References:

REPRESENTATIONS

$$R: G \longrightarrow GL(n, \mathbb{C})$$

$$\boxed{R(g_1 g_2) = R(g_1)R(g_2)}$$

G is Matrix group → Lie Group

Why? Applications → internal: classify semisimple Lie Groups
invariant theory

→ external: PARTICLE PHYSICS

SU(3)

Given a $R: G \longrightarrow GL(n, \mathbb{C})$, we get a Lie algebra representation

$$R_*: \mathfrak{g} \longrightarrow \mathfrak{gl}(n, \mathbb{C}) \quad \text{Linear map}$$

$$R_*[X, Y] = [R_* X, R_* Y]$$

$$R(\exp X) = \exp(R_* X) \quad \forall X \in \mathfrak{g}$$

⇒ R determines R_* by differentiation.

⇒ R^* determines $R(g)$ for all $g \in \exp \mathfrak{g}$

Does this determine $R(g)$ $\forall g \in G$?

If G is a path connected then yes, R is determined by R^* because G is generated as a group by $\exp(\mathfrak{g})$

Given $R_*: \mathfrak{g} \to \mathfrak{gl}(n, \mathbb{C})$ does $R(\exp X) = \exp(R_* X)$ this give well defined R?

Lie's theorem: Yes, if G is simply connected.

Complete Reducibility

<u>Def</u> A decomposition of a rep. $R: G \longrightarrow GL(n, \mathbb{C})$ is a splitting of $\mathbb{C}^n = V_1 \oplus \dots \oplus V_k$ where each $V_i \subseteq \mathbb{C}^n$ is a subrepresentation of $\mathbb{C}^n$. $R(g)v \in V_i$ whenever $v \in V_i$

$$R(g) = \begin{pmatrix} R(g)|_{V_1} & & 0 \\ & \ddots & \\ 0 & & R(g)|_{V_k} \end{pmatrix} \qquad R = R|_{V_1} \oplus \dots \oplus R|_{V_k}$$

direct sum

<u>Def</u> A subrep $V \subseteq \mathbb{C}^n$ is irreducible if it has no proper subreps.

Lemma: If $\mathbb{C}^n$ admits an invariant Hermitian inner product then the representation can be decomposed into irreducible summands.

Proof: Idea: If $V_i \subseteq \mathbb{C}^n$ is not irreducible then it contains a subrep. $U \subsetneq V_i$

Take $U^\perp$. This will be a subrep. & $V_i = U \oplus U^\perp$. This terminates because 1-d reps are irreducible.

orthogonal complement

Def. A Hermitian inner product is a map

$$\langle , \rangle : \mathbb{C}^n \times \mathbb{C}^n \longrightarrow \mathbb{C} \text{ such that}$$

$$v \cdot v = \sum v_k^2 \in \mathbb{C}$$

$$\langle v, v \rangle = \sum \bar{v}_k v_k \rightarrow \text{Real}$$

- $\langle v, v \rangle \in \mathbb{R}$ (positive unless $v=0$)
- $\langle v, u \rangle = \overline{\langle u, v \rangle}$
- $\langle u, av_1 + bv_2 \rangle = a\langle u, v_1 \rangle + b\langle u, v_2 \rangle$
- $\langle au_1 + bu_2, v \rangle = \bar{a}\langle u_1, v \rangle + \bar{b}\langle u_2, v \rangle$

$$\langle u, v\rangle = \sum_{k=1}^{n} \overline{u_k}\, v_k$$

Invariant: $\langle R(g)u, R(g)v\rangle = \langle u, v\rangle$

Claim: Given $\langle\,,\,\rangle$ & subrep. U,

$U^\perp = \{ w : \langle u, w\rangle = 0 \ \ \forall u \in U \}$ is a subrepresentation.

Proof: $w \in U^\perp$ $\quad R(g)\, w \in U^\perp$ ← check

$$\langle u, R(g)w\rangle = \langle R(g^{-1})u, \underbrace{R(g^{-1})R(g)}_{I} w\rangle$$

$$= \langle R(g^{-1})u, w\rangle = 0$$

$R(g^{-1})u$: stays in U $\qquad w$: $U^\perp$

Representation of $U(1)$

<u>Theorem:</u> If $R: U(1) \longrightarrow GL(n, \mathbb{C})$ is a smooth representation, then $\exists$ basis of $\mathbb{C}^n$ with respect to which

$$R(e^{i\theta}) = \begin{pmatrix} e^{i\mu_1\theta} & & 0 \\ & \ddots & \\ 0 & & e^{i\mu_n\theta} \end{pmatrix} \qquad \mu_1, \dots, \mu_n \in \mathbb{Z}$$

"weights" of the representation

i.e $\mathbb{C}^n = V_1 \oplus \dots \oplus V_n$

$R = R_1 \oplus \dots \oplus R_n$ such that $R_k(e^{i\theta}) = e^{i\mu_k\theta}$

We're looking for a basis of simultaneous eigenvectors

<u>Ex</u>

$$R(e^{i\theta}) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in GL(2, \mathbb{C})$$

$$\begin{vmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{vmatrix} = \lambda^2 - 2\lambda\cos\theta + \underbrace{\cos^2\theta + \sin^2\theta}_{1}$$

$$\Rightarrow \lambda = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2} = \cos\theta \pm i\sin\theta = e^{\pm i\theta}$$

weights are -1 & 1

Eigenvectors for $e^{i\theta}, e^{-i\theta}$ are $\begin{pmatrix} i \\ 1 \end{pmatrix}$ & $\begin{pmatrix} -i \\ 1 \end{pmatrix}$ respectively.

$$R(e^{i\theta}) = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

<u>Lemma 1</u> Any rep. of $U(1)$ admits an invariant Hermitian inner product. (Unitarian trick)

<u>Lemma 2</u> Any irreducible rep. of $U(1)$ is 1-dim

$R(e^{i\theta}) = (e^{in\theta})$ for some $n \in \mathbb{Z}$. (Schur's Lemma)

<u>Proof 1.</u> Take any Herm. inner product $\langle , \rangle$ & avg. it over $U(1)$.

$$\langle u, v \rangle_{inv} = \int_0^{2\pi} \langle R(e^{i\theta})u, R(e^{i\theta})v \rangle \frac{d\theta}{2\pi}$$

inv & invariant

Using invariance:

$$\langle R(e^{i\phi})u, R(e^{i\phi})v\rangle_{inv} = \int_0^{2\pi} \langle \underbrace{R(e^{i\theta})R(e^{i\phi})}_{R(e^{i(\theta+\phi)})} u, \underbrace{R(e^{i\theta})R(e^{i\phi})}v\rangle \frac{d\theta}{2\pi}$$

$$= \int_0^{2\pi} \langle R(e^{i(\theta+\phi)})u, R(e^{i(\theta+\phi)})v\rangle \frac{d\theta}{2\pi}$$

Change of variable: $\theta' = \theta+\phi$

$d\theta' = d\theta$

$$= \int_0^{2\pi} \langle R(e^{i\theta'})v, R(e^{i\theta'})v\rangle \frac{d\theta'}{2\pi} = \langle u, v\rangle_{inv}$$

<u>Remark:</u> Works for any compact G.

<u>Proof of 2:</u> Fix $e^{i\theta} \in U(1)$. Consider $R(e^{i\theta}) \in GL(n,\mathbb{C})$

$R(e^{i\theta})$ has an eigenvalue λ & $V_\lambda = \{v : R(e^{i\theta})v = \lambda v\} \neq \{0\}$

<u>Claim:</u> V_λ is a subrep. of $\mathbb{C}^n$

<u>Prf.</u> $R(e^{i\theta})\overbrace{R(e^{i\phi})v} = R(e^{i\phi})\underbrace{R(e^{i\theta})v}_{\lambda v} = \lambda \overbrace{R(e^{i\theta})v}^{\in V_\lambda}$

$v \in V_\lambda$

If $\mathbb{C}^n$ is irreducible then $\mathbb{C}^n = V_\lambda$

$\Rightarrow R(e^{i\theta}) = \lambda I$ ie. $\lambda : U(1) \longrightarrow \mathbb{C}^\times$ s.t

$$R(e^{i\theta}) = \lambda(\theta) I$$

<u>Claim</u> : $\lambda(\theta)$ is $U(1)$

<u>Proof</u>: $\langle R(e^{i\theta})v, R(e^{i\theta})v \rangle = \langle v, v \rangle$

$= \langle \lambda(\theta) v, \lambda(\theta) v \rangle = |\lambda(\theta)|^2 \langle v, v \rangle$

$|\lambda(\theta)|^2 = 1$

$\lambda : U(1) \xrightarrow[\text{homomorphism}]{} U(1) \Rightarrow \lambda(\theta) = e^{im\theta}$ for some $m \in \mathbb{Z}$

$$R(e^{i\theta}) = e^{im\theta} I$$

$\Rightarrow n = 1$ because any $\mathbb{C}$ line in $\mathbb{C}^n$ is a subrep.

$\Rightarrow \mathbb{C}^n = \mathbb{C}$ as $\mathbb{C}^n$ is irreducible.

Representation of SU(2)

Special unitary 2x2 matrices

$$SU(2) = \{ M : M^\dagger M = I, \det M = 1 \}$$

(3-degrees of freedom)
3-dimensional group
$$= \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} : a, b \in \mathbb{C} \quad |a|^2 + |b|^2 = 1 \right\}$$

$$su(2) = \{ X : X^\dagger = -X , \operatorname{Tr} X = 0 \}$$

3-d Lie algebra
$$= \left\{ \begin{pmatrix} ix & y+iz \\ -y+iz & -ix \end{pmatrix} : x, y, z \in \mathbb{R} \right\}$$

$$\underset{v=(x,y,z)}{M_v} = \begin{pmatrix} ix & y+iz \\ -y+iz & -ix \end{pmatrix}$$

Standard Representation:
$$SU(2) \longrightarrow GL(2, \mathbb{C})$$
$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \longrightarrow \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$$

2-d $\mathbb{C}$-representation

$$SU(2) \longrightarrow GL(su(2) \otimes \mathbb{C})$$

3-d representation $\quad x, y, z \in \mathbb{C}$

$$g \longrightarrow (M_v \longrightarrow g M_v g^{-1})$$

$$SU(2) \longrightarrow GL(1, \mathbb{C})$$

1-d $\mathbb{C}$ rep.
(trivial)

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \longrightarrow (1)$$

$$SU(2) \longrightarrow GL(0)$$

0-d rep.

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \longrightarrow \text{id.}$$

Theorem: For any nonnegative integer n, there is an irreducible representation $R_n : SU(2) \longrightarrow GL(n, \mathbb{C})$

Moreover any irrep $R : SU(2) \longrightarrow GL(V)$ is isomorphic to one of these.

Def: Given two reps. $\begin{array}{l} R: G \longrightarrow GL(V) \\ S: G \longrightarrow GL(W) \end{array}$ a morphism of reps $R \to S$

is a linear map $L : V \longrightarrow W$ st.

$$\begin{tikzcd} V \arrow[r, "R(g)"] \arrow[d, "L"'] & V \arrow[d, "L"] \\ W \arrow[r, "S(g)"'] & W \end{tikzcd}$$

$$L \circ R(g) = S(g) \circ L$$

It's an isomorphism if L is invertible.

e.g. If $V = W = \mathbb{C}^n$ & L is iso. then L is a change of basis matrix.

$$S(g) = L \circ R(g) \circ L^{-1}$$

i.e Representations are isomorphic if given by same matrices when a suitable basis is chosen.

$$U(1) \longrightarrow GL(2, \mathbb{C})$$

$$e^{i\theta} \longmapsto \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

diagonalized version

$$U(1) \longrightarrow GL(2, \mathbb{C})$$

$$e^{i\theta} \longrightarrow \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix}$$

are isomorphic. i.e. $\exists\ L : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ s.t

$$R(g) = L \circ S(g) \circ L^{-1}$$

change of basis matrix $L = \begin{bmatrix} i & -i \\ 1 & -1 \end{bmatrix}$ $\quad L^{-1} = \frac{1}{2i}\begin{bmatrix} 1 & i \\ -1 & i \end{bmatrix}$

$$\begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix}\frac{1}{2i}\begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

New Representations from old

$R: G \longrightarrow GL(V)$

$S: G \longrightarrow GL(W)$

Direct sum: $R \oplus S : G \longrightarrow GL(V \oplus W)$

relation with subgroup and reducibility ?

$$(R\oplus S)(g) = \begin{pmatrix} R(g) & 0 \\ 0 & S(g) \end{pmatrix}$$ not irreducible

Tensor Product $R \otimes S : G \longrightarrow GL(V \otimes W)$

$e_1, \dots, e_m$ basis of V $\quad f_1, \dots, f_n$ basis of W

$e_i \otimes f_j$ basis of $V \otimes W$ (mn dimensional)

$$(R\otimes S)(g)(v\otimes w) = (R(g)v)\otimes(S(g)w)$$

<u>Ex</u> $R=S$ the standard rep. of $SU(2)$ i.e $V=W=\mathbb{C}^2$

$\mathbb{C}^2\otimes\mathbb{C}^2$ basis:

$e_1 = f_1 = \begin{pmatrix}1\\0\end{pmatrix}$

$e_2 = f_2 = \begin{pmatrix}0\\1\end{pmatrix}$

$\{e_1\otimes f_1,\ e_1\otimes f_2,\ e_2\otimes f_1,\ e_2\otimes f_2\}$

$$R\begin{pmatrix}a & b\\ -\bar b & \bar a\end{pmatrix} = \begin{pmatrix}a & b\\ -\bar b & \bar a\end{pmatrix} \qquad \begin{pmatrix}a & b\\ -\bar b & \bar a\end{pmatrix}e_1 = \begin{pmatrix}a\\ -\bar b\end{pmatrix} = ae_1 - \bar b e_2$$

$$(R\otimes S)\begin{pmatrix}a & b\\ -\bar b & \bar a\end{pmatrix}(e_1\otimes f_1) = (ae_1 - \bar b e_2)\otimes(af_1 - \bar b f_2)$$

$$= a^2 e_1\otimes f_1 - a\bar b\, e_1\otimes f_2 - a\bar b\, e_2\otimes f_1 + \bar b^2 e_2\otimes f_2$$

$$(R\otimes S)\begin{pmatrix}a & b\\ -\bar b & \bar a\end{pmatrix} = \begin{array}{c} \begin{pmatrix} a^2 & \cdot & \cdot & \cdot\\ -a\bar b & \cdot & \cdot & \cdot\\ -a\bar b & \cdot & \cdot & \cdot\\ \bar b^2 & \cdot & \cdot & \cdot \end{pmatrix} \\ \begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ e_1\otimes f_1 & e_2\otimes f_2 & & \end{array} \end{array}$$

$2^x 3^y$ dim rep.

not irreducible

Symmetric Powers: $R: G \longrightarrow GL(V)$

$R^{\otimes n}: G \longrightarrow GL(V^{\otimes n})$ not irreducible

↓

subrepresentation of symmetric tensors

e.g. $\mathbb{C}^2 \otimes \mathbb{C}^2$ $\quad e_1 \otimes e_1 \quad e_2 \otimes e_2 \quad \underbrace{e_1 \otimes e_2 \quad e_2 \otimes e_1}_{\text{not symmetric}}$

symmetric

$\mathrm{Sym}^2 \mathbb{C}^2$

$e_1 \otimes e_2 + e_2 \otimes e_1 \rightarrow$ symmetric

$e_2 \otimes e_1 + e_1 \otimes e_2$

($e_1 \otimes e_2 - e_2 \otimes e_1 \rightarrow$ anti-symmetric)

$e_2 \otimes e_1 - e_1 \otimes e_2$

$\mathbb{C}^3$

$$\frac{1}{6}\big(e_1 \otimes e_2 \otimes e_3 + e_2 \otimes e_1 \otimes e_3 + e_1 \otimes e_3 \otimes e_2 + e_3 \otimes e_2 \otimes e_1 + e_2 \otimes e_3 \otimes e_1 + e_3 \otimes e_1 \otimes e_2 \big)$$

Averaging map

$$Av: V^{\otimes n} \longrightarrow V^{\otimes n}$$

$$Av: (v_1 \otimes \dots \otimes v_n) = \frac{1}{n!} \sum_{\sigma \in S_n} v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(n)}$$

Def. $\boxed{\mathrm{Sym}^n V := \mathrm{Image}\left(\mathrm{Av}: V^{\otimes n} \longrightarrow V^{\otimes n}\right)}$

If V is a rep then. Av is a morphism of reps $V^{\otimes n} \longrightarrow V^{\otimes n}$

. Image of a morphism is a subrep.

Morphism

$$\left[\, L: V \longrightarrow W \qquad L \circ R(g) = S(g) \circ L \,\right]$$

$$\mathrm{Av}\left(R(g)\right)^{\otimes n}\left(v_1 \otimes \cdots \otimes v_n\right) = R(g)^{\otimes n} \mathrm{Av}\left(v_1 \otimes \cdots \otimes v_n\right)$$

$$= \mathrm{Av}\left(R(g)v_1 \otimes \cdots \otimes R(g)v_n\right) \qquad = R(g)^{\otimes n} \frac{1}{n!} \sum v_{\sigma_1} \otimes \cdots v_{\sigma_n}$$

$$= \frac{1}{n!} \sum R(g) v_{\sigma(1)} \otimes \cdots \otimes R(g) v_{\sigma(n)} \quad = \frac{1}{n!} \sum R(g) v_{\sigma(1)} \otimes \cdots \otimes R(g) v_{\sigma(n)}$$

equal

Av is a morphism

If L is a morphism from $R: G \to GL(V)$ then
$S: G \to GL(W)$

$$\mathrm{image}(L) = \{ \underbrace{L(v)}_{\in W} : v \in V \}$$

$$S(g)\, L(v) = L\left(R(g)(v)\right) \in \mathrm{image}(L)$$

image (L) is a subrepresentation

? $R_n : SU(2) \longrightarrow GL(n, \mathbb{C})$ will be $\mathrm{Sym}^{n-1} \mathbb{C}^2$

Weight Space Decomposition

SU(2) contains a subgroup isomorphic to U(1)

$\cong$

subgroup inside SU(2)

$$T = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} : \theta \in \mathbb{R} \right\}$$

$$R: SU(2) \longrightarrow GL(V)$$

$$R|_T : T \longrightarrow GL(V)$$

$$V = V_1 \oplus \cdots \oplus V_n \quad \text{weight space decomposition}$$

n: $\dim V$

$$R\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} = \begin{pmatrix} e^{i m_1 \theta} & & 0 \\ & \ddots & \\ 0 & & e^{i m_n \theta} \end{pmatrix}$$

$m_1, \dots, m_n$ weights $\in \mathbb{Z}$

V_i: "weightspaces"

<u>Ex</u> Standard rep: $R: SU(2) \longrightarrow GL(2, \mathbb{C})$

$$R\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \longrightarrow \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

weightspaces $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ & $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

weight $\quad 1 \quad -1$

Ex $\quad R: SU(2) \longrightarrow GL(su(2) \otimes \mathbb{C})$

$$R_*: su(2) \longrightarrow gl(3, \mathbb{C})$$

Look at Lie Algebra Part !

$$\begin{pmatrix} ix & y+iz \\ -y+iz & -ix \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & -2z & 2y \\ 2z & 0 & -2x \\ -2y & 2x & 0 \end{pmatrix}$$

$y = z = 0$, Look at:

$$\exp\begin{pmatrix} ix & 0 \\ 0 & -ix \end{pmatrix} = \begin{pmatrix} e^{ix} & 0 \\ 0 & e^{-ix} \end{pmatrix}$$

$$\exp\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2x \\ 0 & 2x & 0 \end{pmatrix} = \begin{pmatrix} 1 \to e^{i0} & 0 & 0 \\ 0 & \cos 2x & -\sin 2x \\ 0 & \sin 2x & \cos 2x \end{pmatrix}$$

$$\underbrace{}_{\begin{pmatrix} e^{i2x} & 0 \\ 0 & e^{-i2x} \end{pmatrix}}$$

isomorphic

weights: $-2, 0, 2$

Ex $\quad Sym^2(\text{standard})$ $\quad e_1$ and e_2 basis for standard

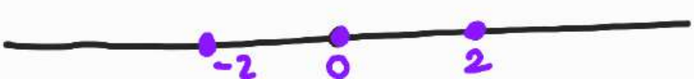
$e_1^2, \; e_1e_2, \; e_2^2$ → HOW WE GOT THESE POLYNOMIALS

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \overset{e_1}{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} = e^{i\theta} \overset{e_1}{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} \qquad e_1^2 \longrightarrow e^{i\theta}e^{i\theta}e_1^2 = e^{i2\theta}e_1$$

$$e_1e_2 \longrightarrow e^{i\theta}e_1e^{-i\theta}e_2 = e_1e_2$$

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \overset{e_2}{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} = e^{-i\theta} \underset{e_2}{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} \qquad e_2^2 \longrightarrow e^{-i\theta}e^{-i\theta}e_2^2 = e^{-i2\theta}e_2$$

weightspaces: $V_1 = \mathbb{C}e_1^2 \quad V_2 = \mathbb{C}e_1e_2 \quad V_3 = \mathbb{C}e_2^2$



X, Y & H

$$R: SU(2) \longrightarrow GL(V) \Rightarrow V = \bigoplus_{j=1}^{n} V_j$$

↳ weight-space decomposition

$$R\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} = \begin{pmatrix} e^{im_1\theta} & & 0 \\ & \ddots & \\ 0 & & e^{im_n\theta} \end{pmatrix}$$

$m_j \in \mathbb{Z}$ "weights"

LOOK AT LIE ALGEBRA

$$R\left[\exp\theta\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\right] = \exp\theta R_*\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$R_*\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} im_1 & & 0 \\ & \ddots & \\ 0 & & im_n \end{pmatrix}$$

$R_*: su(2) \longrightarrow gl(V)$

$$\boxed{su(2) = \left\{ \begin{pmatrix} ix & y+iz \\ -y+iz & -ix \end{pmatrix} : x, y, z \in \mathbb{R} \right\}}$$

$$\sigma_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

we know $R_*(\sigma_1)$ but what are

$R_*(\sigma_2)$, $R_*(\sigma_3)$?

$$\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Pauli Matrices

$$X = \frac{1}{2}(\sigma_2 - i\sigma_3) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\sigma_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

$$Y = -\frac{1}{2}(\sigma_2 + i\sigma_3) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$X, Y \notin su(2)$ $\quad$ $X, Y \in su(2) \otimes \mathbb{C}$

"complexification"

$X, Y \in sl(2, \mathbb{C})$ $\quad \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = iH \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = H$

$R_* : su(2) \longrightarrow gl(v)$ $\quad$ R-linear

$R_*^{\mathbb{C}} : su(2) \otimes \mathbb{C} \longrightarrow gl(v)$ $\quad$ $\mathbb{C}$-linear map

$\sigma_1 \longrightarrow M$

$i\sigma_1 \longrightarrow iM$

<u>Lemma:</u> Write $W_m = \bigoplus_{m_j = m} V_j$

$$\begin{array}{ccc} \bullet & \bullet & \bullet \\ -2 & 0 & 2 \\ W_{-2} & W_0 & W_2 \end{array} \qquad \begin{array}{c} Sym^2 \mathbb{C}^2 \\ \Downarrow \\ Sym^2(\text{Standard}) \end{array}$$

a) $R_*(H)$ acts on W_m as mI

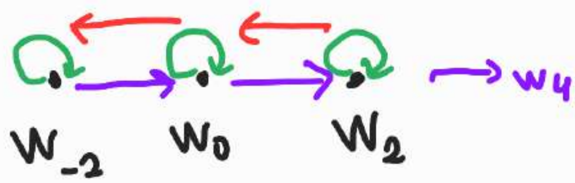
i.e $v \in W_m$ then $R_*(H)v = mv$

b) $R_*(X)$ sends W_m to W_{m+2}

i.e $v \in W_m$ then $R_*(X)v \in W_{m+2}$

c) $R_*(Y)$ sends W_m to W_{m-2}

i.e $v \in W_m$ then $R_*(Y)v \in W_{m-2}$



Proof a) $\sigma_1 = iH \Rightarrow R_*(\sigma_1) = i R_* H = \begin{pmatrix} i\mu_1 & & \\ & \ddots & \\ & & i\mu_n \end{pmatrix}$

$$R_* H = \begin{pmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_n \end{pmatrix}$$

$$R_* H \Big|_{W_m} = m I$$

b) $v \in W_m \qquad R_* X v \in W_{m+2}$

$v \in V_m \Leftrightarrow R_* H v = m v$

$$R_* X v \Leftrightarrow R_* H \underbrace{R_* X v}_{\text{eigenvector}} = \overset{\text{eigenvalue}}{(m+2)} \underbrace{R_* X v}$$

$$\underbrace{R_* X R_* H v} = R_* X m v = \underbrace{m R_* X} v$$

$$[H, X] = 2X \Rightarrow R_* [H, X] = 2 R_* X = [R_* H, R_* X]$$

$$= R_* H R_* X - R_* X R_* H$$

$$\Rightarrow 2 R_* X v = \underbrace{R_* H R_* X v}_{(m+2) R_* X v} - \underbrace{R_* X R_* H v}_{m R_* X v}$$

X, Y, H Example

$$R: SU(2) \longrightarrow GL(V) \;\llcorner \mathbb{C}$$

$$R_*: su(2) \longrightarrow gl(V)$$

$$R_*^{\mathbb{C}}: \underset{\substack{\| \\ su(2)\otimes\mathbb{C}}}{sl(2,\mathbb{C})} \longrightarrow gl(V)$$

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$V = \bigoplus W_m \qquad W_m = \{ v \in V : R_*(H)v = mv \}$$

$$R_*(X) : W_m \longrightarrow W_{m+2}$$

$$R_*(Y) : W_m \longrightarrow W_{m-2}$$

$$R^{\otimes 2}(g)(v_1 \otimes v_2) = R(g)v_1 \otimes R(g)v_2$$

what is $(R^{\otimes 2})_*$? (LIE ALGEBRA ACT)

Claim: $(R^{\otimes n})_*(X)(v_1 \otimes \dots \otimes v_n)$

$$= (R_* X v_1) \otimes v_2 \otimes \dots \otimes v_n$$

$$+ v_1 \otimes (R_* X v_2) \otimes \dots v_n + \dots +$$

$$+ v_1 \otimes v_2 \otimes \dots \otimes (R_* X v_n)$$

$\mathrm{Sym}^2 \mathbb{C}^2$ $\qquad$ $e_1^2 \quad , \quad e_1 e_2 \quad , \quad e_2^2$

(Shorter version of $e_1 \otimes e_1 \, , \, e_1 \otimes e_2 + e_2 \otimes e_1 \, , \, e_2 \otimes e_2$)

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \begin{aligned} e_1 &\longrightarrow e_1 \\ e_2 &\longrightarrow -e_2 \end{aligned}$$

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \begin{aligned} e_1 &\longrightarrow 0 \\ e_2 &\longrightarrow e_1 \end{aligned}$$

$$Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad \begin{aligned} e_1 &\longrightarrow e_2 \\ e_2 &\longrightarrow 0 \end{aligned}$$

eigenvalues

$$\mathrm{Sym}^2 H (e_1^2) = \overbrace{He_1}^{e_1} \otimes e_1 + e_1 \otimes \overbrace{He_1}^{e_1} = 2 e_1^2$$

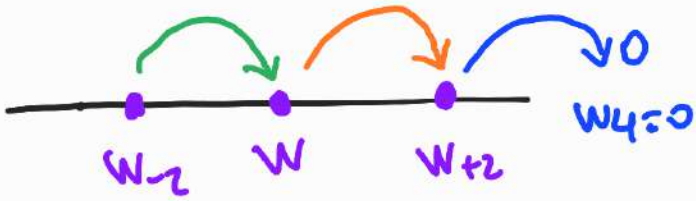
$$\mathrm{Sym}^2 H (e_1 e_2) = \underbrace{He_1}_{e_1} \otimes e_2 + e_1 \otimes \underbrace{He_2}_{-e_2} = 0$$

$$\mathrm{Sym}^2 H (e_2^2) = \underbrace{He_2}_{-e_2} \otimes e_2 + e_2 \otimes \underbrace{He_2}_{-e_2} = -2 e_2^2$$

$$\mathrm{Sym}^2 X(e_1^2) = Xe_1 \otimes e_1 + e_1 \otimes Xe_1 = 0$$

$$\mathrm{Sym}^2 X(e_1e_2) = (Xe_1)\otimes e_2 + e_1 \otimes \underbrace{Xe_2}_{e_1} = e_1^2$$

$$\mathrm{Sym}^2 X(e_2^2) = (Xe_2)\otimes e_2 + e_2 \otimes Xe_2 = 2e_1e_2$$



Proof: $$R^{\otimes n}(\exp(tX)) = \exp t(R_*^{\otimes n} X)$$

$$\hookrightarrow I + t R_*^{\otimes n} X + O(t^2)$$

Apply to $v_1 \otimes \ldots \otimes v_n$

$$\underbrace{R(\exp tX)}\, v_1 \otimes \ldots \otimes R(\exp tX) v_n = \underline{v_1 \otimes \ldots \otimes v_n} + \boxed{t R_*^{\otimes n}(X)(v_1 \otimes \ldots \otimes v_n)} + O(t^2)$$

$$\exp(tR_*X)v_1 \otimes \ldots \otimes \exp(tR_*X)v_n$$

$$(I + tR_*X + \ldots)v_1 \otimes \ldots \otimes (I + tR_*X + \ldots)v_n$$

$$\underline{v_1 \otimes \ldots \otimes v_n} + \boxed{t\big((R_*Xv_1)\otimes v_2 \otimes \ldots v_n + v_1 \otimes (R_*Xv_2) \otimes \ldots \otimes v_n + \ldots + v_1 \otimes v_2 \ldots (R_*Xv_n)\big)} + O(t^2)$$

Classification of $SU(2)$

$\rho : SU(2) \longrightarrow GL(V) \rightsquigarrow \rho_* : sl(2,\mathbb{C}) \longrightarrow gl(V)$

$V = \bigoplus W_m$

$W_m = \{ v \in V : \quad Hv = mv \}$

$X : W_m \longrightarrow W_{m+2}$ $\qquad$ $Y : W_m \longrightarrow W_{m-2}$

Pick $v \in W_m$ for m the highest weight

Take $U \leq V$ to be the subspace spanned by $v, Yv, Y^2 v, \ldots\ldots Y^n v$

<u>Claim:</u> U is a subrep. of V.

$\Rightarrow$ If V is irreducible then $V = U$

$\Rightarrow$ Any irreducible rep. of $SU(2)$ has weight diagram.

All weighted spaces are 1-d.

Proof: Need to show if $v \in U$ then
$Xv \in U$
$Yv \in U$
$Hv \in U$

Check for a basis $v, Yv, \ldots, Y^n v$

Applying Y we get $Yv, Y^2v, \ldots, Y^n v, 0 \in U$

Applying H we get $Hv = mv$ $\quad HYv = (m-2)Yv, \ldots \in U$

Applying X

$\downarrow$

Claim $\boxed{XY^k v = (m+1-k)k\,Y^{k-1} v}$

$$Y^k v \xrightarrow{X} Y^{k-1} v$$

$\Rightarrow$ previous claim

$\Rightarrow$ irreducible rep. with highest weight

m is determined up to isomorphism by m.

(actions of X, Y, H are determined by formulae above)

$$\Rightarrow \mathrm{Sym}^2 \mathbb{C}^2 \cong \left(SU(2) \to GL(su(2) \otimes \mathbb{C})\right)$$

$$\underset{-2}{\bullet} \quad \underset{0}{\bullet} \quad \underset{2}{\bullet} \quad \cong \quad \underset{-2}{\bullet} \quad \underset{0}{\bullet} \quad \underset{2}{\bullet}$$

$\Rightarrow n$ (max. number st. $Y^n v \neq 0$) $= m$

$$XY^k v = (m+1-k)k\,Y^{k-1} v$$

$$XY^{n+1}v = 0 = \underbrace{(m+1-(n+1))}_{=0}\underbrace{(n+1)}_{\neq 0}\underbrace{Y^n v}_{\neq 0}$$

by assumption

$$m+\cancel{1} = (n+\cancel{1}) \Rightarrow \boxed{m=n}$$

$$XY^k v = (m+1-k)k\, Y^{k-1} v$$

<u>Proof:</u> Induction on k. $k=0$: $Xv = 0$

it has already max. weight

Suppose it's true $\forall$ numbers $k>0$

R_x

we've R_x in front of X, Y, H

$$R\ XY - YX = [X,Y] = H$$

apply both sides to $Y^{k-1}v$.

$$XY^k v - YXY^{k-1}v = HY^{k-1}v$$

$$XY^k v = Y\underbrace{XY^{k-1}v} + \overbrace{HY^{k-1}v}^{(m-2k+2)Y^{k-1}v}$$

$$XY^{k-1}v = (m+1-k+1)(k-1)Y^{k-2}v$$

$$= (m+1-k+1)(k-1)Y^{k-1}v + (m-2k+2)Y^{k-1}v$$

$$= (m+1-k)k\, Y^{k-1}v$$

Decomposing into Irreducible Representation

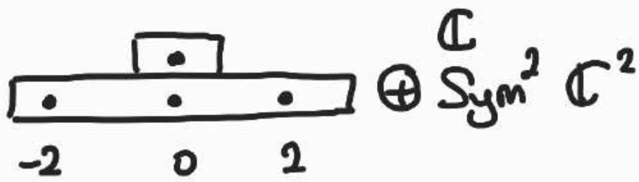
Theorem: Any irrep. of $SU(2)$ is isomorphic to $\mathrm{Sym}^n \mathbb{C}^2$ for some $n \in \{0, 1, 2, \dots\}$

This has weight diagram

$$\underset{-n}{\bullet} \quad \underset{-(n-2)}{\bullet} \quad \bullet \; \cdots\cdots \quad \underset{n-2}{\bullet} \quad \underset{n}{\bullet}$$

Theorem: Any finite dimensional $SU(2)$ rep. splits as a direct sum of irreps.

e.g. $\mathbb{C}^2 \otimes \mathbb{C}^2$

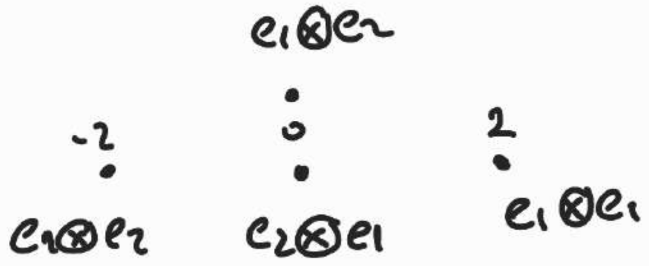


$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$H(e_1 \otimes e_1) = He_1 \otimes e_1 + e_1 \otimes He_1 = 2e_1 \otimes e_1$$
$$H(e_1 \otimes e_2) = He_1 \otimes e_2 + e_1 \otimes He_2 = 0$$
$$H(e_2 \otimes e_1) = He_2 \otimes e_1 + e_2 \otimes He_1 = 0$$
$$H(e_2 \otimes e_2) = He_2 \otimes e_2 + e_2 \otimes He_2 = -2e_2 \otimes e_2$$

$He_1 = e_1$
$He_2 = -e_2$ in $\mathbb{C}^2$

Standard rep.es.



Pick highest weight vector $e_1 \otimes e_1 = v$

v, Yv, Y^2v gives a subrepresentation. $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

Pick an invariant Hermitian inner product:

$U^\perp$ is a complementary subrep. 1-dim.

By inspection, weight diagrams for U & $U^\perp$ are

the same as for $Sym^2 \mathbb{C}^2$ & $\underset{\text{trivial}}{\mathbb{C}}$ $\Rightarrow U \cong Sym^2 \mathbb{C}^2$

$U^\perp \cong \mathbb{C}$

isomorphic

$$\mathbb{C}^2 \otimes \mathbb{C}^2 \cong \underbrace{Sym^2 \mathbb{C}^2}_{} \oplus \mathbb{C}$$

$\sim e_1 \otimes e_2 - e_2 \otimes e_1$ $(\Lambda^2 \mathbb{C}^2)$

$e_1 \otimes e_1$, $e_1 \otimes e_2 + e_2 \otimes e_1$, $e_2 \otimes e_2$

$Y(e_1 \otimes e_1) = e_2 \otimes e_1 + e_1 \otimes e_2$ subrepresentation

exterior square

e.g. $Sym^2(Sym^2 \mathbb{C}^2)$

spanned by

$$Sym^2 \mathbb{C}^2 = \mathbb{C} \cdot (\underset{\alpha}{e_1^2}, \underset{\beta}{e_1 e_2}, \underset{\gamma}{e_2^2})$$

$$Sym^2(Sym^2 \mathbb{C}^2) = \mathbb{C} \cdot (\alpha^2, \alpha\beta, \alpha\gamma, \beta^2, \beta\gamma, \gamma^2)$$

$$H(\alpha^2) = \underbrace{(H\alpha)}_{2\alpha}\alpha + \alpha\underbrace{(H\alpha)}_{2\alpha} = 4\alpha^2$$

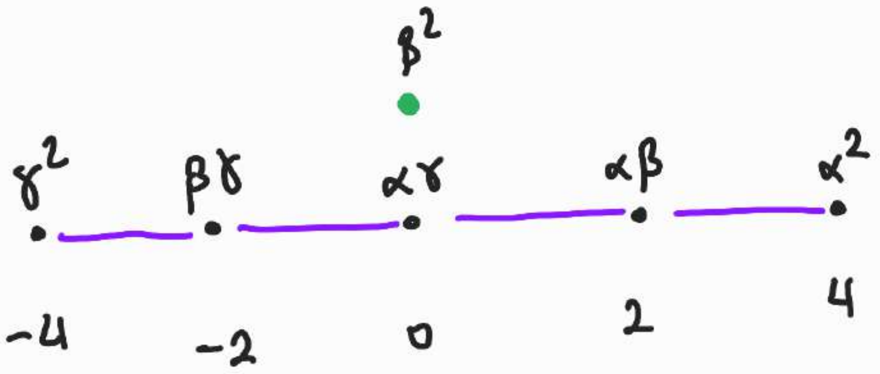
$$H(\alpha\beta) = (\underbrace{H\alpha}_{2\alpha})\beta + \alpha(\underbrace{H\beta}_{0}) = 2\alpha\beta$$

$$H(\alpha\gamma) = (\underbrace{H\alpha}_{+2\alpha})\gamma + \alpha(\underbrace{H\gamma}_{-2\alpha}) = 0$$

$$H(\beta^2) = (\underbrace{H\beta}_{0})\beta + \beta(\underbrace{H\beta}_{0}) = 0$$

$$H(\beta\gamma) = (\underbrace{H\beta}_{0})\gamma + \beta(\underbrace{H\gamma}_{-2\gamma}) = -2\gamma\beta$$

$$H(\gamma^2) = -4\gamma^2$$



$$\mathrm{Sym}^2\,\mathrm{Sym}^1\,\mathbb{C}^2 = \mathrm{Sym}^4\mathbb{C}^2 \oplus \mathbb{C}$$

<u>Claim:</u> $\mathbb{C}$ is spanned by $\beta^2 - \alpha\gamma$

SU(3) Representation

$$T = \text{diagonal matrices in } SU(3) = \left\{ \underbrace{\begin{pmatrix} e^{i\theta_1} & 0 & 0 \\ 0 & e^{i\theta_2} & 0 \\ 0 & 0 & e^{-i(\theta_1+\theta_2)} \end{pmatrix}}_{}^{D(\theta_1,\theta_2)} \;\; \theta_1, \theta_2 \in \mathbb{R} \right\}$$

torus

$U(1) \times U(1)$

$e^{i\theta_1} \quad e^{i\theta_2}$

<u>Lemma:</u> If $R: SU(3) \longrightarrow GL(V)$ is a $\mathbb{C}$-rep then

$$V = \bigoplus W_{k,l}$$

$$W_{k,l} = \left\{ v \in V : R\big(D(\theta_1,\theta_2)\big) v = e^{i(k\theta_1 + l\theta_2)} v \right\}$$

$k, l \in \mathbb{Z}$

<u>Ex</u> $\mathbb{C}^3$ standard rep of $SU(3)$ $\quad e_1, e_2, e_3$

$$R\big(D(\theta_1,\theta_2)\big) = \begin{pmatrix} e^{i\theta_1} & & \\ & e^{i\theta_2} & \\ & & e^{-i(\theta_1+\theta_2)} \end{pmatrix}$$

$e_1 \longrightarrow e^{i\theta_1} e_1$

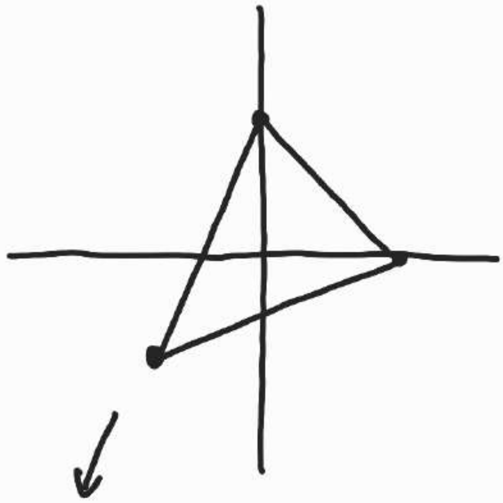
$e_2 \longrightarrow e^{i\theta_2} e_2$

$e_3 \longrightarrow e^{-i(\theta_1+\theta_2)} e_3$

spanned by

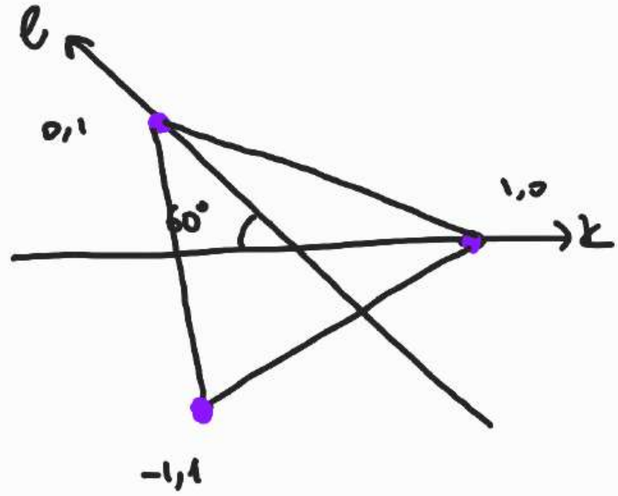
$W_{1,0} = \mathbb{C} \cdot e_1 \qquad W_{0,1} = \mathbb{C} \cdot e_2 \qquad W_{-1,-1} = \mathbb{C} \cdot e_3$

$k=1, l=0$



not equilateral

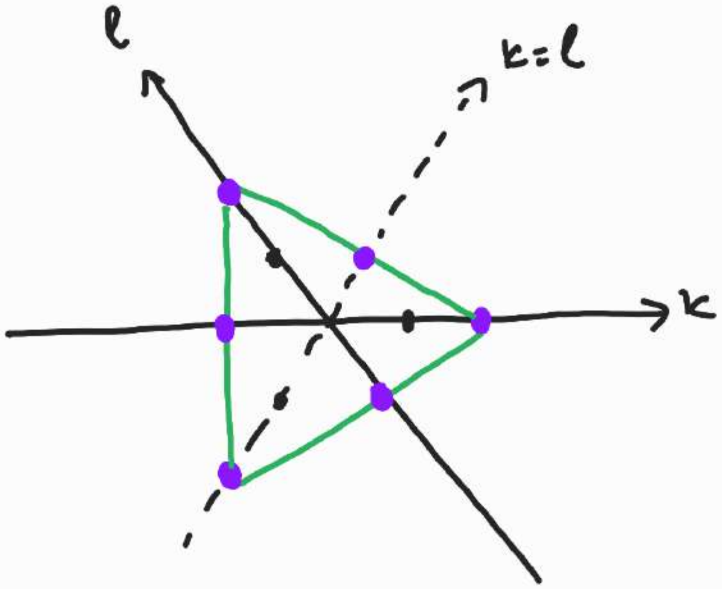
$\rightsquigarrow$ instead



Ex $\mathrm{Sym}^2\mathbb{C}^3$

$R(D(\theta_1,\theta_2))$

	e_1^2	e_1e_2	e_1e_3	e_2^2	e_2e_3	e_3^2
$R(D(\theta_1,\theta_2))$	$\mapsto e^{i2\theta_1}e_1^2$	$\mapsto e^{i(\theta_1+\theta_2)}e_1e_2$	$\mapsto e^{-i\theta_2}e_1e_3$	$\mapsto e^{i2\theta_2}e_2^2$	$\mapsto e^{-i\theta_1}e_2e_3$	$\mapsto e^{-i2(\theta_1+\theta_2)}e_3^2$
$\begin{pmatrix}k\\ l\end{pmatrix}$	$\begin{pmatrix}2\\0\end{pmatrix}$ ✓	$\begin{pmatrix}1\\1\end{pmatrix}$ ✓	$\begin{pmatrix}0\\-1\end{pmatrix}$ ✓	$\begin{pmatrix}0\\2\end{pmatrix}$ ✓	$\begin{pmatrix}-1\\0\end{pmatrix}$ ✓	$\begin{pmatrix}-2\\-2\end{pmatrix}$



<u>Proof of Lemma:</u> $T_1 = \left\{ D(\vartheta_1, 0) = \begin{pmatrix} e^{i\vartheta_1} & & \\ & 1 & \\ & & e^{-i\vartheta_1} \end{pmatrix} \in T \right\} \cong U(1)$

isomorphic

$$V = \bigoplus U_k \qquad U_k = \left\{ v \in V : R(D(\vartheta_1, 0))v = e^{ik\vartheta_1} v \right\}$$

<u>Claim:</u> $T_2 = \left\{ D(\vartheta_2, 0) = \begin{pmatrix} 1 & & \\ & e^{i\vartheta_2} & \\ & & e^{-i\vartheta_2} \end{pmatrix} \in T \right\} \cong U(1)$

preserves each U_k.

This implies the lemma: $U_k = \bigoplus W_{k,\ell}$

$$W_{k,\ell} = \left\{ v \in U_k : R(D(\vartheta, \vartheta_2))v = e^{i\ell\vartheta_2} v \right\}$$

$$v \in U_k \Rightarrow R(D(\vartheta_1, 0))\, v = e^{ik\vartheta_1} v \quad \forall \vartheta_1$$

want to show $R(D(0,\vartheta_2))v \in U_k$ if $v \in U_k$

i.e. $R(D(\vartheta_1, 0)) \underbrace{R(D(0,\vartheta_2))v} = e^{ik\vartheta_1} \underbrace{R(D(0,\vartheta_2))v}$

diagonal matrices commute

$$D(\vartheta_1, 0)\, D(0, \vartheta_2) = D(0, \vartheta_2)\, D(0, \vartheta_1)$$

$$R(D(\vartheta_1, 0))\, R(D(0, \vartheta_2)) = R(D(0, \vartheta_2))\, R(D(\vartheta_1, 0))$$

$$R(D(\vartheta_1,0))\underbrace{R(D(0,\vartheta_2))v} = R(D(0,\vartheta_2))\underbrace{R(D(\vartheta_1,0))v}_{e^{ik\vartheta_1}v \text{ as } v\in U_k}$$

$$= e^{ik\vartheta_1}\underbrace{R(D(0,\vartheta_2))v}_{\in U_k}$$

The Adjoint Representation

<u>Def.</u> Given a Lie group G with Lie algebra of the adjoint rep. is:

$$Ad: G \longrightarrow GL(\mathfrak{g})$$

$$Ad(g)X = gXg^{-1} \in \mathfrak{g}$$

<u>Lemma:</u> $X\in\mathfrak{g}$ & $g\in G \Rightarrow gXg^{-1}\in\mathfrak{g}$

<u>Proof:</u> Need to show $\forall t\in\mathbb{R}$ $\exp(tgXg^{-1})\in G$

$$= I + tgXg^{-1} + \frac{1}{2}t^2 gX\cancel{g^{-1}g}Xg^{-1} + \dots$$

$$= I + tgXg^{-1} + \frac{1}{2}t^2 gX^2g^{-1} + \dots = \underset{\in G}{g}\underbrace{(\exp tX)}_{\in G}\underset{\in G}{g^{-1}} \in G$$

Def. $ad = Ad_* : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$

$$(Ad_* X)Y = \frac{d}{dt}\Big|_{t=0} \left[Ad(\exp tX)\, Y \right]$$

$$= \frac{d}{dt}\Big|_{t=0} \left[(\exp tX)\, Y \exp(-tX) \right]$$

$$= \left[X(\exp tX) Y \exp(-tX) - (\exp tX) Y X \exp(-tX) \right]_{t=0}$$

$$= [XY - YX] = [X,Y]$$

$$\boxed{ad(X)Y = [X,Y]}$$

Ex $sl(2,\mathbb{C})$ $\quad H \quad X \quad Y$

$$ad(H): \left.\begin{aligned} H &\longrightarrow [H,H] = 0 \\ X &\longrightarrow [H,X] = 2X \\ Y &\longrightarrow [H,Y] = -2Y \end{aligned}\right\} \Rightarrow \begin{array}{ccc} Y & H & X \\ \bullet & \bullet & \bullet \\ -2 & 0 & 2 \end{array}$$

$ad \cong Sym^2 \mathbb{C}^2$

Ex $sl(3,\mathbb{C}) = su(3) \otimes \mathbb{C}$

$\downarrow$

8. dim

3×3 Matrices $\to$ 9 entries

1 condition $\to$ $Tr = 0$

Thus 8 degrees of freedom

e_{ij} = zeros except 1 in position i,j

e.g. $E_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ 6 of these

$$H_{13} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad H_\theta : \begin{pmatrix} \theta_1 & & \\ & \theta_2 & \\ & & \theta_3 \end{pmatrix} \qquad \theta_1 + \theta_2 + \theta_3 = 0$$

$$H_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$ad(H_\theta) \quad H_{ij} = [H_\theta, H_{ij}] = 0$$

$$ad(H_\theta) E_{ij} = (\theta_i - \theta_j) E_{ij}$$

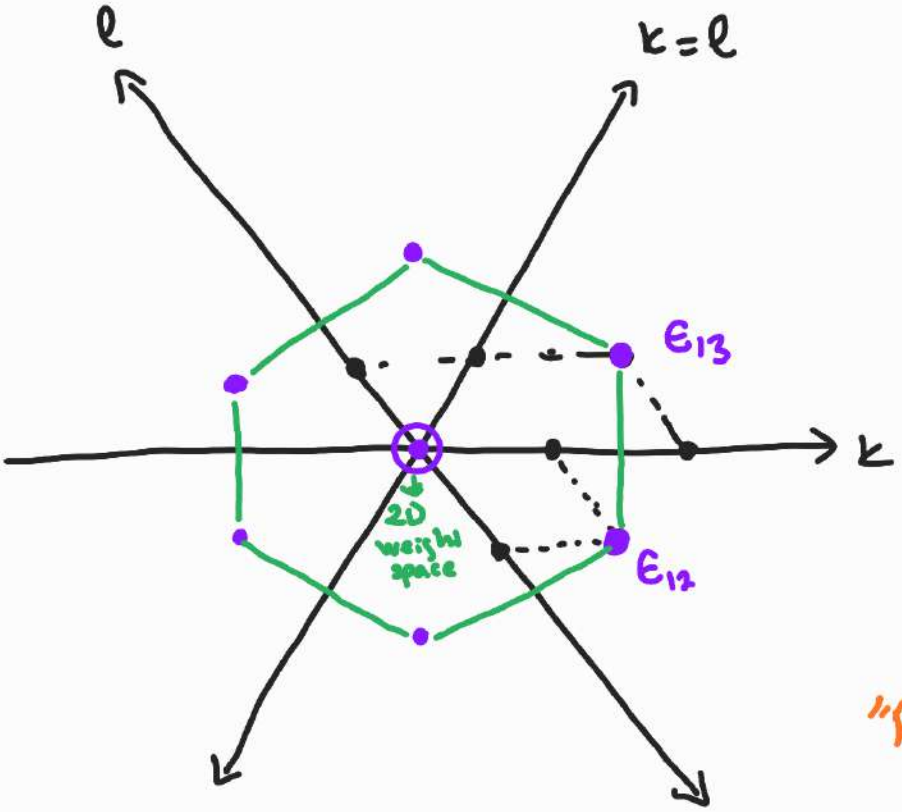
e.g. $$\left[\begin{pmatrix} \theta_1 & & \\ & \theta_2 & \\ & & \theta_3 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right]$$

$$= \begin{pmatrix} 0 & \theta_1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & \theta_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= (\theta_1 - \theta_2) E_{12}$$

$k = 1$ $\qquad$ $l = -1$

Root's Diagram



"Root Spaces"

$(\theta_1 - \theta_3)\,\epsilon_{13} \longrightarrow (2\theta_1 + \theta_2)\,\epsilon_{13}$

where $\theta_3 = -\theta_1 - \theta_2$ $\qquad k = 2,\ l = 1$

$\theta_1 + \theta_1 + \theta_2$

Root Vectors acting on weight spaces

$$R : SU(3) \longrightarrow GL(V)$$

$$\Rightarrow \quad V = \bigoplus W_{k,\ell} \qquad k,\ell \in \mathbb{Z}$$

$$W_{k,\ell} = \left\{ v : R\begin{pmatrix} e^{i\theta_1} & & \\ & e^{i\theta_2} & \\ & & e^{-i(\theta_1+\theta_2)} \end{pmatrix} v = e^{i(k\theta_1+\ell\theta_2)} v \right\}$$

$$= \left\{ v : R_*^{\mathbb{C}} \begin{pmatrix} \theta_1 & & \\ & \theta_2 & \\ & & -\theta_1-\theta_2 \end{pmatrix} v = (k\theta_1 + \ell\theta_2) v \right\}$$

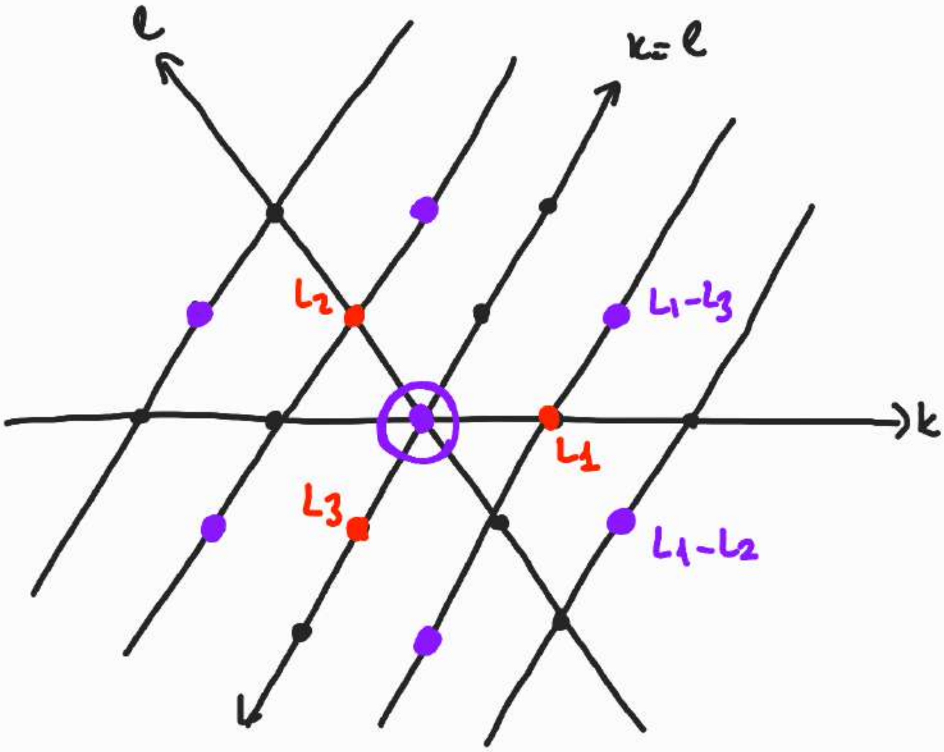
$$W_\lambda = \left\{ v : R_* \begin{pmatrix} \theta_1 & & \\ & \theta_2 & \\ & & -\theta_1-\theta_2 \end{pmatrix} v = \lambda(\theta) v \right\}$$

$$\operatorname{ad}(H_\theta)\, E_{ij} = (\theta_i - \theta_j)\, E_{ij}$$

so $E_{ij} \in W_{L_i - L_j}$

λ could be
$$L_1(\theta) = \theta_1$$
$$L_2(\theta) = \theta_2$$
$$L_3(\theta) = -\theta_1 - \theta_2$$

So $W_{L_1 - L_3} = \mathbb{C}\, E_{13}$



$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

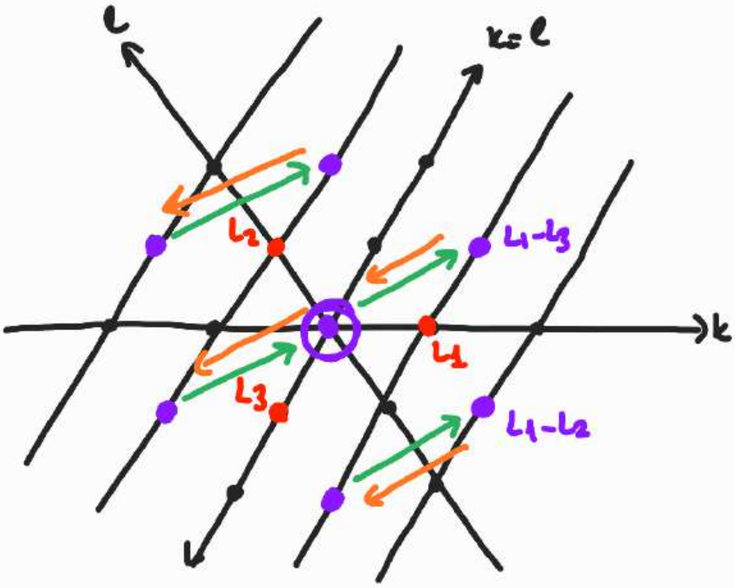
$Y \qquad H \qquad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$\bullet \qquad \bullet \qquad \bullet$

$-2 \qquad 0 \qquad 2$

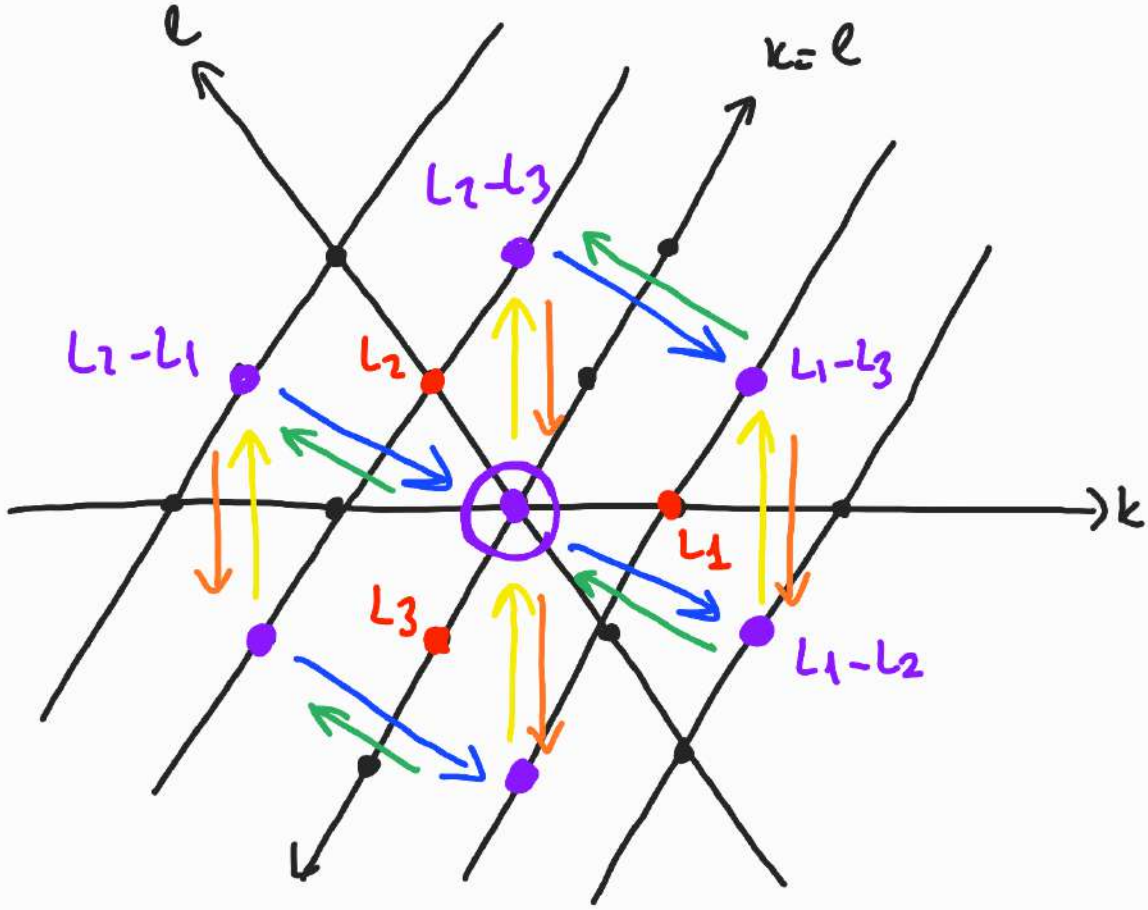
<u>Lemma:</u> Given a Lie group representation

$R: SU(3) \longrightarrow GL(V)$ then $R_*^{\mathbb{C}}(E_{ij}) : W_\lambda \longrightarrow W_{\lambda + L_i - L_j}$



<u>Ex</u> E_{13} acts by translating weight space in the $L_1 - L_3$ direction

E_{31} act $L_3 - L_1$ (opposite)



ϵ_{12} $\quad L_1-L_2$

ϵ_{21} $\quad L_2-L_1$

ϵ_{23} $\quad L_2-L_3$

ϵ_{32} $\quad L_3-L_2$

Proof of Claim: $\quad v \in W_\lambda \quad \Rightarrow \quad R_*^{\mathbb{C}}(\epsilon_{ij})v \in W_{\lambda+L_i-L_j}$

$$\Updownarrow \qquad\qquad\qquad \Updownarrow$$

$$R_*^{\mathbb{C}}(H_\vartheta)\,v = \lambda(\vartheta)v \qquad R_*^{\mathbb{C}}(H_\vartheta)R_*^{\mathbb{C}}(\epsilon_{ij})v = (\lambda+L_i-L_j)(\vartheta)v$$

$$= (\lambda(\vartheta)+\vartheta_i-\vartheta_j)\,v$$

$$[H_\vartheta,\epsilon_{ij}] = \boxed{\mathrm{ad}(H_\vartheta)\epsilon_{ij} = (\vartheta_i-\vartheta_j)\,\epsilon_{ij}} \qquad \epsilon_{ij} \in W^{\mathrm{ad}}_{L_i-L_j}$$

$$R_*^{\mathbb{C}}[H_\theta, \epsilon_{ij}] = [R_*^{\mathbb{C}}(H_\theta), R_*^{\mathbb{C}}(\epsilon_{ij})] = (\vartheta_i - \vartheta_j) R_*^{\mathbb{C}}(\epsilon_{ij})$$

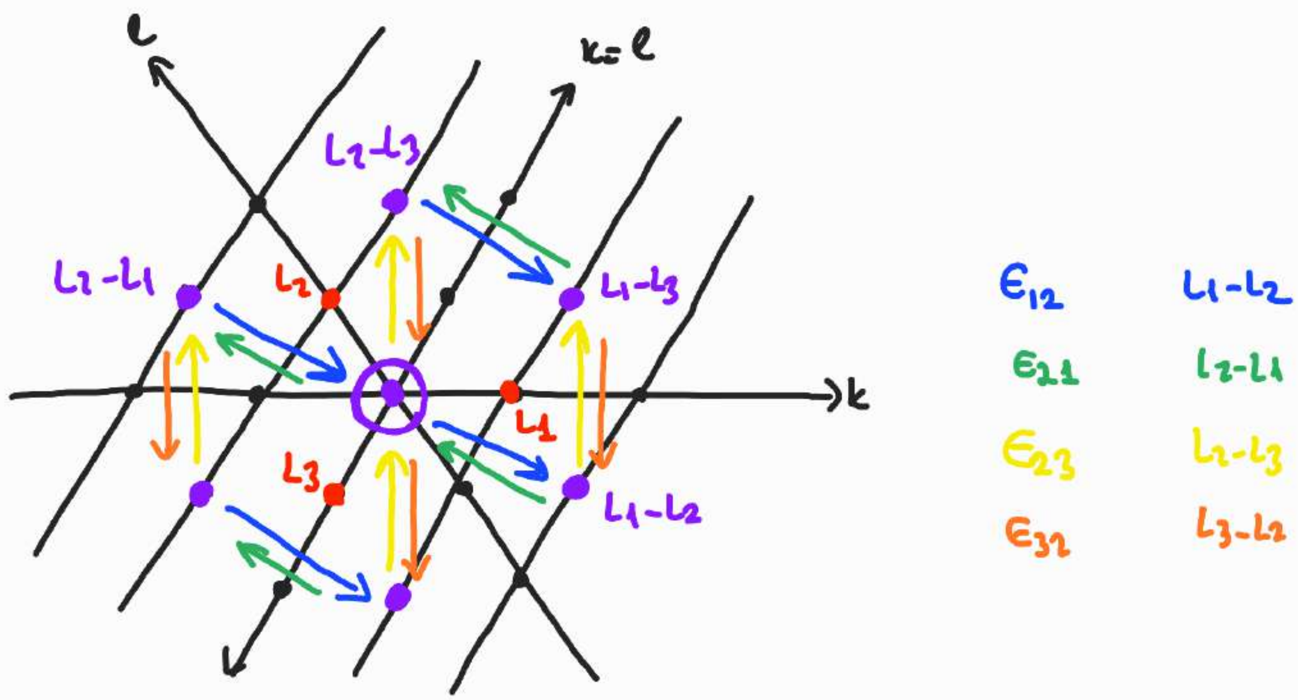
$$= R_*^{\mathbb{C}} H_\theta R_*^{\mathbb{C}} \epsilon_{ij} v - R_*^{\mathbb{C}} \epsilon_{ij} R_*^{\mathbb{C}} H_\theta v$$

$$R_*^{\mathbb{C}} H_\theta R_*^{\mathbb{C}} \epsilon_{ij} v = R_*^{\mathbb{C}} \epsilon_{ij} \underbrace{R_*^{\mathbb{C}} H_\theta v}_{\lambda(\theta) v} + (\vartheta_i - \vartheta_j) R_*^{\mathbb{C}}(\epsilon_{ij}) v$$

$$= \lambda(\theta) R_*^{\mathbb{C}} \epsilon_{ij} v + (\theta_i - \theta_j) R_*^{\mathbb{C}}(\epsilon_{ij}) v$$

$$= (\lambda(\theta) + \theta_i - \theta_j) R_*^{\mathbb{C}} \epsilon_{ij} v$$

Weyl Symmetry

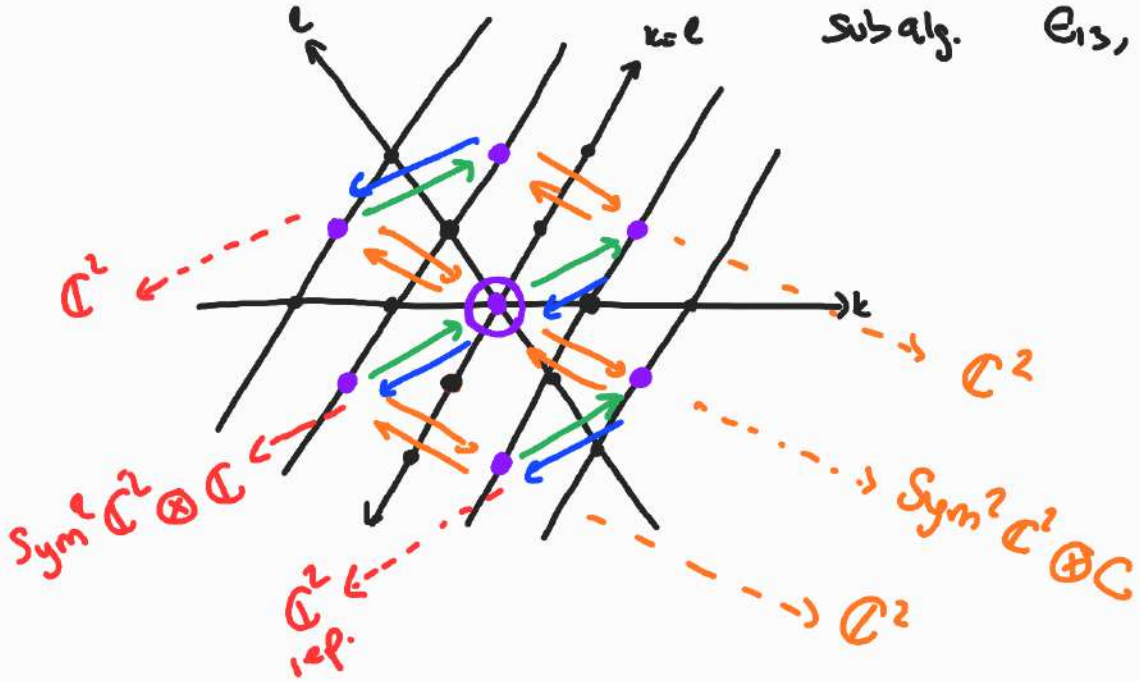


$$\overset{X}{E_{12}}, \overset{Y}{E_{21}} = \left[\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \underset{H}{H_{12}}$$

span a Lie subalgebra of $sl(3,\mathbb{C})$ isomorphic to $sl(2,\mathbb{C})$

Because of this V splits as a sum of $sl(2,\mathbb{C})$ reps in 3 ways.

subalg. $E_{13}, E_{31}, H_{13} = \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}$



Remember : $sl(2,\mathbb{C})$ weight diagrams are symmetric about '0'.

Corollary : Any weight diagram for $SU(3)$-rep has 3 lines of reflection symmetry.

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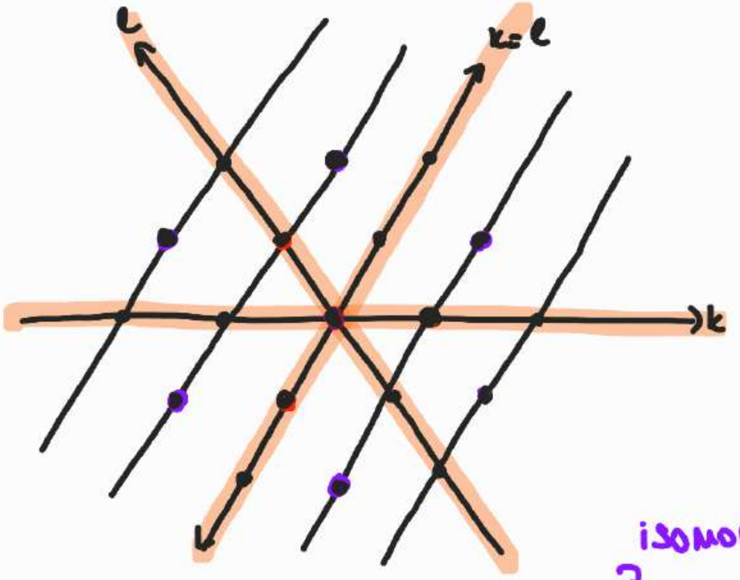
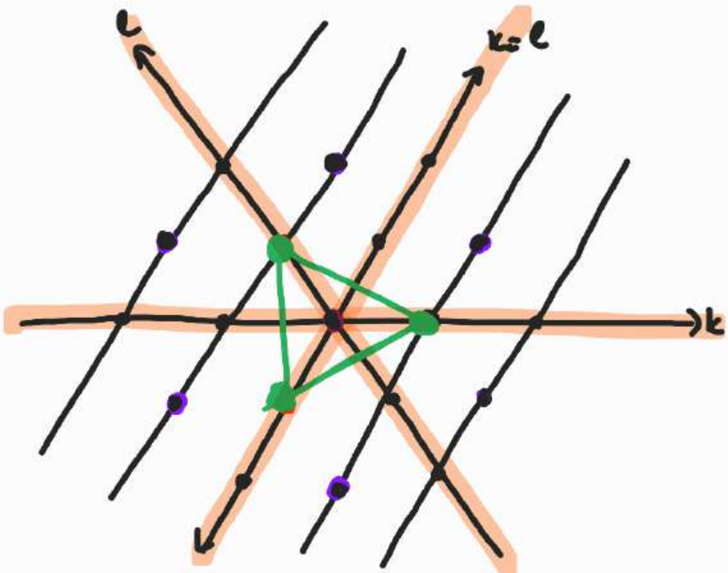
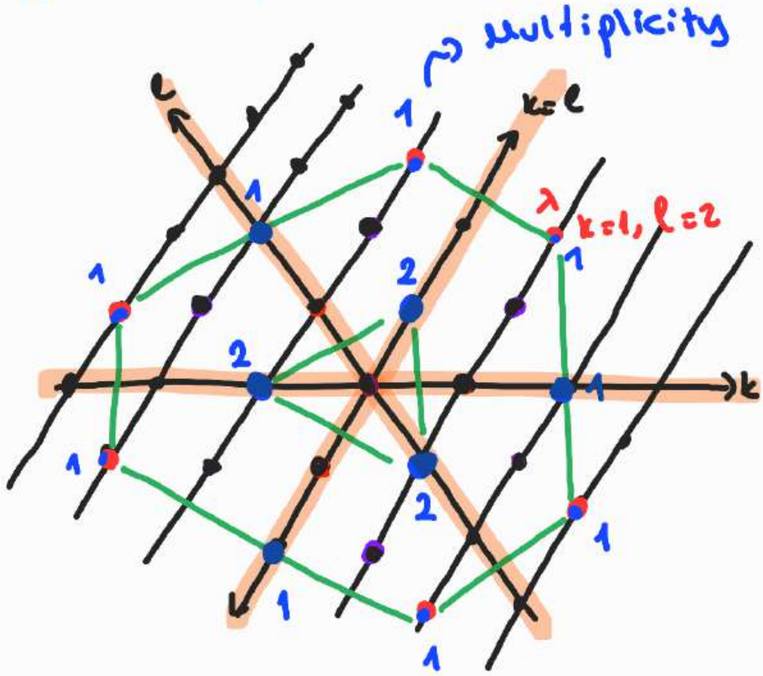


isomorphic

Weyl symmetry (Weyl Group) $\cong S_3$ symmetry group of equilateral triangle



Weight Diagrams for SU(3) - irreducible representation

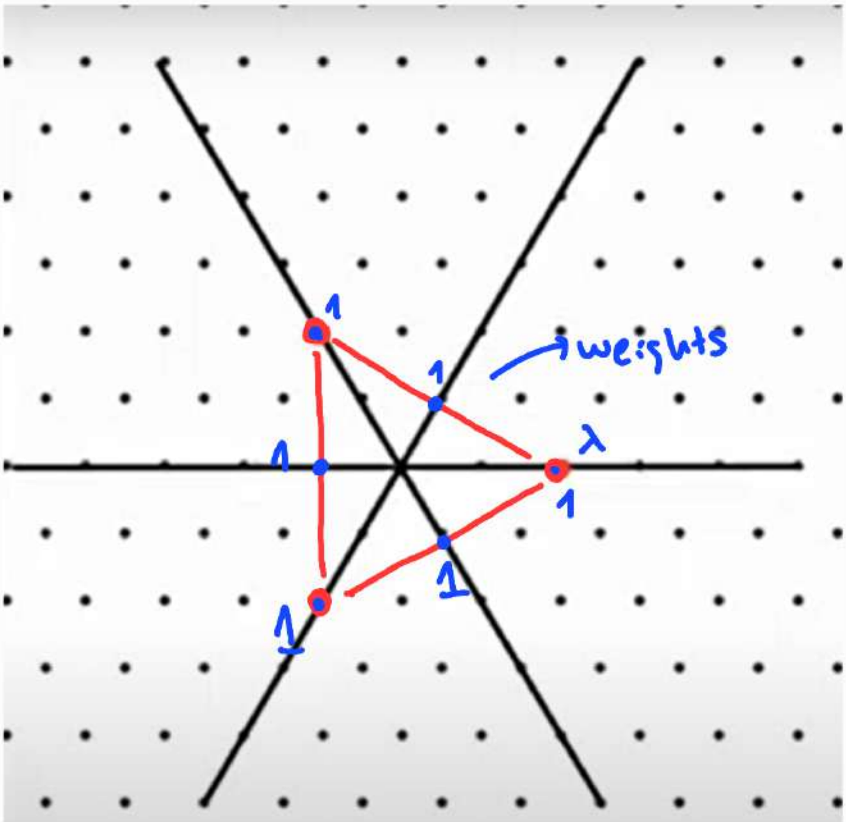


$\lambda = kL_1 - lL_3$

Blue dots are weights $\Gamma_{k,l}$

<u>Theorem:</u> For every $k, l \in \mathbb{N}$, there is an irrep. of SU(3) $\Gamma_{k,l}$, unique up to isomorphism, whose weight diagram is the following. Moreover this is a complete list.

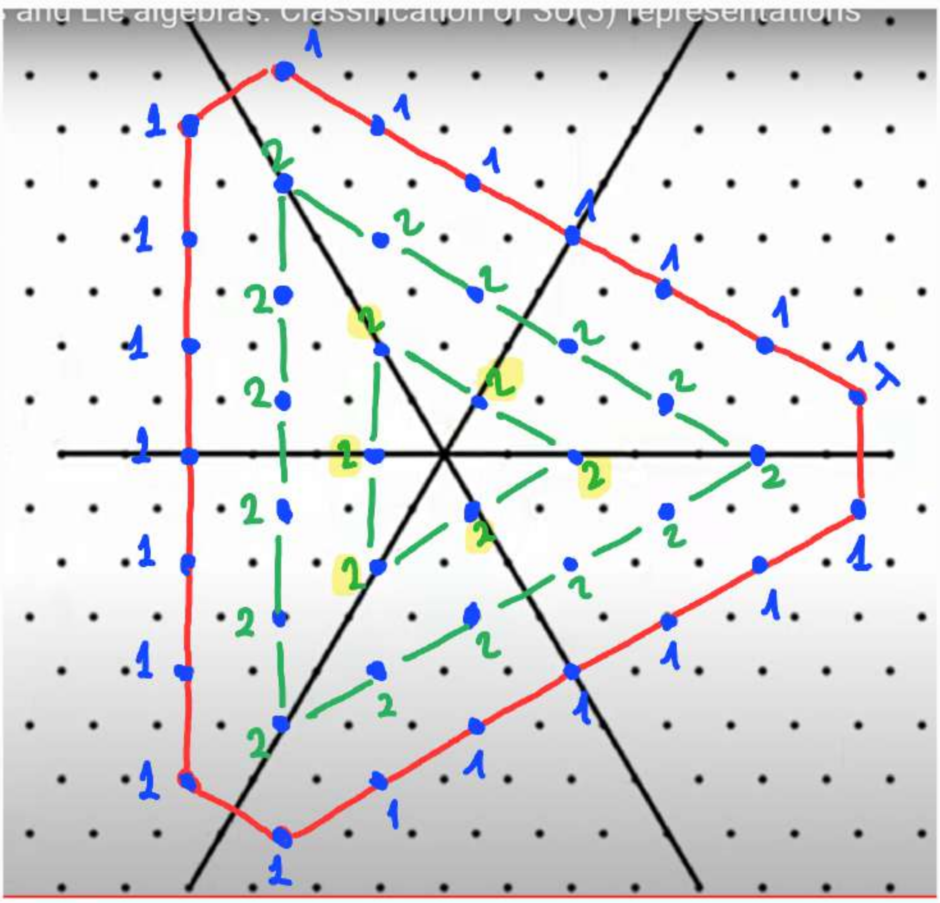
1. Reflect λ using Weyl Group
2. Take the convex hull of these points to get a polygon P.
3. Take as weights the points in P of the form $\lambda + r$ with r in the root lattice.



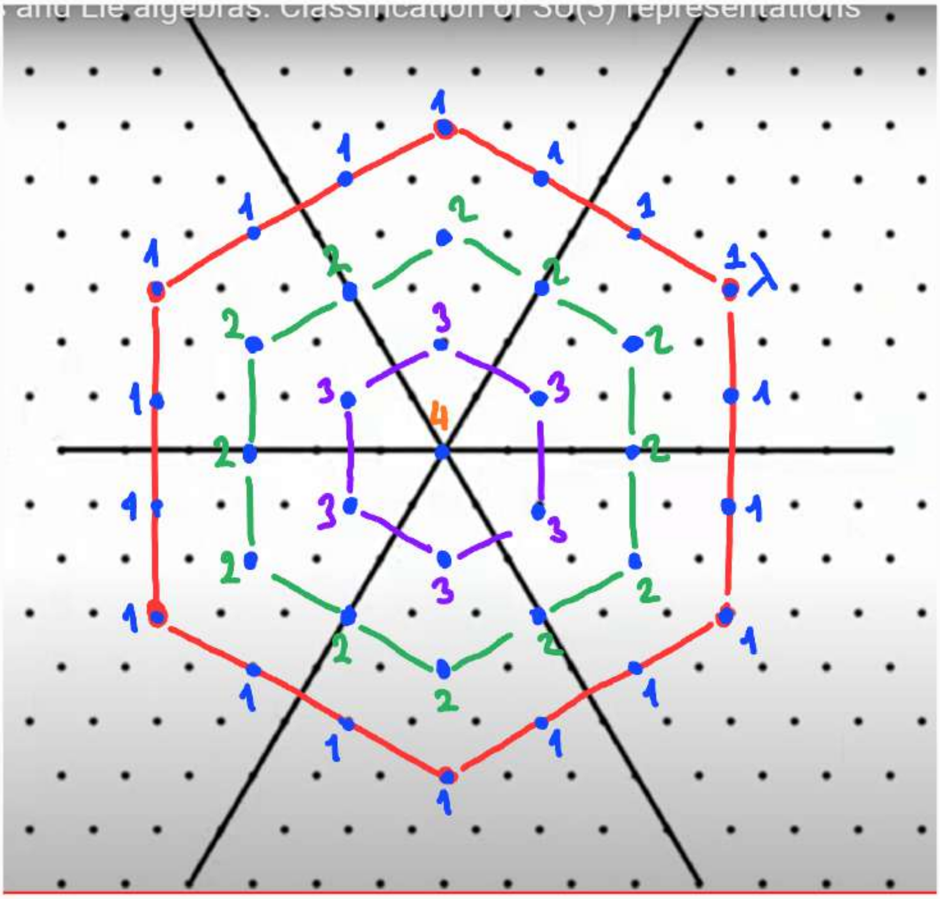
$\lambda = kL_1 - lL_3$

$\Gamma_{2,0}$

$\mathrm{Sym}^2 \mathbb{C}^3$

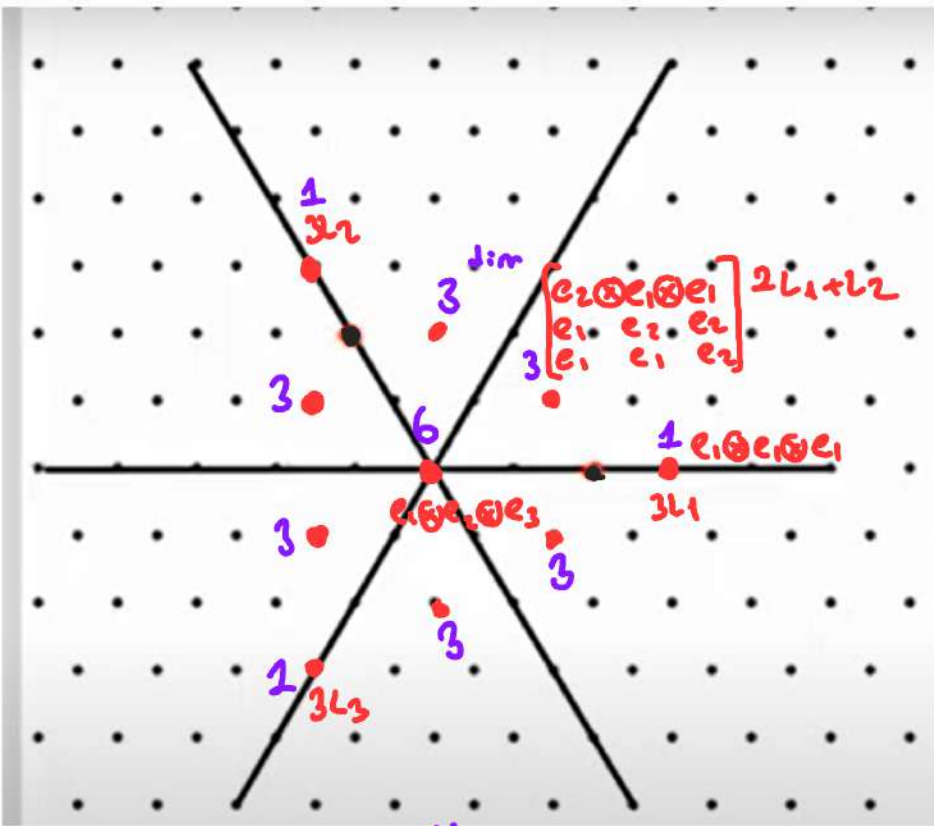


$\Gamma_{6,1}$



$\Gamma_{6,3}$

Decomposing SU(3) Representations

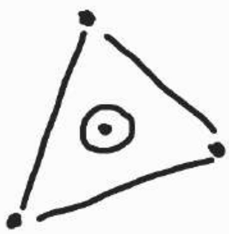


$\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$

standard reps of SU(3)

$\mathbb{C}^3$ $\quad e_1, e_2, e_3$

L_1, L_2, L_3



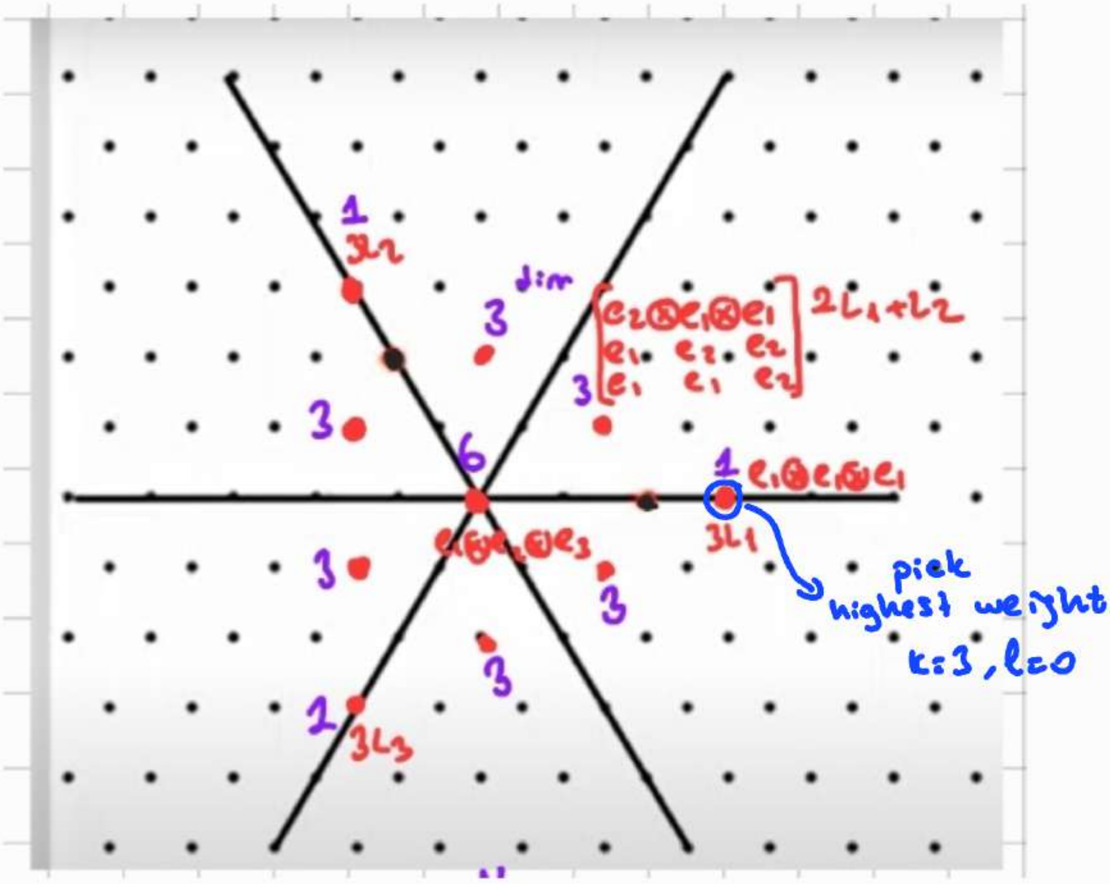
$\Downarrow$

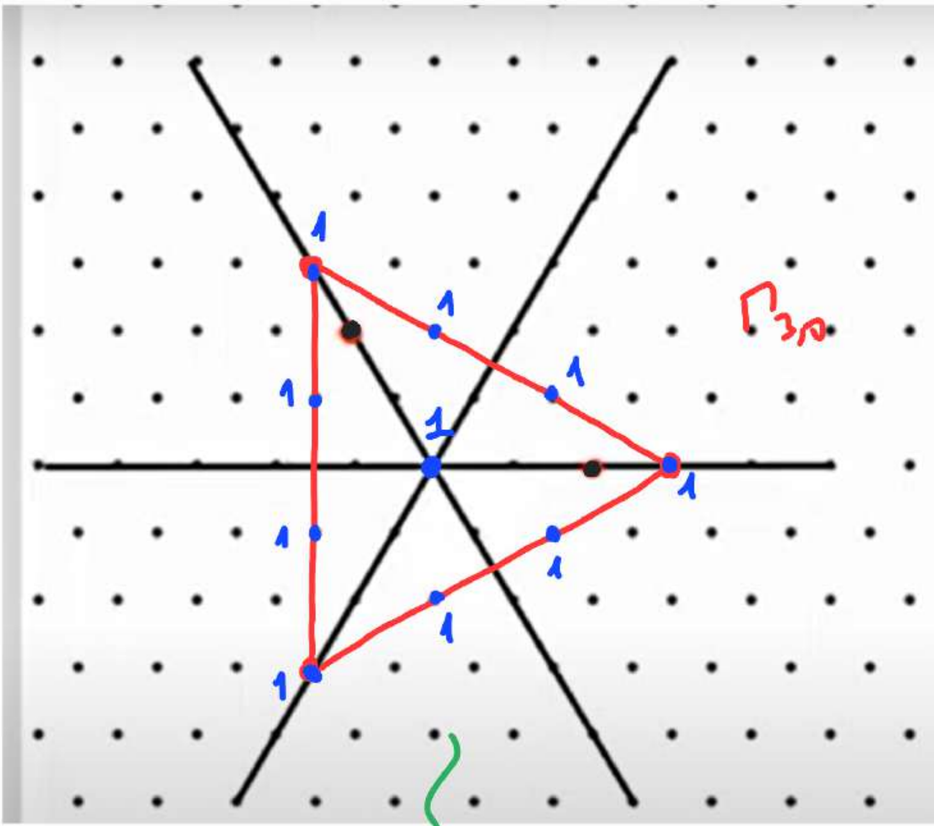
NOT IRREDUCIBLE SO DECOMPOSE IT

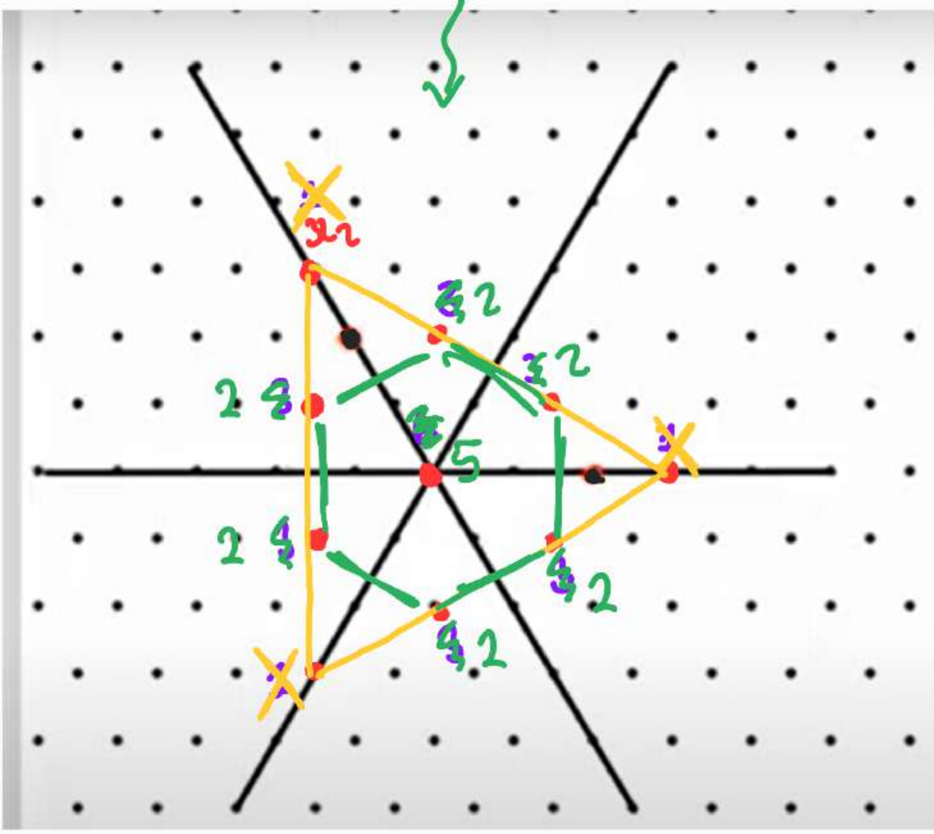
$$e_i \otimes e_j \otimes e_k \in W_{L_i + L_j + L_k}$$

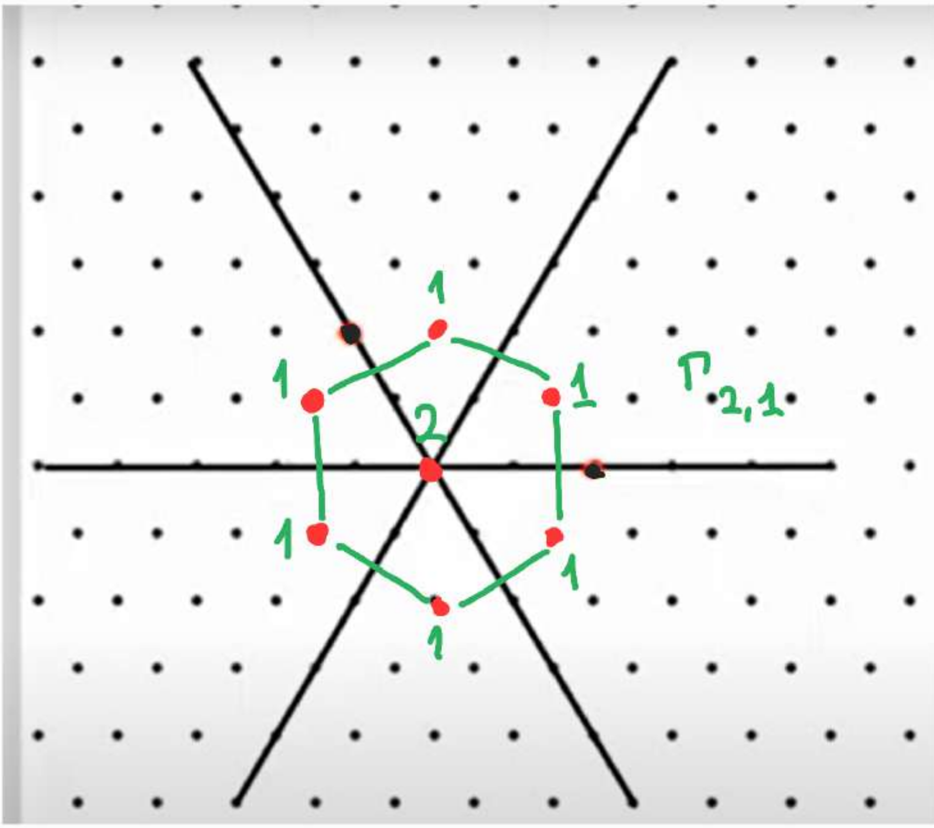
$$H_\theta(e_i \otimes e_j \otimes e_k) = (H_\theta e_i) \otimes e_j \otimes e_k + e_i \otimes (H_\theta e_j) \otimes e_k + e_i \otimes e_j \otimes (H_\theta e_k)$$

$$= (\vartheta_i + \vartheta_j + \vartheta_k)\, e_i \otimes e_j \otimes e_k$$




Γ_{3p}





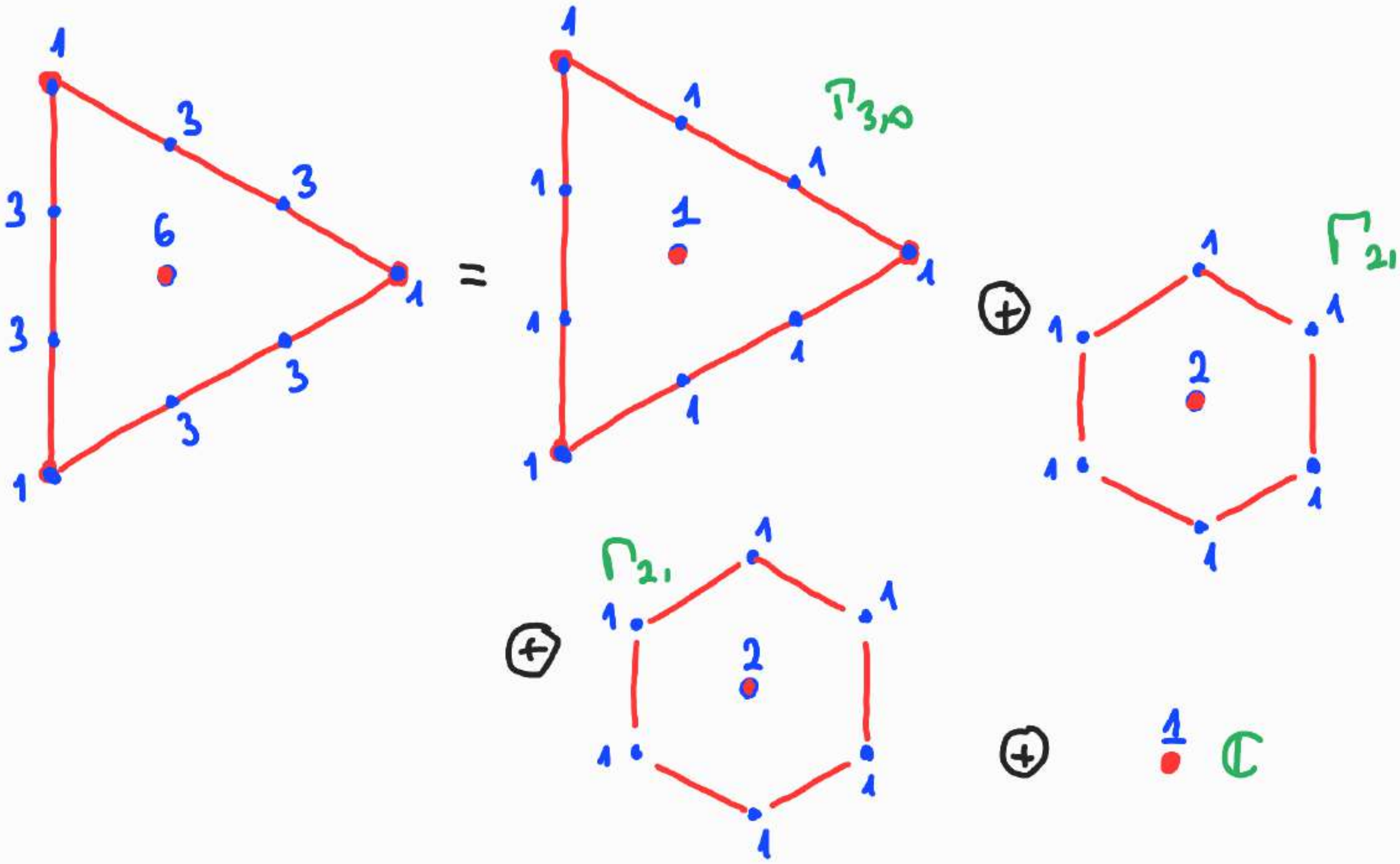
Take 2 copies of it

↓

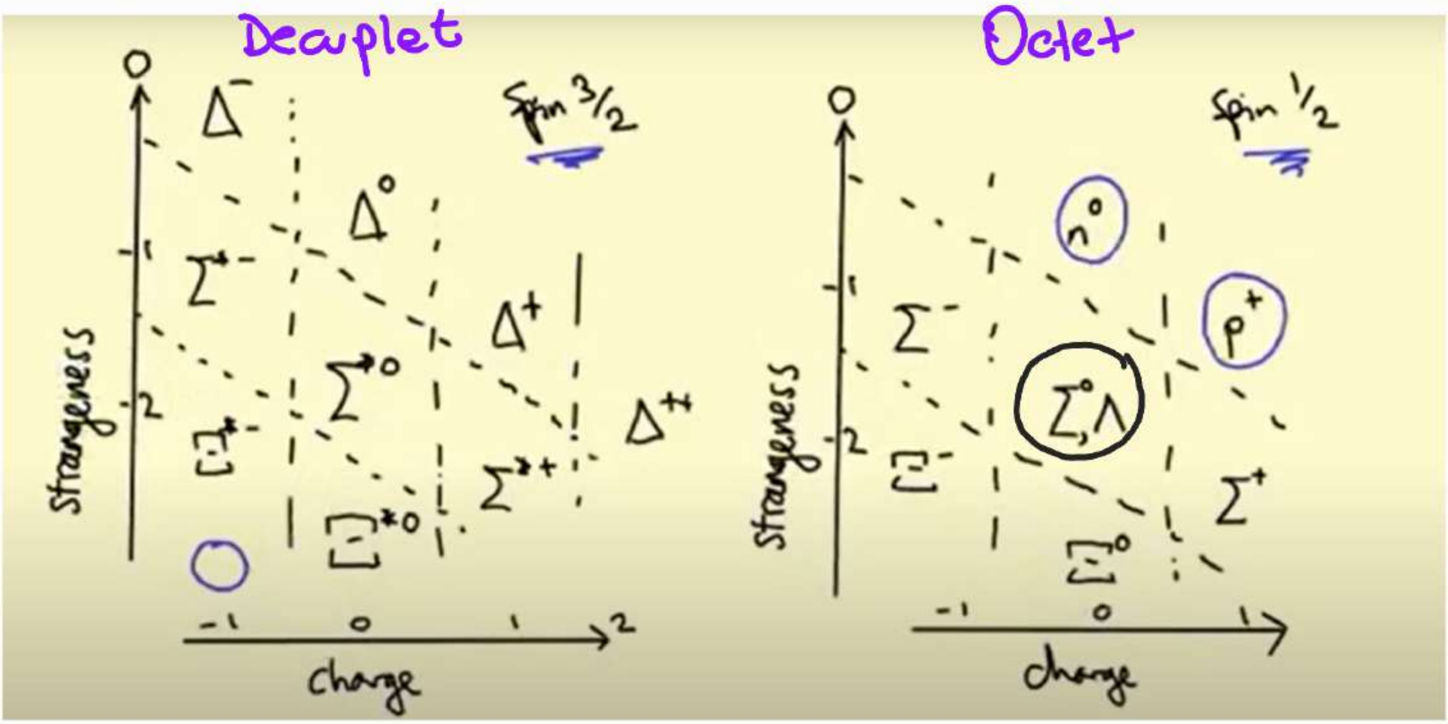
only 1 origin remains

$$\underbrace{}_{}\quad \mathrm{Sym}^3\,\mathbb{C}^3$$

$$\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3 \cong \Gamma_{3,0} \oplus \Gamma_{2,1} \oplus \Gamma_{2,1} \oplus \mathbb{C}$$



The Quark Model

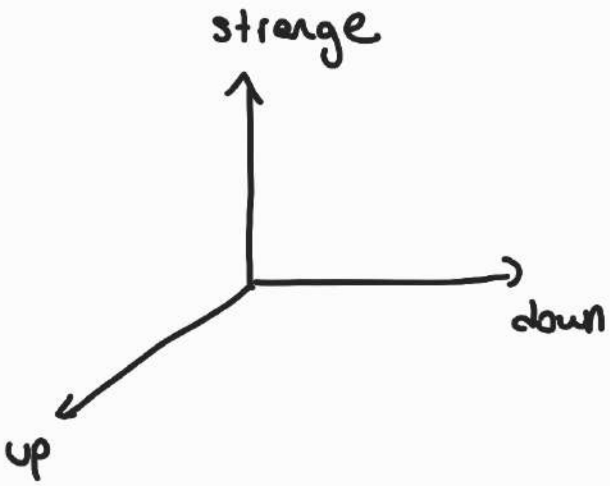


$$\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$$

3 quarks, each with 3 flavours

	up	down	strange
Q :	$\frac{2}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$
S :	0	0	-1

Each quark has state space $\mathbb{C}^3$



SU(3) flavour symmetry

3 quarks have state space $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$

$$u \otimes d \otimes s - d \otimes u \otimes s + d \otimes s \otimes u - u \otimes s \otimes d + s \otimes u \otimes d - s \otimes d \otimes u$$

spans the trivial 1-d subrep $\mathbb{C}$