Week 4 Particle Decay Rates

Cross Sections & Decay Rates

Fermi's Golden Rule

where does it come from?

$$\Gamma_{fi} = 2\pi \, |T_{fi}|^2 \rho(E_f)$$

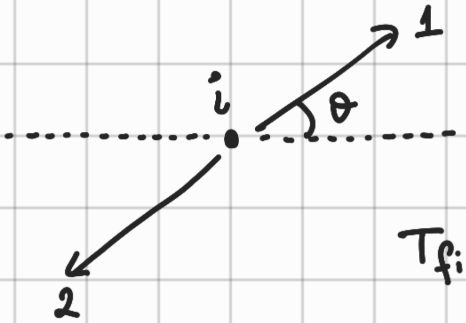
# of transitions per unit time from $|i\rangle$ to final $\langle f|$

transition matrix element with perturbing Hamiltonian $\hat{H}$

$$T_{fi} = \langle f|\hat{H}|i\rangle + \sum_{j \neq i} \frac{\langle f|\hat{H}|j\rangle\langle j|\hat{H}|i\rangle}{E_i - E_j} + \ldots$$

intermediate particle j

$\rho(E_f)$: density of final states $\left|\frac{dn}{dE}\right|_{E_f}$



$i \rightarrow 1 + 2$

$$T_{fi} = \langle \psi_1 \psi_2 | \hat{H} | \psi_i \rangle = \int_V \psi_1^* \psi_2^* \hat{H} \psi_i \, d^3x$$

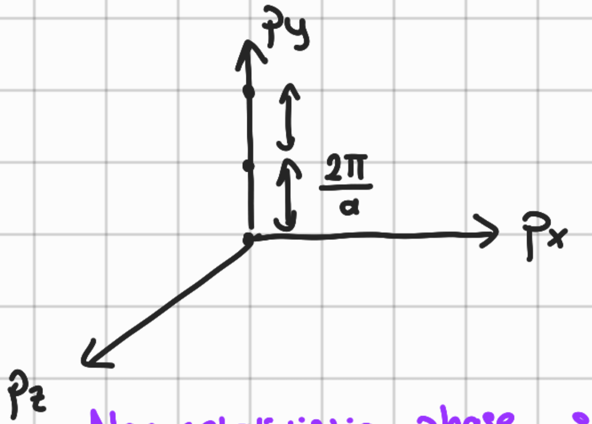
Let's calculate the decay rate in first order perturbation theory using plane-wave description of the particles → Born approxim.

$$\psi_1 = N e^{i(\vec{p}\cdot\vec{r} - ET)} = N e^{-ip.x}$$

?

→ Normalization

Normalization → $\int \psi \psi^* \, dV = N^2 a^3 = 1 \Rightarrow N^2 = 1/a^3$



$$p_x = \frac{2\pi n_x}{a} \qquad p_y = \frac{2\pi n_y}{a} \qquad p_z = \frac{2\pi n_z}{a}$$

$$\psi(x+a, y, z) \downarrow \psi(x, y, z)$$

Non-relativistic phase space

Volume of single state in momentum space:

$$\left(\frac{2\pi}{a}\right)^3 = \frac{(2\pi)^3}{V}$$

$$dn = \frac{d^3\vec{p}}{(2\pi)^3/V} \times \frac{1}{V} = \frac{d^3\vec{p}}{(2\pi)^3} = \frac{dp_x \, dp_y \, dp_z}{(2\pi)^3}$$

# of states

$$\rho(E_f) = \left| \frac{dn}{dE} \right|_{E_f} = \left| \underbrace{\frac{dn}{d|\vec{p}|}} \frac{d|\vec{p}|}{dE} \right|_{E_f}$$

$$d^3\vec{p} = 4\pi p^2 dp$$

$$\frac{dn}{d|\vec{p}|} = \frac{1}{(2\pi)^3} \frac{d^3\vec{p}}{d|\vec{p}|} = \frac{4\pi p^2 dp}{(2\pi)^3 dp} = \frac{4\pi p^2}{(2\pi)^3}$$

$$E^2 = p^2 + m^2$$ → m is constant ?

$$\hookrightarrow \quad 2E\,dE = 2p\,dp$$

$$\frac{dp}{dE} = \frac{E}{p} \approx \frac{1}{\beta} \qquad (1/\beta \gg 1)$$

why ??

$$\rho(E_f) = \frac{4\pi p^2}{(2\pi)^3} \times \frac{1}{\beta}$$

energy conversion

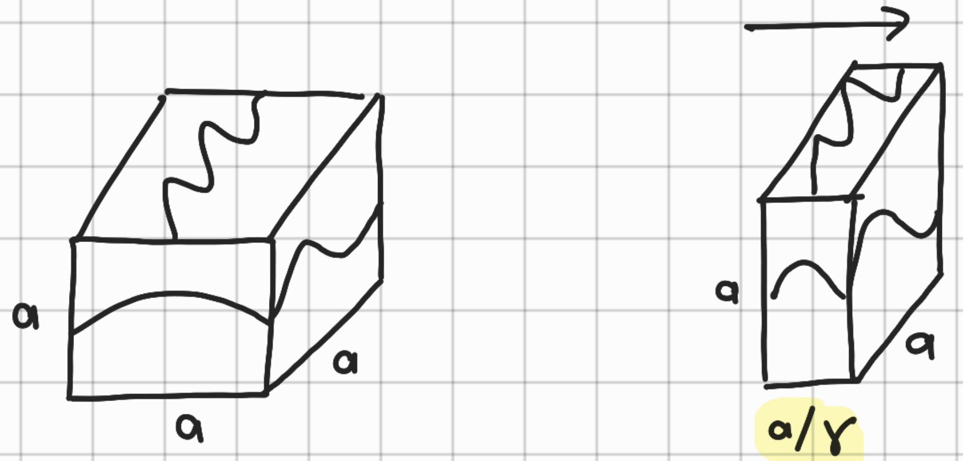
$$\rho(E_f) = \left| \frac{dn}{dE} \right|_{E_f} = \int \frac{dn}{\cancel{dE}} \delta(E-E_i)\,\cancel{dE}$$ since $E_f = E_i$

$$\Gamma_{fi} = 2\pi \int |T_{fi}|^2 \, \delta(E_i - E)\, dn$$

$$\Gamma_{fi} = (2\pi)^4 \int |T_{fi}|^2 \underbrace{\delta(E_i - E_1 - E_2)}_{E \text{ conservation}} \underbrace{\delta^3(p_i - p_1 - p_2)}_{\vec{p} \text{ conservation}} \underbrace{\frac{d^3\vec{p}_1}{(2\pi)^3} \frac{d^3\vec{p}_2}{(2\pi)^3}}_{\text{density of states}}$$

→ for x, y, z

Lorentz Invariant Phase Space



Particle density therefore increases by $\gamma = E/m$

from where?

Usual convention: Normalize to $2E$ particles/ unit volume $\int \psi'^* \psi' dV = 2E$

why?

$$\psi' = (2E)^{1/2} \psi$$

Lorentz invariant matrix element, M_{fi} → How?

$$M_{fi} = \langle \psi_1' \psi_2' \ldots \ldots | \hat{H} | \ldots \ldots \psi_{n-1}' \psi_n' \rangle$$

$$= (2E_1 \, 2E_2 \ldots \ldots 2E_{n-1} \, 2E_n)^{1/2} T_{fi}$$

with ψ_s

$$M_{fi} = \langle \psi_1' \psi_2' | \hat{H} | \psi_i' \rangle = (2E_i . 2E_1 . 2E_2)^{1/2} \langle \psi_1 \psi_2 | \hat{H} | \psi_i \rangle$$

$$= (2E_i . 2E_1 . 2E_2)^{1/2} T_{fi}$$

$$\Gamma_{fi} = \frac{(2\pi)^4}{2E_i} \int |M_{fi}|^2 \delta(E_i - E_1 - E_2)\, \delta^3(\vec{p_i} - p_1 - p_2) \frac{d^3 p_1}{(2\pi)^3 2E_1} \frac{d^3 p_2}{(2\pi)^3 2E_2}$$

$$\frac{d^3\vec{p}}{(2\pi)^3 2E} \rightsquigarrow \text{Lorentz Invariant Phase Space}$$

Two-Body Decay

Since this integral Lorentz invariant

It can be evaluated in any frame

In Center of Mass (CoM) frame: $E^2 = M^2 + p^2$ $\rightarrow E_i = m_i,\ p_i = 0$ $(c = 1)$

$$\Gamma_{fi} = \frac{1}{8\pi^2 E_i} \int |M_{fi}|^2 \delta(m_i - E_1 - E_2)\, \delta^3(\underbrace{p_1 + p_2}_{?}) \frac{d^3\vec{p_1}}{2E_1} \frac{d^3\vec{p_2}}{2E_2}$$

Integrate over p_2 using δ function $\rightarrow p_2 = -p_1$

$$\Gamma_{fi} = \frac{1}{8\pi^2 E_i} \int |M_{fi}|^2 \delta(m_i - E_1 - E_2) \frac{d^3 p_1}{4E_1 E_2}$$

$$d^3 p_1 = p_1^2\, dp_1\, \underbrace{\sin\theta\, d\theta\, d\phi}_{d\Omega}$$

$$E_1^2 = M_1^2 + |\vec{p_1}|^2 \qquad E_2^2 = M_2^2 + |\vec{p_1}|^2$$

$(p_1 = -p_2)$

$$\Gamma_{fi} = \frac{1}{8\pi^2 E_i} \int |M_{fi}|^2 \, \delta\left(M_i - \sqrt{m_1^2 + p_1^2} - \sqrt{M_2^2 + p_1^2}\right) \frac{p_1^2 \, dp_1 \, d\Omega}{E_1 E_2}$$

$\downarrow$ Write in this form

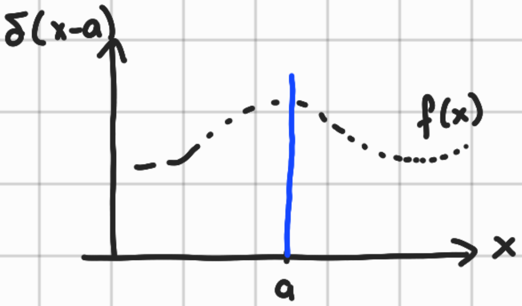
$$\Gamma_{fi} = \frac{1}{32\pi^2 E_i} \int |M_{fi}|^2 \, g(p_1) \, \delta(f(p_1)) \, dp_1 \, d\Omega$$

$$g(p_1) = \frac{p_1^2}{E_1 E_2} = p_1^2 \frac{1}{\sqrt{(m_1^2 + p_1^2)} \cdot \sqrt{(m_2^2 + p_1^2)}}$$

$$f(p_1) = m_i - (m_1^2 + p_1^2)^{1/2} - (m_2^2 + p_1^2)^{1/2}$$

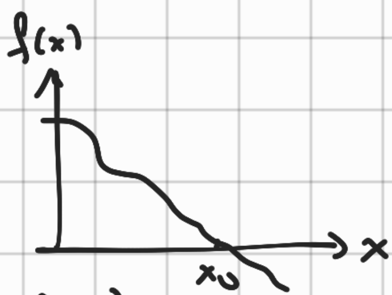
$f(p_1) = 0$ for $p_1 = -p_2 = p^*$ CoM

Dirac δ-function Review



$$\int_{-\infty}^{\infty} \delta(x-a)\, dx = 1$$

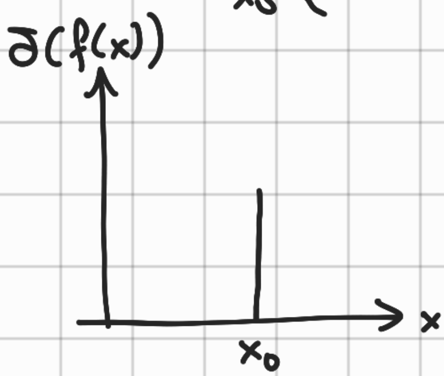
$$\int_{+\infty}^{\infty} f(x)\,\delta(x-a)\, dx = f(a)$$



$$\int_{y_1}^{y_2} \delta(y)\, dy = \begin{cases} 1 & \text{if } y_1 < 0 < y_2 \\ 0 & \text{otherwise} \end{cases}$$

$\downarrow$ $y = f(x)$

$$\int_{x_1}^{x_2} \delta(f(x)) \frac{df}{dx} dx = \begin{cases} 1 & \text{if } x_1 < x_0 < x_2 \\ 0 & \text{oth.} \end{cases}$$



$$\left|\frac{df}{dx}\right| \int_{x_1}^{x_2} \delta(f(x))\,dx = \begin{cases} 1 & \text{if} \quad x_1 < x_0 < x_2 \\ 0 & \text{otherwise} \end{cases}$$

$$\int_{x_1}^{x_2} \delta(f(x))\,dx = \frac{1}{\left|\frac{df}{dx}\right|_{x_0}} \int_{x_1}^{x_2} \delta(x-x_0)\,dx$$

$$\delta(f(x)) = \left|\frac{df}{dx}\right|_{x_0}^{-1} \delta(x-x_0)$$

Using this equation :

$$\int g(p_1)\,\delta(f(p_1))\,dp_1 = \frac{1}{\left|\frac{df}{dp_1}\right|_{p^*}} \int g(p_1)\,\delta(p-p^*)\,dp_1$$

$$= \frac{g(p^*)}{\left|\frac{df}{dp_1}\right|_{p^*}} \quad \text{where } f(p^*) = 0$$

Let's evaluate $\frac{df}{dp_1}$

$$\frac{df}{dp_1} = -\frac{p_1}{(M_1^2+p_1^2)^{1/2}} - \frac{p_1}{(M_2^2+p_1^2)^{1/2}} = -\frac{p_1}{E_1} - \frac{p_1}{E_2} = -p_1\frac{E_1+E_2}{E_1E_2}$$

$$\Gamma_{fi} = \frac{1}{32\pi^2 E_i} \int |M_{fi}|^2 \left| \overbrace{\frac{\cancel{E_1E_2}}{\cancel{p_1}(E_1+E_2)}}^{\left|\frac{df}{dp_1}\right|_{p^*}^{-1}} \; \overbrace{\frac{p_1^{\cancel{2}}}{\cancel{E_1}\cancel{E_2}}}^{g(p_1)} \right|_{p_1=p^*} d\Omega$$

$\downarrow$

$$\Gamma_{fi} = \frac{1}{32\pi^2 E_i} \int |M_{fi}|^2 \left| \frac{p_1}{E_1 + E_2} \right|_{p_1 = p^*} d\Omega$$

M_i

$$\Gamma_{fi} = \frac{|\vec{p}^*|}{32\pi^2 E_i\, m_i} \int |M_{fi}|^2 \, d\Omega$$

M_i

$$\frac{1}{\tau} = \Gamma_{fi} = \frac{|\vec{p}^*|}{32\pi^2 m_i^2} \int |M_{fi}|^2 \, d\Omega$$