Two Level Systems

1) State and Operators

2) Bloch Sphere

2.1) Density Matrix

2.2) Hamiltonian

Harmonic Oscillators

1) Definitions and Notations

$[\hat{x},\hat{p}] = i\hbar \qquad \hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2$

1.1) Ladder Operators

$\hat{H} = \frac{1}{2}\mathbb{I} + \hat{a}^+\hat{a} \quad , \quad [\hat{a},\hat{a}^+] = 1$

2) Dynamics

2.1) Free Harmonic Oscillator

2.2) Driven Oscillator

$\hat{H} = \hbar\omega \hat{a}^+\hat{a} + F_0 \hat{x} \cos(\omega_d t + \varphi)$

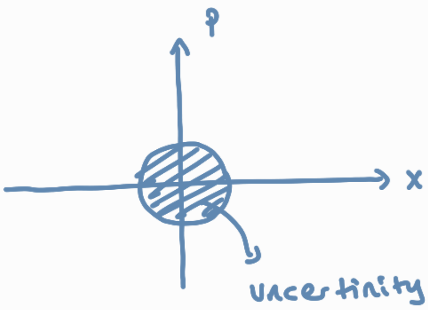
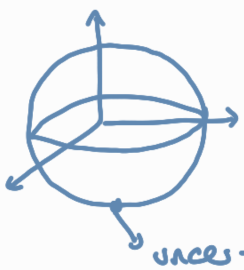
Rotating frame $\rightarrow U_{rot} = e^{i\omega H_0 t}$

driving field

$\hat{a}^+\hat{a}$

2.3) Coherent States

3) Comparison with two-level system

	Harmonic Oscillator	Two-Level System
Spectrum	$\vdots$ $\|2\rangle$ $\|1\rangle$ $\hbar\omega$ $\|0\rangle$	$\|e\rangle$ $\hbar\omega_e$ $\|g\rangle$
Ladder operator	$[\hat{a},\hat{a}^+] = \mathbb{I}$	$[\hat{\sigma}_-,\hat{\sigma}_+] = \hat{\sigma}_z$
Displacement	$\hat{D}(\alpha)$	$\hat{R}_{\vec{n}}(\theta)$
Phase Space	 uncertinity region $[\hat{x},\hat{p}] = i\hbar$	 uncertinity $[\hat{\sigma}_x,\hat{\sigma}_y] = 2i\hat{\sigma}_z$

Quantum State Tomography

Direct Inversion

Assume sample mean of the measurement outcomes equals the quantum mech. mean

$$\underbrace{\langle\langle\sigma_n\rangle\rangle}_{\frac{N_{n\uparrow}-N_{n\downarrow}}{N_{n\uparrow}+N_{n\downarrow}}} = \langle\sigma_n\rangle$$

Bayesian Mean Estimate

$$\hat{\rho}_{BME} = \frac{\int \hat{\rho}\, \mathcal{L}(\hat{\rho})\, D\hat{\rho}}{\int \mathcal{L}(\hat{\rho})\, D\hat{\rho}}$$

→ likelihood

$$\mathcal{L}(\hat{\rho}) = C(\hat{\rho})\, P(N_{x\uparrow}, N_{x\downarrow}, N_{y\uparrow}, N_{y\downarrow}, N_{z\uparrow}, N_{z\downarrow})$$

→ given prior probability

Quantum Measurement

1) Projective Measurement

$\hat{P}_m$ projector onto the eigenspace of m

If m has been observed:

$$\hat{\rho}_M = \frac{\hat{P}_m \hat{\rho} \hat{P}_M}{p(m)}$$

for mixed state

$$p(m) = Tr(\hat{\rho} \hat{P}_m) = E[m] = \langle \hat{M} \rangle$$

2) System-Meter Formulation

$\hat{M}_r$ is not a projector "Measurement" operator

$$\hat{P}(r) = \hat{M}_r^{+} \hat{M}_r$$

$$p(r) = \langle\psi|\hat{P}_r|\psi\rangle = \langle\psi|\hat{M}_r^{+}\hat{M}_r|\psi\rangle$$

3) Generalized Measurements

$$\hat{E}_r = \hat{M}_r^{+}\hat{M}_r \quad \text{st} \quad \sum_r \hat{E}_r = I$$

→ probability operator

Hermitian, positive

↓

Positive operator-valued measurement (POVM)

prob. of outcome

$$p(m) = Tr(\hat{\rho}\hat{E}_r)$$

After meas:

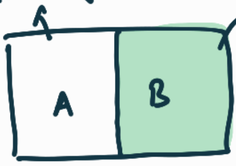
$$\hat{\rho} \rightarrow \hat{\rho}_r = \frac{\hat{M}_r \hat{\rho} \hat{M}_r^{+}}{p(r)}$$

Not observed:

$$\hat{\rho} \rightarrow \sum_r \hat{\rho}_r \cdot p(r) = \sum_r \hat{M}_r \hat{\rho} \hat{M}_r^{+}$$

system of interest

auxilery system



Evolution of Density Matrices

1) Evolution of density Matrices

1.1) Unitary evolutions (closed system)

$$i\hbar \frac{\partial}{\partial t}\hat{\rho} = [\hat{H}, \hat{\rho}] \quad , \quad \hat{\rho}(t) = \hat{U}(t)\hat{\rho}\hat{U}^{+}(t)$$

1.2) Measurements

$$\hat{\rho} \to \hat{\rho}' = \frac{\hat{M}_r \hat{\rho} \hat{M}_r^{+}}{\sqrt{p(r)}}$$ (resuts recorded)

$$\hat{\rho} \to \rho' = \sum_r \hat{M}_r \hat{\rho} \hat{M}_r^{+}$$ (not recorded)

1.3) Unitary evolution on extended space

$$\rho_A' = Tr_B \, \rho' \quad \text{where} \quad \rho' : \hat{U}(\hat{\rho}_A \otimes |0\rangle\langle 0|)\hat{U}^{+}$$

$$= \sum_\mu \langle \mu_B | \hat{U} \, (\hat{\rho}_A \otimes |0\rangle\langle 0|) \, \hat{U}^{+} | \mu_B \rangle$$

$$= \sum_\mu \underbrace{\langle \mu_B | \hat{U} | 0 \rangle}_{M_\mu} \hat{\rho}_A \underbrace{\langle 0 | \hat{U}^{+} | \mu_B \rangle}_{M_\mu^{+}}$$

2) Completely Positive Maps

* Superoperator $\hat{S}$: $\hat{S}(\hat{\rho}) = \sum_\mu \hat{M}_\mu \hat{\rho} \hat{M}_\mu^{+}$

st $\sum_\mu \hat{M}_\mu^{+} M_\mu = I$ (Krauss Theorem)

. In general, a superoperator is not invertible

. Any $\hat{S}$ can be interpreted as a POVM.

Quantum Channels

1) Amplitude Damping Channel

Model for SPONTANEOUS EMISSION

$$|0\rangle \otimes |0\rangle \xrightarrow{\text{nothing happens}} |0\rangle \otimes |0\rangle$$

$$|1\rangle \otimes |0\rangle \longrightarrow \underbrace{\sqrt{p}}_{\text{emission prob}} \, |0\rangle \otimes |1\rangle + \underbrace{\sqrt{1-p}}_{\text{nothing happens}} \, |1\rangle \otimes |0\rangle$$

$$\hat{S} : \hat{\rho_S} \longrightarrow \hat{\rho_S}' = Tr_E[\hat{U} \rho \hat{U}^{+}]$$

Initially $\hat{\rho} = \hat{\rho_S} \otimes |0\rangle\langle 0|_E$

$$\hat{\rho}' = \langle 0_E | \hat{U} \hat{\rho} U^{+} | 0 \rangle_E + \langle 1_E | \hat{U} \hat{\rho} \hat{U}^{+} | 1_E \rangle$$

$$\hat{\rho}' = \hat{M}_0 \hat{\rho}_S \hat{M}_0^{+} + \hat{M}_1 \hat{\rho}_S \hat{M}_1^{+}$$

Bloch Sphere

$\hat{\rho}_S, \hat{M}_0, \hat{M}_1$ operators on $\mathcal{H}_S$ (Hilbert)

$$\hat{M}_0 = \langle 0_E | \hat{U} | 0_E \rangle = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{bmatrix}$$

$$\hat{M}_1 = \langle 1_E | \hat{U} | 0_E \rangle = \begin{bmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{bmatrix}$$

$\hat{M}_0^{+}\hat{M}_0 + \hat{M}_1^{+}\hat{M}_1 = I$

$\hat{M}_0 = |0\rangle\langle 0| + \sqrt{1-p}|1\rangle\langle 1|$

$\hat{M}_1 = \sqrt{p} \, |0\rangle\langle 1|$

$$\langle 1_E | \hat{U} (\hat{\rho}_S \otimes |0\rangle_E\langle 0|_E) \hat{U}^{+} | 1_E \rangle$$

2) Phase Damping Channel

Flipping the phase of the excited state

$$|0\rangle \otimes |0\rangle \to |0\rangle \otimes (\sqrt{p}\,|0\rangle + \sqrt{1-p}\,|1\rangle)$$

$$|1\rangle \otimes |0\rangle \to |1\rangle \otimes (\sqrt{p}\,|0\rangle - \sqrt{1-p}\,|1\rangle)$$

$$\hat{M}_0 = \sqrt{p}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \hat{M}_1 = \sqrt{1-p}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

* Entangling operation

3) Depolarizing channel

Phase damping picks a preffered direction $\hat{z}$

$$\hat{M}_0 = \sqrt{1-p} \; \hat{\mathbb{I}}$$

$$\hat{M}_1 = \sqrt{\frac{p}{3}} \, \hat{\sigma_x} \quad \hat{M}_2 = \sqrt{\frac{p}{3}} \, \hat{\sigma_y} \quad \hat{M}_3 = \sqrt{\frac{p}{3}} \, \hat{\sigma_z}$$

uniform contraction

1) Markov Approximation

For open quantum systems

$i\dot{\hat{\rho}} \neq [\hat{H}, \hat{\rho}]$

Using Markov approximation:

$\hat{\rho}(t+\Delta t) = \hat{\rho}(t) + \hat{O}(\Delta t)$

where $\Delta t \gg \tau$ (memory time)

$\hat{\rho}(t+\Delta t) = S[\hat{\rho}(t)] = \sum_{\mu} \hat{M}_{\mu} \rho(t) \hat{M}_{\mu}^{+}$

2) Derivation of the Lindblad

$\hat{M}_0 = \mathbb{I} - i\hat{K}\Delta t + \hat{O}(\Delta t^2)$

$(\mu \geq 1)$ $\hat{M}_{\mu} = \hat{O}(\sqrt{\Delta t}) = \sqrt{\Delta t}\, \hat{L}_{\mu}$ $\rightarrow M_{\mu}\hat{\rho}M_{\mu}^{+}$ has to be $O(\Delta t)$

with $\hat{H} = \dfrac{\hat{K} + \hat{K}^{+}}{2}$ $\hat{J} = i\,\dfrac{\hat{K} - \hat{K}^{+}}{2}$ $\hat{K} = \hat{H} - i\hat{J}$, $\hat{H}^{+} = \hat{H}$, $\hat{J}^{+} = -\hat{J}$

For unitary part: $\langle \hat{M}_0^{+}\hat{M}_0 \rangle = \hat{\mathbb{I}}$ $\Rightarrow J = 0$

$\hat{M}_0 \hat{\rho} \hat{M}_0^{+} = \hat{\rho} - i\Delta t[\hat{H}, \hat{\rho}]$

For $\mu \geq 1$

$\hat{J} = \frac{1}{2} \sum_{\mu \geq 1} \hat{L}_{\mu}^{+} \hat{L}_{\mu}$

Lindblad Equations

$$\frac{\partial \hat{\rho}}{\partial t} = \underbrace{-i[\hat{H}, \hat{\rho}]}_{\text{unitary part}} + \sum_{\mu \geq 1} \underbrace{\hat{L}_{\mu} \hat{\rho} \hat{L}_{\mu}^{+}}_{\text{POVM (krauss op.)}} \underbrace{- \frac{1}{2} \hat{L}_{\mu}^{+} \hat{L}_{\mu} \rho - \frac{1}{2} \hat{\rho} \hat{L}_{\mu}^{+} \hat{L}_{\mu}}_{\text{Normalization of krauss Rep.}}$$

1) Derivation

↳ 1.1) Model

$\hat{H} = \Delta |e\rangle\langle e| + \frac{\Omega}{2}(|e\rangle\langle g| + |g\rangle\langle e|)$

Spontaneous Emission

ω_e — $|e\rangle$ — $|g\rangle$

$\hat{L} = \sqrt{\Gamma}\,|g\rangle\langle e|$

$Tr(\hat{\rho}\hat{L}^{+}\hat{L}) = \Gamma\, Tr(\hat{\rho}|e\rangle\langle e|)$

$= \Gamma P_e$ → prob. in $|e\rangle$

↳ 1.2) Justification

$\hat{H}_{int} = \hat{d} \otimes \vec{E} = H_{2\text{-level sys}} \otimes H_{environment}$

$\hat{d}$: dipole moment operator

$\vec{E}$: electromagnetic field operator

1.3) Bloch Vector Dynamics

$\dot{\hat{\rho}} = -i[\hat{H}, \hat{\rho}] + \Gamma\left(|g\rangle\langle e|\hat{\rho}|e\rangle\langle g| - \frac{1}{2}\hat{\rho}|e\rangle\langle e|\hat{\rho} - \frac{1}{2}|e\rangle\langle e|\hat{\rho}\right)$

$\dot{\hat{\rho}}_{ee} = \langle e|\dot{\hat{\rho}}|e\rangle = -i\frac{\Omega}{2}(\hat{\rho}_{ge} - \hat{\rho}_{eg}) - \Gamma \rho_{ee}$

"Optical Bloch Equations"

$\dot{a}_z = \Omega a_y - \Gamma(a_z + 1)$

$\dot{a}_x = -\frac{\Gamma}{2} a_x - \Delta a_y$

$\dot{a}_y = -\frac{\Gamma}{2} a_y + \Delta a_x - \Omega a_z$

Optical Bloch Equations

2) Solutions

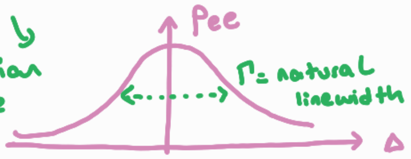
Stationary Solutions $(\dot{\vec{a}} = 0)$

$s = \dfrac{2\Omega^2}{\Gamma^2 + 4\Delta^2}$ → saturation parameter

$P_e = \langle e|\hat{\rho}|e\rangle = \rho_{ee} = \dfrac{1 + a_z}{2} = \dfrac{1}{2}\dfrac{s}{(1+s)}$

For $s \ll 1$ (low saturation)

↳ Lorentzian Profile



For $s \gg 1$ (strong saturation)

$\rho_{ee} = \frac{1}{2}$ → no spontaneous emission

Damped Harmonic Oscillator

1) Introduction

$$\hat{H} = \omega_c \hat{a}^{+}\hat{a} + \sqrt{\kappa}\,\varepsilon\,(\hat{a}\,e^{i\omega_d t} + h.c)$$

$\downarrow$ Rotating frame

$$\hat{H} = \Delta \hat{a}^{+}\hat{a} + \sqrt{\kappa}\,\varepsilon\,(\hat{a} + \hat{a}^{+}) \quad \text{where } \hat{L} = \sqrt{\kappa}\,\hat{a}$$

2) Field Amplitude

$$\langle \hat{a} \rangle = Tr(\hat{\rho}\hat{a})$$

$$\frac{d}{dt}\langle \hat{a}\rangle = -i\Delta\langle \hat{a}\rangle - \frac{\kappa}{2}\langle \hat{a}\rangle - i\sqrt{\kappa}\,\varepsilon$$

For steady state $\langle \dot{a} \rangle = 0$ $\quad \langle \hat{a}\rangle = \dfrac{\sqrt{\kappa}\,\varepsilon}{-\Delta + i\kappa/2}$

$$|\langle \hat{a}\rangle|^2 = \frac{\kappa|\varepsilon|^2}{\Delta^2 + \frac{\kappa^2}{4}} \sim \#\text{ of photons} = \frac{2|\varepsilon|^2}{\kappa}$$

3) Coherent State Decay

$$\hat{L}|\alpha\rangle = \sqrt{\kappa}\,\alpha|\alpha\rangle$$

4) Alternative descriptions

4.1) Photon Number

4.2) Phase - Space Description

Husimi - Q Function

Quantum Trajectories

1) Stochastic Schrödinger Equation

1.1) Intro

$$|\psi_c(t+\Delta t)\rangle = \frac{\hat{M}_\mu|\psi_c(\mu)\rangle}{\sqrt{p_\mu}}$$

1.2) Unraveling

$N_\mu(t) \rightarrow$ classical random variable

$dN_\mu(t) \rightarrow$ classical random process

1.3) Derivation

$$d|\psi_c\rangle = \left(-i\hat{H} - \frac{1}{2}\sum_\mu \hat{L}_\mu^{+}\hat{L}_\mu - \langle\psi_c|\hat{L}_\mu^{+}\hat{L}_\mu|\psi_c\rangle\right)dt|\psi_c\rangle + \sum_\mu dN_\mu\left(\frac{\hat{L}_\mu}{\sqrt{\langle \hat{L}_\mu^{+}\hat{L}_\mu\rangle}} - 1\right)|\psi_c\rangle$$

$\rightarrow$ Stochastic Schrödinger Equation

2) Interpretation

2.1) Lindblad Equation

$$d\hat{\rho} = -i\left[\hat{H}_{eff}\hat{\rho} - \hat{\rho}\hat{H}_{eff}^{+}\right] + \sum_\mu \hat{L}_\mu\hat{\rho}\hat{L}_\mu^{+}$$

$$\hat{H}_{eff} = -i\hat{H} - \frac{1}{2}\sum_\mu \hat{L}_\mu^{+}\hat{L}_\mu$$

2.2) Monte-Carlo Wavefunction Algorithm

3) Weak Continous Measurement

3.1) Homodyning

3.2) Quantum State Difusion

Mechanical Effects of Light

1) Formulation

$$\hat{H} = \frac{\vec{p}^{\,2}}{2m} \otimes \mathbb{I} + \mathbb{I} \otimes \omega_e |e\rangle\langle e| + \frac{\Omega(r)}{2}\left[e^{i(\omega_L t - \vec{k}_L \vec{r})} \otimes |g\rangle\langle e| + h.c \right]$$

2) Recoil and Doppler Effects

2.1) Phase Imprinting

2.2) Shifts

Doppler $= \frac{\vec{p}\cdot\vec{k}_L}{m}$

Recoil $\frac{\hbar^2 k_L^2}{2m} = E_r$

2.3) Jump Operators

$$\dot{\hat{\rho}} = -i[\hat{H}, \hat{\rho}] - \frac{\Gamma}{2}\left(|e\rangle\langle e|\hat{\rho} + \hat{\rho}|e\rangle\langle e|\right) + \sum_{\vec{n}} \hat{L}(\vec{n})\hat{\rho}\hat{L}^{+}(n)$$

3) Semi-classical Forces

3.1) Further simplification

3.2) Spontaneous Emission

3.3) Forces

dipole force

radiation pressure

4) Interpretation

4.1) Dipole force

4.2) Radiation pressure

4.3) Doppler cooling

4.4) Limitations and Extensions

Sideband Cooling