Particle Physics 1: Exercise 6

Exercise 1

Show that

$$[\hat{\mathbf{p}}^{\mathbf{2}}, \hat{\mathbf{r}} \times \hat{\mathbf{p}}] = 0$$

and hence the Hamiltonian of non-relativistic free particle commutes with the angular momentum operator.

Exercise 2

By operating on the Dirac equation,

$$(i\gamma^{\mu}\partial_{\mu} - m)\psi = 0$$

with $\gamma^{\nu}\partial_{\nu}$ prove that the components of ψ satisfy the Klein-Gordon equation.

Exercise 3

Explain what helicity is and verify that the helicity operator:

$$h = \frac{\hat{\boldsymbol{\Sigma}} \cdot \hat{\mathbf{p}}}{2p} = \frac{1}{2p} \begin{pmatrix} \sigma \cdot \hat{\mathbf{p}} & 0 \\ 0 & \sigma \cdot \hat{\mathbf{p}} \end{pmatrix}$$

Commutes with the Dirac Hamiltonian

$$\hat{H}_D = \alpha \cdot \hat{\mathbf{p}} + \beta m$$

Exercise 1

Show that

$$[\hat{\mathbf{p}}^2, \hat{\mathbf{r}} \times \hat{\mathbf{p}}] = 0$$

and hence the Hamiltonian of non-relativistic free particle commutes with the angular momentum operator.

$$[\hat{p}^2, \hat{L}_x] = [\hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2, \hat{y}\hat{p}_z - z\hat{p}_y]$$

$$= [\hat{p}_y^2, \hat{y}\hat{p}_z] - [\hat{p}_z^2, \hat{z}\hat{p}_y]$$

$$= [\hat{p}_y^2, \hat{y}]\hat{p}_z - [\hat{p}_z^2, z]\hat{p}_y$$

$$[y, \hat{p}_y] = \hat{y}\hat{p}_y - \hat{p}_y\hat{y} = i$$

$$[\hat{p}_y^2, \hat{y}] = \hat{p}_y \overbrace{\hat{p}_y\hat{y}}^{\hat{y}\hat{p}_y - i} - \hat{y}\hat{p}_y\hat{p}_y$$

$$= \hat{p}_y(\hat{y}\hat{p}_y - i) - \hat{y}\hat{p}_y\hat{p}_y$$

$$= \underbrace{\hat{p}_y\hat{y}}\hat{p}_y - i\hat{p}_y - \hat{y}\hat{p}_y\hat{p}_y$$

$$= (\hat{y}\hat{p}_y - i)\hat{p}_y - i\hat{p}_y - \hat{y}\hat{p}_y\hat{p}_y$$

$$= -2i\hat{p}_y$$

$$[\hat{p}_z^2, \hat{z}] = -2i\hat{p}_z$$

Thus $[\hat{p}^2, L] = 0$

(non-relativistic free particle)

$$\hat{H}_0 = \hat{p}^2/2M$$

$$[\hat{H}_0, L] = 0$$

$$[\hat{p}^2, \hat{L}_x] = [\hat{p}_y^2, \hat{y}]\hat{p}_z - [\hat{p}_z^2, \hat{z}]\hat{p}_y$$

$$= -2i\hat{p}_y\hat{p}_z + 2i\hat{p}_z\hat{p}_y = 2i[\hat{p}_z, \hat{p}_y] = 0$$

Exercise 2

By operating on the Dirac equation,

$$(i\gamma^{\mu}\partial_{\mu} - m)\psi = 0$$

with $\gamma^{\nu}\partial_{\nu}$ prove that the components of ψ satisfy the Klein-Gordon equation.

Klein- Gordon Equation: $(\partial^{\mu}\partial_{\mu} + m^2)\psi = 0$

$$(i\gamma^{\mu}\partial_{\mu} - m)\psi = 0$$

$$\times(-i) \quad i\gamma^{\nu}\gamma^{\mu}\partial_{\nu}\partial_{\mu}\psi - m\gamma^{\nu}\partial_{\nu}\psi = 0$$

$$\gamma^{\nu}\gamma^{\mu}\partial_{\nu}\partial_{\mu}\psi + m\underbrace{i\gamma^{\nu}\partial_{\nu}\psi}_{m\psi} = 0$$

satisfy Dirac Equation

$$(\underbrace{\gamma^{\nu}\gamma^{\mu}\partial_{\nu}\partial_{\mu}} + m^2)\psi = 0$$

$$\downarrow$$

$$\frac{1}{2}(\gamma^{\nu}\gamma^{\mu} + \gamma^{\nu}\gamma^{\mu})\partial_{\nu}\partial_{\mu} = g^{\mu\nu}\partial_{\nu}\partial_{\mu}$$

$$(\overset{\partial^{\mu}}{g^{\mu\nu}\partial_{\nu}}\partial_{\mu} + m^2)\psi = 0$$

$$(\partial^{\mu}\partial_{\mu} + m^2)\psi = 0$$

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(interpolatation)

Exercise 3

Explain what helicity is and verify that the helicity operator:

$$h = \frac{\hat{\mathbf{\Sigma}} \cdot \hat{\mathbf{p}}}{2p} = \frac{1}{2p}\begin{pmatrix} \sigma \cdot \hat{\mathbf{p}} & 0 \\ 0 & \sigma \cdot \hat{\mathbf{p}} \end{pmatrix}$$

Commutes with the Dirac Hamiltonian

$$\hat{H}_D = \alpha \cdot \hat{\mathbf{p}} + \beta m$$

$$[\hat{h}, \beta m] = 0$$

$\beta m \to$ contains identity matrix

$$[\hat{h}, \alpha \cdot \vec{p}\,] = \frac{1}{2p}\left[\overset{h}{\begin{pmatrix} \vec{\sigma}\cdot\vec{p} & 0 \\ 0 & \vec{\sigma}\cdot\vec{p} \end{pmatrix}} \overset{\vec{\alpha}\cdot\vec{p}}{\begin{pmatrix} 0 & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & 0 \end{pmatrix}} - \begin{pmatrix} 0 & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & 0 \end{pmatrix} \begin{pmatrix} \vec{\sigma}\cdot\vec{p} & 0 \\ 0 & \vec{\sigma}\cdot\vec{p} \end{pmatrix} \right]$$

$$= \frac{1}{2p}\left[\begin{pmatrix} 0 & (\vec{\sigma}\cdot\vec{p})^2 \\ (\vec{\sigma}\cdot\vec{p})^2 & 0 \end{pmatrix} - \begin{pmatrix} 0 & (\vec{\sigma}\cdot\vec{p})^2 \\ (\vec{\sigma}\cdot\vec{p})^2 & 0 \end{pmatrix} \right]$$

$$= 0 \quad \checkmark$$

Thus $\hat{h}$ and $\hat{H}_D$ commute