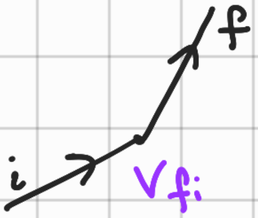
WEEK 8 Interaction by Particle Exchange and QED

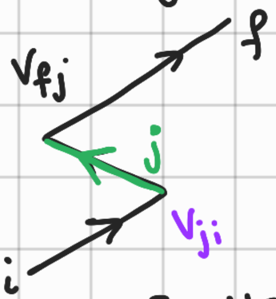
Interaction by particle exchange

Fermi's Golden Rule

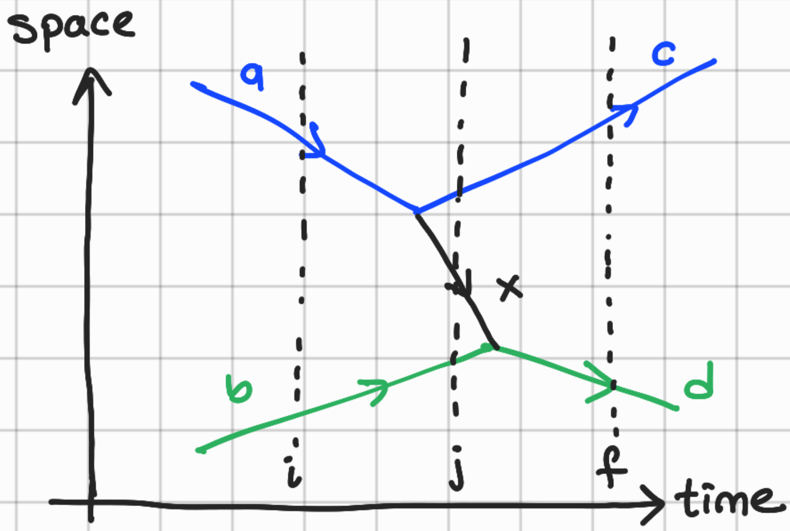
$$\Gamma_{fi} = 2\pi \, |T_{fi}|^2 \, \rho(E_f)$$

$$T_{fi} = \langle f|V|i\rangle + \sum_{j\neq i} \frac{\langle f|V|j\rangle\langle j|V|i\rangle}{E_i - E_j} + \dots$$





Scattering via an intermediate state



One possible picture

$$\underbrace{a+b}_{\text{Initial}} \longrightarrow \underbrace{c+d}_{\text{Final}}$$

$j : c+b+x$

↳ Intermediate state

$a \to c+x$

$b+x \to d$

a emits x before b absorbs it.

$$T_{fi} = \frac{\langle f|V|j\rangle\langle j|V|i\rangle}{E_i - E_j}$$

⇝ $E_i \neq E_j$, allowed by energy-time uncertainity relation

a emits x before b absorbs it

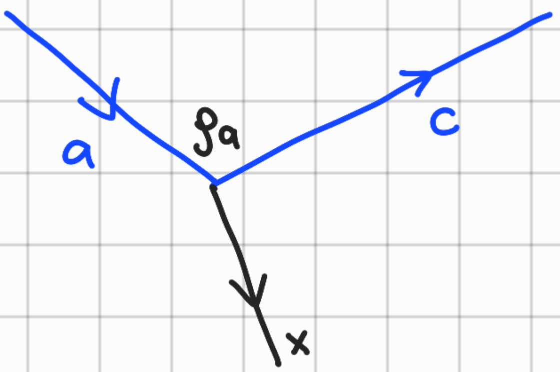
$$T_{fi}^{ab} = \frac{\langle d|V|x+b\rangle\langle c+x|V|a\rangle}{(E_a+E_b) - (E_c+E_x+E_b)}$$

! $V_{ji} = \langle c+x|V|a\rangle$ (and $V_{fi} = \langle d|V|x+b\rangle$ are NON-INVARI. Matrix elements

$$V_{fi} = \prod_k (2E_k)^{-1/2} \mathcal{M}_{fi}$$

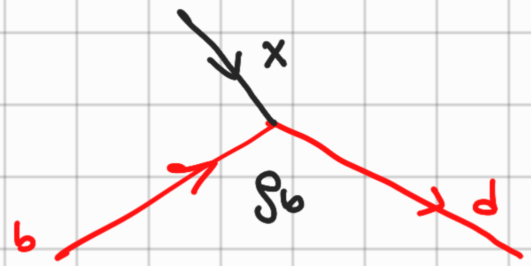
Lorentz invariant

$$\langle c+x|V|a\rangle = \frac{\mathcal{M}(a \to c+x)}{(2E_a 2E_c 2E_x)^{1/2}}$$



scalar L.I quantity g_a

$$\langle c+x|V|a\rangle = \frac{g_a}{(2E_a 2E_c 2E_x)^{1/2}}$$



$$\langle d | V | b+x \rangle = \frac{g_b}{(2E_a \, 2E_d \, 2E_x)^{1/2}}$$

$$T_{fi}^{ab} = \frac{\langle d | V | x+b \rangle \langle c+x | V | a \rangle}{(E_a + E_b) - (E_c + E_x + E_b)}$$

$$\downarrow$$

$$T_{fi}^{ab} = \frac{1}{2E_x} \frac{1}{(2E_a \, 2E_b \, 2E_c \, 2E_d)^{1/2}} \cdot \frac{g_a g_b}{(E_a - E_c - E_x)}$$

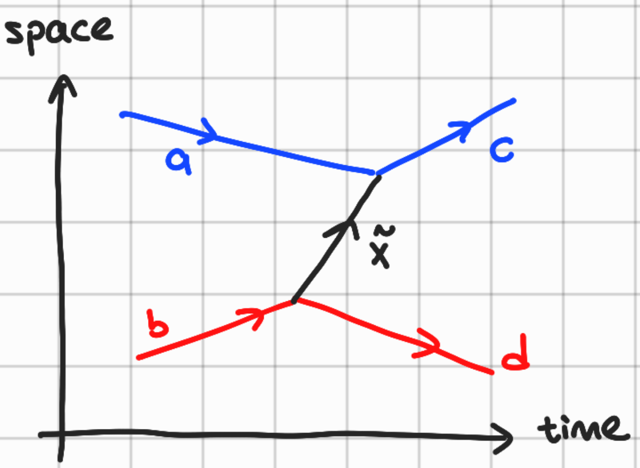
Matrix element for the entire process

$$M_{fi}^{ab} = (2E_a \, 2E_b \, 2E_c \, 2E_d)^{1/2} \, T_{fi}^{ab} = \frac{1}{2E_x} \cdot \frac{g_a g_b}{(E_a - E_c - E_x)}$$

Note that:

Momentum is conserved at each interaction vertex but not energy.

Let's consider the other time orderings



$$\mathcal{M}_{fi}^{ba} = \frac{1}{2E_x} \cdot \frac{g_a g_b}{(E_b - E_d - E_x)}$$

$$E_a + E_b = E_c + E_d$$

$$\boxed{\mathcal{M}_{fi} = \mathcal{M}_{fi}^{ab} + \mathcal{M}_{fi}^{ba}} = \frac{g_a g_b}{2E_x}\left(\frac{1}{E_a - E_c - E_x} + \frac{1}{\underbrace{E_b - E_d}_{-(E_a - E_c)} - E_x}\right)$$

$$= \frac{g_a g_b}{2E_x}\left(\frac{1}{E_a - E_c - E_x} - \frac{1}{E_a - E_c + E_x}\right)$$

$$\cancel{E_a} - \cancel{E_c} + E_x - (\cancel{E_a} - \cancel{E_c} - E_x)$$

$$\mathcal{M}_{fi} = \frac{g_a g_b}{2E_x} \frac{2E_x}{(E_a - E_c)^2 - E_x^2} = \boxed{\frac{g_a g_b}{(E_a - E_c)^2 - E_x^2}}$$

For both time orderings $\rightarrow \vec{p}_x = \vec{p}_b - \vec{p}_d = -(\vec{p}_a - \vec{p}_c)$

$$\boxed{E_x^2 = p_x^2 + M_x^2 = (\vec{p}_a - \vec{p}_c)^2 + M_x^2}$$

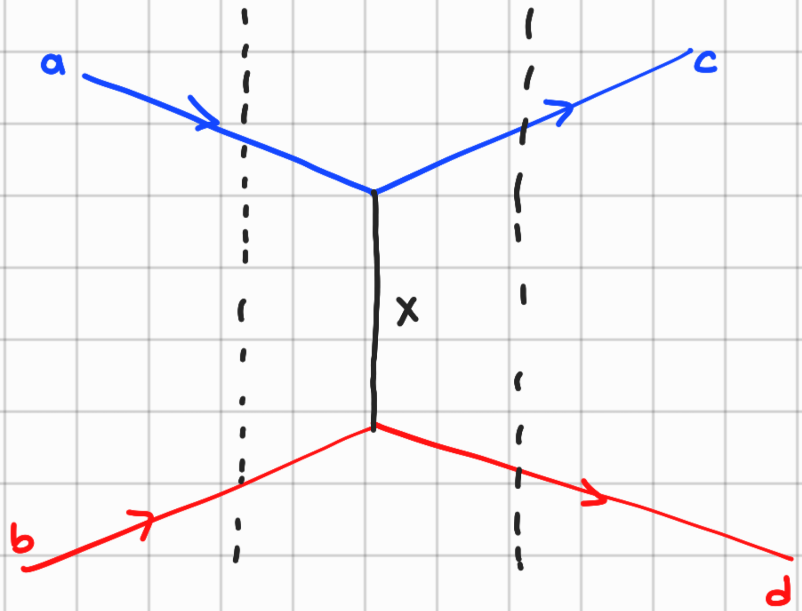
$$M_{fi} = \frac{g_a g_b}{(E_a - E_c)^2 - (\vec{p_a} - \vec{p_c})^2 - M_x^2} = \frac{g_a g_b}{\underbrace{(p_a - p_c)^2}_{\text{4-momentum}} - M_x^2}$$

$$\boxed{M_{fi} = \frac{g_a g_b}{q^2 - m^2}} \quad , \quad q = p_a - p_c$$

Lorentz Invariant

$\frac{1}{q^2 - M^2}$ (propagator) associated to the exchange particle

Note: Feynman Diagrams



Feynman Diagram

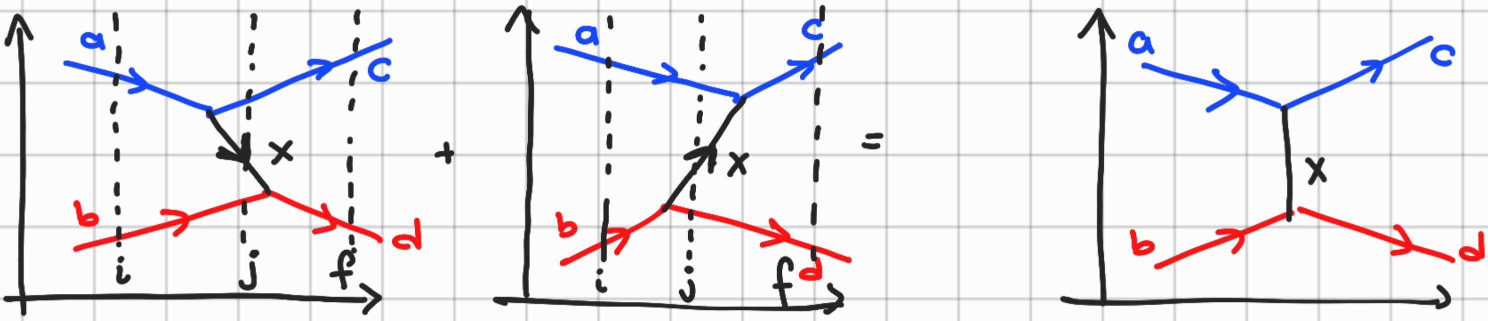
→ Energy and Momentum are conserved at each int. vertex

→ $\frac{1}{q^2 - M^2}$ is the propagator

Virtual Particles

Time-Ordered QM

Feynman Diagram



- momentum conserved @ vertices
- energy NOT conserved @ vertices
- exchanged particle "on mass shell"

$$E_x^2 - |\vec{p}_x|^2 = m_x^2$$

$$M_{fi} = \frac{g_a g_b}{q^2 - m_x^2}$$

- momentum & energy conserved @ vertices
- exchanged particle "off mass shell" virtual particle

$$E_x^2 - |\vec{p}_x|^2 = q^2 \neq m_x^2$$

Quantum Electrodynamics - QED

Interaction of an electron and τ lepton by the exchange of a photon

$$\vec{p} \rightarrow \vec{p} - q\vec{A} \qquad E \longrightarrow E - q\phi \qquad (q \rightarrow \text{charge})$$

in QM: $\vec{p} = -i\vec{\nabla}$ $\qquad E = i\dfrac{\partial}{\partial t}$

Therefore make substitution $i\partial_\mu \longrightarrow i\partial_\mu - qA_\mu$

where $A_\mu = (\phi, -\vec{A})$; $\partial_\mu = \left(\dfrac{\partial}{\partial t}, +\vec{\nabla}\right)$

Dirac Equation:

$$i\gamma^\mu \partial_\mu \Psi - m\Psi = 0 \Rightarrow i\gamma^\mu\partial_\mu\Psi - q\gamma^\mu A_\mu\Psi - m\Psi = 0$$

$$i\gamma^0 \frac{\partial\Psi}{\partial t} + i\vec{\gamma}\,\vec{\nabla}\Psi - q\gamma^\mu A_\mu\Psi - m\Psi = 0$$

$$i\gamma^0\frac{\partial\Psi}{\partial t} = \gamma^0\hat{H}\Psi = m\Psi - i\vec{\gamma}\,\vec{\nabla}\Psi + q\gamma^\mu A_\mu\Psi$$

$\times\gamma^0$ $\qquad \gamma^0\gamma^0 = I$

$$\hat{H}\Psi = \underbrace{(\gamma^0 m - i\gamma^0\vec{\gamma}\,\vec{\nabla})\Psi}_{\text{combined rest mass + KE}} + \underbrace{q\gamma^0\gamma^\mu A_\mu\Psi}_{\text{potential energy}}$$

$$\hat{V}_D = q\gamma^0\gamma^\mu A_\mu$$

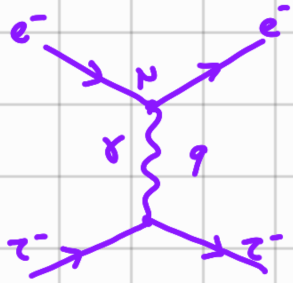
Note that: $q\gamma^0\gamma^0 A_0 = q\phi$

free photon field $\quad A_\mu = \epsilon_\mu^{(\lambda)} e^{i(\vec{p}\cdot\vec{r} - Et)}$

$$\epsilon^{(1)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} x \qquad \epsilon^{(2)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} y$$

* For a real photon, the polarization vector is always transverse to the direction of motion.

Two states of a real photon propagating in z direction



$$\epsilon_- = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -i \\ 0 \end{pmatrix} \qquad \epsilon_+ = -\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ i \\ 0 \end{pmatrix}$$

Circularly Polarized

$$\langle \psi(p_3) | \hat{V}_D | \psi(p_1) \rangle = v_e^+(p_3)\, q_e \gamma^0\gamma^\mu \varepsilon_\mu^{(\lambda)} v_e(p_1)$$

Spinless Interaction:

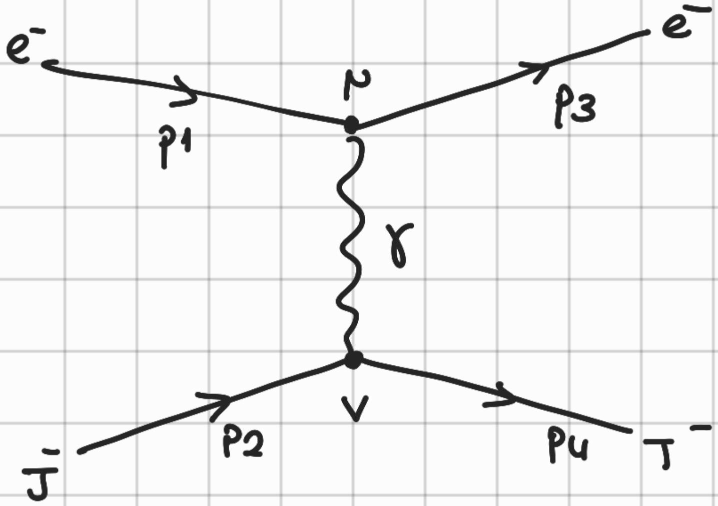
$$\mathcal{M} = \underbrace{\langle \psi_c | V | \psi_A \rangle}_{g_a} \frac{1}{q^2 - M_x^2} \underbrace{\langle \psi_d | V | \psi_b \rangle}_{g_b}$$

$$\langle \psi(p_4) | \hat{V}_D | \psi(p_1) \rangle \rightarrow v_\tau^+(p_4)\, q_\tau \gamma^0\gamma^\nu \varepsilon_\nu^{(\lambda)} v_\tau(p_2)$$

$$\bar{u}_e(p_3) \qquad\qquad\qquad\qquad\qquad\qquad \bar{u}_\tau(p_4)$$

$$\mathcal{M} = \underbrace{\left[u_e^{\dagger}(p_3)\, q_e\, \gamma^0 \gamma^\mu\, u_e(p_1) \right]}_{\text{Interaction of } e^- \text{ with photon}} \underbrace{\sum_\lambda \epsilon_\mu^\lambda \frac{1}{q^2} \epsilon_\nu^{\lambda *}}_{\text{Sum over polarizations}} \underbrace{\left[u_\tau^{*}(p_4)\, q_\tau\, \gamma^0 \gamma^\nu\, u_\tau(p_2) \right]}_{\text{interaction of } \tau^- \text{ with photon}}$$

virtual photons



$$\underbrace{\sum_\lambda \epsilon_\mu^\lambda (\epsilon_\nu^\lambda)^*}_{\text{sum over the polarizations of VIRTUAL PHOTONS}} = -g_{\mu\nu}$$

$\bar{\psi} = \psi^{\dagger} \gamma^0$ (adjoint spinor)

$$\mathcal{M} = \left[\bar{u}_e(p_3)\, q_e\, \gamma^\mu\, u_e(p_1) \right] \frac{-g_{\mu\nu}}{q^2} \left[\bar{u}_\tau(p_4)\, q_\tau\, \gamma^\nu\, u_\tau(p_2) \right]$$

4-vector

$$j_e^\mu = \bar{u}_e(p_3)\, \gamma^\mu\, u_e(p_1) \qquad j_\tau^\nu = \bar{u}_\tau(p_4)\, \gamma^\nu\, u_\tau(p_2)$$

$$\mathcal{M} = -q_e q_\tau \frac{j_e \cdot j_\tau}{q^2}$$

Basic Feynman Rules

- propagator factor for each internal line
  "each virtual particle"

- Dirac spinor for each external line
  "each real incoming or outgoing particle

- vertex factor for each vortex

$$-iM = \text{Product of all factors}$$

external lines

spin 1/2	incoming particle		$u(p)$
	outgoing particle		$\bar{u}(p)$
	incoming antiparticle		$\bar{v}(p)$
	outgoing antiparticle		$v(p)$
spin 1	incoming photon		$\varepsilon^{\mu}(p)$
	outgoing photon		$\varepsilon^{\mu}(p)^{*}$

Internal lines:

spin 1 photon 

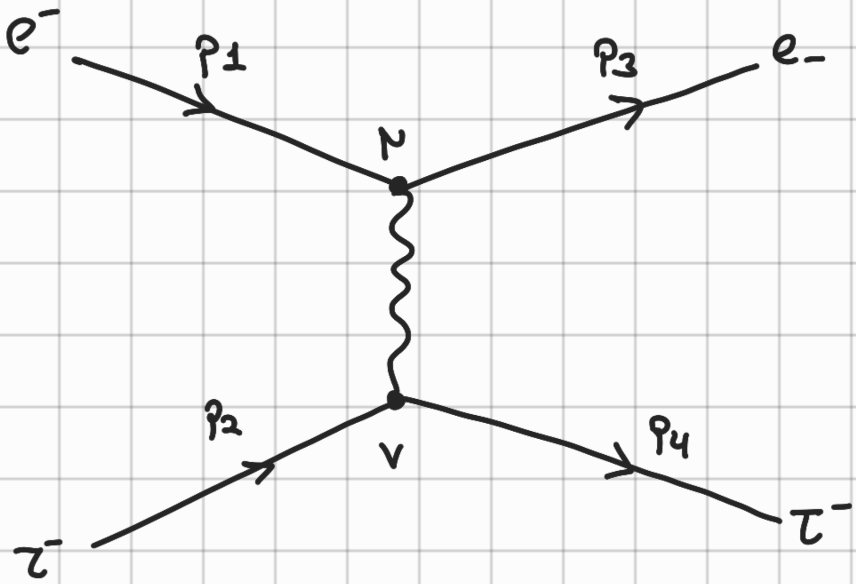
$\frac{-i g_{\mu\nu}}{q^2}$

spin 1/2 fermion  $\frac{i(\gamma^\mu q_\mu + m)}{q^2 - m^2}$

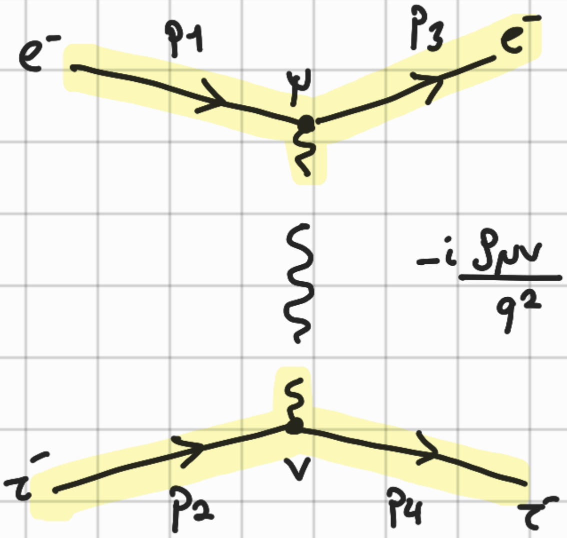
Vertex factors:

spin 1/2 fermion (charge $-|e|$)

$ie\gamma^\mu$

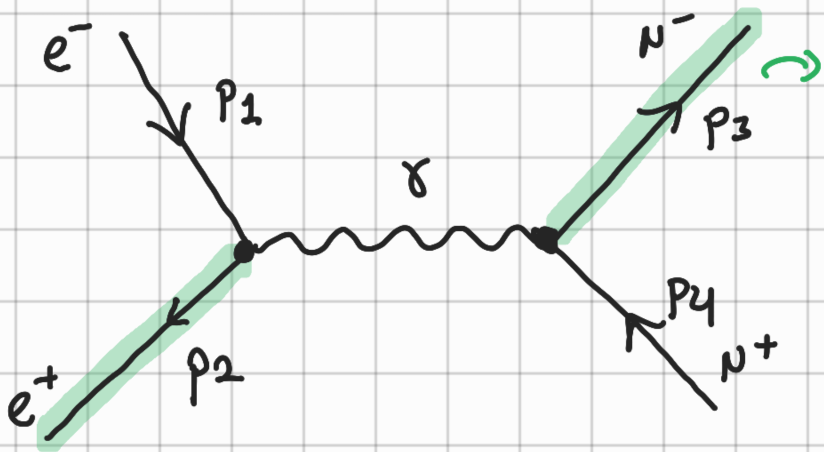


$$\bar{u}_e(p_3)\,[ie\gamma^\mu]\,u_e(p_1)$$



$$\bar{u}_\tau(p_4)\,[ie\gamma^\nu]\,u_\tau(p_2)$$

$$-iM = \left[\bar{u}_e(p_3)\, ie\gamma^\mu u_e(p_1)\right] \frac{-i g_{\mu\nu}}{q^2} \left[\bar{u}_\tau(p_4)\, ie\gamma^\nu u_\tau(p_2)\right]$$



write them as adjoint spinor

$$\left[\bar{v}(p_2)\,[ie\gamma^\mu]\,u(p_1)\right]\left[\frac{-i g_{\mu\nu}}{q^2}\right]\left[\bar{u}(p_3)\,[ie\gamma^\mu]\,u(p_4)\right]$$

v for antiparticles