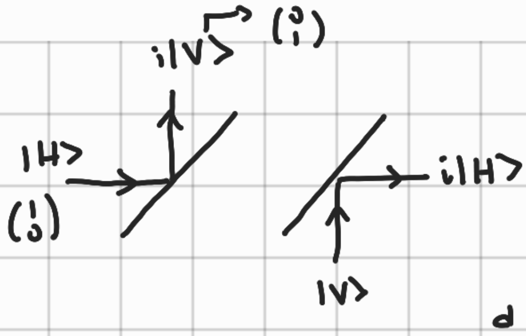
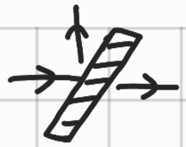


$$S = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$$



$$S|H\rangle = \frac{1}{\sqrt{2}}\left(|H\rangle + i|V\rangle\right) \qquad S|V\rangle = \frac{1}{\sqrt{2}}\left(i|H\rangle + |V\rangle\right)$$

Observable

$$A = \sum_{i=1}^{d} a_i |\varphi_i\rangle\langle\varphi_i|$$

$$E(a_i) = \sum_{i=1}^{d} a_i \, \text{prob}(i) = \sum_{i=1}^{d} a_i \left| \underbrace{\langle\varphi_i|\psi\rangle}_{\langle\psi|\varphi_i\rangle\langle\varphi_i|\psi\rangle} \right|^2$$

$$E(a_i) = \langle\psi| \underbrace{\left(\sum_{i=1}^{d} a_i |\varphi_i\rangle\langle\varphi_i|\right)}_{A} |\psi\rangle$$

Ex

$\Pi_a = |a\rangle\langle a|$ $\qquad$ $\text{Prob}(p=+1) = |\langle\theta|\alpha\rangle|^2 = \langle\theta|\Pi_a|\theta\rangle$

$\Pi_{a_\perp} = |a_\perp\rangle\langle a_\perp|$

$P_a = \underset{p}{(+1)}\,\Pi_a + \underset{p}{(-1)}\,\Pi_{a_\perp}$ $\qquad$ $\text{Prob}(p=-1) = |\langle\theta|\alpha_\perp\rangle|^2 = \langle\theta|\Pi_{a_\perp}|\theta\rangle$

$$E(p) = (+1)\,\text{Prob}(p=+1) + (-1)\,\text{Prob}(p=-1) = \langle\theta|P_a|\theta\rangle$$

$$\langle\theta|P_a^2|\theta\rangle = E(p^2) \rightarrow \text{Var}(p) = E(p^2) - (E(p))^2 = \langle\theta|P_a^2|\theta\rangle - \langle\theta|P_a|\theta\rangle^2$$

Spin 1/2 & Magnetic Moments

$$\vec{M} = \gamma\frac{\hbar}{2}\vec{S}$$

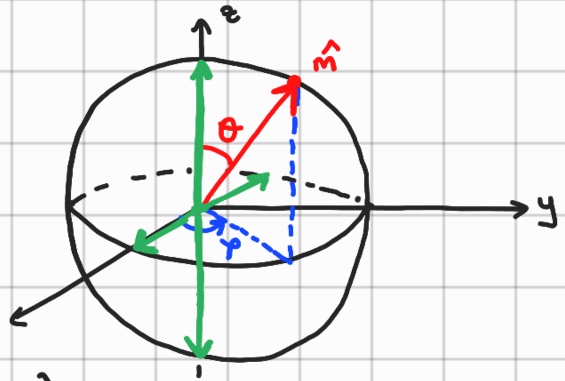
$$\vec{S} = (\sigma_x, \sigma_y, \sigma_z)$$

$$\boxed{H = -\gamma\frac{\hbar}{2}\vec{S}\cdot\vec{B}}$$

$$\overset{\text{observable}}{S_z} = (+1)|\uparrow\rangle\langle\uparrow| + (-1)|\downarrow\rangle\langle\downarrow| = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_x\sigma_y = -\sigma_y\sigma_x$$

$$[\sigma_x, \sigma_y] = 2i\sigma_z$$



Ex $\vec{B} = (0, 0, B)$

$$\hat{H} = -\underbrace{\gamma\frac{\hbar}{2}B}_{-\hbar\omega_0/2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{-\hbar\overset{\text{Larmor Freq.}}{\omega_0}}{2}\left(|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|\right)$$

$$|\psi\rangle = \cos\frac{\theta}{2}|\uparrow\rangle + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)|\downarrow\rangle$$

$$\boxed{i\hbar \frac{d}{dt} U_t = H U_t}$$

Dynamics $|\psi_t\rangle = U_t|\psi_0\rangle$

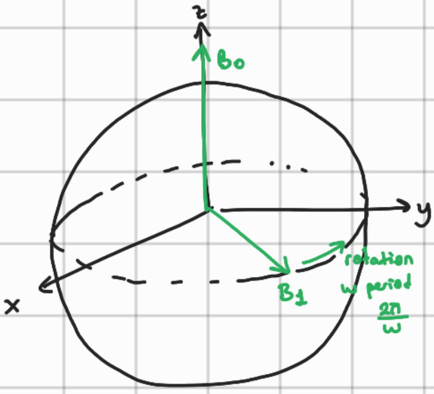
$$U_t = e^{\frac{-itH}{\hbar}}$$ where H is time independent

$$H = \begin{pmatrix} -\frac{\hbar\omega_0}{2} & 0 \\ 0 & +\frac{\hbar\omega_0}{2} \end{pmatrix} \xrightarrow{e^{-\frac{it}{\hbar}H}} \begin{pmatrix} e^{\frac{i\omega_0 t}{2}} & 0 \\ 0 & e^{\frac{i\omega_0 t}{2}} \end{pmatrix}$$

$$|\psi_0\rangle = \cos\left(\frac{\theta}{2}\right)|\uparrow\rangle + e^{i\varphi}\sin\frac{\theta}{2}|\downarrow\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}e^{i\varphi} \end{pmatrix} \xrightarrow{|\psi_t\rangle = U_t|\psi_0\rangle} \begin{pmatrix} \cos\theta/2 \\ \sin\frac{\theta}{2} e^{i(\varphi-\omega_0 t)} \end{pmatrix}$$

Larmor Precession

Dynamics of Spin 1/2



Bloch Sphere

$$H(t) = -\gamma\frac{\hbar}{2}\vec{B}(t)\cdot\vec{\sigma} = -\frac{\gamma\hbar}{2}B_0\sigma_z - \frac{\gamma\hbar}{2}B_1\{\sigma_x\cos\omega t + \sigma_y\sin\omega t\}$$

$$H(t) = \frac{-\hbar\omega_0}{2}\sigma_z - \frac{\hbar\omega_1}{2}\{\sigma_+ e^{-i\omega t} + \sigma_- e^{i\omega t}\}$$

$$i\hbar\frac{d}{dt}\tilde{U}_t = \tilde{H}\tilde{U}_t \quad (\tilde{H} \text{ does not depend on time})$$

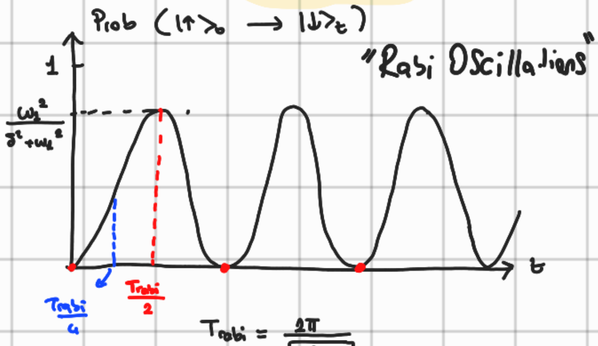
$$\tilde{U}_t = \exp\left(\frac{-it}{\hbar}\tilde{H}\right) \quad \text{"In Rotating Frame"}$$

$$\underset{\text{Rot. Frame}}{|\tilde{\psi}\rangle} = e^{-i\frac{\omega t}{2}\sigma_z}\underset{\text{Lab Frame}}{|\psi\rangle} \xrightarrow{K = \frac{-\hbar\omega}{2}\sigma_z} \underset{\text{Rot. Frame}}{|\tilde{\psi}_t\rangle} = e^{\frac{itK}{\hbar}}|\psi_t\rangle \longrightarrow \tilde{U}_t = e^{\frac{itK}{\hbar}}U_t$$

δ (tuning parameter)

$$\underset{\text{Rot. Frame}}{\tilde{H}} = \frac{-\hbar(\omega_0-\omega)}{2}\sigma_z - \frac{\hbar\omega_1}{2}\sigma_x$$

Time Independent



$$T_{rabi} = \frac{2\pi}{\sqrt{\delta^2+\omega_1^2}}$$

Heisenberg Hamiltonian

Interaction between magnetic moments

$$H = \hbar J\,\vec{\sigma_1}\otimes\vec{\sigma_2}$$

$$H = \hbar J\{\sigma_1^x\otimes\sigma_2^x + \sigma_1^y\otimes\sigma_2^y + \sigma_1^z\otimes\sigma_2^z\}$$

Exchange interaction term

$$H = \hbar J\,\sigma_1^z\otimes\sigma_2^z + 2\hbar J(\underbrace{\sigma_1^+\otimes\sigma_2^-}_{|\uparrow\rangle\langle\downarrow|\ \ |\downarrow\rangle\langle\uparrow|} + \sigma_1^-\otimes\sigma_2^+)$$

$|\uparrow\downarrow\rangle\langle\downarrow\uparrow| + |\downarrow\uparrow\rangle\langle\uparrow\downarrow|$

$$\begin{pmatrix} 0 & & \\ & \boxed{\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}} & \\ & & 0 \end{pmatrix}$$

$$|\uparrow\uparrow\rangle\langle\uparrow\uparrow| + |\downarrow\downarrow\rangle\langle\downarrow\downarrow| - |\uparrow\downarrow\rangle\langle\uparrow\downarrow| - |\downarrow\uparrow\rangle\langle\downarrow\uparrow|$$

$$\begin{array}{c} \uparrow\uparrow \quad \uparrow\downarrow \quad \downarrow\uparrow \quad \downarrow\downarrow \\ \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & 1 \end{pmatrix}_{4\times4} \end{array}$$

$$H = \hbar J\begin{pmatrix} \boxed{1} & & \\ & \boxed{\begin{smallmatrix} -1 & 2 \\ 2 & -1 \end{smallmatrix}} & \\ & & \boxed{1} \end{pmatrix}$$

Eigen Values and EigenVectors

$$|\uparrow\uparrow\rangle,\ |\downarrow\downarrow\rangle,\ \underbrace{\frac{|\uparrow\downarrow\rangle+|\downarrow\uparrow\rangle}{\sqrt{2}}}_{|\Phi_+\rangle},\ \underbrace{\frac{|\uparrow\downarrow\rangle-|\downarrow\uparrow\rangle}{\sqrt{2}}}_{|\Phi_-\rangle}$$

$+\hbar J \quad +\hbar J \quad +\hbar J \quad -3\hbar J$

$\hbar J$

$-3\hbar J$

$$U_t = \begin{pmatrix} e^{-itJ} & & & \\ & e^{-itJ} & & \\ & & e^{-itJ} & \\ & & & e^{+3itJ} \end{pmatrix}$$

in eigen basis

$$U_t = e^{itJ}|\uparrow\uparrow\rangle\langle\uparrow\uparrow| + e^{itJ}|\downarrow\downarrow\rangle\langle\downarrow\downarrow| + e^{itJ}|\Phi_+\rangle\langle\Phi_+| + e^{-3itJ}|\Phi_-\rangle\langle\Phi_-|$$

SWAP Gate has same eigenvectors with H_{Heis}

$$e^{-\frac{it}{\hbar} H_{Heis}} \overset{@\, t=\frac{\pi}{4J}}{=} \underbrace{e^{-i\frac{\pi}{4}}}_{\text{global phase}} \text{SWAP}$$

CNOT $\longrightarrow$ $H_{\text{anisotropic}} = \hbar J \sigma_1^z \otimes \sigma_2^z$

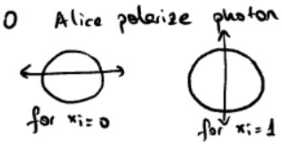
Quantum Key Distribution (BB84)

1) Encoding Phase in A's Lab

$e_1, e_2, \ldots, e_N \in \{0,1\}^N$ $\text{prob}(e_i = {}^0_1) = 1/2$

$x_1, x_2, \ldots, x_N \in \{0,1\}^N$ $\text{prob}(x_i = {}^0_1) = 1/2$

If $e_i = 0$ Alice polarize photon in $|H\rangle$ or $|V\rangle$

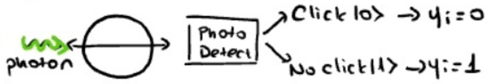


If $e_i = 1$ Alice polarize photon in $|+\rangle, |-\rangle$

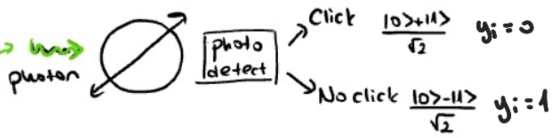
2) Decoding Phase in B's Lab

$d_1, d_2, \ldots, d_N \in \{0,1\}^N$ $\text{prob}(e_i = {}^0_1) = 1/2$

If $d_i = 0$ → Measure in Z basis $\{|0\rangle, |1\rangle\}$



If $d_i = 1$ → Measure in X basis $\left\{\frac{|0\rangle+|1\rangle}{\sqrt{2}}, \frac{|0\rangle-|1\rangle}{\sqrt{2}}\right\}$



3) Public Communication Phase

A&B reveal publicly $(e_1, \ldots, e_N)$ & $(d_1, \ldots, d_N)$

at time instant $i = 1 \ldots N$

$p(e_i = d_i) = 1/2$

$p(e_i \neq d_i) = 1/2$

<u>Lemma</u>:

$P(x_i = y_i \mid e_i = d_i) = 1$

$P(x_i \neq y_i \mid e_i = d_i) = 0$

$P(x_i = y_i \mid e_i \neq d_i) = 1/2$

$P(x_i \neq y_i \mid e_i \neq d_i) = 1/2$

→ if $e_i = d_i$ $(0=0)$ or $(1=1)$

$$\left.\begin{array}{l} |\langle y_i | H^0 | x_i\rangle|^2 \\ |\langle y_i | \underbrace{H^2}_{I} | x_i\rangle|^2 \end{array}\right\} |\langle y_i | x_i\rangle|^2 = \begin{cases} 1, & x_i = y_i \\ 0, & x_i \neq y_i \end{cases}$$

→ if $e_i \neq d_i$ $(0,1)$ or $(1,0)$

$|\langle y_i | H | x_i\rangle|^2 = \frac{1}{2}$

$\langle 0|H|0\rangle = \frac{1}{\sqrt{2}}\langle 0|(|0\rangle+|1\rangle) = 1/\sqrt{2}$ $\langle 1|H|0\rangle = \langle 1|\frac{|0\rangle+|1\rangle}{\sqrt{2}} = 1/\sqrt{2}$

Proof of lemma:

$\underbrace{e_i = 0 \quad H^0|x\rangle = |x\rangle}$ $\underbrace{H^1|x\rangle \quad e_i = 1}$

Alice sends a qubit to Bob : $H^{e_i}|x\rangle \in \left\{|0\rangle, |1\rangle, \frac{|0\rangle+|1\rangle}{\sqrt{2}}, \frac{|0\rangle-|1\rangle}{\sqrt{2}}\right\}$

Bob receives a perfect state and measures it: $\left\{\underbrace{|0\rangle, |1\rangle}_{d_i=0}, \underbrace{\frac{|0\rangle+|1\rangle}{\sqrt{2}}, \frac{|0\rangle-|1\rangle}{\sqrt{2}}}_{d_i=1}\right\}$

$\underbrace{H^{e_i}|x_i\rangle}_{\text{before meas.}} \rightarrow \boxed{\text{Bob}} \rightarrow \underbrace{H^{d_i}|y_i\rangle}_{\text{after meas.}}$

$|(\langle y_i | H^{d_i})(H^{e_i}|x_i\rangle)|^2 = |\langle y_i | H^{d_i} H^{e_i} | x_i\rangle|^2$

Measurement Attack of Eavesdropper

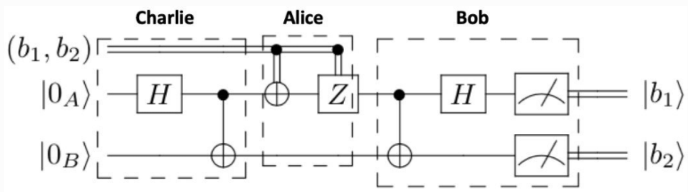
$$\text{Prob}_{Eve}(x_i = y_i \mid e_i = d_i) = \frac{3}{4}$$

$$\text{Prob}(x_i = y_i \mid \underbrace{e_i = d_i, E_i = e_i}_{d_i = E_i}) P(E_i = e_i) + \text{Prob}(x_i = y_i \mid \underbrace{e_i = d_i, E_i \neq e_i}_{d_i \neq E_i}) P(E_i \neq e_i) = 1 + \frac{1}{2}\cdot\frac{1}{2} = \frac{3}{4}$$

One-time pad generation: $\{x_i = y_i \mid \text{with } i \text{ at } e_i = d_i\}$ → Avg. length $= N/2$

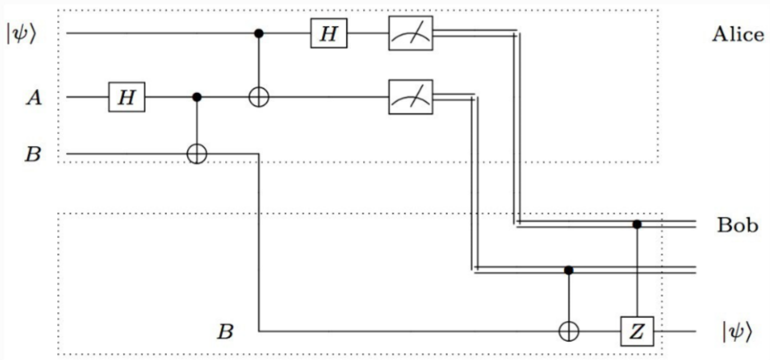
$P(x_i = y_i \mid e_i = d_i) P(e_i = d_i) = 1/2$

Dense Coding



Transmit two classical bits using 1 qubit

Teleportation



Transmit 1 qubit using 2 classical bits

CSHS (Bell Inequality)

For entangled states (Bell)

$$X_{QM} = \cos 2(\alpha-\beta) + \cos 2(\alpha'-\beta) - \cos 2(\alpha-\beta') + \cos 2(\alpha'-\beta')$$

$A\otimes B + A\otimes B' - A'\otimes B + A'\otimes B'$

$$\mathrm{Tr}(\underbrace{\mathcal{B}}_{\text{observable}}\, \rho_{Bell}) = \mathrm{Tr}(\mathcal{B}\, |B_{00}\rangle\langle B_{00}|) = \mathrm{Tr}(\langle B_{00}|\mathcal{B}|B_{00}\rangle)$$

observable

$$A = (+1)|\varphi\rangle\langle\varphi| + (-1)|\varphi_\perp\rangle\langle\varphi_\perp|$$

$$X = \langle\psi|\mathcal{B}|\psi\rangle$$

Average

For $A\otimes B$ $\quad \mathrm{Tr}(\mathcal{B}\rho_{stat}) = \frac{1}{2}\langle 00|\mathcal{B}|00\rangle + \frac{1}{2}\langle 11|\mathcal{B}|11\rangle$

$$\frac{1}{2}\langle 0|A|0\rangle\langle 0|B|0\rangle + \frac{1}{2}\langle 1|A|1\rangle\langle 1|B|1\rangle = \cos(2\alpha)\cos(2\beta)\ldots$$

Density Matrix

$$\rightarrow \rho = \sum_k p_k |e_k\rangle\langle e_k|$$

Partial Traces

$$\rightarrow \mathrm{Tr}_B \underbrace{\rho_{Bell}}_{|B_{00}\rangle\langle B_{00}|} = \frac{1}{2}\{|0\rangle\langle 0|_A + |1\rangle\langle 1|_A\} = \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}_A$$

$$\mathrm{Tr}_A \rho_{Bell} = \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}_B$$

$$\rho = \frac{1}{2}(I + \vec{a}\cdot\vec{\sigma})$$

Reduced Density Matrix

$$\rightarrow \rho_S = \mathrm{Tr}_E\, \rho_{SUE}$$

$$\rho_{SUE}(t) = U_{tot}(t)\, \rho_{SUE}\, U_{tot}(t)^\dagger$$

$$\rho_S(t) = U_S\,(\mathrm{Tr}_E\, \rho_{SUE})\, U_S^\dagger$$

$$\rho_t = U_t \rho_0 U_t^\dagger \rightarrow \rho_S(t) = U_S(t)\rho_S(0)U_S^\dagger(t)$$

$$\rho_A = \sum_{i=1}^{k} \lambda_i |\varphi_i\rangle_A \langle\varphi_i|_A$$

$$\rho_B = \sum_{i=1}^{k} \lambda_i |\psi_i\rangle_B \langle\psi_i|_B$$

$$|\Psi\rangle = \sum_{i=1}^{k} \sqrt{\lambda_i}\, |\varphi_i\rangle_A \otimes |\psi_i\rangle_B$$

Spectral Decomposition:

Orthonormal basis

$$\rho = \sum_{\alpha=1}^{D} \lambda_\alpha |v_\alpha\rangle\langle v_\alpha| \longrightarrow \sum_\alpha \lambda_\alpha = 1$$

$$S(\rho) = -\mathrm{Tr}(\rho\log\rho)$$

For $|B_{00}\rangle \rightarrow$
$$\left.\begin{aligned} S(\rho_A) &= \log 2 \\ S(\rho_B) &= \log 2 \end{aligned}\right\}\ \text{max. local entropy}$$

$S(AB)_{total} = 0 \rightarrow$ since $|B_{00}\rangle$ is pure