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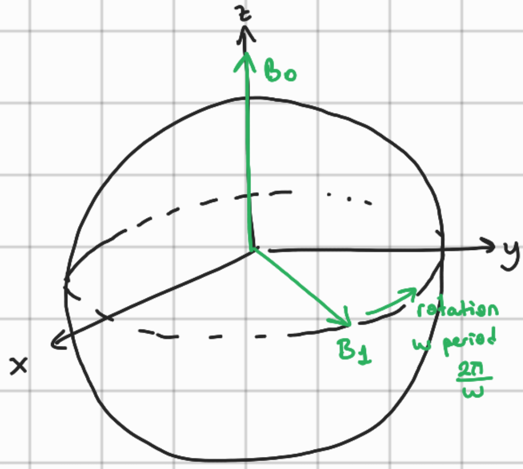
Dynamics of Spin 1/2 in Magnetic field

$(0, 0, B_0)$ // z



add $(B_1 \cos \omega t, B_1 \sin \omega t, 0)$

$$\vec{B} = (B_1 \cos \omega t, B_1 \sin \omega t, B_0)$$

Bloch Sphere

Hamiltonian generating dynamics $H(t) = -\gamma \frac{\hbar}{2} \vec{B}(t) . \vec{\sigma}$ → Pauli Matrices

2x2 matrix

$B_x(t)\sigma_x + B_y(t)\sigma_y + B_z(t)\sigma_z$

time dependent

$i\hbar \frac{d}{dt} U_t = H(t) U_t$ → ODE , $U_{t=0} = \mathbb{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Intuitions and discussion about H(t)

$$H(t) = -\underbrace{\frac{\gamma\hbar}{2} B_0}_{\hbar\omega_0} \sigma_z - \underbrace{\frac{\gamma\hbar}{2} B_1}_{\hbar\omega_1} \{ \sigma_x \cos \omega t + \sigma_y \sin \omega t \}$$

ω_0 = Larmor frequency = γB_0

ω_1 = frequency, measures intensity of $B_1 = \gamma B_1$

$$\sigma_+ = \frac{1}{2}(\sigma_x + i\sigma_y)$$
$$\sigma_- = \frac{1}{2}(\sigma_x - i\sigma_y)$$

$$\Downarrow$$

$$H(t) = -\frac{\hbar\omega_0}{2}\sigma_z - \frac{\hbar\omega_1}{2}\left\{ \sigma_+ e^{-i\omega t} + \sigma_- e^{i\omega t} \right\}$$

$$\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

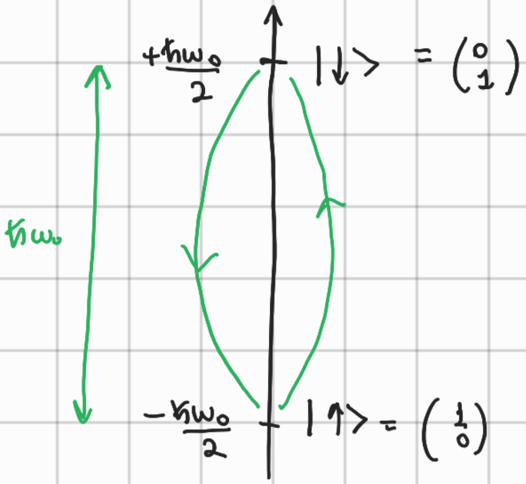
$$\sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

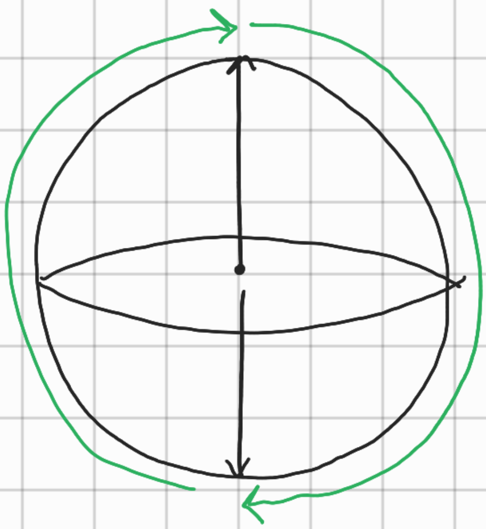
$$\boxed{\begin{array}{ll} \sigma_+|\downarrow\rangle = |\uparrow\rangle & , \quad \sigma_+|\uparrow\rangle = 0 \\ \sigma_-|\downarrow\rangle = |\downarrow\rangle & , \quad \sigma_-|\downarrow\rangle = 0 \end{array}}$$

Raising & Lowering Operators

Since the system is two-level

Energy levels (spectrum) of $H_0 = -\frac{\hbar\omega_0}{2}\sigma_z = \begin{pmatrix} -\frac{\hbar\omega_0}{2} & 0 \\ 0 & +\frac{\hbar\omega_0}{2} \end{pmatrix}$





Goal: $|\Psi_t\rangle$ in detail

$$\text{Prob}\left(\begin{array}{l} \text{spins } |\downarrow\rangle \text{ at time } t \\ \text{when it was } |\uparrow\rangle \text{ at time } 0 \end{array} \right) = \text{Prob}\left(|\uparrow\rangle_{t=0} \to |\downarrow\rangle_t \right)$$

$$= \Big| \langle\downarrow| \underbrace{U_t |\uparrow\rangle_{t=0}}_{|\psi(t)\rangle} \Big|^2$$

Born Rule

Solution of Schrödinger Equation

$$i\hbar \frac{d}{dt} U_t = H(t) U(t)$$

$$U_{t=0} = \mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$H(t) = -\frac{\hbar\omega_0}{2}\sigma_z + \frac{\hbar\omega_1}{2}\left\{\sigma_+ e^{-i\omega t} + \sigma_- e^{i\omega t}\right\}$$

When H is time - independent

you have $U_t = \exp\left(-\frac{it}{\hbar} H\right)$ ⚠

Method of change of Ref- frame

↳ Rotating Reference Frame

idea: attach frame to $(B_1 \cos\omega t, B_1 \sin\omega t, 0)$

Schrödinger Eqn. becomes TIME INDEPENDENT

$$i\hbar \frac{d}{dt} \tilde{U}_t = \tilde{H} \tilde{U}_t$$

↳ does not depend on time

$$\tilde{U}_t = \exp\left(-\frac{it}{\hbar} \tilde{H}\right)$$ in new rotating frame ⚠

Change of frame

Linear transformation in the Hilbert Space $\mathcal{H} \in \mathbb{C}^2$

$$\underset{\text{Rotating Frame}}{|\tilde{\Psi}\rangle} = \underbrace{e^{-i\frac{\omega t}{2}\sigma_z}}_{\text{2x2 Matrix Unitary}} |\Psi\rangle_{\text{Lab frame}}$$

$e^{i\frac{\alpha}{2}\sigma_z}$ = rotation matrix of angle α around z-axis on the Bloch Sphere.

$$K = -\frac{\hbar\omega}{2}\sigma_z \rightarrow \underset{\text{Rot frame}}{|\tilde{\Psi}_t\rangle} = \underbrace{e^{\frac{itK}{\hbar}}}_{\text{change of frame Matrix}} \underset{\text{Lab frame}}{|\Psi_t\rangle}$$

$$\tilde{U}_t |\tilde{\Psi}_0\rangle_{\text{Rot frame}} = e^{\frac{itK}{\hbar}} U_t |\Psi_0\rangle_{\text{Lab frame}}$$

$$\boxed{\tilde{U}_t = e^{\frac{itK}{\hbar}} U_t}$$

Schrödinger Equation

these two matrixes are commuting

$$i\hbar \frac{d}{dt}\tilde{U}_t = -K e^{\frac{itK}{\hbar}} U_t + e^{\frac{itK}{\hbar}} i\hbar \frac{dU_t}{dt} \rightarrow H(t)\,U_t$$

$$\tilde{U}_t$$

$$i\hbar \frac{d}{dt}\tilde{U}_t = -K e^{\frac{itK}{\hbar}} U_t + e^{\frac{itK}{\hbar}} H(t)\,U_t$$

$$\left(e^{-\frac{itK}{\hbar}} \cdot e^{+\frac{itK}{\hbar}}\right)$$

$$i\hbar \frac{d}{dt}\tilde{U}_t = \Big(\underbrace{-K + e^{\frac{itK}{\hbar}} H(t)\, e^{-\frac{itK}{\hbar}}}_{\tilde{H}(t)}\Big)\tilde{U}_t$$

Rotating Frame Hamiltonian

$$e^{\frac{itK}{\hbar}} = e^{-\frac{i\omega t}{2}\sigma_z} = \begin{pmatrix} e^{-\frac{i\omega t}{2}} & 0 \\ 0 & e^{+\frac{i\omega t}{2}} \end{pmatrix}$$

$$= \cos\left(\frac{\omega t}{2}\right)\mathbb{I} - i\,\sigma_z \sin\left(\frac{\omega t}{2}\right)$$

$$\boxed{\tilde{H} = -\frac{\hbar(\omega_0-\omega)}{2}\sigma_z - \frac{\hbar\omega_1}{2}\sigma_x}$$

$$i\hbar\frac{d}{dt}\tilde{U}_t = \tilde{H}\,\tilde{U}_t$$

$t=0 \qquad \tilde{U}_{t=0} = \mathbb{I}$

$\tilde{U}_t = \exp\left(-\frac{it}{\hbar}\tilde{H}\right)$ because $\tilde{H}$ is time independent

$$\tilde{H} = \begin{pmatrix} -\frac{\hbar}{2}(\omega-\omega_0) & -\frac{\hbar\omega_1}{2} \\ -\frac{\hbar\omega_1}{2} & +\frac{\hbar}{2}(\omega-\omega_0) \end{pmatrix}$$

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Rotating Frame

$$|\tilde{\Psi}_t\rangle = e^{\frac{it}{\hbar}K} |\Psi_t\rangle$$

$$\tilde{U}_t = e^{it/\hbar K} U_t$$

$$K = \frac{\hbar\omega_0}{2}\sigma_z$$

$$\Rightarrow \boxed{i\hbar \frac{d}{dt}\tilde{U}_t = \tilde{H} U_t} \quad \text{with} \quad \boxed{\tilde{H}_{\substack{Rot\\frame}} = -\frac{\hbar(\omega_0-\omega)}{2}\sigma_z - \frac{\hbar\omega_1}{2}\sigma_x}$$

Solution of dynamics in rot-frame

tuning parameter

$$\tilde{H} = \frac{\hbar\delta}{2}\sigma_z - \frac{\hbar\omega_1}{2}\sigma_x$$

is time-independent. Hence $\tilde{U}_t = e^{-\frac{it}{\hbar}\tilde{H}}$

Formula: $$e^{ia\,\hat{m}\cdot\vec{\sigma}} = (\cos a)\,\mathbb{I} + i\hat{m}\cdot\vec{\sigma}(\sin a)$$

$\begin{pmatrix}1&0\\0&1\end{pmatrix}$

$a \in \mathbb{R}$

$$\|\hat{m}\| = 1 \qquad \hat{m}\cdot\vec{\sigma} = m_x\sigma_x + m_y\sigma_y + m_z\sigma_z$$

$$U_t = \begin{pmatrix} u_{\uparrow\uparrow}(t) & u_{\uparrow\downarrow}(t) \\ u_{\downarrow\uparrow}(t) & u_{\downarrow\downarrow}(t) \end{pmatrix}$$

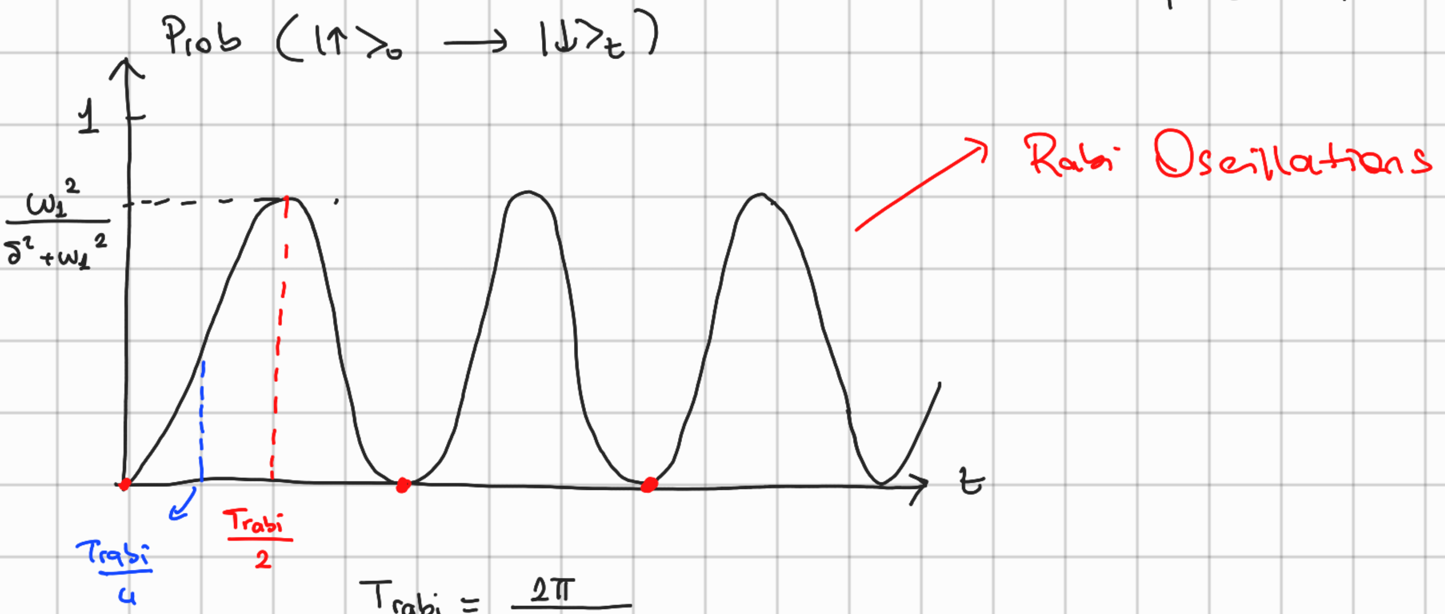
Dirac Notation Form

$$= u_{\uparrow\uparrow}(t)|\uparrow\rangle\langle\uparrow| + u_{\uparrow\downarrow}(t)|\uparrow\rangle\langle\downarrow| + u_{\downarrow\uparrow}(t)|\downarrow\rangle\langle\uparrow| + u_{\downarrow\downarrow}(t)\,|\downarrow\downarrow\rangle$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\text{Prob}\left(|\uparrow\rangle_0 \longrightarrow |\downarrow\rangle_t\right) = \left|\langle\downarrow|\psi_t\rangle\right|^2 = \left|\langle\downarrow|U_t|\uparrow\rangle\right|^2$$

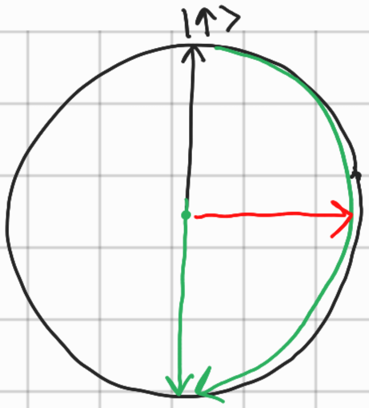
$$= \left|u_{\downarrow\uparrow}(t)\right|^2$$



$$T_{rabi} = \frac{2\pi}{\sqrt{\delta^2+\omega_1^2}}$$

Special case:

- $\delta << \omega_1$ $\quad \left(\frac{\delta}{\omega_1} \to 0\right)$ Tuning $\omega \approx \omega_0$
- Max. prob $\rightarrow 1$
- Trabi, time $\longrightarrow \frac{2\pi}{\omega_1}$



1) $\frac{T_{rabi}}{2}$ where you reach max. prob $\rightarrow \frac{\pi}{\omega_1}$

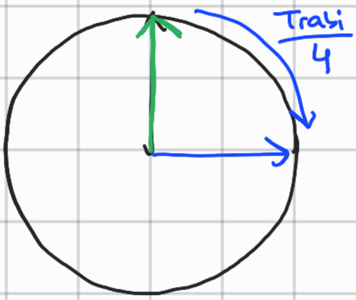
Wait time $\frac{T_{rabi}}{2} = \frac{\pi}{\omega_1}$

$$e^{+i \frac{\pi}{\omega_1 \hbar} \frac{\hbar}{2} \omega_1 \sigma_x} = e^{i \frac{\pi}{2} \sigma_x} = i\sigma_x = i \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\text{NOT gate}}$$

Example

$$\text{NOT}\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}} \right) = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \quad \text{(same!)}$$

2) Waiting $\frac{T_{rabi}}{4}$ time



$$e^{+i \frac{\pi}{4} \sigma_x} = \frac{1}{\sqrt{2}} + i\sigma_x \cdot \frac{1}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \simeq \tilde{U}$$

Hadamard Gate (?)
undirectly

$$\tilde{U}|\uparrow\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{\sqrt{2}} \left(|\uparrow\rangle + i|\downarrow\rangle \right)$$