13/10/22

Quantum Bit

$\mathcal{X}_1$ basis $\{|0\rangle, |1\rangle\}$

n qubits : $\mathcal{X}_n = \underbrace{\mathcal{X}_1 \otimes \mathcal{X}_1 \otimes \cdots\cdots \otimes \mathcal{X}_1}_{\times n}$

$$|\psi\rangle = \sum_{j=0}^{2^n-1} \alpha_j |j\rangle$$

Ex.

$|j\rangle = 0 1 0 0 1 0 0 \ldots\ldots \rightarrow |j\rangle = |0\rangle \otimes |1\rangle \otimes \ldots$

Digital quantum computing

- Process information on qubits
- Finite universal set of elem. operations
- Set the input (initial state)
- Read out the output
- Correct errors (to some extent)

Statement 1. Irreversible classical comp. is reproduced by reversible classical comp.

$\downarrow$

$$f : \{0,1\}^n \longrightarrow \{0,1\}^m \quad \text{with } m < n$$

Define: $\hat{f} : \{0,1\}^{n+m} \longrightarrow \{0,1\}^{n+m}$

$$\hat{f}\,(x\,;\,0^{(m)}) = (x\,;\,f(x))$$

Statement 2 Classical reversible computing is a particular case of Q comp.

Q comp $\quad \hat{U} : X_n \longrightarrow X_n$

for a classical $f : \{0,1\}^n \longrightarrow \{0,1\}^n$

Define $\hat{U} : \{ |000\ldots0\rangle, |000\ldots1\rangle, \ldots\ldots, |111\ldots1\rangle \}$

$$\Rightarrow \{ |f(00\ldots0)\rangle, |f(00\ldots1)\rangle, \ldots\ldots |f(111\ldots1)\rangle \}$$

$$\hat{U} = \begin{pmatrix} 0 & 0 & \\ 0 & 0 & \\ \vdots & \vdots & \\ 0 & 1 & \cdots\cdots \\ 1 & 0 & \\ 0 & \vdots & \\ \vdots & \vdots & \\ 0 & 0 & \end{pmatrix} \rightarrow \text{permutation matrix}$$

One qubit elementary gates

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$Z|0\rangle = |0\rangle$$

$$Z|1\rangle = -|1\rangle$$

$$Z\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) = \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$$

$$[X, Y] = XY - YX = 2iZ$$

$$[X, Z] = -2iY$$

$$[Y, Z] = 2iX$$

Hadamard Gate

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Phase Gate

$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

$\pi/8$ or T Gate

$$T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$$

$$\boxed{\begin{matrix} T^2 = S \\ S^2 = Z \end{matrix}}$$

Rotation Gate:

$$R_\alpha(\theta) = e^{-i\theta_\alpha} \quad \text{where} \quad \alpha = X, Y, Z \; ; \; \theta \in [0, 2\pi]$$

An arbitrary 1-qubit unitary U

$$U = e^{i\varphi} R_{\hat{n}}(\theta) = e^{i\varphi} e^{-i\theta(n_x X + n_y Y + n_z Z)}$$

$$\hat{n} = (n_x, n_y, n_z) \longrightarrow n_x^2 + n_y^2 + n_z^2 = 1 \qquad \theta, \varphi = [0, 2\pi]$$

Theorem: An arbitrary U on 1-qubit can be written as

$$\boxed{U = e^{i\varphi} R_z(\beta) \cdot R_y(\gamma) \cdot R_z(\delta)}$$

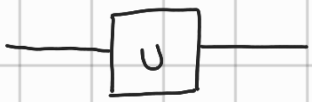
$\varphi, \beta, \gamma, \delta \in [0, 2\pi]$

Corollary: $U = e^{i\varphi} A X B X C$. A, B, C are unitaries

X gate

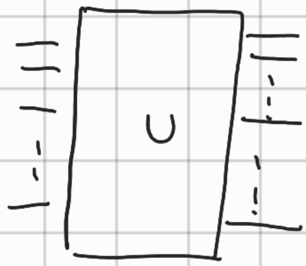
such that $ABC = \mathbb{I}$.

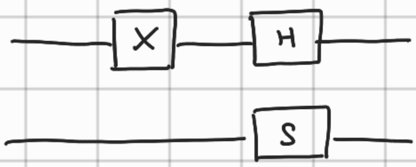
Quantum Circuit Notation



1 qubit unitary U

n qubit unitary





A unitary U is defined by its action on the computational basis.

$|\psi_0\rangle = U|0\rangle$

$|\psi_1\rangle = U|1\rangle$

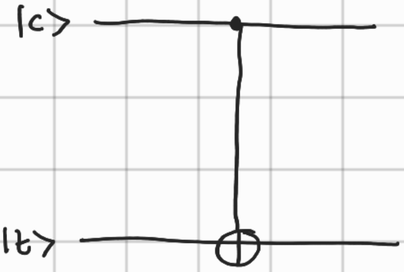
$$U(\alpha|0\rangle + \beta|1\rangle) = \alpha U|0\rangle + \beta U|1\rangle = \alpha|\psi_0\rangle + \beta|\psi_1\rangle$$

$$|\psi_1\rangle = U_1 U_2 |\psi_0\rangle$$

$$= (U_1 \otimes \mathbb{I}_2 \otimes \mathbb{I}_3)(\mathbb{I}_1 \otimes U_2 \otimes \mathbb{I}_3) |\psi_0\rangle$$

$$|\psi_2\rangle = \mathbb{I}_1 \otimes U_{23} |\psi_1\rangle$$

Controlled NOT or C-NOT



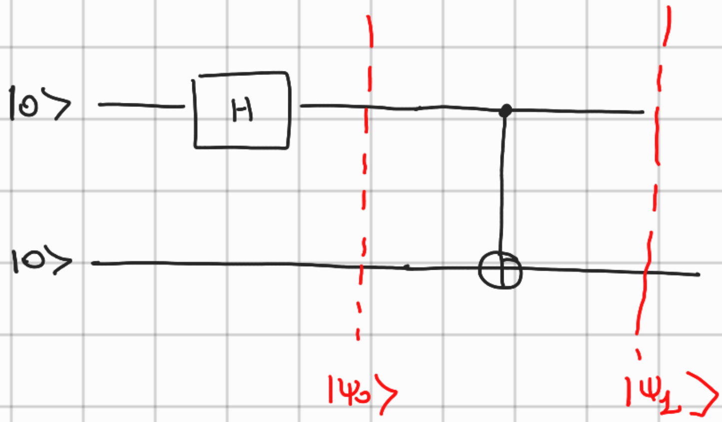
$$\text{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$|c\rangle \otimes |t\rangle \xrightarrow{\text{CNOT}} |c\rangle \otimes |t \oplus c\rangle$$

CNOT is a non-seperable 2-qubit operation

$$\text{CNOT} = |0\rangle\langle 0| \otimes \mathbb{I} + |1\rangle\langle 1| \otimes X$$

$$|\psi_1\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes |0\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |10\rangle)$$



$$\text{CNOT}\left(\frac{1}{\sqrt{2}}\left(|00\rangle + |10\rangle\right)\right) = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

↘ Bell State

$$\hat{U} = e^{-i\hat{H}t/\hbar}$$

$\hat{H}$ = Hamiltonian

If $\hat{H} = \hat{H}_1 \otimes \mathbb{I}_2 + \hat{\mathbb{I}}_1 \otimes \hat{H}_2$

then $\hat{U} = e^{-i\hat{H}t/\hbar} = \hat{U}_1 \otimes \hat{U}_2$

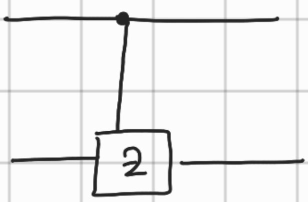
↓
seperable

$$= e^{-i\hat{H}_1 t/\hbar} \otimes e^{-i\hat{H}_2 t/\hbar}$$

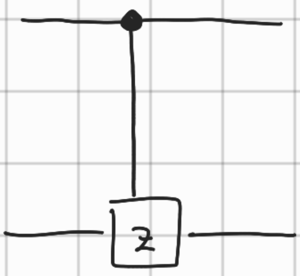
If $\hat{H} = \hat{H}_1 \otimes \mathbb{I}_2 + \hat{\mathbb{I}}_1 \otimes \hat{H}_2 + \overbrace{\hat{A}_1 \otimes \hat{B}_2}^{\text{physical interaction}}$

then U is non-seperable

CZ



$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = |0\rangle\langle 0| \otimes \mathbb{I} + |1\rangle\langle 1| \otimes Z$$



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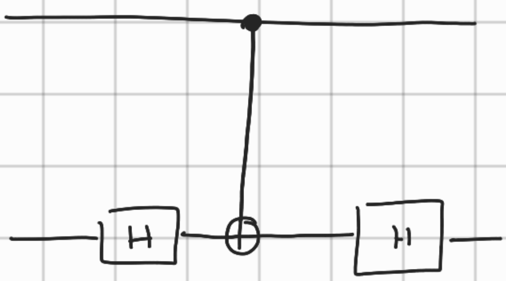
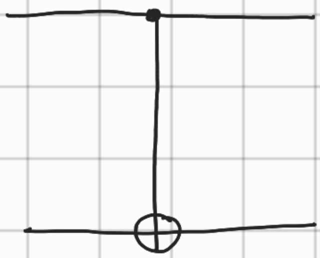
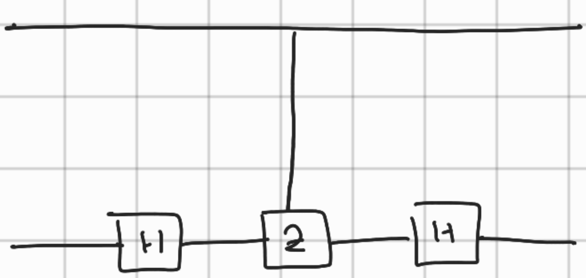


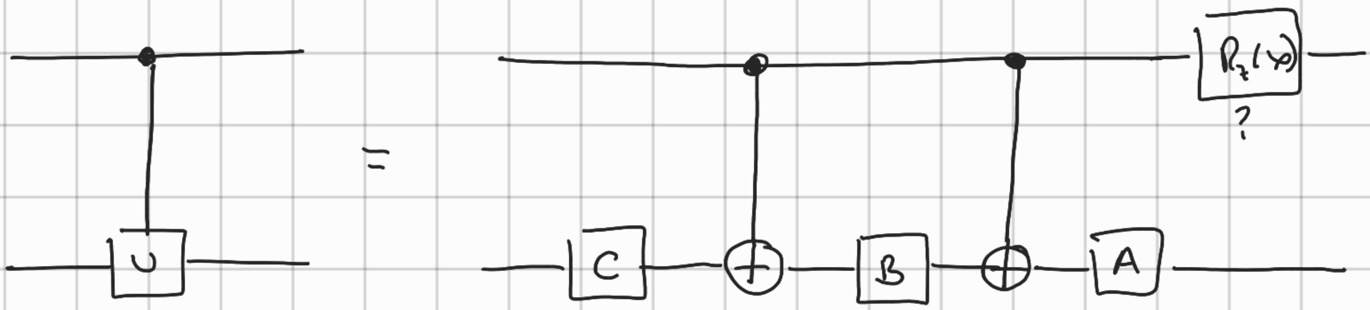




Given a 1-qubit unitary U

Build a $C\text{-}U = |0\rangle\langle 0| \otimes \mathbb{I} + |1\rangle\langle 1| \otimes U$

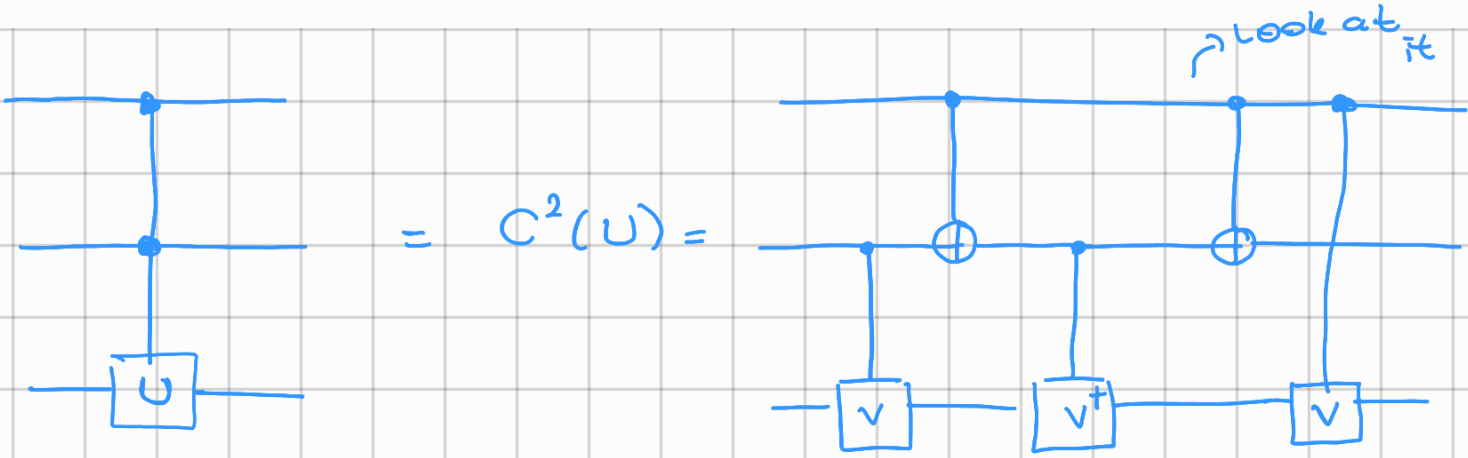
Write U as $U = e^{i\varphi} A X B X C$, $ABC = \mathbb{I}$



$$CU\ |0\rangle \otimes |\psi\rangle = |0\rangle \otimes \underbrace{ABC}_{\mathbb{I}} |\psi\rangle = |0\rangle \otimes |\psi\rangle$$

$$CU\ |1\rangle \otimes |\psi\rangle = e^{-i\varphi} |1\rangle \otimes A X B X C |\psi\rangle$$

$$= |1\rangle \otimes U|\psi\rangle$$

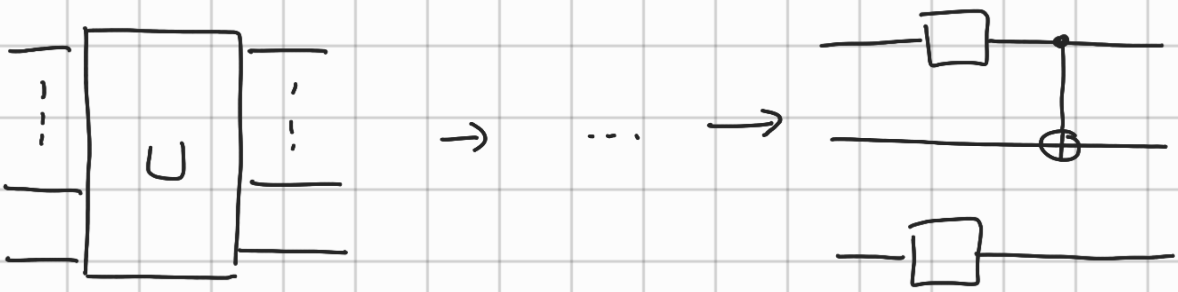


$V^2 = U$

Toffoli Gate

Universal set of elementary gates:

One possible choice is : $\{H, S, T, CNOT\}$



U on n-qubits can be expressed exactly as combination of $O(n^2 4^n)$ CNOTs and arbitrary 1-qubit U's.

Look at Nielsen and Chuang

Solovay - Kitaev Theorem

1-qubit U is approximated to accuracy ε by $O\left(\left(\log \frac{1}{\varepsilon}\right)^c\right)$ elementary ops $\quad c \simeq 2$