$$H(t) = -\frac{1}{2}\nu\left[B(t)\left(\sigma_z^A + \sigma_z^B\right) + b_A(t)\sigma_z^A + b_B(t)\sigma_z^B\right]$$

$$\rho(t) = \langle\langle\langle \rho_{st}(t) \rangle\rangle\rangle$$

$$\rho_{st}(t) = U(t)\,\rho(0)\,U^{\dagger}(t)$$

$$\begin{pmatrix} \rho_{11} & \cdots & \vdots \\ \vdots & \searrow & \vdots \\ \vdots & \cdots & \rho_{44} \end{pmatrix}$$

$$\langle\langle\langle \rho_{st}(t) \rangle\rangle\rangle = \int d_{b_A(t)} P_{b_A(t)} \int d_{B(t)} P_{b_B(t)} \int dB(t)\, P_{B(t)}\, U(t)\,\rho(0)\,U^{\dagger}(t)$$

1. First compute $\rho_{st}(t)$

$$U(t) = e^{-i\int_0^t dt'\, H(t')}$$

$\rightsquigarrow$ since σ_z's are commuting
no need to time-ordered exp. integral

$$U(t) = e^{-i\int_0^t dt' \left[\left[B(t') + b_A(t')\right]\sigma_z^A + \left[B(t') + b_B(t')\right]\sigma_z^B\right]}$$

$$= e^{-i\left(-\frac{\nu}{2}\right)\int_0^t dt' \left[B(t') + b_A(t')\right]\sigma_z^A \otimes I^B} \cdot e^{-i\left(-\frac{\nu}{2}\right)\int_0^t dt' \left[B(t') + b_B(t')\right] I^A \otimes \sigma_z^B}$$

$$\downarrow$$

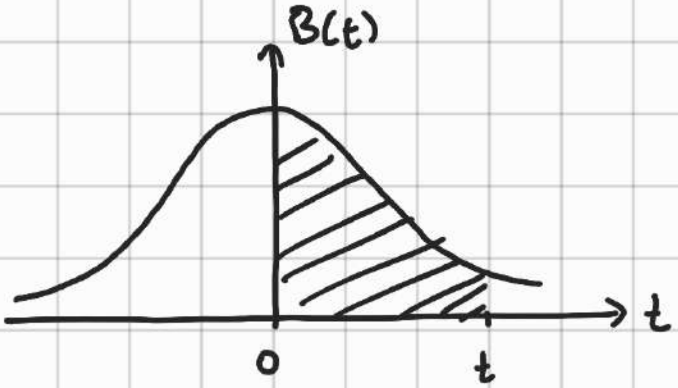
$$e^{-ik\,\sigma_z^A \otimes I^B}$$

$$e^{-ik\sigma_z^A} \otimes e^{-ik I^B} = e^{-ik\sigma_z^A} \otimes I_B$$

$$\cancel{e^{-ik}}\, I$$

global phase factor

$B(t)$, $b_A(t)$, $b_B(t)$ are Gaussian



$$\langle B(t) \rangle = 0$$

$$\langle B(t)\, B(t') \rangle = \frac{\Gamma}{N^2}\, \delta(t-t')$$

$$\int_0^t B(t')\,dt' = ?$$

$$\sigma^2 = \langle B(t) B(t) \rangle - \cancel{\langle B(t) \rangle^2}_{0} = \frac{\Gamma}{N^2}$$

$$\sigma = \sqrt{\frac{\Gamma}{N^2}} = \frac{\sqrt{\Gamma}}{N}$$

$$B(t) = \frac{1}{\frac{\sqrt{\Gamma}}{N}\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{t-0}{\frac{\Gamma}{N^2}}\right)^2} = \frac{N}{\sqrt{2\pi\Gamma}}\, e^{-\frac{1}{2}\left(\frac{N^2 t^2}{\Gamma}\right)}$$

$$\int_0^t B(t')\,dt' = \frac{N}{\sqrt{2\pi\Gamma}} \int_0^t e^{\frac{-N^2 t^2}{2\Gamma}}\, dt \simeq \underset{\text{CDF}}{\Phi}\left(\frac{t}{\sigma}\right) - \Phi(0)$$

$$= \frac{\cancel{N}}{\cancel{\sqrt{2\pi\Gamma}}}\; \frac{\cancel{\sqrt{\pi}}\, \operatorname{erf}\left(\sqrt{\frac{N^2}{2\Gamma}}\, t\right)}{2\sqrt{\cancel{\frac{N^2}{2\Gamma}}}} = \boxed{\frac{1}{2} \operatorname{erf}\left(\overbrace{\frac{N}{\sqrt{2\Gamma}}\, t}^{z}\right)}$$

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}}\left(z - \frac{z^3}{3} + \frac{z^5}{10} \cdots\right) \simeq \frac{1}{\sqrt{\pi}}\, \cancel{\frac{N}{\sqrt{2\Gamma}}\, t}$$

I think it is not allowed

$$\int_0^t B(t')\,dt' \simeq \frac{1}{2}\,\mathrm{erf}\left(\frac{N}{\sqrt{2\Gamma}}\,t\right)$$

$$\int_0^t b_A(t')\,dt' = \frac{1}{2}\,\mathrm{erf}\left(\frac{N}{\sqrt{2\Gamma_A}}\,t\right)$$

$$\int_0^t b_B(t')\,dt' = \frac{1}{2}\,\mathrm{erf}\left(\frac{N}{\sqrt{2\Gamma_B}}\,t\right)$$

let's compute

$$e^{-i\left(-\frac{N}{2}\right)\int_0^t dt'\left[B(t')+b_A(t')\right]\sigma_z^A \otimes I^B} = e^{X}$$

$$X = \left[-i\int_0^t dt' B(t') - i\int_0^t dt' b_A(t')\right]\sigma_z^A \otimes I^B \left(-\frac{N}{2}\right)$$

$$= +i\,\underbrace{\left(\frac{N}{2}\right)\frac{1}{2}\left[\mathrm{erf}\left(\frac{N}{\sqrt{2\Gamma}}\,t\right)+\mathrm{erf}\left(\frac{N}{\sqrt{2\Gamma_A}}\,t\right)\right]}_{K}\,\sigma_z^A \otimes I^B$$

$$e^{iK\sigma_z^A} \otimes I^B = \underbrace{\left[\cos(K)\,I^A + i\sin(K)\,\sigma_z^A\right]} \otimes I^B$$

$$\begin{bmatrix} e^{ik} & 0 & 0 & 0 \\ 0 & e^{ik} & 0 & 0 \\ 0 & 0 & e^{-ik} & 0 \\ 0 & 0 & 0 & e^{-ik} \end{bmatrix} = \begin{bmatrix} \overbrace{\boxed{\cos(K)+i\sin(k)}}^{e^{ik}} & 0 \\ 0 & \underbrace{\boxed{\cos(K)-i\sin(k)}}_{e^{-ik}} \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Let's compute

$$e^{-i\left(\frac{-\nu}{2}\right)\int_0^t dt' \left[B(t') + b_B(t')\right] I^A \otimes \sigma_z^B} = I^A \otimes e^{Y}$$

$$Y = \left[-i \int_0^t dt' B(t') - i\int_0^t dt' \, b_B(t')\right] \sigma_z^B \otimes I^A \left(\frac{-\nu}{2}\right)$$

$$= \underbrace{+i\left(\frac{\nu}{2}\right)\frac{1}{2}\left[\operatorname{erf}\left(\frac{N}{\sqrt{2\Gamma}}t\right) + \operatorname{erf}\left(\frac{\nu}{\sqrt{2\Gamma_B}}t\right)\right]}_{L} \otimes I^A$$

$$e^{iL\sigma_z^B \otimes I^A} = \cos(L)\, I + i \sin(L)\, \sigma_z^B \otimes I^A$$

$$I^A \otimes e^{Y} =$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} e^{iL} & 0 \\ 0 & e^{-iL} \end{pmatrix} \quad \begin{bmatrix} e^{iL} & 0 & 0 & 0 \\ 0 & e^{-iL} & 0 & 0 \\ 0 & 0 & e^{iL} & 0 \\ 0 & 0 & 0 & e^{-iL} \end{bmatrix}_{4\times4}$$

Let's compute

$$U(t) = \begin{bmatrix} e^{ik} & 0 & 0 & 0 \\ 0 & e^{ik} & 0 & 0 \\ 0 & 0 & e^{-ik} & 0 \\ 0 & 0 & 0 & e^{-ik} \end{bmatrix} \begin{bmatrix} e^{iL} & 0 & 0 & 0 \\ 0 & e^{-iL} & 0 & 0 \\ 0 & 0 & e^{iL} & 0 \\ 0 & 0 & 0 & e^{-iL} \end{bmatrix}_{4\times4} .$$

$$U(t) = \begin{bmatrix} e^{i(k+L)} & 0 & 0 & 0 \\ 0 & e^{i(k-L)} & 0 & 0 \\ 0 & 0 & e^{-i(k-L)} & 0 \\ 0 & 0 & 0 & e^{-i(k+L)} \end{bmatrix}_{4\times4}$$

$$U(t) = \begin{pmatrix} e^{i(k+L)} & 0 & 0 & 0 \\ 0 & e^{i(k-L)} & 0 & 0 \\ 0 & 0 & e^{-i(k-L)} & 0 \\ 0 & 0 & 0 & e^{-i(k+L)} \end{pmatrix}_{4\times4}$$

$$U^{+}(t) = \begin{pmatrix} e^{-i(k+L)} & 0 & 0 & 0 \\ 0 & e^{-i(k-L)} & 0 & 0 \\ 0 & 0 & e^{i(k-L)} & 0 \\ 0 & 0 & 0 & e^{i(k+L)} \end{pmatrix}_{4\times4}$$

$$\rho_{st}(t) = U(t)\,\rho(0)\,U^{+}(t)$$

$$\begin{pmatrix} e^{i(k+L)} & 0 & 0 & 0 \\ 0 & e^{i(k-L)} & 0 & 0 \\ 0 & 0 & e^{-i(k-L)} & 0 \\ 0 & 0 & 0 & e^{-i(k+L)} \end{pmatrix}_{4\times4} \underbrace{\begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\ \vdots & \ddots & & \vdots \\ \vdots & & & \vdots \\ \rho_{41} & \cdots & & \rho_{44} \end{pmatrix} \begin{pmatrix} e^{-i(k+L)} & 0 & 0 & 0 \\ 0 & e^{-i(k-L)} & 0 & 0 \\ 0 & 0 & e^{i(k-L)} & 0 \\ 0 & 0 & 0 & e^{i(k+L)} \end{pmatrix}_{4\times4}}$$

$$\begin{pmatrix} e^{i(k+L)} & 0 & 0 & 0 \\ 0 & e^{i(k-L)} & 0 & 0 \\ 0 & 0 & e^{-i(k-L)} & 0 \\ 0 & 0 & 0 & e^{-i(k+L)} \end{pmatrix}_{4\times4} \begin{pmatrix} \rho_{11} e^{-i(k+L)} & \rho_{12} e^{-i(k-L)} & \rho_{13} e^{i(k-L)} & \rho_{14} e^{i(k+L)} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \rho_{41} e^{-i(k+L)} & \cdots & \cdots & \rho_{44} e^{i(k+L)} \end{pmatrix}_{4\times4}$$

$$\rho_{st}(t) = \begin{bmatrix} \rho_{11} & \rho_{12} e^{i2L} & \rho_{13} e^{i2k} & \rho_{14} e^{i2(k+L)} \\ \rho_{21} e^{-i2L} & \rho_{22} & \rho_{23} e^{i2(k-L)} & \rho_{24} e^{i2k} \\ \rho_{31} e^{-i2k} & \rho_{32} e^{-i2(k-L)} & \rho_{33} & \rho_{34} e^{+i2L} \\ \rho_{41} e^{-i2(k+L)} & \rho_{42} e^{-i2k} & \rho_{43} e^{-i2L} & \rho_{44} \end{bmatrix}_{4\times 4}$$

where

$$k = \frac{N}{4}\left[\operatorname{erf}\left(\frac{N}{\sqrt{2\Gamma}} t\right) + \operatorname{erf}\left(\frac{N}{\sqrt{2\Gamma_A}} t\right)\right]$$

$$L = \frac{N}{4}\left[\operatorname{erf}\left(\frac{N}{\sqrt{2\Gamma}} t\right) + \operatorname{erf}\left(\frac{N}{\sqrt{2\Gamma_B}} t\right)\right]$$

2. Compute $\langle\langle\langle \rho_{st}(t) \rangle\rangle\rangle$

$$\langle\langle\langle \rho_{st}(t) \rangle\rangle\rangle = \iiint dt_1\, dt_2\, dt_3\, B(t_1)\, b_A(t_2)\, b_B(t_3)\, \rho_{st}(t)$$

$$= \int dt_1\, B(t_1) \int dt_2\, b_A(t_2) \underbrace{\int dt_3\, b_B(t_3)\, \rho_{st}(t)}_{\langle \rho_{st}(t) \rangle_{b_B}}$$

2.1 Compute $\langle \rho_{st}(t) \rangle_{b_B}$

$$\int dt_3 \, b_B(t_3) \, \rho_{st}(t_3) = \int dt \, b_B(t) \, \rho_{st}(t)$$

$$b_B(t) = \frac{N}{\sqrt{2\pi\Gamma_B}} e^{-\frac{1}{2}\left(\frac{N^2 t^2}{\Gamma_B}\right)}$$

$$= \frac{N}{\sqrt{2\pi\Gamma_B}} \int dt \, e^{-\frac{N^2}{2\Gamma_B} t^2} \rho_{st}(t)$$

For diagonal elements:

$$\rho_{ii} \int_0^t b_B(t') \, dt' = \frac{1}{2} \operatorname{erf}\left(\frac{N}{\sqrt{2\Gamma_B}} t\right) \rho_{ii}$$

For off-diagonal elements: $(i \neq k)$

$$\int b_B(t') dt' \rho_{ik} \, e^{-i(\operatorname{erf}(xt'))} = ?$$

3. Compute the $\rho(t)$ from Kraus Representation

$$\rho(t) = \tilde{\mathcal{E}}(\rho(0)) = \sum_{i,j=1}^{2} \sum_{k=1}^{3} D_k^+ E_j^+ F_i^+ \rho(0) F_i E_j D_k$$

$$\begin{bmatrix} \rho_{11} & \rho_{12} e^{-\frac{t(\Gamma+\Gamma_B)}{2}} & \rho_{13} e^{-\frac{t(\Gamma+\Gamma_A)}{2}} & \rho_{14} e^{-\frac{t(\Gamma_A+\Gamma_B)}{2} - 2t\Gamma} \\ \rho_{21} e^{-\frac{t(\Gamma+\Gamma_B)}{2}} & \rho_{22} & \rho_{23} e^{-\frac{t}{2}(\Gamma_A+\Gamma_B)} & \rho_{24} e^{-\frac{t}{2}(\Gamma+\Gamma_A)} \\ \rho_{31} e^{-\frac{t(\Gamma+\Gamma_A)}{2}} & \rho_{32} e^{-\frac{t(\Gamma_A+\Gamma_B)}{2}} & \rho_{33} & \rho_{34} e^{-\frac{t}{2}(\Gamma+\Gamma_B)} \\ \rho_{41} e^{-\frac{t(\Gamma+\Gamma_A)}{2} - 2t\Gamma} & \rho_{42} e^{-\frac{t}{2}(\Gamma+\Gamma_A)} & \rho_{43} e^{-\frac{t}{2}(\Gamma+\Gamma_B)} & \rho_{44} \end{bmatrix}$$

APPENDIX

$$
\begin{bmatrix}
\rho_{11}(\gamma^2+\omega_1^2) & \rho_{12}(\gamma\gamma_B) & \rho_{13}\gamma\gamma_A & \rho_{14}\gamma_A\gamma_B(\gamma^2+\omega_1\omega_2) \\
\rho_{21}\gamma\gamma_B & \rho_{22}(\gamma_B^2+\omega_B^2) & \rho_{23}\gamma_A\gamma_B & \rho_{24}\gamma\gamma_A(\gamma_B^2+\omega_B^2) \\
\rho_{31}\gamma\gamma_A & \rho_{32}\gamma_A\gamma_B & \rho_{33}(\gamma_A^2+\omega_A^2) & \rho_{34}\gamma\gamma_B(\gamma_A^2+\omega_A^2) \\
\rho_{41}\gamma_A\gamma_B(\gamma^2+\omega_1\omega_2) & \rho_{42}\gamma\gamma_A(\gamma_B^2+\omega_B^2) & \rho_{43}\gamma\gamma_B(\gamma_A^2+\omega_A^2) & (*)
\end{bmatrix}
$$

$$(*) = \rho_{44}\,(\underbrace{\gamma^2+\omega_2^2+\omega_3^2}_{1})\,(\underbrace{\gamma_A^2+\omega_A^2}_{1})\,(\underbrace{\gamma_B^2+\omega_B^2}_{1})$$

$$\gamma(t) = e^{-t/2T_2} \qquad T_2 \to 1/\Gamma$$

$$\gamma_A(t) = e^{-t/2T_2^A}$$

$$\gamma_B(t) = e^{-t/2T_2^B}$$

$$\omega_1(t) = \sqrt{1-e^{-t/T_2}}$$

$$\omega_2(t) = -e^{-t/T_2}\sqrt{1-e^{-t/T_2}}$$

$$\omega_3(t) = \sqrt{(1-e^{-t/T_2})(1-e^{-2t/T_2})}$$

$$\omega_A(t) = \sqrt{1-e^{-t/T_2^A}}$$

$$\omega_B(t) = \sqrt{1-e^{-t/T_2^B}}$$

$$\gamma(t) = e^{-t\Gamma/2}$$

① For D_1, E_1, F_1

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & \gamma_A \end{pmatrix} \otimes I = \begin{bmatrix} 1 &&& \\ & 1 && \\ && \gamma_A & \\ &&& \gamma_A \end{bmatrix} = E_1^{+}$$

$$F_1 = \underset{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}{I} \otimes \begin{pmatrix} 1 & 0 \\ 0 & \gamma_B \end{pmatrix} = \begin{bmatrix} 1 &&& \\ & \gamma_B && \\ && 1 & \\ &&& \gamma_B \end{bmatrix} = F_1^{+}$$

$$D_1 = \begin{bmatrix} \gamma &&& \\ & 1 && \\ && 1 & \\ &&& \gamma \end{bmatrix} = D_1^{+}$$

$$F_1 D_1 = \begin{bmatrix} 1 &&& \\ & \gamma_B && \\ && 1 & \\ &&& \gamma_B \end{bmatrix} \begin{bmatrix} \gamma &&& \\ & 1 && \\ && 1 & \\ &&& \gamma \end{bmatrix} = \begin{bmatrix} \gamma &&& \\ & \gamma_B && \\ && 1 & \\ &&& \gamma\gamma_B \end{bmatrix}$$

$$E_1 F_1 D_1 = \begin{bmatrix} 1 &&& \\ & 1 && \\ && \gamma_A & \\ &&& \gamma_A \end{bmatrix} \begin{bmatrix} \gamma &&& \\ & \gamma_B && \\ && 1 & \\ &&& \gamma\gamma_B \end{bmatrix} = \begin{bmatrix} \gamma &&& \\ & \gamma_B && \\ && \gamma_A & \\ &&& \gamma_A\gamma_B\gamma \end{bmatrix}$$

$$D_1^{+} F_1^{+} E_1^{+} = E_1 F_1 D_1$$ $\rightarrow$ Diagonal matrix multiplication is commutative

$AB = BA$

$$\underbrace{D_1^+ F_1^+ E_1^+}_{Y} \; \rho(0) \; E_1 F_1 D_1 = Y \begin{bmatrix} \rho_{11} & \rho_{12} & \cdots & \rho_{14} \\ \vdots & & \ddots & \vdots \\ \rho_{41} & \cdots & \cdots & \rho_{44} \end{bmatrix} \begin{bmatrix} \gamma & & & \\ & \gamma_B & & \\ & & \gamma_A & \\ & & & \gamma_A\gamma_B\gamma \end{bmatrix}$$

$$\begin{bmatrix} \gamma & & & \\ & \gamma_B & & \\ & & \gamma_A & \\ & & & \gamma_A\gamma_B\gamma \end{bmatrix} \begin{bmatrix} \rho_{11}\gamma & \rho_{12}\gamma_B & \rho_{13}\gamma_A & \rho_{14}\gamma_A\gamma_B\gamma \\ \vdots & & & \\ \rho_{41}\gamma & \cdots & \cdots & \rho_{44}\gamma_A\gamma_B\gamma \end{bmatrix}$$

$$= \begin{bmatrix} \rho_{11}\gamma^2 & \rho_{12}\gamma\gamma_B & \rho_{13}\gamma\gamma_A & \rho_{14}\gamma^2\gamma_A\gamma_B \\ \rho_{21}\gamma\gamma_B & \rho_{22}\gamma_B^2 & \rho_{23}\gamma_B\gamma_A & \rho_{24}\gamma\gamma_A\gamma_B^2 \\ \rho_{31}\gamma\gamma_A & \rho_{32}\gamma_A\gamma_B & \rho_{33}\gamma_A^2 & \rho_{34}\gamma\gamma_A^2\gamma_B \\ \rho_{41}\gamma^2\gamma_A\gamma_B & \rho_{42}\gamma\gamma_A\gamma_B^2 & \rho_{43}\gamma\gamma_A^2\gamma_B & \rho_{44}\gamma^2\gamma_A^2\gamma_B^2 \end{bmatrix} \quad (1)$$

2 D_1, E_1, F_2

$$E_1 = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \gamma_A & \\ & & & \gamma_A \end{bmatrix} \quad F_2 = \begin{bmatrix} 0 & & & \\ & \omega_B & & \\ & & 0 & \\ & & & \omega_B \end{bmatrix} \quad D_1 = \begin{bmatrix} \gamma & & & \\ & 1 & & \\ & & 1 & \\ & & & \gamma \end{bmatrix}$$

$$D_1 E_1 F_2 = \begin{bmatrix} 0 & & & \\ & \omega_B & & \\ & & 0 & \\ & & & \gamma\gamma_A\omega_B \end{bmatrix} = D_1^+ E_1^+ F_2^+$$

$$\begin{pmatrix} q_{11} & & q_{14} \\ & \ddots & \vdots \\ \rho_{41} & \cdots & \rho_{44} \end{pmatrix}$$

$$\underbrace{D_1^+ E_1^+ F_2^+}_{Y} \; \rho(0)\, D_1 E_1 F_2 = Y \begin{bmatrix} 0 & \rho_{12} w_B & 0 & \rho_{14} \gamma \gamma_A w_B \\ 0 & \rho_{22} w_B & 0 & \rho_{24} \gamma \gamma_A w_B \\ 0 & \rho_{32} w_B & 0 & \rho_{34} \gamma \gamma_A w_B \\ 0 & \rho_{42} w_B & 0 & \rho_{44} \gamma \gamma_A w_B \end{bmatrix}$$

$$Y = \begin{bmatrix} 0 & & & \\ & w_B & & \\ & & 0 & \\ & & & \gamma \gamma_A w_B \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \rho_{22} w_B^2 & 0 & \rho_{24} \gamma \gamma_A w_B^2 \\ 0 & 0 & 0 & 0 \\ 0 & \rho_{42} \gamma \gamma_A w_B^2 & 0 & \rho_{44} \gamma^2 \gamma_A^2 w_B^2 \end{bmatrix} \quad (2)$$

③ D_1, E_2, F_1

$$D_1 = \begin{pmatrix} \gamma & & & \\ & 1 & & \\ & & 1 & \\ & & & \gamma \end{pmatrix} \quad E_2 = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & w_A & \\ & & & w_A \end{pmatrix} \quad F_1 = \begin{pmatrix} 1 & & & \\ & \gamma_B & & \\ & & 1 & \\ & & & \gamma_B \end{pmatrix}$$

$$D_1 E_2 F_1 = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & w_A & \\ & & & \gamma \gamma_B w_A \end{pmatrix} = D_1^+ E_2^+ F_1^+$$

$$\underbrace{D_1^+ E_2^+ F_1^+}_{Y} \; \rho(0)\, D_1 E_2 F_1 = \begin{pmatrix} 0 & 0 & \rho_{13} w_A & \rho_{14} \gamma \gamma_B w_A \\ 0 & 0 & \rho_{23} w_A & \rho_{24} \gamma \gamma_B w_A \\ 0 & 0 & \rho_{33} w_A & \rho_{34} \gamma \gamma_B w_A \\ 0 & 0 & \rho_{43} w_A & \rho_{44} \gamma \gamma_B w_A \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \rho_{33}\omega_A^2 & \rho_{34}\gamma\gamma_B\omega_A^2 \\ 0 & 0 & \rho_{43}\gamma\gamma_B\omega_A^2 & \rho_{44}\gamma^2\gamma_B^2\omega_A^2 \end{pmatrix} \quad (3)$$

4 D_1, E_2, F_2

$$D_1 = \begin{pmatrix} \gamma & & & \\ & 1 & & \\ & & 1 & \\ & & & \gamma \end{pmatrix} \quad E_2 = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & \omega_A & \\ & & & \omega_A \end{pmatrix} \quad F_2 = \begin{pmatrix} 0 & & & \\ & \omega_B & & \\ & & 0 & \\ & & & \omega_B \end{pmatrix}$$

$$D_1 E_2 F_2 = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & \gamma\omega_A\omega_B \end{pmatrix} = D_1^\dagger E_2^\dagger F_2^\dagger$$

$$D_1^\dagger E_2^\dagger F_2^\dagger \, \rho(0)\, D_1 E_2 F_2 = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & \gamma\omega_A\omega_B \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & \rho_{14}\gamma\omega_A\omega_B \\ 0 & 0 & 0 & \rho_{24}\gamma\omega_A\omega_B \\ 0 & 0 & 0 & \rho_{34}\gamma\omega_A\omega_B \\ 0 & 0 & 0 & \rho_{44}\gamma\omega_A\omega_B \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho_{44}\gamma^2\omega_A^2\omega_B^2 \end{pmatrix} \quad (4)$$

5) $D_2, \; E_1, F_1$

$$D_2 = \begin{pmatrix} \omega_1 & & & \\ & 0 & & \\ & & 0 & \\ & & & \omega_2 \end{pmatrix} \quad E_1 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \gamma_A & \\ & & & \gamma_A \end{pmatrix} \quad F_1 = \begin{pmatrix} 1 & & & \\ & \gamma_B & & \\ & & 1 & \\ & & & \gamma_B \end{pmatrix}$$

$$D_2 E_1 F_1 = \begin{pmatrix} \omega_1 & & & \\ & 0 & & \\ & & 0 & \\ & & & \omega_2 \gamma_A \gamma_B \end{pmatrix} = D_2^+ E_1^+ F_1^+$$

$$\begin{pmatrix} a_{11} a_{12} a_{13} a_{14} \\ a_{44} \end{pmatrix}$$

$$\underbrace{D_2^+ E_1^+ F_1^+}_{Y} \rho(0) \, D_2 E_1 F_1 = \begin{pmatrix} \omega_1 & & & \\ & 0 & & \\ & & 0 & \\ & & & \omega_2 \gamma_A \gamma_B \end{pmatrix} \begin{pmatrix} \rho_{11}\omega_1 & 0 & 0 & \rho_{14}\omega_2\gamma_A\gamma_B \\ \rho_{21}\omega_1 & 0 & 0 & \vdots \\ \rho_{31}\omega_1 & 0 & 0 & \vdots \\ \rho_{41}\omega_1 & 0 & 0 & \rho_{44}\omega_2\gamma_A\gamma_B \end{pmatrix}$$

$$= \begin{pmatrix} \rho_{11}\omega_1^2 & 0 & 0 & \rho_{14}\,\omega_1\omega_2\,\gamma_A\gamma_B \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \rho_{41}\,\omega_1\omega_2\gamma_A\gamma_B & 0 & 0 & \rho_{44}\,\omega_2^2\gamma_A^2\gamma_B^2 \end{pmatrix} \quad (5)$$

6) $D_2, \; E_1, \; F_2$

$$D_2 = \begin{pmatrix} \omega_1 & & & \\ & 0 & & \\ & & 0 & \\ & & & \omega_2 \end{pmatrix} \quad E_1 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \gamma_A & \\ & & & \gamma_A \end{pmatrix} \quad F_2 = \begin{pmatrix} 0 & & & \\ & \omega_B & & \\ & & 0 & \\ & & & \omega_B \end{pmatrix}$$

$$D_2 E_1 F_2 = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & \omega_2\omega_B\gamma_A \end{pmatrix} = D_2^+ E_1^+ F_2^+$$

$$D_2^+ E_1^+ F_2^+ \rho(0) F_2 E_1 D_2 = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & w_2 w_B \gamma_A \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & \rho_{14} w_2 w_B \gamma_A \\ 0 & 0 & 0 & \rho_{24} w_2 w_B \gamma_A \\ 0 & 0 & 0 & \rho_{34} w_2 w_B \gamma_A \\ 0 & 0 & 0 & \rho_{44} w_2 w_B \gamma_A \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho_{44} w_2^2 w_B^2 \gamma_A^2 \end{pmatrix} \quad (6)$$

7) $D_1 \; E_2 F_1$

$$D_2 = \begin{pmatrix} w_1 & & & \\ & 0 & & \\ & & 0 & \\ & & & w_2 \end{pmatrix} \qquad E_2 = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & w_A & \\ & & & w_A \end{pmatrix} \qquad F_1 = \begin{pmatrix} 1 & & & \\ & \gamma_B & & \\ & & 1 & \\ & & & \gamma_B \end{pmatrix}$$

$$D_2^+ E_2^+ F_1^+ = D_2 E_2 F_1 = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & w_2 w_A \gamma_B \end{pmatrix}$$

$$= \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & \rho_{44} w_2^2 w_A^2 \gamma_B^2 \end{pmatrix} \quad (7)$$

8) D_2, E_2, F_2

$$D_2 = \begin{pmatrix} \omega_1 & & & \\ & 0 & & \\ & & 0 & \\ & & & \omega_2 \end{pmatrix} \quad E_2 = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & \omega_A & \\ & & & \omega_A \end{pmatrix} \quad F_2 = \begin{pmatrix} 0 & & & \\ & \omega_B & & \\ & & 0 & \\ & & & \omega_B \end{pmatrix}$$

$$D_2^+ E_2^+ F_2^+ = D_2 E_2 F_2 = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & \omega_A \omega_B \omega_2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & \rho_{44}\, \omega_2^2 \omega_A^2 \omega_B^2 \end{pmatrix} \quad (8)$$

9) D_3, E_1, F_1

$$D_3 = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & \omega_3 \end{pmatrix} \quad E_1 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \gamma_A & \\ & & & \gamma_A \end{pmatrix} \quad F_1 = \begin{pmatrix} 1 & & & \\ & \gamma_B & & \\ & & 1 & \\ & & & \gamma_B \end{pmatrix}$$

$$D_3 E_1 F_1 = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & \omega_3 \gamma_A \gamma_B \end{pmatrix} = D_3^+ E_1^+ F_1^+$$

$$D_3^+ E_1^+ F_1^+ \rho(0) D_3 E_1 F_1 = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & \rho_{44}\, \omega_3^2 \gamma_A^2 \gamma_B^2 \end{pmatrix} \quad (9)$$

10) $D_3 E_1 F_2$

$$D_3^+ E_1^+ F_2^+ = D_3 E_1 F_2 = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & \rho_{44}\,\omega_3^2 \gamma_A^2 \omega_B^2 \end{pmatrix} \quad (10)$$

11) $D_3 E_2 F_1$

$$D_3^+ E_2^+ F_1^+ = D_3 E_2 F_1 = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & \rho_{44}\,\omega_3^2 \omega_A^2 \gamma_B^2 \end{pmatrix} \quad (11)$$

12) D_3, E_2, F_2

$$D_3^+ E_2^+ F_2^+ = D_3 E_2 F_2 = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & \rho_{44}\,\omega_3^2 \omega_A^2 \omega_B^2 \end{pmatrix}$$

Sum

$$\begin{bmatrix} \rho_{11}\gamma^2 & \rho_{12}\gamma\gamma_B & \rho_{13}\gamma\gamma_A & \rho_{14}\gamma^2\gamma_A\gamma_B \\ \rho_{21}\gamma\gamma_B & \rho_{22}\gamma_B^2 & \rho_{23}\gamma_B\gamma_A & \rho_{24}\gamma\gamma_A\gamma_B^2 \\ \rho_{31}\gamma\gamma_A & \rho_{32}\gamma_A\gamma_B & \rho_{33}\gamma_A^2 & \rho_{34}\gamma\gamma_A^2\gamma_B \\ \rho_{41}\gamma^2\gamma_A\gamma_B & \rho_{42}\gamma\gamma_A\gamma_B^2 & \rho_{43}\gamma\gamma_A^2\gamma_B & \rho_{44}\gamma^2\gamma_A^2\gamma_B^2 \end{bmatrix} \quad (1)$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \rho_{22}\omega_B^2 & 0 & \rho_{24}\gamma\gamma_A\omega_B^2 \\ 0 & 0 & 0 & 0 \\ 0 & \rho_{42}\gamma\gamma_A\omega_B^2 & 0 & \rho_{44}\gamma^2\gamma_A^2\omega_B^2 \end{bmatrix} \quad (2)$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \rho_{33}\omega_A^2 & \rho_{34}\gamma\gamma_B\omega_A^2 \\ 0 & 0 & \rho_{43}\gamma\gamma_B\omega_A^2 & \rho_{44}\gamma^2\gamma_B^2\omega_A^2 \end{pmatrix} \quad (3)$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho_{44}\gamma^2\omega_A^2\omega_B^2 \end{pmatrix} \quad (4)$$

$$\begin{pmatrix} \rho_{11} w_1^2 & 0 & 0 & \rho_{14} w_1 w_2 \gamma_A \gamma_B \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \rho_{41} w_1 w_2 \gamma_A \gamma_B & 0 & 0 & \rho_{44} w_2^2 \gamma_A^2 \gamma_B^2 \end{pmatrix} \quad (5)$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho_{44} w_2^2 w_B^2 \gamma_A^2 \end{pmatrix} \quad (6)$$

$$\begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & \rho_{44} w_2^2 w_A^2 \gamma_B^2 \end{pmatrix} \quad (7)$$

$$\begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & \rho_{44} w_2^2 w_A^2 w_B^2 \end{pmatrix} \quad (8)$$

$$\begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & \rho_{44} w_3^2 \gamma_A^2 \gamma_B^2 \end{pmatrix} \quad (9)$$

$$\begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & \rho_{44} w_3^2 \gamma_A^2 w_B^2 \end{pmatrix} \quad (10)$$

$$\begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & \rho_{44} w_3^2 w_A^2 \gamma_B^2 \end{pmatrix} \quad (11)$$

$$\begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & \rho_{44} w_3^2 w_A^2 w_B^2 \end{pmatrix} \quad (12)$$

$$
\begin{bmatrix}
\rho_{11}(\gamma^2+\omega_1^2) & \rho_{12}(\gamma\gamma_B) & \rho_{13}\gamma\gamma_A & \rho_{14}\gamma_A\gamma_B(\gamma^2+\omega_1\omega_2) \\
\rho_{21}\gamma\gamma_B & \rho_{22}(\gamma_B^2+\omega_B^2) & \rho_{23}\gamma_A\gamma_B & \rho_{24}\gamma\gamma_A(\gamma_B^2+\omega_B^2) \\
\rho_{31}\gamma\gamma_A & \rho_{32}\gamma_A\gamma_B & \rho_{33}(\gamma_A^2+\omega_A^2) & \rho_{34}\gamma\gamma_B(\gamma_A^2+\omega_A^2) \\
\rho_{41}\gamma_A\gamma_B(\gamma^2+\omega_1\omega_2) & \rho_{42}\gamma\gamma_A(\gamma_B^2+\omega_B^2) & \rho_{43}\gamma\gamma_B(\gamma_A^2+\omega_A^2) & (*)
\end{bmatrix}
$$

$$
\begin{array}{lllll}
(*) & \rho_{44}\gamma^2\gamma_A^2\gamma_B^2 & \rho_{44}\gamma^2\gamma_A^2\omega_B^2 & \rho_{44}\gamma^2\gamma_B^2\omega_A^2 & \rho_{44}\gamma^2\omega_A^2\omega_B^2 \\
& \rho_{44}\omega_2^2\gamma_A^2\gamma_B^2 & \rho_{44}\omega_2^2\omega_B^2\gamma_A^2 & \rho_{44}\omega_2^2\omega_A^2\gamma_B^2 & \rho_{44}\omega_2^2\omega_A^2\omega_B^2 \\
& \rho_{44}\omega_3^2\gamma_A^2\gamma_B^2 & \rho_{44}\omega_3^2\gamma_A^2\omega_B^2 & \rho_{44}\omega_3^2\omega_A^2\gamma_B^2 & \rho_{44}\omega_3^2\omega_A^2\omega_B^2 \\
& + & + & + & + \\
\hline
& \rho_{44}\gamma_A^2\gamma_B^2(\gamma^2+\omega_2^2+\omega_3^2) & \gamma_A^2\omega_B^2(\gamma^2+\omega_2^2+\omega_3^2) & \omega_A^2\gamma_B^2(\gamma^2+\omega_2^2+\omega_3^2) & \omega_A^2\omega_B^2(\gamma^2+\omega_2^2+\omega_3^2)
\end{array}
$$

$$
= \rho_{44}(\gamma^2+\omega_2^2+\omega_3^2)\left[\gamma_A^2(\gamma_B^2+\omega_B^2)+\omega_A^2(\gamma_B^2+\omega_B^2)\right]
$$

$$
(*) = \rho_{44}(\gamma^2+\omega_2^2+\omega_3^2)(\gamma_A^2+\omega_A^2)(\gamma_B^2+\omega_B^2)
$$