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Prof. Dr. Jean-Philippe Brantut
Dr. Rohit Prasad Bhatt
Francesca Orsi

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Problem Set 3 : Quantum Measurement

For questions contact: francesca.orsi@epfl.ch

I. PROJECTIVE MEASUREMENT AND POSITIVE OPERATOR VALUED MEASUREMENT (POVM) ON A SINGLE PHOTON QUBIT

In this exercise, we would like to get a better feeling for generalized quantum measurements by looking at a single qubit encoded in the polarization state of a photon.

$$|\theta\rangle = \cos(\theta)|H\rangle + \sin(\theta)|V\rangle \tag{1}$$

The simplest possible measurement apparatus for detecting the polarization state of the photon consists of a polarizer with an angle ϕ w.r.t to the horizontal axis H and a photon detector such that our measurement is described by the measurement operator

$$\hat{M}_\phi = |0\rangle\langle\phi| = |0\rangle(\cos(\phi)\langle H| + \sin(\phi)\langle V|) \tag{2}$$

An orthogonal measurement on a single polarization qubit can be defined by measuring at the angles $\phi = 0, \pi/2$, which can give us the projection on the horizontal (H) and vertical (V) polarization. The corresponding measurement operators are $\hat{M}_H = |0\rangle\langle H|$, $\hat{M}_V = |0\rangle\langle V|$. As we will see in the exerecise, if we want to distinguish non-orthogonal states in our measurement, we have to drop the requirement of having only orthogonal measurement operators. Note that this measurement is already a POVM, which destroys the photon and therefore cannot be repeated like a projective measurement. A general POVM for single photon polarization requires a much more complicated measurement setup as for example proposed in S. Ahnert et al. Phys. Rev. A 71, 012330 (2005).

Reminder:
We now define a quantum measurement to be described by a collection of measurement operators $\{\hat{M}_m\}$ acting on the state space of the system being measured. The index m refers to the measurement outcomes that occur in the experiment. The probability of observing m is $p(m) = \langle\psi|\hat{E}_m|\psi\rangle$ with probability operators $\hat{E}_m := \hat{M}_m^\dagger \hat{M}_m$, which satisfy the completeness relation $\sum_m \hat{E}_m = 1$. After a measurement the system will be in a state

$$\frac{\hat{M}_m|\psi\rangle}{\sqrt{\langle\psi|\hat{M}_m^\dagger \hat{M}_m|\psi\rangle}} \tag{3}$$

for an initial state $|\psi\rangle$.

Suppose, we prepare the photon in one of the two orthogonal states $|H\rangle$ or $|V\rangle$ and then try to distinguish these states by performing a simple projective measurement using our polarizer set to $\phi = 0, \pi/2$ and a photon counter.

✓ 1.) Show that this projective measurement is a 2-element POVM. Calculate the probabilities for measuring $|H\rangle$ and $|V\rangle$?

We now set the polarizer to $\phi = 0, \pi/4$ and measure $\hat{M}_H, \hat{M}_{\pi/4}$.

✓ 2.) Calculate the probabilities for the new measurement with $\hat{M}_H, \hat{M}_{\pi/4}$. Can we form a POVM using those measurement operators?

Suppose we now create a more complicated measurement setup e.g. the one mentioned in S. Ahnert et al. Phys. Rev. A 71, 012330 (2005), which allows us to measure a 3-element POVM. We now want to create a proper POVM by modifying our previous measurement operators to:

$$\hat{E}_1 := \lambda |H\rangle \langle H| \tag{4}$$

$$\hat{E}_2 := \frac{\lambda}{2}(|H\rangle + |V\rangle)(\langle H| + \langle V|) \tag{5}$$

$$\hat{E}_3 := \mathbb{1} - \hat{E}_1 - \hat{E}_2. \tag{6}$$

3.) Under which constraints for λ does this form a proper POVM? What are the probabilities to measure the states given above using this POVM.

II. MEASUREMENT-BASED PREPARATION OF SCHRÖDINGER CAT STATES WITH DISPERSIVE ATOM-FIELD INTERACTION.

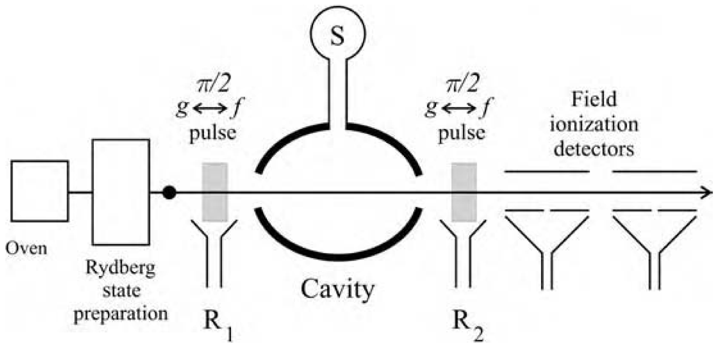


FIG. 1: Schematic of the experiment designed to generate the Schrödinger cat states in the Haroche group in Paris.

In this exercise, we want to explore the possibility to prepare Schrödinger Cat states of a microwave cavity field by using a measurement based preparation with Rydberg atoms. This experiment was performed in the Haroche group in Paris. The setup is as shown in figure 1 above and constitutes a form of a Ramsey interferometer. In the sequence atoms exit an oven and are prepared in a Rydberg state $|g\rangle$. In the two Ramsey zones R_1 and R_2, we apply two $\pi/2$ pulses to the atoms, which are resonant with the $|g\rangle \to |f\rangle$ transition. The central cavity is initially prepared in the coherent state α, which is off-resonant with any atomic transition and therefore only dispersively interacts with the atom given by the interaction Hamiltonian:

$$\hat{H} = -\hbar\chi\hat{a}^{\dagger}\hat{a}\,|g\rangle\langle g| \tag{7}$$

with the interaction strength χ and the photon annihilation operator $\hat{a}$.

1.) Show that the interaction Hamiltonian gives a phase shift to the cavity field if the atom is in the state $|g\rangle$ i.e.

$$e^{-i\chi\hat{a}^{\dagger}\hat{a}t}|\alpha\rangle = |\alpha e^{i\chi t}\rangle \tag{8}$$

In general a resonant Rabi pulse rotates the atomic states in the following way

$$|g\rangle \to \cos(\theta/2)|g\rangle + \sin(\theta/2)|f\rangle \tag{9}$$

$$|f\rangle \to \cos(\theta/2)|f\rangle - \sin(\theta/2)|g\rangle \tag{10}$$

2.) Compute the outcome of the full sequence of first preparing the atom in superposition state using a $\pi/2$-pulse in R_1, then letting it interacting with the cavity for an interaction time $\chi t = \pi$ and finally closing the interferometer by applying a second $\pi/2$-pulse in R_2. If we now projectively measure the state of our atom using an field ionizing detector, what state do we get for the cavity field?

III. SIMULTANEOUS QUANTUM MEASUREMENT OF A PAIR OF CONJUGATE OBSERVABLES

The Hamiltonian formulation of classical mechanics assigns simultaneous, arbitrarily accurate, values for the canonically conjugate position and momentum to distinguishable particles. In classical mechanics these simultaneous values are regarded as properties of the particles themselves and measurement simply reveals these values and need not add uncertainty to their determination. In quantum mechanics the conjugate position and momentum are represented by non-commuting operators, which have to satisfy the Heisenberg uncertainty principle. This does not mean, however, that simultaneous measurements of position and momentum are impossible, only that such simultaneous measurements cannot be made arbitrarily accurate.
In this exercise we explore the simultaneous quantum measurement of a pair of conjugate variables of our system coupled to two detectors, which interact instantaneously with the system. The system consists of a particle with dimensionless position $\hat{X}$ and momentum $\hat{P}$, satisfying $[\hat{X}, \hat{P}] = i$. Our detectors are two other particles denoted by d_i with position $\hat{X}_i$ and momentum $\hat{P}_i$, $i = 1, 2$. The interaction of the system with the detectors couples the momentum of one of the detectors to $\hat{X}$ and the other to $\hat{P}$ of the system.
Therefore, the unitary interaction is

$$\hat{U} = \exp\left(-i\left(\hat{X}\hat{P}_1 + \hat{P}\hat{P}_2\right)\right). \tag{11}$$

The detectors are prepared in the initial states $|d_i\rangle$

$$\langle x_i|d_i\rangle = \left(\frac{2}{\pi}\right)e^{-x_i^2}. \tag{12}$$

Before the interaction the system and detectors are assumed to be uncorrelated. The interaction operator then couples the system to the measurement apparatus and entangles the two. The result of the measurement is encoded in the measurement of the positions $\hat{X}_1$, $\hat{X}_2$ of both detectors.
The measurement operator is therefore given by

$$\hat{M}(X_1, X_2) := \langle X_1, X_2|\hat{U}|d_1, d_2\rangle \tag{13}$$

A. Projection of the System on a Coherent State

In the first part of the exercise, we would like to show that the measurement projects the system into a "coherent state".

1.) Show that the measurement operator $\hat{M}(X_1, X_2)$ takes the form:

$$\hat{M}(X_1, X_2) = \frac{1}{2\pi}\int dP_1 dP_2\, \langle P_1, P_2|d_1, d_2\rangle \exp\left(-i\left(P_1(\hat{X} - X_1) + P_2(\hat{P} - X_2)\right)\right) \tag{14}$$

2.) Now show that

$$\hat{M}(X_1, X_2) = \hat{D}(X_1, X_2)\hat{M}(0,0)\hat{D}(X_1, X_2)^\dagger \tag{15}$$

with the displacement operator

$$\hat{D}\left(\mu, \nu\right) := \exp\left(-i\left(\mu\hat{P} - \nu\hat{X}\right)\right) \tag{16}$$

and

$$\hat{M}(0,0) := \frac{1}{2\pi}\int dP_1 dP_2\, \langle P_1, -P_2|d_1, d_2\rangle\, \hat{D}\left(P_2, P_1\right)^\dagger. \tag{17}$$

Hint: Make use of the Baker-Campbell Hausdorff formula $e^{\hat{X}}e^{\hat{Y}} = e^{\hat{X}+\hat{Y}+1/2[\hat{X},\hat{Y}]}$

3.) Rewrite $\hat{M}(0,0)$ as

$$\hat{M}\left(0,0\right) = \frac{1}{\sqrt{2\pi}\pi}\int d^2\rho\, \langle\rho|0\rangle\, \hat{D}\left(\rho\right)^\dagger \tag{18}$$

in terms of the annihilation operator $\hat{a} := \frac{1}{\sqrt{2}}(\hat{X} + i\hat{P})$, which satisfies the usual commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$ and a new displacement operator $\hat{D}(\rho) := \exp(\rho\hat{a}^\dagger - \rho^*\hat{a})$ with variable $\rho = \frac{1}{\sqrt{2}}(P_2 + iP_1)$.

4.) In the next step, show that

$$\hat{M}(0,0) = \frac{1}{\sqrt{2\pi}} : e^{-\hat{a}^\dagger\hat{a}} :, \tag{19}$$

by calculating $\langle\alpha|\hat{M}(0,0)|\beta\rangle$. Here $: \cdot :$ denotes normal ordering of the creation/annihilation operators i.e. all the creation operators stand to the left of the annihilation operators.

For the calculation make use of the following relations for displacement operators and coherent states:

$$\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle \tag{20}$$

$$\hat{D}(\beta)\,|\alpha\rangle = \exp\left(\frac{1}{2}(\alpha^*\beta - \alpha\beta^*)\right)|\alpha+\beta\rangle \tag{21}$$

$$\langle\alpha|\beta\rangle = \exp\left(-\frac{1}{2}|\alpha|^2 - \frac{1}{2}|\beta|^2 + \alpha^*\beta\right) \tag{22}$$

$$1 = \pi^{-1}\int d^2\alpha\,|\alpha\rangle\langle\alpha|\,. \tag{23}$$

5.) In the last step, show that $\hat{M}(X_1, X_2)$ is actually the projector on the coherent state labelled by $\chi := \frac{1}{\sqrt{2}}(X_1 + iX_2)$ by proofing $: e^{-\hat{a}^\dagger\hat{a}} := |0\rangle\langle 0|$.

B. Measurement Results and Interpretation

In the second part, we would like to find out about our system by looking at the first and second moment of our detector positions.

6.) Show that $\hat{E}(X_1, X_2) := \hat{M}(X_1, X_2)\hat{M}(X_1, X_2)$ is a probability operator.

7.) Compute

$$\int \chi\hat{E}(\chi)d^2\chi \tag{24}$$

and

$$\int |\chi|^2\hat{E}(\chi)d^2\chi. \tag{25}$$

8.) Now we define the probability density $\text{Prob}(\chi) := Tr(\hat{E}(X_1, X_2)\hat{\rho})$, where $\hat{\rho}$ is the density matrix of the system. Compute the first and second moment of the detector positions i.e. $\langle X_1\rangle$, $\langle X_2\rangle$, $\langle X_1^2\rangle$, $\langle X_2^2\rangle$ and interpret the result.

Suppose, we prepare the photon in one of the two orthogonal states $|H\rangle$ or $|V\rangle$ and then try to distinguish these states by performing a simple projective measurement using our polarizer set to $\phi = 0, \pi/2$ and a photon counter.

1.) Show that this projective measurement is a 2-element POVM. Calculate the probabilities for measuring $|H\rangle$ and $|V\rangle$?

We now set the polarizer to $\phi = 0, \pi/4$ and measure $\hat{M}_H, \hat{M}_{\pi/4}$.

In POVM: $\hat{E}_r = \hat{M}_r^\dagger \hat{M}_r$ st $\sum_r \hat{E}_r = I$

For $\hat{E}_H = \hat{M}_H^\dagger \hat{M}_H = \hat{P}_H = |H\rangle\langle H|$

$|H\rangle\langle 0|$ $\quad$ $|0\rangle\langle H|$

$E_i = E_i^\dagger$

V, H

Hermitian and orthogonal projectors

$\hat{E}_V = \hat{M}_V^\dagger \hat{M}_V = \hat{P}_V = |V\rangle\langle V|$

and $\sum_{i=V,H} \hat{E}_i = |H\rangle\langle H| + |V\rangle\langle V| = I \rightsquigarrow$ completeness

Also for any $|\psi\rangle \rightarrow \langle\psi| E_i |\psi\rangle \geq 0$

Since it's hermitian matrix, it's positive definite

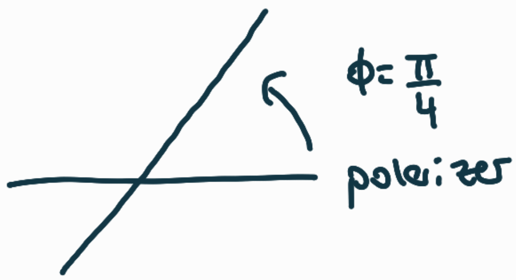
Measurement probabilities

$P_H(H) = \langle H| \hat{E}_H |H\rangle = 1$ | $P_V(H) = \langle V| E_H |V\rangle = 0$

$P_H(V) = \langle H| \hat{E}_V |H\rangle = 0$ | $P_V(V) = \langle V| E_V |V\rangle = 1$

for the two orthogonal states, the projective measurement tell us with certinity in which polarization state we are.

2.) Calculate the probabilities for the new measurement with $\hat{M}_H, \hat{M}_{\pi/4}$. Can we form a POVM using those measurement operators?



$$|\phi\rangle = \underbrace{\cos\phi}_{1/\sqrt{2}}|H\rangle + \underbrace{\sin\phi}_{1/\sqrt{2}}|V\rangle$$

$$P_\phi = \frac{1}{2}(|H\rangle + |V\rangle)(\langle H| + \langle V|)$$

$$\hat{E}_H = \hat{P}_H$$

$$\hat{E}_\phi = \hat{P}_\phi$$

$$P_H(H) = \langle H|\hat{E}_H|H\rangle$$

Prob.

$$P_H(H) = 1$$

$$P_H(\phi) = \langle\phi|\underbrace{\hat{P}_H}_{|H\rangle\langle H|}|\phi\rangle = \frac{1}{2}$$

$$P_\phi(H) = \frac{1}{2}$$

$$P_\phi(\phi) = 1$$

Probabilities do not add up to 1.

Since $\sum_{i=H,\phi} \hat{E}_i \neq 1$ we can not construct a 2-element POVM.

Suppose we now create a more complicated measurement setup e.g. the one mentioned in S. Ahnert et al. Phys. Rev. A 71, 012330 (2005), which allows us to measure a 3-element POVM. We now want to create a proper POVM by modifying our previous measurement operators to:

$$\hat{E}_1 := \lambda |H\rangle\langle H| \tag{4}$$

$$\hat{E}_2 := \frac{\lambda}{2}(|H\rangle + |V\rangle)(\langle H| + \langle V|) \tag{5}$$

$$\hat{E}_3 := \mathbb{1} - \hat{E}_1 - \hat{E}_2. \tag{6}$$

3.) Under which constraints for λ does this form a proper POVM? What are the probabilities to measure the states given above using this POVM?

To create POVM one should satisfy $\sum_i E_i = 1$

Ensure that $\langle\psi|\hat{E}_3|\psi\rangle \geq 0$, Already known that $0 \leq \lambda \leq 1$

$$\hat{E}_1 = \begin{array}{cc} |H\rangle & |V\rangle \\ \end{array}$$
$$\hat{E}_1 = \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix} \qquad E_2 = \begin{pmatrix} \frac{\lambda}{2} & \frac{\lambda}{2} \\ \frac{\lambda}{2} & \frac{\lambda}{2} \end{pmatrix} \qquad |H\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |V\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$E_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \frac{3\lambda}{2} & \frac{\lambda}{2} \\ \frac{\lambda}{2} & \frac{\lambda}{2} \end{pmatrix} = \begin{pmatrix} 1 - \frac{3\lambda}{2} - a & -\frac{\lambda}{2} \\ -\frac{\lambda}{2} & 1 - \frac{\lambda}{2} - a \end{pmatrix}$$

$$\det(\hat{E}_3 - aI) = 0$$

$$\left(\left(1 - \frac{3\lambda}{2} - a\right)\left(1 - \frac{\lambda}{2} - a\right) - \frac{\lambda^2}{4} \right) = 0$$

$$\lambda^2 + (2a-2)\lambda + \left(\frac{a^2}{2} - 2\lambda + 1 \right) = 0$$

$$\lambda_{1,2} = (1-a) \pm \frac{a}{\sqrt{2}}$$

$$P_H(1) = \frac{\sqrt{2}}{1+\sqrt{2}} \simeq 0.58$$

$$P_H(2) = \frac{1}{2} \frac{\sqrt{2}}{1+\sqrt{2}} \simeq 0.29$$

$$P_H(3) = 1 - P_H(1) - P_H(2) \simeq 0.12$$

$$P_\phi(1) = \frac{1}{2} \frac{\sqrt{2}}{1+\sqrt{2}}$$

$$P_\phi(2) = \frac{\sqrt{2}}{1+\sqrt{2}}$$

$$P_\phi(3) = 1 - P_{\phi 1} - P_{\phi 2}$$

Exc II)

1.) Show that the interaction Hamiltonian gives a phase shift to the cavity field if the atom is in the state $|g\rangle$ i.e.

$$e^{-i\chi\hat{a}^\dagger\hat{a}t}|\alpha\rangle = |\alpha e^{i\chi t}\rangle \quad (8)$$

$$\hat{U}\hat{a}|\alpha\rangle = \alpha\hat{U}|\alpha\rangle \quad \text{where} \quad \hat{U} = e^{-i\chi\hat{a}^\dagger\hat{a}t}$$

No $|g\rangle\langle g|$ since atom in $|g\rangle$ state

$$\hat{U}\hat{a}\underbrace{\hat{U}^\dagger\hat{U}}_{I}|\alpha\rangle = \hat{U}\hat{a}\hat{U}^\dagger(\hat{U}|\alpha\rangle)$$

Backer-Campbell Hausdorff Formula

$$\hat{U}\hat{a}U^\dagger = \sum_m^\infty \frac{1}{m!}[-i\chi\hat{a}^\dagger\hat{a}, \hat{a}]_m$$

$$\text{with} \quad [\hat{a}^\dagger\hat{a}, \hat{a}]_0 = \hat{a}^\dagger\underbrace{[\hat{a},\hat{a}]}_{0} + \underbrace{[\hat{a}^\dagger,\hat{a}]}_{-1}\hat{a} = -1.\hat{a}$$

$$\vdots$$

$$[\hat{a}^\dagger\hat{a}, \hat{a}]_m = [\hat{a}^\dagger\hat{a}, [\hat{a}^\dagger\hat{a}, \hat{a}]_{m-1}]$$

$$U\hat{a}U^\dagger = \underbrace{\sum_m^\infty \frac{1}{m!}(i\chi t)^m}_{e^{i\chi t}}\hat{a} = e^{i\chi t}\hat{a}$$

Coming back to term $\underbrace{(\hat{U}\hat{a}U^\dagger)}_{e^{i\chi t}\hat{a}}(\hat{U}|\alpha\rangle) = \alpha\hat{U}|\alpha\rangle$

$$e^{i\chi t}\hat{a}\hat{U}|\alpha\rangle = \alpha\hat{U}|\alpha\rangle$$

$$\boxed{\hat{a}\hat{U}|\alpha\rangle = e^{-i\chi t}\alpha\hat{U}|\alpha\rangle}$$

$$|\alpha\rangle \longrightarrow |\alpha e^{-i\chi t}\rangle$$

Cavity field is phase shifted by χt

In general a resonant Rabi pulse rotates the atomic states in the following way

$$|g\rangle \rightarrow \cos(\theta/2)\,|g\rangle + \sin(\theta/2)\,|f\rangle \tag{9}$$
$$|f\rangle \rightarrow \cos(\theta/2)\,|f\rangle - \sin(\theta/2)\,|g\rangle \tag{10}$$

2.) Compute the outcome of the full sequence of first preparing the atom in superposition state using a $\pi/2$-pulse in R_1, then letting it interacting with the cavity for an interaction time $\chi t = \pi$ and finally closing the interferometer by applying a second $\pi/2$-pulse in R_2. If we now projectively measure the state of our atom using an field ionizing detector, what state do we get for the cavity field?

Preparation of Schrödinger Cat State

Step 1:

In R_1: $\frac{\pi}{2}$ Pulse on $|g\rangle \leftrightarrow |f\rangle$ transition prepares atom in

$$\theta = \frac{\pi}{2} \rightarrow \frac{1}{\sqrt{2}}\left(|g\rangle + |f\rangle\right)$$

Step 2: Before atom-cavity interaction

$$\downarrow$$

$$|\Psi_{init}\rangle = \frac{1}{\sqrt{2}}\left(|g\rangle + |f\rangle\right) \otimes \underbrace{|\alpha\rangle}_{\text{cavity}}$$

Dispersive interaction:

$$|\Psi(t)_c\rangle = e^{-i\hat{H}t/\hbar}\,|\Psi_{init}\rangle$$

$$= \frac{1}{\sqrt{2}}\left(e^{-i\chi\hat{a}^\dagger\hat{a}t|g\rangle\langle g|}\,|g\rangle|\alpha\rangle + |f\rangle|\alpha\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(|g\rangle\,\underbrace{e^{-i\chi\hat{a}^\dagger\hat{a}}\,|\alpha\rangle}_{\substack{|\alpha e^{-i\chi t}\rangle \\ \text{from Part 1}}} + |f\rangle|\alpha\rangle\right)$$

Step 3: $R_2 \to \frac{\pi}{2}$ pulse

$$|g\rangle \to \frac{1}{\sqrt{2}}(|g\rangle + |f\rangle)$$

$$|f\rangle \to \frac{1}{\sqrt{2}}(|f\rangle - |g\rangle)$$

$$|\Psi_{final}\rangle = \frac{1}{2}\Big[(|g\rangle+|f\rangle)|\alpha e^{-i\chi t}\rangle + (|f\rangle - |g\rangle)|\alpha\rangle\Big]$$

$$= \frac{1}{2}\Big[(|\alpha e^{-i\chi t}\rangle - |\alpha\rangle)|g\rangle + (|\alpha e^{-i\chi t}\rangle + |\alpha\rangle)|f\rangle$$

where $\chi t = \pi$

$$|\Psi_{final}\rangle = \frac{1}{2}\Big((|-\alpha\rangle - |\alpha\rangle)|g\rangle + (|-\alpha\rangle + |\alpha\rangle)\Big)|f\rangle$$

Exc III)

A. Projection of the System on a Coherent State

In the first part of the exercise, we would like to show that the measurement projects the system into a "coherent state".

1.) Show that the measurement operator $\hat{M}(X_1, X_2)$ takes the form:

$$\hat{M}(X_1,X_2) = \frac{1}{2\pi}\int dP_1 dP_2 \langle P_1, P_2|d_1, d_2\rangle \exp\left(-i\left(P_1(\hat{X}-X_1)+P_2(\hat{P}-X_2)\right)\right) \tag{14}$$

$$\hat{M}(x_1,x_2) = \langle x_1|\langle x_2|\, \hat{U} \int dP_1\, |P_1\rangle\langle P_1| \int dP_2\, |P_2\rangle\langle P_2|\, |d_1\rangle|d_2\rangle$$

$$= \int dP_1 dP_2 \exp\left(-i(\hat{x}P_1 + \hat{p}P_2)\right) \langle P_1, P_2|d_1, d_2\rangle \langle x_1|P_1\rangle \langle x_2|P_2\rangle$$

where $\langle x_i|P_i\rangle = \dfrac{e^{iP_i x_i}}{\sqrt{2\pi}}$

2.) Now show that

$$\hat{M}(X_1, X_2) = \hat{D}(X_1, X_2)\hat{M}(0,0)\hat{D}(X_1, X_2)^\dagger \tag{15}$$

with the displacement operator

$$\hat{D}(\mu,\nu) := \exp\left(-i\left(\mu\hat{P} - \nu\hat{X}\right)\right) \tag{16}$$

and

$$\hat{M}(0,0) := \frac{1}{2\pi}\int dP_1 dP_2 \langle P_1, -P_2 | d_1, d_2\rangle \hat{D}(P_2, P_1)^\dagger\,. \tag{17}$$

Hint: Make use of the Baker-Campbell Hausdorff formula $e^{\hat{X}}e^{\hat{Y}} = e^{\hat{X}+\hat{Y}+1/2[\hat{X},\hat{Y}]}$

$$\hat{M}(x_1,x_2) = \hat{D}(x_1,x_2)\,\hat{M}(0,0)\,\hat{D}(x_1,x_2)^\dagger$$

$$= \overbrace{\exp(-i(x_1\hat{p} - x_2\hat{x}))\frac{1}{2\pi}\int dP_1 dP_2 \langle P_1,-P_2|d_1,d_2\rangle \overline{\exp(+i(P_2\hat{p}-P_1\hat{x}))\, \cdot\, \exp(i(x_1\hat{p}-x_2\hat{x}))}^{\mathrm{I}}}^{\mathrm{II})}$$

BCH $\rightarrow$ $e^{\hat{x}}e^{\hat{y}} = e^{\hat{x}+\hat{y}+\frac{1}{2}[\hat{x},\hat{y}]}$

I) $[i\,(P_2\hat{p} - P_1\hat{x}),\ i(x_1\hat{p} - x_2\hat{x})]$

$= -[-P_1\hat{x}, x_1\hat{p}] - [P_2\hat{p}, -x_2\hat{x}]$

$= P_1 x_1 i - i P_2 x_2$

II) $[-i\,(x_1\hat{p} - x_2\hat{x}),\ i(P_2\hat{p} - P_1\hat{x}) + i(x_1\hat{p} - x_2\hat{x})]$

$= i\,x_1 P_1 - i x_2 P_2$

3.) Rewrite $\hat{M}(0,0)$ as

$$\hat{M}(0,0) = \frac{1}{\sqrt{2\pi}\pi} \int d^2\rho \, \langle \rho | 0 \rangle \, \hat{D}(\rho)^\dagger \tag{18}$$

in terms of the annihilation operator $\hat{a} := \frac{1}{\sqrt{2}}(\hat{X} + i\hat{P})$, which satisfies the usual commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$ and a new displacement operator $\hat{D}(\rho) := \exp(\rho \hat{a}^\dagger - \rho^* \hat{a})$ with variable $\rho = \frac{1}{\sqrt{2}}(P_2 + iP_1)$.

$$\hat{a} = \frac{1}{\sqrt{2}}(\hat{X} + i\hat{P}) \quad , \quad [\hat{a}, \hat{a}^+] = 1$$

$$\rho = \frac{1}{\sqrt{2}}(P_2 + iP_1) \quad , \quad D(\rho)^+ = \exp(\rho\hat{a}^+ + \rho^*\hat{a})$$

$$\hat{M}(0,0) = \frac{1}{2\pi} \int dP_1 dP_2 \, \langle P_1, -P_2 | d_1, d_2 \rangle \exp(-i(P_2\hat{P} - P_1\hat{X}))$$

use $\displaystyle \langle P_i, d_i \rangle = \int dx_j \, \langle P_j | x_j \rangle \left(\frac{2}{\pi}\right)^{1/4 \to ?} e^{-x_j^2}$

$$= \int dx_j \, e^{-(x_j + iP_j/2)^2} e^{-P_j^2/4} \frac{1}{2^{1/4}\pi^{3/4}}$$

$$= \frac{1}{2^{1/4}\pi^{3/4}} e^{-P_j^2/4} \underbrace{\int dx_j \, e^{-(x_j - \frac{iP_j}{2})^2}}_{\sqrt{\pi}} = \frac{1}{(2\pi)^{1/4}} e^{-P_j^2/4}$$

$$\Downarrow$$

$$\hat{M}(0,0) = \frac{1}{\sqrt{2\pi}\,\pi} \int d^2\rho \, \langle \rho | 0 \rangle \, \hat{D}(\rho)^+$$

4.) In the next step, show that

$$\hat{M}(0,0) = \frac{1}{\sqrt{2\pi}} : e^{-\hat{a}^\dagger \hat{a}} :, \tag{19}$$

by calculating $\langle \alpha | \hat{M}(0,0) | \beta \rangle$. Here $: \cdot :$ denotes normal ordering of the creation/annihilation operators i.e. all the creation operators stand to the left of the annihilation operators.

For the calculation make use of the following relations for displacement operators and coherent states:

$$\hat{a} \, |\alpha\rangle = \alpha \, |\alpha\rangle \tag{20}$$

$$\hat{D}(\beta) \, |\alpha\rangle = \exp\left(\frac{1}{2}(\alpha^* \beta - \alpha \beta^*)\right) |\alpha + \beta\rangle \tag{21}$$

$$\langle \alpha | \beta \rangle = \exp\left(-\frac{1}{2}|\alpha|^2 - \frac{1}{2}|\beta|^2 + \alpha^* \beta\right) \tag{22}$$

$$1 = \pi^{-1} \int d^2\alpha \, |\alpha\rangle \langle \alpha| \, . \tag{23}$$

$$\langle \alpha | \hat{M}(0,0) | \beta \rangle = \frac{1}{\sqrt{2\pi}\,\pi} \int d^2\rho \, \langle \rho | 0 \rangle \langle \alpha | \hat{D}^+(\rho) | \beta \rangle$$

$$= \frac{1}{\sqrt{2\pi}\,\pi} \int d^2\rho \, \langle 0 | \rho \rangle \langle \alpha + \rho | \beta \rangle \, e^{\frac{1}{2}(\alpha \rho^* - \alpha^* \rho)}$$

$$= \frac{1}{\sqrt{2\pi}} \int \frac{d^2\rho}{\pi} \langle 0 | \rho \rangle \langle \rho | \hat{D}^+(\alpha) | \beta \rangle \, e^{\frac{1}{2}(\alpha \rho^* - \alpha^* \rho) + \frac{1}{2}(\alpha \rho^* - \alpha^* \rho)}$$

$$= \frac{1}{\sqrt{2\pi}} \int \frac{d^2\rho}{\pi} \langle 0 | \rho \rangle \langle \rho | \hat{D}^+(\alpha) | \beta \rangle \, e^{\alpha \rho^* - \alpha^* \rho}$$

$$= \frac{1}{\sqrt{2\pi}} \langle 0 | e^{-\alpha^* \hat{a}} \overbrace{\int \frac{d^2\rho}{\pi} |\rho\rangle\langle\rho|}^{\mathbb{I}} e^{\alpha \hat{a}^+} \hat{D}^+(\alpha) | \beta \rangle$$

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$$= \frac{1}{\sqrt{2\pi}} \langle 0 | \hat{D}(\alpha) \, \hat{D}^+(\alpha) | \beta \rangle \, e^{-|\alpha|^2/2}$$

$$= e^{-|\alpha|^2/2 - |\beta|^2/2} = \frac{1}{\sqrt{2\pi}} \langle \alpha | \beta \rangle \, e^{-\alpha^* \beta} \quad (?)$$

5.) In the last step, show that $\hat{M}(X_1, X_2)$ is actually the projector on the coherent state labelled by $\chi := \frac{1}{\sqrt{2}}(X_1 + iX_2)$ by proofing : $e^{-\hat{a}^\dagger \hat{a}} := |0\rangle\langle 0|$.

$$1 = \langle 0|0\rangle\langle 0|0\rangle$$

$$= \langle 0|\hat{\mathbb{I}}|0\rangle\langle 0|\hat{\mathbb{I}}|0\rangle$$

$$= \langle 0| \int \frac{d^2\alpha\, d^2\beta}{\pi \cdot \pi} |\alpha\rangle\langle\alpha| e^{-\hat{a}^\dagger \hat{a}} |\beta\rangle\langle\beta|0\rangle$$

$$= \int \frac{d^2\alpha}{\pi} \frac{d^2\beta}{\pi} \langle 0|\alpha\rangle\langle\alpha|\beta\rangle e^{-\alpha^* \beta} \langle\beta|0\rangle$$

$$= \int \frac{d^2\alpha}{\pi} \frac{d^2\beta}{\pi} e^{-|\alpha|^2} e^{-|\beta|^2} = 1$$

$$\hat{M}(x_1, x_2) = \hat{D}(\chi)\hat{M}(0,0)\hat{D}(x_1, x_2)^\dagger = \frac{1}{\sqrt{2\pi}} |\chi\rangle\langle\chi|$$

B. Measurement Results and Interpretation

In the second part, we would like to find out about our system by looking at the first and second moment of our detector positions.

6.) Show that $\hat{E}(X_1, X_2) := \hat{M}(X_1, X_2)\hat{M}(X_1, X_2)$ is a probability operator.

7.) Compute

$$\int \chi \hat{E}(\chi) d^2\chi \tag{24}$$

and

$$\int |\chi|^2 \hat{E}(\chi) d^2\chi. \tag{25}$$

8.) Now we define the probability density $\text{Prob}(\chi) := Tr(\hat{E}(X_1, X_2)\hat{\rho})$, where $\hat{\rho}$ is the density matrix of the system. Compute the first and second moment of the detector positions i.e. $\langle X_1\rangle$, $\langle X_2\rangle$, $\langle X_1^2\rangle$, $\langle X_2^2\rangle$ and interpret the result.

6) $\hat{E}(x_1,x_2) = \hat{U}^+(x_1,x_2)\,\hat{U}(x_1,x_2) = \frac{1}{2\pi}|x\rangle\langle x|$

$$\int dx_1 dx_2\, \hat{E}(x_1,x_2) = \frac{1}{\pi}\int \frac{dx_1}{\sqrt{2}}\frac{dx_2}{\sqrt{2}}|x\rangle\langle x|$$

$$= \frac{1}{\pi}\int d^2x\, |x\rangle\langle x| = \hat{\mathbb{I}}$$

$\hat{\mathbb{1}}\,?$

7) $\langle\alpha|\hat{a}|\alpha\rangle \rightarrow \langle x_1\rangle = \langle\hat{x}\rangle \quad , \langle x_2\rangle = \langle\hat{p}\rangle$

8) $\langle\alpha|\hat{a}\hat{a}^+|\alpha\rangle \rightarrow \langle x_1^2\rangle = \langle\hat{x}^2\rangle + \frac{1}{2}$

$$\langle x_2^2\rangle = \langle\hat{p}^2\rangle + \frac{1}{2}$$

*Using 2 meters one can measure $\{\hat{x},\hat{p}\}$ simultaneously.