from Youtube Channel: The Curiosity Effect

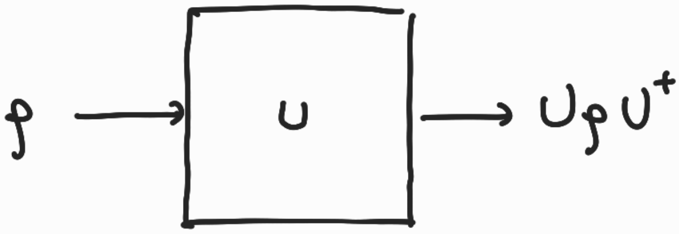
Deriving the Markovian Lindblad Master Equation from Quantum Operators

$$\frac{d\rho}{dt} = -i[H,\rho] - \sum_p L_p^\dagger L_p \rho - \sum_p \rho L_p^\dagger L_p + \sum_p 2 L_p \rho L_p^\dagger$$

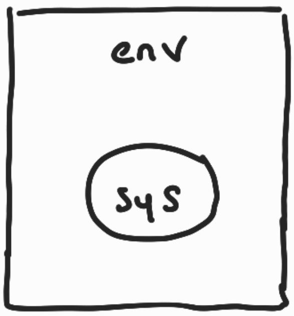
Quantum Operation

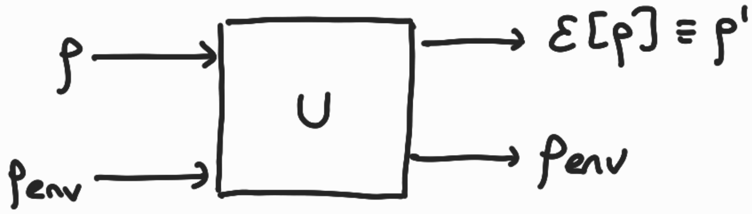
Closed Sys.

$|\psi_{in}\rangle \rightarrow U|\psi_{in}\rangle = |\psi_{out}\rangle$



Open Sys.





$$\rho' = \mathrm{Tr}_{env}\left[U(\rho \otimes \rho_{env}) U^\dagger \right]$$

Assumption 1

The initial state can be written as a product state. There's no correlation initially.

$$\rho_{env} = \sum_k \sum_{k'} |e_k\rangle\langle e_{k'}| \, \alpha_{kk'}, \qquad \langle e_{k'} | e_k \rangle = \delta_{k',k}$$

Assumption 2

Initially $\rho_{env} = |e_0\rangle\langle e_0|$ in a pure state

Then,

$$\rho' = \sum_k \langle e_k | U \, \rho \otimes |e_0\rangle\langle e_0| \, U^{+} | e_k \rangle$$

$$\rho' = \sum_k \langle e_k | U \overbrace{\sum_{k'} |e_{k'}\rangle\langle e_{k'}|}^{\text{IS IT "1"? } \checkmark \ I} \rho \, |e_0\rangle\langle e_0| U^{+} | e_k \rangle$$

$$\rho' = \sum_k \sum_{k'} \langle e_k | U | e_{k'} \rangle \, \delta_{k',0} \, \rho \, \langle e_0 | U^{+} | e_k \rangle$$

$$\boxed{\rho' = \sum_k E_k \, \rho \, E_k^{+}} \qquad \text{where } E_k = \langle e_k | U | e_0 \rangle$$

$$\rho' = \rho(t) \qquad \rho = \rho(0) \quad \Rightarrow \quad \rho(t) = \sum_k E_k(t) \, \rho(0) \, E_k^{+}(t)$$

Quantum Operation describes a change of state from $t=0$ to t. It is not continious evolution. But in Lindblad we need continious evolution.

$$\rho(t+\Delta t) = \rho(t) + \Delta t \frac{d\rho}{dt} + O(\Delta t^2)$$

We should look for a quantum operation

$$\rho(t+\Delta t) = \sum_k E_k(\Delta t)\,\rho(t)\,E_k^{+}(\Delta t) + O(\Delta t^2)$$

going back to :

$$\rho(t+\Delta t) = Tr_{env}\left[U \rho_{s,env}(t)\, U^{+}\right] \text{ where}$$

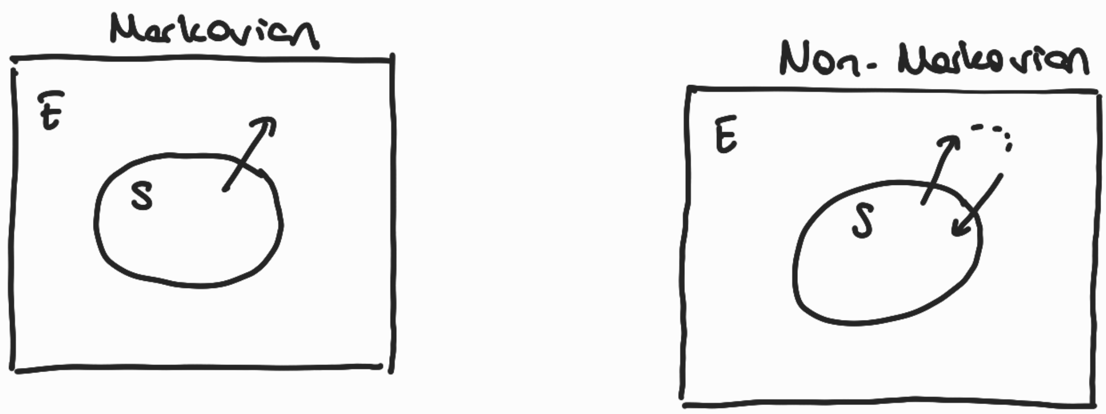
$$\rho_{s,env}(0) = \rho_s \otimes \rho_{env}$$

Markov Approximation

$\rho(t+\Delta t)$ is completely determined by $\rho(t)$.

Non-Markovian Approximation

$\rho(t+\Delta t)$ is determined by $\rho(t)$ and the environment. Since the environment can pass information back to the system.



Time that E remembers S = δt_E
After δt_E, E forgets what S did.

Assumption 3

Timescale of the S evolution is δt_S

$$\delta t_S \gg \delta t_E$$

Assumption 4

The environment doesn't couple strongly to S

$$\rho_{S,env}(t) = \rho_S(t) \otimes \rho_{env} \quad , \quad \rho_{env} \simeq |e_0\rangle\langle e_0| \text{ always in a pure state}$$

$$\rho(t+\Delta t) = \mathrm{Tr}_{env}\left[U \rho(t) \otimes \rho_{env} U^\dagger\right]$$

$$\rho(t+\Delta t) = \sum_k \langle e_k|U|e_0\rangle \rho(t) \langle e_0|U^\dagger|e_k\rangle$$

$$\rho(t+\Delta t) = \sum_k M_k(\Delta t) \rho(t) M_k^\dagger(\Delta t)$$

$$M_k(\Delta t) = M_k^{(0)} + \sqrt{\Delta t}\, M_k^{(1)} + \Delta t\, M_k^{(2)} + \dots$$

$$= M_0(\Delta t) \rho(t) M_0^\dagger(\Delta t) + \sum_{k\neq 0} M_k(\Delta t) \rho(t) M_k^\dagger(\Delta t)$$

$$M_0(0) = 1 \quad , \quad M_k(0) = 0$$

$$\boxed{M_0(\Delta t) = 1 + \Delta t\, G} \qquad \boxed{M_k(\Delta t) = \sqrt{\Delta t}\, L_k + O(\Delta t)}$$

HOW?

$$\rho(t+\Delta t) = (1 + \Delta t G)\, \rho(t)\, (1 + \Delta t\, G^{+}) + \Delta t\, L_k\, \rho(t)\, L_k^{+} + O(\Delta t^2)$$

$$\rho(t+\Delta t) = \rho(t) + \Delta t\left(G\rho(t) + \rho(t) G^{+}\right) + \Delta t\, L_k\, \rho(t)\, L_k^{+} + O(\Delta t^2)$$

$$\frac{d\rho}{dt} = \frac{\rho(t+\Delta t) - \rho(t)}{\Delta t} = G\rho(t) + \rho(t) G^{+} + L_k\, \rho(t)\, L_k^{+}$$

To get Lindblad form $G = K - iH$, $G^{+} = K + iH^{+}$

$$\frac{d\rho}{dt} = K\rho(t) + \rho(t) K - iH\rho(t) - i\rho(t) H^{+} + L_k\, \rho(t)\, L_k^{+}$$

If $H^{+} = H$ (Hermitian), $K = -\frac{1}{2} L_k^{+} L_k$

$$\boxed{\frac{d\rho}{dt} = -i[H, \rho] - \frac{1}{2} L_k^{+} L_k\, \rho(t) - \frac{1}{2} \rho(t)\, L_k^{+} L_k - L_k\, \rho(t)\, L_k^{+}}$$

- Quantum jumps (?)
- Quantum Trajectories
- POVM
- Spontaneous emission of a 2-level atom

From Preskill's Notes: (Chapter 3)

Master equation for open quantum systems

For a closed system:

$$|\Psi(t+dt)\rangle = (I - i\,dt\,\hat{H})\,|\Psi(t)\rangle$$

$$\frac{|\Psi(t+dt)\rangle - |\Psi(t)\rangle}{dt} = \frac{d\Psi}{dt} = -i\,\underbrace{\hat{H}|\Psi\rangle}_{i\frac{d|\Psi\rangle}{dt}}$$

$$\rho(t+dt) = \rho(t) - i\,dt\,[\hat{H}, \rho(t)]$$

For an open system:

Markovian evolution for the infinitesimal time interval dt may be expressed as:

$$\rho(t+dt) = \underbrace{\mathcal{E}_{dt}}_{\text{quantum channel}}(\rho(t))$$

$$\mathcal{E}_{dt} = I + dt\,\mathcal{L}$$

$$\rho(t+dt) = \rho(t) + dt\,\mathcal{L}(\rho)$$

$$\frac{\rho(t+dt) - \rho(t)}{dt} = \mathcal{L}(\rho)$$

$$\boxed{\dot{\rho} = \mathcal{L}(\rho)}$$ "Lindbladian"

This evolution equation has the formal solution:

$$\rho(t) = \lim_{n\to\infty}\left(1+\frac{\mathcal{L}t}{n}\right)^n(\rho(0)) = e^{\mathcal{L}t}(\rho(0))$$

where L is time dependent.

The channel has an operator-sum representation

$$\boxed{\rho(t+dt) = \mathcal{E}_{dt}(\rho(t)) = \sum_a M_a\, \rho(t)\, M_a^\dagger = \rho(t) + O(dt)} \qquad (*)$$

$$\boxed{\begin{aligned} M_0 &= I + (-iH + K)\,dt \\ M_a &= \sqrt{dt}\; L_a \end{aligned}}$$

From the kraus-operator completeness relation:

$$I = \sum_a M_a^\dagger M_a = I + dt\Big(\underbrace{2K + \sum_{a>0} L_a^\dagger L_a}\Big) + \dots$$

$$K = -\frac{1}{2}\sum_{a>0} L_a^\dagger L_a$$

$M_0^\dagger M_0$

$$\left(I + (+i\hat{H} + \hat{K})dt\right)\cdot\left(I + (-i\hat{H} + \hat{K})\,dt\right)$$

$$I + dt\,\underbrace{(-i\hat{H} + \hat{K}) + (i\hat{H} + \hat{K})}_{2\hat{K}dt}dt + dt^2(\_\_\_)^2$$

Substituting into (*)

$$\dot{\rho} = \mathcal{L}(\rho) = -i[H,\rho] + \sum_{a>0}\left(L_a\rho L_a^\dagger - \frac{1}{2}L_a^\dagger L_a\rho - \frac{1}{2}\rho L_a^\dagger L_a\right)$$