30/11/22

Density Matrix → describes statistical mixture of states

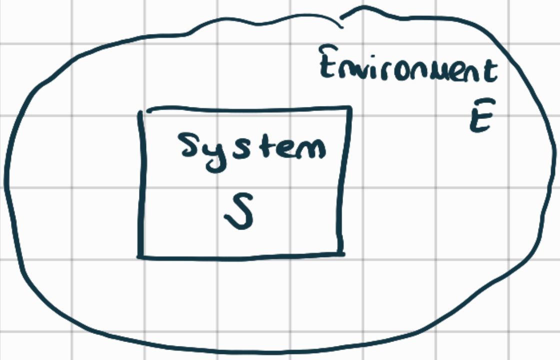
$|\varphi_1\rangle, \ldots, |\varphi_k\rangle$: $p_1, \ldots, p_k$

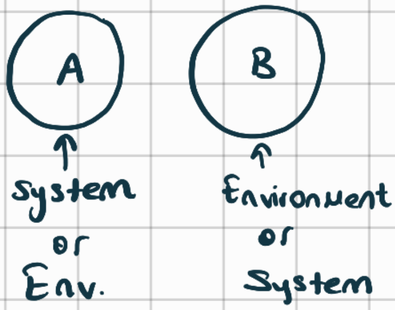
$0 \leqslant p_i \leqslant 1$, $\sum_i p_i = 1$ $\qquad \rho = \sum_k p_k |\varphi_k\rangle\langle\varphi_k|$

convex combination of projectors or pure states

$\lambda_i \geqslant 0$, $\lambda_1 + \lambda_2 + \ldots + \lambda_n = 1 \leftarrow \lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n$

System that is in "contact" with some environment





Notion of Partial Trace

Hilbert Space $\mathcal{H}_S \otimes \mathcal{H}_E$

vector $|\Psi\rangle = \sum_{j,l} c_{j,l} \, |u_j\rangle_S \otimes |v_l\rangle_E$

Matrix $A = \sum a_{jl;j'l'} \left(|u_j\rangle_S \otimes |v_l\rangle_E \right) \underbrace{\left(\langle v_{j'}|_S \otimes \langle v_{l'}|_E \right)}$

line of matrix — column of matrix

$\left(|u_j\rangle_S \langle u_{j'}|_S \right) \otimes \left(|v_l\rangle_E \langle v_{l'}|_E \right)$

Partial Trace of A over $\mathcal{H}_E$

$$Tr_E(A) = Tr_E \left(|u_j\rangle\langle u_{j'}|_S \right) \otimes \left(|v_l\rangle_E \langle v_{l'}|_E \right) \overset{def}{\equiv} |u_j\rangle_S \langle u_{j'}|_S \, \underbrace{Tr_E \left(|v_l\rangle_E \langle v_{l'}|_E \right)}_{\langle v_l | v_{l'} \rangle = \delta_{ll'}}$$

by cyclicity

$$Tr_E(A) = \sum_{j,j'} \left\{ \sum_l a_{jl,j'l} \right\} |u_j\rangle_S \langle u_{j'}|_S$$

→ Smaller matrix over $\mathcal{H}_S$

Partial Trace of A over $\mathcal{H}_S$

Yield a smaller matrix over $\mathcal{H}_E$

$$\mathrm{Tr}_S\, A = \sum_{j,j',e,e'} a_{je,j'e'}\, \mathrm{Tr}_S \left(|u_j\rangle\langle u_{j'}| \otimes |\sigma_e\rangle\langle\sigma_{e'}| \right)$$

$$\mathrm{Tr}_S \underbrace{|u_j\rangle\langle u_{j'}|}_{\langle u_j | u_{j'}\rangle = \delta_{jj'}}$$

$$\mathrm{Tr}_S\, A = \sum_{e,e'} \left\{ \sum_j a_{je,je'} \right\} |v_e\rangle_E \langle v_e|_E$$

Remark: $M = |\varphi\rangle_A \langle\varphi'|_A \otimes |\psi\rangle_B \langle\psi'_B|$

$\mathcal{H}_A \otimes \mathcal{H}_B$

Matrix acts on $\mathcal{H}_A$

$d_A \times d_A$

dimension

$$\mathrm{Tr}_B\, M = \left(|\varphi\rangle_A \langle\varphi'|_A \right) \langle\psi'|\psi\rangle_B$$

acts on $\mathcal{H}_B$

$d_B \times d_B$

$$\mathrm{Tr}_A\, M = \langle\varphi'|\varphi\rangle_A \left(|\psi\rangle_B \langle\psi'|_B \right)$$

Remark: Array formalism

$$M = \begin{array}{c} \\ \left(\begin{array}{cccc} M_{0000} & M_{0001} & M_{0010} & M_{0011} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{array} \right) \end{array} \begin{array}{c} AB \\ 00 \\ 01 \\ 10 \\ 11 \end{array}$$

columns: 00, 01, 10, 11; blocks: C, D

$\mathbb{C}^2 \otimes \mathbb{C}^2$

$$\mathrm{Tr}_B\, M = \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix}$$

$$\mathrm{Tr}_A\, M = \left(\boxed{\begin{matrix} \cdot & \cdot \\ \cdot & \cdot \end{matrix}} \right)$$

C+D

Important Example

4x4 matrix $\quad \rho_{Bell} = |B_{00}\rangle\langle B_{00}| \underset{\substack{\downarrow \\ \text{in comp. basis}}}{=} \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)\frac{1}{\sqrt{2}}\left(\langle 00| + \langle 11|\right)$

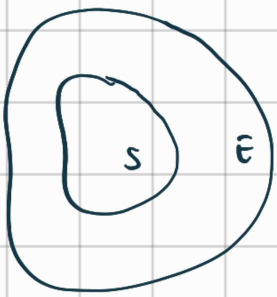
$$= \frac{1}{2}\left\{ \overset{AB}{|00\rangle}\overset{AB}{\langle 00|} + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11| \right\}$$

$$= \frac{1}{2}\left\{ |0\rangle_A\langle 0|_A \otimes \underbrace{|0\rangle_B\langle 0|_B}_{\langle 0|0\rangle} + |0\rangle_A\langle 1|_A \otimes \cancel{\underbrace{|0\rangle_B\langle 1|_B}_{\underbrace{\langle 1|0\rangle}_{0}}} + \cancel{\cdots} \right.$$

$$\boxed{\mathrm{Tr}_B\, \rho_{Bell} = \frac{1}{2}\left\{ |0\rangle\langle 0|_A + |1\rangle\langle 1|_A \right\}} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_A$$

$$\mathrm{Tr}_A\, \rho_{Bell} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_B$$

Reduced Density Matrix of a system S interact with E



$\mathcal{H}_{total} = \mathcal{H}_S \otimes \mathcal{H}_E$

Assume that state of $S \cup E$ is $|\Psi\rangle_{S\cup E}$

$\rho_{S\cup E} = |\Psi\rangle_{S\cup E}\langle\Psi|_{S\cup E}$

RDM of S is by definition $\quad \boxed{\rho_S = \mathrm{Tr}_{\mathcal{H}_E}\, \rho_{S\cup E}}$

Generalization $\qquad \mathcal{H}_{tot} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$

Total state ρ_{ABC}

$$\rho_A = \mathrm{Tr}_{BC}\, \rho_{ABC} \qquad \rho_{AB} = \mathrm{Tr}_C\, \rho_{ABC}$$

$$\rho_B = \mathrm{Tr}_{AC}\, \rho_{ABC} \qquad \rho_{AC} = \mathrm{Tr}_B\, \rho_{ABC}$$

$$\rho_C = \mathrm{Tr}_{AB}\, \rho_{ABC} \qquad \rho_{AB} = \mathrm{Tr}_C\, \rho_{ABC}$$

Example with 3 qubits

$$|GHZ\rangle = \frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(|0\rangle_A \otimes |0\rangle_B \otimes |0\rangle_C + |1\rangle_A \otimes |1\rangle_B \otimes |1\rangle_C\right) \qquad \text{"entangled"}$$

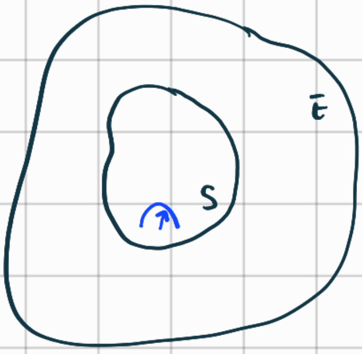
$\rho_{GHZ} = |GHZ\rangle\langle GHZ| \qquad$ 8x8 Matrices $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$

$$\rho_{GHZ} = \frac{1}{2}\left\{ |000\rangle\langle 000| + |000\rangle\langle 111| + |111\rangle\langle 000| + |111\rangle\langle 111| \right\}$$

$$\rho_A = \mathrm{Tr}_{BC}\, \rho_{GHZ} = \frac{1}{2}\left\{ |0\rangle_A\langle 0|_A + |1\rangle_A\langle 1|_A \right\}$$

$$\rho_{AB} = \mathrm{Tr}_C\, \rho_{GHZ} = \frac{1}{2}\left\{ |00\rangle\langle 00| + |11\rangle\langle 11| \right\}$$

Local experiments can be described by RDM



$$\rho_{S\cup E} \rightsquigarrow \rho_S = \mathrm{Tr}_E\, \rho_{S\cup E}$$

Suppose I measure observable: $A_S \otimes \mathbb{1}_E$

Average value of measurement is

$$Av(A_S \otimes 1_E) = \mathrm{Tr}_{S\cup E}\, \rho_{S\cup E}\, A_S \otimes 1_E$$

$$= \mathrm{Tr}_S \left\{ (\underbrace{\mathrm{Tr}_E\, \rho_{S\cup E}}_{\rho_S}) A_S \right\} = \mathrm{Tr}_S (\rho_S A_S)$$

* Suppose S manipulates qubits by local unitary operation

$$U_{tot}(t) = U_S(t) \otimes 1_E$$

$$\rho_{S\cup E}(t) = U_{tot}(t)\, \rho_{S\cup E}\, U_{tot}(t)^\dagger$$

$$\rho_S(t) = \mathrm{Tr}_E\, U_S \otimes \mathbb{I}_E\, \rho_{S\cup E}\, U_S^\dagger \otimes 1_E$$

$$= U_S (\mathrm{Tr}_E\, \rho_{S\cup E}) U_S^\dagger = U_S(t)\, \rho_S\, U_S^\dagger(t)$$