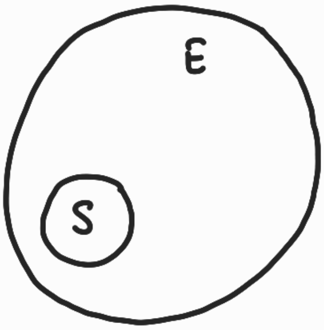
Ref: Theory of Quantum Noise and Decoherence, Lecture 7

THE MASTER EQUATION



$$\mathcal{E}_S(\rho_S) = Tr_E\left(U_{SE}\, \rho_S(0) \otimes \rho_E(0)\, U_{SE}^{+}\right)$$

↳ completely positive map

Assumption: At t_0, $\boxed{\rho_{SE}(0) = \rho_S(0) \otimes \rho_E(0)}$

E.g. Either → System has been insulated from E design at $t = t_0$

→ System and environment are weakly coupled.

Hilbert Space : $\mathcal{H}_S \otimes \mathcal{H}_E$

$\mathcal{H}_S$ ↓ often finite dimensional

$\mathcal{H}_E$ → often infinite # of harmonic oscillators

Dynamics: $H_{tot} = \underbrace{H_S + H_E}_{H_0} + V$

H_0 "free part"

V ↳ interaction btw the system and the env.

Assume, "reservoir" or environment is in termal equilibrium at temperature T:

$$\rho_E(0) = \frac{e^{-\frac{H_E}{k_B T}}}{Tr\left(e^{-H_E/k_B T}\right)} = \frac{e^{-\beta H_E}}{Tr\left(e^{-\beta H_E}\right)}$$

Total state of S+E @ time t:

$$\rho_{tot}(t) = e^{-iH_{tot}t}\left(\rho_S(0)\otimes\rho_E(0)\right)e^{iH_{tot}t}$$

$$\boxed{\frac{d}{dt}\rho_{tot}(t) = -i\left[H_0+V,\ \rho_{tot}(t)\right]}$$

$$\rho_S(t) = \mathrm{Tr}_E\left(\rho_{tot}(t)\right)$$

Assume $\underbrace{H_0 = H_S + H_E}$ is solvable

H_S → finite dim. H_E → quasi-free ?

Move to rotating frame: (interaction picture)

$$V_{int}(t) = e^{iH_0t}\,V\,e^{+iH_0t}$$

$$\rho_{int,tot}(t) = e^{-iH_0t}\,\rho_{tot}(t)\,e^{+iH_0t}$$

$$\boxed{\frac{d}{dt}\rho_{int,tot}(t) = -i\left[V_{int}(t),\ \rho_{int,tot}(t)\right]} \quad (*)$$

$$\frac{d}{dt}\rho_{int,tot}(t) = +i\left(e^{-iH_0t}\left[H_0,\rho_{int,tot}(t)\right]e^{iH_0t}\right) \quad ?\ \text{How?}$$

$$-i\left(e^{-iH_0t}\left[H_0+V,\rho_{int,tot}(t)\right]e^{iH_0t}\right)$$

We drop the "int" subscript. Now $(*)$

$$\rho_{tot}(t) = \int_0^t \frac{d}{ds}\rho_{tot}(s)\, ds + \rho_{tot}(0)$$

$$\rho_{tot}(t) = -i\int_0^t [V(s), \rho_{tot}(s)]\, ds + \rho_{tot}(0) \quad (**)$$

Substituting $(**) \longrightarrow (*)$

$-i$ (?)

$$\frac{d}{dt}\rho_{tot}(t) = -i\left[V(t), \rho_{tot}(0)\right] - \int_0^t dt_1 \left[V(t), [V(t_1), \rho_{tot}(t_1)]\right]$$

$$\frac{d}{dt}\rho_S(t) = -i\, \mathrm{Tr}_E\left([V(t), \rho_{tot}(0)]\right) - \int_0^t dt_1\, \mathrm{Tr}_E\left[V(t), [V(t), \rho_{tot}(t_1)]\right] \quad (***)$$

Assumption 1: $\rho_{tot}(0) = \rho_S(0) \otimes \rho_E(0)$

Split $V(t)$ into $V(t) = V_S(t) + V_{SE}(t)$ where $V_S = V_S \otimes I_E$

and $\mathrm{Tr}_E(V_{SE}(t)\, \rho_{tot}(0)) = 0$

Assumption 2: $V = \lambda H_I$ with λ small

Assumption 3 (Born Approximation)

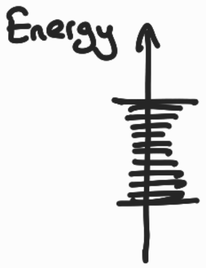
We can replace $\rho_{tot}(t_1)$ in $(***)$ by

$$\rho_{tot}(t_1) = \rho_S(t_1) \otimes \rho_E(t_1) \cong \rho_S(t_1) \otimes \rho_E(0)$$

$$\frac{d}{dt}\rho_S(t) = -i\left[V_S(t), \rho_S(0)\right] - \int_0^t \mathrm{Tr}_E\left(\left[V_{SE}(t), \left[V_{SE}(t_1), \rho_S(t_1) \otimes \rho_E(0)\right]\right]\right) dt_1$$

Nonlocal in time

Assumption 4: The physics of bath is such that system couples roughly equally to many closely spaced energy levels of bath.



$$\rho_S(t_1) \simeq \rho_S(t)$$

$$\frac{d}{dt}\rho_S(t) = -i\left[V_S(t), \rho_S(0)\right] - \int_0^t dt_1\, \mathrm{Tr}_E\left(\left[V_S(t), \left[V_{SE}(t_1), \rho_S(t) \otimes \rho_E(0)\right]\right]\right)$$

"Red-Field Equation"

Assumption 5: (Markov Approximation - Memoryless)

Assume we have memoryless bath and replace lower limit of integral by $-\infty$:

$$\frac{d}{dt}\rho_S(t) = -i\left[V_S(t), \rho_S(0)\right] - \int_{-\infty}^t dt_1\, \mathrm{Tr}_E\left(\left[V_{SE}(t), \left[V_{SE}(t_1), \rho_S(t) \otimes \rho_E(0)\right]\right]\right)$$

"Born-Markov Master Equation"

(Here, energy levels are continous)

Requires H_E to have continous spectra in relevant energy

The radiative-damping master equation:

Following Bohr & Einstein, Wigner & Weisskopf explained how radiative decay of atom from excited state could be explained in Q.M.

Environment : Free-Field

$$H_E = \sum_k \omega_k \hat{b}_k^+ \hat{b}_k \quad \text{with } [\hat{b}_k, \hat{b}_\ell^+] = \delta_{k,\ell}$$

System: $H_S = \omega_a \frac{\sigma^z}{2}$

Interaction : "dipole coupling"

$$V = \sum_k (g_k \hat{b}_k + g_k \hat{b}_k^+)(\sigma^+ + \sigma^-) \quad \text{where}$$

$\sigma^+ = |0\rangle\langle 1|$, $\sigma^- = |1\rangle\langle 0|$ with g_k proportional to dipole matrix element for transition: they depend on structure of mode k:

(?)

$$g_k = \frac{1}{\sqrt{V_k}} \qquad V_k = \text{volume of mode } k$$

$$V_{int}(t) = \sum_k \left(g_k \hat{b}_k e^{-i\omega_k t} + g_k \hat{b}_k^+ e^{i\omega_k t} \right) \left(\sigma^+ e^{i\omega_a t} + \sigma^- e^{-i\omega_a t} \right)$$

$\omega_a \sim 10^{15} s^{-1}$. We approximate Vint by:

$$V_{int}(t) \cong \sum \Big(\underbrace{g_k \hat{b}_k \sigma^+ e^{-i(\omega_k - \omega_a)t}}_{V_+} + \underbrace{g_k \hat{b}_k^+ \sigma^- e^{i(\omega_k - \omega_a)t}}_{V_-} \Big)$$

Rotating Wave Approximation (RWA)

Now choose $\rho_E(0) = |\Omega\rangle\langle\Omega|$ $\rightarrow$ vacuum, no photon

In Born Approximation:

hermitian conjugate

gamma

$$\frac{d}{dt} \rho_s(t) = -\int_0^t dt_1 \left\{ \Gamma(t-t_1) \left(\sigma^+ \sigma^- \rho_s(t_1) - \sigma^- \rho_s(t_1) \sigma^+ \right) + h.c \right\}$$