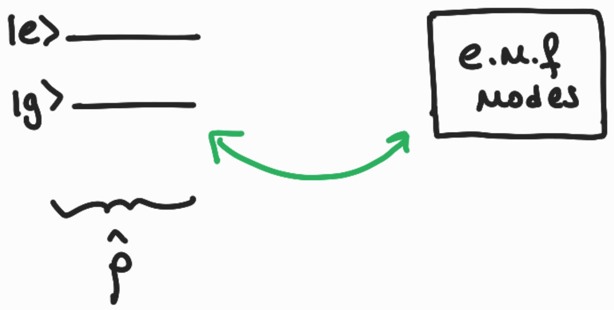
OPTICAL BLOCH EQUATIONS

Two level system + environment



I. Derivation

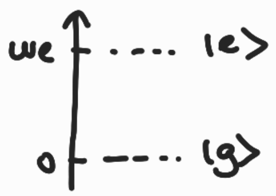
1) Model

can be anything

$$\boxed{\hat{H} = \Delta |e\rangle\langle e| + \frac{\Omega}{2}\left(\overbrace{|e\rangle\langle g| + |g\rangle\langle e|}^{\hat{\sigma}_x}\right)}$$

$$\hat{H} = \frac{\Delta}{2}\hat{\mathbb{I}} + \vec{h}\cdot\vec{\sigma}$$

$\Delta = \omega_e - \omega_{drive}$



- $\hat{H}$ in the frame rotating at the drive frequency + RWA

- We have chosen the phase of the external drive

$$\vec{h} \in xz \text{ plane}$$

In general, $\frac{\Omega}{2}\left(|e\rangle\langle g| e^{i\phi} + |g\rangle\langle e| e^{-i\phi}\right)$, ϕ arbitrary

Spontaneous emission

$$\hat{L} = \sqrt{\Gamma}\, |g\rangle\langle e|$$

jump operator

- jumps produce transitions $|e\rangle \to |g\rangle$
- Note: $\mathrm{Tr}(\hat{\rho}\hat{L}^{+}\hat{L}) = \Gamma .\ \mathrm{Tr}(\hat{\rho}\, |e\rangle\langle e|) = \Gamma\, p_e$

prob. being in $|e\rangle$ state

2. Justification

(fock space)

Extended Hilbert Space : $\mathcal{H} = \mathcal{H}_{\substack{2\text{-lev}\\ \text{sys}}} \otimes \mathcal{H}_{emf}$

Light-Matter interaction : $\hat{H}_{int} = \hat{d} \otimes \hat{\vec{E}}$

(dipole) moment operator

(electro magnetic field operator)

$$\hat{\vec{d}} = \vec{e}_x\, d_0\, \big(|e\rangle\langle g| + |g\rangle\langle e| \big)$$ (Dipolar approximation)

polarization vector

reduced dipole moment

$$\vec{E} = -i \sum_{\vec{k},\lambda} \sqrt{\frac{\hbar \omega_{\vec{k}}}{2\varepsilon_0 V}}\, \vec{\varepsilon}_\lambda \left(e^{-i(\vec{k}.\vec{r} - \omega_{\vec{k}} t)}\, \hat{a}_{\vec{k},\lambda} - \text{hermit conj} \right)$$

$\omega_{\vec{k}} = c|\vec{k}|$

V quantization volume

Fermi Golden Rule $\Gamma_{\vec{k},\lambda} = \frac{2\pi}{\hbar} \left| \langle \{0\}_{\vec{k},\lambda}, e | \hat{\vec{d}} \otimes \hat{\vec{E}} | \{1\}_{\vec{k}}, g \rangle \right|^2$

$$= \delta(\omega_e - \omega_{\vec{k}})$$

Lindblad Equation

$$\dot{\hat{\rho}} = -i[\hat{H}, \hat{\rho}] + \sum_{\vec{k},\lambda} \left(\hat{L}_{\vec{k},\lambda} \hat{\rho} \hat{L}^{+}_{k,\lambda} - \frac{1}{2} \hat{L}^{+}_{k\lambda} \hat{L}_{k,\lambda} \hat{\rho} - \frac{1}{2} \hat{\rho} \hat{L}^{+}_{k,\lambda} \hat{L}_{k,\lambda} \right)$$

Here $\hat{L}_{\vec{k},\lambda} = \sqrt{\Gamma_{k,\lambda}}\, |g\rangle\langle e|$

$$\dot{\hat{\rho}} = -i[\hat{H}, \hat{\rho}] + \left(\sum_{k,\lambda} \Gamma_{k,\lambda} \right) |g\rangle\langle e| \hat{\rho} |e\rangle\langle g| - \frac{1}{2} |e\rangle\langle e| \hat{\rho} - \frac{1}{2} \hat{\rho} |e\rangle\langle e|$$

Replace all $L_{\vec{k},\lambda}$ by $\hat{L} = \sqrt{\Gamma}\, |g\rangle\langle e|$, $\Gamma = \sum_{\vec{k},\lambda} \Gamma_{\vec{k},\lambda}$

Remark Momentum is not conserved ($\hbar\vec{k}$ is taken by the field)

3. Bloch vector dynamics

$$\dot{\hat{\rho}} = -i[\hat{H}, \hat{\rho}] + \Gamma \left(|g\rangle\langle e| \hat{\rho} |e\rangle\langle g| - \frac{1}{2} \hat{\rho} |e\rangle\langle e| - \frac{1}{2} |e\rangle\langle e| \hat{\rho} \right)$$

Matrix element $\dot{\rho}_{ee} = \langle e | \dot{\hat{\rho}} | e \rangle$

$$= -i \langle e | \hat{H}\hat{\rho} - \hat{\rho}\hat{H} | e \rangle - \Gamma \rho_{ee}$$

$$= -i \frac{\Omega}{2} (\rho_{ge} - \rho_{eg}) - \Gamma \rho_{ee}$$

$$\dot{\rho}_{eg} = -i\langle e|\hat{H}\hat{\rho} - \hat{\rho}\hat{H}|g\rangle - \frac{\Gamma}{2}\rho_{eg}$$

$$= -i\left(\Delta\rho_{eg} + \frac{\Omega}{2}(\rho_{gg} - \rho_{ee})\right) - \frac{\Gamma}{2}\rho_{eg}$$

$$\rho_{ee} + \rho_{gg} = 1 \quad \Rightarrow \quad \dot{\rho}_{ee} = -\dot{\rho}_{gg}$$

$$\rho_{ge} = \rho_{eg}^{*}$$

?

$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$|e\rangle\langle e| \quad |g\rangle\langle g| \quad \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\hat{\rho} = \frac{1}{2}(\mathbb{I} + \vec{a}\cdot\vec{\sigma})$$

$$a_z = \rho_{ee} - \rho_{gg} = 2\rho_{ee} - 1$$

$$a_x = \rho_{eg} - \rho_{ge}$$

$$a_y = i(\rho_{eg} - \rho_{ge})$$

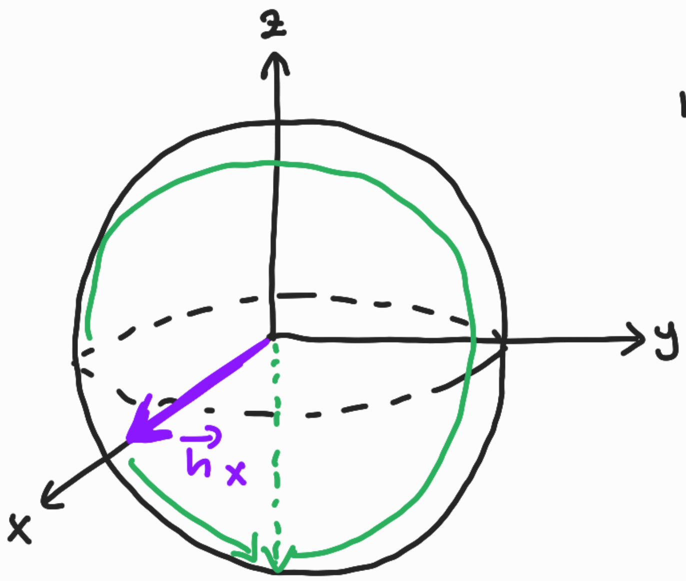
$$\dot{a}_z = \Omega a_y - \Gamma(a_z + 1)$$

$$\dot{a}_x = -\frac{\Gamma}{2}a_x - \underbrace{\Delta}_{\text{detuning}} a_y$$

$$\dot{a}_y = -\frac{\Gamma}{2}a_y + \Delta a_x - \underbrace{\Omega a_z}_{\text{external drive}}$$

"Optical Bloch" Equation

$\star$ choice of phase appears in the a_y equation



$\vec{h}_x$ causes rotation in yz plane

II. Solutions

Stationary solutions ($\dot{\vec{a}}=0$)

Let $s = \dfrac{2\Omega^2}{\Gamma^2+4\Delta^2}$ = saturation parameter (dimensionless)

$$a_x = -\frac{2\Delta}{\Omega}\frac{s}{1+s}$$

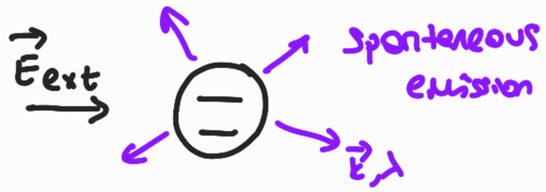
$$a_y = \frac{\Gamma}{\Omega}\frac{s}{(1+s)}$$

$$a_z = \frac{-1}{(1+s)}$$

Interpretation: $p_e = \langle e|\hat{\rho}|e\rangle = \rho_{ee} = \dfrac{1+a_z}{2} = \dfrac{1}{2}\dfrac{s}{(1+s)}$

prob. being in $|e\rangle$

$\Leftrightarrow$ Rate of photon scattering

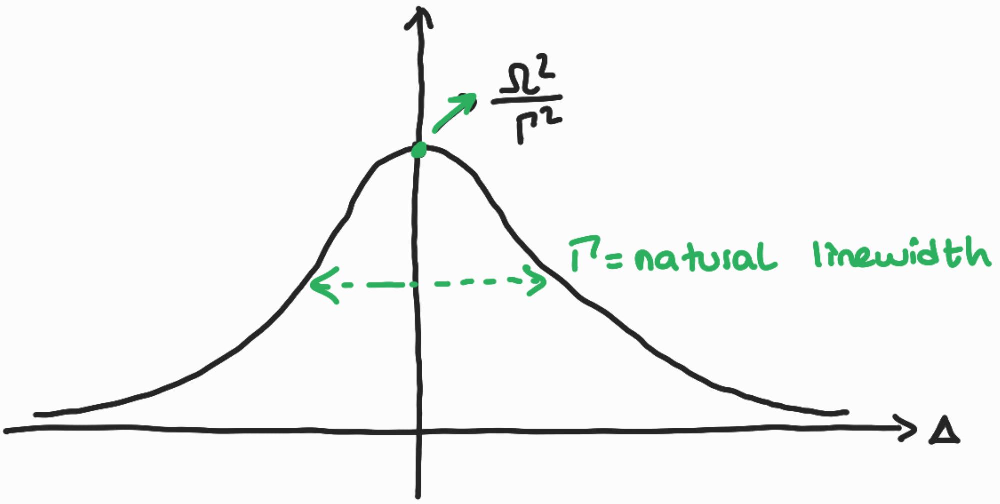


$\Gamma \rho_{ee}$ = rate of spontaneous emission = photon scattering

. $s \ll 1$: low saturation $\nearrow$ either $\Omega \ll \Gamma, \Delta \ldots$

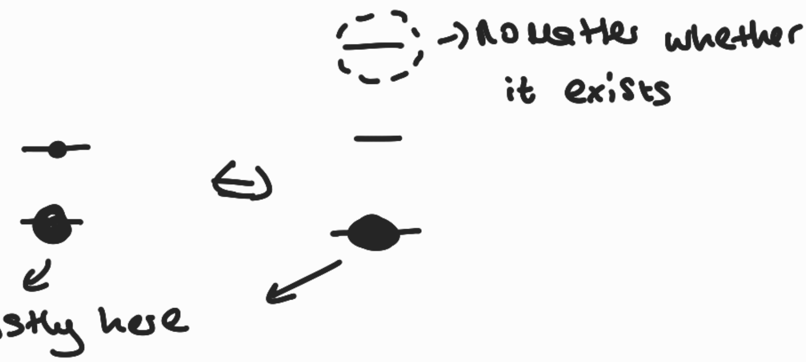
$\searrow$ large Δ

Then $\rho_{ee} = \frac{s}{2} = \frac{\Omega^2}{\Gamma^2 + 4\Delta^2} \ll 1$ "Lorentzian Profile"



Remark: Classical harmonic oscillator with damping rate Γ

Low s $\Rightarrow$ low ρ_{ee}



. $s \gg 1$ Strong saturation $\rho_{ee} = \frac{1}{2}$, $\Omega \gg \Gamma, \Delta$

forget Γ : at the scale of $T_{rabi} = \frac{2\pi}{\Omega}$,

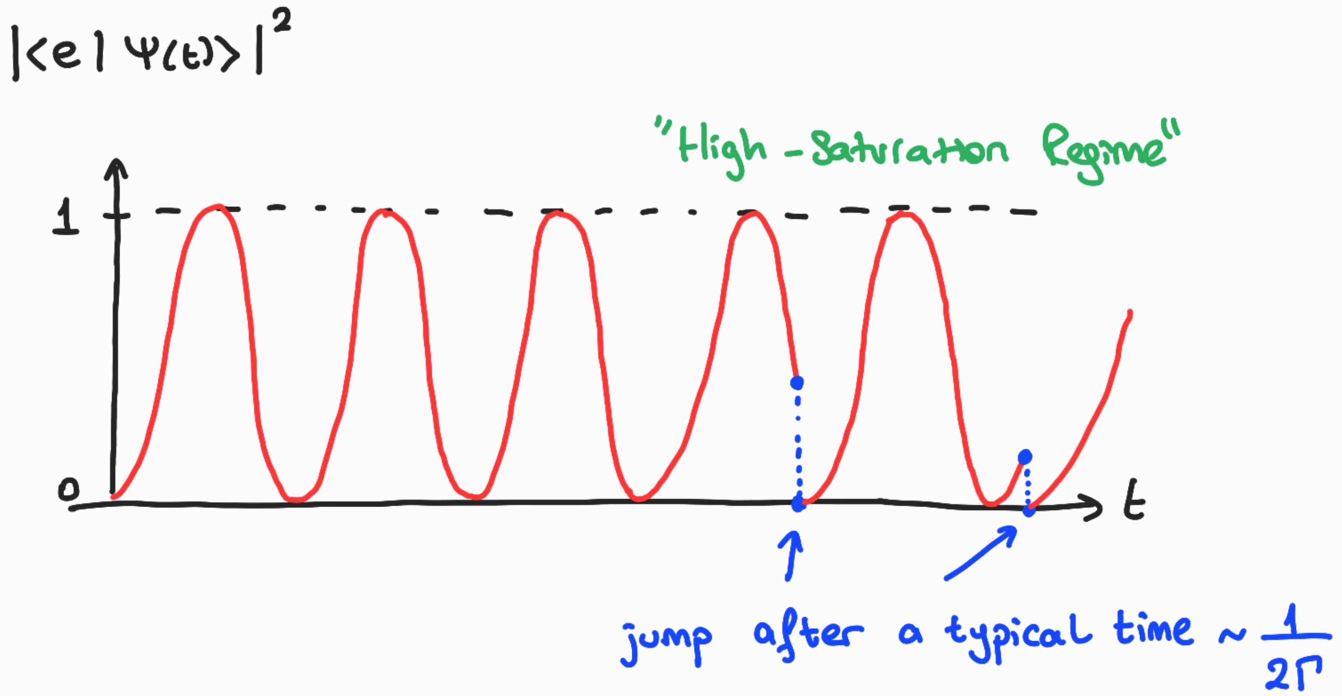
there is no spontaneous emission.

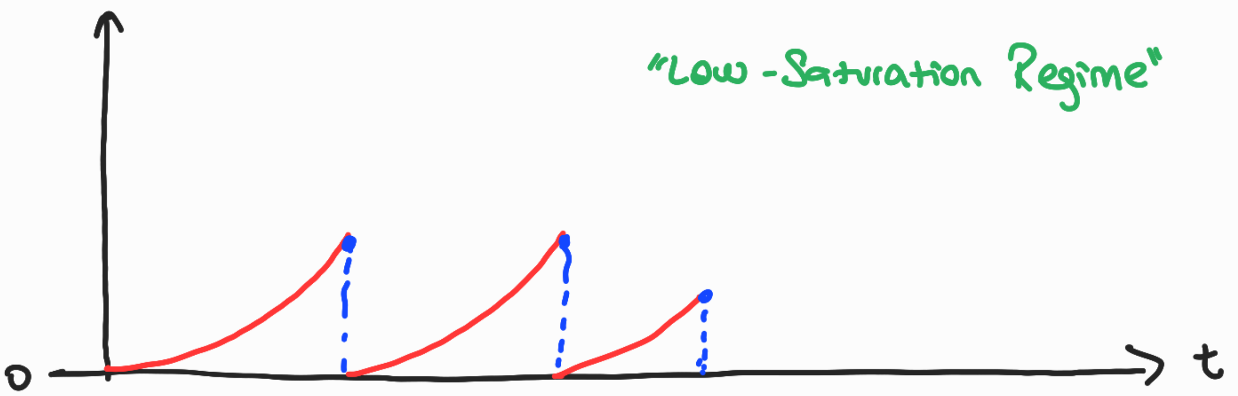
$\frac{1}{2}$ = time average of the Rabi oscillation

. Intermediate region: $\rho_{ee} = \frac{\overset{?}{4}\Omega^2}{4\Delta^2 + \Gamma'^2}$, $\Gamma' = \sqrt{\Gamma^2 + 2\Omega^2}$

power brodening

Quantum Trajectory Interpretation





Remarks: We can include other processes than sp. emission

$$\hat{L}_1 = \sqrt{\Gamma_1}\, |g\rangle\langle e|$$

$$\hat{L}_2 = \sqrt{\Gamma_2}\, |e\rangle\langle e| \qquad \text{Dephasing}$$

then: $$\dot{\rho}_{ee} = -i\langle e|\hat{H}\hat{\rho} - \hat{\rho}\hat{H}|e\rangle - \Gamma_1 \rho_{ee}$$

$$\dot{\rho}_{eg} = -i\langle e|\hat{H}\hat{\rho} - \hat{\rho}\hat{H}|g\rangle - \frac{\Gamma_1}{2}\rho_{eg} - \frac{\Gamma_2}{2}\rho_{eg}$$

In general, T_1 time $\longrightarrow$ decay time of ρ_{ee}

T_2 time $\longrightarrow$ " " " ρ_{eg}

Without dephasing : $T_2 = 2T_1$ $\rightarrow$ performance criteria