3 Decay rates and cross sections

This chapter describes the methodology for the calculations of cross sections and decay rates in relativistic quantum mechanics. In particular, it introduces the ideas of Lorentz-invariant phase space, the Lorentz-invariant matrix element and the treatment of kinematics in particle decays and interactions. The end product is a set of master formulas which, once the quantum mechanical matrix element for a process is known, can be used to obtain expressions for decays rates and cross sections. Provided the main concepts are understood, it is possible to skip the details of the derivations.

3.1 Fermi's golden rule

Much of particle physics is based on the experimental measurements of particle decay rates and particle interaction cross sections. These experimentally observable phenomena represent transitions between different quantum mechanical states. In *non-relativistic* quantum mechanics, transition rates are obtained using Fermi's golden rule, which was derived in Section 2.3.6. Fermi's golden rule for the transition rate Γ_{fi} from an initial state $|i\rangle$ to a final state $|f\rangle$ is usually expressed as

$$\Gamma_{fi} = 2\pi |T_{fi}|^2 \rho(E_i), \tag{3.1}$$

where T_{fi} is the transition matrix element and $\rho(E_i)$ is the *density of states*. The transition matrix element is determined by the Hamiltonian for the interaction which causes the transitions $\hat{H}'$. In the limit where the perturbation is weak, the transition matrix element is given by a perturbation expansion in terms of the interaction Hamiltonian,

transition matrix

$$T_{fi} = \langle f|\hat{H}'|i\rangle + \sum_{j\neq i} \frac{\langle f|\hat{H}'|j\rangle\langle j|\hat{H}'|i\rangle}{E_i - E_j} + \cdots .$$

The transition rate of (3.1) depends on the density of states $\rho(E_i)$,

$$\rho(E_i) = \left|\frac{\mathrm{d}n}{\mathrm{d}E}\right|_{E_i},$$

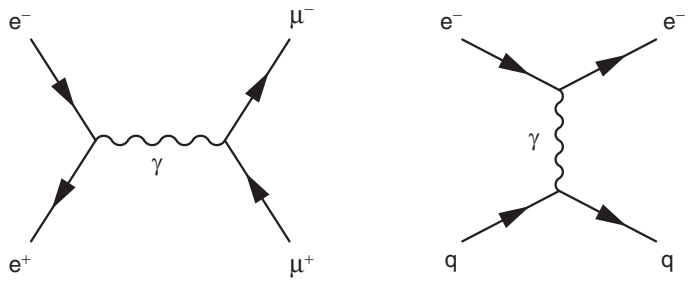

Fig. 3.1 Feynman diagrams for $e^+e^- \to \mu^+\mu^-$ annihilation and $e^-q \to e^-q$ scattering.

where dn is the number of accessible states in the energy range $E \to E + \mathrm{d}E$. Alternatively, the density of states can be written as an integral over *all* final-state energies using the Dirac delta-function to impose energy conservation,

$$\left|\frac{\mathrm{d}n}{\mathrm{d}E}\right|_{E_i} = \int \frac{\mathrm{d}n}{\mathrm{d}E}\delta(E_i - E)\,\mathrm{d}E,$$

giving the alternative form of Fermi's golden rule

$$\Gamma_{fi} = 2\pi \int |T_{fi}|^2 \delta(E_i - E)\,\mathrm{d}n, \tag{3.2}$$

which appeared as an intermediate step in the derivation of Fermi's golden rule, see (2.49).

The transition rate between two states depends on two components, (i) the *transition matrix element*, which contains the fundamental particle physics, and (ii) the *density of accessible states*, which depends on the kinematics of the process being considered. The aim of the first part of this book is to develop the methodology for the calculation of decay rates and interaction cross sections for particle annihilation and scattering processes such as those represented by the Feynman diagrams of Figure 3.1. In modern particle physics the most complete theoretical approach to such calculations is to use quantum field theory. Nevertheless, the same results can be obtained using perturbation theory in relativistic quantum mechanics (RQM). This requires a relativistic formulation of Fermi's golden rule where the density of states is based on relativistic treatments of phase space and the normalisation of the plane waves used to represent the particles.

3.2 Phase space and wavefunction normalisation

Before discussing the relativistic wavefunction normalisation and phase space, it is worth briefly reviewing the non-relativistic treatment. In non-relativistic quantum mechanics, the decay rate for the process $a \to 1 + 2$ can be calculated using

Fermi's golden rule. To first order in perturbation theory, the transition matrix element is

$$T_{fi} = \langle \psi_1 \psi_2 | \hat{H}' | \psi_a \rangle \tag{3.3}$$

$$= \int_V \psi_1^* \psi_2^* \hat{H}' \psi_a \mathrm{d}^3\mathbf{x}. \tag{3.4}$$

In the Born approximation, the perturbation is taken to be small and the initial- and final-state particles are represented by plane waves of the form

$$\psi(\mathbf{x}, t) = A e^{i(\mathbf{p}\cdot\mathbf{x} - Et)}, \tag{3.5}$$

where A determines the wavefunction normalisation. The integral in (3.4) extends over the volume in which the wavefunctions are normalised. It is usual to adopt a scheme where each plane wave is normalised to one particle in a cubic volume of side a. Using the non-relativistic expression for probability density $\rho = \psi^*\psi$, this is equivalent to writing

$$\int_0^a \int_0^a \int_0^a \psi^* \psi \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z = 1,$$

which implies that the normalisation constant in (3.5) is given by

$$A^2 = 1/a^3 = 1/V,$$

where V is the volume of the box.

The normalisation of one particle in a box of volume a^3 implies that the wavefunction satisfies the periodic boundary conditions[1]

$$\psi(x + a, y, z) = \psi(x, y, z), \text{ etc.},$$

as illustrated in Figure 3.2. The periodic boundary conditions on the wavefunction, for example $e^{ip_x x} = e^{ip_x(x+a)}$, imply that the components of momentum are quantised to

$$(p_x, p_y, p_z) = (n_x, n_y, n_z)\frac{2\pi}{a},$$

[1] In terms of counting the number of states, the periodic boundary conditions are equivalent to requiring that the wavefunction is zero at the boundaries of the volume. This condition implies that the wavefunction consists of standing waves of the form $\psi(x, y, z) = A \sin(p_x x) \sin(p_y y) \sin(p_z z)$, with p_x, p_y and p_z such that there are a half-integer number of wavelengths along each side of the box. Since $\sin(p_x x) = (e^{ip_x x} - e^{-ip_x x})/2i$, the wavefunction expressed in this way has forward-going and backward-going components and the integration over phase space is restricted to positive values of p_x, p_y and p_z. The same number of states are obtained with periodic boundary conditions, with an integer number of wavelengths in each direction. In this case, the phase space integral extends over both positive and negative values of p_x, p_y and p_z.

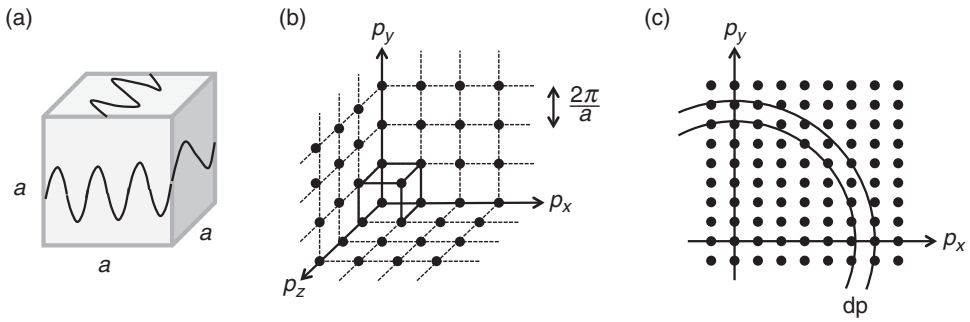

Fig. 3.2 The non-relativistic treatment of phase space: (a) the wavefunction of a particle confined to a box of side a satisfies the periodic boundary conditions such that there are an integer number of wavelengths in each direction; (b) the allowed states in momentum space; and (c) the number of states in a range $\mathrm{p} \longrightarrow \mathrm{p} + \mathrm{dp}$ in two dimensions.

where n_x, n_y and n_z are integers. This restricts the allowed momentum states to the discrete set indicated in Figure 3.2b. Each state in momentum space occupies a cubic volume of

$$\mathrm{d}^3\mathbf{p} = \mathrm{d}p_x \mathrm{d}p_y \mathrm{d}p_z = \left(\frac{2\pi}{a}\right)^3 = \frac{(2\pi)^3}{V}.$$

As indicated in Figure 3.2c, the number of states dn with magnitude of momentum in the range $\mathrm{p} \longrightarrow \mathrm{p} + \mathrm{dp}$, is equal to the momentum space volume of the spherical shell at momentum p with thickness dp divided by the average volume occupied by a single state, $(2\pi)^3/V$, giving

$$\mathrm{d}n = 4\pi \mathrm{p}^2 \mathrm{dp} \times \frac{V}{(2\pi)^3},$$

and hence

$$\frac{\mathrm{d}n}{\mathrm{dp}} = \frac{4\pi \mathrm{p}^2}{(2\pi)^3} V.$$

The density of states in Fermi's golden rule then can be obtained from

$$\rho(E) = \frac{\mathrm{d}n}{\mathrm{d}E} = \frac{\mathrm{d}n}{\mathrm{dp}} \left|\frac{\mathrm{dp}}{\mathrm{d}E}\right|.$$

The density of states corresponds to the number of momentum states accessible to a particular decay and increases with the momentum of the final-state particle. Hence, all other things being equal, decays to lighter particles, which will be produced with larger momentum, are favoured over decays to heavier particles.

The calculation of the decay rate will *not* depend on the normalisation volume; the volume dependence in the expression for phase space is cancelled by the factors of V associated with the wavefunction normalisations that appear in the square of transition matrix element. Since the volume will not appear in the final result, it

is convenient to normalise to one particle per *unit* volume by setting $V = 1$. In this case, the number of accessible states for a particle associated with an infinitesimal volume in momentum space $\mathrm{d}^3\mathbf{p}_i$ is simply

$$\mathrm{d}n_i = \frac{\mathrm{d}^3\mathbf{p}_i}{(2\pi)^3}.$$

For the decay of a particle to a final state consisting of N particles, there are $N-1$ independent momenta in the final state, since the momentum of one of the final-state particles can always be obtained from momentum conservation. Thus, the number of independent states for an N-particle final state is

$$\mathrm{d}n = \prod_{i=1}^{N-1} \mathrm{d}n_i = \prod_{i=1}^{N-1} \frac{\mathrm{d}^3\mathbf{p}_i}{(2\pi)^3}.$$

This can be expressed in a more democratic form including the momentum space volume element for the Nth particle $\mathrm{d}^3\mathbf{p}_N$ and using a three-dimensional delta-function to impose momentum conservation

$$\mathrm{d}n = \prod_{i=1}^{N-1} \frac{\mathrm{d}^3\mathbf{p}_i}{(2\pi)^3} \delta^3\left(\mathbf{p}_a - \sum_{i=1}^{N} \mathbf{p}_i\right) \mathrm{d}^3\mathbf{p}_N, \tag{3.6}$$

where $\mathbf{p}_a$ is the momentum of the decaying particle. Therefore the general non-relativistic expression for N-body phase space is

$$\mathrm{d}n = (2\pi)^3 \prod_{i=1}^{N} \frac{\mathrm{d}^3\mathbf{p}_i}{(2\pi)^3} \delta^3\left(\mathbf{p}_a - \sum_{i=1}^{N} \mathbf{p}_i\right). \tag{3.7}$$

3.2.1 Lorentz-invariant phase space

The wavefunction normalisation of one particle per unit volume is not Lorentz invariant since it only applies to a particular frame of reference. In a different reference frame, the original normalisation volume will be Lorentz contracted by a factor of $1/\gamma$ along its direction of relative motion, as shown in Figure 3.3. Thus, the original normalisation of one particle per unit volume corresponds to a normalisation of $\gamma = E/m$ particles per unit volume in the boosted frame of reference. A Lorentz-invariant choice of wavefunction normalisation must therefore be proportional to E particles per unit volume, such that the increase in energy accounts for the effect of Lorentz contraction. The usual convention is to normalise to $2E$ particles per unit volume. The reason for this particular choice is motivated in Section 3.2.3 and also in Chapter 4.

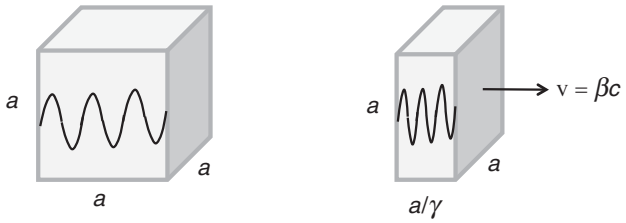

Fig. 3.3 The normalisation volume in a particular frame is length contracted along the direction of motion for a general rest frame.

The wavefunctions ψ appearing in the transition matrix element T_{fi} of Fermi's golden rule are normalised to one particle per unit volume,

$$\int_V \psi^* \psi \, \mathrm{d}^3\mathbf{x} = 1.$$

Wavefunctions with the appropriate Lorentz-invariant normalisation, here written as ψ', are normalised to $2E$ particles per unit volume

$$\int_V \psi'^* \psi' \mathrm{d}^3\mathbf{x} = 2E,$$

and therefore

$$\psi' = (2E)^{1/2}\psi.$$

For a general process, $a + b + \cdots \to 1 + 2 + \cdots$, the *Lorentz-invariant matrix element*, using wavefunctions with a Lorentz-invariant normalisation, is defined as

$$\mathcal{M}_{fi} = \langle \psi'_1 \psi'_2 \cdots | \hat{H}' | \psi'_a \psi'_b \cdots \rangle. \tag{3.8}$$

The Lorentz-invariant matrix element is therefore related to the transition matrix element of Fermi's golden rule by

$$\mathcal{M}_{fi} = \langle \psi'_1 \psi'_2 \cdots | \hat{H}' | \psi'_a \psi'_b \cdots \rangle = (2E_1 \cdot 2E_2 \cdots 2E_a \cdot 2E_b \cdots)^{1/2} T_{fi}, \tag{3.9}$$

where the product on the RHS of (3.9) includes all intial- and final-state particles.

3.2.2 Fermi's golden rule revisited

For a two-body decay $a \to 1 + 2$, the quantum mechanical transition rate is given by Fermi's golden rule, which in the form of (3.2) can be written

$$\Gamma_{fi} = 2\pi \int |T_{fi}|^2 \delta(E_a - E_1 - E_2) \, \mathrm{d}n,$$

where dn is given by (3.7), and hence

$$\Gamma_{fi} = (2\pi)^4 \int |T_{fi}|^2 \delta(E_a - E_1 - E_2)\delta^3(\mathbf{p}_a - \mathbf{p}_1 - \mathbf{p}_2)\frac{\mathrm{d}^3\mathbf{p}_1}{(2\pi)^3}\frac{\mathrm{d}^3\mathbf{p}_2}{(2\pi)^3}. \tag{3.10}$$

Using the relation between the transition matrix element and the Lorentz invariant matrix element of (3.9), this can be written as

$$\Gamma_{fi} = \frac{(2\pi)^4}{2E_a} \int |\mathcal{M}_{fi}|^2 \delta(E_a - E_1 - E_2)\delta^3(\mathbf{p}_a - \mathbf{p}_1 - \mathbf{p}_2)\frac{\mathrm{d}^3\mathbf{p}_1}{(2\pi)^3 2E_1}\frac{\mathrm{d}^3\mathbf{p}_2}{(2\pi)^3 2E_2}, \tag{3.11}$$

with $|\mathcal{M}_{fi}|^2 = (2E_a 2E_1 2E_2)|T_{fi}|^2$. One consequence of using wavefunctions with a Lorentz invariant normalisation, is that the phase space integral over $\mathrm{d}^3\mathbf{p}/(2\pi)^3$ has been replaced by an integral over terms like

$$\frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3 2E},$$

which is known as the Lorentz-invariant phase space factor. To prove this is Lorentz invariant, consider a Lorentz transformation along the z-axis, where the element $\mathrm{d}^3\mathbf{p}$ transforms to $\mathrm{d}^3\mathbf{p}'$ given by

$$\mathrm{d}^3\mathbf{p}' \equiv \mathrm{d}p'_x \mathrm{d}p'_y \mathrm{d}p'_z = \mathrm{d}p_x \mathrm{d}p_y \cdot \frac{\mathrm{d}p'_z}{\mathrm{d}p_z}\mathrm{d}p_z = \frac{\mathrm{d}p'_z}{\mathrm{d}p_z}\mathrm{d}^3\mathbf{p}. \tag{3.12}$$

From the Einstein energy–momentum relation, $E^2 = p_x^2 + p_y^2 + p_z^2 + m^2$, and the Lorentz transformation of the energy–momentum four-vector,

$$p'_z = \gamma(p_z - \beta E) \quad \text{and} \quad E' = \gamma(E - \beta p_z),$$

it follows that

$$\frac{\mathrm{d}p'_z}{\mathrm{d}p_z} = \gamma\left(1 - \beta\frac{\partial E}{\partial p_z}\right) = \gamma\left(1 - \beta\frac{p_z}{E}\right) = \frac{1}{E}\gamma(E - \beta p_z) = \frac{E'}{E},$$

which when substituted into (3.12) demonstrates that

$$\frac{\mathrm{d}^3\mathbf{p}'}{E'} = \frac{\mathrm{d}^3\mathbf{p}}{E},$$

and hence $\mathrm{d}^3\mathbf{p}/E$ is Lorentz invariant.

The matrix element $\mathcal{M}_{fi}$ in (3.11) is defined in terms of wavefunctions with a Lorentz-invariant normalisation, and the elements of integration over phase space $\mathrm{d}^3\mathbf{p}_i/E_i$ are also Lorentz invariant. Consequently, the integral in (3.11) is Lorentz invariant and thus (3.11) expresses Fermi's golden rule in a Lorentz-invariant form. This is an important result, it is exactly the required relativistic treatment of transition rates needed for the calculation of decay rates. The resulting transition rate for the decay $a \to 1 + 2$ given in (3.11) is inversely proportional to the energy of the decaying particle in the frame in which it is observed, $E_a = \gamma m_a$, as expected from relativistic time dilation.

3.2.3 *Lorentz-invariant phase space

The expression for the decay rate given (3.11) can be extended to an N-body decay, $a \to 1 + 2 + \cdots + N$. In this more general case, the phase space integral involves the three-momenta of all final-state particles

$$\mathrm{d}LIPS = \prod_{i=1}^{N} \frac{\mathrm{d}^3\mathbf{p}_i}{(2\pi)^3 2E_i},$$

where $\mathrm{d}LIPS$ is known as the element of Lorentz-invariant phase space (LIPS). The factors $1/2E_i$ can be rewritten in terms of a delta-function using (A.6) of Appendix A and the constraint from the Einstein energy–momentum relationship, $E_i = \mathbf{p}_i^2 + m_i^2$, which implies that

$$\int \delta(E_i^2 - \mathbf{p}_i^2 - m^2)\,\mathrm{d}E_i = \frac{1}{2E_i}.$$

Hence, the integral over Lorentz-invariant phase space can be written as

$$\int \cdots \mathrm{d}LIPS = \int \cdots \prod_{i=1}^{N} (2\pi)^{-3} \delta(E_i^2 - \mathbf{p}_i^2 - m_i^2)\,\mathrm{d}^3\mathbf{p}_i\,\mathrm{d}E_i,$$

which, in terms of the four-momenta of the final-state particles is

$$\int \cdots \mathrm{d}LIPS = \int \cdots \prod_{i=1}^{N} (2\pi)^{-3} \delta(p_i^2 - m_i^2)\,\mathrm{d}^4 p_i.$$

Similarly, the transition rate for the two-body decay $a \to 1 + 2$, given in (3.11), can be written as

$$\Gamma_{fi} = \frac{(2\pi)^4}{2E_a} \int (2\pi)^{-6} |\mathcal{M}_{fi}|^2 \delta^4(p_a - p_1 - p_2)\delta(p_1^2 - m_1^2)\delta(p_2^2 - m_2^2)\,\mathrm{d}^4 p_1 \mathrm{d}^4 p_2.$$

The integral now extends over *all* values of the energies and momenta of each of the final-state particles. The delta-functions ensure that the decay rate only has contributions from values of the four-momenta of the final-state particles compatible with overall energy and momentum conservation and the Einstein energy–momentum relation $p_i^2 = m_i^2$. This form of the expression for the decay rate elucidates clearly the point that all the fundamental physics lives in the matrix element. It also provides a deeper insight into the origin of the phase space integral.

3.3 Particle decays

In general, a given particle may decay by more than one decay mode. For example, the tau-lepton can decay into a number of final states, $\tau^- \to e^- \bar{\nu}_e \nu_\tau$, $\tau^- \to \mu^- \bar{\nu}_\mu \nu_\tau$ and $\tau^- \to \nu_\tau + \text{hadrons}$. The transition rate for *each* decay mode j can be calculated independently using Fermi's golden rule. The individual transition rates Γ_j are referred to as *partial decay rates* or, for reasons that will become apparent later, *partial widths*.

The total decay rate is simply the sum of the decay rates for the individual decay modes. For example, if there are N particles of a particular type, the number that decay in time δt is given by the sum of the numbers of decays into each decay channel,

$$\delta N = -N\Gamma_1 \delta t - N\Gamma_2 \delta t - \cdots = -N \sum_j \Gamma_j \, \delta t = -N\Gamma \delta t, \tag{3.13}$$

where the total decay rate per unit time Γ is the sum of the individual decay rates,

$$\Gamma = \sum_j \Gamma_j.$$

The number of particles remaining after a time t is obtained by integrating (3.13) to give the usual exponential form

$$N(t) = N(0)\, e^{-\Gamma t} = N(0)\, \exp\left(-\frac{t}{\tau}\right),$$

where the lifetime of the particle in its rest frame τ is referred to as its proper lifetime and is determined from the total decay rate

lifetime ← $\tau = \dfrac{1}{\Gamma}.$

The relative frequency of a particular decay mode is referred to as the *branching ratio* (or branching fraction). The branching ratio for a particular decay mode $BR(j)$ is given by the decay rate to the mode j relative to the total decay rate

$$BR(j) = \frac{\Gamma_j}{\Gamma}.$$

For example, the branching ratio for the tau-lepton decay $\tau^- \to e^- \bar{\nu}_e \nu_\tau$ is 0.17, which means that on average 17% of the time a τ^- will decay to $e^- \bar{\nu}_e \nu_\tau$. By definition, the branching ratios for all decay modes of a particular particle sum to unity.

3.3.1 Two-body decays

The transition rate for each decay mode of a particle can be calculated by using the relativistic formulation of Fermi's golden rule given in (3.11). The rate depends on

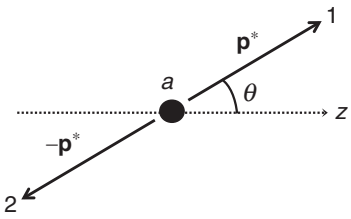

Fig. 3.4 The two-body decay $a \longrightarrow 1 + 2$ in the rest frame of particle a.

the matrix element for the process and the phase space integral. The matrix element depends on the nature of the decay and needs to be evaluated for each process. In contrast, the form of the phase space integral depends only on the number of particles in the final state. Furthermore, since the integral of (3.11) is Lorentz invariant, it can be evaluated in any frame.

Consider the two-body decay $a \rightarrow 1 + 2$, shown in Figure 3.4. In the centre-of-mass frame, the decaying particle is at rest, $E_a = m_a$ and $\mathbf{p}_a = \mathbf{0}$, and the two daughter particles are produced back to back with three-momenta $\mathbf{p}^*$ and $-\mathbf{p}^*$. In this frame, the decay rate is given by (3.11),

$$\Gamma_{fi} = \frac{1}{8\pi^2 m_a} \int |\mathcal{M}_{fi}|^2 \delta(m_a - E_1 - E_2) \delta^3(\mathbf{p}_1 + \mathbf{p}_2) \frac{\mathrm{d}^3\mathbf{p}_1}{2E_1} \frac{\mathrm{d}^3\mathbf{p}_2}{2E_2}. \tag{3.14}$$

It is not straightforward to evaluate the phase space integral in this expression, but fortunately the calculation applies to all two-body decays and has to be performed only once. The $\delta^3(\mathbf{p}_1 + \mathbf{p}_2)$ term in (3.14) means that the integral over $\mathrm{d}^3\mathbf{p}_2$ has the effect of relating the three-momenta of the final-state particles giving $\mathbf{p}_2 = -\mathbf{p}_1$ and hence

$$\Gamma_{fi} = \frac{1}{8\pi^2 m_a} \int |\mathcal{M}_{fi}|^2 \frac{1}{4E_1 E_2} \delta\left(m_a - E_1 - E_2\right) \mathrm{d}^3\mathbf{p}_1, \tag{3.15}$$

where E_2 is now given by $E_2^2 = (m_2^2 + \mathrm{p}_1^2)$. In spherical polar coordinates,

$$\mathrm{d}^3\mathbf{p}_1 = \mathrm{p}_1^2 \mathrm{dp}_1 \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi = \mathrm{p}_1^2 \, \mathrm{dp}_1 \mathrm{d}\Omega,$$

and (3.15) can be written

$$\Gamma_{fi} = \frac{1}{8\pi^2 m_a} \int |\mathcal{M}_{fi}|^2 \delta\left(m_a - \sqrt{m_1^2 + \mathrm{p}_1^2} - \sqrt{m_2^2 + \mathrm{p}_1^2}\right) \frac{\mathrm{p}_1^2}{4E_1 E_2} \, \mathrm{dp}_1 \mathrm{d}\Omega. \tag{3.16}$$

At first sight this integral looks quite tricky. Fortunately the Dirac delta-function does most of the work. Equation (3.16) has the functional form

$$\Gamma_{fi} = \frac{1}{8\pi^2 m_a} \int |\mathcal{M}_{fi}|^2 g(\mathrm{p}_1) \, \delta(f(\mathrm{p}_1)) \, \mathrm{dp}_1 \mathrm{d}\Omega, \tag{3.17}$$

with

$$g(\mathrm{p}_1) = \frac{\mathrm{p}_1^2}{4E_1E_2}, \tag{3.18}$$

and

$$f(\mathrm{p}_1) = m_a - E_1 - E_2 = m_a - \sqrt{m_1^2 + \mathrm{p}_1^2} - \sqrt{m_2^2 + \mathrm{p}_1^2}. \tag{3.19}$$

The Dirac delta-function $\delta(f(\mathrm{p}_1))$ imposes energy conservation and is only non-zero for $\mathrm{p}_1 = \mathrm{p}^*$, where p^* is the solution of $f(\mathrm{p}^*) = 0$. The integral over $\mathrm{d}\mathrm{p}_1$ in (3.17) can be evaluated using the properties of the Dirac delta-function (see Appendix A), whereby

$$\int |\mathcal{M}_{fi}|^2 \, g(\mathrm{p}_1)\, \delta\left(f(\mathrm{p}_1)\right) \, \mathrm{d}\mathrm{p}_1 = |\mathcal{M}_{fi}|^2 \, g(\mathrm{p}^*) \left|\frac{\mathrm{d}f}{\mathrm{d}\mathrm{p}_1}\right|^{-1}_{\mathrm{p}^*}. \tag{3.20}$$

The derivative $\mathrm{d}f/\mathrm{d}p_1$ can be obtained from (3.19),

$$\left|\frac{\mathrm{d}f}{\mathrm{d}\mathrm{p}_1}\right| = \frac{\mathrm{p}_1}{(m_1^2 + \mathrm{p}_1^2)^{1/2}} + \frac{\mathrm{p}_1}{(m_2^2 + \mathrm{p}_1^2)^{1/2}} = \mathrm{p}_1 \left(\frac{E_1 + E_2}{E_1E_2}\right),$$

which, when combined with the expression for $g(\mathrm{p}_1)$ given in (3.18), leads to

$$g(\mathrm{p}^*) \left|\frac{\mathrm{d}f}{\mathrm{d}\mathrm{p}_1}\right|^{-1}_{\mathrm{p}_1=\mathrm{p}^*} = \frac{\mathrm{p}^{*2}}{4E_1E_2} \cdot \frac{E_1E_2}{\mathrm{p}^*(E_1 + E_2)} = \frac{\mathrm{p}^*}{4m_a}.$$

Thus, the integral of (3.20) is

$$\int |\mathcal{M}_{fi}|^2 \, g(\mathrm{p}_1)\, \delta(f(\mathrm{p}_1)) \, \mathrm{d}\mathrm{p}_1 = \frac{\mathrm{p}^*}{4m_a} |\mathcal{M}_{fi}|^2,$$

and therefore,

$$\int |\mathcal{M}_{fi}|^2 \delta(m_a - E_1 - E_2) \delta^3(\mathbf{p}_1 + \mathbf{p}_2) \frac{\mathrm{d}^3\mathbf{p}_1}{2E_1} \frac{\mathrm{d}^3\mathbf{p}_2}{2E_2} = \frac{\mathrm{p}^*}{4m_a} \int |\mathcal{M}_{fi}|^2 \mathrm{d}\Omega, \tag{3.21}$$

and hence (3.14) becomes

$$\Gamma_{fi} = \frac{\mathrm{p}^*}{32\pi^2 m_a^2} \int |\mathcal{M}_{fi}|^2 \, \mathrm{d}\Omega. \tag{3.22}$$

Equation (3.22) is the general expression for *any* two-body decay. The fundamental physics is contained in the matrix element and the additional factors arise from the phase space integral. The matrix element, which may depend on the decay angle, remains inside the integral. The centre-of-mass frame momentum of the final-state particles p^* can be obtained from energy conservation, or equivalently $f(\mathrm{p}^*) = 0$, and is given by (see Problem 3.2)

$$\mathrm{p}^* = \frac{1}{2m_a} \sqrt{\left[(m_a^2 - (m_1 + m_2)^2\right]\left[m_a^2 - (m_1 - m_2)^2\right]}.$$

3.4 Interaction cross sections

The calculation of interaction rates is slightly more complicated than that for particle decays because it is necessary to account for the flux of initial-state particles, where flux is defined as the number of particles crossing a unit area per unit time. In the simplest case, one can imagine a beam of particles of type a, with flux ϕ_a, crossing a region of space in which there are n_b particles per unit volume of type b. The interaction rate per target particle r_b will be proportional to the incident particle flux and can be written

$$r_b = \sigma\phi_a. \tag{3.23}$$

The fundamental physics is contained in σ, which has dimensions of area, and is termed the interaction *cross section*. Sometimes it is helpful to think of σ as the *effective* cross sectional area associated with each target particle. Indeed, there are cases where the cross section is closely related to the physical cross sectional area of the target, for example, neutron absorption by a nucleus. However, in general, the cross section is simply an expression of the underlying quantum mechanical probability that an interaction will occur.

The definition of the cross section is illustrated by the situation shown in Figure 3.5a, where a single incident particle of type a is travelling with a velocity $\mathbf{v}_a$ in a region defined by the area A, which contains n_b particles of type b per unit volume moving with a velocity $\mathbf{v}_b$ in the opposite direction to $\mathbf{v}_a$. In time δt, the particle a crosses a region containing $\delta N = n_b(\mathrm{v}_a + \mathrm{v}_b)A\delta t$ particles of type b. The interaction probability can be obtained from the *effective* total cross sectional area of the δN particles divided by the area A, which can be thought of as the probability that the incident particle passes through one of the regions of area σ drawn around each of the δN target particles, as shown in Figure 3.5b. The interaction probability δP is therefore

$$\delta P = \frac{\delta N\,\sigma}{A} = \frac{n_b(\mathrm{v}_a + \mathrm{v}_b)A\,\sigma\delta t}{A} = n_b \mathrm{v}\sigma\delta t,$$

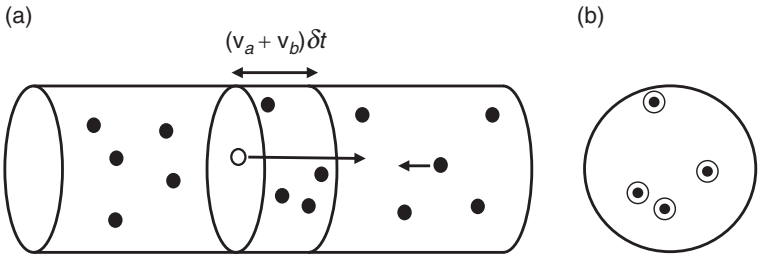

Fig. 3.5 The left-hand plot (a) shows a single incident particle of type a traversing a region containing particles of type b. The right-hand plot (b) shows the projected view of the region traversed by the incident particle in time δt.

where $\mathrm{v} = \mathrm{v}_a + \mathrm{v}_b$. Hence the interaction rate for each particle of type a is

$$r_a = \frac{\mathrm{d}P}{\mathrm{d}t} = n_b \mathrm{v} \sigma.$$

For a beam of particles of type a, with number density n_a confined to a volume V, the total interaction rate is

$$\text{rate} = r_a \, n_a V = (n_b \mathrm{v} \sigma) \, n_a V. \tag{3.24}$$

The expression of (3.24) can be rearranged into

$$\text{rate} = (n_a \mathrm{v})(n_b V)\sigma = \phi \, N_b \, \sigma.$$

Thus the total rate is equal to

$$\text{rate} = \text{flux} \times \text{number of target particles} \times \text{cross section},$$

which is consistent with the definition of (3.23). More formally, the cross section for a process is defined as

$$\sigma = \frac{\text{number of interactions per unit time per target particle}}{\text{incident flux}}.$$

It should be noted that the flux ϕ accounts for the *relative* motion of the particles.

3.4.1 Lorentz-invariant flux

The cross section for a particular process can be calculated using the relativistic formulation of Fermi's golden rule and the appropriate Lorentz-invariant expression for the particle flux. Consider the scattering process $a + b \to 1 + 2$, as observed in the rest frame where the particles of type a have velocity $\mathbf{v}_a$ and those of type b have velocity $\mathbf{v}_b$, as shown in Figure 3.6. If the number densities of the particles are n_a and n_b, the interaction rate in the volume V is given by

$$\text{rate} = \phi_a \, n_b V \, \sigma = (\mathrm{v}_a + \mathrm{v}_b) \, n_a \, n_b \, \sigma \, V, \tag{3.25}$$

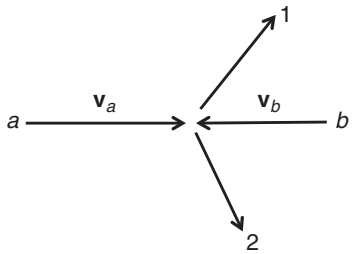

Fig. 3.6 The two-body scattering process $a + b \to 1 + 2$.

where ϕ_a is the flux of particles of type a through a plane moving at velocity $\mathbf{v}_b$,

$$\phi_a = n_a(\mathrm{v}_a + \mathrm{v}_b).$$

Normalising the wavefunctions to one particle in a volume V, gives $n_a = n_b = 1/V$, for which the interaction rate in the volume V is

$$\Gamma_{fi} = \frac{(\mathrm{v}_a + \mathrm{v}_b)}{V}\sigma. \tag{3.26}$$

Because the factors of V in the expression for the flux will ultimately be cancelled by the corresponding factors from the wavefunction normalisation and phase space, the volume V will not appear in the final result and it is again convenient to adopt a normalisation of one particle per unit volume. With this choice, the cross section is related to the transition rate by

$$\sigma = \frac{\Gamma_{fi}}{(\mathrm{v}_a + \mathrm{v}_b)}.$$

The transition rate Γ_{fi} is given by Fermi's golden rule, which in the form of (3.10) gives

$$\sigma = \frac{(2\pi)^4}{(\mathrm{v}_a + \mathrm{v}_b)} \int |T_{fi}|^2 \delta(E_a + E_b - E_1 - E_2)\delta^3(\mathbf{p}_a + \mathbf{p}_b - \mathbf{p}_1 - \mathbf{p}_2)\frac{\mathrm{d}^3\mathbf{p}_1}{(2\pi)^3}\frac{\mathrm{d}^3\mathbf{p}_2}{(2\pi)^3}.$$

This can be expressed in a Lorentz-invariant form by writing T_{fi} in terms of the Lorentz-invariant matrix element $\mathcal{M}_{fi} = (2E_1\,2E_2\,2E_3\,2E_4)^{1/2}T_{fi}$,

$$\sigma = \frac{(2\pi)^{-2}}{4\,E_aE_b(\mathrm{v}_a + \mathrm{v}_b)} \int |\mathcal{M}_{fi}|^2 \delta(E_a + E_b - E_1 - E_2)\delta^3(\mathbf{p}_a + \mathbf{p}_b - \mathbf{p}_1 - \mathbf{p}_2)\frac{\mathrm{d}^3\mathbf{p}_1}{2E_1}\frac{\mathrm{d}^3\mathbf{p}_2}{2E_2}. \tag{3.27}$$

The integral in (3.27) is now written in a Lorentz-invariant form. The quantity $F = 4E_aE_b(\mathrm{v}_a + \mathrm{v}_b)$ is known as the Lorentz-invariant flux factor. To demonstrate the Lorentz invariance of F, first write

$$\begin{aligned} F &= 4E_aE_b(\mathrm{v}_a + \mathrm{v}_b) = 4E_aE_b\left(\frac{\mathrm{p}_a}{E_a} + \frac{\mathrm{p}_b}{E_b}\right) = 4(E_a\mathrm{p}_b + E_b\mathrm{p}_a), \\ \Rightarrow \qquad F^2 &= 16(E_a^2\mathrm{p}_b^2 + E_b^2\mathrm{p}_a^2 + 2E_aE_b\mathrm{p}_a\mathrm{p}_b), \end{aligned} \tag{3.28}$$

and then note that, for the case where the incident particle velocities are collinear,

$$(p_a\cdot p_b)^2 = (E_aE_b + \mathrm{p}_a\mathrm{p}_b)^2 = E_a^2E_b^2 + \mathrm{p}_a^2\mathrm{p}_b^2 + 2E_aE_b\mathrm{p}_a\mathrm{p}_b. \tag{3.29}$$

Substituting the expression for $2E_aE_b\mathrm{p}_a\mathrm{p}_b$ from (3.29) into (3.28) then gives

$$F^2 = 16\left[(p_a\cdot p_b)^2 - (E_a^2 - \mathrm{p}_a^2)(E_b^2 - \mathrm{p}_b^2)\right].$$

Thus, F can be written in the manifestly Lorentz-invariant form

$$F = 4\left[(p_a \cdot p_b)^2 - m_a^2 m_b^2\right]^{\frac{1}{2}}.$$

Since both F and the integral in (3.27) are Lorentz invariant, it can be concluded that the cross section for an interaction is itself Lorentz invariant.

3.4.2 Scattering in the centre-of-mass frame

Because the interaction cross section is a Lorentz-invariant quantity, the cross section for the process $a + b \to 1 + 2$ can be calculated in any frame. The most convenient choice is the centre-of-mass frame where $\mathbf{p}_a = -\mathbf{p}_b = \mathbf{p}_i^*$ and $\mathbf{p}_1 = -\mathbf{p}_2 = \mathbf{p}_f^*$, and the centre-of-mass energy is given by $\sqrt{s} = (E_a^* + E_b^*)$. In the centre-of-mass frame, the Lorentz-invariant flux factor is

$$\begin{aligned} F &= 4E_a^* E_b^* (\mathrm{v}_a^* + \mathrm{v}_b^*) = 4E_a^* E_b^* \left(\frac{\mathrm{p}_i^*}{E_a^*} + \frac{\mathrm{p}_i^*}{E_b^*}\right) = 4\mathrm{p}_i^* (E_a^* + E_b^*) \\ &= 4\mathrm{p}_i^* \sqrt{s}. \end{aligned}$$

Using this expression and the constraint that $\mathbf{p}_a + \mathbf{p}_b = \mathbf{0}$, (3.27) becomes

$$\sigma = \frac{1}{(2\pi)^2} \frac{1}{4\mathrm{p}_i^* \sqrt{s}} \int |\mathcal{M}_{fi}|^2 \delta\left(\sqrt{s} - E_1 - E_2\right) \delta^3(\mathbf{p}_1 + \mathbf{p}_2) \frac{\mathrm{d}^3\mathbf{p}_1}{2E_1} \frac{\mathrm{d}^3\mathbf{p}_2}{2E_2}. \tag{3.30}$$

The integral in (3.30) is the same as that of (3.21) with m_a replaced by $\sqrt{s}$. Therefore, applying the results from Section 3.3.1 immediately leads to

$$\sigma = \frac{1}{16\pi^2 \mathrm{p}_i^* \sqrt{s}} \times \frac{\mathrm{p}_f^*}{4\sqrt{s}} \int |\mathcal{M}_{fi}|^2 \mathrm{d}\Omega^*,$$

where the solid angle element has been written as $\mathrm{d}\Omega^*$ to emphasise that it refers to the centre-of-mass frame. Hence the cross section for *any* two-body $\to$ two-body process is given by

$$\sigma = \frac{1}{64\pi^2 s} \frac{\mathrm{p}_f^*}{\mathrm{p}_i^*} \int |\mathcal{M}_{fi}|^2 \mathrm{d}\Omega^*. \tag{3.31}$$

3.5 Differential cross sections

In many cases it is not only the total cross section that is of interest, but also the distribution of some kinematic variable. For example, Figure 3.7 shows the inelastic

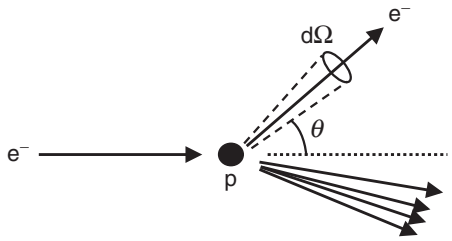

Fig. 3.7 An example of $e^- p \to e^- p$ scattering where the electron is scattered into a solid angle $d\Omega$.

scattering process $e^- p \to eX$ where the proton breaks up. Here, the angular distribution of the scattered electron provides essential information about the fundamental physics of the interaction. In this case, the relevant experimental measurement is the *differential* cross section for the scattering rate into an element of solid angle $d\Omega = d(\cos\theta)d\phi$,

$$\frac{d\sigma}{d\Omega} = \frac{\text{number of particles scattered into } d\Omega \text{ per unit time per target particle}}{\text{incident flux}}.$$

The integral of the differential cross section gives the total cross section,

$$\sigma = \int \frac{d\sigma}{d\Omega} d\Omega.$$

Differential cross sections are not restricted to angular distributions. In some situations, it is the energy distribution of the scattered particle that is sensitive to the underlying fundamental physics. In other situations one might be interested in the joint angular and energy distribution of the scattered particles. In each case, it is possible to define the corresponding differential cross section, for example

$$\frac{d\sigma}{dE} \quad \text{or} \quad \frac{d^2\sigma}{dE d\Omega}.$$

3.5.1 Differential cross section calculations

Differential cross sections can be calculated from the differential form of (3.31),

$$d\sigma = \frac{1}{64\pi^2 s} \frac{p_f^*}{p_i^*} |\mathcal{M}_{fi}|^2 d\Omega^*. \tag{3.32}$$

The simplest situation is where the laboratory frame corresponds to the centre-of-mass frame, for example e^+e^- annihilation at LEP or pp collisions at the LHC.

In this case, the differential cross section expressed in terms of the angles of one of the final-state particles is immediately obtained from (3.32)

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega^*} = \frac{1}{64\pi^2 s}\frac{\mathrm{p}_f^*}{\mathrm{p}_i^*}|\mathcal{M}_{fi}|^2. \tag{3.33}$$

In fixed-target experiments, such as $\mathrm{e}^-\mathrm{p} \to \mathrm{e}^-\mathrm{p}$ elastic scattering, where the target proton is at rest, the laboratory frame is not the centre-of-mass frame and the calculation is more involved. Here, the differential cross section is most useful when expressed in terms of the observable laboratory frame quantities, such as the angle through which the electron is scattered, θ. The differential cross section with respect to the laboratory frame electron scattering angle can be obtained by applying the appropriate coordinate transformation to (3.32).

The transformation from the differential cross section in the centre-of-mass frame to that in the laboratory frame is most easily obtained by first writing (3.32) in a Lorentz-invariant form, which is applies in all frames. This is achieved by expressing the element of solid angle $\mathrm{d}\Omega^*$ in terms of the Mandelstam variable $t = p_1 - p_3$. For $\mathrm{e}^-\mathrm{p} \to \mathrm{e}^-\mathrm{p}$ scattering, t is a function of the initial- and final-state electron four-momenta. Using the definitions of the particle four-momenta shown in Figure 3.8,

$$\begin{aligned} t = (p_1^* - p_3^*)^2 &= p_1^{*2} + p_3^{*2} - 2p_1^*\cdot p_3^* \\ &= m_1^2 + m_3^2 - 2(E_1^*E_3^* - \mathbf{p}_1^*\cdot\mathbf{p}_3^*) \\ &= m_1^2 + m_3^2 - 2E_1^*E_3^* + 2\mathrm{p}_1^*\mathrm{p}_3^*\cos\theta^*. \end{aligned} \tag{3.34}$$

In the centre-of-mass frame, the magnitude of the momenta and the energies of the final-state particles are fixed by energy and momentum conservation and the only free parameter in (3.34) is the electron scattering angle θ^*, thus

$$\mathrm{d}t = 2\mathrm{p}_1^*\mathrm{p}_3^*\,\mathrm{d}(\cos\theta^*),$$

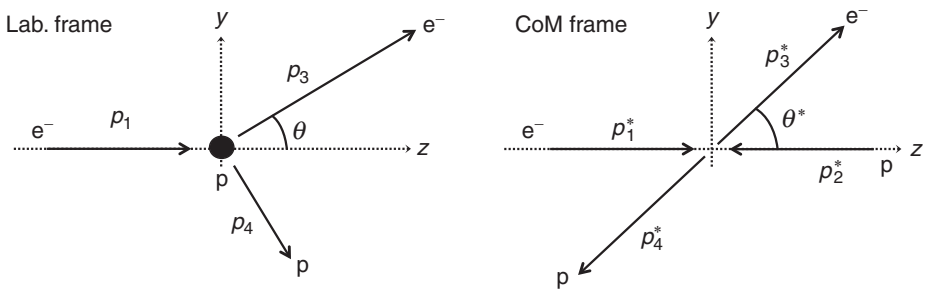

Fig. 3.8 The process of $\mathrm{e}^-\mathrm{p} \to \mathrm{e}^-\mathrm{p}$ elastic scattering shown in the laboratory (left) and centre-of-mass (right) frames.

and therefore

$$\mathrm{d}\Omega^* \equiv \mathrm{d}(\cos\theta^*)\,\mathrm{d}\phi^* = \frac{\mathrm{d}t\,\mathrm{d}\phi^*}{2\mathrm{p}_1^*\mathrm{p}_3^*}. \tag{3.35}$$

Writing p_1^* and p_3^* respectively as p_i^* and p_f^*, and substituting (3.35) into (3.32) leads to

$$\mathrm{d}\sigma = \frac{1}{128\pi^2 s\,\mathrm{p}_i^{*2}}|\mathcal{M}_{fi}|^2\mathrm{d}\phi^*\mathrm{d}t. \tag{3.36}$$

Assuming that matrix element is independent of the azimuthal angle, the integration over $\mathrm{d}\phi^*$ just introduces a factor of 2π and therefore

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{1}{64\pi s\,\mathrm{p}_i^{*2}}|\mathcal{M}_{fi}|^2. \tag{3.37}$$

The magnitude of the momentum of the initial-state particles in the centre-of-mass frame can be shown to be

$$\mathrm{p}_i^{*2} = \frac{1}{4s}[s-(m_1+m_2)^2][s-(m_1-m_2)^2]. \tag{3.38}$$

Since σ, s, t and $|\mathcal{M}_{fi}|^2$ are all Lorentz-invariant quantities, Equation (3.37) gives a general Lorentz-invariant expression for the differential cross section for the two-body $\rightarrow$ two-body scattering process.

3.5.2 Laboratory frame differential cross section

Because (3.37) is valid in all rest frames, it can be applied directly to the example of $\mathrm{e}^-\mathrm{p} \rightarrow \mathrm{e}^-\mathrm{p}$ elastic scattering in the laboratory frame, shown in Figure 3.8. In the limit where the incident and scattered electron energies are much greater than the electron rest mass, the laboratory frame four-momenta of the particles are

$$\begin{aligned} & p_1 \approx (E_1, 0, 0, E_1), \\ & p_2 = (m_\mathrm{p}, 0, 0, 0), \\ & p_3 \approx (E_3, 0, E_3\sin\theta, E_3\cos\theta), \\ \text{and} \quad & p_4 = (E_4, \mathbf{p}_4). \end{aligned}$$

The momenta of the initial-state particles in the centre-of-mass frame are given by (3.38) and since $m_\mathrm{e} \ll m_\mathrm{p}$,

$$\mathrm{p}_i^{*2} \approx \frac{(s-m_\mathrm{p}^2)^2}{4s}, \tag{3.39}$$

where s is given by

$$s = (p_1 + p_2)^2 = p_1^2 + p_2^2 + 2p_1 \cdot p_2 \approx m_\mathrm{p}^2 + 2p_1 \cdot p_2 \\ = m_\mathrm{p}^2 + 2E_1 m_\mathrm{p},$$

and therefore

$$\mathrm{p}_i^{*2} = \frac{E_1^2 m_\mathrm{p}^2}{s}. \tag{3.40}$$

The differential cross section in terms of the laboratory frame scattering angle of the electron can be obtained from

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\mathrm{d}\sigma}{\mathrm{d}t}\left|\frac{\mathrm{d}t}{\mathrm{d}\Omega}\right| = \frac{1}{2\pi}\frac{\mathrm{d}t}{\mathrm{d}(\cos\theta)}\frac{\mathrm{d}\sigma}{\mathrm{d}t}, \tag{3.41}$$

where the factor 2π arises from the integral over $\mathrm{d}\phi$ (again assuming azimuthal symmetry). An expression for $\mathrm{d}t/\mathrm{d}(\cos\theta)$ can be obtained by writing the Mandelstam variable t in terms of the laboratory frame four-momenta, defined above,

$$t = (p_1 - p_3)^2 \approx -2E_1 E_3(1 - \cos\theta), \tag{3.42}$$

where E_3 is itself a function of θ. Conservation of energy and momentum imply that $p_1 + p_2 = p_3 + p_4$, and t can also be expressed in terms of the four-momenta of the initial and final-state proton,

$$t = (p_2 - p_4)^2 = 2m_\mathrm{p}^2 - 2p_2 \cdot p_4 = 2m_\mathrm{p}^2 - 2m_\mathrm{p}E_4 = -2m_\mathrm{p}(E_1 - E_3), \tag{3.43}$$

where the last step follows from energy conservation, $E_4 = E_1 + m_\mathrm{p} - E_3$. Equating (3.42) and (3.43) gives the expression for E_3 as a function of $\cos\theta$,

$$E_3 = \frac{E_1 m_\mathrm{p}}{m_\mathrm{p} + E_1 - E_1\cos\theta}. \tag{3.44}$$

Because E_1 is the fixed energy of the initial-state electron, differentiating (3.43) with respect to $\cos\theta$ gives

$$\frac{\mathrm{d}t}{\mathrm{d}(\cos\theta)} = 2m_\mathrm{p}\frac{\mathrm{d}E_3}{\mathrm{d}(\cos\theta)}. \tag{3.45}$$

Differentiating the expression for E_3 of (3.44), gives

$$\frac{\mathrm{d}E_3}{\mathrm{d}(\cos\theta)} = \frac{E_1^2 m_\mathrm{p}}{(m_\mathrm{p} + E_1 - E_1\cos\theta)^2} = \frac{E_3^2}{m_\mathrm{p}},$$

which when substituted into (3.45) leads to

$$\frac{\mathrm{d}t}{\mathrm{d}(\cos\theta)} = 2E_3^2. \tag{3.46}$$

Substituting (3.46) into (3.41), and using the Lorentz-invariant expression for the differential cross section of (3.37) gives

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{2\pi} 2E_3^2 \frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{E_3^2}{64\pi^2 s\, \mathrm{p}_i^{*2}} |\mathcal{M}_{fi}|^2.$$

The momentum of the intial-state particles in the centre-of-mass frame can be eliminated using (3.40) and thus

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2} \left(\frac{E_3}{m_\mathrm{p} E_1} \right)^2 |\mathcal{M}_{fi}|^2. \tag{3.47}$$

Finally, the energy of the scattered electron E_3 can be expressed in terms of $\cos\theta$ alone using (3.44). Therefore the differential cross section can be written as an explicit function of $\cos\theta$ and the energy of the incident electron

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2} \left(\frac{1}{m_\mathrm{p} + E_1 - E_1 \cos\theta} \right)^2 |\mathcal{M}_{fi}|^2. \tag{3.48}$$

The same calculation including the mass of the electron is algebraically more involved, although the steps are essentially the same.

Summary

The general expression for the decay rate $a \to 1 + 2$ is

$$\Gamma = \frac{\mathrm{p}^*}{32\pi^2 m_a^2} \int |\mathcal{M}_{fi}|^2 \mathrm{d}\Omega, \tag{3.49}$$

where p^* is the magnitude of the momentum of the final-state particles in the rest frame of the decaying particle, which is given by

$$p^* = \frac{1}{2m_i} \sqrt{\left[(m_i^2 - (m_1 + m_2)^2 \right] \left[m_i^2 - (m_1 - m_2)^2 \right]}.$$

The expression for the differential cross section for the process $a + b \to c + d$ in the centre-of-mass frame is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega^*} = \frac{1}{64\pi^2 s} \frac{\mathrm{p}_f^*}{\mathrm{p}_i^*} |\mathcal{M}_{fi}|^2, \tag{3.50}$$

where p_i^* and p_f^* are respectively the magnitudes of the initial- and final-state momenta in the centre-of-mass frame. In the limit where the electron mass can

be neglected, the differential cross section for $e^- p \to e^- p$ elastic scattering, in the proton rest frame is

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\left(\frac{E_3}{m_p E_1}\right)^2 |\mathcal{M}_{fi}|^2, \tag{3.51}$$

where E_3 is a function of the electron scattering angle.

Problems

3.1 Calculate the energy of the μ^- produced in the decay at rest $\pi^- \to \mu^- \bar{\nu}_\mu$. Assume $m_\pi = 140$ GeV, $m_\mu = 106$ MeV and take $m_\nu \approx 0$.

3.2 For the decay $a \to 1 + 2$, show that the momenta of both daughter particles in the centre-of-mass frame p^* are

$$p^* = \frac{1}{2m_a}\sqrt{\left[(m_a^2 - (m_1 + m_2)^2\right]\left[m_a^2 - (m_1 - m_2)^2\right]}.$$

3.3 Calculate the branching ratio for the decay $K^+ \to \pi^+\pi^0$, given the partial decay width $\Gamma(K^+ \to \pi^+\pi^0) = 1.2 \times 10^{-8}$ eV and the mean kaon lifetime $\tau(K^+) = 1.2 \times 10^{-8}$ s.

3.4 At a future e^+e^- linear collider operating as a Higgs factory at a centre-of-mass energy of $\sqrt{s} = 250$ GeV, the cross section for the process $e^+e^- \to HZ$ is 250 fb. If the collider has an instantaneous luminosity of 2×10^{34} cm^{-2} s^{-1} and is operational for 50% of the time, how many Higgs bosons will be produced in five years of running?

Note: 1 femtobarn $\equiv 10^{-15}$ b.

3.5 The total $e^+e^- \to \gamma \to \mu^+\mu^-$ annihilation cross section is $\sigma = 4\pi\alpha^2/3s$, where $\alpha \approx 1/137$. Calculate the cross section at $\sqrt{s} = 50$ GeV, expressing your answer in both natural units and in barns (1 barn $= 10^{-28}$ m^2). Compare this to the total pp cross section at $\sqrt{s} = 50$ GeV which is approximately 40 mb and comment on the result.

3.6 A 1 GeV muon neutrino is fired at a 1 m thick block of iron ($^{56}_{26}$Fe) with density $\rho = 7.874 \times 10^3$ kg m^{-3}. If the average neutrino–nucleon interaction cross section is $\sigma = 8 \times 10^{-39}$ cm^2, calculate the (small) probability that the neutrino interacts in the block.

3.7 For the process $a + b \to 1 + 2$ the Lorentz-invariant flux term is

$$F = 4\left[(p_a \cdot p_b)^2 - m_a^2 m_b^2\right]^{\frac{1}{2}}.$$

In the non-relativistic limit, $\beta_a \ll 1$ and $\beta_b \ll 1$, show that

$$F \approx 4 m_a m_b \left|\mathbf{v}_a - \mathbf{v}_b\right|,$$

where $\mathbf{v}_a$ and $\mathbf{v}_b$ are the (non-relativistic) velocities of the two particles.

3.8 The Lorentz-invariant flux term for the process $a + b \to 1 + 2$ in the centre-of-mass frame was shown to be $F = 4p_i^* \sqrt{s}$, where p_i^* is the momentum of the intial-state particles. Show that the corresponding expression in the frame where b is at rest is

$$F = 4 m_b p_a.$$

3.9 Show that the momentum in the centre-of-mass frame of the initial-state particles in a two-body scattering process can be expressed as

$$\mathrm{p}_i^{*2} = \frac{1}{4s}[s - (m_1 + m_2)^2][s - (m_1 - m_2)^2].$$

3.10 Repeat the calculation of Section 3.5.2 for the process $\mathrm{e}^- \mathrm{p} \longrightarrow \mathrm{e}^- \mathrm{p}$ where the mass of the electron is no longer neglected.

(a) First show that

$$\frac{\mathrm{d}E_3}{\mathrm{d}(\cos\theta)} = \frac{\mathrm{p}_1 \mathrm{p}_3^2}{\mathrm{p}_3(E_1 + m_\mathrm{p}) - E_3 \mathrm{p}_1 \cos\theta}.$$

(b) Then show that

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2} \cdot \frac{\mathrm{p}_3^2}{\mathrm{p}_1 m_\mathrm{p}} \cdot \frac{1}{\mathrm{p}_3(E_1 + m_\mathrm{p}) - E_3 \mathrm{p}_1 \cos\theta} \cdot |\mathcal{M}_{fi}|^2,$$

where (E_1, p_1) and (E_3, p_3) are the respective energies and momenta of the initial-state and scattered electrons as measured in the laboratory frame.