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Abstract

General sets of coherent states are constructed for quantum systems admitting a nondegenerate infinite discrete energy spectrum. They are eigenstates of an annihilation operator and satisfy the usual properties of standard coherent states. The application of such a construction to the quantum optics Jaynes–Cummings model leads to a new understanding of the properties of this model.

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1 Introduction

The study of coherent states (CS) for a quantum mechanical system has received a lot of attention [1, 2, 3, 4] and the definition, applications, generalizations of such states have been the subject of many papers. In the recent years, they were discussed in connection with exactly solvable models and nonlinear algebras [5, 6, 7, 8, 9, 10] as well as deformed algebras [11]. They were also produced using supersymmetric methods [12, 13].

A common starting point of all these approaches is the observation of the properties of the original CS for the harmonic oscillator. It is well-known [3] that they are described equivalently as eigenstates of the usual bosonic annihilation operator, from a displacement operator acting on a fundamental state and as minimum uncertainty states. What we observe in the different generalizations proposed is that the preceding definitions are no longer equivalent and only some of the properties of the harmonic oscillator CS are preserved.

In this respect, our approach is new since we propose a definition of CS, for a general quantum system, as eigenstates of an annihilation operator which maintain all the properties listed before in a sense that will be clarified in the paper. In fact, the quantum system under consideration must admit a factorization [14] in terms of annihilation and creation operators to be able to realize our construction. This means that we are restricting ourselves to such systems that have a nondegenerate infinite discrete energy spectrum. Let us note that our CS coïncide with the ones proposed by Gazeau and Klauder [15] (see also [16]) where a set of four requirements for such states has been imposed, i.e. continuity, resolution of unity, temporal stability and action identity.

A relevant application of our construction of CS is the Jaynes–Cummings (JC) model, describing, in its simplest version, the interaction of a cavity mode with a two level system [17]. The question of analysing the behaviour of dynamical quantities in CS for this model has been first asked by Narozhny & al [18] where the atomic inversion presents a time evolution consisting of Rabi oscillations. This model has received after a lot of attention [19, 20] and the connection with SUSY [21, 22], in particular with the SUSY harmonic oscillator, has led to new sets of coherent states [23].

A deep analysis of the energy spectrum of the JC leads to a selection of the parameters to avoid problems of degeneracy not allowed by our construction. The new set of CS is then used to compute physical quantities like the number of photons, the dispersion of the energy and the atomic inversion.

The paper is organized as follows. In Section 2, the factorization of a given Hamiltonian with a nondegenerate discrete energy spectrum leads to the introduction of creation and annihilation operators. Two explicit examples help to illustrate our construction of such operators. In Section 3, the three definitions of CS are used to describe what we call "generalized CS" (GCS). Other properties are given such as stability in time, over completeness and resolution of the identity. In Section 4, a new set of CS for the JC model is introduced from our preceding considerations. The emphasis is made on the nondegenerate energy spectrum case and the behaviour of some physical quantities is given.

2 Creation and annihilation operators for an arbitrary quantum mechanical system

We start with general considerations on the construction of creation and annihilation operators from the factorization of a given Hamiltonian admitting a nondegenerate discrete infinite energy spectrum.

Let us assume that the Hamiltonian H of a quantum system is given and admits a nondegenerate discrete infinite spectrum of energy $\{E_n, n = 0, 1, 2, \ldots\}$ such that the fundamental energy $E_0 = 0$ and the others are in increasing order, i.e.

$$E_0 = 0 < E_1 < E_2 < \cdots < E_{n-1} < E_n < \cdots . \tag{2.1}$$

The corresponding energy eigenstates are denoted by $|\psi_n\rangle$, for $n = 0, 1, 2, \ldots$, and are assumed to satisfy the orthogonality and completeness conditions

$$\langle\psi_m|\psi_n\rangle = \delta_{mn}, \quad \sum_{n=1}^{\infty} |\psi_n\rangle\langle\psi_n| = I, \tag{2.2}$$

where I is the identity operator. We also have

$$H|\psi_n\rangle = E_n|\psi_n\rangle. \tag{2.3}$$

Creation and annihilation operators a^+ and a^- may then be defined from their action on the eigenstates as

$$a^+|\psi_n\rangle = (E_{n+1})^{1/2} e^{-i(E_{n+1}-E_n)\alpha}|\psi_{n+1}\rangle, \tag{2.4}$$
$$a^-|\psi_n\rangle = (E_n)^{1/2} e^{i(E_n-E_{n-1})\alpha}|\psi_{n-1}\rangle. \tag{2.5}$$

An explicit formula for these operators will follow. The exponential factor appearing in all these expressions produces only a phase factor, since $\alpha \in \mathbb{R}$, and will be significant for the temporal stability of the CS we will construct in the next section. An arbitrary eigenstate $|\psi_n\rangle$ may then be constructed from the ground state $|\psi_0\rangle$ of H which satisfies $a^-|\psi_0\rangle = 0$. We get

$$|\psi_n\rangle = \big(E(n)\big)^{-1/2} e^{iE_n\alpha} (a^+)^n |\psi_0\rangle, \quad n > 0, \tag{2.6}$$

where we have defined

$$E(n) = E_1 E_2 \dots E_n. \tag{2.7}$$

Let us define also the diagonal (state labelling) operator N such that

$$N|\psi_n\rangle = n|\psi_n\rangle. \tag{2.8}$$

The existence of creation and annihilation operators a^+ and a^- satisfying the relations (2.4) and (2.5) implies that the Hamiltonian H may be factorized as

$$H = H(N) = a^+a^-. \tag{2.9}$$

Another consequence is that the set of operators $a^+,\ a^-,\ N$ generate a so-called generalized oscillator algebra [24] with commutation relations

$$[N, a^+] = a^+, \quad [N, a^-] = -a^-, \tag{2.10}$$

$$[a^-, a^+] = f(N), \tag{2.11}$$

where $f(N)$ is given by

$$f(N) = H(N+1) - H(N). \tag{2.12}$$

$H(N)$ is the original Hamiltonian given in eq. (2.9) and $H(N+1) = a^-a^+$ is known as the supersymmetric partner of H [5, 6]. For any diagonal operator $g(N)$ we have, due to eq. (2.10),

$$a^- g(N) = g(N+1)\ a^-, \quad a^+ g(N+1) = g(N)\ a^+. \tag{2.13}$$

Let us give an explicit realization of our operators in the complete set of states satisfying eq. (2.2). We can write easily

$$N = \sum_{n=1}^{\infty} n |\psi_n\rangle\langle\psi_n|, \tag{2.14}$$

so that, for any diagonal operator $g(N)$ with eigenvalues $g(n)$, we have

$$g(N) = \sum_{n=0}^{\infty} g(n) |\psi_n\rangle\langle\psi_n|, \tag{2.15}$$

and, in particular, we get:

$$H = H(N) = \sum_{n=1}^{\infty} E_n |\psi_n\rangle\langle\psi_n|. \tag{2.16}$$

The operators a^+ and a^- are given by

$$a^+ \quad = \quad \sqrt{H(N)}\ e^{-if(N-1)\alpha} \sum_{n=0}^{\infty} |\psi_{n+1}\rangle\langle\psi_n|, \tag{2.17}$$

$$a^- \quad = \quad \sqrt{H(N+1)}\ e^{if(N)\alpha} \sum_{n=0}^{\infty} |\psi_n\rangle\langle\psi_{n+1}|. \tag{2.18}$$

To give an illustration of such a construction, we take two examples. The first one is the standard harmonic oscillator Hamiltonian which has an energy spectrum linear in n. It can seem trivial but helps us to fix the notation that will be used in the following when the Jaynes-Cummings will be studied. The second one is the Pöschl-Teller Hamiltonian, taken as in [9, 10], which admits an energy spectrum quadratic in n.

For the harmonic oscillator, we take

$$H_0 = H(N_0) = a_0^+ a_0^- = N_0, \tag{2.19}$$

so that the ground state energy is zero. The energy eigenstates generate the usual Fock space

$$\mathcal{F}_b = \{|n\rangle, \quad n = 0, 1, 2, \ldots\} \tag{2.20}$$

and the number operator is N_0 with

$$N_0|n\rangle = n|n\rangle. \tag{2.21}$$

From eq. (2.12), we get $f(N_0) = 1$ and the usual creation and annihilation operators are given by

$$a_0^+ = \sqrt{N_0} \sum_{n=0}^{\infty} |n+1\rangle\langle n|, \tag{2.22}$$

$$a_0^- = \sqrt{N_0+1} \sum_{n=0}^{\infty} |n\rangle\langle n+1|, \tag{2.23}$$

where we have set $\alpha = 0$ in the definitions (2.17) and (2.18). Eqs. (2.4) and (2.5) thus become

$$a_0^+|n\rangle = \sqrt{n+1}\ |n+1\rangle, \quad a_0^-|n\rangle = \sqrt{n}\ |n-1\rangle, \tag{2.24}$$

which are the usual action of the creation and annihilation on the states $|n\rangle$. From eqs. (2.22) and (2.23), we easily deduce that

$$\sum_{n=0}^{\infty} |n+1\rangle\langle n| = \frac{1}{\sqrt{N_0}} a_0^+, \tag{2.25}$$

$$\sum_{n=0}^{\infty} |n\rangle\langle n+1| = \frac{1}{\sqrt{N_0+1}} a_0^-, \tag{2.26}$$

expressions which will be useful in the definitions of creation and annihilation operators for the Jaynes-Cummings model.

The Pöschl-Teller Hamiltonian is given by [9, 10]

$$H_{\mathrm{PT}} = \frac{p^2}{2m} + \frac{\epsilon\nu(\nu-1)}{cos^2 kx}, \tag{2.27}$$

where $\epsilon = \frac{\hbar^2}{2m}k^2$, k is a real parameter and $\nu > 0$. The energy spectrum is

$$E_{PT,n} = \epsilon(n+\nu)^2, \ n = 0, 1, 2, \ldots \tag{2.28}$$

and the energy eigenstates denoted here by ϕ_n are well-known [25] and may be expressed in terms of Jacobi functions. They are given by [10, 25]

$$\phi_n(x) = \left(\frac{k(n+\nu)\Gamma(n+2\nu)}{n!}\right)^{1/2} cos^{1/2}(kx)\ P^{1/2-\nu}_{n+\nu-1/2}(sin(kx)), \tag{2.29}$$

We see that $E_{PT,0} = \epsilon\nu^2$ and our considerations apply if we take

$$H(N) = a^+_{\mathrm{PT}} a^-_{\mathrm{PT}} = H_{\mathrm{PT}} - \epsilon\nu^2, \tag{2.30}$$

which has a ground state ϕ_0 with zero energy. In terms of the number operator N , we write explicitly

$$H(N) = \epsilon((N+\nu)^2 - \nu^2) = \epsilon N(N+2\nu), \tag{2.31}$$

so that the function $f(N)$, given by eq. (2.12), becomes

$$f_{\mathrm{PT}}(N) = \epsilon(2(N+\nu)+1). \tag{2.32}$$

The creation and annihilation operators a^+_{PT} and a^-_{PT} thus take the form

$$a^+_{\mathrm{PT}} = \sqrt{\epsilon(N)(N+2\nu)}\ e^{-i\alpha\epsilon(2(N+\nu)-1)} \sum_{n=0}^{\infty} |\phi_{n+1}\rangle\langle\phi_n|, \tag{2.33}$$

$$a^-_{\mathrm{PT}} = \sqrt{\epsilon(N+1)(N+2\nu+1)}\ e^{i\alpha\epsilon(2(N+\nu)+1)} \sum_{n=0}^{\infty} |\phi_n\rangle\langle\phi_{n+1}|. \tag{2.34}$$

An interesting connection with the generalized oscillator algebra is now given. From eqs. (2.30) and (2.31), we can write N as a function of H_{PT} and then express all the operators a^+_{PT}, a^-_{PT} and $f_{\mathrm{PT}}(N)$ in terms of H_{PT}. Indeed, we have:

$$N + \nu = \sqrt{\frac{H_{\mathrm{PT}}}{\epsilon}}, \tag{2.35}$$

so that,

$$f_{\mathrm{PT}}(N) = 2\sqrt{\epsilon H_{\mathrm{PT}}} + \epsilon. \tag{2.36}$$

From eqs. (2.11), (2.13) and (2.36), the following commutation relations are easily proved

$$[a^+_{\mathrm{PT}}, H_{\mathrm{PT}}] = -(2\sqrt{\epsilon H_{\mathrm{PT}}} - \epsilon)\ a^+_{\mathrm{PT}}, \quad [a^-_{\mathrm{PT}}, H_{\mathrm{PT}}] = a^-_{\mathrm{PT}}(2\sqrt{\epsilon H_{\mathrm{PT}}} - \epsilon), \tag{2.37}$$

$$[a^-_{\mathrm{PT}}, a^+_{\mathrm{PT}}] = 2\sqrt{\epsilon H_{\mathrm{PT}}} + \epsilon. \tag{2.38}$$

This non-linear algebra is similar to the one obtained in the paper of Quesne [10], in the sense that the following creation and annihilation operators are satisfying the Quesne non-linear algebra:

$$b^+ = g^2(H_{\mathrm{PT}})\ a^+_{\mathrm{PT}}, \quad b^- = a^-_{\mathrm{PT}}, \tag{2.39}$$

with

$$g(H_{\mathrm{PT}}) = \sqrt{\frac{1}{\epsilon}\left(1 + \frac{\nu(1-\nu)\ \epsilon}{(H_{\mathrm{PT}} - \nu^2\epsilon)(\sqrt{\epsilon H_{\mathrm{PT}}} - \epsilon)}\right)}. \tag{2.40}$$

So we have essentially an equivalent set of creation and annihilation operators for this model.

Let us mention that this last example has been taken to prove that our procedure is working for Hamiltonians other than the usual harmonic oscillator. It is beyond the scope of this paper to consider the GCS of the Pöschl-Teller model. They could be studied in another contribution and related to a recent approach by Antoine et al[26]. Let us insist here on the fact that another non-trivial example, which would be our central model in this paper, will be the Jaynes-Cummings model that exhibit an energy spectrum irrational in n.

3 Generalized coherent states

GCS will be constructed for arbitrary quantum systems as described in Section 2 by using the different definitions of CS. Let us recall that the first definition deals with the construction of eigenstates of the annihilation operator of the system under consideration. For the system governed by the Hamiltonian $H =$ (2.9), such states are labelled by $|z, \alpha\rangle$, $z \in \mathbb{C}$, $\alpha \in \mathbb{R}$ (where α is the parameter entering in the definitions (2.4) and (2.5)) and they are assumed to satisfy

$$a^-|z, \alpha\rangle = z|z, \alpha\rangle. \tag{3.1}$$

Once we decompose $|z, \alpha\rangle$ in the basis $\big\{|\psi_n\rangle = (2.6)\big\}$ such that

$$|z, \alpha\rangle = \sum_{n=0}^{\infty} C_n |\psi_n\rangle, \tag{3.2}$$

and insert this expression in eq. (3.1) using eq. (2.5), we find

$$C_n = \frac{z^n}{(E(n))^{1/2}} e^{-iE_n\alpha} C_0, \quad n > 0, \tag{3.3}$$

with $E(n) =$ (2.7). For $n = 0$, we already know that $|\psi_0\rangle$ is an eigenstate of a^- with eigenvalue 0. Finally, the GCS $|z, \alpha\rangle$ take the form

$$|z, \alpha\rangle = C_0 \sum_{n=0}^{\infty} \frac{z^n e^{-iE_n\alpha}}{(E(n))^{1/2}} |\psi_n\rangle, \tag{3.4}$$

if we have set $E(0) = 1$. The constant C_0 will be fixed by imposing the normalization to unity. We get

$$|C_0| = \left(\sum_{n=0}^{\infty} \frac{|z|^{2n}}{E(n)}\right)^{-1/2}. \tag{3.5}$$

Following the second definition, we introduce a displacement operator $D(z)$ that acts on the fundamental state $|\psi_0\rangle$. To be able to get the state $|z,\alpha\rangle$ = (3.4), we must construct the operator $D(z)$ which can be non-unitary. Indeed, we first adopt a procedure known as a linearisation of a nonlinear algebra [7, 8]. This means that we modify the operators a^- and a^+ satisfying eqs. (2.10) and (2.11) for new ones which will be called A^- and A^+, but which are not the adjoint to each other. They must satisfy

$$[A^-, A^+] = 1 \tag{3.6}$$

together with

$$[N, A^+] = A^+, \quad [N, A^-] = -A^-. \tag{3.7}$$

A solution which is valid on the Hilbert space $\mathcal{H}$ of all energy eigenstates of H is

$$A^- = a^-, \quad A^+ = \left(\frac{N}{H(N)}\right)a^+. \tag{3.8}$$

The non-unitary displacement operator $D(z)$ is now

$$D(z) = \exp(zA^+ - \bar{z}A^-) = \exp\left(-\frac{1}{2}|z|^2\right)\exp zA^+ \exp(-\bar{z}A^-), \tag{3.9}$$

because of eqs. (3.6). The GCS are then obtained by acting with $D(z)$ on the fundamental state $|\psi_0\rangle$ which satisfies $a^-|\psi_0\rangle = A^-|\psi_0\rangle = 0$. We get

$$D(z)|\psi_0\rangle = |z,\alpha\rangle. \tag{3.10}$$

Now let us discuss the third definition of coherent states as states minimizing a certain uncertainty relation. First we recall that the mean value and dispersion of an operator Z in a normalized GCS $|z,\alpha\rangle$ are respectively given by

$$\langle Z\rangle = \langle z,\alpha|Z|z,\alpha\rangle, \quad \Delta Z = \sqrt{\langle Z^2\rangle - \langle Z\rangle^2}. \tag{3.11}$$

Second we construct two hermitian operators

$$X = \frac{1}{\sqrt{2}}(a^+ + a^-), \quad P = \frac{i}{\sqrt{2}}(a^+ - a^-), \tag{3.12}$$

which satisfy, due to eq. (2.11), the commutation relation

$$[X, P] = if(N) = i(H(N+1) - H(N)). \tag{3.13}$$

It is then well-known [27] that, our GCS being eigenstates of a^-, they minimize the Heisenberg uncertainty relation

$$(\Delta X)^2(\Delta P)^2 \geq \frac{1}{4}\big\langle i\big[X,P\big]\big\rangle^2 = \frac{1}{4}\big\langle f(N)\big\rangle^2 \tag{3.14}$$

with the equal values of $(\Delta X)^2$ and $(\Delta P)^2$, i.e.

$$(\Delta X)^2 = (\Delta P)^2 = \frac{1}{2}(\langle H(N+1)\rangle - \langle H(N)\rangle) = \frac{1}{2}\left[\left(|C_0|^2\sum_{n=0}^{\infty}\frac{|z|^{2n}}{E(n)}E_{n+1}\right) - |z|^2\right]. \tag{3.15}$$

Let us end this section by giving some other properties of the coherent states. We see that they are continuous in $z \in \mathbb{C}$ and $\alpha \in \mathbb{R}$. Moreover, the presence of the phase factor in the definitions (2.4) and (2.5) of the action of a^- and a^+ leads to temporal stability of the CS. Indeed, we have

$$e^{itH}|z,\alpha\rangle = |z,\alpha+t\rangle. \tag{3.16}$$

The analysis of completeness (in fact, the overcompleteness) requires to compute the resolution of the identity [4, 15, 16], that is

$$I = \int d\mu(z)|z,\alpha\rangle\langle z,\alpha|, \tag{3.17}$$

where the measure $d\mu(z)$ has to be determined. Note that the integral is over the disk $\big\{z \in \mathbb{C}\colon |z| < R\big\}$ where the radius of convergency R is

$$R = \lim_{n\to\infty}\sqrt[n]{E(n)}, \tag{3.18}$$

with $E(n)$ given by eqs. (2.7). Using the definition (3.4) of GCS, we can write eq. (3.17) as

$$I = |\psi_0\rangle\langle\psi_0| + \sum_{n=1}^{\infty} |\psi_n\rangle\langle\psi_n| \left[\frac{1}{E(n)} \int d\,\mu(z)|C_0|^2|z|^{2n}\right]. \tag{3.19}$$

If we suppose that $d\mu(z)$ depends only on $|z|$, we can take $z = re^{i\varphi}$ and write

$$\pi|C_0|^2|z|^{2n}d\mu(z) = h(r^2)r^{2n+1}dr\,d\varphi = \frac{1}{2}h(u)u^n du\,d\varphi,$$

for $u = r^2$. The resolution of the identity is then equivalent to the determination of the function $h(u)$ satisfying

$$\int_0^{R^2} h(u)u^n du = E(n). \tag{3.20}$$

For $R \to \infty$, it is clear that $h(u)$ is the inverse of the Mellin transform of $E(n)$. Note that the calculation thus requires the explicit knowledge of the spectrum of the system under consideration. A special application with the JC energy spectrum will be treated in the next section.

A final comment can be made in connection with the work of Gazeau and Klauder [15]. In fact, our states satisfy all the requirements given in their approach but they are more general since we are working with $z \in \mathbb{C}$ and $\alpha \in \mathbb{R}$. They also satisfy additional properties. Indeed, they are eigenstates of an annihilation operator, we have been able to introduce in a general context, and they may be obtained from the action of a displacement operator.

4 New set of coherent states for the Jaynes–Cummings model

In this section, the Jaynes–Cummings model will be re-analysed with the view of constructing the coherent states as discussed in the preceding section. We begin (§A) by a discussion of the energy spectrum of the model. It will be the occasion to put the emphasis on possible problems of degeneracy which, at our knowledge, have never been mentioned. Next, we introduce annihilation and creation operators in the case where there is no degeneracy (§B) and we give the new set of coherent states. The case of degeneracy is also discussed (§C) and we insist on the fact that some restrictions on the parameters of the model have to be made in order that our construction be valid. Finally (§D), the behaviour of mean values and dispersion of physical quantities is analysed in the case where the energy spectrum presents increasing energy levels.

4.1 The Jaynes–Cummings model and energy spectrum

It is well-known [17] that the JC model describes a spin 1/2-fermion in interaction with a one-mode magnetic field having an oscillating component along the x-axis and a constant component along the z-axis. In the rotating-wave approximation, it may be described by the Hamiltonian

$$H_{\mathrm{JC}} = \omega\left(a_0^+a_0^- + \frac{1}{2}\right)\sigma_0 + \frac{\omega_0}{2}\sigma_3 + \kappa(a_0^+\sigma_- + a_0^-\sigma_+), \tag{4.1}$$

where ω is the field mode frequency, ω_0 the atomic frequency and κ a coupling constant. Moreover the operators a_0^+ and a_0^- are the usual creation and annihilation operators for the radiation field, $\sigma_\pm$ and σ_3 are associated to the usual Pauli matrices and σ_0 is the identity matrix.

Let us recall that the Hamiltonian $H_{\mathrm{JC}} =$ (4.1) can also be written as a linear combination of generators of the superalgebra $u(1/1)$, which thus corresponds to the dynamical superalgebra of H_{JC} [21, 22]. Indeed, we have

$$H_{\mathrm{JC}} = \frac{1}{2}(\omega + \omega_0)\mathcal{N} - \frac{\Delta}{2}\mathcal{M} + \kappa(Q_0^+ + Q_0^-), \tag{4.2}$$

where

$$\mathcal{N} = \left(a_0^+a_0^- + \frac{1}{2}\right)\sigma_0 + \frac{1}{2}\sigma_3, \quad \mathcal{M} = -\left(a_0^+a_0^- + \frac{1}{2}\right)\sigma_0 + \frac{1}{2}\sigma_3, \tag{4.3}$$

$$Q_0^+ = a_0^+\sigma_-, \quad Q_0^- = a_0^-\sigma_+, \tag{4.4}$$

and $\Delta = \omega - \omega_0$ is the detuning which is assumed to be a positive quantity. In the absence of the oscillating component of the magnetic field ($\kappa = 0$) and for the exact resonance ($\Delta = 0$), we get the SUSY harmonic oscillator Hamiltonian (with $\omega = 1$)[6]:

$$H_{\mathrm{SUSY}}^{\lambda_0} = \{Q_0^+, Q_0^-\} = \begin{pmatrix} a_0^- a_0^+ & 0 \\ 0 & a_0^+ a_0^- \end{pmatrix}. \tag{4.5}$$

Let us also mention that a diagonalisation of $H_{\mathrm{JC}} = (4.1)$ has been performed [23] and gave rise to

$$H_D = O^\dagger H_{\mathrm{JC}} O = \begin{pmatrix} \omega(N_0+1) - \kappa r(N_0+1) & 0 \\ 0 & \omega N_0 + \kappa r(N_0) \end{pmatrix}, \tag{4.6}$$

with $N_0 = (2.19)$ associated with the photon number. Note that the set $\{N_0, a_0^+, a_0^-\}$ generates the usual (linear) oscillator algebra. The operator O is given by

$$O = \begin{pmatrix} \frac{1}{R(N_0+1)}\left(\frac{\Delta}{2} + \kappa r(N_0+1)\right) & \frac{\kappa}{R(N_0+1)} a_0^- \\ -a_0^+ \frac{\kappa}{R(N_0+1)} & \frac{1}{R(N_0)}\left(\frac{\Delta}{2} + \kappa r(N_0)\right) \end{pmatrix}, \tag{4.7}$$

where

$$R(N_0) = \left[\left(\frac{\Delta}{2} + \kappa r(N_0)\right)^2 + \kappa^2 N_0\right]^{1/2}, \tag{4.8}$$

and

$$r(N_0) = \left(\left(\frac{\Delta}{2\kappa}\right)^2 + N_0\right)^{1/2}. \tag{4.9}$$

Using such a diagonalisation, it is very easy to describe the energy spectrum and eigenstates of H_{JC}. Indeed, we see immediately that the energy eigenvalues are

$$\varepsilon_n^- = \omega n + \kappa r(n), \quad \varepsilon_n^+ = \omega(n+1) - \kappa r(n+1), \tag{4.10}$$

and the corresponding eigenstates are easily computed. We find

$$|\varepsilon_0^-\rangle_{\mathrm{JC}} = |0,-\rangle, \tag{4.11}$$

$$|\varepsilon_{n+1}^-\rangle_{\mathrm{JC}} = O|n+1,-\rangle = \frac{1}{R(n+1)}\left(\kappa\sqrt{n+1}|n,+\rangle + \left(\frac{\Delta}{2} + \kappa r(n+1)\right)|n+1,-\rangle\right), \tag{4.12}$$

$$|\varepsilon_n^+\rangle_{\mathrm{JC}} = O|n,+\rangle = \frac{1}{R(n+1)}\left(\left(\frac{\Delta}{2} + \kappa r(n+1)\right)|n,+\rangle - \kappa\sqrt{n+1}|n+1,-\rangle\right), \tag{4.13}$$

where we are working in the Fock space

$$\mathcal{F} = \mathcal{F}_b \otimes \mathcal{F}_f = \left\{|n,-\rangle = \begin{pmatrix} 0 \\ |n\rangle \end{pmatrix}, |n,+\rangle = \begin{pmatrix} |n\rangle \\ 0 \end{pmatrix},\ n = 0,1,2,\ldots\right\}. \tag{4.14}$$

Since we want to apply our preceding considerations to this model, it is important to analyse more deeply the energy spectrum of the component Hamiltonians of H_D. Let us write $H_D = (4.6)$ as

$$H_D = \begin{pmatrix} H_{D(+)} & 0 \\ 0 & H_{D(-)} \end{pmatrix}, \tag{4.15}$$

with

$$H_{D(+)} = \omega(N_0+1) - \kappa r(N_0+1),\ H_{D(-)} = \omega N_0 + \kappa r(N_0). \tag{4.16}$$

For $H_{D(-)}$, the energies are given by ε_n^- as in eq. (4.10) and satisfy

$$\varepsilon_{n+1}^- - \varepsilon_n^- = \omega + \frac{\kappa}{D(n)}, \tag{4.17}$$

where

$$D(n) = (\delta + n + 1)^{1/2} + (\delta + n)^{1/2}, \tag{4.18}$$

if we have set

$$\delta = \left(\frac{\Delta}{2\kappa}\right)^2. \tag{4.19}$$

Since Δ is positive, the energies are such that

$$\varepsilon_{n+1}^- > \varepsilon_n^-. \tag{4.20}$$

For $H_{D(+)}$, the energies are given by ε_n^+ as in eq. (4.10) and satisfy

$$\varepsilon_{n+1}^+ - \varepsilon_n^+ = \omega - \frac{\kappa}{D(n+1)}. \tag{4.21}$$

We see that they are not necessarily strictly increasing, i.e. for some values of the parameters κ, ω, Δ, the quantity (4.21) may be negative. More than that the energy spectrum of $H_{D(+)}$ may present some degeneracies, and may have some negative energy levels. Indeed, we can see (Appendix Appendix A) that three possibilities exist, i.e.

(i) if $0 \leq \kappa/\omega \leq 2\sqrt{\delta+1}$, the energies ε_n^+ are strictly increasing for all values of n and there is no degeneracy;

(ii) if $\kappa/\omega > 2\sqrt{\delta+1}$ and there is no entire value of n and k such that $\varepsilon_n^+ = \varepsilon_k^+$, there is no degeneracy but the energies are not strictly increasing;

(iii) if $\kappa/\omega > 2\sqrt{\delta+1}$ and there exists some n and $k(n \neq k)$ such that

$$\varepsilon_n^+ = \varepsilon_k^+, \quad n \leq \left(\frac{\kappa}{2\omega}\right)^2 - (1+\delta) \leq k, \tag{4.22}$$

there is a degeneracy for the states $|\varepsilon_n^+\rangle$ and $|\varepsilon_k^+\rangle$. We can show that only double degeneracy levels occur but we may have many of them in this case.

4.2 General set of coherent states for the nondegenerated case

Let us first consider the case (i) where the Hamiltonian components of $H_D =$ (4.15) have strictly increasing energy spectrum, that is for $\kappa/\omega \leq 2\sqrt{\delta+1}$. We are then in the hypotheses of Section 2 and we can factorised H_D on the form

$$H_D = \begin{pmatrix} a_{(+)}^+ a_{(+)}^- + \varepsilon_0^+ & 0 \\ 0 & a_{(-)}^+ a_{(-)}^- + \varepsilon_0^- \end{pmatrix} = \begin{pmatrix} H_{(+)}(N) + \varepsilon_0^+ & 0 \\ 0 & H_{(-)}(N) + \varepsilon_0^- \end{pmatrix}. \tag{4.23}$$

Once we define the annihilation and creation operators

$$a_D^- = \begin{pmatrix} a_{(+)}^- & 0 \\ 0 & a_{(-)}^- \end{pmatrix}, \quad a_D^+ = \begin{pmatrix} a_{(+)}^+ & 0 \\ 0 & a_{(-)}^+ \end{pmatrix}, \tag{4.24}$$

which lead to

$$H_D = a_D^+ a_D^- + \begin{pmatrix} \varepsilon_0^+ & 0 \\ 0 & \varepsilon_0^- \end{pmatrix}, \tag{4.25}$$

we can construct the GCS $|z,\alpha\rangle_D$ as eigenstates of a_D^-, from our preceding considerations. The use of the unitary operator O will then give the annihilation operator for the Jaynes–Cummings model as

$$a_{\mathrm{JC}}^- = O a_D^- O^\dagger, \tag{4.26}$$

and the corresponding GCS

$$|z,\alpha\rangle_{\mathrm{JC}} = O|z,\alpha\rangle_D. \tag{4.27}$$

The factorised Hamiltonians $H_{(\pm)}(N) = a_{(\pm)}^+ a_{(\pm)}^-$ have energy eigenvalues

$$E_{n,(\pm)} = \varepsilon_n^\pm - \varepsilon_0^\pm, \tag{4.28}$$

with $\varepsilon_n^\pm$ given in eq. (4.10) for $n \in \mathbb{N}$. We write explicitly

$$E_{n,(+)} = \omega n + \kappa\big((\delta+1)^{1/2} - (\delta+n+1)^{1/2}\big), \quad E_{n,(-)} = \omega n + \kappa\big((\delta+n)^{1/2} - \delta^{1/2}\big). \tag{4.29}$$

The creation $a_{(\pm)}^+$ and annihilation $a_{(\pm)}^-$ operators satisfy the relations (2.4) and (2.5) respectively. Since the energy eigenstates are now $|\psi_n\rangle = |n\rangle$, the operator N is nothing else then $N_0 = a_0^+ a_0^-$. Moreover,

$$\begin{aligned} H_{(+)}(N_0) &= a_{(+)}^+ a_{(+)}^- = \omega N_0 + \kappa\big((\delta+1)^{1/2} - (\delta+N_0+1)^{1/2}\big), && (4.30)\\ H_{(-)}(N_0) &= a_{(-)}^+ a_{(-)}^- = \omega N_0 - \kappa\big(\delta^{1/2} - (\delta+N_0)^{1/2}\big). && (4.31) \end{aligned}$$

The commutator (2.11) thus write

$$[a^-_{(+)}, a^+_{(+)}] = f_{(+)}(N_0) = \omega - \frac{\kappa}{D(N_0+1)}, \tag{4.32}$$

$$[a^-_{(-)}, a^+_{(-)}] = f_{(-)}(N_0) = \omega + \frac{\kappa}{D(N_0)}. \tag{4.33}$$

Following eqs. (2.17) and (2.18) and the relations (2.25) and (2.26), we get

$$a^+_{(\pm)} = \sqrt{\frac{H_{(\pm)}(N_0)}{N_0}}\; e^{-if_{(\pm)}(N_0-1)\alpha} a^+_0, \tag{4.34}$$

$$a^-_{(\pm)} = \sqrt{\frac{H_{(\pm)}(N_0+1)}{N_0+1}}\; e^{if_{(\pm)}(N_0)\alpha} a^-_0. \tag{4.35}$$

Similarly to the case of the Pöschl-Teller Hamiltonian, we can show that the sets $\{H_{D(\pm)}, a^+_{(\pm)}, a^-_{(\pm)}\}$ generate non-linear algebras explicitly given by:

$$[a^+_{(\pm)}, H_{D(\pm)}] = \ a^+_{(\pm)}\ f_{(\pm)}(N_0), \quad [H_{D(\pm)}, a^-_{(\pm)}] = -f_{(\pm)}(N_0)\ a^-_{(\pm)}, \tag{4.36}$$

together with eqs.(4.32) and (4.33), where N_0 is now seen as a function of $H_{D(\pm)}$. Indeed, from eq.(4.16), the fact that $N_0|0\rangle = 0$ and $\kappa/\omega \leq 2\sqrt{\delta+1}$, we get:

$$N_0 = -1 + \frac{H_{D(+)}}{\omega} + \frac{1}{2\omega^2}(\kappa + \sqrt{\kappa^4 + 4\omega^2\kappa^2 H_{D(+)} + \omega^2\Delta^2}), \tag{4.37}$$

$$= \frac{H_{D(-)}}{\omega} + \sqrt{\kappa^4 - 4\omega^2\kappa^2 H_{D(-)} + \omega^2\Delta^2}. \tag{4.38}$$

Let us now study special cases of interest. First, for $\kappa = 0$, the energy spectrum is linear in n and the energy levels are equally spaced by the frequency ω. We have

$$H_{(\pm)}(N_0) = \omega N_0, \quad f_{(\pm)}(N_0) = \omega, \tag{4.39}$$

so that

$$a^+_{(\pm)} = \sqrt{\omega}\ e^{-i\omega\alpha} a^+_0,\ a^-_{(\pm)} == \sqrt{\omega}\ e^{i\omega\alpha} a^-_0, \tag{4.40}$$

which are essentially the creation and annihilation operators for the usual harmonic oscillator.

Second we consider the case of weak coupling limit where we keep at most terms of order 2 in κ. We expand $E_{n,(\pm)}$ in eq. (4.29) to get

$$E_{n,(\pm)}(\kappa \lll) = \omega_\pm(\kappa) n, \tag{4.41}$$

where we have taken

$$\omega_\pm(\kappa) = \left(\frac{\omega \mp \kappa^2}{\Delta}\right). \tag{4.42}$$

It also leads to an energy spectrum linear in n but now the energy levels are equally spaced by the quantity $\omega_\pm(\kappa)$ and we find

$$H_{(\pm)}(N_0) = \omega_\pm(\kappa) N_0, \quad f_{(\pm)}(N_0) = \omega_\pm(\kappa) \tag{4.43}$$

and

$$a^+_{(\pm)} = \sqrt{\omega_\pm(\kappa)}\ e^{-i\omega_\pm(\kappa)\alpha} a^+_0,\ a^-_{(\pm)} == \sqrt{\omega_\pm(\kappa)}\ e^{i\omega_\pm(\kappa)\alpha} a^-_0. \tag{4.44}$$

Let us finally mention that for $\kappa \neq 0$ and exact resonance ($\Delta = 0$) or $\delta = 0$, we have an energy spectrum of the form

$$E_{n,(+)}(\Delta = 0) = \omega n - \kappa\sqrt{n+1} + \kappa, \tag{4.45}$$

$$E_{n,(-)}(\Delta = 0) = \omega n + \kappa\sqrt{n}. \tag{4.46}$$

The energy levels are thus not equally spaced. In this case, the operators $H_{(\pm)}(N_0)$ in eqs. (4.30) and (4.31) become

$$H_{(+)}(N_0) = \omega N_0 - \kappa\sqrt{N_0+1} + \kappa, \tag{4.47}$$

$$H_{(-)}(N_0) = \omega N_0 + \kappa\sqrt{N_0} \tag{4.48}$$

and we have
$$f_{(+)}(N_0) = \omega - \frac{\kappa}{\sqrt{N_0+2}+\sqrt{N_0+1}}, \ f_{(-)}(N_0) = \omega + \frac{\kappa}{\sqrt{N_0+1}+\sqrt{N_0}}. \tag{4.49}$$
The creation and annihilation operators are then given by
$$a^+_{(+)} = \sqrt{\omega + \frac{\kappa}{N_0}(1-\sqrt{N_0+1})}\, e^{-if_{(+)}(N_0-1)\alpha} a_0^+, \tag{4.50}$$
$$a^-_{(+)} = \sqrt{\omega + \frac{\kappa}{N_0+1}(1-\sqrt{N_0+2})}\, e^{if_{(+)}(N_0)\alpha} a_0^- \tag{4.51}$$

and

$$a^+_{(-)} = \sqrt{\omega + \frac{\kappa}{\sqrt{N_0}}}\, e^{-if_{(-)}(N_0-1)\alpha} a_0^+, \tag{4.52}$$
$$a^-_{(-)} = \sqrt{\omega + \frac{\kappa}{\sqrt{N_0+1}}}\, e^{if_{(-)}(N_0)\alpha} a_0^-. \tag{4.53}$$

Following the considerations of Section 3, the GCS $|z,\alpha\rangle_D$ and $|z,\alpha\rangle_{\rm JC}$ may be obtained easily. Indeed, we get

$$|z,\alpha\rangle_D = C_{+,D}\sum_{n=0}^{\infty}\frac{z^n e^{-iE_{n,(+)}\alpha}}{(E_{(+)}(n))^{1/2}}|n,+\rangle + C_{-,D}\sum_{n=0}^{\infty}\frac{z^n e^{-iE_{n,(-)}\alpha}}{(E_{(-)}(n))^{1/2}}|n,-\rangle, \tag{4.54}$$

where $C_{\pm,D}$ are normalization constants and $E_{(\pm)}(n)$ have been defined as in eq. (2.7) from the energies $E_{n,(\pm)}$ given in (4.29) (let us recall that we have set $E_{(\pm)}(0) = 1$). Since the states $|z,\alpha\rangle_{\rm JC}$ are obtained from the action of O on $|z,\alpha\rangle_D$ (see eq.(4.27)), we can write them immediately from (4.54) by substituting $|\varepsilon_n^{\pm}\rangle$ to $|n,\pm\rangle$as given by eqs.(4.11)–(4.13). Let us denote by $|z,\alpha\rangle_{\rm JC}^{(\pm)}$ the pure states,

$$|z,\alpha\rangle_{\rm JC}^{(\pm)} = C_{\pm}\sum_{n=0}^{\infty}\frac{z^n e^{-iE_{n,(\pm)}\alpha}}{(E_{(\pm)}(n))^{1/2}}|\varepsilon_n^{\pm}\rangle, \tag{4.55}$$

where the normalization constants are given by

$$|C_{\pm}| = \left(\sum_{n=0}^{\infty}\frac{|z|^{2n}}{E_{(\pm)}(n)}\right)^{-1/2}. \tag{4.56}$$

A general GCS $|z,\alpha\rangle_{\rm JC}$ will then be written as

$$|z,\alpha\rangle_{\rm JC} = \cos\left(\frac{\theta}{2}\right)|z,\alpha\rangle_{\rm JC}^{(-)} + \sin\left(\frac{\theta}{2}\right)e^{i\phi}|z,\alpha\rangle_{\rm JC}^{(+)}. \tag{4.57}$$

Let us recall that the resolution of the identity asks to solve the inverse Mellin transform of the functions $E_{(\pm)}(n)$. Indeed, we have to find the functions $h_{(\pm)}(u)$ given by

$$h_{(\pm)}(u) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} E_{(\pm)}(s)u^{-(s+1)}\, ds, \quad c\in\mathbb{R}, \tag{4.58}$$

where the functions $E_{(\pm)}(s)$ correspond to analytic prolongation of

$$E_{\pm}(n) = E_{n,(\pm)}E_{n-1,(\pm)}\dots E_{1,(\pm)}. \tag{4.59}$$

This problem is solved for the general case in the Appendix Appendix B. Here we give the explicit results for the special cases mentioned earlier. For the weak coupling limit and the absence of coupling, we observe that the energy spectrum is linear in n, and we write

$$E_{n,(\pm)} = \omega_{\pm}(\kappa)n, \tag{4.60}$$

where $\omega_{\pm}(\kappa)$ is given in eq. (4.42), so that

$$E_{(\pm)}(s) = \big(\omega_{\pm)}(\kappa)\big)^s\Gamma(s+1), \tag{4.61}$$

and the solution of eq. (4.58) is

$$h_\pm(u) = \frac{1}{\omega_\pm(\kappa)} e^{-u/\omega_\pm(\kappa)}. \tag{4.62}$$

In absence of coupling ($\kappa = 0$), we see that $h_+ = h_-$ and are the same as for the SUSY harmonic oscillator. The case of weak coupling leads to different functions h_+ and h_- and so to different measures.

Let us now consider the case (ii) where $\kappa/\omega > 2\sqrt{\delta+1}$ and the energy levels are not degenerated. We consider here only the case of $H_{(+)}$ because there is no problem with $H_{(-)}$. For the integer $n \in [0, x_1]$, where

$$x_1 = \left(\frac{\kappa}{\omega} - \sqrt{\delta+1}\right)^2 - (1+\delta), \tag{4.63}$$

the energies are not in increasing order with n. Moreover, the fundamental energy level is no longer the level ε_0^+. A simple example is given when $\delta = 0$, $\omega = 1$ and $\kappa = 2\sqrt{2}$. Indeed, we see that

$$\varepsilon_1^+ = -2 < \varepsilon_2^+ = -1.88 < \varepsilon_0^+ = -1.83 < \varepsilon_3^+ = -1.6 < \varepsilon_4^+ < \varepsilon_5^+ < \cdots. \tag{4.64}$$

Other examples are shown in the Appendix Appendix A.

For such a situation, the problem may be solved easily by renaming the levels such that they appear in increasing order. The preceding considerations on the construction of GCS may thus be directly adapted.

4.3 Coherent states for the degenerated case

When the energy spectrum of $H_{(+)}$ presents degeneracies, it is impossible to do the preceding construction of GCS because the Hamiltonian cannot be factorised in terms of unique creation and annihilation operators.

The GCS may be constructed retaining only the four criterion given by Gazeau and Klauder [15], i.e. continuity, temporal stability, resolution of unity and action identity.

4.4 Calculation of physical quantities in the coherent states

It will be relevant to examine the behaviour of different physical quantities in our coherent states and compare them with other approaches.

Let us first introduce the notations. In the pure states (4.55), let us write the mean value of an operator Z as

$$\langle Z\rangle_{\mathrm{JC}}^{(\pm)} = {}_{\mathrm{JC}}^{(\pm)}\langle z,\alpha|Z|z,\alpha\rangle_{\mathrm{JC}}^{(\pm)}. \tag{4.65}$$

When it will be calculated in the diagonal states $|z,\alpha\rangle_D^{(\pm)}$, we will write it as $\langle Z\rangle_D^{(\pm)}$. If we are considering a general state $|z,\alpha\rangle_{\mathrm{JC}} =$ (4.57), we use the formula

$$\langle Z\rangle_{\mathrm{JC}} = \frac{1}{2}\left[(1-\cos\theta)\langle Z\rangle_{\mathrm{JC}}^{(+)} + (1+\cos\theta)\langle Z\rangle_{\mathrm{JC}}^{(-)} + \sin\theta\left(e^{i\phi}\langle Z\rangle_{\mathrm{JC}}^{(+-)} + e^{-i\phi}\langle Z\rangle_{\mathrm{JC}}^{(-+)}\right)\right], \tag{4.66}$$

where, evidently, we have

$$\langle Z\rangle_{\mathrm{JC}}^{(\pm\mp)} = {}_{\mathrm{JC}}^{(\pm)}\langle z,\alpha|Z|z,\alpha\rangle_{\mathrm{JC}}^{(\mp)}. \tag{4.67}$$

It is sometimes useful to compute the physical quantities in the diagonal states. Let us recall that, because of the relation (4.27), we have

$$\langle Z\rangle_{\mathrm{JC}}^{(\pm)} = \langle O^\dagger Z O\rangle_D^{(\pm)}. \tag{4.68}$$

The dispersion of the operator Z in the normalized states is computed from the mean values as

$$(\Delta Z^2)_{\mathrm{JC}}^{(\pm)} = \langle Z^2\rangle_{\mathrm{JC}}^{(\pm)} - \left(\langle Z\rangle_{\mathrm{JC}}^{(\pm)}\right)^2. \tag{4.69}$$

Let us also introduce the new parameter $x = |z|^2$ to shorten our expressions and because it is a good approximation of the number of photons in the weak coupling limit of our model.

We begin with the operator $\mathcal{N}$ given in eq. (4.3). It is a constant of motion and represents the total number of particles. Since it is invariant under the transformation by O, we have

$$\langle\mathcal{N}\rangle_{\mathrm{JC}}^{(\pm)} = \langle\mathcal{N}\rangle_D^{(\pm)}. \tag{4.70}$$

The calculation will be done with the diagonal coherent states because we know the action of N_0 on the states $|n,\pm\rangle$. We easily find

$$\langle\mathcal{N}\rangle^{(+)}_{\mathrm{JC}} = |C_+|^2 \sum_{n=0}^{\infty} \frac{x^n}{E_{(+)}(n)}(n+1), \quad \langle\mathcal{N}\rangle^{(-)}_{\mathrm{JC}} = |C_-|^2 \sum_{n=0}^{\infty} \frac{x^n}{E_{(-)}(n)} n. \tag{4.71}$$

For the case of weak coupling, we have a compact expression for $E_{(\pm)}(n)$, i.e.

$$E_{(\pm)}(n) = \big(\omega_\pm(\kappa)\big)^n n!, \tag{4.72}$$

where $\omega_\pm(\kappa) =$ (4.42). The formula (4.71) thus simplifies to give

$$\langle\mathcal{N}\rangle^{(+)}_{\mathrm{JC}}(\kappa \lll) = 1 + \frac{x}{\omega_+(\kappa)}, \quad \langle\mathcal{N}\rangle^{(-)}_{\mathrm{JC}}(\kappa \lll) = \frac{x}{\omega_-(\kappa)}. \tag{4.73}$$

Note that $\mathcal{N}$ is nothing else than the SUSY harmonic oscillator Hamiltonian and, in the weak coupling limit, the GCS are the usual CS of the harmonic oscillator but with a frequency equal to $\omega_{(\pm)}(\kappa)$.

The dispersion is also easy to compute

$$(\Delta\mathcal{N})^{2(+)}_{\mathrm{JC}} = |C_+|^2 \sum_{n=0}^{\infty} \frac{x^n}{E_{(+)}(n)}(n+1)^2 - |C_+|^4 \left(\sum_{n=0}^{\infty} \frac{x^n}{E_{(+)}(n)}(n+1)\right)^2, \tag{4.74}$$

$$(\Delta\mathcal{N})^{2(-)}_{\mathrm{JC}} = |C_-|^2 \sum_{n=0}^{\infty} \frac{x^n}{E_{(-)}(n)} n^2 - |C_-|^4 \left(\sum_{n=0}^{\infty} \frac{x^n}{E_{(-)}(n)} n\right)^2. \tag{4.75}$$

Again, the expressions simplify in the weak coupling limit and we find as expected

$$(\Delta\mathcal{N})^{2(\pm)}_{\mathrm{JC}}(\kappa \lll) = \frac{x}{\omega_\pm(\kappa)}. \tag{4.76}$$

Now, we want to evaluate the mean value of $N_0 = a_0^+ a_0$, which is the total number of photon. The calculation is direct via the expression of the energy eigenstates $|\varepsilon_n^{(\pm)}\rangle$ in the basis $|n,\pm\rangle$ (see eqs. (4.11)–(4.13)), we get

$$\langle N_0\rangle^{(+)}_{\mathrm{JC}} = |C_+|^2 \sum_{n=0}^{\infty} \frac{x^n}{E_{(+)}(n)} \left(n + \frac{k^2(n+1)}{(R(n+1))^2}\right), \tag{4.77}$$

$$\langle N_0\rangle^{(-)}_{\mathrm{JC}} = |C_-|^2 \sum_{n=0}^{\infty} \frac{x^n}{E_{(-)}(n)} \left(n - \frac{\kappa^2 n}{(R(n))^2}\right). \tag{4.78}$$

For the case of weak coupling, we find

$$\langle N_0\rangle^{(+)}_{\mathrm{JC}}(\kappa \lll) = \left(1 + \frac{2\kappa^2}{\Delta^2}\right)\frac{x}{\omega_+(\kappa)} + \frac{2\kappa^2}{\Delta^2}, \quad \langle N_0\rangle^{(-)}_{\mathrm{JC}}(\kappa \lll) = \left(1 - \frac{2\kappa^2}{\Delta^2}\right)\frac{x}{\omega_-(\kappa)}. \tag{4.79}$$

For the mean value and dispersion of the energy, we see that

$$\langle H_{\mathrm{JC}}\rangle_{\mathrm{JC}} = \langle H_D\rangle_D, \tag{4.80}$$

and similarly for $\langle H^2_{\mathrm{JC}}\rangle_{\mathrm{JC}}$, so all the calculations are done using the form (4.6) of H_D. We find

$$\langle H_{\mathrm{JC}}\rangle^{(\pm)}_{\mathrm{JC}} = \langle H_D\rangle^{(\pm)}_D = x + \varepsilon_0^\pm, \tag{4.81}$$

and

$$\langle H^2_{\mathrm{JC}}\rangle^{(\pm)}_{\mathrm{JC}} = (\varepsilon_0^\pm)^2 + 2\varepsilon_0^\pm x + |C_\pm|^2 \sum_{n=0}^{\infty} \frac{x^n}{E_{(\pm)}(n)} E^2_{n,\pm}. \tag{4.82}$$

The dispersion thus take the form

$$(\Delta H^2_{\mathrm{JC}})^{(\pm)}_{\mathrm{JC}} = |C_\pm|^2 \sum_{n=0}^{\infty} \frac{x^n}{E_{(\pm)}(n)} (E^2_{n,(\pm)} - x^2), \tag{4.83}$$

where $|C_\pm|$ is given by eq. (4.56).

In the weak coupling limit, it is easy to see that

$$(\Delta H^2_{\rm JC})^{(\pm)}_{\rm JC}(\kappa \lll) = \frac{x}{\omega_{\pm}(\kappa)}, \tag{4.84}$$

which is an expected result because our GCS have been constructed in close connection with the harmonic oscillator ones.

When the exact resonance is considered, we write separately the dispersions for the states $|z,\alpha\rangle^{(+)}$ and $|z,\alpha\rangle^{(-)}$. Indeed, let us recall that, for the case of $|z,\alpha\rangle^{(-)}$, they are well defined for all values of the coupling constant. For $\omega = 1$, we have

$$(\Delta H^2_{\rm JC})^{(-)}_{\rm JC}(\Delta = 0) = \left(\sum_{n=0}^{\infty} C_-(n,\kappa)\right)^{-1}\left(\sum_{n=0}^{\infty} C_-(n,\kappa)\big(n(\sqrt{n}+\kappa)^2 - x^2\big)\right), \tag{4.85}$$

where $C_-(0,\kappa) = 1$ and

$$C_-(n,\kappa) = \frac{x^n}{\sqrt{n!}\prod_{i=1}^{n}(\sqrt{i}+\kappa)}, \quad \text{for } n \neq 0. \tag{4.86}$$

For the case of $(\Delta H^2_{\rm JC})^{(+)}_{\rm JC}$, we write for $\omega = 1$,

$$(\Delta H^2_{\rm JC})^{(+)}_{\rm JC}(\Delta = 0) = \left(\sum_{n=0}^{\infty} C_+(n,\kappa)\right)^{-1}\left(\sum_{n=0}^{\infty} C_+(n,\kappa)\big((n - \kappa(\sqrt{n+1}-1))\big)^2 - x^2\right), \tag{4.87}$$

where $C_+(0,\kappa) = 1$ and

$$C_+(n,\kappa) = \frac{x^n}{\prod_{i=1}^{n}(i - \kappa(\sqrt{i+1}-1))}, \quad \text{for } n \neq 0. \tag{4.88}$$

We see immediately here the problem already mentioned because eq. (4.88) may become singular for some values of κ. These are precisely the ones for which the energy level ε_0^+ is degenerated (it is not the lower one in this case). So, to be valid, our considerations must exclude the values of κ such that

$$\kappa = \sqrt{i+1}+1, \quad i = 1,2,\ldots. \tag{4.89}$$

This is precisely the case here since $\kappa \leq 2$.

Let us finally mention that the expressions (4.85) and (4.87) as functions of $x = |z|^2$ show a behaviour which is closed to be linear. This result was expected by our construction of GCS which maintain all the properties of the standard CS for the harmonic oscillator.

The atomic inversion is an important quantity in the Jaynes–Cummings model. It has been shown that, when the system is prepared in the CS of the radiation field [18], the temporal behavior of this atomic inversion consists of Rabi oscillations. Let us show that it is again the case for the GCS we have constructed in the preceding section. The degree of atomic inversion is measured by the function

$$\big(I(t)\big)_{\psi} = \langle\psi|\sigma_3(t)|\psi\rangle, \tag{4.90}$$

where

$$\sigma_3(t) = e^{-itH_{\rm JC}}\sigma_3 e^{itH_{\rm JC}}. \tag{4.91}$$

It is easy to show that in the pure states (4.55), it is a constant in time and we have

$$\big(I(t)\big)^{(+)}_{\rm JC} = |C_+|^2\sum_{n=0}^{\infty}\frac{x^n}{E_{(+)}(n)}\left(1 - \frac{2\kappa^2(n+1)}{(R(n+1))^2}\right), \tag{4.92}$$

$$\big(I(t)\big)^{(-)}_{\rm JC} = |C_-|^2\left(\sum_{n=1}^{\infty}\frac{x^n}{E_{(-)}(n)}\left(\frac{2\kappa^2 n}{R(n))^2} - 1\right) - 1\right). \tag{4.93}$$

In the general state, we use the formula (4.66) and the time evolution is reflected in the mixed terms $\langle\sigma_3\rangle^{(+-)}$ and $\langle\sigma_3\rangle^{(-+)}$. We find the general expression

$$\big\langle I(t)\big\rangle_{\rm JC} = \frac{1}{2}\Big[(1-\cos\theta)\big\langle I(t)\big\rangle^{(+)}_{\rm JC} + (1+\cos\theta)\big\langle I(t)\big\rangle^{(-)}_{\rm JC} \tag{4.94}$$

$$+2\sin\theta|C_+||C_-|x^{1/2}\sum_{n=0}^{\infty}\frac{x^n}{(E_{(-)}(n-1)E_{(+)}(n))^{1/2}}\frac{\sqrt{n+1}}{r(n+1)}\cos\varphi_n(t)\Big], \tag{4.95}$$

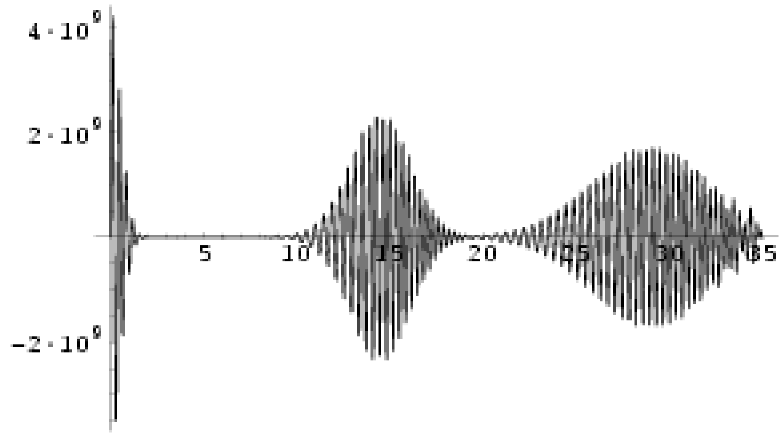

Figure 1: The collapse and revivals of the atomic inversion for $x = 20$ and $k = 2$ as a function of t.

where

$$\varphi_n(t) = (\alpha + t)(E_{n+1,(-)} - E_{n,(+)}) - \phi = (\alpha + t)\omega\left[1 - \frac{\Delta}{2\omega} + \frac{\kappa}{\omega}\big(2r(n+1)\big) - \sqrt{\left(\frac{\Delta}{2\kappa}\right)^2 + 1}\right] - \phi. \tag{4.96}$$

In fact, we are interested in the explicit expression of $\langle I(t)\rangle_{\rm JC}$ for the exact resonance. So if we take $\Delta = 0$, $\alpha = 0$, $\theta = \phi = \pi/2$ and $\omega = 1$, we find

$$\big\langle I(t)\big\rangle_{\rm JC} = -\frac{1}{2}|C_-|^2 + |C_+||C_-|x^{1/2}\sum_{n=0}^{\infty}\frac{x^n}{(E_{(-)}(n-1)E_{(+)}(n))^{1/2}}\sin\big[t\big(1+\kappa(2\sqrt{n+1}-1)\big)\big]. \tag{4.97}$$

The Rabi oscillations appear due to the presence of the last term in $\big\langle I(t)\big\rangle_{\rm JC}$. Indeed, Fig. 1 shows the revivals that characterize the atomic inversion. Let us insist on the fact that our results are valid for $\kappa < 2$, where neither singularity appear in $\big\langle I(t)\big\rangle_{\rm JC}$ nor square root of negative energy. It is in accordance with our discussion of the energy spectrum of $H_{\rm JC}$.

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Appendix A

Let us discuss the energy spectrum of $H_{D(+)}$ as given by eq. (4.16). From eq. (4.10) with eqs. (4.9) and (4.19), we can write the energy levels as

$$\varepsilon_n^+ = \omega(n+1) - \kappa\sqrt{\delta + n + 1}, \tag{A.1}$$

where $n = 0, 1, 2, \ldots$. As mentioned before, these levels may appear to be not in increasing order, present some degeneracies or be negative for some values of n, depending on the values of the parameters ω, κ and δ, as given by eq. (4.19). To show that let us consider the function

$$f(x) = \omega(x+1) - \kappa\sqrt{\delta + x + 1}, \quad \text{for } x \geq 0 \tag{A.2}$$

which coincides with ε_n^+ for integer values of x. Moreover, we take

$$F(x,y) = f(x) - f(y), \quad x, y \geq 0, \tag{A.3}$$

from which we will make the analysis of the degeneracies.

The function $f(x)$ is continuous in x and admits a minimum value at $x_0 = (\kappa/2\omega)^2 - (1+\delta)$ which is

$$f(x_0) = -\left(\frac{\kappa^2}{4\omega} + \delta\omega\right), \tag{A.4}$$

and takes only negative values since all the physical parameters are assumed to be positive. As a first consequence, if x_0 is a positive integer, $f(x_0)$ represents the fundamental energy level of $H_{D(+)}$ and even of $H_{\rm JC}$ (since $\varepsilon_0^- = \Delta/2 \geq 0$). If x_0 is not a positive integer, the fundamental level is obtained for $n = [x_0]$ or $[x_0]+1$ where $[x_0]$ is the integer part of x_0.

Another consequence is that if $x_0 \leq 0$, or equivalently, if

$$\frac{\kappa}{\omega} \leq 2\sqrt{1+\delta}, \tag{A.5}$$

the first energy level is $f(0)$ and, since the function is strictly increasing for positive values of x, we have a strictly increasing spectrum beginning with

$$\varepsilon_0^+ = \omega\left(1 - \frac{\kappa}{\omega}\sqrt{\delta+1}\right). \tag{A.6}$$

Let us note again that this quantity may be negative. We also see, by shifting the energy spectrum by ε_0^+, i.e. by taking

$$E_{n,(+)} = \varepsilon_n^+ - \varepsilon_0^+, \tag{A.7}$$

that we find a strictly increasing spectrum beginning with the zero fundamental energy. This is the case where we are satisfying all the hypotheses for the construction of our GCS.

The case where $x_0 > 0$, or equivalently

$$\frac{\kappa}{\omega} > 2\sqrt{1+\delta}, \tag{A.8}$$

is the one where problems may occur. Indeed, we see that for integer values $n > x_1$, where x_1 is the unique solution of $f(x_1) = f(0)$, the function is strictly increasing and no degeneracy occurs. But now for $n \in [0, x_1]$, the function is not monotonic and we could have degeneracies of some of the corresponding energy levels. The possible degeneracies occur if $F(p,q) = 0$ for some integers $p \in [0, x_0[$ and $q \in]x_0, x_1]$ so we will examine the behavior of $F(x,y) =$ (A.4). First, let us determine the value x_1 by solving $F(x_1, 0) = 0$. We immediately find

$$x_1 = \left(\frac{\kappa}{\omega} - \sqrt{1+\delta}\right)^2 - (1+\delta). \tag{A.9}$$

Now, we have

$$F(x,y) = \omega(\sqrt{x+1+\delta} - \sqrt{y+1+\delta})\left(\sqrt{x+1+\delta} + \sqrt{y+1+\delta} - \frac{\kappa}{\omega}\right) = \omega F_1(x,y)F_2(x,y). \tag{A.10}$$

If $x = y$, it is trivially zero and if $x \neq y$, the discussion of the zeros of $F_2(x,y)$ will give the admissible pairs of integers (p,q) for which the degeneracy is $\varepsilon_p^+ = \varepsilon_q^+$. It is equivalent to find all integers p and q, with $p \in [0, x_0[$ and $q \in]x_0, x_1]$ such that

$$q = \left(\frac{\kappa}{\omega} - \sqrt{p+1+\delta}\right)^2 - (1+\delta). \tag{A.11}$$

This means that, if we have degenerated states, they all appear in pairs and it can exist many pairs of degenerated states. Let us illustrate this fact by some simple examples where we are in the condition (A.8). Taking $\delta = 0$ and $\omega = 1$, we have $\kappa > 2$. The following values of κ are taken to illustrate four spectra, i.e. one without degeneracy, one for which the fundamental level is degenerated, one with two pairs of degenerated levels and, finally, one with three pairs of degenerated levels.

1) $\kappa = 2\sqrt{5}$: there is no degeneracy but the levels ε_k^+ for $k \in [0, 12[$ must be reordered to have an increasing spectrum. Indeed the list of values of ε_k^+ for $k = 0, \ldots, 19$ is

$$\{ \quad -3.47214, -4.32456, -4.74597, -4.94427, -5., -4.95445, -4.83216, -4.64911, -4.41641, -4.14214, -3.8324, -3.49193, -3.12452, -2.7332, -2.32051, -1.88854, -1.43909, -0.973666, -0.493589, 0\};$$

2) $\kappa = 2 + \sqrt{5}$: the fundamental level is degenerated and corresponds to $\varepsilon_3^+ = \varepsilon_4^+ = -2\sqrt{5}$. The other levels are not degenerated but must be reordered again. Indeed, the values of ε_k^+ for $k = 0, \ldots$ 17 are given by

$$\{ \quad -3.23607, -3.9907, -4.33708, -4.47214, -4.47214, -4.37621, -4.20758, -3.98141, -3.7082, -3.39562, -3.04945, -2.67417, -2.27336, -1.84992, -1.40622, -0.944272, -0.465756, -0.0278856\};$$

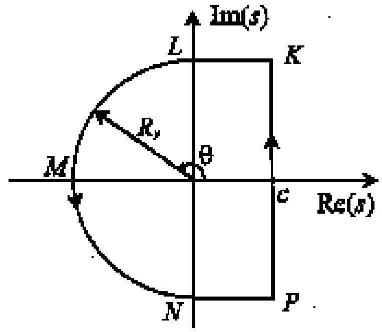

Figure 2: The integration contour for the inverse Mellin transform.

3) $\kappa = 6$: two pairs of levels are degenerated since $\varepsilon_0^+ = \varepsilon_{24}^+ = -5$ and $\varepsilon_3^+ = \varepsilon_{15}^+ = -8$. The list of values of ε_k^+ for $k = 0, \ldots, 24$ is

{ −5., −6.48528, −7.3923, −8., −8.41641, −8.69694, −8.87451, −8.97056, −9., −8.97367, −8.89975, −8.78461, −8.63331, −8.44994, −8.2379, −8., −7.73863, −7.45584, −7.15339, −6.83282, −6.49545, −6.14249, −5.77499, −5.39388, −5.}

4) $\kappa = 8$: three pairs of levels are degenerated since we have $\varepsilon_0^+ = \varepsilon_{48}^+ = -7$, $\varepsilon_3^+ = \varepsilon_{35}^+ = -12$ and $\varepsilon_8^+ = \varepsilon_{24}^+ = -15$.

Appendix B

In this appendix, we evaluate the functions $h_{(\pm)}(u)$ given in eq. (4.58), i.e.,

$$h_{(\pm)}(u) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} E_{(\pm)}(s)u^{-(s+1)}\,ds, \quad c \in \mathbb{R}, \tag{B.1}$$

where the functions $E_{(\pm)}(s)$ are defined in eq. (4.59).

Let us first consider the computation of $h_{(+)}(u)$. This requires to make an analytic prolongation of $E_+(n)$ to cover all the complex plane. Indeed, using the fact that

$$E_{(+)}(n) = E_+^0(n) = E_{n,(+)}E_{n-1,(+)}\ldots E_{n-p+1,(+)}E_{(+)}^0(n-p), \tag{B.2}$$

we define the analytic prolongation as the function

$$E_{(+)}^p(s) = \frac{E_{(+)}^0(s+p)}{E_{s+p,(+)}E_{s+p-1,(+)}\ldots E_{s+1,(+)}}, \tag{B.3}$$

which is well defined for every $s \in \mathbb{C}$ satisfying the conditions

$$E_{s+k,(+)} \neq 0, \quad k = p,\ p-1, \ldots, 1. \tag{B.4}$$

Except from these points, the function $E_+^p(s)$ coïncides with $E_+^0(s)$ and is analytic. It has $2p$ simple poles at the points

$$s \equiv s_{k,(+)}^+ = -k \quad \text{and} \quad s \equiv s_{k,(+)}^- = -k + \frac{\kappa^2}{\omega^2} - \frac{2\kappa}{\omega}\sqrt{1+\delta},\ k \in \mathbb{N} - \{0\}. \tag{B.5}$$

To evaluate $h_+(u)$, let us consider the contour of integration given in Fig. 2, where the radius R_p is given by

$$R_p = \left|\frac{\kappa^2}{\omega^2} - \frac{2\kappa}{\omega}\sqrt{1+\delta}\right| + p + \frac{1}{2}. \tag{B.6}$$

The function $E_+^p(s)$ has no singularity along this contour. Then, according to the residue theorem, we have

$$\frac{1}{2\pi i}\oint E_{(+)}^p(s)u^{-(s+1)}\,ds = \sum_{k=1}^{p} \mathrm{Res}\big[E_{(+)}^p(s)u^{-(s+1)}, s = s_k^+\big] + \mathrm{Res}\big[E_{(+)}^p(s)u^{-(s+1)}, s = s_k^-\big]. \tag{B.7}$$

By taking the limit as $p \to \infty$, we obtain

$$\text{Sum of residues} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} E^{p}_{(+)}(s)u^{-(s+1)}\,ds + \frac{1}{2\pi i}\lim_{p\to\infty}\int_{\mathrm{KLMNP}} E^{p}_{(+)}(s)u^{-(s+1)}\,ds. \tag{B.8}$$

By the same considerations as in [6], one can prove that

$$\lim_{p\to\infty}\int_{\mathrm{KLMNP}} E^{p}_{(+)}(s)u^{-(s+1)}\,ds = 0 \tag{B.9}$$

and consequently we have $h_+(u)$ is given by the sum of residues of $E^p_+(s)$, when $p \to \infty$, at the poles $s^+_{k,(+)}$ and $s^-_{k,(+)}$ given by eq. (B.5). Note that we can write

$$\mathrm{Res}\big[E_{(+)}(s)u^{-(s+1)}, s = s^{\pm}_{k,(+)}\big] = B^{\pm}_{k,(+)}u^{-(s^{\pm}_{k,(+)}+1)}, \tag{B.10}$$

where

$$B^{\pm}_{k,(+)} = \lim_{s\to s^{\pm}_{k,(+)}} (s - s^{\pm}_{k,(+)})E_{(+)}(s). \tag{B.11}$$

To evaluate the coefficients $B^{\pm}_{k,(+)}$, we factorise the inverse of $E_+(s)$ under the form

$$\frac{1}{E_{(+)}(s)} = e^{c_+ s}\prod_{k=1}^{\infty}\left(1 - \frac{s}{s^+_{k,(+)}}\right)e^{s/s^+_{k,(+)}}\prod_{k=1}^{\infty}\left(1 - \frac{s}{s^-_{k,(+)}}\right)e^{s/s^-_{k,(+)}}, \quad c_+ \in \mathbb{R}. \tag{B.12}$$

The measure may then be written as

$$h_+(u) = \sum_{k=1}^{\infty}\big[B^+_{k,(+)}u^{-(s^-_{k,(+)}+1)} + B^+_{k,(+)}u^{-(s^-_{k,(+)}+1)}\big], \tag{B.13}$$

where the expressions of $B^+_{k,(+)}$ and $B^-_{k,(+)}$ are given below.

It is easy to see that the computation of the measure $h_-(u)$ is similar and we get

$$h_-(u) = \sum_{k=1}^{\infty}\big[B^+_{k,(-)}u^{-(s^-_{k,(-)}+1)} + B^-_{k,(-)}u^{-(s^-_{k,(-)}+1)}\big], \tag{B.14}$$

where

$$s^+_{k,(-)} = -k + \frac{\kappa^2}{\omega^2} + \frac{\Delta}{\omega}, \quad s^-_{k,(-)} = -k, \qquad k \in \mathbb{N} - \{0\}. \tag{B.15}$$

Now, we get for the coefficients $B^+_{k,(-)}$ and $B^-_{k,(-)}$ which can be summarized as

$$B^+_{k,(\pm)} = -s^+_{k,(\pm)}e^{-(c_{(\pm)}s^+_{k,(\pm)}+1)} \times \prod_{l\neq k}^{\infty}\frac{s^+_{l,(\pm)}e^{-(s^+_{k,(\pm)}/s^+_{l,(\pm)})}}{s^+_{l,(\pm)} - s^+_{k,(\pm)}}\prod_{l=1}^{\infty}\frac{s^-_{l,(\pm)}e^{-(s^+_{k,(\pm)}/s^-_{l,(\pm)})}}{s^-_{l,(\pm)} - s^+_{k,(\pm)}}, \tag{B.16}$$

$$B^-_{k,(\pm)} = -s^-_{k,(\pm)}e^{-(c_{(\pm)}s^-_{k,(\pm)}+1)} \times \prod_{l=1}^{\infty}\frac{s^+_{l,(\pm)}e^{-(s^-_{k,(\pm)}/s^+_{l,(\pm)})}}{s^+_{l,(\pm)} - s^-_{k,(\pm)}}\prod_{l\neq k}^{\infty}\frac{s^-_{l,\pm}e^{-(s^-_{l,(\pm)}/s^-_{k,(\pm)})}}{s^-_{l,(\pm)} - s^-_{k,(\pm)}}, \quad c_{(\pm)} \in \mathbb{R}. \tag{B.17}$$

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