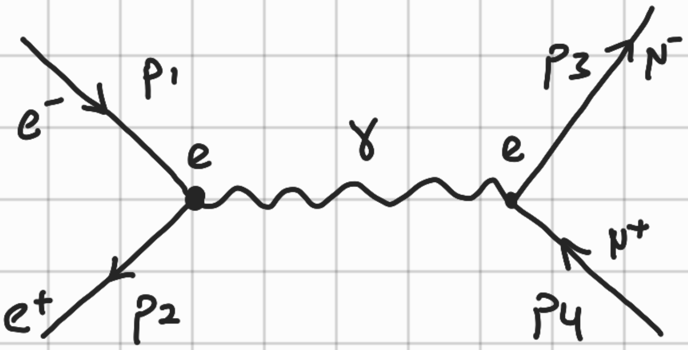
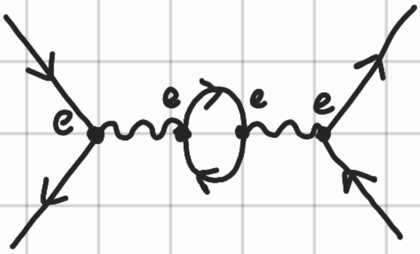
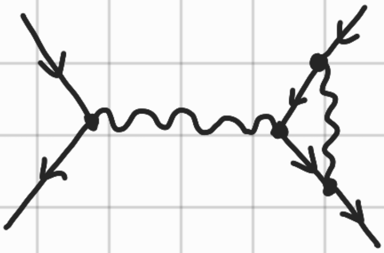
WEEK 9 ELECTRON - POSITRON ANNIHILATION

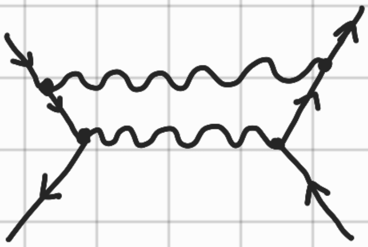


$$|M|^2 \propto \alpha^2$$

$$\alpha = e^2/4\pi$$







$$\mathcal{M}_{fi} = \alpha M_{LO} + \alpha^2 \sum_j M_{1,j} + \dots$$

$$|\mathcal{M}_{fi}|^2 = \left(\alpha M_{LO} + \alpha^2 \sum_j M_{1,j} + \dots\right)\left(\alpha M_{LO}^* + \alpha^2 \sum_k M_{1,k}^* + \dots\right)$$

$$|\mathcal{M}_{fi}|^2 = \alpha^2 |M_{LO}|^2 + \alpha^3 \sum_j \left(M_{LO} M_{1,j}^* + M_{LO}^* M_{1,j}\right) + \alpha^4 \sum_{jk} M_{1,j} M_{1,k}^* + \dots$$

$\alpha \approx 1/137$

converges rapidly

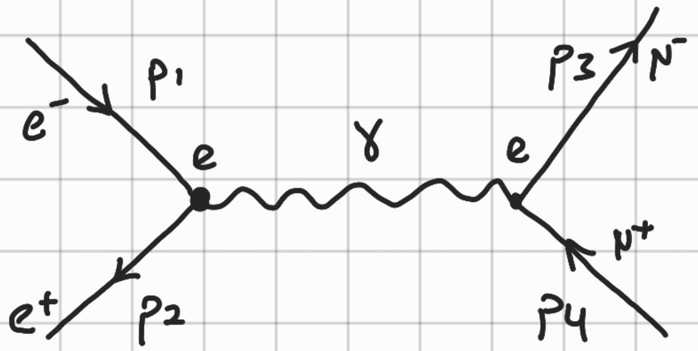
Electron-positron annihilation

$$\mathcal{M} = -\frac{e^2}{q^2} g_{\mu\nu} \left[\bar{v}(p_2)\gamma^\mu u(p_1)\right]\left[\bar{u}(p_3)\gamma^\nu v(p_4)\right]$$

$$= \frac{-e^2}{q^2} g_{\mu\nu}\, j_e^{\mu}\, j_\mu^{\nu}$$

electron 4-vector

muon 4-vector



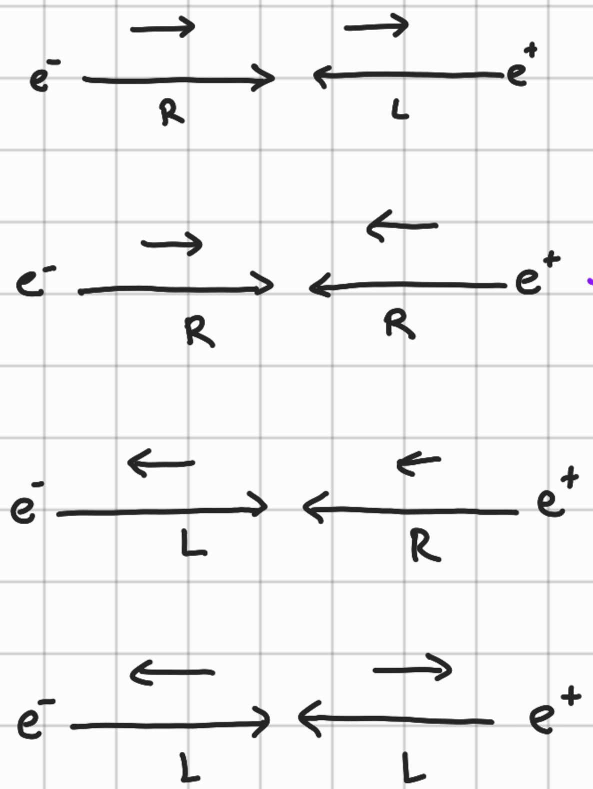
$$q = p_1 + p_2 = p_3 + p_4$$

$$q^2 = (p_1+p_2)^2 = s = (E_1^* + E_2^*)^2$$

$$\mathcal{M} = -\frac{e^2}{s}\, j_e \cdot j_\mu \quad \text{where } \sqrt{s} = 2E_{beam}$$

if e^- and e^+ beams have equal energy

Spin Sums



Possible helicity combinations in the e^+e^- initial state

$$\sum |M_{RR}|^2 = |M_{RR\to RR}|^2 + |M_{RR\to RL}|^2 + |M_{RR\to LR}|^2 + |M_{RR\to LL}|^2$$

In most e^+e^- colliders, beams are unpolarised, # numbers of (+) and (-) helicity states :

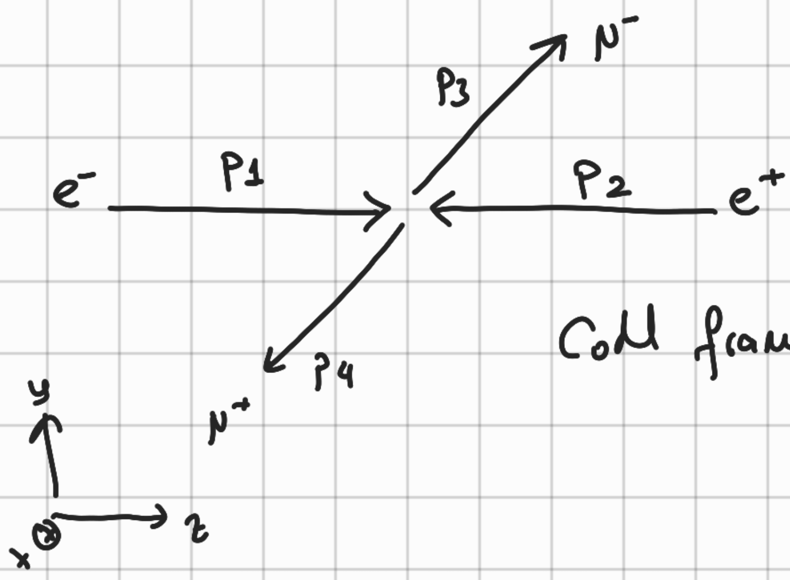
$$\langle |M_{fi}|^2 \rangle = \frac{1}{4}\left(|M_{RR}|^2 + |M_{RL}|^2 + |M_{LR}|^2 + |M_{LL}|^2 \right)$$

$$\langle |M_{fi}|^2 \rangle = \frac{1}{4}\left(|M_{RR\to RR}|^2 + |M_{RR\to RL}|^2 + \dots \dots + |M_{RL\to RR}|^2 + \dots \right)$$

$$\langle |M_{fi}|^2 \rangle = \frac{1}{4} \sum_{\text{spins}} |M|^2$$

Helicity Amplitudes

$\sqrt{s} >> m_\mu \rightarrow$ particle mass can be neglected



$p_1 = (E, 0, 0, E)$

$p_2 = (E, 0, 0, -E)$

$p_3 = (E, E\sin\theta, 0, E\cos\theta)$

$p_4 = (E, -E\sin\theta, 0, -E\cos\theta)$

$$u_\uparrow = \sqrt{E}\begin{pmatrix} c \\ se^{i\phi} \\ c \\ se^{i\phi} \end{pmatrix} \quad u_\downarrow = \sqrt{E}\begin{pmatrix} -s \\ ce^{i\phi} \\ s \\ -ce^{i\phi} \end{pmatrix} \quad v_\uparrow = \sqrt{E}\begin{pmatrix} s \\ -ce^{i\phi} \\ -s \\ ce^{i\phi} \end{pmatrix} \quad v_\downarrow = \sqrt{E}\begin{pmatrix} c \\ se^{i\phi} \\ c \\ se^{i\phi} \end{pmatrix}$$

where $c = \cos\frac{\theta}{2}$ $\quad s = \sin\frac{\theta}{2}$

Initial state e^- $(\theta = 0, \phi = 0)$ $\qquad$ Final state μ^- $(\theta, 0)$

Initial state e^+ $(\theta = \pi, \phi = \pi)$ $\qquad$ Final state μ^+ $(\pi - \theta, \pi)$

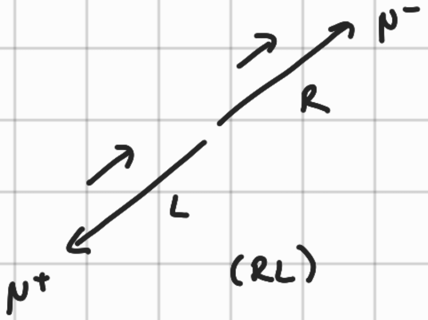
$$u_\uparrow(p_1) = \sqrt{E}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad u_\downarrow(p_1) = \sqrt{E}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \quad v_\uparrow(p_2) = \sqrt{E}\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \quad v_\downarrow(p_2) = \sqrt{E}\begin{pmatrix} 0 \\ -1 \\ 0 \\ -1 \end{pmatrix}$$

$$u_\uparrow(p_3) = \sqrt{E}\begin{pmatrix} c \\ s \\ c \\ s \end{pmatrix} \quad u_\downarrow(p_3) = \sqrt{E}\begin{pmatrix} -s \\ c \\ s \\ -c \end{pmatrix} \quad v_\uparrow(p_4) = \sqrt{E}\begin{pmatrix} c \\ s \\ -c \\ s \end{pmatrix} \quad v_\downarrow(p_4) = \sqrt{E}\begin{pmatrix} s \\ -c \\ s \\ -c \end{pmatrix}$$

The muon and electron currents

$$\overline{\psi} \gamma^\mu \phi = \psi^\dagger \gamma^0 \gamma^\mu \phi$$

$$\overline{\psi}\gamma^0\phi = \psi^\dagger\gamma^0\gamma^0\phi = \psi_1^*\phi_1 + \psi_2^*\phi_2 + \psi_3^*\phi_3 + \psi_4^*\phi_4,$$
$$\overline{\psi}\gamma^1\phi = \psi^\dagger\gamma^0\gamma^1\phi = \psi_1^*\phi_4 + \psi_2^*\phi_3 + \psi_3^*\phi_2 + \psi_4^*\phi_1,$$
$$\overline{\psi}\gamma^2\phi = \psi^\dagger\gamma^0\gamma^2\phi = -i(\psi_1^*\phi_4 - \psi_2^*\phi_3 + \psi_3^*\phi_2 - \psi_4^*\phi_1),$$
$$\overline{\psi}\gamma^3\phi = \psi^\dagger\gamma^0\gamma^3\phi = \psi_1^*\phi_3 - \psi_2^*\phi_4 + \psi_3^*\phi_1 - \psi_4^*\phi_2.$$



For example, for the RL combination

$$\mu_\uparrow(p_3) \text{ and } v_\downarrow(p_4)$$

Muon currents are:

$$j_\mu^0 = \bar{u}_\uparrow(p_3)\gamma^0 v_\downarrow(p_4) = 0$$

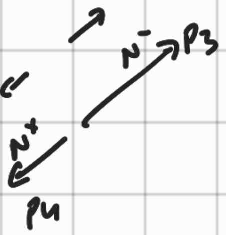
$$j_\mu^1 = \bar{u}_\uparrow(p_3)\gamma^1 v_\downarrow(p_4) = -2E\cos\theta$$

$$j_\mu^2 = \bar{u}_\uparrow(p_3)\gamma^2 v_\downarrow(p_4) = 2iE$$

$$j_\mu^3 = \bar{u}_\uparrow(p_3)\gamma^3 v_\downarrow(p_4) = 4Esc = 2E\sin\theta$$

Thus $j_{\mu,RL} = \bar{u}_\uparrow(p_3)\gamma^\nu v_\downarrow(p_4) = 2E(0, -\cos\theta, i, \sin\theta)$

Repeating the procedure for the other three helicity combinations gives:



$$j_{\mu,RL} = \bar{u}_\uparrow(p_3)\,\gamma^\nu\, v_\uparrow(p_4) = 2E\,(0, -\cos\theta, i, \sin\theta)$$

$$j_{\mu,RR} = \bar{u}_\uparrow(p_3)\,\gamma^\nu\, v_\downarrow(p_4) = (0,0,0,0)$$

$$j_{\mu,LL} = \bar{u}_\downarrow(p_3)\,\gamma^\nu\, v_\uparrow(p_4) = (0,0,0,0)$$

$$j_{\mu,LR} = \bar{u}_\downarrow(p_3)\,\gamma^\nu\, v_\downarrow(p_4) = 2E\,(0, -\cos\theta, -i, \sin\theta)$$

$$\Big[\underbrace{\bar{u}(p_3)\,\gamma^\mu\, v(p_4)}_{\text{muon current}}\Big]^\dagger = \Big[\, u(p_3)^\dagger\,\gamma^0\gamma^\mu\, v(p_4)\Big]^\dagger$$

$(AB)^\dagger = B^\dagger A^\dagger$

$\gamma^{0\dagger} = \gamma^0$

$\gamma^{\mu\dagger}\gamma^0 = \gamma^0\gamma^\mu$

$$= v(p_4)^\dagger\,\gamma^{\mu\dagger}\,\gamma^{0\dagger}\, u(p_3)$$

$$= v(p_4)^\dagger\,\gamma^{\mu\dagger}\gamma^0\, u(p_3)$$

$$= \underbrace{v(p_4)^\dagger\gamma^0}\,\gamma^\mu\, u(p_3)$$

$$= \bar{v}(p_4)\,\gamma^\mu\, u(p_3)$$

$$\bar{v}_\downarrow(p_4)\,\gamma^\mu\, u_\uparrow(p_3) = \big[\,\bar{u}_\uparrow(p_3)\,\gamma^\mu\, v_\downarrow(p_4)\big]^* = 2E\,(0, -\cos\theta, -i, \sin\theta)$$

$$\bar{v}_\uparrow(p_4)\,\gamma^\mu\, u_\downarrow(p_3) = \big[\,\bar{u}_\downarrow(p_3)\,\gamma^\mu\, v_\uparrow(p_4)\big]^* = 2E\,(0, -\cos\theta, i, \sin\theta)$$

$\boxed{\text{If } \theta = 0}$

$$j_{e,RL} = \overset{L}{\bar{v}_\downarrow}(p_2)\,\gamma^\mu\, \overset{R}{u_\uparrow}(p_1) = 2E\,(0, -\overbrace{\cos(0)}^{-1}, -i, \overbrace{\sin(0)}^{0})$$

$$j_{e,LR} = \underset{R}{\bar{v}_\uparrow}(p_2)\,\gamma^\mu\, \underset{L}{u_\downarrow}(p_1) = 2E\,(0, -1, i, 0)$$

The $e^+e^- \longrightarrow \mu^+\mu^-$ cross section

In the limit $E \gg m$, only two of the four helicity combinations for both the initial and the final states are non-zero.

$$\mathcal{M} = -\frac{e^2}{s}\, j_e \cdot j_\mu$$

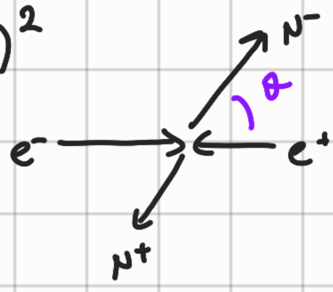
For example, $\mathcal{M}_{RL \to RL} = -\frac{e^2}{s}\left[2E(0,-1,i,0)\right] \cdot \left[2E(0,-\cos\theta,i,\sin\theta)\right]$

$$s = (2E)^2 = 4E^2 \qquad = -\frac{e^2}{s} \cdot 4E^2 \,(0 + \cos\theta - 1 + 0)$$

$$\alpha = \frac{e^2}{4\pi} \qquad = -\frac{e^2}{\cancel{s}}\cancel{4E^2}\left[-(1-\cos\theta)\right] = e^2(1+\cos\theta)$$

$$= 4\pi\alpha\,(1+\cos\theta)$$

$$|\mathcal{M}_{RL\to RL}|^2 = |\mathcal{M}_{LR\to LR}|^2 = (4\pi\alpha)^2(1+\cos\theta)^2$$



$$|\mathcal{M}_{RL\to LR}|^2 = |\mathcal{M}_{LR\to RL}|^2 = (4\pi\alpha)^2(1-\cos\theta)^2$$

$$\langle|\mathcal{M}_{fi}|^2\rangle = \frac{1}{4} \times \left(|\mathcal{M}_{RL\to RL}|^2 + |\mathcal{M}_{RL\to LR}|^2 + |\mathcal{M}_{LR\to RL}|^2 + |\mathcal{M}_{LR\to LR}|^2\right)$$

$$= \frac{1}{4} e^4\left[2(1+\cos\theta)^2 + 2(1-\cos\theta)^2\right]$$

$$= \frac{1}{4} e^4 \cdot 4\,(1+\cos^2\theta) = e^4(1+\cos^2\theta)$$

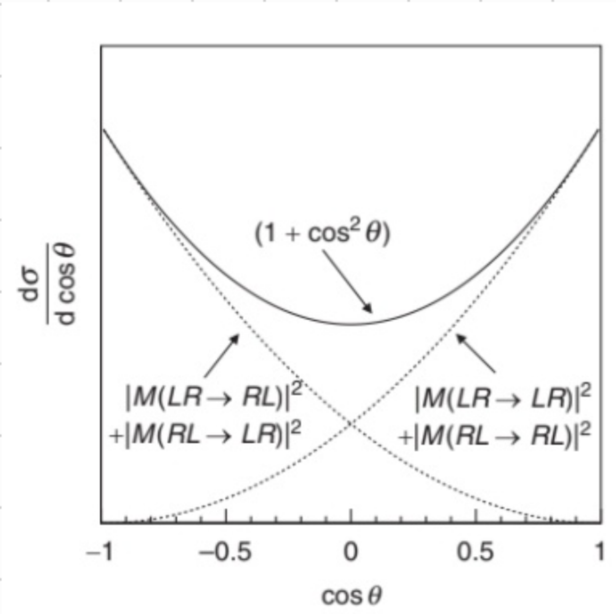
General cross section formula :

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s} \frac{|\vec{p_f}^*|}{|\vec{p_i}^*|} |M_{fi}|^2$$

$\hookrightarrow p_i^* = p_f^* = E$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s} e^4 (1 + \cos^2\theta)$$

where $\alpha = \frac{e^2}{4\pi}$ $\quad \alpha^2 = \frac{e^4}{16\pi^2}$ $\rightarrow$ $\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}(1+\cos^2\theta)$



$$\int (1+\cos^2\theta)\, d\Omega = 2\pi \int_{-1}^{+1} (1+\cos^2\theta)\, d(\underbrace{\cos\theta}_{u}) = \frac{16\pi}{3}$$

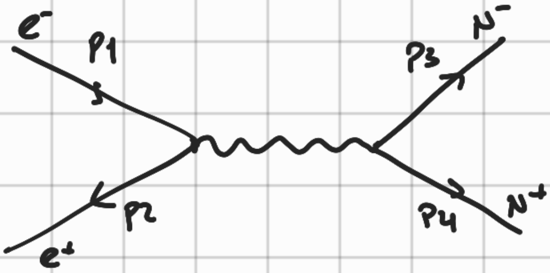
$\swarrow$

$d\phi\, d(\cos\theta)$

$$\sigma = \frac{\alpha^2}{\cancel{4}s} \cdot \frac{\cancel{16}^{4}\pi}{3} = \frac{4\pi\alpha^2}{3s}$$

Lorentz Invariant Form

$$\langle |M_{fi}|^2 \rangle = e^4 (1+\cos^2\theta)$$



$$p_1 = (E, 0, 0, E)$$

$$p_2 = (E, 0, 0, -E)$$

$$p_3 = (E, E\sin\theta, 0, E\cos\theta)$$

$$p_4 = (E, -E\sin\theta, 0, -E\cos\theta)$$

$$p_1 p_2 = 2E^2 \qquad p_1 p_3 = E^2(1-\cos\theta) \qquad p_1 p_4 = E^2(1+\cos\theta)$$

$$p_1^\mu p_{2\mu} = (E,0,0,E)\cdot(E,-0,-0,+E)$$

$$\langle |M_{fi}|^2 \rangle = 2e^4 \, \frac{(p_1\cdot p_3)^2 + (p_1\cdot p_4)^2}{(p_1\cdot p_2)^2}$$

Express in terms of the Mandelstam variables

$$s = (p_1+p_2)^2 = \underset{\substack{\downarrow \\ \text{4-vectors}}}{p_1^2} + 2p_1\cdot p_2 + p_2^2 = M_1^2 + M_2^2 + 2p_1\cdot p_2$$

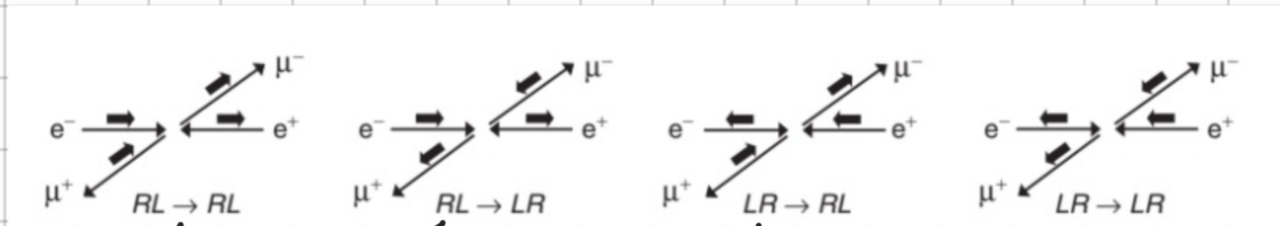
If mass can be neglected : $s = +2\,p_1\cdot p_2$

$$t = -2p_1\cdot p_3$$

$$u = -2p_1\cdot p_4$$

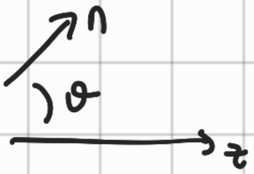
$$\langle |M_{fi}|^2\rangle = 2e^4\left(\frac{t^2+u^2}{s^2}\right)$$

Spin in electron-positron annihilation



$|S, S_z\rangle = |1,+1\rangle_z$ $\qquad\qquad$ $|S,S_z\rangle = |1,-1\rangle_z$

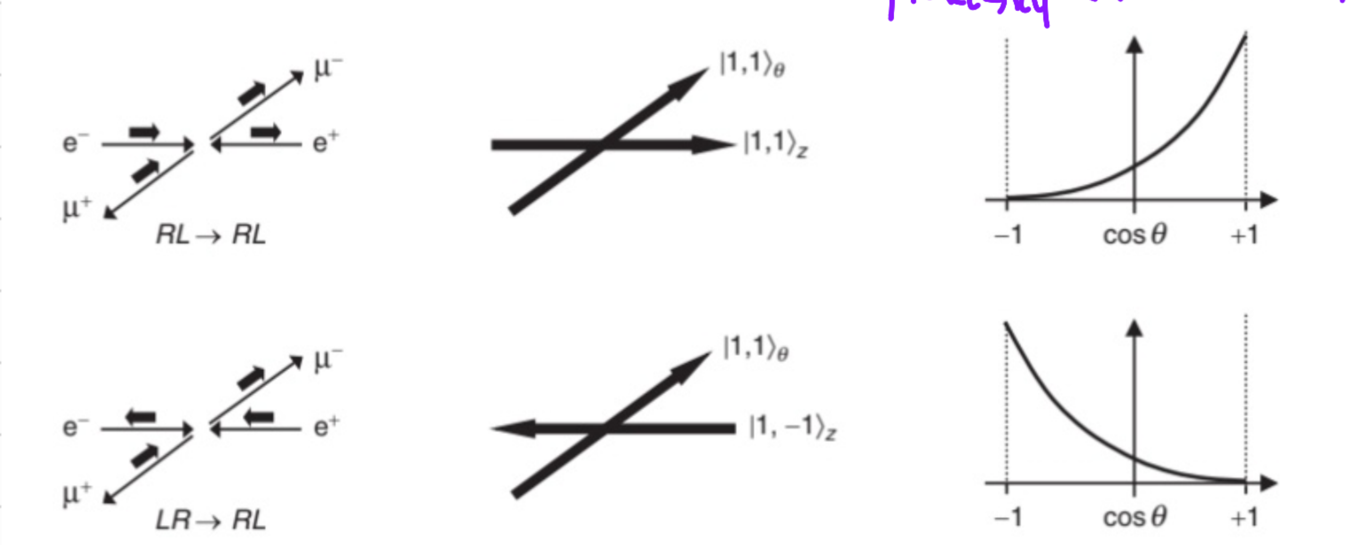
$\mu^+\mu^- \rightsquigarrow |1,\pm 1\rangle_\theta$



$$\hat{S}_n = \frac{1}{2}\,\vec{n}\cdot\vec{\sigma}$$

$$|1,+1\rangle_\theta = \frac{1}{2}(1-\cos\theta)|1,-1\rangle + \frac{1}{\sqrt{2}}\sin\theta\,|1,0\rangle + \frac{1}{2}(1+\cos\theta)|1,+1\rangle$$

$|M_{RL\to RL}|^2 = (4\pi\alpha)^2(1+\cos\theta)^2$



$|M_{RL\to LR}|^2 = (4\pi\alpha)^2(1-\cos\theta)^2$

$$M_{RL \to RL} \propto \langle 1,+1 | 1,+1 \rangle_\theta = \frac{1}{2}(1+\cos\theta)$$

$$M_{LR \to RL} \propto \langle 1,-1 | 1,+1 \rangle_\theta = \frac{1}{2}(1-\cos\theta)$$

Chirality

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

$$(\gamma^5)^2 = 1$$

$$\gamma^{5\dagger} = \gamma^5$$

$$\gamma^5 \gamma^\mu = -\gamma^\mu \gamma^5$$

$$\gamma^5 u_\uparrow = +u_\uparrow \qquad \gamma^5 u_\downarrow = -u_\downarrow \qquad \gamma^5 v_\uparrow = -v_\uparrow \qquad \gamma^5 v_\downarrow = +v_\downarrow$$

$\downarrow$

Helicity eigenstates

In general:

$$\gamma^5 u_R = +u_R \qquad \gamma^5 u_L = -u_L \qquad \gamma^5 v_R = -v_R \qquad \gamma^5 v_L = +v_L$$

★ Solutions to the Dirac Equation which are also eigen states of γ^5 are identical to the massless helicity eigen states.

Chiral projection Operators

$$P_R = \frac{1}{2}(1+\gamma^5) \rightarrow \frac{1}{2}\left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right)$$

$$P_L = \frac{1}{2}(1-\gamma^5) \rightarrow \frac{1}{2}\left(\begin{array}{cc|cc} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ \hline -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{array}\right)$$

$$P_R + P_L = 1 \qquad P_R P_R = P_R \qquad P_L P_L = P_L \qquad P_L P_R = 0$$

$$\boxed{\begin{array}{ll} P_R u_R = u_R & P_R v_R = 0 \\ P_R u_L = 0 & P_R v_L = v_L \end{array}}$$

$$P_L u_R = 0 \qquad P_L v_R = v_R$$

$$P_L u_L = u_L \qquad P_L v_L = 0$$

$$u = a_R u_R + a_L u_L = \frac{1}{2}(1+\gamma^5)u + \frac{1}{2}(1-\gamma^5)u$$

complex coefficient

Chirality in QED

$$i \underbrace{Q_f}_{\text{fermion's charge}} e \, \overline{\psi} \gamma^\mu \phi$$

★ Any 4-vector current can be decomposed into contributions from left and right handed chiral states.

$$\overline{\psi} \gamma^\mu \phi = \left(a_R^* \overline{\psi}_R + a_L^* \overline{\psi}_L \right) \gamma^\mu \left(b_R \phi_R + b_L \phi_L \right)$$

$$= a_R^* b_R \overline{\psi}_R \gamma^\mu \phi_R + a_R^* b_L \overline{\psi}_R \gamma^\mu \phi_L + a_L^* b_R \overline{\psi}_L \gamma^\mu \phi_R$$

$$+ a_L^* b_L \overline{\psi}_L \gamma^\mu \phi_L$$

where $\boxed{u_R(p') = P_R u_R(p')}$

and $\bar{u}_L(p) = [u_L(p)]^\dagger \gamma^0 = [P_L u_L(p)]^\dagger \gamma^0 = \left[\frac{1}{2}(1-\gamma^5) u_L(p) \right]^\dagger \gamma^0$

$$= [u_L(p)]^\dagger \frac{1}{2}(1-\gamma^5)\gamma^0 \quad \text{where} \quad \gamma^5 = \gamma^{5\dagger}$$

$$= [u_L(p)]^\dagger \gamma^0 \frac{1}{2}(1+\gamma^5) \quad \text{where} \quad \gamma^0\gamma^5 = -\gamma^5\gamma^0$$

$$\boxed{\bar{u}_L(p) = \overline{u_L}(p) \, P_R}$$

Thus, $\bar{u}_L(p)\gamma^\mu u_R(p') = \bar{u}_L(p) \, P_R \gamma^\mu \, P_R \, u_R(p') = 0$

$\gamma^\mu P_L P_R = 0$

$$P_R \gamma^\mu = \frac{1}{2}(1+\gamma^5)\gamma^\mu = \gamma^\mu \frac{1}{2}(1-\gamma^5) = \gamma^\mu P_L$$

$$\bar{u}_L \gamma^\mu u_R = 0$$

$$\bar{u}_R \gamma^\mu u_L = 0$$

$$\bar{v}_L \gamma^\mu v_R = 0$$

$$\bar{v}_R \gamma^\mu v_L = 0$$

$$\bar{v}_L \gamma^\mu u_L = 0$$

$$\bar{v}_R \gamma^\mu u_R = 0$$

Helicity and Chirality

$$u_\uparrow(p,\theta,\phi) = N \begin{pmatrix} c \\ se^{i\phi} \\ \kappa c \\ \kappa s e^{i\phi} \end{pmatrix} \quad \text{with} \quad \kappa = \frac{p}{E+m} \quad \text{and} \quad N = \sqrt{E+m}$$

$$P_R u_\uparrow = \frac{1}{2}(1+\kappa)\, N \begin{pmatrix} c \\ se^{i\phi} \\ c \\ se^{i\phi} \end{pmatrix} \quad \text{and} \quad P_L u_\uparrow = \frac{1}{2}(1-\kappa)\, N \begin{pmatrix} c \\ se^{i\phi} \\ -c \\ -se^{i\phi} \end{pmatrix}$$

$$u_\uparrow(p,\theta,\phi) = \frac{1}{2}(1+\kappa)\, N \begin{pmatrix} c \\ se^{i\phi} \\ c \\ se^{i\phi} \end{pmatrix} + \frac{1}{2}(1-\kappa) N \begin{pmatrix} c \\ se^{i\phi} \\ -c \\ -se^{i\phi} \end{pmatrix}$$

$$\propto \frac{1}{2}(1+\kappa)\, u_R + \frac{1}{2}(1-\kappa)\, u_L$$

Trace techniques

Completeness Relations

$$\sum_{s=1}^{2} u_s(p)\,\bar{u}_s(p) = u_1(p)\,\bar{u}_1(p) + u_2(p)\,\bar{u}_2(p)$$

$$u_s(p) = \sqrt{E+m}\begin{pmatrix} \phi_s \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\,\phi_s \end{pmatrix} \quad \text{with} \quad \phi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \phi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\bar{u}_s(p) = u_s^{\dagger}\gamma^0 = \sqrt{E+m}\left(\phi_s^T \quad \phi_s^T \frac{(\vec{\sigma}\cdot\vec{p})^{\dagger}}{E+m} \right)\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

$$= \sqrt{E+m}\left(\phi_s^T \quad -\phi_s^T \frac{(\vec{\sigma}\cdot p)}{E+m} \right)$$

$$\sum_{s=1}^{2} u_s(p)\,\bar{u}_s(p) = (E+m)\sum_{s=1}^{2}\begin{pmatrix} \phi_s\phi_s^T & -\frac{\sigma\cdot p}{E+m}\phi_s\phi_s^T \\ \frac{\sigma\cdot p}{E+m}\phi_s\phi_s^T & -\frac{(\sigma\cdot p)^2}{(E+m)^2}\phi_s\phi_s^T \end{pmatrix}$$

Using:

$$\sum_{s=1}^{2}\phi_s\phi_s^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad (\sigma\cdot p)^2 = p^2 = (E+m)(E-m)$$

$$\sum_{s=1}^{2} u_s(p)\,\bar{u}_s(p) = \begin{pmatrix} (E+m)I & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & (-E+m)I \end{pmatrix}$$

$$\sum_{s=1}^{2} u_s\bar{u}_s = (\gamma^{\mu}p_{\mu} + mI) = \not{p} + m$$

$$\gamma^{\mu}p_{\mu} = E\gamma^0 - p_x\gamma^1 - p_y\gamma^2 - p_z\gamma^3$$

$$\sum_{r=1}^{2} v_r \bar{v}_r = (\gamma^\mu p_\mu - mI) = \not{p} - m$$

For antiparticles

Spin sums and the trace formalism

$$\bar{u}(p)\,\underbrace{\Gamma}_{\gamma^\mu}\,u(p') = \bar{u}(p)_j \Gamma_{ji} u(p')_i$$

Ex. Summation over repeated indices:

$$(c_1, c_2)\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = c_1 B_{11} a_1 + c_1 B_{12} a_2 + c_2 B_{21} a_1 + c_2 B_{22} a_2$$

$$\boxed{c^T B a = c_j B_{ji} a_i}$$

$$M_{fi} = \frac{-e^2}{q^2}\left[\bar{v}(p_2)\gamma^\mu u(p_1)\right] g_{\mu\nu} \left[\bar{u}(p_3)\gamma^\nu v(p_4)\right]$$

γ_μ

$$= \frac{-e^2}{q^2}\left[\bar{v}(p_2)\gamma^\mu u(p_1)\right]\left[\bar{u}(p_3)\gamma_\mu v(p_4)\right]$$

$$M_{fi}^{\dagger} = \frac{e^2}{q^2}\left[\bar{v}(p_2)\gamma^\nu u(p_1)\right]^{\dagger}\left[\bar{u}(p_3)\gamma_\nu v(p_4)\right]^{\dagger}$$

$$|M_{fi}|^2 = \frac{e^4}{q^4}\left[\bar{v}(p_2)\gamma^\mu u(p_2)\right]\left[\bar{v}(p_2)\gamma^\nu u(p_1)\right]^{\dagger} \times \left[\bar{u}(p_3)\gamma_\mu v(p_4)\right]\left[\bar{u}(p_3)\gamma_\nu v(p_4)\right]^{\dagger}$$

$$\langle |M_{fi}|^2 \rangle = \frac{1}{4} \sum_{\text{spins}} |M_{fi}|^2$$

$$= \frac{e^4}{4q^4} \Bigg[\sum_{s,r} \left[\bar{v}^r(p_2) \gamma^\mu u^s(p_1) \right] \left[\bar{v}^r(p_2) \gamma^\nu u^s(p_1) \right]^\dagger$$

$$\times \sum_{s',r'} \left[\bar{u}^{s'}(p_3) \gamma_\mu v^{r'}(p_4) \right] \left[\bar{u}^{r'}(p_3) \gamma_\nu v^{s'}(p_4) \right]^\dagger$$

$$\sum_{\text{spins}} \left[\bar{\psi} \Gamma_1 \phi \right] \left[\bar{\psi} \Gamma_2 \phi \right]^\dagger$$

$$\Gamma_1 \to \gamma^\mu \qquad \Gamma_2 \to \gamma^\nu$$

$$[\bar{\psi} \Gamma \phi]^\dagger = [\psi^\dagger \gamma^0 \Gamma \phi]^\dagger = \phi^\dagger \Gamma^\dagger \gamma^{0\dagger} \psi = \underbrace{\phi^\dagger \gamma^0}_{\bar{\phi}} \overbrace{\gamma^0}^{I} \Gamma^\dagger \gamma^0 \psi$$

$\bar{\phi} \to$ adjoint spinor

$$[\bar{\psi} \Gamma \phi]^\dagger = \bar{\phi} \gamma^0 \Gamma^\dagger \gamma^0 \psi$$

$$[\bar{\Psi} \Gamma \phi]^\dagger = \bar{\phi} \bar{\Gamma} \psi \quad \text{with} \quad \bar{\Gamma} = \gamma^0 \Gamma^\dagger \gamma^0$$

$\gamma^0 \gamma^{\mu\dagger} \gamma^0 = \gamma^\mu$ for all μ where $\Gamma = \gamma^\mu$

$$\bar{\Gamma} = \Gamma$$

$$[\bar{\psi} \Gamma \phi]^\dagger = \bar{\phi} \Gamma \psi$$

$$\sum_{\text{spins}} |M_{fi}|^2 = \frac{e^4}{4q^4}\left[\sum_{s,r} \left[\bar{v}^r(p_2)\gamma^\mu u^s(p_1)\right]\left[\bar{v}^r(p_2)\gamma^\nu u^s(p_1)\right]^\dagger\right.$$

$$\times \sum_{s',r'} \left[\bar{u}^{s'}(p_3)\gamma_\mu v^{r'}(p_4)\right]\left[\bar{u}^{r'}(p_3)\gamma_\nu v^{s'}(p_4)\right]$$

Using $[\bar{\psi}\,\Gamma\,\phi]^\dagger = \bar{\phi}\,\Gamma\,\psi$

$$\sum_{\text{spins}} |M_{fi}|^2 = \frac{e^4}{4q^4}\left[\sum_{s,r} \left[\bar{v}^r(p_2)\gamma^\mu u^s(p_1)\right]\left[\bar{u}^s(p_1)\gamma^\nu v^r(p_2)\right]\right.$$

$$\times \sum_{s',r'} \left[\bar{u}^{s'}(p_3)\gamma_\mu v^{r'}(p_4)\right]\left[\bar{v}^{r'}(p_4)\gamma_\nu u^{s'}(p_3)\right]$$

$$\mathcal{L}_{(e)}^{\mu\nu} = \sum_{s,r=1}^{2} \bar{v}_j^r(p_2)\,\gamma^\mu_{ji}\,u_i^s(p_1)\,\bar{u}_n^s(p_1)\,\gamma^\nu_{nm}\,v_m^r(p_2)$$

$$\mathcal{L}_{(e)}^{\mu\nu} = \left[\sum_{r=1}^{2} v_m^r(p_2)\,\bar{v}_j^r(p_2)\right]\left[\sum_{s=1}^{2} u_i^s(p_1)\,\bar{u}_n^s(p_1)\right]\gamma^\mu_{ji}\,\gamma^\nu_{nm}$$

$$\mathcal{L}_{(e)}^{\mu\nu} = (\not{p}_2 - m)_{mj}\,(\not{p}_1 + m)_{in}\,\gamma^\mu_{ji}\,\gamma^\nu_{nm}$$

$$\mathcal{L}_{(e)}^{\mu\nu} = (\not{p}_2 - m)_{nj}\, \gamma^{\mu}_{ji}\, (\not{p}_1 + m)_{in}\, \gamma^{\nu}_{nm}$$

$$\mathcal{L}_{(e)}^{\mu\nu} = \left[(\not{p}_2 - m)\, \gamma^{\mu}\, (\not{p}_1 + m)\, \gamma^{\nu} \right]_{mm}$$

$$\mathcal{L}_{(e)}^{\mu\nu} = \mathrm{Tr}\left([\not{p}_2 - m]\, \gamma^{\mu}\, [\not{p}_1 + m]\, \gamma^{\nu} \right)$$

μ mass $= M$

$$\mathcal{L}^{(\mu)}_{\mu\nu} = \mathrm{Tr}\left([\not{p}_3 + M]\, \gamma_{\mu}\, [\not{p}_4 - M]\, \gamma_{\nu} \right)$$

$$\sum_{\text{spins}} |M_{fi}|^2 = \frac{e^4}{q^4}\, \mathcal{L}_{(e)}^{\mu\nu}\, \mathcal{L}^{(\mu)}_{\mu\nu}$$

Trace Theorems

$$\mathrm{Tr}(A+B) = \mathrm{Tr}(A) + \mathrm{Tr}(B)$$

$$\mathrm{Tr}(AB \ldots XYZ) = \mathrm{Tr}(ZYX \ldots BA) \rightarrow \text{Cycling}$$

$$\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2 g^{\mu\nu} I$$

$$\mathrm{Tr}(\gamma^{\mu}\gamma^{\nu}) + \mathrm{Tr}(\gamma^{\nu}\gamma^{\mu}) = 2 g^{\mu\nu}\, \mathrm{Tr}(I)$$

$|$

$$\downarrow$$

$$\hookrightarrow \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$Tr(\gamma^\mu \gamma^\nu) = g^{\mu\nu} \underbrace{Tr(I)}_{4}$$

$$Tr(\gamma^\mu \gamma^\nu) = 4 g^{\mu\nu}$$

$$Tr(\gamma^\mu \gamma^\nu \gamma^\rho) = Tr(\gamma^5 \gamma^5 \gamma^\mu \gamma^\nu \gamma^\rho)$$ cyclical

$$= Tr(\gamma^5 \gamma^\mu \gamma^\nu \gamma^\rho \gamma^5)$$ $\gamma^5 \gamma^\mu = -\gamma^\mu \gamma^5$

$$= -Tr(\underbrace{\gamma^5 \gamma^5}_{I} \gamma^\mu \gamma^\nu \gamma^\rho)$$

$$Tr(\gamma^\mu \gamma^\nu \gamma^\rho) = -Tr(\gamma^\mu \gamma^\nu \gamma^\rho)$$

$$0$$

for any odd number of γ matrices $(1,3,5)$

using $\gamma^a \gamma^b = 2g^{ab} - \gamma^b \gamma^a$

$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma = 2g^{\mu\nu} \gamma^\rho \gamma^\sigma - \gamma^\nu \gamma^\mu \gamma^\rho \gamma^\sigma$$

$$= 2g^{\mu\nu} \gamma^\rho \gamma^\sigma - 2g^{\mu\rho} \gamma^\nu \gamma^\sigma + \gamma^\nu \gamma^\rho \gamma^\mu \gamma^\sigma$$

$$= 2g^{\mu\nu} \gamma^\rho \gamma^\sigma - 2g^{\mu\rho} \gamma^\nu \gamma^\sigma + 2g^{\mu\sigma} \gamma^\nu \gamma^\rho - \gamma^\nu \gamma^\rho \gamma^\sigma \gamma^\mu$$

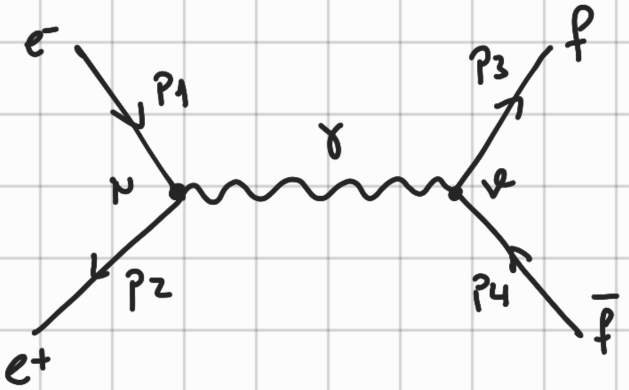
$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma + \gamma^\nu \gamma^\rho \gamma^\sigma \gamma^\mu = 2g^{\mu\nu} \gamma^\rho \gamma^\sigma - 2g^{\mu\rho} Tr(\gamma^\nu \gamma^\sigma) + 2g^{\mu\sigma} Tr(\gamma^\nu \gamma^\rho)$$

$$2\,Tr(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 2g^{\mu\nu}\gamma^\rho\gamma^\sigma - 2g^{\mu\rho}\,Tr(\gamma^\nu\gamma^\sigma) + 2g^{\mu\sigma}\,Tr(\gamma^\nu\gamma^\rho)$$

Using $Tr(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}$

$$Tr(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4g^{\mu\nu}g^{\rho\sigma} - 4g^{\mu\rho}g^{\nu\sigma} + 4g^{\mu\sigma}g^{\nu\rho}$$

Revisit the $e^- e^+$ annihilation



$$e^+e^- \longrightarrow f\bar{f}$$

↳ fermion

$$\langle |M_{fi}|^2 \rangle = \frac{1}{4}\sum_{spins} |M_{fi}|^2 =$$

$$= \frac{Q_f^2 e^4}{4q^4}\, \underbrace{Tr(\not{p}_2\gamma^\mu\not{p}_1\gamma^\nu)}_{m_e=0}\, Tr\left([\not{p}_3 + m_f]\gamma_\mu[\not{p}_4 - m_f]\gamma_\nu\right)$$

$$Tr(\not{p}_2\gamma^\mu\not{p}_1\gamma^\nu) = p_{2\rho}\,p_{1\sigma}(\gamma^\rho\gamma^\mu\gamma^\sigma\gamma^\nu)$$

$$\begin{bmatrix} \not{p}_1 = \gamma^\sigma p_{1\sigma} \\ \not{p}_2 = \gamma^\rho p_{2\rho} \end{bmatrix}$$

$$= 4\,p_{2\rho}\,p_{1\sigma}(g^{\rho\mu}g^{\sigma\nu} - g^{\rho\sigma}g^{\mu\nu} + g^{\rho\nu}g^{\mu\sigma})$$

$$= 4p_2^\mu p_1^\nu - 4g^{\mu\nu}(p_1\cdot p_2) + 4p_2^\nu p_1^\mu$$

$$\mathrm{Tr}\left([\not{p}_3 + m_f]\gamma_\mu [\not{p}_4 - m_f]\gamma_\nu \right) = \mathrm{Tr}(\not{p}_3 \gamma_\mu \not{p}_4 \gamma_\nu) - m_f^2\,\mathrm{Tr}(\gamma_\mu \gamma_\nu)$$

$$= 4 p_{3\mu} p_{4\nu} - 4 g_{\mu\nu}(p_3 \cdot p_4) + 4 p_{3\nu} p_{4\mu} - 4 m_f^2 g_{\mu\nu}$$

$$\langle |M_{fi}|^2 \rangle = 16 \frac{Q_f^2 e^4}{4 q^4} \left[p_2^\mu p_1^\nu - g^{\mu\nu}(p_1 \cdot p_2) + p_2^\nu p_1^\mu \right]$$

$$\times \left[p_{3\mu} p_{4\nu} - g_{\mu\nu}(p_3 \cdot p_4) + p_{3\nu} p_{4\mu} - m_f^2 g_{\mu\nu} \right]$$

Using:

$$g^{\mu\nu} g_{\mu\nu} = 4 \quad , \quad p_2^\mu p_1^\nu g_{\mu\nu} = (p_1 \cdot p_2) \qquad p_2^\mu p_1^\nu p_{3\mu} p_{4\nu} = (p_2 \cdot p_3)(p_1 \cdot p_4)$$

$\downarrow$ simplify:

$$\langle |M_{fi}|^2 \rangle = \cancel{4} \frac{Q_f^2 e^4}{q^4} \left[\cancel{2}(p_1 \cdot p_3)(p_2 \cdot p_4) + \cancel{2}(p_1 \cdot p_4)(p_2 \cdot p_3) + \cancel{2} m_f^2 (p_1 \cdot p_2) \right]$$

$$q^2 = (p_1 + p_2)^2 = p_1^2 + p_2^2 + 2 p_1 \cdot p_2 = \underbrace{\cancel{m_1^2}}_{0} + \underbrace{\cancel{m_2^2}}_{0} + \cancel{2}(p_1 \cdot p_2)$$

$p = \beta E$

$\beta = \frac{v}{c}$

$$p_1 = (E, 0, 0, +E)$$

$$p_2 = (E, 0, 0, -E)$$

$$p_3 = (E, \beta E \sin\theta, 0, +\beta E \cos\theta)$$

$$p_4 = (E, -\beta E \sin\theta, 0, -\beta E \cos\theta)$$

$$p_1 \cdot p_3 = p_2 \cdot p_4 = E^2(1-\beta\cos\theta)$$

$$p_1 \cdot p_4 = p_2 \cdot p_3 = E^2(1+\beta\cos\theta)$$

$$p_1 p_2 = 2E^2$$

$$\langle |M_{fi}|^2 \rangle = 2\,\frac{Q_f^2 e^4}{4\cancel{E^4}} \left[\cancel{E^4}(1-\beta\cos\theta)^2 + \cancel{E^4}(1+\beta\cos\theta)^2 + 2\cancel{E^2}\frac{M_f^2}{E^2} \right]$$

$$= Q_f^2 e^4 \left(1 + \beta^2\cos^2\theta + \frac{E^2 - p^2}{E^2} \right)$$

$$= Q_f^2 e^4 \left(2 + \beta^2\cos^2\theta - \beta^2 \right)$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s} \frac{|\vec{p_f}^*|}{|\vec{p_i}^*|} |M_{fi}|^2$$

$$p_f^* = p = \beta E$$

$$p_i^* = \sqrt{E^2 - \cancel{m^2}} \simeq E$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s} \frac{p}{E} \langle |M_{fi}|^2 \rangle$$

$$= \frac{1}{4s} \beta\, Q_f^2\, \alpha^2 \left(2 + \beta^2\cos^2\theta - \beta^2 \right)$$

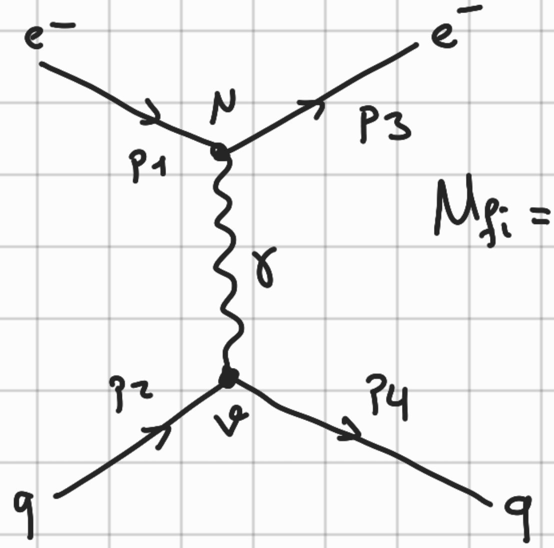
$$\left(\alpha = \frac{e^2}{4\pi} \right)$$

$\Big\downarrow$ Integrate over $d\Omega$

$$\sigma(e^+e^- \to f\bar{f}) = \frac{4\pi\alpha^2 Q_f^2}{3s}\,\beta\left(\frac{3-\beta^2}{2}\right) \quad \text{with}$$

$$\beta^2 = \left(1 - \frac{4m_f^2}{s}\right)$$

Electron - Quark Scattering



$$M_{fi} = \frac{Q_q e^2}{q^2}\left[\bar{u}(p_3)\gamma^\mu u(p_1)\right] g_{\mu\nu} \left[\bar{u}(p_4)\gamma^\nu u(p_2)\right]$$

t-channel electron-quark scattering

$p_3^\mu p_1^\nu - g^{\mu\nu}(p_1 \cdot p_3) + p_1^\mu p_3^\nu + m_e^2 g^{\mu\nu})$

$$\sum_{\text{spins}} |M_{fi}|^2 = \frac{Q_q^2 e^4}{q^4}\,\mathrm{Tr}\left([\not{p}_3 + m_e]\gamma^\mu[\not{p}_1 + m_e]\gamma^\nu\right)$$

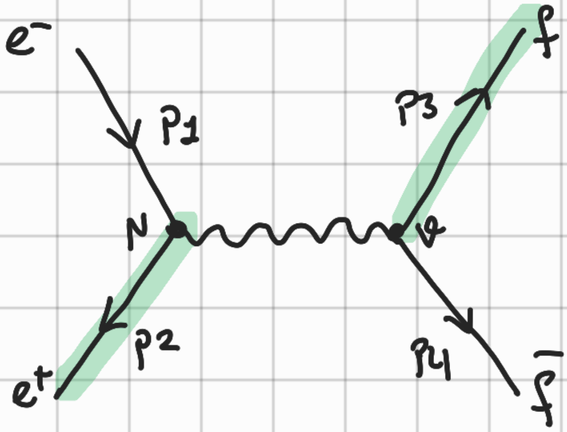
$$\times\, \mathrm{Tr}\left([\not{p}_4 + m_q]\gamma_\mu[\not{p}_2 + m_q]\gamma^\nu\right)$$

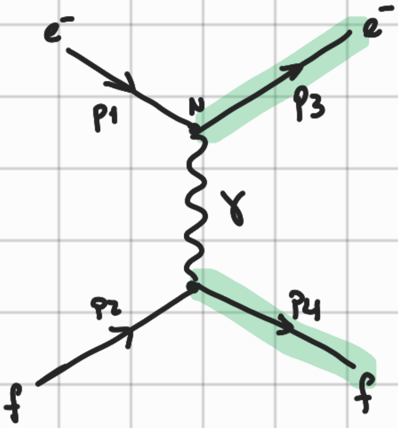
$p_{4\mu} p_{2\nu} - g_{\mu\nu}(p_2 \cdot p_4) + p_{2\mu} p_{4\nu} + m_q^2 g_{\mu\nu}$

$\Big\downarrow$ simplify

$$\langle |M_{fi}|^2 \rangle = 2 Q_q^2 e^4 \left(\frac{s^2 + u^2}{t^2}\right)$$

Crossing Symmetry





$$j_e^{\mu} = \bar{v}(p_2)\gamma^{\mu}u(p_1) \qquad\qquad j_e^{\mu} = \bar{u}(p_3)\gamma^{\mu}u(p_1)$$

$$j_f^{\nu} = \bar{u}(p_3)\gamma^{\nu}v(p_4) \qquad\qquad j_f^{\nu} = \bar{u}(p_4)\gamma^{\nu}u(p_2)$$

$$\left.\begin{aligned} u(p_1) &\rightarrow u(p_1) \\ \bar{v}(p_2) &\rightarrow \bar{u}(p_3) \\ \bar{u}(p_3) &\rightarrow \bar{u}(p_4) \\ v(p_4) &\rightarrow u(p_2) \end{aligned}\right\}$$

$M_{e^-f \rightarrow e^-f}$ can be obtained from $M_{e^+e^- \rightarrow f\bar{f}}$ by those substi.

$p_1 \rightarrow p_1$, $p_2 \rightarrow -p_3$

$p_3 \rightarrow p_4$, $p_4 \rightarrow -p_2$

antiparticle   particle

$$\boxed{s^2 \rightarrow t^2 \quad , \quad t^2 \rightarrow u^2 \quad , \quad u^2 \rightarrow s^2}$$

$$\langle |M_{fi}|^2 \rangle_s = 2Q_f^2 e^4 \left(\frac{t^2 + u^2}{s^2} \right) \longleftrightarrow \langle |M_{fi}|^2 \rangle_t = 2Q_f^2 e^4 \left(\frac{u^2 + s^2}{t^2} \right)$$