots\dots = p_K = \frac{1}{K} \rightsquigarrow \text{Maximal}$$

<u>Lemma</u>: $0 \leq H(X) \leq \log_2 K$

Quantum system described by a DM ρ

$$S(\rho) = -\mathrm{Tr}(\rho \log \rho)$$

In practise $\rho = \sum_{\alpha=1}^{D} \lambda_\alpha |u_\alpha\rangle\langle u_\alpha|$ → orthonormal basis

spectral decomposition

$\rho|u_\alpha\rangle = \lambda_\alpha |u_\alpha\rangle$

$0 \leq \lambda_\alpha \leq 1$

$\sum_\alpha \lambda_\alpha = 1$

$$F(\rho) = \sum_{\alpha=1}^{D} F(\lambda_\alpha)\,|u_\alpha\rangle\langle u_\alpha|$$

↘
any function of ρ

$$\rho\log(\rho) = \sum_{\alpha} \left(\lambda_\alpha \log(\lambda_\alpha)\right) |u_\alpha\rangle\langle u_\alpha|$$

$$\boxed{S(\rho) = -\,\mathrm{Tr}\,(\rho\log\rho)}$$

$S(\rho)$ quantifies degree of uncertainity in the quantum mixed state ρ.

Example $\rho = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$ $\quad S(\rho) = \log_2 2 = 1$

$\rho = |\psi\rangle\langle\psi|$ → eigenvalues $\lambda = 1, \underbrace{0 \ldots 0}_{\text{dim of } \mathcal{H}}$

↓
pure states
(no uncertinity)

$$S(\rho) = -1.\log 1 - 0.\log 0 = 0$$

Entangled State: $|B_{00}\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$

$$S\left(|B_{00}\rangle\langle B_{00}|\right) = 0$$

! "0" global entropy

$$\rho_A = Tr_B |B_{00}\rangle\langle B_{00}| = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_A \qquad S(\rho_A) = \log 2$$

→ Max. local entropy

$$\rho_B = Tr_A |B_{00}\rangle\langle B_{00}| = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_B \qquad S(\rho_B) = \log 2$$