POSTULATES OF QM

SUMMARY

① a quantum system is described by its state vector $|\psi\rangle$, which is an element of an Hilbert space $\mathcal{H}$

Dirac notation

ⓐ vectors: we label a set of eigenvectors

$$|i\rangle \; ; \; i = 1, 2, \ldots, n$$

for a single qubit $\mathcal{H} = \text{Span}\{|0\rangle, |1\rangle\}$

$$|0\rangle \longleftrightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix} \; ; \; |1\rangle \longleftrightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

ⓑ Inner product: $(|\psi\rangle, |\phi\rangle) \Longrightarrow$ complex number

"BRA" vector: $\langle\psi|$: $\langle 0| \longleftrightarrow [1, 0]$

$\langle 1| \longleftrightarrow [0, 1]$

$$|\psi\rangle \longleftrightarrow \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \; ; \; \langle\psi| \longleftrightarrow [a_1^*, a_2^*]$$

$$\langle\psi|\phi\rangle = [a_1^*, a_2^*] \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = a_1^* b_1 + a_2^* b_2$$

ⓒ Outer product

$(|\psi\rangle, |\phi\rangle) \longrightarrow$ MATRIX

$$|\psi\rangle\langle\phi| = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} [b_1^*, b_2^*] \equiv \begin{bmatrix} a_1 b_1^* & a_1 b_2^* \\ a_2 b_1^* & a_2 b_2^* \end{bmatrix}$$

(2) For every measurable physical properties there exist a Hermitian operator which acts on the system Hilbert Space

(3) The eigenvalues of the operator are the measurable values of the physical property $\hat{A}|i\rangle = A_i|i\rangle$ $\{A_1, A_2, \dots A_N\}$

$A^\dagger = A \Longrightarrow \langle Ax|y\rangle \equiv \langle x|Ay\rangle \rightarrow$ real eigenvalues

↳ transpose + Complex conjugate

(4) The probability of obtaini A_i when measuring $\hat{A}$ is: $P(A_i) = |\langle i|\psi\rangle|^2 =$

$= \langle\psi|i\rangle\langle i|\psi\rangle.$

After the measurement the system is left in the state corresponding to the measured result.

"wavefunction collapse"

What is the probability of being in the $|0\rangle$ state? $\langle\psi|0\rangle\langle 0|\psi\rangle$ $M_0 = |0\rangle\langle 0|$ projection operator

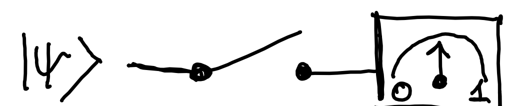
if $|\psi\rangle = |0\rangle \Rightarrow \langle 0|0\rangle\langle 0|0\rangle = 1$

if $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \Rightarrow \langle\psi|0\rangle\langle 0|\psi\rangle = \frac{1}{2}$

BRIEF RECAP OF QUANTUM MECHANICS

- The instantaneous state of any quantum system is given by its wavefunction, also equivalent to a complex-valued vector in the same abstract Hilbert space.
- All observables (i.e. physical quantities that we are interested in measuring) are described by Hermitian operators, defined in the same Hilbert space.

Since we cannot direcly observe the quantum system we need to perform a <u>measurement</u>; this consists in a classical apparatus that interacts with the quantum system:



The classical measurement apparatus presents many degrees of freedoms that during the interaction time with the quantum system get entangled with it. This causes the collaps of the system's wavefunction...

AXIOMS OF QUANTUM THEORY

(1) The state of a quantum system at time "t" is described by a normalized vector $|\psi(t)\rangle$ belonging to the Hilbert space $\mathcal{H}$, specific of the system under study.

(2) [Schrödinger equation] The state evolves in time according to

$i\hbar|\dot{\psi}(t)\rangle = H|\psi\rangle$, where H is the Hamiltonian; it is an Hermitian operator associated with the energy of the system

(3) [Collapse of the wavefunction] The measured value of an observable is always one of the eigenvalues of its operator $\hat{A}$ (a_j) whatever the state of the system was before the measurement, immediately after it the state vector of the system is the corresponding normalized eigenvector of $\hat{A}$, $|a_j\rangle$: $\hat{A}|a_j\rangle = a_j|a_j\rangle$

(4) [Born's rule] The probability of measuring a particular eigenvalue, a_j, of the observable $\hat{A}$ (collapsing the wavefunction $|\psi(t)\rangle$ into a given eigen-vector $|a_j\rangle$) is given by the square's modulus of the former's projection on the latter: $P_j(t) = |\langle a_j|\psi(t)\rangle|^2$

OBSERVATIONS:

- special role played by time and energy (Hamiltonian) in the evolution of the system.
- linear, reversible, unitary quantum evolution determined by the Schrödinger Equation.
- non-linear, irreversible, non-unitary measurement process

$$|\psi(t)\rangle = U(t)\,|\psi(0)\rangle \; ; \; U(t) = \exp\left[-\frac{i}{\hbar} H t\right]$$

and for H hermitian $\Rightarrow$ the evolution operator $U(t)$ is unitary

$$U(t)^{+}\, U(t) = U(t)\, U(t)^{+} = \mathbb{I}$$

- The unitarity of the evolution operator ensures that the normalization of the state vector is preserved.
- Axioms (3) + (4) form the "projection postulate":
  - the measurement of the observable $\hat{A}$ projects the state vector $|\psi(t)\rangle$ on the eigenvectors of $\hat{A}$.
  - the square modulus of the projection gives the probability with which the system is likely to be found.

OBSERVATIONS: Physical quantity $A \rightarrow$ observation $\hat{A}$

$\hat{A}|u_n\rangle = \lambda_n |u_n\rangle$ To understand what it means to measure $\hat{A}$ in Q.M., the key equation that we should consider is the Eigenvalues equation.

POSTULATE III

- The results of a measurement of a physical quantity is one of the eigenvalues of the associated observable.
- This means that when we want to measure property $\hat{A}$ we first have to solve the eigenvalue equation for $\hat{A}$, which allows us to find all eigenvalues $\hat{A}$: $\lambda_1, \lambda_2, \ldots, \lambda_n \ldots$
- This postulate says that whatever is the state of our system, when we measure $\hat{A}$ we can only get its eigenvalues as outcome of the measurement [IRRESPECTIVE OF THE STATE OF THE SYSTEM]
- The key question become: if we measure $\hat{A}$ in a system characterized by the state $|\psi\rangle$, which of the eigenvalues we will get?

Postulate III tells us that we get one of the λ_i eigenvalues of $\hat{A}$ (5)
but it does not tell us which one. For this we need to look at the next postulate

POSTULATE IV: The measurement of $\hat{A}$ in a system in the state $|\psi\rangle$ (normalized) gives λ_n with probability

$$P(\lambda_n) = |\langle u_n|\psi\rangle|^2$$

independent from $|\psi\rangle$ of the system.

$$\hat{A}: \quad \lambda_1, \quad \lambda_2, \quad \cdots, \quad \lambda_N, \cdots$$

$$|u_1\rangle, \quad |u_2\rangle, \quad \cdots, \quad |u_N\rangle, \cdots$$

$$|\psi\rangle: \quad |\langle u_1|\psi\rangle|^2, \quad |\langle u_2|\psi\rangle|^2, \cdots \quad |\langle u_N|\psi\rangle|^2, \cdots$$

But when we measure $\hat{A}$, we will measure in a system with specific state $|\psi\rangle$

So what we get when we measure $\hat{A}$ depends on 2 things:

1. the intrinsic properties of $\hat{A}$ (λ_i, $|\lambda_i\rangle$) and
2. the specific state $|\psi\rangle$ of our system.

This introduces the famous $\boxed{\text{probabilistic nature of Q.M.}}$
Rather than telling us the precise outcome of a measurement, it tells us the probability associated with any given outcome.

- All we can just predict is the probability of getting a particular outcome. We have to actually measure $\hat{A}$ to find out the answer. This is in strong contrast with classical mechanics, where we can always, in principle, predict a priory the outcome of a measurement, even before doing it.

- Practically this means that we should consider many copies of our system in the state $|\psi\rangle$; we will measure $\hat{A}$ for all of these systems obtaining one of the possible eigenvalue each time.

$$
\begin{array}{ccccl}
|\psi\rangle & |\psi\rangle & |\psi\rangle & |\psi\rangle & \text{-- N copies} \\
\downarrow \hat{A} & \downarrow \hat{A} & \downarrow \hat{A} & \downarrow \hat{A} & \\
\lambda_{10} & \lambda_{1} & \lambda_{99} & \lambda_{10} &
\end{array}
$$

- Let's say that we get eigenvalue λ_m a total of p_m times

$$N \longrightarrow \infty : \frac{p_m}{N} \longrightarrow P(\lambda_m)$$

probability of getting λ_m from the postulate IV

But if we have just 1 copy of the system, all we can know is the probability to get a particular outcome.

• If we write $|\psi\rangle$ in the complete basis of the eigenstates (7) of the Hermitian operator $\hat{A}$, $|\psi\rangle = \sum_n c_n |u_n\rangle$, these expantion coefficient "c_n" (called "the representation") of $|\psi\rangle$ in the $|u_n\rangle$ basis) are given by the projection of $|\psi\rangle$ in the $|u_n\rangle$ basis: $c_n = \langle u_n | \psi \rangle$

• So $\underbrace{P(\lambda_n)}_{\text{probability of measuring the eigenvalue } \lambda_n} = |\langle u_n|\psi\rangle|^2 = |c_n|^2$

$\hookrightarrow$ c_n tells also already how likely is to measure the λ_n if we are in the state $|\psi\rangle = \sum_n c_n |u_n\rangle$

• A special case is when $|\psi\rangle = |u_n\rangle$ eigenvector of $\hat{A}$.
In this case we already know, before measuring, with absolute certainty what the outcome of the measurement will be $P(\lambda_n) = |c_n|^2 = 1$. We will get λ_n with probability 1.

We can describe this process in a more pictorial manner

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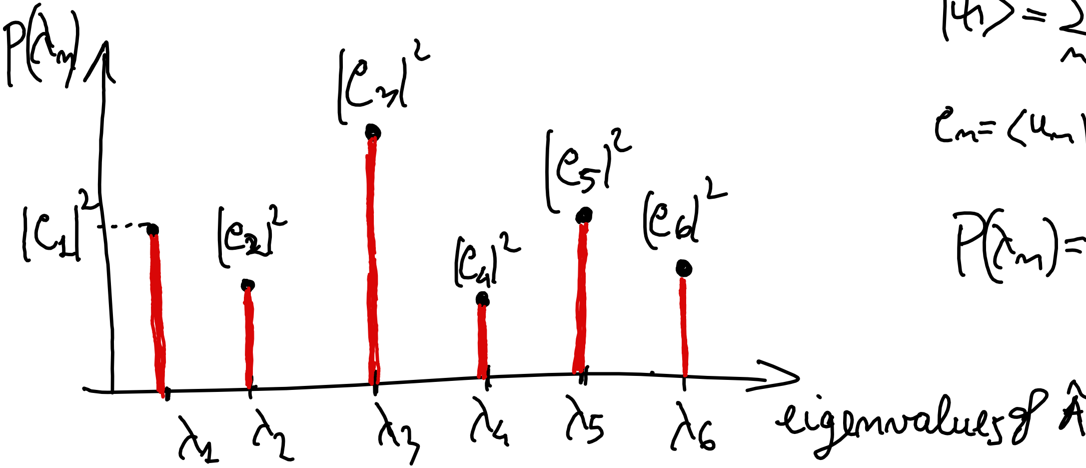

$|\psi\rangle = \sum_n c_n |u_n\rangle$ ← bases of eigenstates of $\hat{A}$

$c_n = \langle u_n|\psi\rangle$

$P(\lambda_n) = |\langle u_n|\psi\rangle|^2 = |c_n|^2$

- This diagram shows what is called a probability distribution. The higher is a line $|c_n|^2 \Longrightarrow$ the higher is the likelihood that we will measure that eigenvalue λ_n.
- Postulate IV teaches us about the measurement process. It tells us precisely the hights of these probability distributions.
- It tells us precisely the probability distribution of the different outcomes. If we have a large number N of copies of the system, the fraction of time that we will get any given outcome approaches the distribution as $N \to \infty$

Conceptual understanding of quantum measurements (9)

For any individual measurement we cannot known in advance what the outcome of the measurement will be. Unlike classical physics, in the quantum world we cannot predict precise outcome of the measurement of a physical quantity. Instead, what Q.M. tells us is the precise probability distribution of all possible outcomes of that measurement.

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OBSERVATION ABOUT THE POSTULATE (5) [see the following page]

- The spirit of Quantum Computing is to design systems that evolves according to specific Hamiltonians: $|\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle \quad (\hbar=1)$ in order to produce specific transformation of the states.

5) The evolution of a closed system is governed by a unitary operator : $|\psi(t)\rangle = U(t,t_0)|\psi(t_0)\rangle$, where $U = e^{-i\hat{H}t/\hbar}$; $U^{+} = U$

The evolution is determined by the Schrödinger equation:

$$i\hbar \frac{\partial |\psi(t)\rangle}{\partial t} = \hat{H}\,|\psi(t)\rangle$$

6) The state space of a composite system is the tensor product of the subsystems.

$$H^{\otimes n} = H \otimes H \otimes \dots \otimes H$$

For 1-qubit , the basis states are : $\{|0\rangle, |1\rangle\}$; $|0\rangle = \begin{bmatrix}1\\0\end{bmatrix}$; $|1\rangle = \begin{bmatrix}0\\1\end{bmatrix}$

For 2-qubits , " " " " : $\{|0\rangle \otimes |0\rangle ; |0\rangle \otimes |1\rangle , |1\rangle \otimes |0\rangle , |1\rangle \otimes |1\rangle\}$

example $|0\rangle \otimes |1\rangle = \begin{bmatrix}1\\0\end{bmatrix} \otimes \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\begin{bmatrix}0\\1\end{bmatrix}\\ 0\begin{bmatrix}0\\1\end{bmatrix}\end{bmatrix} = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}$

So the base states are

$$\begin{bmatrix}1\\0\\0\\0\end{bmatrix} ; \begin{bmatrix}0\\1\\0\\0\end{bmatrix} ; \begin{bmatrix}0\\0\\1\\0\end{bmatrix} ; \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

OBSERVATIONS:

- special role played by time and energy (Hamiltonian) in the evolution of the system;
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$$|\psi(t)\rangle = U(t)|\psi(0)\rangle \quad ; \quad U(t) = \exp\left[-\frac{i}{\hbar} H t\right]$$

and for H hermitian $\Rightarrow$ the evolution operator is unitary

$U(t)^{\dagger} U(t) = U(t) U(t)^{\dagger} = I \Rightarrow$ · preserve the length of the vector in the H space
· the norm of the vector is preserved

- The unitarity of the evolution operator ensures that the normalization of the state vector is preserved.
- Axioms (3) + (4) form the "projection postulate":
  - the measurement of the observable $\hat{A}$ projects the state vector $|\psi(t)\rangle$ on the eigenvectors of $\hat{A}$.
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ENTANGLEMENT:

example $|\psi_1\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |01\rangle\right) =$

$= \frac{1}{\sqrt{2}} |0\rangle \left(|0\rangle + |1\rangle\right)$

$|\psi_2\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$

Those two states are very different!

Let's measure the state of qubit A: $M_0^{(A)} = \overset{A}{|0\rangle\langle 0|} \otimes \overset{B}{\mathbb{I}}$

$$\langle\psi_1| M_0^{(A)} |\psi_1\rangle = 1 \quad ; \quad \langle\psi_2| M_0^{(A)} |\psi_2\rangle = \frac{1}{2}$$

After the measurement

$$|\psi_1\rangle \longrightarrow M_0^{(A)} |\psi_1\rangle = |0\rangle\left(|0\rangle + |1\rangle\right)$$

$$|\psi_2\rangle \longrightarrow M_0^{(A)} |\psi_2\rangle = |00\rangle$$

So for the $|\psi_2\rangle$, by measuring A we will known also the state of qubit B. This happen because $|\psi_2\rangle$ in an <u>entangled state</u>.

<u>More precisely</u> : if we cannot write the two-qubit states (13)
as a tensor product of two single qubit states, this state is defined ENTANGLED.

$|\psi\rangle = |\alpha\rangle \otimes |\beta\rangle$ where $|\alpha\rangle = a|0\rangle + b|1\rangle$
$|\beta\rangle = c|0\rangle + d|1\rangle$

$|\psi\rangle = ac|00\rangle + ad|01\rangle + bc|10\rangle + bd|11\rangle$

if we would like to get $|\psi\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$, we cannot find any a, b, c, d that satisfy this.

- We can try to quantify how much entangled 2 states are:

Degree of entanglement : $|\psi\rangle = p|\psi_{ent}\rangle + \sqrt{1-p^2}\, e^{i\varphi}|\psi_{sep}\rangle$

maximally entangled ← ; separable ←

We can define the "concurrence":

$$\mathcal{C}(|\psi\rangle) \equiv 2\left|C_{00}C_{11} - C_{01}C_{10}\right| \quad \text{for } |\psi\rangle = C_{00}|00\rangle + C_{01}|01\rangle + \ldots$$

$|\phi^+\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$ $\qquad |\psi^+\rangle = 1/\sqrt{2}\left(|01\rangle + |10\rangle\right)$

$|\phi^-\rangle = 1/\sqrt{2}\left(|00\rangle - |11\rangle\right)$ $\qquad |\psi^-\rangle = 1/\sqrt{2}\left(|01\rangle - |10\rangle\right)$

N.B. being in a product state means that you can ask the question: what is the state of each subsystem?

Instead, entangled states, are all of those states for which you cannot answer this question [not separable].

It is not possible to say what is, for example, the state of the first qubit, because the first qubit, for istance, is $|0\rangle$ but only if the second is $|0\rangle$ too. There is a strong correlation!

Because of the linear superposition principle, entangled states are the majority of all possible states of several qubits.

- Entanglement is a resource for quantum computing.

It is possible to demonstrate that without entanglement you cannot do Q.C. If you remove entanglement, what you can do with a quantum computer can be simulated classically.

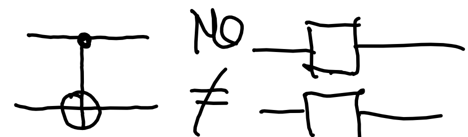
SEPARABLE GATES

Are just single qubit gates sufficient to build a Q.C.? NO

Suppose that you want to carry out operations using only single qubit gates (of any form). This means that the unitary that you apply of at least 2 qubits $U = U_1 \otimes U_2$ is just of a single tensor product of unitary gates on each qubits.

So this unitary operations is separable. The problem is that if the input is separable, by applying a separable unitary gate also the output will be necessary separable.

In other words, if you use a circuit with only single qubit gates then the result is not entangled [starting with separable states is a sort of assumption in Q.C.] $\Rightarrow$ thus can be simulated with a probabilistic classical algorithm.



NO $\neq$

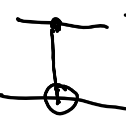
CNOT and CZ are not separable

It is necessary to have at least a gate that generates entanglement!

$|\psi_1\rangle = a|0\rangle + b|1\rangle$

$\otimes$

$|\psi_2\rangle = |0\rangle$



$|\psi'\rangle = a|00\rangle + b|11\rangle$

$\neq |\psi'_1\rangle \otimes |\psi'_2\rangle$

N.B. Thanks to CNOT you can transform a separable state into an entangled state.

A non-separable gate is what we need in order that the quantum computation paradigm will be more efficient of the classical computation.

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READOUT

Reading out a qubit means asking if the qubit is in $|0\rangle$ or $|1\rangle$ state. In quantum mechanics this means to find an observable $\hat{M}$ that is diagonal on the computational basis: $\hat{M} = m_0 |0\rangle\langle 0| + m_1 |1\rangle\langle 1|$.

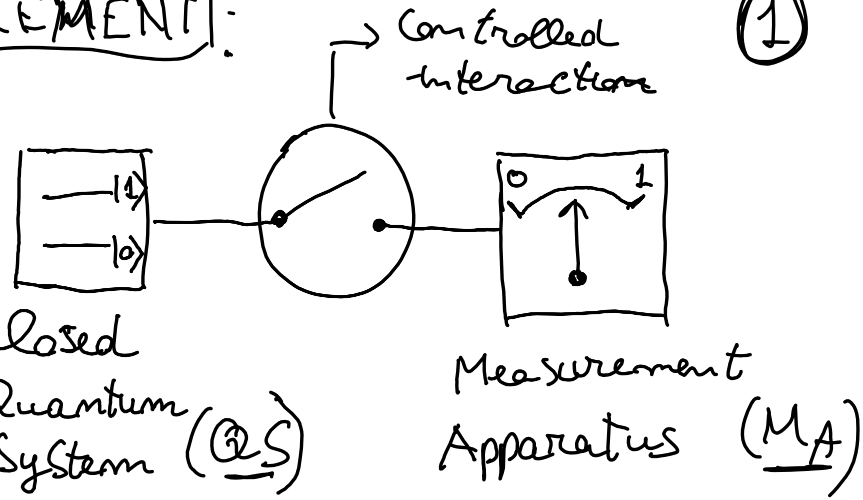
Therefore, if we measure m_0 we have measured the $|0\rangle$ state...

We restrict our computer to readout only in the computational basis. If you want to readout on a different basis we need first to apply a rotation.

Principle of deferred measurement $\left\{ \begin{array}{l} \text{Measurement can be always} \\ \text{be moved at the end} \end{array} \right.$

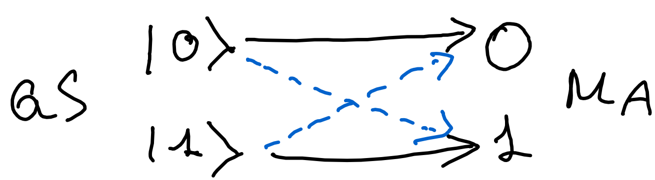
Quantum MEASUREMENT:

- generic setup



- goal: faithful reconstruction of qubit state

- Desired properties of MA:
  - HIGH ON/OFF RATIO : no interaction of MA with QS when OFF; strong interaction when ON
  - HIGH FIDELITY of mapping of QS into the states of the MA (classical states).



infidelity (misindentification)

- FAST MA in comparison to qubit relaxation/coherence
- Quantum-Non-Demolition (QND) measurement:
  - repeatibility of measurement with the same outcome.

$|\psi\rangle$ — $\boxed{M}$ — $|\psi_m\rangle$ - - - - - - - - - $|\psi_m\rangle$ — $\boxed{M}$ —

outcome "m" · free evolution · same outcome "m"

MEASUREMENT POSTULATE

- measurement result "m" with qubit state $|\psi\rangle$ occurs with probability $P_m = \langle\psi|\hat{M}_m^+ M_m|\psi\rangle$, with a set of measurement operators $\{\hat{M}_m\}$ acting on the qubit states $|\psi\rangle$ that is complete

$$\sum_m P_m = 1 \Longleftrightarrow \sum_m \hat{M}_m^+ M_m = \mathbb{I}$$

- post-measurement qubit state $|\psi'\rangle = \dfrac{\hat{M}_m|\psi\rangle}{\sqrt{P_m}}$ normalization

Applying P_m corresponds to asking the question: "is the system in state $|m\rangle$?"

$$P_\uparrow|\uparrow\rangle = |\uparrow\rangle\langle\uparrow|\uparrow\rangle = 1\,|\uparrow\rangle \quad \text{answer YES!}$$

$$P_\uparrow|\downarrow\rangle = |\uparrow\rangle\langle\uparrow|\downarrow\rangle = 0 \quad \text{" NO!}$$

MEASUREMENT of QUBIT STATE in computational basis

- define measurement operators:

$$\left.\begin{aligned}\hat{M}_0 &= |0\rangle\langle 0| = \begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\\ \hat{M}_1 &= |1\rangle\langle 1| = \begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix} = \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\end{aligned}\right\} \text{complete } \sum_m M_m^\dagger M_m = \mathbb{I}$$

<u>example</u>: measurement of $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

$$P_0 = \langle\psi|\hat{M}_0^\dagger \hat{M}_0|\psi\rangle = \alpha^*\alpha = |\alpha|^2$$

$$P_1 = \langle\psi|\hat{M}_1 M_1|\psi\rangle = \beta^*\beta = |\beta|^2$$

NOTE: - single preparation of state $|\psi\rangle$ with single measurement $\hat{M}_m$ results in single outcome "m" with probability p_m

- to determine p_m, $|\psi\rangle$ has to be prepared and measured repeatibilly (to determine $|\alpha|^2$ and $|\beta|^2$)
- full knoledge of state requires α, β to be known

e.g. $\hat{\sigma}_z$ for ground state:

$$\langle 0|\hat{\sigma}_z|0\rangle = \langle 0|\left(|1\rangle\langle 1| - |0\rangle\langle 0|\right)|0\rangle =$$

$$= \langle 0|1\rangle\langle 1|0\rangle - \langle 0|0\rangle\langle 0|0\rangle = -1$$

$\hat{\sigma}_z$ for the excited state: $\langle 1|\hat{\sigma}_z|1\rangle = 1$

$\hat{\sigma}_z$ for a superposition state: $|\psi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$

$$\langle\psi|\hat{\sigma}_z|\psi\rangle = \frac{1}{\sqrt{2}}\left(\langle 0| + \langle 1|\right)\hat{\sigma}_z\left(|0\rangle + |1\rangle\right)\frac{1}{\sqrt{2}} =$$

$$= \frac{1}{2}\left(\langle 0| + \langle 1|\right)\left(|1\rangle\langle 1| - |0\rangle\langle 0|\right)\left(|0\rangle + |1\rangle\right) = \frac{1}{2}(-1+1) = 0$$

- $\hat{\sigma}_z$ measures projection of Bloch vectors onto z-axis.

- post measurement state

$$|\psi_0\rangle = \frac{\hat{M}_0|\psi\rangle}{\sqrt{p_0}} = \frac{\alpha}{|\alpha|}|0\rangle$$

$$|\psi_1\rangle = \frac{\hat{M}_1|\psi\rangle}{\sqrt{p_1}} = \frac{\beta}{|\beta|}|1\rangle$$

- repeated measurements

$$p_{00} = \langle\psi_0|\hat{M}_0^\dagger \hat{M}_0|\psi_0\rangle = 1$$

↓ probability of results of second measurement to be $m=0$ provided that first result was $m=0$

$$p_{01} = 0$$

$$p_{10} = 0$$

$$p_{11} = 1$$

NOTE: any projective measurement should fulfill the above properties.