# 1. Proof of Schmidt Theorem

$\mathcal{H}_{tot} = \mathcal{H}_A \otimes \mathcal{H}_B$ $\quad\quad$ $\dim \mathcal{H}_A = D_A$, $\quad \dim \mathcal{H}_B = D_B$

Let the system be a pure vector state $|\Psi\rangle \in \mathcal{H}_{tot}$

Let $\begin{cases} \rho_A = Tr_B |\Psi\rangle\langle\Psi| \\ \rho_B = Tr_A |\Psi\rangle\langle\Psi| \end{cases}$

a) Non-zero eigenvalues of $\rho_A$ and $\rho_B$ are identical

possibly they have zero eigenvalues but in differing number

As a consequence:

$$S(\rho_A) = S(\rho_B)$$

b)
$$\left.\begin{aligned} \rho_A &= \sum_{i=1}^{k} \lambda_i \, |\varphi_i\rangle_A \langle \varphi_i|_A \\ \rho_B &= \sum_{i=1}^{k} \lambda_i \, |\psi_i\rangle_B \langle \psi_i|_B \end{aligned}\right\} \text{Spectral Decomposition}$$

$\boxed{\lambda_i \neq 0}$

with $\{|\varphi_i\rangle_A\}$ orthonormal & $\{|\psi_i\rangle_B\}$ are orthonormal

c) $$\underbrace{|\Psi\rangle}_{\text{pure state}} = \sum_{i=1}^{k} \sqrt{\lambda_i} \, |\varphi_i\rangle_A \otimes |\psi_i\rangle_B$$

Proof:

First: we check that b implies a

$\#(\lambda_i \neq 0)$ are the same in $\rho_A$ & $\rho_B$

$$S(\rho_A) = S(\rho_B) = -\sum_{i=1}^{k} \lambda_i \log \lambda_i$$

Second remark: c implies b

$$|\psi\rangle\langle\psi| = \sum_{i,j} \sqrt{\lambda_i}\sqrt{\lambda_j}\, |\varphi_i\rangle_A \otimes |\psi_i\rangle_B \langle\varphi_j|_A \otimes \langle\psi_j|_B$$

$$\rho_A = Tr_B\big(|\psi\rangle\langle\psi|\big) = \sum_{i,j} \sqrt{\lambda_i}\sqrt{\lambda_j}\, |\varphi_i\rangle_A \langle\varphi_j|_A \underbrace{Tr_B\left(|\psi_i\rangle\langle\psi_j|_B\right)}_{Tr_B \underbrace{\langle\psi_j|\psi_i\rangle_B}_{\delta_{ji} = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}}}$$

$$\boxed{\rho_A = \sum_i \lambda_i |\varphi_i\rangle_A \langle\varphi_i|_A}$$

$$\boxed{\rho_B = \sum_i \lambda_i |\psi_i\rangle_B \langle\psi_i|_B}$$

Prove c:

$$|\psi\rangle = \sum_{i=1}^{k} \lambda_i \, |\varphi_i\rangle_A \otimes |\psi_i\rangle_B$$

$$|\psi\rangle \in \underbrace{\mathcal{H}_A}_{D_A} \otimes \underbrace{\mathcal{H}_B}_{D_B}$$

Dimension

Expand $|\psi\rangle$ on tensor product basis:

$$|\psi\rangle = \sum_{\nu,\nu'} \underbrace{c_{\nu\nu'}}_{\in \mathbb{C}} \underbrace{|\nu\rangle_A}_{\text{basis of } \mathcal{H}_A} \otimes \underbrace{|\nu'\rangle_B}_{\text{basis of } \mathcal{H}_B}$$

$\nu = 1 \dots D_A$

$\nu' = 1 \dots D_B$

$c_{\nu\nu'}$ is a $D_A \times D_B$ matrix

From linear algebra singular value decomposition:

$$C = U \Sigma V^{\dagger} \quad , \; V^{\dagger} = V^{T,*}$$

$U$: $D_A \times D_A$ unitary matrix

$V^{\dagger}$: $D_B \times D_B$ unitary matrix

$\Sigma$: $D_A \times D_B$ matrix

contains the singular values of $C$

Structure of Matrix $\Sigma$ :

→ $D_A = D_B$

$$\Sigma = \begin{pmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_D \end{pmatrix} \; ; \; \sigma_i \geq 0$$

$\sigma_1 \dots \sigma_k > 0 \qquad \sigma_{k+1} \dots \sigma_D = 0$

it would be that $k = D$ → no zero singular values

→ $D_A > D_B$

$$\Sigma = \underbrace{\begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_{D_B} \\ \hline & 0 & \end{bmatrix}}_{D_B} \Big\updownarrow D_A$$

$\sigma_1, \dots \sigma_k > 0 \quad ; \quad \sigma_{k+1} \dots \sigma_{D_B} = 0$

(it would be that $k = D_B$)

→ $D_B > D_A$

$$\Sigma = \underbrace{\left[\begin{array}{ccc|c} \sigma_1 & & 0 & \\ & \ddots & & 0 \\ 0 & & \sigma_k & \end{array}\right]}_{D_B} \Big\updownarrow D_A$$

$\sigma_1 \dots \sigma_k > 0 \qquad \sigma_{k+1} \dots \sigma_{D_A} = 0$

$$|\psi\rangle = \sum_{\mu,\mu'} c_{\mu,\mu'} \, |\mu\rangle_A \otimes |\mu'\rangle_B \; ?$$

$$= \sum_{\mu,\mu'=1}^{D_A, D_B} \sum_{i=1}^{K} \underbrace{U_{\mu i} \, \sigma_i \, (V^+)_{i\mu'}} \; |\mu\rangle_A \otimes |\mu'\rangle_B$$

$$= \sum_{i=1}^{k} \sigma_i \sum_{\mu,\mu'=1}^{D_A, D_B} U_{\mu i} \, (V^+)_{i\mu'} \, |\mu\rangle_A \otimes |\mu'\rangle_B$$

$$= \sum_{i=1}^{K} \sigma_i \underbrace{\left\{ \sum_{\mu=1}^{D_A} U_{\mu i} \, |\mu\rangle_A \right\}}_{|\varphi_i\rangle_A} \otimes \underbrace{\left\{ \sum_{\mu'=1}^{D_B} \underbrace{(V^+)_{i\mu'}}_{V^*_{\mu' i}} \, |\mu'\rangle_B \right\}}_{|\psi_i\rangle_B}$$

$$|\psi\rangle = \sum_{i=1}^{K} \sigma_i \, |\varphi_i\rangle_A \otimes |\psi_i\rangle_B$$

$$\sigma_i = \sqrt{\lambda_i}$$

This explains many examples of bipartite systems where we noticed $S(\rho_A) = S(\rho_B)$ for

$$|\varphi\rangle = |B_{00}\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B\right)$$

$\rightarrow \rho_A = \frac{1}{\sqrt{2}}(|0\rangle\langle 0|_A + |1\rangle\langle 1|_A)$

$\hookrightarrow \rho_A = \rho_B$

$$|W\rangle = \frac{1}{\sqrt{3}}\left(|001\rangle + |010\rangle + |100\rangle\right)$$

$$\rho_A = Tr_{BC}\, |W\rangle\langle W|$$

$$\rho_{BC} = Tr_A |W\rangle\langle W|$$

we computed eigenvalue of $\rho_A$ & $\rho_{BC}$ $\lambda_i = \frac{1}{3}, \frac{2}{3}$

$$S(\rho_A) = S(\rho_B) = +\frac{1}{3}\log 3 + \frac{2}{3}\log\frac{3}{2}$$

$$|W\rangle = \sqrt{\frac{1}{3}}\,|\varphi_0\rangle_A \otimes |\psi_0\rangle_{BC} + \sqrt{\frac{2}{3}}\,|\varphi_1\rangle_A \otimes |\psi_1\rangle_{BC}$$

Schmidt Decomposition

## 2. Properties of von Neumann Entropy

Recall that $\varrho \longrightarrow S(\varrho) = -\mathrm{Tr}(\varrho \log \varrho)$

Spectral Decomposition

$$\varrho^{+} = \varrho \quad ; \quad \varrho \geq 0 \quad : \quad \mathrm{Tr}(\varrho) = 1$$

- Eigenvalue $\lambda_i \in R$, $\lambda_i \geq 0$, $\sum_i \lambda_i = 1$

- $S(\varrho) = -\sum_i \lambda_i \log \lambda_i$

= Shannon Entropy for prob. distribution defined by the set of $\{\lambda_i\}$.

Following Properties:

a) $0 \leq S(\varrho) \leq \log(\dim \mathcal{H})$

b) Concavity $\quad S(t\varrho + (1-t)\sigma) \leq t S(\varrho) + (1-t) S(\sigma)$
$0 \leq t \leq 1$

c) Subadditivity $\quad S(\varrho_{AB}) \leq S(\varrho_A) + S(\varrho_B)$

d) Araki-Lieb inequality $\quad |S(\varrho_A) - S(\varrho_B)| \leq S(\varrho_{AB})$

e) Strong sub additivity

a) $0 \leq S(\rho) \leq \log(\dim \mathcal{H})$

obvious

$$S(\rho) = -\sum_{i=1}^{D} \lambda_i \log \lambda_i$$

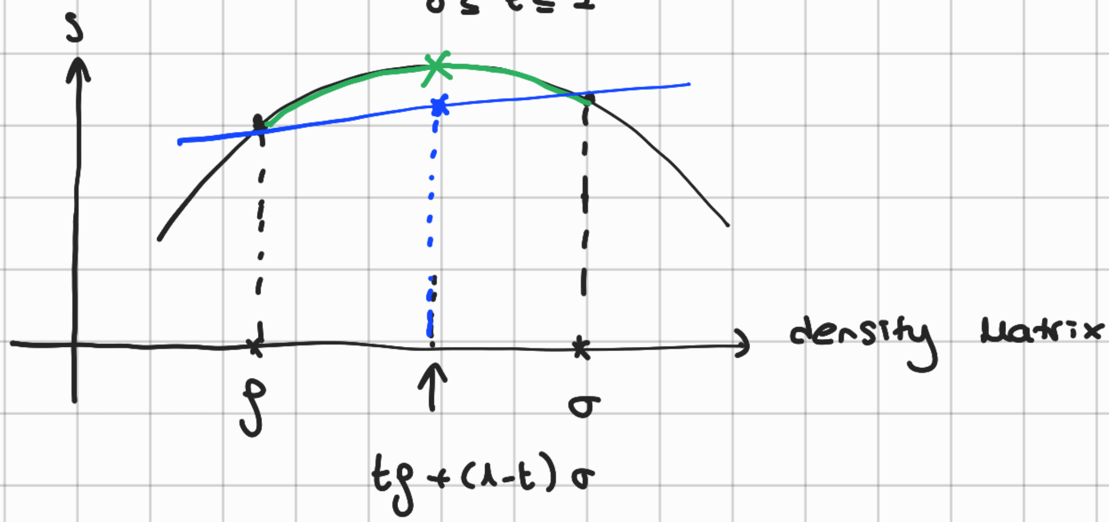
$D = \dim \mathcal{H}$

This is maximized for $\lambda_i = \frac{1}{D}$

$$S_{max} = -\sum_{i=1}^{D} \frac{1}{D} \log \frac{1}{D} = \log D$$

b) Concavity : $S(t\rho + (1-t)\sigma) \geqslant t S(\rho) + (1-t) S(\sigma)$

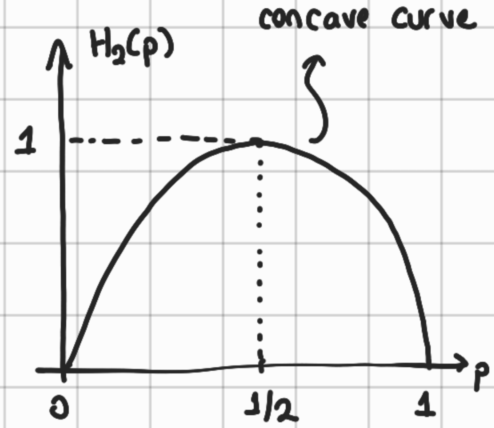
$0 \leq t \leq 1$




The same kind of concavity for the binary Shannon entropy of Ber(p) variable

Bernoulli

$$H_2(p) = -p\log_2 p - (1-p)\log_2(1-p)$$



c) Subadditivity $S(\rho_{AB}) \leq S(\rho_A) + S(\rho_B)$

(follows from concavity)

d) Araki - Lieb Inequality

$$S(\rho_{AB}) \geq |S(\rho_A) - S(\rho_B)|$$

Examples:

$\underbrace{\rho_{AB} = |B_{00}\rangle\langle B_{00}|}_{\text{pure state vector}}$ : $\underbrace{\rho_A = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{S(A) = \log 2}$ $\underbrace{\rho_B = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{S(B) = \log 2}$

$S(\rho_{AB}) = 0$

$0 \geq |\log 2 - \log 2|$

Also Remark: $S(\rho_{AB}) \leq S(A)$ or $S(B)$ (not always true for all quantum system)

In classical info. theory, it is NEVER TRUE

Hall mark of entanglement in AB system

Classically $\rightarrow$ $H(X,Y) \geq H(X)$ or $H(Y)$

# Proof of Araki-Lieb Inequality:

## Technique of purification:

→ Take $\rho_{AB}$ density matrix, I construct a bigger state on bigger Hilbert Space $\mathcal{H}_{AB} \otimes \mathcal{H}_R$

$|\psi\rangle_{ABR}$ which is pure s.t:

$$\rho_{AB} = \mathrm{Tr}_R |\psi\rangle_{ABR} \langle\psi|_{ABR}$$

→ why does $|\psi\rangle_{ABR}$ exist? (Not unique)

take spectral decomposition of $\rho_{AB} = \sum_i \mu_i |x_i\rangle\langle x_i|$

$$|\psi\rangle_{ABR} = \sum_i \sqrt{\mu_i}\, |x_i\rangle_{AB} \otimes |x_i\rangle_R$$

easy to check $\rho_{AB} = \mathrm{Tr}_R |\psi\rangle_{ABR} \langle\psi|_{ABR}$

→ Use Schmidt Theorem on the purified state & subadditivity to prove Araki Lieb inequality

$S(ABR) = 0$ since $|\psi\rangle_{ABR}\langle\psi|_{ABR}$

is rank-one Density Matrix (pure state)

$S(A) = S(BR)$ $S(B) = S(AR)$ $S(R) = S(AB)$

$\downarrow$ Schmidt $\downarrow$ Schmidt $\downarrow$ Schmidt

Suppose without loss of generality $S(A) \geq S(B)$

We want to prove $S(AB) \geq S(A) - S(B)$

$\Leftrightarrow$ $S(A) \leq S(AB) + S(B)$

$\qquad\quad \| \qquad\qquad \|$

$\Leftarrow$ $\underline{S(BR) = S(R) + S(B)}$

$\downarrow$

sub additivity