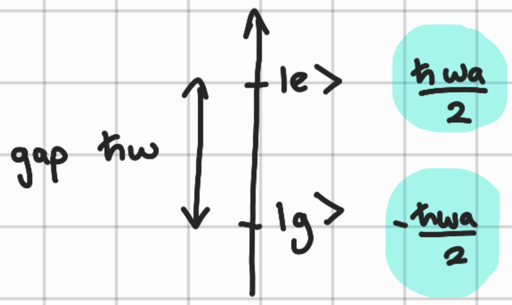
WEEK 14 Jaynes-Cummings Hamiltonian

Hamiltonian 3 terms

$$\hat{H} = H_{\substack{\text{two level}\\ \text{system}}} + H_{\substack{\text{harmonic}\\ \text{oscillator}}} + H_{\substack{\text{coupling}\\ \text{(interaction)}}}$$

→ Two level system



$$H_{\substack{\text{two level}\\ \text{system}}} = -\frac{\hbar\omega_a}{2}|g\rangle\langle g| + \frac{\hbar\omega_a}{2}|e\rangle\langle e|$$

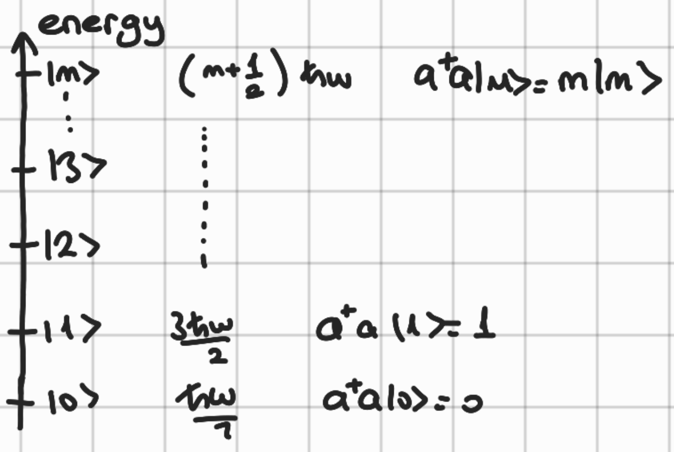
$$= \frac{\hbar\omega_a}{2}\left(|e\rangle\langle e| - |g\rangle\langle g|\right)$$

$$= \frac{\hbar\omega_a}{2}\begin{array}{c c} & \begin{array}{cc} e & g \end{array} \\ & \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{array}{c} e \\ g \end{array} \end{array}$$

$$H_{\text{two-level}} = \frac{\hbar\omega_a}{2}\sigma_z$$

→ Harmonic Osc. Mode

$$H_{h.o} = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)$$



→ Interaction $\boxed{H_{int} = \frac{\hbar\Omega}{2}\left(a^{+}\sigma_{-} + a\,\sigma_{+}\right)}$

$$\sigma_{+} = \begin{array}{cc} e & g \\ \end{array}$$
$$\sigma_{+} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = |e\rangle\langle g| \quad \text{where } |g\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ |e\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\sigma_{-} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = |g\rangle\langle e|$$

$$\sigma_{+}|g\rangle = |e\rangle \qquad \sigma_{+}|e\rangle = 0$$

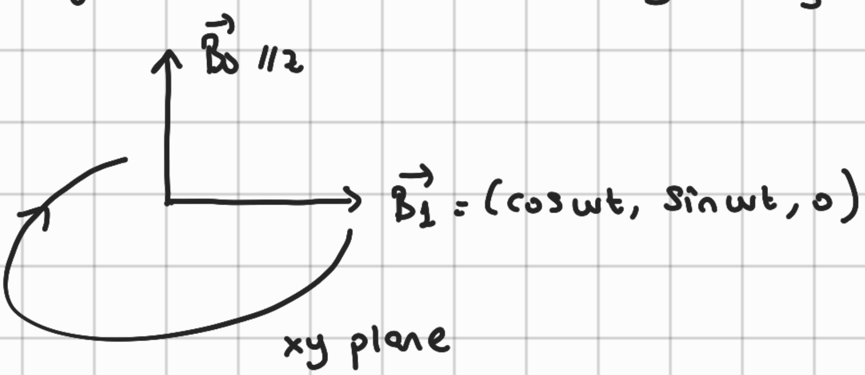
$$\sigma_{-}|e\rangle = |g\rangle \qquad \sigma_{-}|g\rangle = 0$$

absorption of photon
one photon is annihileted

Jaynes-Cummings Hamiltonian

$$\hat{H} = \frac{\hbar\omega_a}{2}\sigma_z + \hbar\omega\left(a^{+}a + \frac{1}{2}\right) + \frac{\hbar\Omega}{2}\left(a^{+}\sigma_{-} + a\,\sigma_{+}\right)$$

Remark: Dynamics spin in rotating magnetic field



$$\underbrace{-\frac{\hbar\omega_{Larmor}}{2}\sigma_z}_{\text{from } B_0} + \underbrace{\frac{\hbar\omega_1}{2}\left(\sigma_+ e^{-i\omega t} + \sigma_- e^{+i\omega t}\right)}_{\text{from } B_1}$$

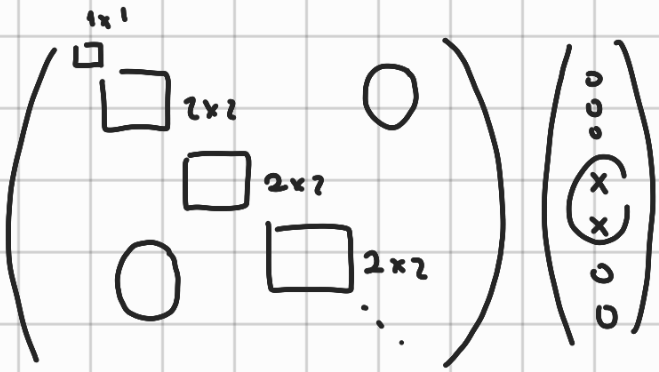
(Similar to James Cummings Hamiltonian)

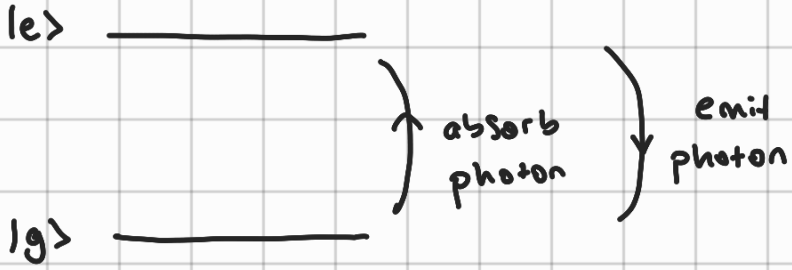
$$\hat{H} = \frac{\hbar\omega_a}{2}\overset{2\times2}{\sigma_z} + \hbar\omega\left(\overset{\text{infinite}\times\text{infinite}}{a^\dagger a} + \frac{1}{2}\right) + \frac{\hbar\Omega}{2}\left(\overset{\text{inf}}{a^\dagger}\,\overset{2\times2}{\sigma_-} + \overset{\text{inf}}{a}\,\overset{2\times2}{\sigma_+}\right)$$

$$\underbrace{L^2(\mathbb{R})}_{\text{harmonic osc}} \otimes \underbrace{\mathbb{C}^2}_{\text{qubit on two-level system}}$$

Infact this inf. x inf. matrix has $\underbrace{\text{Block structure}}_{2\times2 \text{ block}}$

Need to find invariant subspaces ?





# of excitations which is an invariant

$$\underbrace{\#\text{ of photons}}_{a^\dagger a} + \underbrace{\#\text{ exc of atom}}_{|e\rangle\langle e|}$$

$$N_{exc} = a^{\dagger}a + \sigma_+ \sigma_-$$

$\sigma_+ = |e\rangle\langle g|$

$\sigma_- = |g\rangle\langle e|$

$\sigma_+ \sigma_- = |e\rangle\underbrace{\langle g|g\rangle}_{1}\langle e| = |e\rangle\langle e|$

$|g,m\rangle = |g\rangle \otimes |m\rangle \qquad \mathbb{C}^2 \otimes L^2(\mathbb{R})$

Notation

$N_{exc}|g,m\rangle = a^{\dagger}a|g,m\rangle + \sigma_+\sigma_-|g,m\rangle$

$N_{exc}|g,m\rangle = m|g,m\rangle + 0$

$\rightarrow N_{exc}|g,m+1\rangle = (m+1)|g,m+1\rangle$

$\rightarrow N_{exc}|e,m\rangle = (m+1)|e,m\rangle$

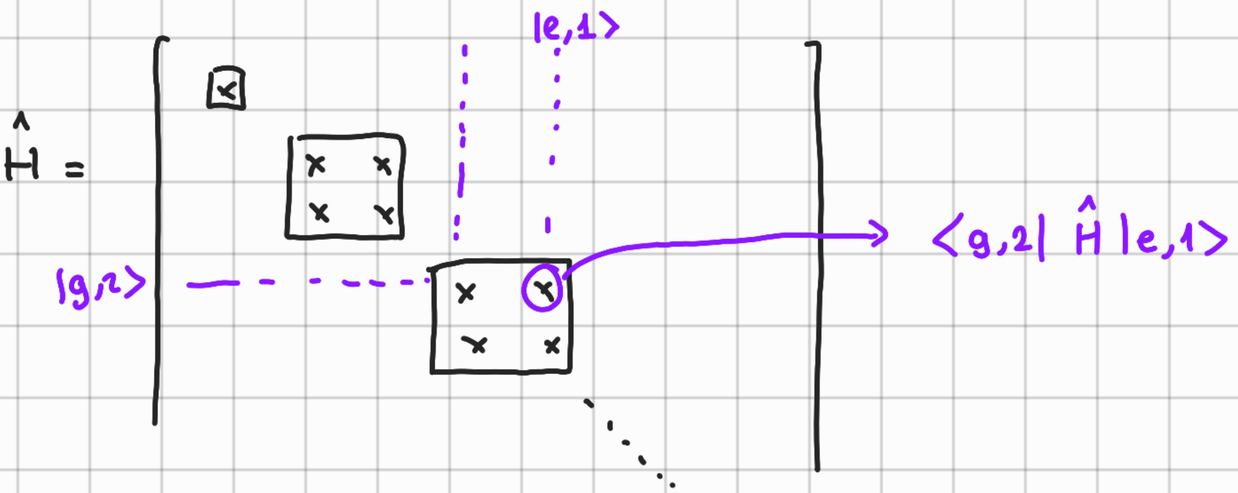
eigenstates of N_{exc} with eigenvalues ★

$$\begin{pmatrix} a^{\dagger}a|e,m\rangle + \sigma_+\sigma_-|e,m\rangle \\ m|e,m\rangle + |e,m\rangle \end{pmatrix}$$

$$\begin{array}{c|cccccc} & |g,0\rangle & |g,1\rangle & |e,0\rangle & |g,2\rangle & |e,1\rangle & \cdots \\ \hline |g,0\rangle & \boxed{0} & & & & & \\ |g,1\rangle & & 1 & 0 & & & \\ |e,0\rangle & & 0 & 1 & & & \\ |g,2\rangle & & & & 2 & 0 & \\ |e,1\rangle & & & & 0 & 2 & \\ \vdots & & & & & & \ddots \end{array}$$

$$H = \frac{\hbar\omega_a}{2}\sigma_z + \hbar\omega\, a^{+}a + \frac{h\Omega}{2}\left(a^{+}\sigma_- + a\,\sigma_+\right)$$

Check with commutation relation $[\hat{H}, N_{exc}]$



$$
\begin{array}{c|cc}
 & |g,m\rangle & |e,m-1\rangle \\
\hline
 & \left[\begin{array}{ccc}
\boxed{\langle g,0|\hat{H}|g,0\rangle} & & \\
 & \ddots & \\
 & & \boxed{\begin{array}{cc} \langle g,m|H|g,m\rangle & \langle g,m|H|e,m-1\rangle \\ \langle e,m-1|H|g,m\rangle & \langle e,m-1|H|e,m-1\rangle \end{array}}_{2\times 2} \\
 & & \ddots
\end{array}\right] & \\
|g,m\rangle & & \\
|e,m-1\rangle & &
\end{array}
$$

$$
= \left[\begin{array}{ccc}
-\frac{\hbar\omega_a}{2} & & \\
 & \ddots & \\
 & & \boxed{\begin{array}{cc} \overset{\text{Coupling term}=0}{-\frac{\hbar\omega_a}{2}+\hbar\omega m} & \overset{\text{Only coupling}}{\frac{\hbar\Omega}{2}\sqrt{m}} \\ \underset{\text{Only coupling}}{\frac{\hbar\Omega}{2}\sqrt{m}} & \underset{\text{Coupling term}=0}{+\frac{\hbar\omega_a}{2}+\hbar\omega(m-1)} \end{array}}_{2\times 2} \\
 & & \ddots
\end{array}\right]
$$

$$
\langle e,m-1|a\,\sigma_+|g,m\rangle = \sqrt{m}\,\langle e,m-1|e,m-1\rangle = \sqrt{m}
$$

$m \geq 0$

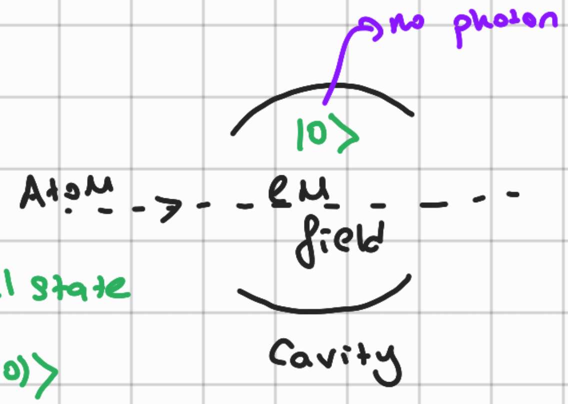
<u>Eigenvalues</u> $\quad -\frac{\hbar\omega_a}{2} \; ; \cdots\cdots ; \; E_\pm(m) = \hbar\omega\left(m+\frac{1}{2}\right) \pm \frac{\hbar}{2}\sqrt{\delta^2+\Omega^2(m+1)}$

$\delta = \omega_a - \omega$

<u>Eigenstates</u> $\quad |g,0\rangle \; ; \cdots\cdots ; \; |\pm,m\rangle = \cos\theta_m \, |e,m\rangle + \sin\theta_m \, |g,m+1\rangle$

$$\tan\theta_m = -\frac{\Omega\sqrt{m+1}}{\delta}$$

Example: Rabi Oscillations in the "vacuum"



Initial state

$|\psi(0)\rangle$

$|\psi(0)\rangle = |e,0\rangle = |e\rangle \otimes |0\rangle$

$|\psi(t)\rangle = e^{-\frac{it}{\hbar}H_{JC}} \, |e,0\rangle$

express this in terms of:

$\cos\theta_m \, |+,m\rangle + \sin\theta_m \, |-,m\rangle$

$$|\psi(t)\rangle = e^{-\frac{it}{\hbar}E_+(m)} \cos\theta_m \, |+,m\rangle + e^{-\frac{it}{\hbar}E_-(m)} \sin\theta_m \, |-,m\rangle$$

Go back to $|e,m\rangle$, $|g,m+1\rangle$ basis

Look at tuning $\delta = \omega_a - \omega = 0$

$$|\psi(t)\rangle \xrightarrow[\delta \to 0]{} \left(\cos\frac{\Omega t}{2}\right)|e,0\rangle - i\left(\sin\frac{\Omega t}{2}\right)|g,1\rangle$$