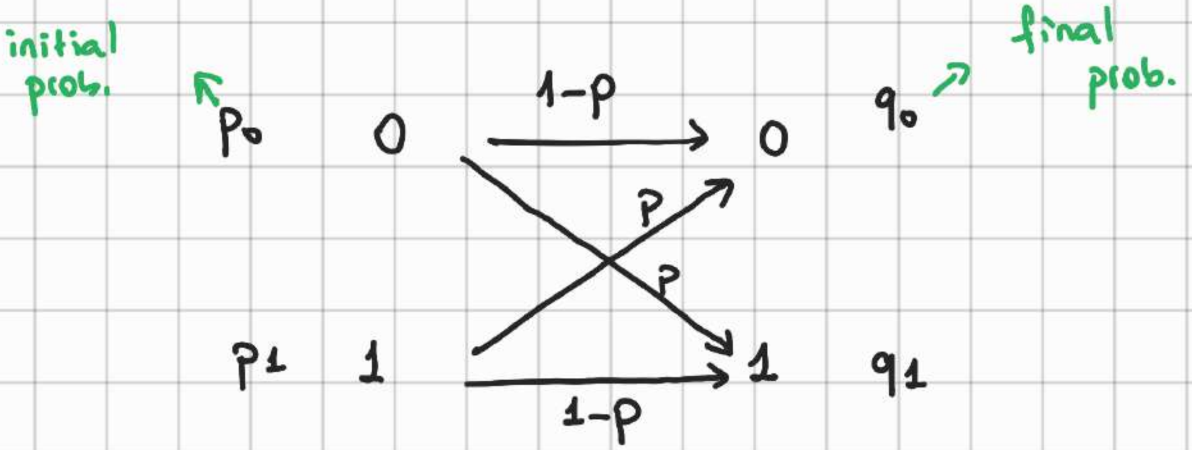
Quantum Noise and Quantum Operations

8.1 Classical noise and Markov processes



After a long time a bit on a hard disk drive may flip with probability p.

$$\begin{bmatrix} q_0 \\ q_1 \end{bmatrix} = \begin{bmatrix} 1-p & p \\ p & 1-p \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \end{bmatrix}$$

$$\vec{q} = \underbrace{E}_{\text{evolution matrix}} \vec{p}$$

Summarizing, the key features of classical noise are as follows: there is a linear relationship between input and output probabilities, described by a transition matrix with non-negative entries (*positivity*) and columns summing to one (*completeness*). Classical noise processes involving multiple stages are described as Markov processes, provided the noise is caused by independent environments. Each of these key features has important analogues in the theory of quantum noise. Of course, there are also some surprising new features of quantum noise!

8.2 Quantum Operations

$$\rho' = \mathcal{E}(\rho)$$

density operator

map $\mathcal{E}$ is quantum operation

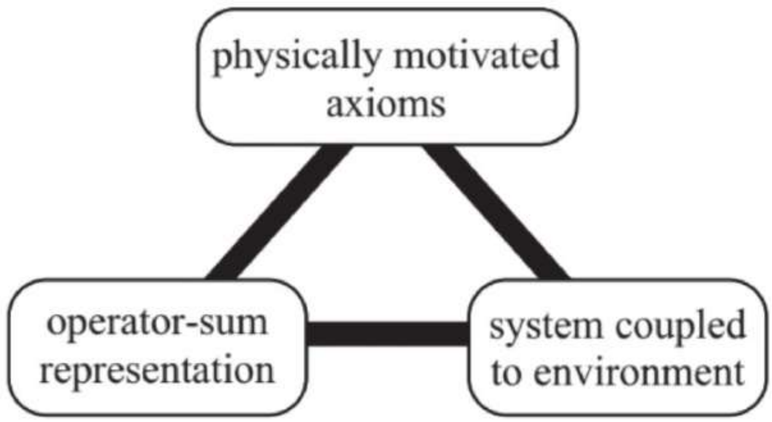


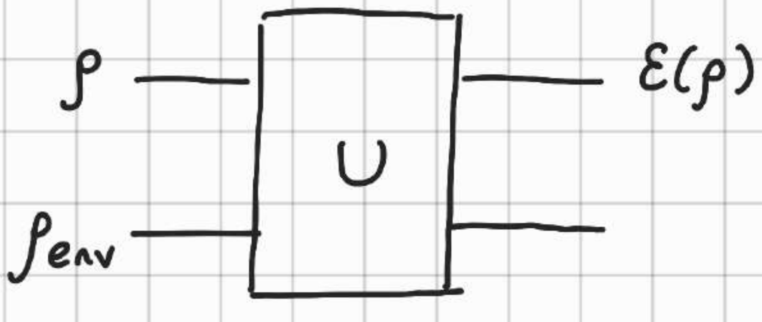
Figure 8.2. Three approaches to quantum operations which are equivalent, but offer different advantages depending upon the intended application.

8.2 Environments and Quantum Operations

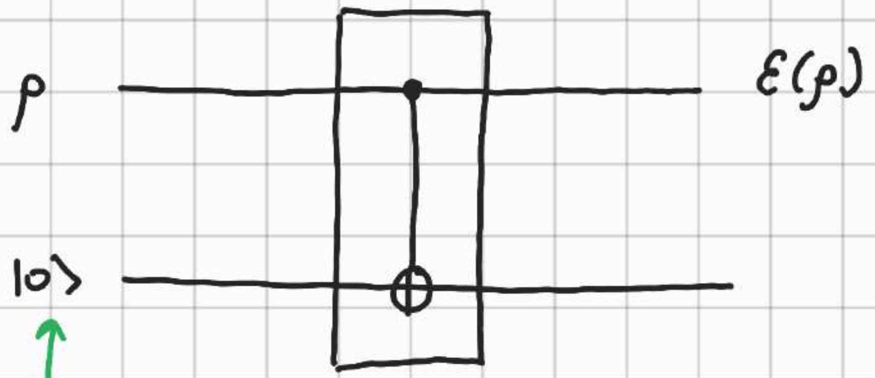
$$\mathcal{E}(\rho) = Tr_{env}\left[U (\rho \otimes \rho_{env}) U^{\dagger} \right]$$

system

important assumption!



Ex



Assume initially $\rho_{env} = |0\rangle\langle 0|$

$$\mathcal{E}(\rho) = Tr_{env}\left[\underbrace{U}_{CNOT}\left(\rho \otimes |0\rangle\langle 0|\right)\underbrace{U^{\dagger}}_{CNOT^{\dagger}}\right]$$

?

$$\mathcal{E}(\rho) = P_0 \rho P_0 + P_1 \rho P_1$$

$P_0 = |0\rangle\langle 0| \qquad P_1 = |1\rangle\langle 1|$

8.2.3 Operator-sum Representation

$$\mathcal{E}(\rho) = Tr_{env}\left[U\left(\rho \otimes \rho_{env}\right)U^{\dagger}\right]$$

$$\mathcal{E}(\rho) = \sum_k \langle e_k| U\left[\rho \otimes |e_0\rangle\langle e_0|\right] U^{\dagger} |e_k\rangle$$

$$= \sum_k \underbrace{E_k}_{\langle e_k|U|e_0\rangle} \rho E_k^{\dagger}$$

$$1 = \mathrm{Tr}\left[\mathcal{E}(\rho)\right]$$

$$= \mathrm{Tr}\left[\sum_k E_k \rho E_k^\dagger\right]$$

$$= \mathrm{Tr}\left[\sum_k E_k^\dagger E_k \rho\right]$$

$$\sum_k E_k^\dagger E_k = I$$

Physical interpretation of the operator-sum representation

There is a nice interpretation that can be given to the operator-sum representation. Imagine that a measurement of the environment is performed in the basis $|e_k\rangle$ after the unitary transformation U has been applied. Applying the principle of implicit measurement, we see that such a measurement affects only the state of the environment, and does not change the state of the principal system. Let ρ_k be the state of the principal system given that outcome k occurs, so

$$\rho_k \propto \mathrm{Tr}_E\left(|e_k\rangle\langle e_k| U(\rho \otimes |e_0\rangle\langle e_0|)U^\dagger |e_k\rangle\langle e_k|\right)$$

$$\rho_k = E_k \rho E_k^\dagger$$

Normalize $$\rho_k = \frac{E_k \rho E_k^\dagger}{\mathrm{Tr}(E_k \rho E_k^\dagger)}$$

prob. of outcome k

$$p(k) = \mathrm{Tr}\left[|e_k\rangle\langle e_k| U(\rho \otimes |e_0\rangle\langle e_0|)U^\dagger |e_k\rangle\langle e_k|\right)$$

$$= \mathrm{Tr}(E_k \rho E_k^\dagger)$$

$$\mathcal{E}(\rho) = \sum_k p(k)\,\rho_k = \sum_k E_k \rho E_k^{\dagger}$$

$$\frac{E_k \rho E_k^{\dagger}}{\mathrm{Tr}(E_k \rho E_k^{\dagger})}$$

This gives us a beautiful physical *interpretation* of what is going on in a quantum operation with operation elements $\{E_k\}$. The action of the quantum operation is equivalent to taking the state ρ and randomly replacing it by $E_k\rho E_k^\dagger/\mathrm{tr}(E_k\rho E_k^\dagger)$, with probability $\mathrm{tr}(E_k\rho E_k^\dagger)$. In this sense, it is very similar to the concept of noisy communication channels used in classical information theory; in this vein, we shall sometimes refer to certain quantum operations which describe quantum noise processes as being noisy quantum channels.

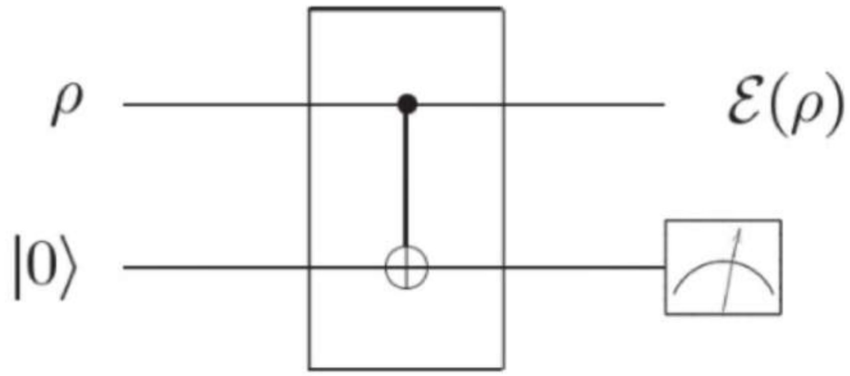


Figure 8.5. Controlled-NOT gate as an elementary model of single qubit measurement.

$$U = |0_P\, 0_E\rangle\langle 0_P\, 0_E| + |0_P$$

principal system