Exercise set #1

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Exercise 1:

Given the following qubits:

$$|\Psi_1\rangle = |0\rangle$$
$$|\Psi_2\rangle = |0\rangle - i\,|1\rangle$$
$$|\Psi_3\rangle = \sqrt{2}\,|0\rangle + i\sqrt{2}\,|1\rangle$$

2+2= 4

a) Verify if the three cubits are normalized and if not, normalize them.

b) Show that $|\Psi_2\rangle$ and $|\Psi_3\rangle$ are orthogonal.

c) Show that $|\Psi_1\rangle$ and $|\Psi_2\rangle$ are not orthogonal.

d) What is the probability that if we measured $|\Psi_3\rangle$ in the computational basis $\{|0\rangle, |1\rangle\}$ we would get $|1\rangle$?

e) What is the probability that if we measured $|\Psi_1\rangle$ in the $\{|\Psi_2\rangle, |\Psi_3\rangle\}$ basis we would get $|\Psi_2\rangle$?

$|\psi_1\rangle = |0\rangle$

$|\psi_2\rangle = |0\rangle - i|1\rangle$

$|\psi_3\rangle = \sqrt{2}|0\rangle + i\sqrt{2}|1\rangle$

a) $\rightarrow \langle\psi_1|\psi_1\rangle \overset{?}{=} 1$

$\langle 0|0\rangle = \begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = 1$ ✓

$\rightarrow \langle\psi_2|\psi_2\rangle \overset{?}{=} 1$

$|\psi_2\rangle = |0\rangle - i|1\rangle = \begin{pmatrix}1\\0\end{pmatrix} - i\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\-i\end{pmatrix}$

$\langle\psi_2|\psi_2\rangle = \begin{pmatrix}1 & +i\end{pmatrix}\begin{pmatrix}1\\-i\end{pmatrix} = 1 - \underbrace{i^2}_{-1} = 2$ × not normalized

It can be $|\psi_2\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}i|1\rangle$

$\rightarrow \langle\psi_3|\psi_3\rangle \overset{?}{=} 1$

$|\psi_3\rangle = \sqrt{2}|0\rangle + i\sqrt{2}|1\rangle = \sqrt{2}\left(|0\rangle + i|1\rangle\right) = \sqrt{2}\begin{pmatrix}1\\i\end{pmatrix}$

$\langle\psi_3|\psi_3\rangle = \sqrt{2}\begin{pmatrix}1 & -i\end{pmatrix}\sqrt{2}\begin{pmatrix}1\\i\end{pmatrix} = 2\left(1 - \underbrace{i^2}_{-1}\right) = 4$ × not normalized

$|\psi_3\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}i|1\rangle$

b) $\langle\psi_2|\psi_3\rangle \overset{?}{=} 0$ (orthogonal)

$|\psi_2\rangle = \begin{pmatrix}1\\-i\end{pmatrix}$ $\langle\psi_2|\psi_3\rangle = \sqrt{2}\begin{pmatrix}1 & i\end{pmatrix}\begin{pmatrix}1\\i\end{pmatrix} = \left(1 + \underbrace{i^2}_{-1}\right)\sqrt{2} = 0$

$|\psi_3\rangle = \sqrt{2}\begin{pmatrix}1\\i\end{pmatrix}$

c) $\langle\psi_1|\psi_3\rangle \neq 0$ (not orthogonal)

$|\psi_1\rangle = |0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ $\langle\psi_1|\psi_3\rangle = \sqrt{2}\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}1\\i\end{pmatrix} = \sqrt{2}\cdot 1 \neq 0$

$|\psi_3\rangle = \sqrt{2}\begin{pmatrix}1\\i\end{pmatrix}$

d) $|\psi_3\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle$

$\swarrow$

normalized

$$P(|1\rangle) = \left|\frac{i}{\sqrt{2}}\right|^2 = \frac{1}{2}$$

e) $\boxed{|\psi_1\rangle = \alpha_2|\psi_2\rangle + \alpha_3|\psi_3\rangle}$

$$|0\rangle = \alpha_2 \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix} + \alpha_3\frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}$$

$$\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}\alpha_2+\alpha_3\\ i(\underbrace{-\alpha_2+\alpha_3}_{0})\end{pmatrix} \qquad \frac{\sqrt{2}\cancel{2}\alpha_2}{\cancel{\sqrt{2}}} = 1 \rightarrow \alpha_2 = \frac{1}{\sqrt{2}}$$

$$\Downarrow$$

$$\alpha_2 = \alpha_3$$

$$P(|\psi_2\rangle) = |\alpha_2|^2 = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}\,//$$

Exercise 2:

Given the single cubit quantum gates:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

and the qubits:

$$|\Psi_1\rangle = |1\rangle$$
$$|\Psi_2\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)$$

a) Show that X, Y, Z and H are unitary and hermitian.

b) Show that $XYZ = iI$.

c) Show that $HXH = Z$.

d) Show that $HZH = X$ without explicitly multiplying the matrices.

e) Show that $HXHZHXH = Z$ without explicitly multiplying the matrices.

f) What is the result of applying Y to $|\Psi_2\rangle$?

g) What is the result of applying H to $|\Psi_1\rangle$?

h) How can we prepare $|\Psi_2\rangle$ from $|\Psi_1\rangle$? $|\Psi_2\rangle = H X |\Psi_1\rangle$

a) Unitary and Hermitian

$UU^{\dagger} = U^{\dagger}U = I$ $\qquad$ $\rightarrow$ $U = U^{\dagger}$

$$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad Y^{\dagger} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad Y = Y^{\dagger}$$

Since X, Z, H are hermitian.

b) $XYZ = iI$

c) $HXH = Z$

d) $HZH = X$

From c) $\quad \underbrace{HH}_{I} X \underbrace{HH}_{I} = HZH$

e) $\underbrace{HX}\ \underbrace{H}_{I}Z\ \underbrace{HXH}_{Z} = Z$

$\boxed{HXH = Z}$

f) $$Y|\psi_2\rangle = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \frac{1}{\sqrt{2}}(-i|0\rangle + i|1\rangle)$$

$$Y|\psi_2\rangle = \frac{1}{\sqrt{2}}(-i|0\rangle + i|1\rangle) = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

e) $$H|\psi_1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

h) $|\psi_2\rangle = HX|\psi_1\rangle$

$\rightarrow$ one way for solution

Exercise 3:

If $\hat{n} = (n_X, n_Y, n_Z)$ is a real unit vector in three dimensional space, we can define a rotation by θ around the $\hat{n}$ axis as:

$$R_{\hat{n}}(\theta) = \cos\left(\frac{\theta}{2}\right) I - i \sin\left(\frac{\theta}{2}\right)(n_X X + n_Y Y + n_Z Z)$$

a) Show that $R_{\hat{x}}(\pi) = -iX$, where $\hat{x} = (1, 0, 0)$.

b) What is the result of applying $R_{\hat{z}}(\pi)$, where $\hat{z} = (0, 0, 1)$, to $|\Psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$?

c) Find $\hat{n}$ so that $H = iR_{\hat{n}}(\pi)$.

d) Show that $H = iR_{\hat{x}}(\pi)R_{\hat{y}}(\frac{\pi}{2})$, where $\hat{x} = (1, 0, 0)$ and $\hat{y} = (0, 1, 0)$.

e) Show that $R_{\hat{z}}(\theta) Y R_{\hat{z}}(\theta)^\dagger = \cos\theta Y - \sin\theta X$.

a) $R_{\hat{x}}(\pi) = \underbrace{\cos\left(\frac{\pi}{2}\right)}_{0} I - i \underbrace{\sin\left(\frac{\pi}{2}\right)}_{1} X = -iX$ ✓

b) $R_{\hat{z}}(\pi) = \underbrace{\cos\left(\frac{\pi}{2}\right)}_{0} I - i\underbrace{\sin\left(\frac{\pi}{2}\right)}_{1} (Z) = -iZ$

$-iZ|\Psi\rangle = \frac{-i}{\sqrt{2}}\left(Z|0\rangle + iZ|1\rangle\right) = \frac{-i}{\sqrt{2}}\left(|1\rangle - i|0\rangle\right)$

$Z|0\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |1\rangle$

$Z|1\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -|0\rangle$

$= \frac{1}{\sqrt{2}}\left(-|0\rangle - i|1\rangle\right)$

$= - \frac{1}{\sqrt{2}}\left(|0\rangle + i|1\rangle\right)$

global phase

c) $H = i R_{\hat{n}}(\pi)$

$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ $\quad R_{\hat{n}}(\pi) = \underbrace{\cos\left(\frac{\pi}{2}\right)}_{0} I - i\underbrace{\sin\left(\frac{\pi}{2}\right)}_{1}\hat{n}$

$i R_{\hat{n}}(\pi) = -i\hat{n}\, i$

$H = i R_{\hat{n}}(\pi) = \hat{n} = \frac{1}{\sqrt{2}}(1, 0, 1)$

↳ since we don't have any imaginary part in Hadamard Gate

d) $H = i R_x(\pi) R_y\left(\frac{\pi}{2}\right) = i\,(-iX)\left(\frac{1}{\sqrt{2}} I - \frac{i}{\sqrt{2}} Y\right) = H$

e) $R_z(\theta)\, Y\, R_z(\theta)^\dagger = \cos\theta\, Y - \sin\theta\, X$

$= \left(\cos\left(\frac{\theta}{2}\right) I - i\sin\left(\frac{\theta}{2}\right) Z\right) Y \left(\cos\frac{\theta}{2} I + i\sin\frac{\theta}{2} Z\right)$

$= \left(\cos\frac{\theta}{2} Y - i\sin\frac{\theta}{2} ZY\right)\left(\cos\frac{\theta}{2} I + i\sin\left(\frac{\theta}{2}\right) Z\right)$

$= \left(\cos^2\frac{\theta}{2} Y + \sin^2\frac{\theta}{2}\overset{-Y}{ZYZ} + i\left(\sin\frac{\theta}{2}\cos\frac{\theta}{2} YZ - \sin\frac{\theta}{2}\cos\frac{\theta}{2} Z\cancel{X}\right)\right)$

$XYZ = iI$ $\quad \boxed{YZ = iX}$ $\quad YZ = -ZY$

$ZYZ = -ZZY = -Y$

$= \cos\theta\, Y - \sin\theta\, X$