THE WEAK INTERACTION

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1. The Weak Charged-current interaction

Both QED and QCD are mediated by massless neutral spin-1 bosons.

Spinor part of the QED and QCD interaction vertices have the same $\bar{u}(p')\gamma^{\mu}u(p)$ form.

* The charged-current weak interaction differs in almost all respects.

  - Mediated by massive charged $W^{\pm}$ bosons
  - couples together fermions differing by 1 unit of e^- charge
  - Only place in the Standard Model where parity is not conserved

2. Parity

Spatial inversion through the origin $x \longrightarrow -x$

$$\psi(x,t) \longrightarrow \psi'(x,t) = \hat{P}\psi(x,t) = \psi(-x,t)$$

parity operator

The original wavefunction is clearly recovered if the parity operator is applied twice:

$$\hat{P}\hat{P}\psi(x,t) = \hat{P}\psi(-x,t) = \psi(x,t)$$

$$\hat{P}\hat{P} = I$$

* If physics is invariant under parity transformations, then the parity operation must be unitary

$$\hat{P}^{+}\hat{P} = I$$

Thus:

$$\hat{P}\hat{P} = I \quad \& \quad \hat{P}^{+}\hat{P} = I \Rightarrow \hat{P}^{+}\hat{P}$$

$\hat{P}$ is a Hermitian operator

* Since $\hat{P}$ is a Hermitian operator that corresponds to an observable property of a quantum-mechanical system.

* If the interaction Hamiltonian commutes with $\hat{P}$, parity is an observable conserved quantity in the interaction.
In this case, if $\psi(x,t)$ is an eigenstate of the Hamiltonian, it is also an eigenstate of the parity operator with an eigenvalue P

$$\hat{P}\psi(x,t) = P\,\psi(x,t)$$

$$\underbrace{\hat{P}\hat{P}}_{I}\psi(x,t) = P\,\hat{P}\psi(x,t) = \underbrace{P^2}_{1}\psi(x,t)$$

Since $\hat{P}$ is Hermitian, its eigenvalues are real.

$$\boxed{P = \pm 1}$$

2.1 Intrinsic Parity

Fundamental particles possess an intrinsic parity.

Parity operator for Dirac spinors is γ^0

$$\hat{P} = \gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

For spin 1/2 particles by convention:

$$P(e^-) = P(\mu^-) = P(\tau^-) = P(\nu) = P(q) = +1$$

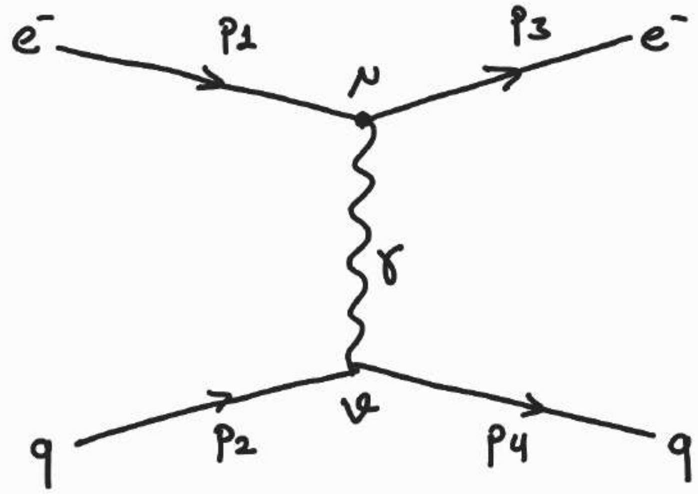
$$P(e^+) = P(\mu^+) = P(\tau^+) = P(\bar{\nu}) = P(\bar{q}) = -1$$

Spin-1 bosons:

$$P(\gamma) = P(g) = P(W^+) = P(W^-) = P(Z) = -1$$

from gauge field theory ?

2.2 Parity Conservation in QED



$e^-q \rightarrow e^-q$

QED t-channel electron-quark scattering process

$$\mathcal{M} = \frac{Q_q e^2}{q^2} j_e \cdot j_q$$

$$j_e^{\mu} = \bar{u}(p_3)\gamma^{\mu} u(p_1)$$

$$j_q^{\nu} = \bar{u}(p_4)\gamma^{\nu} u(p_2)$$

The equivalent matrix element for the parity transformed process,

$$\boxed{u \xrightarrow{\hat{P}} \hat{P} u = \gamma^0 u}$$

Adjoint spinors:

$$\bar{u} = u^{\dagger}\gamma^0 \xrightarrow{\hat{P}} (\underbrace{\hat{P}u}_{\gamma^0 u})^{\dagger}\gamma^0 = u^{\dagger}\gamma^{0\dagger}\gamma^0 = \overbrace{u^{\dagger}\gamma^0}^{\bar{u}}\gamma^0 = \bar{u}\gamma^0$$

$$\boxed{\bar{u} \xrightarrow{\hat{P}} \bar{u}\gamma^0}$$

$$j_e^\mu = \bar{u}(p_3)\gamma^\mu u(p_1) \xrightarrow{\hat{P}} \bar{u}(p_3)\gamma^0\gamma^\mu\gamma^0 u(p_1)$$

→ Time-like component of the current ($\mu=0$)

$$j_e^0 \xrightarrow{\hat{P}} \bar{u}\gamma^0\underbrace{\gamma^0\gamma^0}_{I} u = \bar{u}\gamma^0 u = j_e^0$$

→ Space-like components of j^μ, with indices $k=1,2,3$

$$j_e^k \xrightarrow{\hat{P}} \bar{u}\gamma^0\gamma^k\gamma^0 u = -\bar{u}\gamma^k\underbrace{\gamma^0\gamma^0}_{I} u = -\bar{u}\gamma^k u = -j_e^k$$

$$(\gamma^0\gamma^k = -\gamma^k\gamma^0)$$

4-vector scalar product:

$$j_e \cdot j_q = j_e^0 j_q^0 - j_e^k j_q^k \xrightarrow{\hat{P}} j_e^0 j_q^0 - (-j_e^k)(-j_q^k) = j_e j_q$$

★ This invariance implies that parity is conserved

★ The conservation of parity in strong and electromagnetic interactions needs to be taken into account when considering particle decays.

✓ $\rho^0(1^-) \longrightarrow \pi^+(0^-) + \pi^-(0^-)$ & $\eta(0^-) \to \pi^+(0^-) + \pi^-(0^-)$ ✗

J^P values

P → parity

J → total angular momentum

The total parity of the 2-body final state is the product of the intrinsic parities of the particles and the parity of the orbital wavefunction, which is given by $(-1)^\ell$ where ℓ is the orbital angular momentum in the final state.

$$P(\rho^0) = P(\pi^+)\,P(\pi^-)\,(-1)^{\ell=1} \Rightarrow -1 = (-1)(-1)(-1) \checkmark$$

$$P(\eta) = P(\pi^+)\,P(\pi^-)\,(-1)^{\ell=0} \Rightarrow \underbrace{-1}_{-1} = \underbrace{(-1)(-1)}_{+1}(+1) \times$$

This decay does not occur.

parity is not conserved

Scalars, pseudoscalars, vectors and axial vectors

Physical quantities can be classified acc to their rank (dimensionality) and parity inversion properties.

→ Scalar quantities like mass, temperature are invariant under parity transformation

→ Vector quantities like position, momentum change sign

→ Axial vector (pseudovector) do not change sign.

↳ cross product of two vector

$$\underset{(+)}{\vec{L}} = \underset{(-)}{\vec{x}} \times \underset{(-)}{\vec{p}} \qquad \underset{(+)}{d\vec{B}} \propto \underset{(-)}{\vec{j}} \times \underset{(-)}{d^3\vec{x}}$$

"Biot-Savart Law"

★ Scalar quantities can be formed out of scalar products of two vectors or two axial vectors.

$$p^2 = \vec{p} \cdot \vec{p}$$

magnitude squared

→ Pseudoscalar are formed from the product of a vector and an axial vector, change sign under the parity operation

$$\text{Helicity} \quad h \propto \vec{S} \cdot \vec{p}$$

spin is an axial vector

2.3 Parity Violation in Nuclear β-decay

1957 : Chien-Shiung Wu et al studied β-decay of polarized ^{60}Co nuclei:

$$^{60}Co \longrightarrow {}^{60}Ni^{*} + e^{-} + \bar{\nu}_e$$

$$^{60}Ni^{*} \longrightarrow {}^{60}Ni + 2\gamma$$

β-decay:

β particles

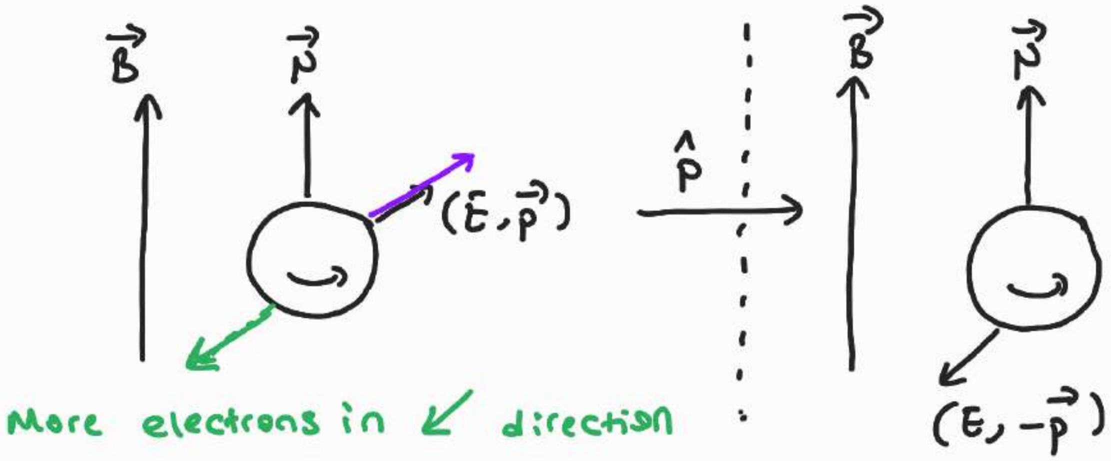
$$n \longrightarrow p + e^{-} + \bar{\nu}_e$$

They specifically focused on the emission direction of the beta particles.

If the weak force conserves parity symmetry, then the laws of physics should be the same regardless of whether we observe the process directly or through a mirror reflection.

If we were to observe the beta decay in a mirror, the electrons should preferentially be emitted in the opposite direction.

However, the emitted electrons may exhibit a preferred direction, breaking the mirror symmetry.



Thus **PARITY IS VIOLATED**

! The weak interaction does not have 4-vector currents of the form $j^\mu = \bar{u}(p')\gamma^\mu u(p)$.

3. V-A structure of the weak interaction

In general, there are only 5 possible combinations of two spinors and γ-matrices that form Lorentz-invariant currents, called "bilinear covariants":

	Form	Components	Boson spin
Scalar	$\bar{\Psi}\phi$ ($\gamma^0 = I$)	1	0
Pseudoscalar	$\bar{\Psi}\gamma^5\phi$	1	0
Vector	$\bar{\Psi}\gamma^\mu\phi$	4	1
Axial vector	$\bar{\Psi}\gamma^\mu\gamma^5\phi$	4	1
Tensor	$\bar{\Psi}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)\phi$	6	2

1. Scalar current: electric charge density

2. Pseudoscalar current: Axial charge density

3. Vector currents: Electromagnetic current (flow of electric charge) or conserved energy-momentum current in relativistic field theories

4. Axial vector current: weak axial charge

5. Tensor current: Flow or density that possesses directional information beyond that of a vector

"Current = flow of a conserved quantity"

↳ In relativistic field theories, various conserved quantities are associated with specific currents. These currents can be scalar, vector, axial vector, or tensor quantities

In QED, the factor $g_{\mu\nu}$ in the matrix element arises from the $(2J+1)+1$ polarisation states of the $J^P = 1^-$ virtual photon

$$(2\times 1+1)+1 = 4 \text{ degrees of freedom}$$

For spin-2 boson $(J=2)$ $\rightarrow (2J+1)+1 = 6$ polarisation states for the spin-2 virtual particles

☆ The most general form for the interaction btw a fermion and a boson is a linear combination of bilinear covariants

If it is restricted to the exchange of a spin-1 (vector) boson, the most general form for the interaction is a linear combination of vector and axial vector currents.

$$j^\mu \propto \bar{u}(p')(g_V\gamma^\mu + g_A\gamma^\mu\gamma^5)u(p) = g_V j_V^\mu + g_A j_A^\mu$$

where $j_V^\mu = \bar{u}(p')\gamma^\mu u(p)$ & $j_A^\mu = \bar{u}(p')\gamma^\mu\gamma^5 u(p)$

Parity transformation properties of j_A^μ:

$$j_A^\mu = \bar{u}\gamma^\mu\gamma^5 u \xrightarrow{\hat{P}} \bar{u}\gamma^0\gamma^\mu\gamma^5\gamma^0 u = -\bar{u}\gamma^0\gamma^\mu\gamma^0\gamma^5 u$$

$$\gamma^5\gamma^0 = -\gamma^0\gamma^5$$

Time-like component of the axial vector current transforms as

$$j_A^0 = \xrightarrow{\hat{P}} -\bar{u}\gamma^0\underbrace{\gamma^0\gamma^0}_{I}\gamma^5 u = -\bar{u}\gamma^0\gamma^5 u = -j_A^0$$

And the space-like component transform as:

$$j_A^k = \xrightarrow{\hat{P}} -\bar{u}\gamma^0\gamma^k\gamma^0\gamma^5 u = +\bar{u}\gamma^k\gamma^5 u = +j_A^k$$

Thus, the scalar product of two axial vector currents is invariant under parity transformations:

$$j_1\cdot j_2 = j_1^0 j_2^0 - j_1^k j_2^k \xrightarrow{\hat{P}} (-j_1^0)(-j_2^0) - j_1^k j_2^k = j_1\cdot j_2$$

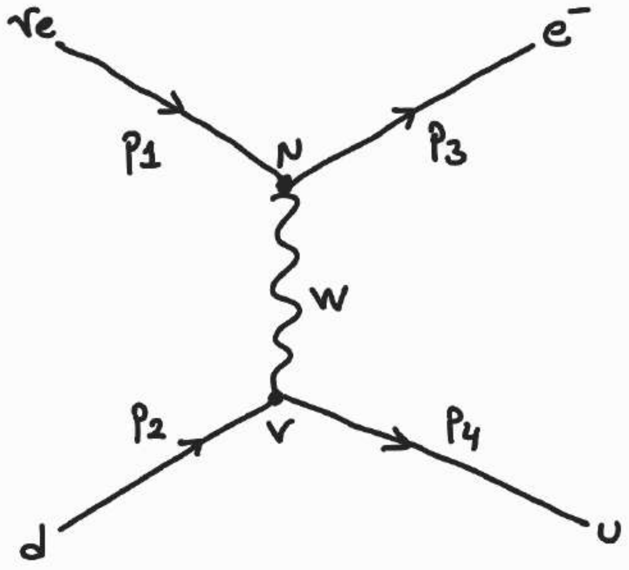
$$j_V^0 \xrightarrow{\hat{P}} +j_V^0 \;;\; j_V^k \xrightarrow{\hat{P}} -j_V^k$$

$$j_A^0 \xrightarrow{\hat{P}} -j_A^0 \;;\; j_A^k \xrightarrow{\hat{P}} +j_A^k$$

★ Whilst the scalar products of two vector currents or two axial vector currents are unchanged in a parity transformation the scalar product $j_V\cdot j_A$ transforms to $-j_V\cdot j_A$

↓

used in observation of parity violation



$\nu_e\, d \longrightarrow e^- u$

Inverse β decay

$$j_{\nu e}^{\mu} = \bar{u}(p_3)\left(g_V \gamma^{\mu} + g_A \gamma^{\mu}\gamma^{5}\right)u(p_1) = g_V\, j_{\nu e}^{V} + g_A\, j_{\nu e}^{A}$$

$$j_{du}^{\nu} = \bar{u}(p_4)\left(g_V \gamma^{\nu} + g_A \gamma^{\nu}\gamma^{5}\right)u(p_2) = g_V\, j_{du}^{V} + g_A\, j_{du}^{A}$$

matrix element

$$\boxed{\mathcal{M}_{fi} \propto j_{\nu e}\cdot j_{du}} = g_V^2\, \underbrace{j_{\nu e}^{V}\cdot j_{du}^{V}} + g_A^2\, \underbrace{j_{\nu e}^{A}\cdot j_{du}^{A}} + g_V g_A\,\underbrace{\left(j_{\nu e}^{A}\cdot j_{du}^{V} + j_{\nu e}^{V}\cdot j_{du}^{A}\right)}_{\text{sign change}}$$

do not change sign under parity transformation

$$j_{\nu e}\cdot j_{du} \xrightarrow{\hat{P}} g_V^2\, j_{\nu e}^{V}\cdot j_{du}^{V} + g_A^2\, j_{\nu e}^{A}\cdot j_{du}^{A} - g_V g_A\left(j_{\nu e}^{V}\cdot j_{du}^{A} + j_{\nu e}^{A}\cdot j_{du}^{V}\right)$$

Thus the relative strength of the parity violating part of the matrix element compared to the parity conserving part is given by:

$$\frac{g_V g_A}{g_V^2 + g_A^2}$$

→ If either g_V or g_A is "0", parity is conserved in the interaction

→ $|g_V| = |g_A|$, maximal parity violation occurs

→ From the experiment, it is known the weak charged current due to the exchange of $W^\pm$ boson is a vector minus axial vector (V−A) interaction of the form $\gamma^\mu - \gamma^\mu\gamma^5$, with a vertex factor of

$$-\frac{i g_W}{\sqrt{2}} \frac{1}{2} \gamma^\mu (1 - \gamma^5)$$

↳ these factors will be explained later

Corresponding four-vector current is given by:

$$\boxed{j^\mu = \frac{g_W}{\sqrt{2}} \bar{u}(p') \frac{1}{2} \gamma^\mu (1 - \gamma^5) u(p)}$$

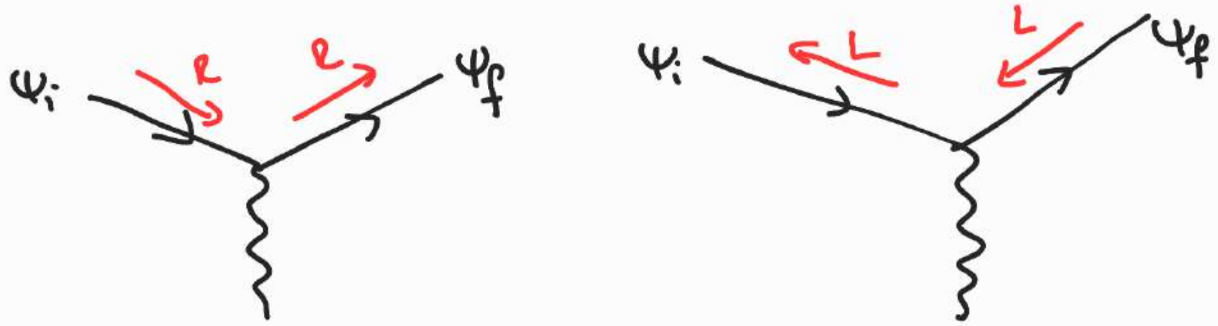
4. Chiral Structure of the weak interaction

$$\left.\begin{aligned} P_R &= \frac{1}{2}(1+\gamma^5) \\ P_L &= \frac{1}{2}(1-\gamma^5) \end{aligned}\right\}$$

☆ Any spinor can be decomposed into left and right-handed chiral components.

$$u = \frac{1}{2}(1+\gamma^5)u + \frac{1}{2}(1-\gamma^5)u = P_R u + P_L u = a_R u_R + a_L u_L$$

☆ In the ultra-relativistic limit only two helicity combinations are non-zero. (RR-LL)



For the weak interaction, the V-A vertex factor of j^μ already includes the left-handed chiral projection operator

$$j^\mu = \frac{g_W}{\sqrt{2}}\bar{u}(p')\frac{1}{2}\gamma^\mu(1-\gamma^5)u(p)$$

$$j^{\mu}_{RR} = \frac{g_W}{\sqrt{2}} \bar{u}_R(p') \gamma^{\mu} \overbrace{\frac{1}{2}(1-\gamma^5)}^{P_L} u_R(p)$$

$$j^{\mu}_{RR} = \frac{g_W}{\sqrt{2}} \bar{u}_R(p') \gamma^{\mu} P_L u_R(p) = 0$$

★ Hence only left-handed chiral particle states participate in the charged-current weak interaction.

! For anti-particles P_L projects out right-handed chiral states:

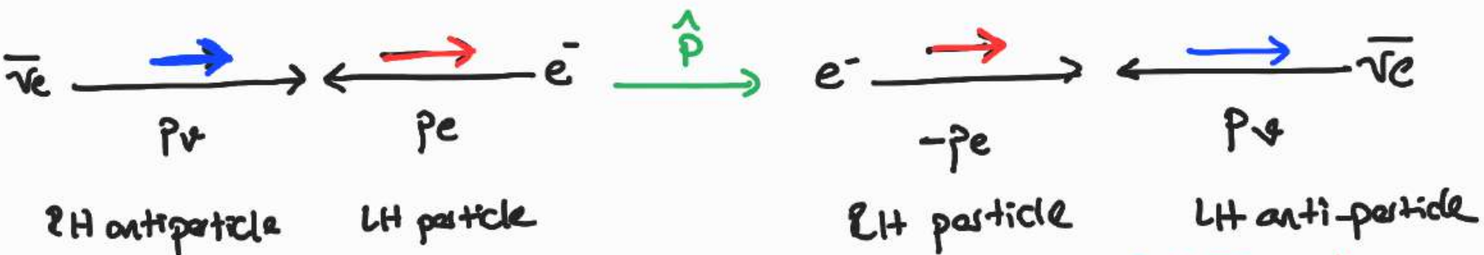
$$\frac{1}{2}(1-\gamma^5) v = v_R$$

★ Thus only the left-handed chiral particle & right-handed chiral antiparticles participate charged-current weak interaction.

★ The helicity dependence of the weak interaction

$\Updownarrow$

$\boxed{\text{parity violation}}$



vector $p \to -p$, spin (axial vector) does not change

IT IS NOT ALLOWED!

5. The W-boson propagator

For QED, the exchange of the massless spin-1 photon:

$$\frac{-i g_{\mu\nu}}{q^2}$$

W-boson is massive ($m_W \sim 80$ GeV)

$$\frac{1}{q^2 - m_W^2}$$

Corresponding sum over the polarisation states of the exchanged virtual massive spin-1 boson:

$$\sum_\lambda \epsilon_\mu^{\lambda *} \epsilon_\nu^{\lambda} = -g_{\mu\nu} + \frac{q_\mu q_\nu}{m_W^2}$$

Feynman rule associated with the exchange of a virtual W:

$$\frac{-i}{q^2 - m_W^2}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{m_W^2} \right)$$

In the limit where $q^2 \ll m_W^2$, the $q_\mu q_\nu$ term is small and the propagator can be taken to be:

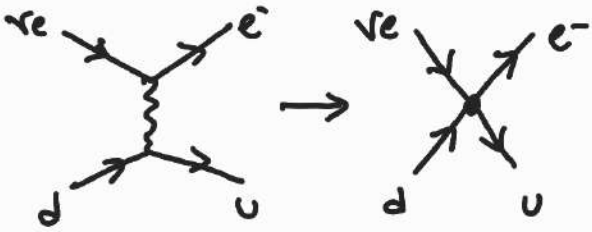
$$\frac{-i g_{\mu\nu}}{q^2 - m_W^2}$$

5.1 Fermi theory

For most low-energy weak interactions, such as majority of particle decays, $|q|^2 << m_W^2$; thus W boson propagator can be approximated by:

$$\frac{i g_{\mu\nu}}{m_W^2}$$

No longer q^2 dependence. Physically, this corresponds to replacing the propagator with an interaction which occurs at a single point in space-time.



(1934)
Fermi formulated the weak interaction before the discovery of the parity violation and the matrix element for β decay:

$$M_{fi} = G_F\, g_{\mu\nu} \left[\bar{\psi}_3 \gamma^\mu \psi_1\right]\left[\bar{\psi}_4 \gamma^\nu \psi_2\right]$$

strength of the weak interaction Fermi constant

After the discovery of parity violation by Wu (1957):

$$M_{fi} = \frac{1}{\sqrt{2}} G_F\, g_{\mu\nu} \left[\bar{\psi}_3 \gamma^\mu (1-\gamma^5)\psi_1\right]\left[\bar{\psi}_4 \gamma^\nu (1-\gamma^5)\psi_2\right]$$

$$M_{fi} = -\left[\frac{g_W}{\sqrt{2}}\bar{\psi}_3 \frac{1}{2}\gamma^\mu (1-\gamma^5)\psi_1\right] \cdot \left[\frac{g_{\mu\nu} - q_\mu q_\nu / m_W^2}{q^2 - m_W^2}\right]\left[\frac{g_W}{\sqrt{2}}\bar{\psi}_4 \frac{1}{2}\gamma^\nu (1-\gamma^5)\psi_2\right]$$

In the limit $q^2 \ll m_W^2$ reduces to

$$M_{fi} = \frac{g_W^2}{8 m_W^2} g_{\mu\nu} \left[\bar{\psi}_3 \gamma^\mu (1-\gamma^5) \psi_1 \right] \left[\bar{\psi}_4 \gamma^\nu (1-\gamma^5) \psi_2 \right]$$

Hence :

$$\boxed{\frac{G_F}{\sqrt{2}} = \frac{g_W^2}{8 m_W^2}}$$

5.2 Strength of the weak interaction

The strength of the weak interaction is most precisely determined from low-energy measurement, and in particular from the muon lifetime.

For $m_\mu \ll m_W$, Fermi Theory can be used

$$\Gamma(\mu^- \rightarrow e^- \nu_\mu \bar{\nu}_e) = \frac{1}{\tau_\mu} = \frac{G_F^2 m_\mu^5}{192\pi^3}$$

Measurement of muon life-time and mass, provide a precise determination of the Fermi constant

$$G_F = 1.16638 \times 10^{-5} \text{ GeV}^{-2}$$

(Ch.16)

Using G_F and measurement of M_W, g_W can be obtained.

$$\alpha_W = \frac{g_W^2}{4\pi} = \frac{8 m_W^2 G_F}{4\sqrt{2}\pi} \simeq \frac{1}{30}$$

$\alpha_W > \alpha$, The weak interaction is in fact intrinsically stronger than the electromagnetic interaction.

$$P_{QED} \sim \frac{1}{q^2} \qquad P_W \sim \frac{1}{q^2 - M_W^2}$$

for $|q^2| \ll M_W^2 \longrightarrow P_W \simeq -\frac{1}{M_W^2}$

Therefore, weak interaction decay rates (which are proportional to $|\mathcal{M}|^2$, are suppressed by a factor q^4/M_W^4 relative to QED decay rates.

In high energy limits where $|q^2| > m_W^2$, weak and electromagnetic interaction have similer strength.

6. Helicity in pion decay

Charged pions ($\pi^{\pm}$) are the $J^P = 0^-$ meson states from $u\bar{d}$ and $d\bar{u}$. They are the lightest mesons with $m(\pi^{\pm}) \sim 140$ MeV : thus, can not decay via the strong interaction. They can only decay through the weak interaction to final states with lighter funda fermions.

Hence, charged pions can only decay to final states with with either electrons or muons.

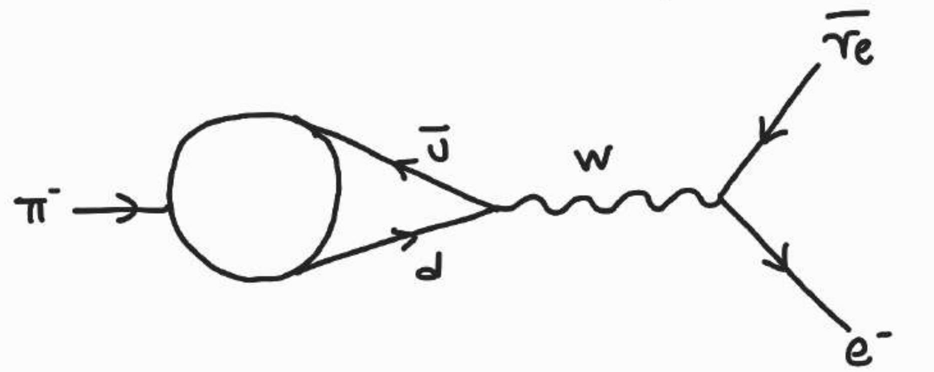
$$\pi^- \longrightarrow e^- \bar{\nu}_e$$

expected to be greater. (due to momentum of the decay products)

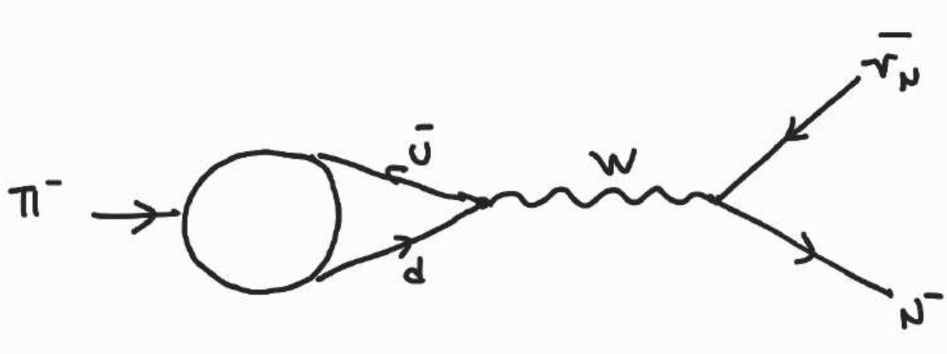
$$\pi^- \longrightarrow \mu^- \bar{\nu}_\mu$$

dominating

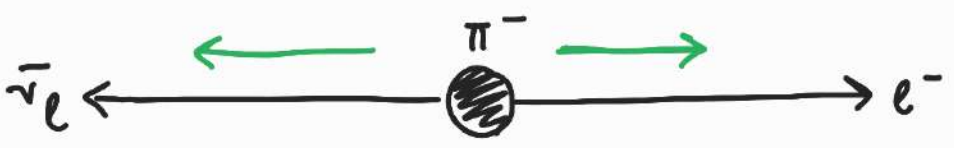
$$\pi^- \longrightarrow \mu^- \bar{\nu}_\mu \gamma$$



$$\frac{\Gamma(\pi^- \to e^- \bar{\nu}_e)}{\Gamma(\pi^- \to \mu^- \bar{\nu}_\mu)} = 1.230 \times 10^{-4}$$



✱ Manifestation of the chiral structure of the weak interaction and provides a clear illustration of the difference btw. helicity and chirality



$\ell = e^-$ or μ^-

The weak interaction only couples to LH chiral particle & RH chiral antiparticle states.

Since neutrinos are effectively massless ($m_\nu \ll E$), the neutrino chiral states are equivalent to the helicity states. Thus, anti-neutrino is always produced in a RH helicity.

Pion is a spin-0 particle, lepton-neutrino system must be in the spin-0 singlet state, with the charged lepton and neutrino spins in opposite direction.

Thus, since the neutrino is RH, conservation of angular momentum implies that the charged lepton is also produced in a RH helicity state.

However weak interaction vertex is non-zero only for LH particles.

In some sense wrong helicity. If the charged lepton was also massless, decay would not occur.

↓

CHIRAL & HELICITY STATES ARE NOT EQUIVALENT

RH helicity particle

$$u_\uparrow \equiv \frac{1}{2}\left(1+\frac{p}{E+m}\right)u_R + \underbrace{\frac{1}{2}\left(1-\frac{p}{E+m}\right)}u_L$$

In the weak-interaction vertex only u_L component will give a non-zero contribution to the matrix element.

$$\mathcal{M} \sim \frac{1}{2}\left(1-\frac{p_e}{E_e+m_e}\right)$$

Taking the mass of the neutrino to be zero:

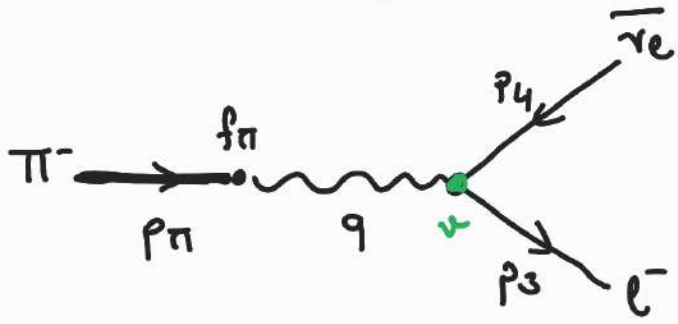
$$E_e = \frac{m_\pi^2+m_e^2}{2m_\pi} \qquad p_e = \frac{m_\pi^2-m_e^2}{2m_\pi}$$

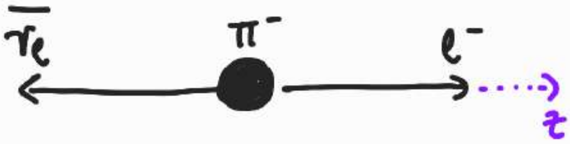
$$\frac{p_e}{E_e} = \frac{m_\pi - m_L}{m_\pi + m_L}$$

$$\mathcal{M} \sim \frac{m_e}{m_\pi + m_e}$$

Since $\frac{m_\mu}{m_e} \simeq 200$, pion decays to electrons are strongly suppressed with cmp. to muons.

6.1 Pion decay rate





$$p_\pi = (m_\pi, 0, 0, 0)$$

$$p_\ell = p_3 = (E_\ell, 0, 0, p)$$

$$p_{\bar{\nu}} = p_4 = (p, 0, 0, -p)$$

The weak leptonic current ass. with the $\ell^- \bar{\nu}_\ell$ vertex:

$$j_\ell^\nu = \frac{g_w}{\sqrt{2}} \bar{u}(p_3) \frac{1}{2} \gamma^\nu (1-\gamma^5) v(p_4)$$

✲ Since pion is a bound $q\bar{q}$ state, the hadronic current cannot be expressed in terms of free particle Dirac spinors.

However pion current has to be 4-vector quantity. Then replace $\bar{u}\gamma^\mu(1-\gamma^5)u$ with $f_\pi p_\pi^\mu$ where f_π is a constant

$$\mathcal{M}_{fi} = \left[\frac{g_w}{\sqrt{2}} \cdot \frac{1}{2} f_\pi p_\pi^\mu \right] \times \left[\frac{g_{\mu\nu}}{m_w^2} \right] \times \left[\frac{g_w}{\sqrt{2}} \bar{u}(p_3) \gamma^\nu \frac{1}{2} (1-\gamma^5) v(p_4) \right]$$

$$= \frac{g_w^2}{4 m_w^2} g_{\mu\nu} f_\pi p_\pi^\mu \bar{u}(p_3) \gamma^\nu \frac{1}{2} (1-\gamma^5) v(p_4)$$

* In the pion rest frame, only the time-like component of the pion 4-momentum is non-zero. $\boxed{p_\pi^0 = m_\pi}$

$$\mathcal{M}_{fi} = \frac{g_W^2}{4 m_W^2} f_\pi\, m_\pi\, \bar{u}(p_3)\, \gamma^0\, \frac{1}{2}(1-\gamma^5)\, v(p_4)$$

Since $\bar{u}\gamma^0 = u^\dagger \underbrace{\gamma^0\gamma^0}_{I} = u^\dagger$

$$\mathcal{M}_{fi} = \frac{g_W^2}{4 m_W^2} f_\pi\, m_\pi\, u^\dagger(p_3)\, \overbrace{\frac{1}{2}(1-\gamma^5)\, v(p_4)}^{v_\uparrow(p_4)}$$

For neutrino $\rightarrow$ ($m \ll E$) helicity eigenstates are equivalent to the chiral states

$$\mathcal{M}_{fi} = \frac{g_W^2}{4 m_W^2} f_\pi\, m_\pi\, u^\dagger(p_3)\, v_\uparrow(p_4)$$

It was calculated before:

$$u_\uparrow(p_3) = \sqrt{E_\ell + m_\ell}\begin{pmatrix} 1 \\ 0 \\ \frac{p}{E_\ell + m_\ell} \\ 0 \end{pmatrix} \qquad u_\downarrow(p_3) = \sqrt{E_\ell + m_\ell}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -\frac{p}{E_\ell + m_\ell} \end{pmatrix}$$

$(\theta = 0, \phi = 0)$ $\qquad$ $(\theta = 0, \phi = 0)$

$$v_\uparrow(p_4) = \sqrt{p}\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$$

$(\theta = \pi, \phi = \pi)$

One can immediately see that $u_{\downarrow}^{\dagger}(p_3)\, v_{\uparrow}(p_4) = 0$.

Thus only non-zero matrix element comes from the case in which both the charged lepton and the antineutrino are in RH helicity states.

$$\mathcal{M}_{fi} = \frac{g_W^2}{4m_W^2} f_\pi m_\pi \sqrt{E_\ell + m_\ell}\, \sqrt{p} \left(1 - \frac{p}{E_\ell + m_\ell} \right)$$

$$\mathcal{M}_{fi} = \frac{g_W^2}{4m_W^2} f_\pi m_\pi \frac{m_\pi + m_\ell}{\sqrt{2m_\pi}} \left(\frac{m_\pi^2 - m_\ell^2}{2m_\pi} \right)^{1/2} \frac{2m_\ell}{m_\pi + m_\ell}$$

$$\mathcal{M}_{fi} = \left(\frac{g_W}{2m_W} \right)^2 f_\pi m_\ell \left(m_\pi^2 - m_\ell^2 \right)^{1/2}$$

$$\langle |\mathcal{M}_{fi}|^2 \rangle = |\mathcal{M}_{fi}|^2 = 2\, G_F^2\, f_\pi^2 m_\ell^2 \left(m_\pi^2 - m_\ell^2 \right)$$

In Ch.3

$$\Gamma = \frac{4\pi}{32\pi^2 m_\pi^2}\, p \langle |\mathcal{M}_{fi}|^2 \rangle = \frac{G_F^2}{8\pi m_\pi^3} f_\pi^2 \left[m_\ell \left(m_\pi^2 - m_\ell^2 \right) \right]^2$$

(decay rate)

$$\frac{\Gamma(\pi^- \longrightarrow e^- \bar{\nu}_e)}{\Gamma(\pi^- \longrightarrow \mu^- \bar{\nu}_\mu)} = \left[\frac{m_e(m_\pi^2 - m_e^2)}{m_\mu(m_\pi^2 - m_\mu^2)}\right]^2 = 1.26 \times 10^{-4}$$