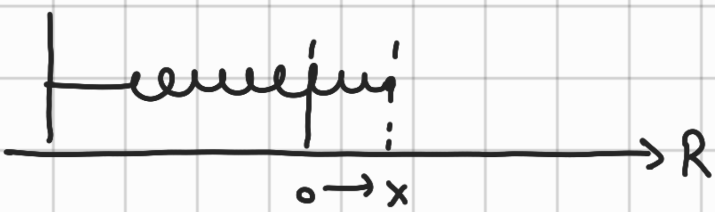
# WEEK 13 HARMONIC OSCILLATOR




$F = -k \cdot x$

Electromagnetic field is a collection of harmonic

$k = \frac{2\pi}{\lambda}$ , $\omega$ $(\omega = ck)$

and the photons are in fact excitations of these modes

## Hamiltonian (energy fct) of classical harmonic oscillator

Kinetic energy $\frac{1}{2} m v^2 = \frac{p^2}{2m}$ where $p = mv$ is momentum

Potential energy $\frac{1}{2} k x^2 = V(x)$ Force: $-\frac{d}{dx} V(x) = -kx$

$$H(x,p) = \frac{p^2}{2m} + \frac{1}{2} k x^2$$

Equations of motion: $\boxed{m\ddot{x} = -kx}$

$$\boxed{\begin{aligned} \dot{x} &= \frac{p}{m} = \frac{\partial H}{\partial p} \\ \dot{p} &= -kx = -\frac{\partial H}{\partial x} \\ &\downarrow \\ m\ddot{x} & \end{aligned}}$$

Solutions:

$$x(t) = A\cos(\omega t) + B\sin(\omega t)$$

$$p(t) = -mA\omega\sin(\omega t) + mB\,\omega\cos(\omega t)$$

$A$ & $B$ depend on $x(0)$, $p(0)$

$x(t) \sim \cos(\omega t)$

$\ddot{x}(t) = -\omega^2\cos(\omega t)$

$\boxed{m\omega^2 = k} \longrightarrow \omega = \sqrt{\frac{k}{m}}$ → Frequency of oscillations

Continous and unbounded degree of freedom : $x \in \mathbb{R}$

Hilbert space is infinite dim. space of "functions"

$\downarrow$

wave functions

$$\mathcal{H} = L^2(\mathbb{R}) = \left\{ \psi : \mathbb{R} \to \mathbb{C},\ x \to \psi(x) \quad \text{s.t} \int_{-\infty}^{\infty} |\psi(x)|^2 < \infty \right\}$$

State vector is a fct. $\psi$ or $|\psi\rangle$

And the value $\psi(x) = \langle x|\psi\rangle$

(Here $|x\rangle$ is a sort of basis of $\mathcal{H} = L^2(\mathbb{R})$)

inf. dim. matrix

$H(x,p) = \frac{p^2}{2m} + \frac{1}{2}kx^2$ is replaced by an operator

acting on functions of $L^2(R)$

How do we construct or guess the operator(s)

Basic objects are <u>position</u> & <u>momentum</u>

correspondence principle

$\hat{x}$ by definition $(\hat{x}\psi)(x) \equiv x\,\psi(x)$ → multiplication operator

$\hat{p}$ by definition $(\hat{p}\psi)(x) \equiv -i\hbar\frac{\partial}{\partial x}\psi(x)$

1920's

De Broglie $p = mv \rightarrow \lambda = \frac{h}{p} \Leftrightarrow \frac{2\pi}{k} = \frac{h}{p} \Rightarrow \boxed{p = \hbar k}$

Let $\psi(x) = e^{ikx}$

$(\hat{p}\psi)(x) = \hbar k\,\psi(x)$

Apply the Correspondence Principle to Hamiltonian

$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}k\hat{x}^2$ operator that acts on $\psi \in L^2(\mathbb{R})$

$\hat{p} = -i\hbar\frac{d}{dx} \Rightarrow \hat{p}^2 = -\hbar^2\frac{d^2}{dx^2}$

$\hat{x}\,\psi(x) = x\,\psi(x) \Rightarrow \hat{x}^2\psi(x) = x^2\psi(x)$

→ value in R

$$\hat{H}\Psi(x) = \frac{-\hbar^2}{2m}\frac{d^2}{dx^2}\Psi(x) + \frac{1}{2}k x^2 \Psi(x)$$

$$\boxed{\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}k\hat{x}^2}$$

## Commutation Relation $\hat{x}$, $\hat{p}$

$$\hat{x}\hat{p} - \hat{p}\hat{x} = i\hbar \, \mathbb{I}$$

Notation $[A,B] \equiv AB - BA$

$$[\hat{x},\hat{p}] = i\hbar$$

$$[\hat{x},\hat{p}]\Psi = i\hbar\Psi$$

$$\hat{x}(\hat{p}\Psi)(x) - \hat{p}(\hat{x}\Psi)(x) = x(-i\hbar\Psi'(x)) - (-i\hbar)\frac{d}{dx}(x\Psi(x))$$

$$= \cancel{-i\hbar x \Psi'(x)} + i\hbar \underline{\underline{\Psi(x)}} + \cancel{i\hbar x \Psi'(x)}$$

## Algebraic approach with creation & annihilation operators

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}k\hat{x}^2 = \underset{\text{Unit of energy}}{\hbar\omega}\left\{\frac{\hat{p}^2}{2m\hbar\omega} + \frac{1}{2}\underbrace{\frac{k}{\hbar\omega}}_{\frac{m\omega}{2\hbar}}{}^2\hat{x}^2\right\}$$

Recall $\omega = \sqrt{\frac{k}{m}}$ $\quad k = m\omega^2$

$$\hat{H} = \hbar\omega \Big(\underbrace{\frac{\hat{p}}{\sqrt{2m\hbar\omega}} + i\sqrt{\frac{m\omega}{2\hbar}}\,\hat{x}}_{a^{+}\text{ (creation)}}\Big)\Big(\underbrace{\frac{\hat{p}}{\sqrt{2m\omega\hbar}} - i\sqrt{\frac{m\omega}{2\hbar}}\,\hat{x}}_{a\text{ (annihilation)}}\Big) -$$

$$i\hbar\omega\sqrt{\frac{m\omega}{2\hbar}}\,\frac{1}{\sqrt{2m\hbar\omega}}\,(\underbrace{\hat{x}\hat{p} - \hat{p}\hat{x}}_{i\hbar\mathbb{I}}) \rightarrow \frac{\hbar\omega}{2}$$

$$\boxed{\hat{H} = \hbar\omega\left(a^{+}a + \frac{1}{2}\right)}$$

$$\boxed{[a, a^{+}] = aa^{+} - a^{+}a = 1}$$

Solve algebrically the eigenvalue problem :

$$\hat{H}\,\Psi_{\mu} = E_{m}\Psi_{\mu}$$

$\mu$: label or quantum number

$$a^{+}a\,\Psi_{m} = \underbrace{\left(\frac{E_m}{\hbar\omega} - \frac{1}{2}\right)}_{\mu_m}\Psi_m$$

$$\boxed{a^{+}a\,\Psi_m = \mu_m \Psi_{\mu}}$$

Remarks $\rightarrow$ $a^{+}a$ has only $\geq 0$ eigenvalues

$$\langle\psi|a^{+}a|\psi\rangle = \||a|\psi\rangle\|^2 \geq 0$$

$\rightarrow$ If $\Psi_m$ is an eigenvector with eigenvalue $\mu_\mu \Rightarrow$

$a\Psi_m$ e.v with e.val. $\mu_\mu - 1$

$\rightarrow$ If $\psi_m$ is an eigenvector with eigenvalue $\mu_M \Rightarrow$

$a^{\dagger}\psi_m$ e.v with e.val. $\mu_M + 1$

$a\psi_0 = 0$ such that $\mu_0 = 0$ , $\mu_M = M$ $(M \geq 1)$

Spectrum of $(a^{\dagger}a)\psi = \mu\,\psi$ $\rightarrow$ integers $\in \mathbb{N}$

$$\begin{array}{c|l}
\vdots & \vdots \\
4 & |4\rangle \propto a^{\dagger}|3\rangle \propto a|5\rangle \\
3 & |3\rangle \propto a^{\dagger}|2\rangle \propto a|4\rangle \\
2 & |2\rangle \propto a^{\dagger}|1\rangle \propto a|3\rangle \\
1 & |1\rangle \propto a^{\dagger}|0\rangle \propto a|2\rangle \\
0 & |0\rangle \propto a|1\rangle
\end{array}$$

$\boxed{a\psi_0 = 0}$ Lowest state must be annihilated by a

$E_m = \hbar\omega\left(m + \frac{1}{2}\right) \rightarrow$ "Spectrum of Harmonic Oscillator"

# of photons

✱ Remark: $E_0 = \frac{\hbar\omega}{2}$ = residual quantum fluctuation of Ground State

Proof of Ladder Operator:

$$\boxed{a^\dagger a\,\psi = \mu\,\psi} \Rightarrow a(a^\dagger a\,\psi) = \mu\, a\,\psi$$

$$(a^\dagger a + 1)\, a\,\psi = \mu\, a\,\psi$$

$$\boxed{(a^\dagger a)(a\psi) = (\mu - 1)(a\psi)}$$

$$\boxed{\begin{aligned} a^\dagger |m\rangle &= \sqrt{m+1}\,|m+1\rangle \\ a\,|m\rangle &= \sqrt{m}\,|m-1\rangle \\ a\,|0\rangle &= 0 \\ a^\dagger a\,|m\rangle &= m\,|m\rangle \end{aligned}}$$

Number Operator

Position Space Representation

$$\langle x|m\rangle = \psi_m(x)$$

$$\langle x|0\rangle = \psi_0(x)$$

by applying $a^\dagger$

$$\boxed{|m\rangle = \frac{(a^\dagger)^m}{\sqrt{m!}}|0\rangle}$$

Defining eqn G.S : $a|0\rangle = 0$

In position space:

$$\left( \frac{-i}{\sqrt{2m\hbar\omega}} \left( -i\hbar \frac{\partial}{\partial x} \right) + \sqrt{\frac{m\omega}{2\hbar}}\, x \right) \psi_0(x) = 0$$

$$\sqrt{\frac{\hbar}{2m\omega}}\, \psi_0'(x) + \sqrt{\frac{m\omega}{2\hbar}}\, x\, \psi_0(x) = 0$$

$$\boxed{\psi_0'(x) = -m\omega\, x\, \psi_0(x)}$$

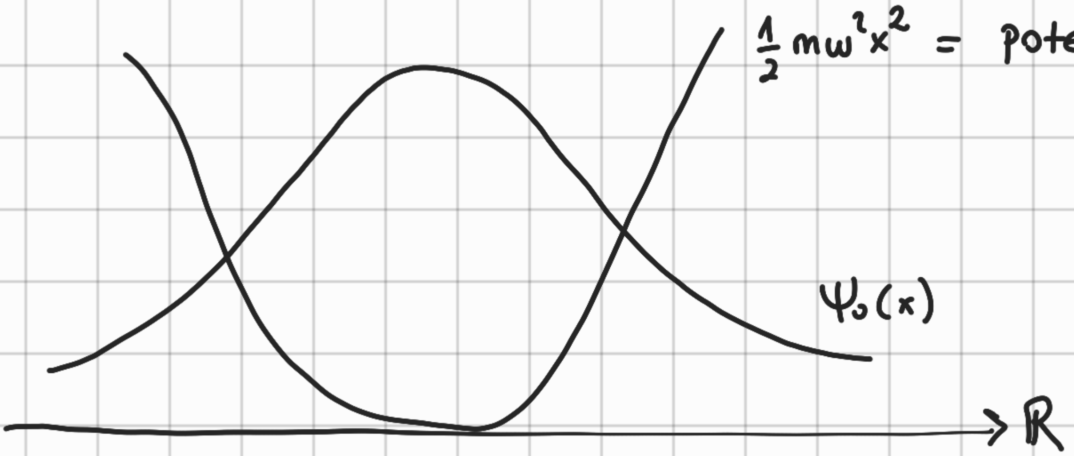
$\langle x|0\rangle$

$$\psi_0(x) = C \exp\left( -\frac{1}{2} m\omega x^2 \right)$$

Gaussian

From Normalisation condition: $\int dx\, |\psi(x)|^2 = 1$

$\frac{1}{2} m\omega^2 x^2$ = potential energy

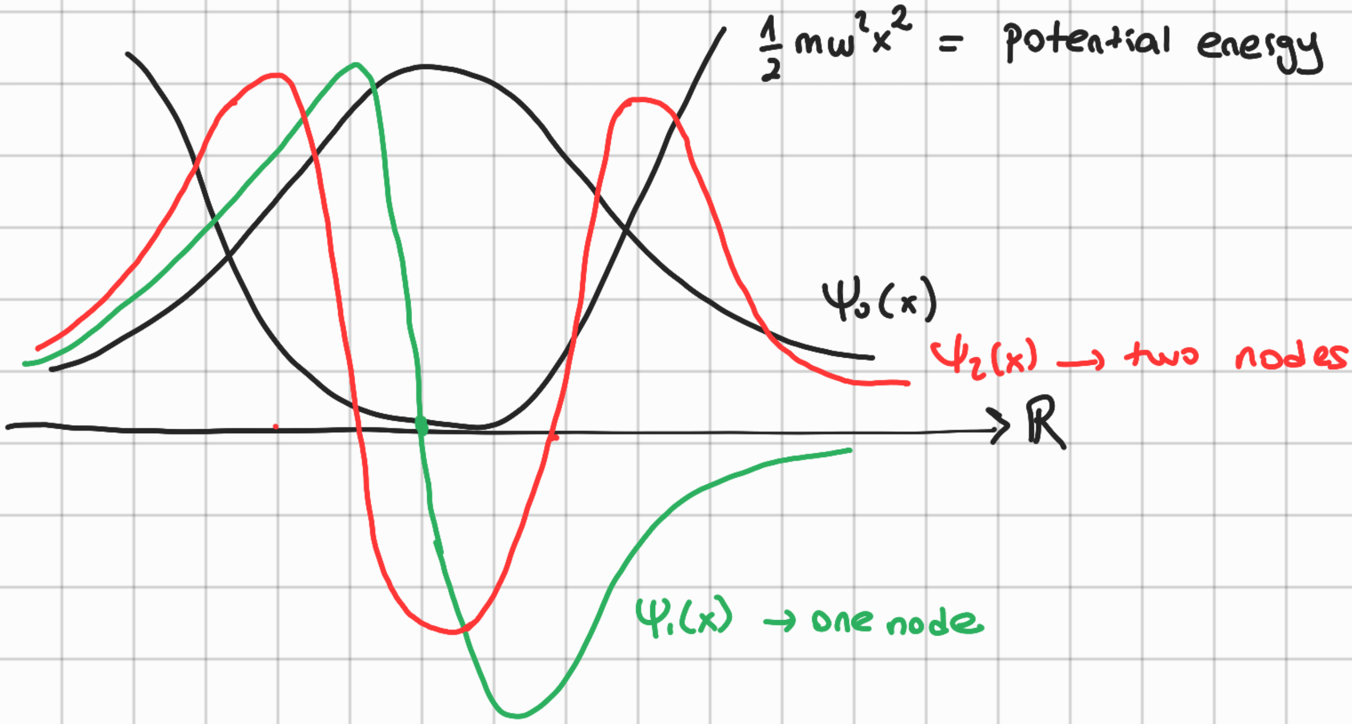


$$\langle x|1\rangle = \psi_1(x)$$
$$\langle x|0\rangle = \psi_0(x)$$
$\uparrow a^+$

$$|1\rangle = \frac{a^+|0\rangle}{\sqrt{1!}}$$

$$\psi_1(x) = \frac{1}{\sqrt{2m\hbar\omega}}(-i\hbar)\,\psi_0'(x) - i\sqrt{\frac{m\omega}{2\hbar}}\,x\,\psi_0(x)$$

$$\psi_1(x) \propto x\,\psi_0(x)$$




$$|2\rangle = \frac{a^+}{\sqrt{2}}|1\rangle$$

$$\psi_2(x) = \left(\dots x + \dots\left(-i\hbar\frac{\partial}{\partial x}\right)\right)\underbrace{\psi_1(x)}_{x\,\psi_0(x)} \propto x^2\psi_0(x) + \underbrace{\frac{\partial}{\partial x}(x\,\psi_0(x))}_{\psi_0(x)+x^2\psi_0(x)}$$

$$\boxed{\psi_2(x) = (A + Bx^2)\,\psi_0(x)} \rightarrow \text{Hermite Polynomials}$$