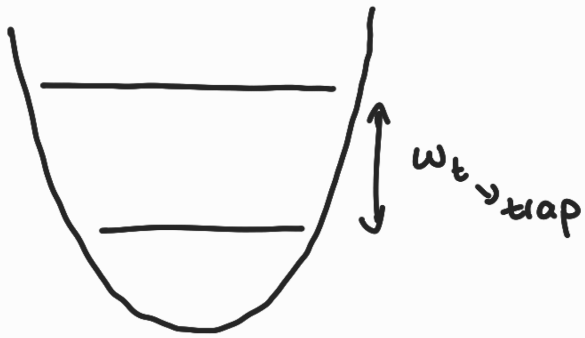
Sideband Cooling

So far, free space : $|\vec{k}\rangle$, $\vec{k}$ was continously distributed

Variations of $\vec{k}$ are continous $\Rightarrow$ Force
(continous change of momentum)

Discrete spectrum:



Harmonic trap

Physical realizations
* atoms / ions in traps
* Mechanical oscillators

I- Motional Sidebands

• Hilbert space: $\mathcal{H} = \mathcal{H}_{com} \otimes \mathcal{H}_{internal}$

Field: driven mode $\rightarrow$ coherent state (laser)
vacuum considered as a resavoir.

Two-level atom: $\mathcal{H}_{int} = \mathrm{Sp}\{|g\rangle, |e\rangle\}$

$\mathcal{H}_{com} = \mathrm{Sp}\{|n\rangle, n \in \mathbb{N}\}$

eigenstates of the harmonic trap

- <u>Hamiltonian:</u>

$\hat{z}$ on $\mathcal{H}_{com}$

$$\omega_e |e\rangle\langle e| \otimes \mathbb{I} + \omega_t \hat{\mathbb{I}} \otimes \hat{a}^+\hat{a} + \frac{\Omega}{2}\left(|e\rangle\langle g| \otimes e^{-i(k\hat{z}-\omega_L t)} + h.c\right)$$

$\hat{a}$: annihilates vibrational quanta ("phonons")

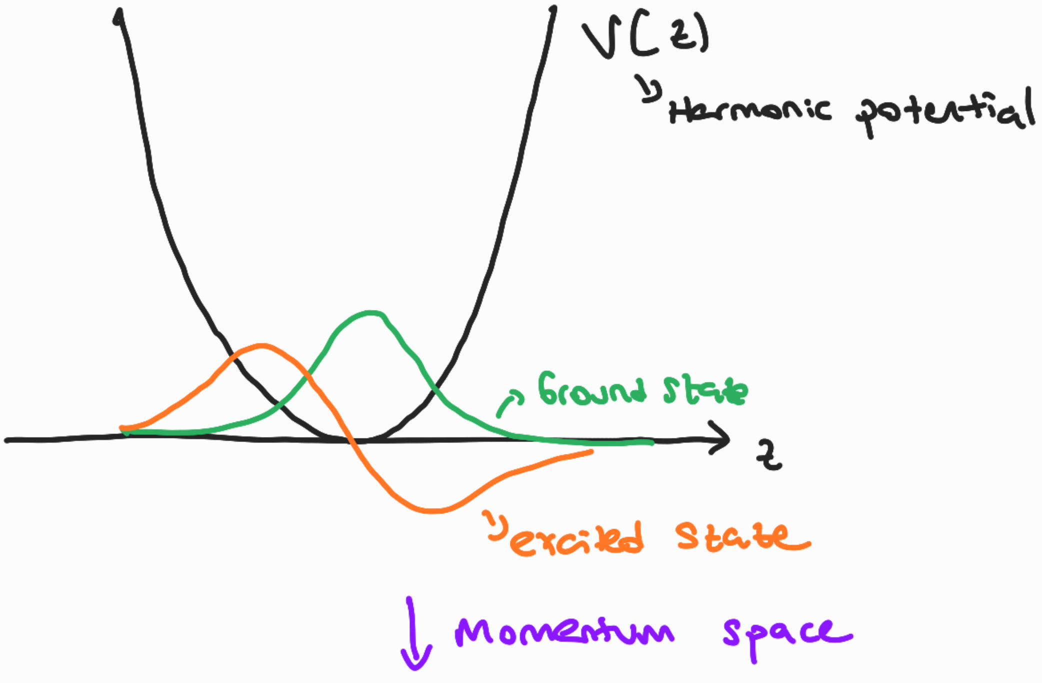
1) Lamb-Dicke parameter

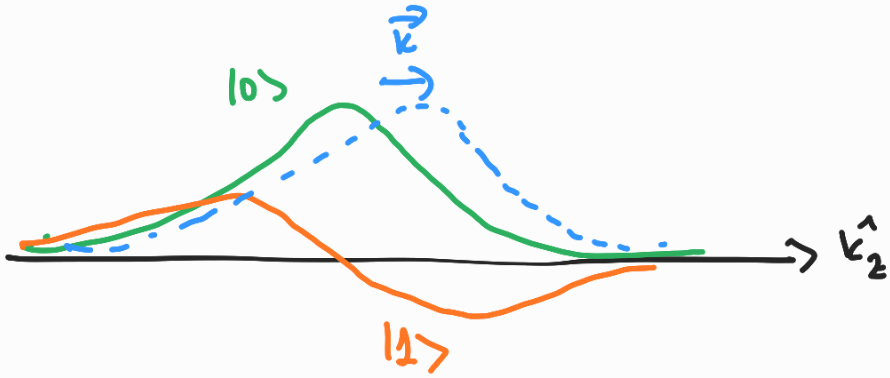
$\hat{z} = z_0(\hat{a} + \hat{a}^+)$ with $z_0 = \sqrt{\frac{\hbar}{m\omega_t}}$ harmonic length

$$e^{ik_L\hat{z}} = e^{ikz_0(\hat{a}+\hat{a}^+)}$$

$\boxed{\eta = kz_0}$

"Lamb-Dicke parameter





Photon absorption: $|\vec{p}\rangle \longrightarrow |\vec{p}+\vec{k}\rangle$ translation in k space

η measures this displacement w.r to the size of the wavefunction.

2) Sideband Spectrum

$$e^{i\overset{\eta}{\widetilde{kz_0}}(\hat{a}+\hat{a}^+)} \underset{\substack{\downarrow \\ \text{since} \\ \hat{a}\ \&\ \hat{a}^+ \text{ are not} \\ \text{commuting}}}{=} e^{i\eta\hat{a}^+} e^{i\eta\hat{a}} e^{-\eta^2/2}$$

↗ because of the commutator

$$= \sum_{n,m} \frac{(i\eta)^{n+m}}{n!\,.m!} \hat{a}^{+(m)} \hat{a}^{(n)} e^{-\eta^2/2}$$

Group terms by q, number of quanta exchanges

$$= e^{-\eta^2/2} \Bigg\{ \sum_m (-1)^m \eta^{2m} \frac{\hat{a}^{+(m)}\hat{a}^{(m)}}{(m!)^2}$$

$$+ \sum_q \underbrace{\boxed{\sum_m (-1)^m \eta^{2m} \frac{\hat{a}^{+(m)}\hat{a}^{(m)}}{m!\,(m+q)!}}}_{f_q(\hat{a}^+\hat{a})} \hat{a}^{(q)} \underbrace{(i\eta)^q}_{q \text{ dependent part}}$$

$$+ \sum_m (-1)^m \eta^{2m}\, a^{+q} (i\eta)^q \frac{\hat{a}^{+m}\hat{a}^m}{m!\,(m+q)!} \Bigg\}$$

for $f_q(x) = \sum_m (-1)^m \eta^{2m} \frac{x^{(m)}}{m!\,(m+q)!}$

$$= e^{-\eta^2/2} \Bigg\{ f_0(\hat{a}^+\hat{a}) + \sum_{q>0} (i\eta)^q \Big[f_q(\hat{a}^+\hat{a})\hat{a}^q + \hat{a}^{+q} f_q(\hat{a}^+\hat{a}) \Big] \Bigg\}$$

3) Full Hamiltonian

$$\hat{H} = \omega_e |e\rangle\langle e| + \omega_t \hat{a}^\dagger \hat{a}$$

$$+ \frac{\Omega}{2} e^{-\eta^2/2} \left\{ f_0(\hat{a}^\dagger \hat{a}) |e\rangle\langle g| + \sum_{q>0} \dots |e\rangle\langle g| \right\} e^{-i\omega_L t} + h.c.$$

trap freq.

Interaction picture $\hat{a} \longrightarrow \hat{a}\, e^{-i\omega_t t}$

$$|e\rangle \longrightarrow e^{-i\omega_e t} |e\rangle$$

$$\hat{H} = \frac{\Omega}{2} e^{-\eta^2/2} \Big\{ f_0(\hat{a}^\dagger \hat{a}) e^{i(\omega_e - \omega_L)t} |e\rangle\langle g| \quad \rightarrow \text{Carrier}$$

$$+ \sum_q f_q(\hat{a}^\dagger \hat{a})\, e^{i(\omega_e - \omega_L - q\omega_t)t} (i\eta)^q \hat{a}^q |e\rangle\langle g|$$

→ Red

$$+ \sum_q e^{i(\omega_e - \omega_L + q\omega_t)t} (i\eta)^q \hat{a}^q |e\rangle\langle g| f_q(\hat{a}^\dagger \hat{a})$$

→ Blue

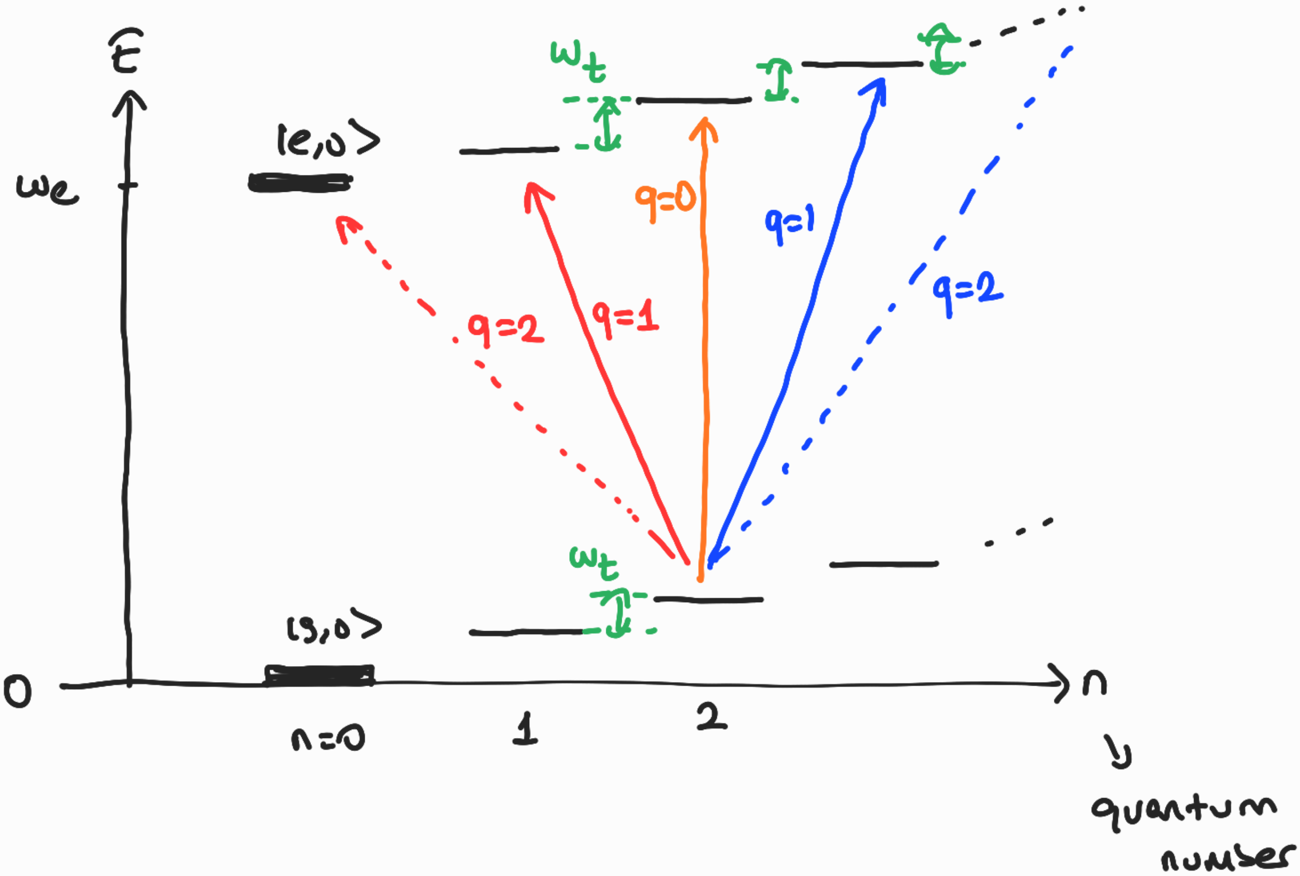
$$+ h.c. \Big\}$$

Each term is called a motional sideband

Carrier: $\omega_e - \omega_L$ → resonant when $\omega_L = \omega_e$

Red sidebands: $\omega_e - \omega_L - q\omega_t$ → resonant when $\omega_L = \omega_e - q\omega_t$
$q > 0$

Blue sidebands: $\omega_e - \omega_L + q\omega_t$ → " " $\omega_L = \omega_e + q\omega_t$



4) Addressing sidebands

- When $\Omega \ll \omega_t$: rotating wave approximation

  neglect terms oscillating at ω_t or higher

- Choosing $\omega_L = \omega_e \pm \omega_q t$

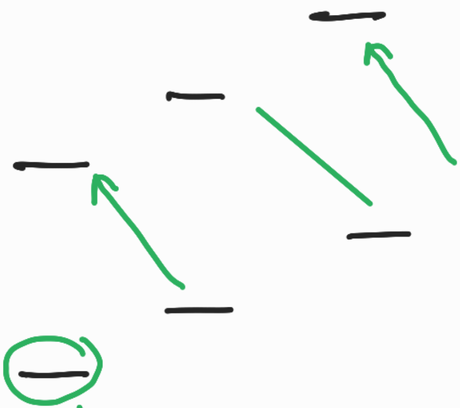
• Lamb-Dicke regime: $\eta \ll 1$, $f_q(x) = \frac{1}{q!}$, $e^{-\eta^2/2} = 1$

Carrier: $H_c = \frac{\Omega}{2}\left(|g\rangle\langle e| + |e\rangle\langle g| \right)$

Red-Sideband: $H_r = \frac{\Omega}{2}(i\eta)\left(\hat{a}\,|e\rangle\langle g| - \hat{a}^{+}|g\rangle\langle e|\right)$

Jaynes-Cummings Hamiltonian

Remark: There exists a "dark" state



$|n=0\rangle$ not coupled to light

Blue-sideband: $\hat{H}_b$ = "Anti"- Jaynes Cumming Hamiltonian

"No dark states"

II. Cooling

1) Cooling cycle

Lindblad Equation: $\dot{\hat{\rho}} = -i[\hat{H},\hat{\rho}] - \frac{\Gamma}{2}\left(|e\rangle\langle e|\hat{\rho} + \hat{\rho}|e\rangle\langle e|\right) + \sum_{\vec{n}} \Gamma_{\vec{n}} \hat{L}_{\vec{n}} \hat{\rho} \hat{L}_{\vec{n}}^{+}$

with $L_{\vec{n}} = e^{ik\vec{n}.\vec{r}} |g\rangle\langle e|$

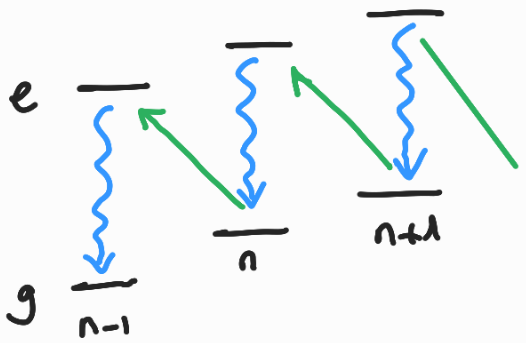
in 1D: $\hat{L}_{\pm} = e^{\pm ik\hat{z}} |g\rangle\langle e| = e^{ikz_0(\hat{a}+\hat{a}^{+})} |g\rangle\langle e|$

$= (\hat{\mathbb{I}} \pm i\eta^{\pm(\hat{a}+\hat{a}^{+})} + \ldots) |g\rangle\langle e|$

Same as free space:

$$\sum_{\vec{n}} \Gamma_{\vec{n}} \hat{L}_{\vec{n}} \hat{\rho} \hat{L}_{\vec{n}}^{+} \sim \sum_{\vec{n}} \Gamma_{\vec{n}} |g\rangle\langle e| \hat{\rho} |e\rangle\langle g| + O(\eta^2)$$

in the Lamb-Dicke regime $\eta \ll 1$: Spontaneous emission does not change n.



Driving the red sideband + sp. emission

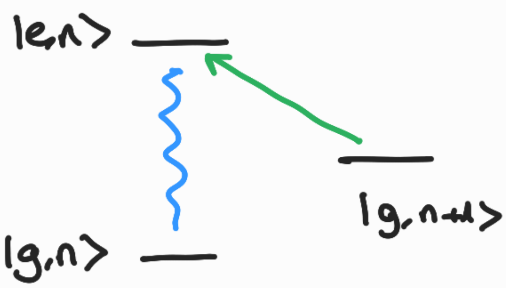
$|n\rangle \longrightarrow |n-1\rangle$ Down to $|n=0\rangle$

cooling!

(In practise : limited $O(\eta^2)$)

2) Quantative description

Method of adiabatic elimination :



spon. emission rate

3 levels problem for $\Omega_r \ll \Gamma$

red-side band Rabi freq

$\rho_{ee} = 0$

population in $|e\rangle$ state

Effective 2-level theory :

$$\dot{\hat\rho} = -i[\hat H, \hat\rho] - \frac{\Gamma}{2}\left(|e\rangle\langle e|\hat\rho + \hat\rho|e\rangle\langle e|\right) + \Gamma \underbrace{\langle e|\hat\rho|e\rangle}_{\text{operator on } \mathcal{H}_{com}} |g\rangle\langle g|$$

Hilbert space

$$\langle e|\dot{\hat\rho}|e\rangle = \dot{\hat\rho}_{ee} = -i\langle e|\left[i\frac{\Omega_r \eta}{2}\left(\hat a|e\rangle\langle g| - \hat a^{+}|g\rangle\langle e|\right), \hat\rho\right]|e\rangle - \Gamma\hat\rho_{ee}$$

$$= \frac{\Omega_r\eta}{2}\left(\hat a \underbrace{\langle g|\hat\rho|e\rangle}_{\hat\rho_{ge}} + \underbrace{\langle e|\hat\rho|g\rangle}_{\hat\rho_{eg}}\hat a^{+}\right) - \Gamma\hat\rho_{ee}$$

$$\langle g|\dot{\hat\rho}|g\rangle = \dot{\hat\rho}_{gg} = \frac{\Omega_r \eta}{2}\left(\hat\rho_{ge}\,\hat a + \hat a^{+}\hat\rho_{eg}\right) + \Gamma\,\hat\rho_{ee}$$

$$\langle g|\dot{\hat\rho}\,|e\rangle = \dot{\hat\rho}_{eg} = \frac{\Omega_r \eta}{2}\left(\hat a\,\hat\rho_{gg} + \hat\rho_{ee}\,\hat a^{+}\right) - \frac{\Gamma}{2}\hat\rho_{eg}$$

★ Internal degress of freedom are fast!

Steady state is reached : $\hat\rho_{eg} = \frac{\Omega_r \eta}{\Gamma}\,\hat a\,\hat\rho_{gg}$, $\hat\rho_{ee} \sim 0$

Then $\dot{\hat\rho}_{gg} = -\frac{\Omega_r^2\eta^2}{2\Gamma}\left(\hat a^{+}\hat a\,\hat\rho_{gg} + \hat\rho_{gg}\,\hat a^{+}\hat a\right) + \frac{\Omega_r^2\eta^2}{\Gamma}\,\hat a\,\hat\rho_{gg}\,\hat a^{+}$

(No $\hat\rho_{ee}$ dependence anymore)

Effective Lindblad Eq for $|g\rangle$:

$$\boxed{\dot{\hat\rho} = -\frac{\Gamma_1}{2}\left(\hat a^{+}\hat a\,\hat\rho + \hat\rho\,\hat a^{+}\hat a\right) + \Gamma_1\,\hat a\,\hat\rho\,\hat a^{+}}$$

$\hat L_{eff} = \sqrt{\Gamma_1}\,\hat a$

$\Gamma_1 = \frac{\Omega_r^2\eta^2}{\Gamma} \ll \Gamma, \Omega_r$