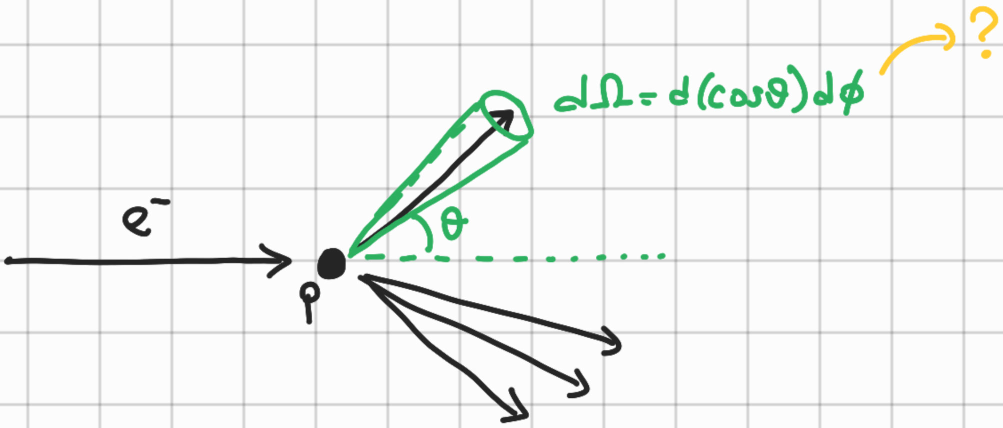
Week 5 Particle Scattering

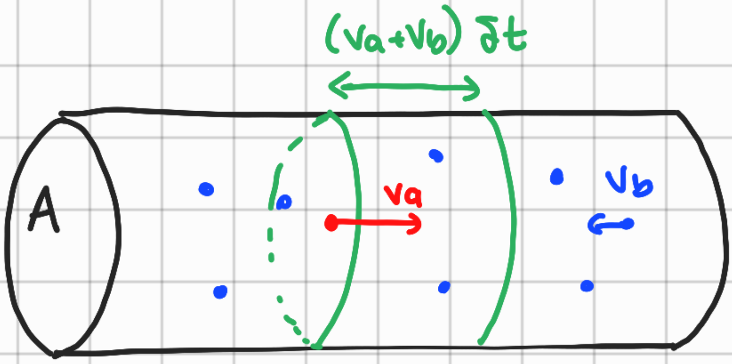
Cross Sections

$$\sigma = \frac{\text{\# of interactions per unit time per target}}{\text{incident flux}}$$

$$\frac{d\sigma}{d\Omega} = \frac{\text{\# of interactions per unit time per target into } d\Omega}{\text{incident flux}}$$



$$\rightarrow \sigma = \int \frac{d\sigma}{d\Omega}\, d\Omega$$



in time δt a single particle a traverses region containing $n_b (v_a + v_b) A \delta t$ particles of type b

$\downarrow$

Interaction probability $= \dfrac{n_b \overbrace{(v_a + v_b)}^{v} \cancel{A}\, \delta t\ \sigma}{\cancel{A}}$

$= \boxed{n_b\, v\, \delta t\ \sigma}$

Rate for a single particle a $= n_b v \sigma$

Consider volume V, total reaction rate:

$$(n_b v \sigma) \cdot (n_a V) = (n_b V)(n_a v)\sigma$$

$$= N_b \phi_a \sigma$$

Rate $= (\#\text{ of targets})_b \cdot (\text{Flux})_a \cdot (\text{Cross Section})$

Cross Section Calculations

"Scattering Process"

$1 + 2 \longrightarrow 3 + 4$

why only p_3 & p_4?

Fermi's Golden Rule:

$$\Gamma_{fi} = (2\pi)^4 \int |T_{fi}|^2\, \delta(E_1 + E_2 - E_3 - E_4)\, \delta^3(\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - \vec{p}_4)\, \frac{d^3\vec{p}_3}{(2\pi)^3} \frac{d^3\vec{p}_4}{(2\pi)^3}$$

$$\frac{\text{Rate}}{\text{Volume}} = (\#\,\text{Density of } 2)\cdot(\text{Flux of } 1)\times\sigma = n_2 \cdot n_1(v_1+v_2)\cdot\sigma$$

For 1 target particle per unit volume

$$\downarrow$$

$$\text{Rate} = (v_1+v_2)\,\sigma \implies \sigma = \frac{\Gamma_{fi}}{v_1+v_2}$$

$$\sigma = \frac{(2\pi)^4}{v_1+v_2}\int |T_{fi}|^2\,\delta(E_1+E_2-E_3-E_4)\,\delta^3(p_1+p_2-p_3-p_4)\,\frac{d^3p_3}{(2\pi)^3}\frac{d^3p_4}{(2\pi)^3}$$

Lorentz Invariant Parts

To obtain a Lorentz invariant form use wave-functions

normalised to $2E$ particles per unit volume $\psi' = (2E)^{1/2}\,\psi$

Again, define $M_{fi} = (2E_1\,2E_2\,2E_3\,2E_4)^{1/2}\,T_{fi}$

$$\sigma = \frac{(2\pi)^{-2}}{2E_1 \cdot 2E_2\,(v_1+v_2)}\int |M_{fi}|^2\,\delta(E_1+E_2-E_3-E_4)\,\delta^3(p_1+p_2-p_3-p_4)\,\frac{d^3p_3}{2E_3}\,\frac{d^3p_4}{2E_4}$$

Lorentz-Invariant form

$$F = 2E_1\,2E_2\,(v_1+v_2) \xrightarrow[\text{of 4-vector scalar product}]{\text{in the form}} F = 4\left[(p_1^{\mu}p_{2\mu})^2 - m_1^2 m_2^2\right]^{1/2}$$

Two Special cases of Lorentz Invariant Flux

1. CoM frame: $F = 4E_1E_2(v_1+v_2)$ How?

$$= 4E_1E_2\left(\frac{|\vec{p}^*|}{E_1}+\frac{|\vec{p}^*|}{E_2}\right)$$

$$= 4|\vec{p}^*|(E_1+E_2) = 4|\vec{p}^*|\sqrt{s}$$

Mandelstam Variable "s" $= (p_1+p_2)^2$

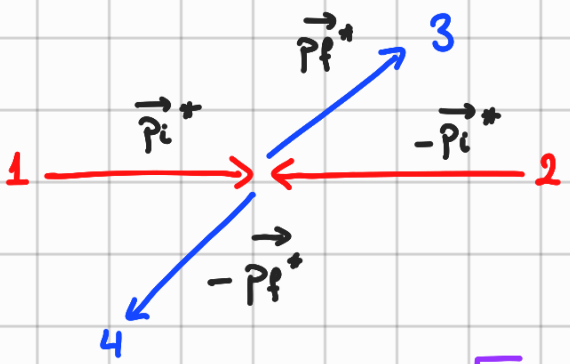
2. Target particle (particle 2) at rest $\rightarrow E_2 = m_2$, $v_2 = 0$

$$F = 4E_1E_2(v_1+v_2)$$

$$= 4E_1 m_2 v_1$$

$$= 4\cancel{E_1} m_2 \frac{|p_1|}{\cancel{E_1}} = 4m_2|\vec{p}_1|$$

$2 \rightarrow 2$ Body Scattering in CoM frame



$$\sigma = \frac{(2\pi)^{-2}}{\underbrace{(2E_1)(2E_2)(v_1+v_2)}_{4|\vec{p}_i^*|\sqrt{s}}}\int |M_{fi}|^2\,\delta(\overbrace{E_1+E_2}^{\sqrt{s}}-E_3-E_4)\,\delta^3(\overbrace{\vec{p}_1+\vec{p}_2}^{0}-\vec{p}_3-\vec{p}_4)\,\frac{d^3p_3}{2E_3}\,\frac{d^3p_4}{2E_4}$$

$$\vec{p}_1+\vec{p}_2 = 0 \longrightarrow E_1+E_2 = \sqrt{s}$$

$$\sigma = \frac{(2\pi)^{-2}}{4\,|\vec{p}_i^{\,*}|\sqrt{s}} \underbrace{\int |M_{fi}|^2\, \delta(\sqrt{s} - E_3 - E_4)\, \delta^3(\vec{p}_3 + \vec{p}_4)\, \frac{d^3p_3}{2E_3}\, \frac{d^3p_4}{2E_4}}$$

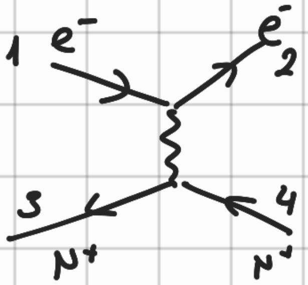
similar to case in particle decay calculation

$$\sigma = \frac{(2\pi)^{-2}}{4\,|\vec{p}_i^{\,*}|\sqrt{s}} \overbrace{\frac{|\vec{p}_f^{\,*}|}{4\, m_i} \int |M_{fi}|^2\, d\Omega^*}$$

$\uparrow \sqrt{s}$

$$\sigma = \frac{1}{64\pi^2 s} \frac{|\vec{p}_f^{\,*}|}{|\vec{p}_i^{\,*}|} \int |M_{fi}|^2\, d\Omega^*$$

For an elastic scattering $|\vec{p}_i^{\,*}| = |\vec{p}_f^{\,*}|$



$$\sigma_{elastic} = \frac{1}{64\pi^2 s} \int |M_{fi}|^2\, d\Omega^*$$

total cross section

$$d\sigma = \frac{1}{64\pi^2 s} \frac{|\vec{p}_f^{\,*}|}{|\vec{p}_i^{\,*}|} |M_{fi}|^2\, \underbrace{d\Omega^*}$$

$d(\cos\theta^*)\, d\phi^*$

in the C.o.M frame

! Let's find a L.I expression for $d\sigma$

Mandelstam "t"

$$t = q^2 = (p_1 - p_3)^2 = p_1^2 + p_3^2 - 2p_1.p_3 = m_1^2 + m_3^2 - 2p_1.p_3$$

$$p_1^{*\mu} = (E_1^*, 0, 0, |\vec{p_1}^*|)$$

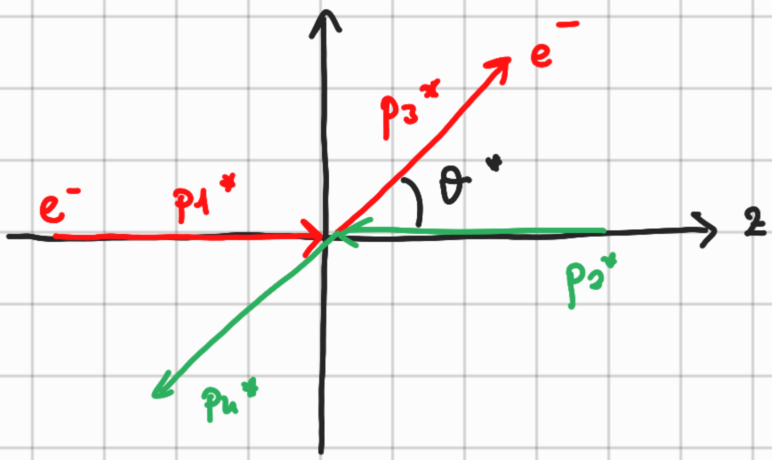
$$p_3^{*\mu} = (E_3^*, |\vec{p_3}^*| \sin\theta^*, 0, |\vec{p_3}^*| \cos\theta^*)$$

$$p_1^\mu p_{3\mu} = E_1^* E_3^* - |\vec{p_1}^*||\vec{p_3}^*| \cos\theta^*$$

only free parameter

$$t = m_1^2 + m_3^2 - 2E_1^* E_3^* + 2p_1^* p_3^* \cos\theta^*$$

$$dt = 2p_1^* p_3^* \, d(\cos\theta^*)$$



$$d\Omega^* = d(\cos\theta^*)\, d\phi^*$$

$$d\Omega^* = \frac{dt\, d\phi^*}{2|\vec{p_1}^*||\vec{p_3}^*|}$$

Mandelstam "t"

$$d\sigma = \frac{1}{64\pi^2 s} \frac{|\vec{p_f}^*|}{|\vec{p_i}^*|} |M_{fi}|^2 d\Omega^* = \frac{1}{2.\, 64\pi^2 s.\, |\vec{p_i}^*|^2} |M_{fi}|^2 d\phi^* dt$$

Assume no ϕ^* dependence of $|M_{fi}|^2$

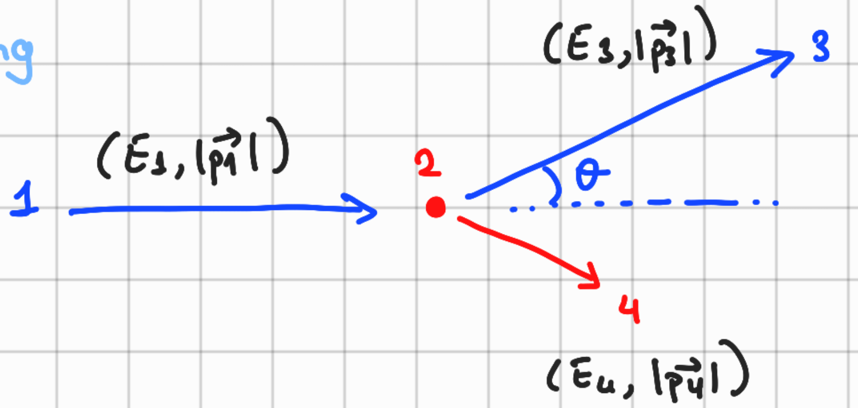
Integrate over $d\phi^*$,
introduces a factor of 2π

$$\boxed{\frac{d\sigma}{dt} = \frac{1}{64\pi s\, p_i^{*2}} |\mathcal{M}_{fi}|^2}$$

Lorentz Invariant

$$p_i^{*2} = \frac{1}{4s}\left[s-(m_1+m_2)^2\right]\left[s-(m_1-m_2)^2\right]$$

Scattering



$m_1 = m_3 = 0$

$m_2 = m_4 = M$

$$d\Omega = 2\pi\, d(\cos\theta)$$

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma}{dt}\frac{dt}{d\Omega} = \frac{1}{2\pi}\frac{t}{d(\cos\theta)}\frac{d\sigma}{dt}$$

(?)

$$p_1 = (E_1, 0, 0, E_1)$$

$$p_2 = (M, 0, 0, 0)$$

$$p_3 = (E_3, E_3\sin\theta, 0, E_3\cos\theta)$$

$$p_4 = (E_4, \vec{p_4})$$

Look at them later!

$$t = (p_1 - p_3)^2 = -2p_1 \cdot p_3 = -2E_1E_3(1-\cos\theta)$$

$\downarrow$

from $(E, \vec{p})$ conversion $p_1 + p_2 = p_3 + p_4$

equate these

$$t = (p_2 - p_4)^2 = 2M^2 - 2p_2 p_4 = 2M^2 - 2ME_4$$

$$t = 2M^2 - 2M(\underbrace{E_1 + M}_{E_2} - E_3) = -2M(E_1 - E_3)$$

E_1 is a constant (the 'E1' energy of the incoming particle)

$$\frac{dt}{d(\cos\theta)} = 2M \frac{dE_3}{d(\cos\theta)}$$

$$E_3 = \frac{E_1 M}{M + E_1 - E_1 \cos\theta}$$

$\swarrow$

$$\frac{dE_3}{d\cos\theta} = \frac{E_1^2 M}{(M + E_1 - E_1\cos\theta)^2} = E_1^2 M \left(\frac{E_3}{E_1 M}\right)^2 = \frac{E_3^2}{M}$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{2\pi} \frac{dt}{d(\cos\theta)} \frac{d\sigma}{dt} = \frac{1}{2\pi} 2M \frac{E_3^2}{M} \frac{d\sigma}{dt}$$

$$= \frac{E_3^2}{\pi} \frac{d\sigma}{dt} = \frac{E_3^2}{\pi} \frac{1}{16\pi(s - M^2)^2} |M_{fi}|^2$$

?

)

since $p_2^2 = 0$

$$s = (p_1 + p_2)^2 = M^2 + 2 p_1 p_2 = M^2 + 2 M E_1$$

$$(s - M^2) = 2 M E_1$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \left(\frac{E_3}{M E_1} \right)^2 |M_{fi}|^2$$

in the limit of $m_1 \to 0$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \left(\frac{1}{M + E_1 - E_1 \cos\theta} \right)^2 |M_{fi}|^2$$

Summary

Particle decay: $$\Gamma = \frac{|\vec{p}^*|}{32\pi^2 m_i^2} \int |M_{fi}|^2 \, d\Omega$$

where $$p^* = \frac{1}{2m_i}\sqrt{\left[m_i^2 - (m_1+m_2)^2\right]\left[m_i^2 - (m_1-m_2)^2\right]}$$?

$$\sigma = \frac{1}{64\pi^2 s} \frac{|\vec{p}_f^{\,*}|}{|\vec{p}_i^{\,*}|} \int |M_{fi}|^2 d\Omega^*$$

Lab frame $(m_1 = 0)$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\left(\frac{E_3}{M E_1}\right)^2 |M_{fi}|^2 \iff \frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\left(\frac{1}{M + E_1 - E_1\cos\theta}\right)^2 |M_{fi}|^2$$

Lab frame $(m_1 \neq 0)$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \cdot \frac{1}{|\vec{p}_1| m_2} \cdot \frac{|\vec{p}_3|^2}{|\vec{p}_3|(E_1 + m_2) - E_3|\vec{p}_1|\cos\theta} \cdot |M_{fi}|^2$$

$$E_1 + m_2 = \sqrt{|\vec{p}_3|^2 + m_3^2} + \sqrt{|\vec{p}_1|^2 + |\vec{p}_3|^2 - 2|p_1||p_3|\cos\theta + m_4^2}$$?