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MICRO-435 HW1

a) Verify if the three qubits are normalized and if not, normalize them.

$|\psi_1\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \qquad \langle\psi_1|\psi_1\rangle \stackrel{?}{=} 1$

$\langle\psi_1|\psi_1\rangle = \frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}\langle 0|0\rangle + \frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}\langle 1|1\rangle = \frac{1}{2}\cdot 1 + \frac{1}{2}\cdot 1 = 1 \checkmark$

$|\psi_2\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle \qquad \langle\psi_2|\psi_2\rangle \stackrel{?}{=} 1$

$\langle\psi_2|\psi_2\rangle = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\langle 0|0\rangle - \frac{1}{\sqrt{2}}\cdot\left(-\frac{1}{\sqrt{2}}\right)\langle 1|1\rangle = \frac{1}{2}\cdot 1 + \frac{1}{2}\cdot 1 = 1 \checkmark$

$|\psi_3\rangle = \frac{1}{4}|0\rangle + \frac{3}{4}|1\rangle \qquad \langle\psi_3|\psi_3\rangle \stackrel{?}{=} 1$

$\langle\psi_3|\psi_3\rangle = \frac{1}{4}\cdot\frac{1}{4}\langle 0|0\rangle + \frac{3}{4}\cdot\frac{3}{4}\langle 1|1\rangle = \frac{1}{16}\cdot 1 + \frac{9}{16}\cdot 1 = \frac{10}{16} \neq 1 \times$

not normalized

$|\psi_3\rangle = \frac{\alpha}{4}|0\rangle + \frac{3\alpha}{4}|1\rangle \longrightarrow \frac{\alpha^2}{16} + \frac{9\alpha^2}{16} = 1 \qquad \frac{10\alpha^2}{16} = 1 \rightarrow \sqrt{\alpha^2} = \sqrt{\frac{16}{10}} = \frac{4}{\sqrt{10}}$

$|\psi_3\rangle = \frac{1}{\sqrt{10}}|0\rangle + \frac{3}{\sqrt{10}}|1\rangle \leftarrow$ normalized version

b) Show that $|\psi_1\rangle$ and $|\psi_2\rangle$ are orthogonal

$|\psi_1\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \qquad |\psi_2\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$

$\langle\psi_1|\psi_2\rangle \stackrel{?}{=} 0$

$\langle\psi_1| = \frac{1}{\sqrt{2}}\langle 0| + \frac{1}{\sqrt{2}}\langle 1| \longrightarrow \langle\psi_1|\psi_2\rangle = \frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}\langle 0|0\rangle - \frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}\langle 1|1\rangle = \frac{1}{2}\cdot 1 - \frac{1}{2}\cdot 1 = 0$

c) $P(|\psi_3\rangle \rightarrow |0\rangle)$ in the basis $\{|0\rangle, |1\rangle\}$

$|\psi_3\rangle = \frac{1}{\sqrt{10}}|0\rangle + \frac{3}{\sqrt{10}}|1\rangle$

$P(|\psi_3\rangle \rightarrow |0\rangle) = \left|\frac{1}{\sqrt{10}}\right|^2 = \frac{1}{10}$

d) $P(|\psi_3\rangle \to |\psi_2\rangle)$ in the basis of $\{|\psi_1\rangle, |\psi_2\rangle\}$

$$|\psi_3\rangle = \alpha|\psi_1\rangle + \beta|\psi_2\rangle$$

$$\frac{1}{\sqrt{10}}|0\rangle + \frac{3}{\sqrt{10}}|1\rangle = \alpha\left(\frac{1}{\sqrt2}|0\rangle + \frac{1}{\sqrt2}|1\rangle\right) + \beta\left(\frac{1}{\sqrt2}|0\rangle - \frac{1}{\sqrt2}|1\rangle\right)$$

$$\frac{1}{\cancel{\sqrt{10}}\sqrt5} = \frac{\alpha}{\sqrt2} + \frac{\beta}{\cancel{\sqrt2}} \to (\alpha+\beta) = \frac{1}{\sqrt5}$$

$$\frac{3}{\sqrt{10}} = \frac{\alpha}{\sqrt2} - \frac{\beta}{\sqrt2} \to (\alpha-\beta) = \frac{3}{\sqrt5}$$

$+$

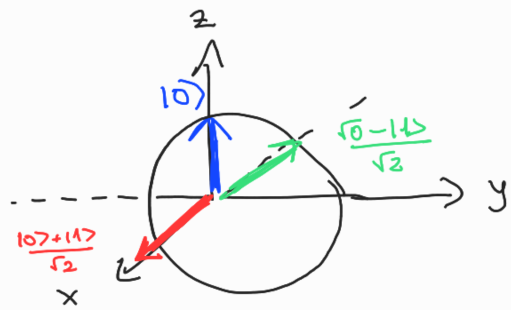
$$2\alpha = \frac{4}{\sqrt5} \to \alpha = \frac{2}{\sqrt5} \qquad \beta = -\frac{1}{\sqrt5}$$

$$|\psi_3\rangle = \frac{2}{\sqrt5}|\psi_1\rangle - \frac{1}{\sqrt5}|\psi_2\rangle$$

$$P(|\psi_3\rangle \to |\psi_2\rangle) = \left|-\frac{1}{\sqrt5}\right|^2 = \frac{1}{5}$$

e) How can we prepare $|\psi_1\rangle$ and $|\psi_2\rangle$ starting from $|0\rangle$ using only X, Y, Z, H

$$|\psi_1\rangle = \frac{1}{\sqrt2}|0\rangle + \frac{1}{\sqrt2}|1\rangle$$



$|\psi_1\rangle =$ $|0\rangle$ —[H]— $\frac{|0\rangle+|1\rangle}{\sqrt2}$ $= H|0\rangle$

$$|\psi_2\rangle = \frac{1}{\sqrt2}|0\rangle - \frac{1}{\sqrt2}|1\rangle$$

$|0\rangle$ —[H]—[Z]— $\frac{|0\rangle-|1\rangle}{\sqrt2}$ $= ZH|0\rangle$

$\longrightarrow$
time

check $\quad ZH = \frac{1}{\sqrt2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} = \begin{pmatrix}1 & 1\\ -1 & +1\end{pmatrix}\frac{1}{\sqrt2}$

$$|\psi_2\rangle = ZH|0\rangle = \frac{1}{\sqrt2}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt2}\begin{pmatrix}1\\-1\end{pmatrix} = \frac{1}{\sqrt2}(|0\rangle - |1\rangle)$$

f) Show that S is unitary

$$S = \begin{pmatrix}1 & 0\\ 0 & i\end{pmatrix} \qquad S^\dagger = \begin{pmatrix}1 & 0\\ 0 & -i\end{pmatrix}$$

$SS^\dagger = \begin{pmatrix}1 & 0\\ 0 & i\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -i\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = I$ ✓

$-i^2 = 1$

$S^\dagger S = \begin{pmatrix}1 & 0\\ 0 & -i\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & i\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = I$ ✓

$\Big\}$ S is unitary

g) $|\psi_3\rangle - \boxed{S} - \boxed{H} - \underbrace{\boxed{Z} - \underbrace{\boxed{H} - \boxed{X} - \boxed{H}}}_{Z = Z^{+}} - |\psi_{out}\rangle$ I

From exercise 1, we proved $HXH = Z$ and $Z = Z^{+}$ (hermitian)

$Z \cdot Z^{+} = I$

It can be simplified to

$|\psi_3\rangle - \boxed{S} - \boxed{H} - |\psi_{out}\rangle \longrightarrow \psi_{out} = H\,S\,|\psi_3\rangle$

$\xrightarrow{\text{time}}$

$$HS = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \quad ; \quad |\psi_3\rangle = \frac{1}{\sqrt{10}}|0\rangle + \frac{3}{\sqrt{10}}|1\rangle = \frac{1}{\sqrt{10}}\begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$|\psi_{out}\rangle = HS\,|\psi_3\rangle = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{10}}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \frac{1}{\sqrt{20}}\begin{pmatrix} 1+3i \\ 1-3i \end{pmatrix} = \frac{(1+3i)}{\sqrt{20}}|0\rangle + \frac{(1-3i)}{\sqrt{20}}|1\rangle$$

h) Simplify the circuit

From part g $\boxed{|\psi_{out}\rangle = H\,S\,|\psi_3\rangle}$