DAMPED HARMONIC OSCILLATOR

I. Introduction

Single mode of the em field 

★ Hamiltonian $\hat H = \omega_c \hat a^\dagger \hat a + \sqrt{\kappa}\,\mathcal{E}\,(\hat a\, e^{i\omega_e t} + \text{hermit conj.})$

in a rotating frame at ω_e $\quad U(t) = e^{i\omega_e t\,(\hat a^\dagger \hat a)}$

$$\boxed{\hat H = \Delta \hat a^\dagger \hat a + \sqrt{\kappa}\,\mathcal{E}\,(\hat a + \hat a^\dagger)} \qquad \Delta = \omega_c - \omega_e$$

★ jump operator $\hat L = \sqrt{\kappa}\,\hat a$

Lindblad Equation $\dot{\hat\rho} = -i[\hat H, \hat\rho] + \kappa\left(\hat a \hat\rho \hat a^\dagger - \frac{1}{2}\hat\rho \hat a \hat a^\dagger - \frac{1}{2}\hat a^\dagger \hat a \hat\rho\right)$

II. Field Amplitude

Cyclicity
$Tr(ABC) = Tr(CAB)$

$$\langle \hat a \rangle = Tr(\hat\rho \hat a)$$

$$\frac{d}{dt}\langle \hat a \rangle = \frac{d}{dt} Tr[\hat\rho \hat a] = Tr(\dot{\hat\rho}\hat a)$$

$$Tr\left[\hat a \left(-i[\hat H, \hat\rho] + \kappa\left(\hat a \hat\rho \hat a^\dagger - \frac{1}{2}\hat\rho \hat a^\dagger \hat a - \frac{1}{2}\hat a^\dagger \hat a \hat\rho\right)\right)\right]$$

$$-i\,Tr(\hat a[\hat H, \hat\rho]) = -i\,Tr(\hat a \hat H \hat\rho - \hat a \hat\rho \hat H) = -i\,Tr((\underbrace{\hat a \hat H - \hat H \hat a}) \hat\rho) = -i\langle[\hat a, \hat H]\rangle$$

$$= -i\langle[\hat a, \Delta \hat a^\dagger \hat a + \sqrt{\kappa}\,\mathcal{E}(\hat a + \hat a^\dagger)]\rangle$$

$$= -i\Delta\langle \hat a \rangle - i\sqrt{\kappa}\,\mathcal{E}$$

$$\boxed{K\,\mathrm{Tr}\left[\hat a\,\hat a\hat\rho\hat a^{+} - \frac{1}{2}\hat a\hat\rho\hat a^{+}\hat a - \frac{1}{2}\hat a\hat a^{+}\hat a\hat\rho\right]}$$

$$= K\,\mathrm{Tr}\left(\hat a^{+}\cancel{\hat a\hat a}\hat\rho\right) - \frac{K}{2}\mathrm{Tr}\left(\hat a^{+}\cancel{\hat a\hat a}\hat\rho\right) - \frac{K}{2}\mathrm{Tr}\left(\underbrace{\cancel{\hat a\hat a^{+}}}_{\hat a^{+}\hat a+1}\hat a\hat\rho\right)$$

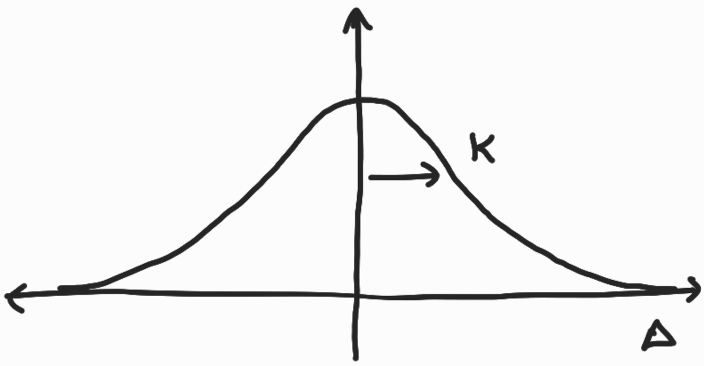
$$= \frac{K}{2}\langle\hat a\rangle$$

Thus $$\boxed{\langle\dot{\hat a}\rangle = -i\,\Delta\langle\hat a\rangle - \frac{K}{2}\langle\hat a\rangle - i\sqrt{K}\,\mathcal{E}}$$

Steady state $\langle\dot a\rangle = 0$ $\Rightarrow$ $\langle a\rangle = \dfrac{\sqrt{K}\,\mathcal{E}}{-\Delta + iK/2}$

Intensity

$$|\langle a\rangle|^2 = \frac{K\,|\mathcal{E}|^2}{\Delta^2 + \frac{K^2}{4}}$$



$$|\langle\hat a\rangle|^2 \sim \#\text{ of photons} = \frac{2|\mathcal{E}|^2}{K}$$

$|\mathcal{E}|^2$: photon flux $\Leftrightarrow$ laser intensity

$\mathcal{E}$ is the laser amplitude

Remark $\hat H \longrightarrow \hat H_{eff} = \left(\Delta - \frac{iK}{2}\right)\hat a^{+}\hat a$ not Hermitian

III. Coherent State Decay

Consider $|\alpha\rangle$: $\hat{L}|\alpha\rangle = \sqrt{\kappa}\,\alpha|\alpha\rangle$

Extended Hilbert Space: $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_{env}$

$$\hat{U}(|\alpha\rangle \otimes |0\rangle) = \big(\underbrace{\hat{M}_0|\alpha\rangle}_{\text{no jump}}\big)\otimes|0\rangle + \big(\underbrace{\hat{M}_1|\alpha\rangle}_{\text{jump}}\big)\otimes|1\rangle$$

$$= \left[\left(\mathbb{I} - i\Delta t\hat{H} - i\Delta t\hat{J}\right)|\alpha\rangle\right]\otimes|0\rangle + \left(\hat{M}_1|\alpha\rangle\right)\otimes|1\rangle$$

$$= \left[\left(\mathbb{I} - i\Delta t\left(\omega_c - \frac{i\kappa}{2}\right)\hat{a}^\dagger\hat{a}\right)|\alpha\rangle\right]\otimes|0\rangle + \left(\sqrt{\Delta t\kappa}\,\alpha|\alpha\rangle\right)\otimes|1\rangle$$

Measure the environment

→ 1 is obtained $|\alpha\rangle \longrightarrow |\alpha\rangle$

→ 0 is obtained $\underbrace{\left(\hat{\mathbb{I}} - i\Delta t\left(\omega_c - \frac{i\kappa}{2}\right)\hat{a}^\dagger\hat{a}\right)}_{}|\alpha\rangle$

(Δt is sufficiently small) $e^{-i\Delta t\left(\omega_c - \frac{i\kappa}{2}\right)\hat{a}^\dagger\hat{a}}\,|\alpha\rangle$

if $\kappa = 0$: $e^{-i\Delta t \omega_c \hat{a}^\dagger \hat{a}} |\alpha\rangle \underset{\downarrow}{=} |\alpha e^{-i\Delta t \omega_c}\rangle$

free evolution

with κ finite: $e^{-\Delta t \frac{\kappa}{2} \hat{a}^\dagger \hat{a}} |\alpha(t)\rangle$

IV Alternative descriptions

1) Photon Number

$|n\rangle$ fock state $\mathcal{H}_s = \mathrm{Sp}\{ |n\rangle, n \in \mathbb{N} \}$

$\langle n | \hat{\rho} | m \rangle = \rho_{nm}$

No external drive: $\hat{H} = \omega_c \hat{a}^\dagger \hat{a}$

$$\dot{\rho}_{nm} = \frac{d}{dt} \langle n | \hat{\rho} | m \rangle = \langle n | \dot{\hat{\rho}} | m \rangle$$

$$\dot{\rho}_{nm} = i\omega_c (n-m) \rho_{nm} - \frac{\kappa}{2}(n+m)\rho_{nm} + \kappa \sqrt{n+1}\sqrt{m+1}\, \rho_{n+1}\, \rho_{m+1}$$

Population $\quad p(n) = \rho_{nn}$

$$\Rightarrow \dot{p}(n) = - \underline{\kappa n\, p(n)} + \underline{\kappa (n+1)\, p(n+1)}$$

Remark: Rate of decay of $p(n)$ is $n\kappa$

Fock states with large n are highly unstable

# of photons $\langle \dot{\hat n} \rangle = \frac{d}{dt}\langle n \rangle = \mathrm{Tr}(\dot{\hat\rho}\, n) = \sum_n \dot{\rho}(n)\, n = -\kappa \langle n \rangle$

Notice $\left(\langle \hat a \rangle = -i\left(\omega_c - \frac{\kappa}{2}\right)\langle a \rangle \right)$?

2- Phase space description

Husimi Q function: $Q(\alpha) = \frac{1}{\pi}\langle \alpha | \hat\rho | \alpha \rangle$

Lindblad Equation $\rightarrow$ $$\frac{\partial Q}{\partial t} = -i\omega_c\left(\alpha^* \frac{\partial Q}{\partial \alpha^*} \cdot \alpha \frac{\partial Q}{\partial \alpha}\right) + \frac{\kappa}{2}\left(\frac{\partial(\alpha Q)}{\partial \alpha} + \frac{\partial(\alpha^* Q)}{\partial \alpha^*}\right) + \kappa \frac{\partial^2 Q}{\partial \alpha \partial \alpha^*}$$

(Statistical Physics IV) Fokker-Planck equation

V. Extensions

1. Finite temperature

unitary evolution on $\mathcal{H}_S \otimes \mathcal{H}_E$

$$\hat U(|\psi\rangle \otimes |N\rangle) = (\hat M_0 |\psi\rangle) \otimes |N\rangle + (\hat M_- |\psi\rangle) \otimes |N+1\rangle + (\hat M_+ |\psi\rangle) \otimes |N+1\rangle$$

$$\lambda(\hat{a}\hat{b}^{+} + \underset{\hat{a}^{+}\hat{b}}{\underbrace{hc}}) = \hat{H}_{SE}$$

(sys-env coupling)

$$\hat{M}_{-} \sim \sqrt{k\Delta t}\,\sqrt{N+1}\,\hat{a} \quad \rightarrow \quad \hat{L}_{-} = \sqrt{k}\,\sqrt{N+1}\,\hat{a}$$

$$\hat{M}_{+} \sim \sqrt{k\Delta t}\,\sqrt{N}\,\hat{a}^{+} \quad \rightarrow \quad \hat{L}_{+} = \sqrt{k}\,\sqrt{N}\,\hat{a}^{+}$$

At temperature $T = \frac{1}{\beta}$ $\quad N = \dfrac{1}{e^{\beta w_c} - 1}$ so that @ $T=0$: $\hat{L}_{+} = 0$

2) Phase Damping

$\hat{L} = \gamma \hat{a}^{+}\hat{a}$ (preserves photon number, get rid of coherences)

Environment "measuring the # of photons"

Fock basis: $\quad \dot{\rho}_{nm}(t) = \left[-i\,w_c\,(n-m) - \frac{\gamma}{2}(n+m)^2\right]\rho_{nm}$

ρ_{nm} for $n \neq m \longrightarrow 0$ exponentially fast