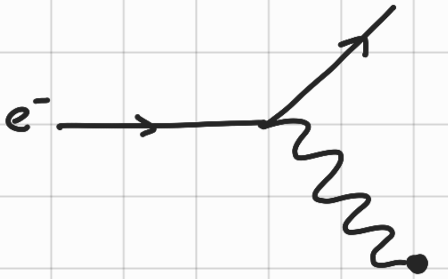
WEEK 10 Electron - proton elastic Scattering

Probing the structure of the proton

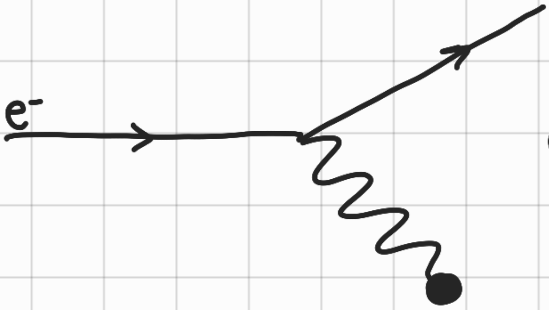
1)



$\lambda >> r_p$

At very low energies $e^-p \to e^-p$ process can be describ in terms of elastic scattering of the e^- in the static potential of an point-like proton

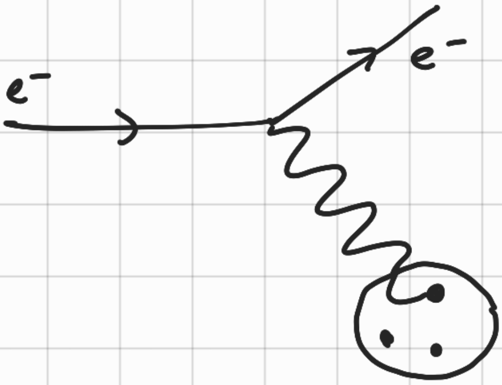
2)



$\lambda \sim r_p$

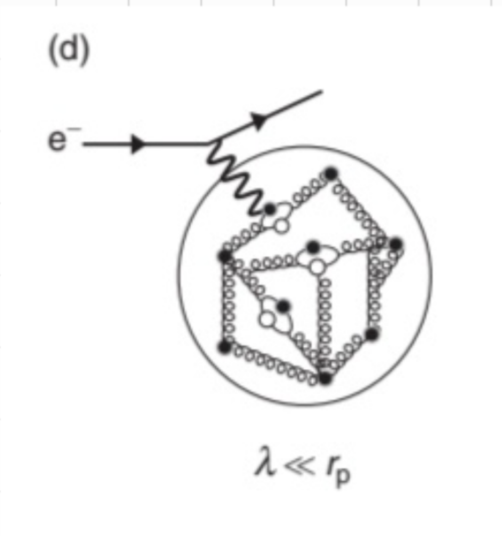
Scattering process is no longer purely electrostatic in nature

3)

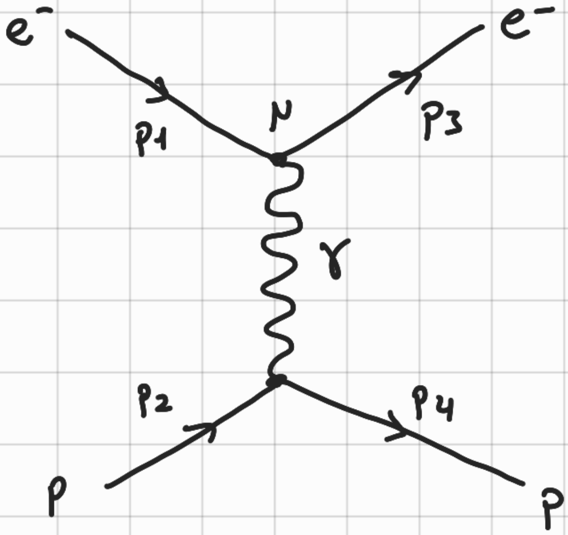


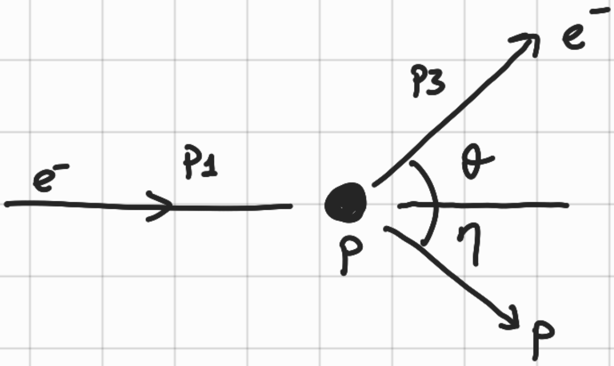
Inelastic scattering

4)



Rutherford and Mott scattering





Proton is at rest in the lab frame

$$\mathcal{M}_{fi} = \frac{Q_q e^2}{q^2} \left[\bar{u}(p_3) \gamma^\mu u(p_1) \right] g_{\mu\nu} \left[\bar{u}(p_4) \gamma^\nu u(p_2) \right]$$

$$u_\uparrow = N_e \begin{pmatrix} c \\ s e^{i\phi} \\ \kappa c \\ \kappa s e^{i\phi} \end{pmatrix} \qquad u_\downarrow = N_e \begin{pmatrix} -s \\ c e^{i\phi} \\ \kappa s \\ -\kappa c e^{i\phi} \end{pmatrix}$$

where $N_e = \sqrt{E + m_e}$ $\quad s = \sin(\theta/2) \quad c = \cos(\theta/2)$

If $v_{p,\text{scattered}}$ is small (its E_k can be neglected) and assume E_{e^-} does not change $\kappa_{\text{final}} = \kappa_{\text{initial}}$

$$\kappa = \frac{p}{E + m_e} = \frac{\beta_e \gamma_e \cancel{m_e}}{(\gamma_e + 1)\cancel{m_e}} = \frac{\beta_e \gamma_e}{\gamma_e + 1}$$

$\downarrow$ take θ, and $\phi = 0$

$$u_\uparrow(p_1) = N_e \begin{pmatrix} 1 \\ 0 \\ \kappa \\ 0 \end{pmatrix}, \; u_\downarrow(p_1) = N_e \begin{pmatrix} 0 \\ 1 \\ 0 \\ -\kappa \end{pmatrix} \text{ and } u_\uparrow(p_3) = N_e \begin{pmatrix} c \\ s \\ \kappa c \\ \kappa s \end{pmatrix} \; u_\downarrow(p_3) = N_e \begin{pmatrix} -s \\ c \\ \kappa s \\ -\kappa c \end{pmatrix}$$

The e^- currents for the 4 possible helicity states:

$$j_{e\uparrow\uparrow} = \bar{u}_\uparrow(p_3)\gamma^\mu u_\uparrow(p_1) = (E+m_e)\left[(k^2+1)c,\ 2ks,\ +2iks,\ 2kc\right]$$

$$j_{e\downarrow\downarrow} = \bar{u}_\downarrow(p_3)\gamma^\mu u_\downarrow(p_1) = (E+m_e)\left[(k^2+1)c,\ 2ks,\ -2iks,\ 2kc\right]$$

$$j_{e\downarrow\uparrow} = \bar{u}_\uparrow(p_3)\gamma^\mu u_\downarrow(p_1) = (E+m_e)\left[(1-k^2)s,\ 0,\ 0,\ 0\right]$$

$$j_{e\uparrow\downarrow} = \bar{u}_\downarrow(p_3)\gamma^\mu u_\uparrow(p_1) = (E+m_e)\left[(k^2-1)s,\ 0,\ 0,\ 0\right]$$

- In the relativistic limit ($k \approx 1$), only 2 helicity comb. give non-zero matrix elements.

- At lower energies ($k < 1$), 4 helicity combinations give non-zero matrix elements

↳ Helicity eigenstates $\neq$ Chiral eigenstates

For the proton: $\rightarrow \beta_p \ll 1 \rightsquigarrow k \approx 0$, $\theta_p = \eta$, $\phi_p = \pi$

$$u_\uparrow(p_2) = \sqrt{2m_p}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \equiv u_1(p_2) \qquad u_\downarrow(p_2) = \sqrt{2m_p}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} \equiv u_2(p_2)$$

$$u_\uparrow(p_4) \approx \sqrt{2m_p}\begin{pmatrix}c_\eta\\-s_\eta\\0\\0\end{pmatrix} \qquad u_\downarrow(p_4) \approx \sqrt{2m_p}\begin{pmatrix}-s_\eta\\-c_\eta\\0\\0\end{pmatrix}$$

$$j_{p\uparrow\uparrow} = -j_{p\downarrow\downarrow} = 2m_p\,[c_n, 0, 0, 0]$$

$$j_{p\uparrow\downarrow} = j_{p\downarrow\uparrow} = -2m_p\,[s_n, 0, 0, 0]$$

$$\mathcal{M}_{fi} = \frac{e^2}{q^2}\, j_e \cdot j_p$$

$$\langle |\mathcal{M}_{fi}^2| \rangle = \frac{1}{4} \sum |\mathcal{M}_{fi}^2|$$

$$= \frac{1}{4}\frac{e^4}{q^4} \times 4m_p^2 (E+m_e)^2 . \big[\underbrace{c_n^2 + s_n^2}_{1}\big] . \left[4(1+k^2)^2 c^2 + 4(1-k^2)^2 s^2\right]$$

$$= \frac{4m_p^2 m_e^2 e^4 (\gamma_e+1)^2}{q^4} \left[(1-k^2)^2 + 4k^2c^2\right]$$

Simplify $k = \dfrac{\beta_e \gamma_e}{\gamma_e + 1}$ and $\gamma_e^2 = \dfrac{1}{(1-\beta_e^2)}$

$\downarrow$

$$\langle |\mathcal{M}_{fi}^2| \rangle = \frac{16 m_p^2 m_e^2 e^4}{q^4} \left[1 + \beta_e^2 \gamma_e^2 \cos^2\frac{\theta}{2}\right]$$

$\hookrightarrow$ $q^2 = (p_1 - p_3)^2$ $\rightsquigarrow$ t-channel

$$E_1 = E_3 = E \qquad \text{and} \qquad p_1 = p_3 = p$$

$$q = (p_1 - p_3) = (0, \vec{p}_1 - \vec{p}_3)$$

$$q^2 = (0, \vec{p}_1 - \vec{p}_3)^2 = -2p^2 (1 - \cos\theta) = -4p^2 \sin^2(\theta/2)$$

Substitute q^2 into $\langle |M_{fi}^2| \rangle$ equation

$$\langle |M_{fi}^2| \rangle = \frac{M_p^2 M_e^2 e^4}{p^4 \sin^4(\theta/2)} \left[1 + \beta_e^2 \gamma_e^2 \cos^2 \frac{\theta}{2} \right]$$

provided the proton recoil can be neglected, this Matrix is equally applicable for non-relativistic or relativistic.

Rutherford Scattering

→ Proton recoil is neglected

$E_1 \sim m_e \ll M_p$

→ Electron is non-relativistic ($\beta_e \gamma_e \ll 1$)

$$\langle |M_{fi}^2| \rangle = \frac{M_p^2 M_e^2 e^4}{p^4 \sin^4(\theta/2)}$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \left(\frac{1}{M_p + E_1 - E_1 \cos\theta} \right)^2 \langle |M_{fi}^2| \rangle$$

"Lab Frame Differental Cross Section"

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 M_p^2} \langle |M^2_{fi}| \rangle = \frac{M_e^2 e^4}{64\pi^2 p^4 \sin^4(\theta/2)}$$

$$E_k = p^2/2m_e \longrightarrow (p^2)^2 = (2M_e E_k)^2$$

$$(e^2)^2 = (4\pi\alpha)^2$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{Rutherford} = \frac{\alpha^2}{16 E_k^2 \sin^4(\theta/2)}$$

★ In the non-relativistic limit, only the interaction between the electric charges of the electron and proton contribute to the scattering process

There's no significant contribution from the magnetic (spin-spin) interaction

Mott Scattering

→ Proton recoil still is neglected

→ Electron is relativistic.

$$\left.\begin{matrix} \\ \\ \end{matrix}\right\} M_e \ll E \ll M_p$$

$k \approx 1$ and 4 possible electron current are zero.

$$\langle |M_{fi}|^2 \rangle = \frac{M_p^2 m_e^2 e^4}{p^4 \sin^4(\theta/2)} \left[1 + \overbrace{\beta_e^2 \gamma_e^2 \cos^2\frac{\theta}{2}}^{\beta_e^2 \gamma_e^2 \cos^2\frac{\theta}{2}} \right]$$

$E \approx p$

$E = \gamma_e m_e$

$$\frac{M_p^2 \cancel{m_e^2} e^4}{E^4 \sin^4(\theta/2)} \cdot \left(\cancel{\beta_e^2}^{\,\to 1} \cancel{\gamma_e^2} \cos^2\frac{\theta}{2} \right)$$

$$E^4 \to E^2 \cdot E^2 \to \cancel{\gamma_e^2 m_e^2}$$

$$\langle |M_{fi}|^2 \rangle \approx \frac{M_p^2 e^4}{E^2 \sin^4(\theta/2)} \cos^2\frac{\theta}{2}$$

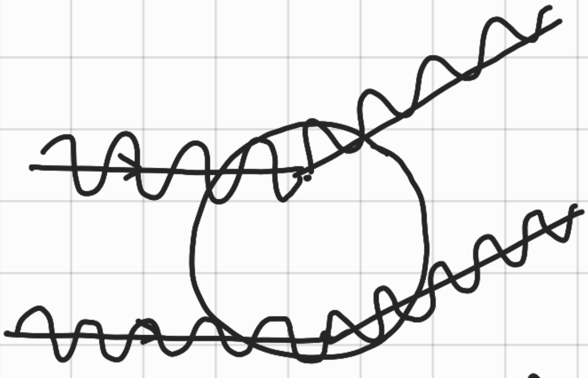
$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \left(\frac{1}{M_p + \cancel{E_1} - \cancel{E_1 \cos\theta}} \right)^2 \langle |M_{fi}|^2 \rangle$$

$M_p >> E_1$

$$\boxed{\left(\frac{d\sigma}{d\Omega}\right)_{Mott} = \frac{\alpha^2}{4E^2 \sin^4(\theta/2)} \cos^2\frac{\theta}{2}}$$

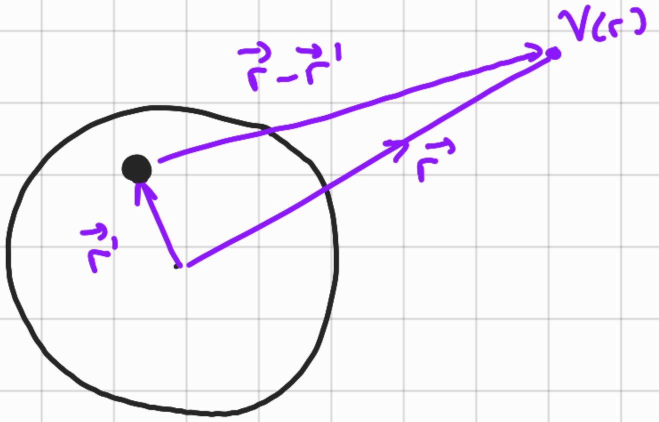
Again no spin-spin interaction (negligible)

Form Factors



$\lambda < r_p$

Origin of the form factor in elastic scattering



Q is the total charge, $\rho(r')$ is the charge distribution

$$\int \rho(r')\, d^3 r' = 1$$

$$V(r) = \int \frac{Q\ \rho(\vec{r}\,')}{4\pi\, |\vec{r}-\vec{r}\,'|}\, d^3 \vec{r}\,'$$

$$\mathcal{M}_{fi} = \langle \psi_f | V(r) | \psi_i \rangle = \int e^{-i \vec{p}_3 \cdot \vec{r}}\, V(r)\, e^{i \vec{p}_1 \cdot \vec{r}}\, d^3 \vec{r}$$

$$e^{i(\vec{p}_3 \cdot \vec{r} - Et)} \qquad e^{i(\vec{p}_1 \cdot \vec{r} - Et)}$$

Define $\vec{q} = \vec{p}_1 - \vec{p}_3$

$$\mathcal{M}_{fi} = \int\int e^{i \vec{q} \cdot \vec{r}}\, \frac{Q\ \rho(\vec{r}\,')}{4\pi\, |\vec{r}-\vec{r}\,'|}\, d^3 \vec{r}\,'\, d^3 r$$

$$= \int\int e^{i \vec{q} \cdot \overbrace{(\vec{r}-\vec{r}\,')}^{R}} \cdot e^{i \vec{q} \cdot \vec{r}\,'}\ Q \cdot \frac{\rho(\vec{r}\,')}{4\pi\, |\underbrace{\vec{r}-\vec{r}\,'}_{R}|}\, d^3 \vec{r}\,'\, d^3 \vec{r}$$

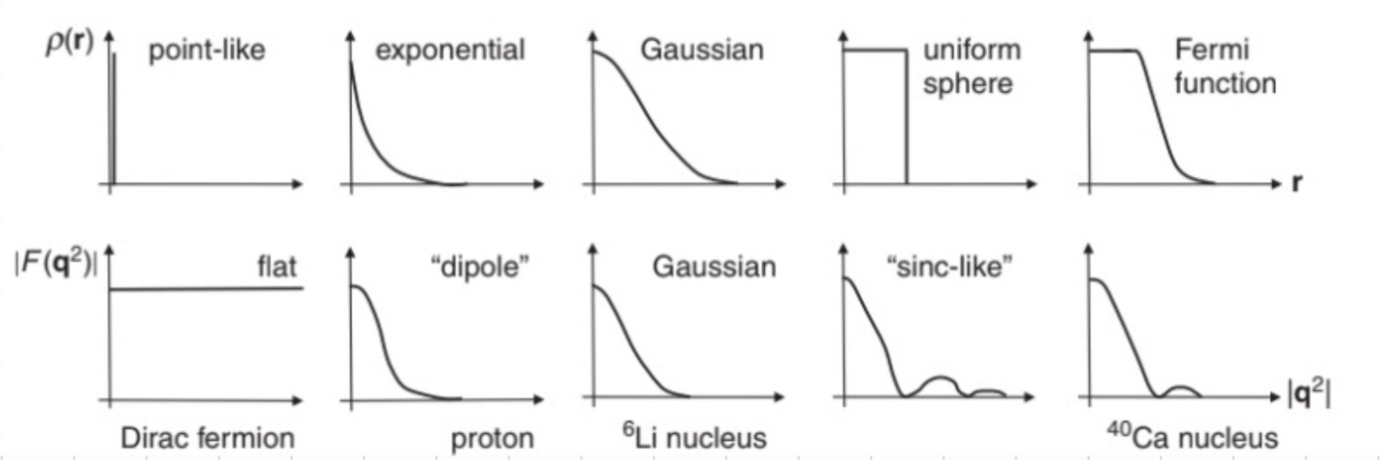
$R = |\vec{r} - \vec{r}\,'|$

$$M_{fi} = \underbrace{\int e^{i\vec{q}\cdot\vec{R}} \frac{Q}{4\pi|R|} d^3\vec{R}}_{M_{fi}^{pt}} \underbrace{\int \rho(\vec{r}') e^{i\vec{q}\cdot\vec{r}'} d^3\vec{r}'}_{F(\vec{q}^2)}$$

M_{fi}^{pt} "Scattering from a potential" due to a point charge which was found earlier

$F(\vec{q}^2)$ "Form Factor" → 3-dimensional Fourier Transform

$$\boxed{M_{fi} = M_{fi}^{pt} F(\vec{q}^2)}$$



Relativistic electron - proton elastic Scattering

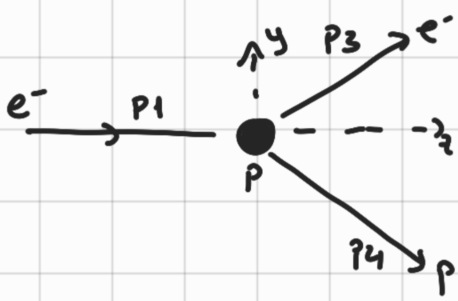
At high energies, magnetic spin-spin interaction becomes important

$p_1 = (E_1, 0, 0, E_1)$

$p_2 = (m_p, 0, 0, 0)$

$p_3 = (E_3, 0, E_3 \sin\theta, E_3 \cos\theta)$

$p_4 = (E_4, \vec{p_4})$



Assume $E_e \gg m_e$

$$\langle |M_{fi}^2| \rangle = \frac{8e^4}{(p_1-p_3)^4}\left[(p_1\cdot p_2)(p_3\cdot p_4) + (p_1\cdot p_4)(p_2\cdot p_3) - m_p^2 (p_1\cdot p_3)\right]$$

Scattering kinematics

$p_4 = p_1 + p_2 - p_3$ from momentum conservation

$p_2\cdot p_3 = E_3 m_p$, $p_1\cdot p_2 = E_1 m_p$ $\quad p_1 p_3 = E_1 E_3 (1-\cos\theta)$

$$p_3\cdot p_4 = p_3\cdot p_1 + p_3\cdot p_2 - \underbrace{\cancel{p_3\cdot p_3}}_{m_e^2} = E_1 E_3 (1-\cos\theta) + E_3 m_p$$

$$p_1\cdot p_4 = \underbrace{\cancel{p_1\cdot p_1}}_{m_e^2} + p_1\cdot p_2 - p_1\cdot p_3 = E_1 m_p - E_1 E_3 (1-\cos\theta)$$

$$\langle |M_{fi}^2| \rangle = \frac{8e^4}{(p_1-p_3)^4} m_p E_1 E_3 \left[(E_1-E_3)(1-\cos\theta) + m_p (1+\cos\theta)\right]$$

$$= \frac{8e^4}{(p_1-p_3)^4} 2 m_p E_1 E_3 \left[(E_1-E_3) \sin^2\frac{\theta}{2} + m_p \cos^2\frac{\theta}{2}\right]$$

$$q^2 = (p_1-p_3)^2 = p_1^2 + p_3^2 - 2p_1 p_3 \approx -2E_1 E_3 (1-\cos\theta)$$

$$q^2 \approx -4E_1 E_3 \sin^2\frac{\theta}{2}$$

Since q^2 is negative, let's define $Q^2 = -q^2 = 4E_1E_3 \sin^2\frac{\theta}{2}$

>0

$$q \cdot p_2 = (p_1 - p_3) \cdot p_2 = M_p(E_1 - E_3)$$

$\hookrightarrow q = p_4 - p_2 \rightarrow p_4^2 = (q+p_2)^2 = q^2 + 2q \cdot p_2 + p_2^2$

4-vector

$p_2^2 = p_4^2 = M_p^2$

$$q^2 + 2q \cdot p_2 + \cancel{p_2^2} = \cancel{p_4^2}$$

$$q^2 = -2q \cdot p_2$$

$$q \cdot p_2 = -\frac{q^2}{2}$$

$$(E_1 - E_3) = \frac{q \cdot p_2}{m_p} = -\frac{q^2}{2m_p} = \frac{Q^2}{2M_p}$$

$$\langle |M_{fi}|^2 \rangle = \frac{M_p^2 e^4}{E_1 E_3 \sin^4(\theta/2)} \left[\cos^2\frac{\theta}{2} + \frac{Q^2}{2M_p^2} \sin^2\frac{\theta}{2} \right]$$

$$\frac{d\sigma}{d\Omega} \approx \frac{1}{64\pi^2} \left(\frac{E_3}{M_p E_1} \right)^2 \langle |M_{fi}|^2 \rangle$$

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2 \sin^4(\theta/2)} \frac{E_3}{E_1} \left(\cos^2\frac{\theta}{2} + \frac{Q^2}{2M_p^2} \sin^2\frac{\theta}{2} \right]$$

)

" Here Q^2 and E_3 can be expressed in terms of the scattering angle θ."

$$q^2 = -2m_p(E_1 - E_3) = -2E_1E_3(1-\cos\theta)$$

$$E_3 = \frac{E_1 m_p}{\underbrace{m_p + E_1(1-\cos\theta)}}$$

$$q^2 \approx -2E_1E_3(1-\cos\theta)$$

$$Q^2 = \frac{2m_p E_1^2(1-\cos\theta)}{m_p + E_1(1-\cos\theta)}$$

In the limit of $Q^2 \ll m_p^2$ and $E_3 \approx E_1$, expression can be reduced to Mott scattering cross section.

with additional terms

$$\frac{E_3}{E_1} \quad \& \quad + \sin^2\theta/2$$

energy lost due to the proton recoil

from magnetic spin-spin interaction

The Rosenbluth Formula

It is found that:

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2 \sin^4(\theta/2)} \frac{E_3}{E_1} \left(\cos^2\frac{\theta}{2} + \frac{Q^2}{2M_p^2} \sin^2\frac{\theta}{2} \right)$$ ⇝ for a "point-like" proton

$G_E(\hat{Q}^2)$: Form factor of the charge distribution of the proton

$G_M(\hat{Q}^2)$: Form factor related to the magnetic moment distribution of the proton

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2 \sin^4(\theta/2)} \frac{E_3}{E_1} \left(\frac{G_E^2 + \tau G_M^2}{(1+\tau)} \cos^2\frac{\theta}{2} + 2\tau G_M^2 \sin^2\frac{\theta}{2} \right)$$

where $\tau = \frac{Q^2}{4M_p^2}$

✱ $G_E(Q^2)$ and $G_M(Q^2)$ are functions of the 4-momentum squared of the virtual photon

Unlike the form factor $F(\vec{q}^{\,2})$ → 3-momentum squared

$$Q^2 = -q^2 = \vec{q}^{\,2} - (E_1 - E_3)^2$$

$\frac{Q^2}{2M_p}$

$$Q^2\left(1 + \frac{Q^2}{4M_p^2}\right) = \vec{q}^{\,2}$$

↓

In the limit where $Q^2 << 4 m_p^2 \rightsquigarrow Q^2 \approx \vec{q}^{\,2}$

$$G_E(Q^2) = G_E(\vec{q}^{\,2}) = \int e^{i\vec{q}\cdot\vec{r}} \rho(\vec{r})\, d^3\vec{r}$$

$$G_M(Q^2) \simeq G_M(\vec{q}^{\,2}) = \int e^{i\vec{q}\cdot\vec{r}} \mu(\vec{r})\, d^3\vec{r}$$

$$\vec{\mu} = \frac{q}{m}\vec{S}$$ "Magnetic moment of a point-like Dirac particle"

$$\vec{\mu} = 2.79\,\frac{e}{m_p}\vec{S}$$ "Anomalous magnetic moment of the proton"

$$G_E(0) = \int \rho(r)\, d^3r = 1$$

$$G_M(0) = \int \mu(r)\, d^3r = +2.79$$

Measuring $G_E(Q^2)$ and $G_M(Q^2)$

$$\frac{d\sigma}{d\Omega} = \left(\frac{G_E^2 + \tau G_M^2}{(1+\tau)} + 2\tau G_M^2 \tan^2\frac{\theta}{2} \right) \cdot \left(\frac{d\sigma}{d\Omega}\right)_0$$

where $$\left(\frac{d\sigma}{d\Omega}\right)_0 = \frac{\alpha^2}{4E_1^2 \sin^4(\theta/2)} \left(\frac{E_3}{E_1}\right) \cos^2\frac{\theta}{2}$$

$$\frac{d\sigma}{d\Omega} \Big/ \left(\frac{d\sigma}{d\Omega}\right)_0 \approx G_E^2 = |F(\vec{q})|^2$$

for $\tau \ll 1$ (at low Q^2)

$$\frac{d\sigma}{d\Omega} \Big/ \left(\frac{d\sigma}{d\Omega}\right)_0 \approx \left(1 + 2\tau \tan^2\frac{\theta}{2}\right) G_M^2$$

for $\tau \gg 1$ (at high Q^2)