LIE GROUP & LIE ALGEBRAS

1. Matrix Exponential

$$\exp(A) = I + A + \frac{1}{2}A^2 + \frac{1}{3!}A^3 + \dots = \sum_{n=0}^{\infty} \frac{1}{n!}(A)^n$$

$$A = \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix} \qquad A^2 = \begin{pmatrix} -\theta^2 & 0 \\ 0 & -\theta^2 \end{pmatrix} = -\theta^2 I$$

$$A^3 = A^2 A = -\theta^2 A = \begin{pmatrix} 0 & \theta^3 \\ -\theta^3 & 0 \end{pmatrix}$$

$$A^4 = \theta^4 I$$

$$A^5 = \theta^4 A = \begin{pmatrix} 0 & -\theta^5 \\ \theta^5 & 0 \end{pmatrix}$$

$$\exp(A) = \begin{pmatrix} 1 - \frac{1}{2}\theta^2 + \frac{1}{4}\theta^4 \dots & -\theta + \frac{1}{3!}\theta^3 \dots\dots \\ \theta - \frac{1}{3!}\theta^3 \dots & 1 - \frac{1}{2}\theta^2 + \frac{1}{4}\theta^4 \dots\dots \end{pmatrix}$$

$$\exp(A) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

2D Rotation Matrix

2. Matrix Exponential Properties

$$\exp(A) = \sum_{n \geq 0} \frac{1}{n!} A^n \quad , \; A^0 = I$$

Lemma: a) $\exp(A)$ converges: absolutely $\to$ reorder terms

uniformly along with partial derivatives on any bounded set of As.

$\downarrow$

differentiate exp

b) $\frac{d}{dt}\left(\exp(tA)\right) = A \exp(tA)$

c) If $AB = BA$ then $\exp(A)\exp(B) = \exp(A+B)$

d) $\exp(A)$ is invertible with inverse $\exp(-A)$

Proof b: $\frac{d}{dt} \exp(tA) = \frac{d}{dt} \sum_{n \geq 0} \frac{1}{n!} (tA)^n = \sum_{n \geq 0} \frac{1}{n!} \frac{d}{dt}(t^n A^n)$

$$= \sum \frac{1}{n!} n t^{n-1} A^n = A \sum \frac{1}{(n-1)!} t^{n-1} A^{n-1}$$

$$= A \sum_{n=1}^{\infty} \frac{1}{(n-1)!} (tA)^{n-1}$$

$$A \sum_{m=0}^{\infty} \frac{1}{m!} (tA)^m = A \exp(tA)$$

c $\Rightarrow$ d $\quad A(-A) = (-A)A \overset{c}{\Longrightarrow} \exp(A)\exp(-A) = \exp(0) = I$

Proof of c: $\quad \exp A \exp B = \sum_{m \geq 0} \frac{1}{m!} A^m \sum_{n \geq 0} \frac{1}{n!} B^n$

$$= \sum_m \sum_n \frac{1}{m!\,n!} A^m B^n$$

	$m=0$	1	2
$n=0$	I	A	$\frac{1}{2}A^2$
$n=1$	B	AB	
2	$\frac{1}{2}B^2$		

$k=0$, $k=1$, $k=2$

Cauchy-Product Formula

$$= I + (A+B) + \left(\frac{1}{2}A^2 + AB + \frac{1}{2}B^2\right) + \cdots$$

$$= \sum_{k \geq 0} \underbrace{\sum_{m=0}^{k} \frac{1}{m!\,(k-m)!} A^m B^{k-m}}_{\text{look like binomial expansion}} \quad \leftarrow \text{use } AB = BA$$

$n = k-m$

$$= \sum_{k \geq 0} \frac{1}{k!} \underbrace{\sum_{m=0}^{k} \frac{k!}{m!\,(k-m)!} A^m B^{k-m}}_{(A+B)^k} = \sum_k \frac{1}{k!}(A+B)^k$$

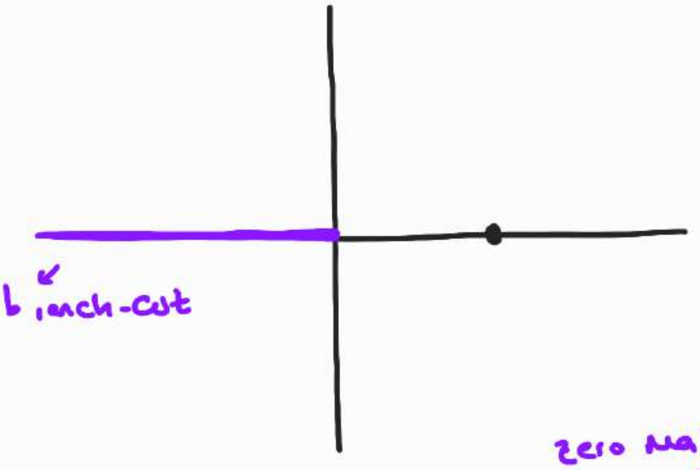
$$= \exp(A+B)$$

$$(A+B)^2 = A^2 + AB + BA + B^2 \overset{AB=BA}{=} A^2 + 2AB + B^2$$

3. Local Logarithm

$$\exp : \underset{\mathfrak{gl}(n,\mathbb{R})}{\text{all } n\times n \text{ matrices}} \underset{\log}{\overset{\longrightarrow}{\longleftarrow}} \underset{GL(n,\mathbb{R})}{\text{invertible } n\times n \text{ matrices}}$$

general linear group



$\log 1 = 0$, $\log(e^{2\pi i}) = 0$

$\log(e^{4\pi i}) = 0$

Theorem: There exists neighborhoods

$0 \in U \subseteq \mathfrak{gl}(n,\mathbb{R})$ & $I \in V \subseteq GL(n,\mathbb{R})$

zero matrix

identity matrix

such that $\exp|_U : U \to V$ is bijective

(inverse) exists

log

Theorem: (Inverse function theorem) Suppose $F : \mathbb{R}^N \to \mathbb{R}^N$ is a differentiable map such that

$$d_0F = \begin{pmatrix} \frac{\partial F_1(0)}{\partial x_1} & \cdots & \frac{\partial F_1(0)}{\partial x_N} \\ \vdots & & \vdots \\ \frac{\partial F_N(0)}{\partial x_1} & \cdots & \frac{\partial F_N(0)}{\partial x_N} \end{pmatrix} \qquad \begin{pmatrix} F_1(x_1,\dots,x_N) \\ \vdots \\ F_N(x_1,\dots,x_N) \end{pmatrix}$$

derivative @ origin

d_0F is invertible. Then, if $y = F(0)$, then there exist neighborhoods $0 \in U \subseteq \mathbb{R}^N$ & $y \in V \subseteq \mathbb{R}^N$ s.t.

$F|_U : U \to V$ is bijection and F^{-1} is differentiable.

$$\boxed{d_y(F^{-1}) = (d_0 F)^{-1}}$$

$$\begin{array}{ccc} \exp: \mathfrak{gl}(n,\mathbb{R}) & \longrightarrow & GL(n,\mathbb{R}) \\ \parallel & & \subseteq \\ \mathbb{R}^{n^2} & \longrightarrow & \mathbb{R}^{n^2} \end{array}$$

$$\begin{bmatrix} \frac{\partial (\exp A)_{11}}{\partial A_{11}}(0) & \frac{\partial (\exp A)_{12}}{\partial A_{12}}(0) & \cdots & \frac{\partial (\exp A)_{n}}{\partial A_{nn}}(0) \\ \frac{\partial (\exp A)_{21}}{\partial A_{11}}(0) & & & \vdots \\ \vdots & & & \\ \frac{\partial (\exp A)_{nn}}{\partial A_{nn}}(0) & \cdots & & \frac{\partial (\exp A)_{nn}}{\partial A_{nn}}(0) \end{bmatrix}$$

$$\exp A = \cancel{I} + A + \frac{1}{2}A^2 \cancel{+} \cdots$$

differentiate nee to "0" yields 0

$$= \begin{bmatrix} \frac{\partial A_{11}}{\partial A_{11}} & \cdots & \frac{\partial A_{n1}}{\partial A_{nn}} \\ \vdots & & \vdots \\ \frac{\partial A_{m}}{\partial A_{11}} & \cdots & \frac{\partial A_{nn}}{\partial A_{nn}} \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

$$F(x) = F(0) + d_0 F(x) + \cdots$$

↓

Taylor Expansion

$d_0 \exp(A) = A$

$d_0 \exp(I) = I$

↓

identity

$$\exp(A) = I + \underbrace{d_0 \exp(A)}_{A} + \cdots$$

$$\log(I + A) = A - \frac{1}{2}A^2 + \frac{1}{3}A^3 - \frac{1}{4}A^4 \cdots$$

4. Baker - Campbell - Handsorff Formula

$$\begin{array}{ccc} \exp: \; gl(n,\mathbb{R}) & \longrightarrow & GL(n,\mathbb{R}) \\ \text{IU} & & \text{IU} \\ U & \xleftarrow{\log} & V \\ \text{neighborhood of origin} & & \text{neigh of identity} \end{array}$$

$$\log(I+X) = X - \frac{1}{2}X^2 + \frac{1}{3}X^3 - \dots$$

$$AB = BA \Rightarrow \exp(A)\exp(B) = \exp(A+B)$$

what if $AB \neq BA$?

$$\log(\exp(A)\exp(B)) = \log\left(\left(1 + A + \frac{1}{2}A^2 + \dots\right)\left(1 + B + \frac{B^2}{2} + \dots\right)\right)$$

$$= \log\left(1 + \overbrace{\left[A + B + AB + \frac{A^2}{2} + \frac{B^2}{2} + \frac{AB^2}{2} + \frac{A^2B}{2} + \dots\right]}^{X}\right)$$

$$= X - \frac{1}{2}X^2 + \frac{1}{3}X^3 - \dots$$

$$= \left(A + B + AB + \cancel{\frac{A^2}{2}} + \cancel{\frac{B^2}{2}} + \dots\right) - \frac{1}{2}\left(\cancel{A^2} + AB + BA + \cancel{B^2} + \dots\right) + \dots$$

$$= A + B + AB - \frac{1}{2}(AB + BA) + \dots$$

$$= A + B + \frac{1}{2}(AB - BA) + \dots$$

"Lie brackets"
→ commutator

$$[A,B] = AB - BA$$

Cubic term $\frac{1}{12}[A,[A,B]] - \frac{1}{12}[B,[A,B]]$

BCH Formula:

$$\exp(A)\exp(B) = \exp\left(A+B+\frac{1}{2}[A,B] + \cdots\cdots \right)$$

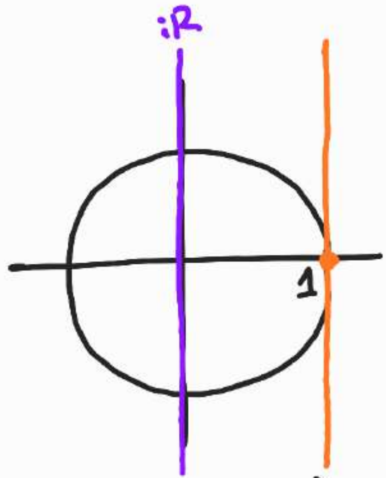
5. Lie Algebra of a Matrix Group

$$\exp: gl(n,\mathbb{R}) \longrightarrow GL(n,\mathbb{R})$$

- locally invertible
- $\exp A \exp B = \exp($ smt. determined by $A, B, [\,,\,])$

Goal: $\exp: g \longrightarrow G$ & $[\,,\,]$ on g

Ex $G = U(1) = \{ z \in \mathbb{C}, |z| = 1 \}$



$z \in U(1)$ can be written as $e^{i\theta}$

$$\exp: i\mathbb{R} \longrightarrow U(1)$$
$$i\theta \longrightarrow e^{i\theta}$$

$$e^{i\theta_1} e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$$

$$[i\theta_1, i\theta_2] = i\theta_1 i\theta_2 - i\theta_2 i\theta_1 = 0$$

$i\mathbb{R}$ is parallel to target line to $U(1)$ at identity (1)

Theorem: Let G be a topologically closed subgroup of $GL(n,\mathbb{R})$. Define $\mathfrak{g} = \{ X \in \mathfrak{gl}(n,\mathbb{R}) : \exp(tX) \in G \ \forall t \in \mathbb{R} \}$.

$\to$ LIE ALGEBRA of G.

Then:

1. $\mathfrak{g}$ is a vector space.

2. $X, Y \in \mathfrak{g} \Rightarrow [X,Y] \in \mathfrak{g}$

3. $\mathfrak{g}$ is parallel to tangent space of G at I.

4. $\exp : \mathfrak{g} \longrightarrow G$ is locally invertible.

Remark: 1. Topologically closed subgroups of $GL(n,\mathbb{R})$ are Lie Groups. These are examples of Lie Groups which are not Matrix groups but most interesting examples are.

2. Topologically closed means if $g_1, g_2, \dots$ is a sequence of elements of G such that g_k converges in $GL(n,\mathbb{R})$ then $\lim g_k \in G$

Ex $GL(n,\mathbb{Q})$ (rational) $\to$ NOT CLOSED

$$\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 3.1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 3.14 & 0 \\ 0 & 1 \end{pmatrix}, \dots, \begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix}$$

(irrational)

3. If $G \subseteq GL(n,\mathbb{R})$ is a subgroup then $\bar{G}$ is a top closed subgroup.

5.1 O(n), Orthogonal Matrices

$\{A : A^TA = I\}$

$\langle v, w\rangle = v^Tw \qquad (Av)^TAw = v^T\underbrace{A^TA}_{I}w = v^Tw$

$o(n) = \{X : \exp(tX) \in O(n) \ \forall t \in \mathbb{R}\}$

$(\exp(tX))^T \cdot \exp(tX) = I$

$\exp(tX^T) \cdot \exp(tX) = I$

$\exp(tX)^{-1} = \exp(-tX)$

$\exp(tX^T) = \exp(-tX) \ \forall t \iff X \in o(n)$

If $X^T = -X$ then $\exp(tX^T) = \exp(-tX)$ so $X \in o(n)$

→ If t is very small then tX^T & tX are near to the zero matrix. So $\exp(tX^T)$ & $\exp(-tX)$ are near to the identity (I). So taking logs, we get $-tX = tX^T$

$\Downarrow$

$-X = X^T$

$\Rightarrow o(n) = \{X : X^T = -X\}$ anti symmetric matrices

$$\exp\underbrace{\begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix}}_{X} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in O(2)$$

$$X^T = -X$$

other way rather then using logs:

$$\exp(tX^T)\exp(tX) = I$$

$$\frac{dI}{dt} = 0 = \frac{d}{dt}\exp(tX^T)\exp(tX) = X^T\exp(tX^T)\exp(tX) + \exp(tX^T)X\exp(tX) \quad \forall t$$

$t=0 \quad 0 = X^T + X \longrightarrow X^T = -X$

1. $O(n)$ is topologically closed. $F: gl(n,\mathbb{R}) \longrightarrow gl(n,\mathbb{R})$

$A \longrightarrow A^T A$

$$O(n) = F^{-1}(I)$$

F is continous so if $A_k \in O(n)$ then $F(A_k) = I$

so $\lim_{k\to\infty} F(A_k) = I = F(\lim_{k\to\infty} A_k)$

2. $X, Y \in o(n) \Rightarrow [X,Y] = XY - YX \in o(n)$

5.2 $SL(2,\mathbb{C})$

S = "special" $\Leftrightarrow$ $\det A = 1$

$$sl(2,\mathbb{C}) = \left\{ \begin{pmatrix} a, b \\ c, d \end{pmatrix} : \det \exp\left(t \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = 1 \quad \forall t \right\}$$

$$\det \exp t \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \det \left[\underbrace{\begin{pmatrix} 1+ta & tb \\ tc & 1+td \end{pmatrix}}_{I + X + \frac{1}{2}\cancel{X^2}} + \underbrace{O(t^2)}_{\text{higher terms order}} \right]$$

$$= (1+ta)(1+td) - t^2 cb + O(t^2)$$

$$= 1 + \underbrace{t(a+d) + O(t^2)}_{\text{they should vanish}} = 1 \quad \forall t$$

$$\left.\frac{d}{dt}\right|_{t=0} \det \exp\left(t \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = (a+d) + \underbrace{\dots \cancel{t}}_{t=0} = (a+d) = \frac{dI}{dt} = 0$$

$$\boxed{a+d=0}$$

$$\Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in sl(2,\mathbb{C}) \Leftrightarrow a+d=0$$

So $sl(2,\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a+d=0 \right\}$

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$aH + bX + cY \rightarrow$ BASIS

$$* \ [H, X] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$$

$$[H, X] = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = 2X$$

$$* \ [H, Y] = \begin{pmatrix} 0 & 0 \\ -2 & 0 \end{pmatrix} = -2Y$$

$$* \ [X, Y] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = H$$

$\exp(tH) \in SL(2, \mathbb{C}) \quad \forall t$

$$\exp\begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \qquad \det \exp(tH) = e^t . e^{-t} = 1 \quad \forall t$$

$$sl(2, \mathbb{C}) = \{ \text{tracefree } 2\times 2 \text{ Matrices} \}$$

6. Lie Algebras

<u>Def:</u> A Lie algebra is vector space $\mathfrak{g}$ equipped with a map $[\,,\,]: \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ such that

1. $[\,,\,]$ is bilinear i.e.

$$[aX + bY, Z] = a[X,Z] + b[Y,Z]$$

$$[X, aY + bZ] = a[X,Y] + b[X,Z]$$

2. $[X,X] = 0 \quad \forall X \in \mathfrak{g}$

3. Jacobi Identity $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$

$$\forall X, Y, Z \in \mathfrak{g}$$

All three hold for $[A,B] = AB - BA$ on matrices

2. $\rightarrow$ $[X,Y] = -[Y,X]$

$$\underbrace{[X+Y, X+Y]}_{0} = \cancel{[X,X]}_{0} + \cancel{[Y,Y]}_{0} + \underbrace{[X,Y] + [Y,X]}_{} = 0$$

<u>Def:</u> A Lie subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ is a subspace st

$$\forall X, Y \in \mathfrak{h} \quad [X,Y] \in \mathfrak{h}$$

All our examples are subalgebras of $\mathfrak{gl}(n, \mathbb{R})$.