Homework 11
Quantum Information Processing

Exercise 1 *Useful identity for realisation of NOT gate by NMR or quantum circuits.*

We consider two qubits (for example: spins 1/2 or general two-level systems) and the operations:

- Rotations of angle $\frac{\pi}{2}$ around z axis for each spin

$$R_1 = \exp\left(-i\frac{\pi}{2}\frac{\sigma_1^z}{2}\right) \text{ et } R_2 = \exp\left(-i\frac{\pi}{2}\frac{\sigma_2^z}{2}\right)$$

- Hadamard gate $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.
- Unitary evolution $U = \exp\left(-i\frac{t}{\hbar}\mathcal{H}\right)$ associated to the (anitropic) Heisenberg hamiltonian

$$\mathcal{H} = \hbar J \sigma_1^z \otimes \sigma_2^z.$$

We let the system evolve for a time $t = \frac{\pi}{4J}$.

a) Calculate the product

$$(I \otimes H)\, U\, (R_1 \otimes R_2)\, (I \otimes H)$$

where I is the 2×2 identity matrix.

b) Show that this product is equivalent to a 4×4 CNOT gate defined by

$$\text{CNOT}|x\rangle \otimes |y\rangle = |x\rangle \otimes |y \oplus x\rangle$$

Here $x, y \in \{0, 1\}$ and $\oplus$ addition modulo 2. The bit $|x\rangle$ is called control bit. The second qubit is flipped iff the control bit is in the state $|1\rangle$.

Exercise 2 *Refocusing technique*

Consider the hamiltonian of Heisenberg type (here only the “z” term is present).

$$\mathcal{H} = \hbar J \sigma_1^z \otimes \sigma_2^z$$

for the interaction between two qubits. Let

$$R_1 = \exp\left(i\pi\frac{\sigma_1^x}{2}\right),$$

the π-pulse (or rotation around x) acting on the first spin. This is a one-qubit opertaion and can be realized by NMR techniques seen in class comme vu au cours (constant + rotating field).

We consider the Heisenberg evolution of the two spins for a time $\frac{t}{2}$, followed by a π-pulse, followed by the Heisenberg evolution for a time $\frac{t}{2}$, followed by a final π-pulse. The total evolution is

$$U_{tot} = (R_1 \otimes \mathbb{I}_2)\, e^{-i\frac{t}{2}\frac{\mathcal{H}}{\hbar}}\, (R_1 \otimes \mathbb{I}_2)\, e^{-i\frac{t}{2}\frac{\mathcal{H}}{\hbar}}$$

a) Show the general identity valid for all times t:

$$(R_1 \otimes \mathbb{I}_2)\, e^{-\frac{it}{\hbar}\mathcal{H}}\, (R_1 \otimes \mathbb{I}_2)\, e^{-\frac{it}{\hbar}\mathcal{H}} = \mathbb{I}_1 \otimes \mathbb{I}_2$$

b) In practice $J << 1$. Can you say a few words on the physical interpretation of this identity ?

Exercise 1

a) Calculate the product

$$(I \otimes H)\,U\,\overbrace{(R_1 \otimes R_2)\,(I \otimes H)}^{A}$$

where I is the 2×2 identity matrix. $\rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad R_1 = \exp\left(-i\frac{\pi}{2}\frac{\sigma_1^z}{2}\right), \quad U = \exp\left(\frac{-it}{\hbar}\mathcal{H}\right) \quad \text{where } t = \frac{\pi}{4J}$$

$$\mathcal{H} = \hbar J \sigma_1^z \otimes \sigma_2^z$$

$$\exp\left(i\frac{\alpha}{2}\, n\cdot\sigma\right) = \left(\cos\frac{\alpha}{2}\right) I + i\left(\sin\frac{\alpha}{2}\right) n\cdot\vec{\sigma}$$

$$R_1 = \cos\left(\frac{\pi}{4}\right) I + i\sin\left(\frac{\pi}{4}\right)\sigma_1^z \quad \sigma_1^z \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1+i & 0 \\ 0 & 1-i \end{pmatrix}$$

$$R_1 \otimes R_2 = \frac{1}{2}\begin{pmatrix} 2i & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2i \end{pmatrix} = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}$$

$$(1+i)^2 = \cancel{1} + 2i + \cancel{i^2} = 2i$$

$$(1+i)(1-i) = 1 - \underbrace{i^2}_{-1} = 2$$

$$\underbrace{(R_1 \otimes R_2)(I \times H)}_{A} = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}\frac{1}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} i & i & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -i & i \end{pmatrix}$$

$$U\left(t = \frac{\pi}{4J}\right) = \exp\left(-i\frac{\pi}{4\cancel{J}}\cancel{J}\,\sigma_1^z \otimes \sigma_2^z\right)$$

$$U\left(t = \frac{\pi}{4J}\right) = \cos\frac{\pi}{4}\, I - i\sin\frac{\pi}{4}\,\sigma_1^z \otimes \sigma_2^z$$

$$I \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^z \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_1^z \otimes \sigma_2^z \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1-i & 0 & 0 & 0 \\ 0 & 1+i & 0 & 0 \\ 0 & 0 & 1+i & 0 \\ 0 & 0 & 0 & 1-i \end{pmatrix}$$

$$UA = \frac{1}{\sqrt{2}}\begin{pmatrix} 1-i & 0 & 0 & 0 \\ 0 & 1+i & 0 & 0 \\ 0 & 0 & 1+i & 0 \\ 0 & 0 & 0 & 1-i \end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} i & i & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -i & i \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1+i & 1+i & 0 & 0 \\ 1+i & -(1+i) & 0 & 0 \\ 0 & 0 & (1+i) & (1+i) \\ 0 & 0 & -(1+i) & i+1 \end{pmatrix}$$

$$(1-i)\,i = i - \underbrace{i^2}_{-1} = i+1$$

$$-(1-i)\,i = -(i - \underbrace{i^2}_{-1}) = -(i+1)$$

$$= \frac{(i+1)}{2}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

$$(I \otimes H)\,UA = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \frac{(i+1)}{2}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

$$= \frac{(i+1)}{2\sqrt{2}}\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{pmatrix} = \underbrace{\frac{(i+1)}{\sqrt{2}}}_{e^{i\pi/4}}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$e^{i\pi/4} \rightarrow$ global phase

$\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$

b)

$$CNOT = \begin{array}{c} \begin{array}{cccc} 00 & 01 & \overset{11}{\uparrow}\,10 & \overset{10}{\uparrow}\,11 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

they're same

Exercise 2

a) Show the general identity valid for all times t:

$$\underbrace{(R_1 \otimes \mathbb{I}_2)}_{C}\,\underbrace{e^{-\frac{it}{\hbar}\mathcal{H}}}_{B}\,\underbrace{(R_1 \otimes \mathbb{I}_2)\, e^{-\frac{it}{\hbar}\mathcal{H}}}_{A} = \mathbb{I}_1 \otimes \mathbb{I}_2$$

$\underbrace{BA}_{I_1 \otimes X_2}$

$$\mathcal{H} = \hbar J \sigma_1^z \otimes \sigma_2^z$$

$$R_1 = \exp\left(i\pi \frac{\sigma_1^x}{2}\right) = \underbrace{\cos\left(\frac{\pi}{2}\right)}_{0} I + \underbrace{i\sin\left(\frac{\pi}{2}\right)}_{1} \sigma_1^x = i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$\sigma_1^x \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$U = e^{-\frac{it}{\cancel{\hbar}}\cancel{\hbar} J \sigma_1^z \otimes \sigma_2^z} = \underbrace{\cos(Jt) I}_{\begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}} - i\sin(Jt)\, \underbrace{\sigma_1^z \otimes \sigma_2^z}_{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}$$

$$U = \begin{bmatrix} \cos(Jt) - i\sin(Jt) & 0 & 0 & 0 \\ 0 & \cos(Jt) + i\sin(Jt) & 0 & 0 \\ 0 & 0 & \cos(Jt) + i\sin(Jt) & 0 \\ 0 & 0 & 0 & \cos(Jt) - i\sin(Jt) \end{bmatrix}$$

$$\boxed{U = \begin{bmatrix} e^{-iJt} & & & 0 \\ & e^{iJt} & & \\ & & e^{iJt} & \\ 0 & & & e^{-iJt} \end{bmatrix}}$$

$$\underbrace{(R_1 \otimes I_2)\, e^{-it\frac{\mathcal{H}}{\hbar}}}_{A} = i\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} e^{-iJt} & 0 & 0 & 0 \\ 0 & e^{iJt} & 0 & 0 \\ 0 & 0 & e^{iJt} & 0 \\ 0 & 0 & 0 & e^{-iJt} \end{pmatrix} = i\begin{bmatrix} 0 & e^{iJt} & 0 & 0 \\ e^{-iJt} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-iJt} \\ 0 & 0 & e^{iJt} & 0 \end{bmatrix}$$

$R_1 \otimes I_2 \downarrow$

$$i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= i\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$BA = i\begin{bmatrix} e^{-iJt} & 0 & 0 & 0 \\ 0 & e^{iJt} & 0 & 0 \\ 0 & 0 & e^{iJt} & 0 \\ 0 & 0 & 0 & e^{-iJt} \end{bmatrix}\begin{bmatrix} 0 & e^{iJt} & 0 & 0 \\ e^{-iJt} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-iJt} \\ 0 & 0 & e^{iJt} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = (I_1 \otimes X_2)$$

$$CBA = i\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$= i\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I_1 \otimes I_2$$

global phase?

* Heisenberg Evolution for time t, followed by a P pulse,

Heisenberg Evolution for time t, followed by a final P pulse

=

Identity

b) In practise $J \ll 1$. What is the physical interpretation

× $J \to$ frequency $\to$ gate change

High frequency yields decrease in gate fidelity.

For a plausable circuit, gate shouldn't change easily. :)

Solution:

evolutions of nuclear spin ?

$J \ll 1$ $\quad \tau = \frac{\pi}{4J} \gg \pi$ The π-pulses of NMR are much faster than the evolution of nuclear spins. Thus by injecting two π-pulses at instants $\frac{\tau}{2}$ and τ we recover the initial state and everything looks as if the spins had not evolved.