QUANTUM CHROMODYNAMICS

1. The Local Gauge Principle
2. Colour in QCD

1. The Local Gauge Principle

Local gauge symmetry is a fundamental concept in physics that describes the invariance of physical physical laws under transformations that vary from point to point in spacetime.

$\phi \rightarrow$ scalar potential

$\vec{A} \rightarrow$ vector potential

$$\left.\begin{aligned} \phi &\longrightarrow \phi' = \phi - \frac{\partial \chi}{\partial t} \\ \vec{A} &\longrightarrow \vec{A'} = \vec{A} + \nabla \chi \end{aligned}\right\} \quad A_\mu \longrightarrow A_\mu' = A_\mu - \partial_\mu \chi$$

$(\phi, -\vec{A})$ $\quad (\partial_0, \nabla)$

★ Suppose that there's a fundamental symmetry of the Universe that requires the invariance of physics under local phase transformation defined by:

$$\psi(x) \longrightarrow \psi'(x) = \hat{U}(x)\,\psi(x) = e^{iq_\chi(x)}\,\psi(x)$$

This is similar to U(1) global phase transformation of $\psi \rightarrow \psi' = e^{i\phi}\psi$. But here the phase $q_\chi(x)$ can be different at all points in space-time

For this local U(1) phase transformation of the free-particle Dirac Equation:

$$i\gamma^\mu \partial_\mu \psi = m\psi \qquad (1)$$

$$i\gamma^\mu \underbrace{\partial_\mu (e^{iq\chi(x)}\psi)}_{e^{iq\chi(x)}\partial_\mu\psi \; + \; iq(\partial_\mu \chi)\, e^{iq\chi(x)}\psi} = m\, e^{iq\chi(x)}\psi$$

$$\cancel{e^{iq\chi}}\; i\gamma^\mu (\partial_\mu \psi + iq(\partial_\mu \chi)\psi) = \cancel{e^{iq\chi}}\, m\psi$$

$$i\gamma^\mu (\partial_\mu + iq\partial_\mu \chi)\psi = m\psi \qquad (2)$$

Equation (1) differs from (2) by the term $-q\gamma^\mu(\partial_\mu \chi)\psi$

★ Thus, the free-particle Dirac Equation does NOT possess the invariance under a U(1) local phase transformation.

Local phase invariance is not possible for a free theory, without interactions.

★★ The required invariance can be established only by modifying the Dirac Equation to include a new degree of freedom A_μ such that:

$$i\gamma^{\mu}(\partial_{\mu} + iqA_{\mu})\psi - m\psi = 0$$

field corresponding to
a massless gauge boson

This equation is invariant under the local phase transformation $A_{\mu} \rightarrow A_{\mu}' = A_{\mu} - \partial_{\mu}\chi$ in order to cancel the unwanted $-q\gamma^{\mu}(\partial_{\mu}\chi)\psi$ term in (2)

$$q\gamma^{\mu}A_{\mu}\psi \rightarrow \text{Interaction Term}$$

★ The requirement that physics is invariant under local U(1) phase transformations implies the existence of a gauge field which couples to Dirac particles in exactly same way as the photon.

★ All of QED, including ultimately Maxwell's equations, can be derived by requiring the invariance of physics under local U(1) transformations of the form $\hat{U} = e^{iq\chi(x)}$.

1.1 From QED to QCD

QED $\rightarrow$ U(1) local phase transformation

QCD $\rightarrow$ SU(3) local phase transformation

$$\psi(x) \longrightarrow \psi'(x) = \left[e^{i g_s \alpha(x) \cdot \hat{T}} \right] \psi(x)$$

$\hat{T} = \{ T^a \}$ $\rightarrow$ 8 generators of the SU(3) symmetry

Gell-Mann Matrices

$$T^a = \frac{1}{2} \lambda^a$$

$\alpha^a(x)$ $\rightarrow$ 8 functions of space-time coordinate x

☆ Because the generators of SU(3) are represented by 3x3 matrices, the wavefunction ψ must now include three additional degrees of freedom that can be represented by a three component vector analogous to the representation of u, d, s quarks in SU(3) flavour symmetry

This new degree of freedom is termed "colour" with red, blue and green

The Dirac Equation becomes:

$$i\gamma^\mu \left[\partial_\mu + i g_s (\partial_\mu \alpha) \cdot \hat{T} \right] \psi = m\psi$$

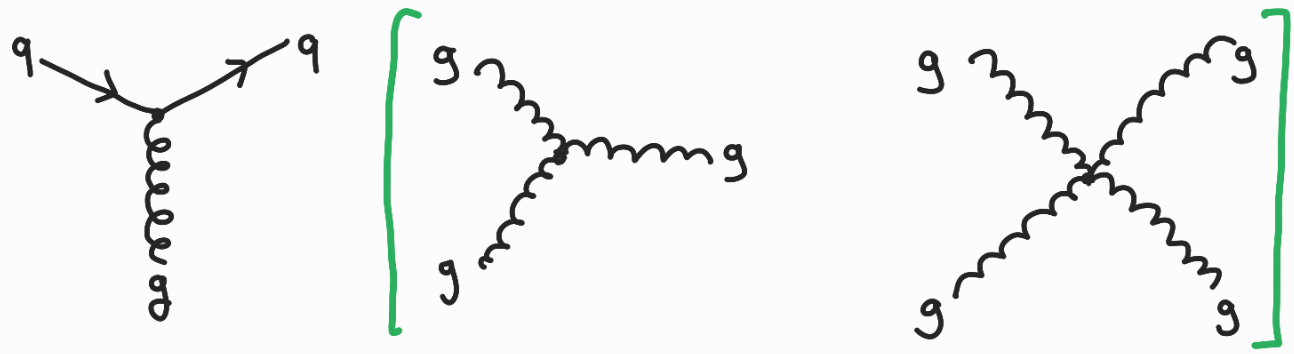
$$i\gamma^\mu \left[\partial_\mu + i g_s G_\mu^a T^a \right] \psi - m\psi = 0$$

It is invariant under local SU(3) phase transformations provided the new fields transform as:

$$G_\mu^k \longrightarrow G_\mu^{k\,\prime} = G_\mu^k - \partial_\mu \alpha_k - g_s f_{ijk} \alpha_i G_\mu^j$$

Since the generators of SU(3) symmetry do not commute and the f_{ijk} are the structure constants of the SU(3) group, defined by the commutation relations
$[\lambda_i, \lambda_j] = 2i f_{ijk} \lambda_k$

Because the generators SU(3) do not commute, QCD is known as a non-Abelian gauge theory and the presence of the additional term gives rise to gluon self-interactions



qqg interaction vertex: $g_s T^a \gamma^\mu G_\mu^a \psi = g_s \frac{1}{2} \lambda^a \gamma^\mu G_\mu^a \psi$

2. Colour in QCD

The theory of strong interaction, QCD is very similar to QED but with 3 conserved "color" charges:

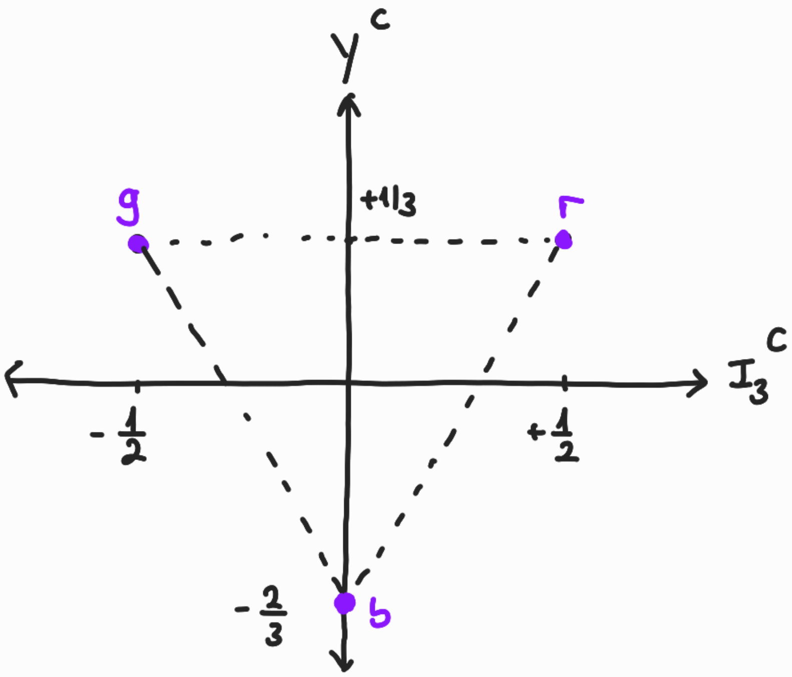
QED

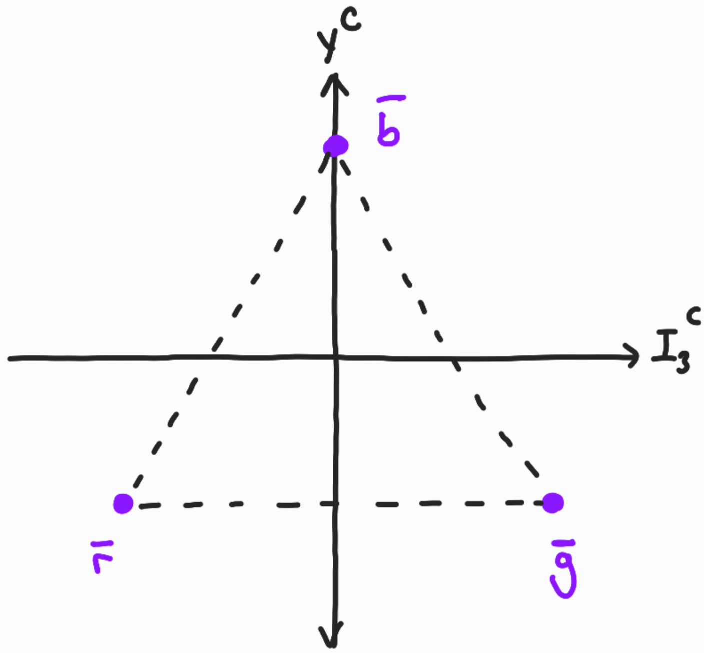
- The electron carries one unit of charge $-e$
- The antielectron " " " " $+e$
- The force is mediated by a massless "gauge boson" - the photon

QCD

- Quarks carry color charge r, g, b
- Antiquarks carry anti-charge $\bar{r}, \bar{g}, \bar{b}$
- The force is mediated by a massless gluons

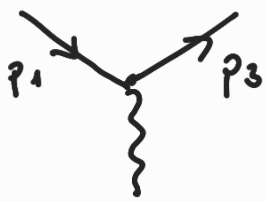
$$r = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad g = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad b = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

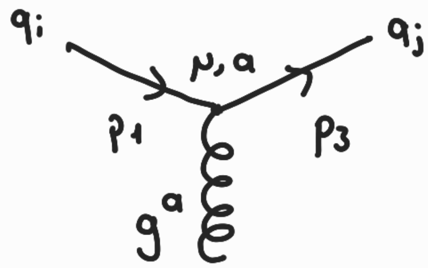




2.1 The quark-gluon vertex

$$\underbrace{-iq\gamma^\mu A_\mu \psi}_{\text{QED Interaction Term}} \longrightarrow \underbrace{-ig_s \frac{1}{2}\lambda^a \gamma^\mu G_\mu^a \psi}_{\text{QCD Interaction Term}}$$





Vertex factor: $ie\gamma^\mu = iq\gamma^\mu \longrightarrow ig_s \gamma^\mu \frac{1}{2}\lambda^a$

Quark wavefunction $u(p) \longrightarrow c_i\, u(p)$

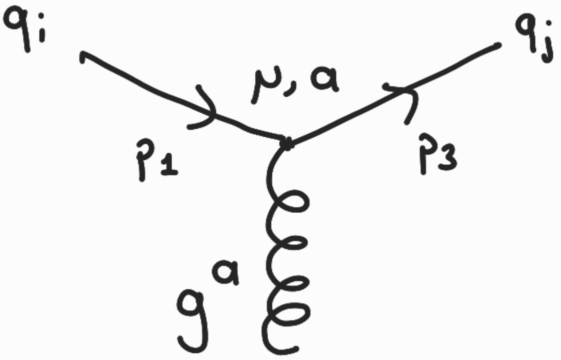
$u(p)$: Dirac spinor

c_i: colour states

$$c_1 = r = \begin{pmatrix}1\\0\\0\end{pmatrix} \qquad c_2 = g = \begin{pmatrix}0\\1\\0\end{pmatrix} \qquad c_3 = b = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

Quark current

$$j_q^{\mu} = \bar{u}(p_3)\, c_j^{\dagger} \left\{ -\frac{1}{2} i g_s \lambda^a \gamma^{\mu} \right\} c_i\, u(p_1)$$



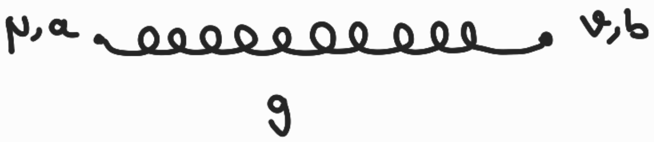
$$\bar{u}(p_3)\, c_j^{\dagger} \left\{ -\frac{1}{2} i g_s \lambda^a \gamma^{\mu} \right\} c_i\, u(p_1) = -\frac{1}{2} i g_s \left[c_j^{\dagger} \lambda^a c_i \right] \left[\bar{u}(p_3) \gamma^{\mu} u(p_1) \right]$$

The factorised color part of the interaction is:

$$c_j^{\dagger} \lambda^a c_i = c_j^{\dagger} \begin{pmatrix} \lambda^a_{1i} \\ \lambda^a_{2i} \\ \lambda^a_{3i} \end{pmatrix} = \lambda^a_{ji}$$

Hence, the qqg vertex can be written as:

$$-\frac{1}{2} i g_s \lambda^a_{ji} \left[\bar{u}(p_3) \gamma^{\mu} u(p_1) \right]$$

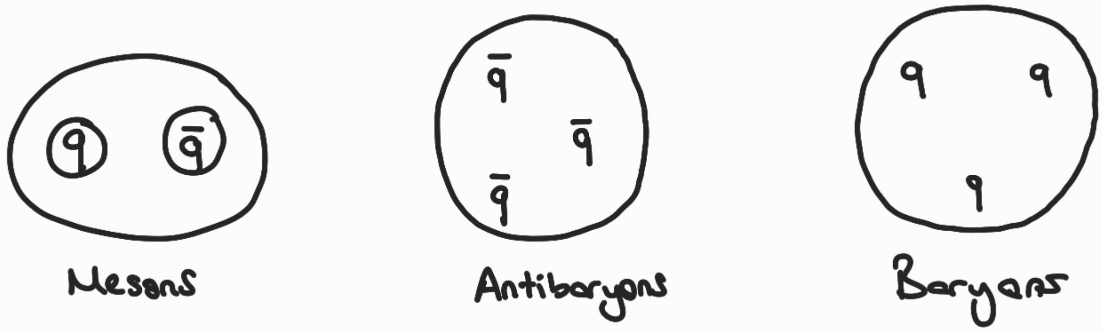


Gluon propagator $= -i \dfrac{g_{\mu\nu}}{q^2} \delta^{ab}$

Delta function (δ) ensures that the gluon of type a emitted at the vertex labelled μ is the same as that which is absorbed at vertex ν.

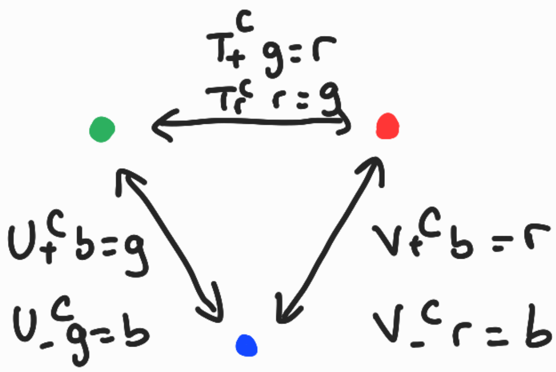
3. Color in QCD

Why we have only mesons, baryons and antibaryons states in nature?



- Non-observation of free quarks which would carry color charge

- Quarks are always found in bound states - colorless hadrons



3.1 Color confinement and hadronic states

Gluon self-interactions are believed to result in colour confinement

Remind the spin states of 2 spin-1/2 particles

4-spin combinations $\uparrow\uparrow$ $\uparrow\downarrow$ $\downarrow\uparrow$ $\downarrow\downarrow$

Gives 4 eigenstates of $\hat{S}^2$, $\hat{S}_z$ as $2 \otimes 2 = 3 \oplus 1$

$$|1,+1\rangle = \uparrow\uparrow$$

$$|1,0\rangle = \frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow) \quad \oplus \quad |0,0\rangle = \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)$$

$$|1,-1\rangle = \downarrow\downarrow$$

Spin-0 singlet

Spin-1 triplet

The singlet state is "spinless"

is invariant under $SU(2)$, spin ladder operators yield 0:

$$S_\pm |0,0\rangle = 0$$

In the same way color singlets are "colorless":

- They're 0 color quantum numbers $I_3^C = 0$, $Y^C = 0$
- Invariant under SU(3) color transformation

★★ - Ladder operators yield 0