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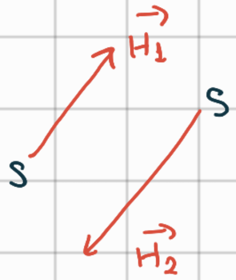
Heisenberg Hamiltonian

Interaction between magnetic moments (spin 1/2)

Realization of two qubit gates (CNOT, SWAP)



$$E = c\,\vec{M}_1 \vec{M}_2$$

Quantum Situation

for an Nucleus 1 $\rightarrow$ intrinsic magnetic moment

effective two level $\rightarrow$ effective magnetic moment

$$\vec{\sigma}_1 = (\sigma_1^x, \sigma_1^y, \sigma_1^z)$$

$$\vec{\sigma}_2 = (\sigma_2^x, \sigma_2^y, \sigma_2^z)$$

$$\boxed{H = \hbar J\, \vec{\sigma}_1 . \vec{\sigma}_2} \rightarrow \mathcal{H}_H = \mathbb{C}_1^2 \otimes \mathbb{C}_2^2$$

$\left[\frac{1}{s}\right]$

In literature they use that (But it is a tensor) product

H is a 4x4 Matrices

$$\langle \psi | \vec{\sigma}_1 \cdot \vec{\sigma}_2 | \psi \rangle$$

Study the matrix H $\left\{ \begin{array}{l} \text{algebra of Pauli matrices} \\ \text{Dirac Notation} \\ \text{Component from } 4\times4 \text{ array} \end{array} \right.$

$$H = \hbar J \left\{ \underbrace{\sigma_1^x \otimes \sigma_2^x}_{4\times4} + \underbrace{\sigma_1^y \otimes \sigma_2^y}_{4\times4} + \underbrace{\sigma_1^z \otimes \sigma_2^z}_{4\times4} \right\}$$

$\sigma^+ = \frac{1}{2}(\sigma^x + i\sigma^y)$ $\quad \sigma^+|\uparrow\rangle = 0$ $\quad \sigma^+|\downarrow\rangle = |\uparrow\rangle$

$\sigma^- = \frac{1}{2}(\sigma^x - i\sigma^y)$ $\quad \sigma^-|\uparrow\rangle = |\downarrow\rangle$ $\quad \sigma^-|\downarrow\rangle = 0$

$$\boxed{H = \hbar J \sigma_1^z \otimes \sigma_2^z + 2\hbar J(\sigma_1^+ \otimes \sigma_2^- + \sigma_1^- \otimes \sigma_2^+)}$$

Remark: $\quad |\Psi\rangle = |\uparrow\rangle_1 \otimes |\downarrow\rangle_2$

$$(\sigma_1^+ \otimes \sigma_2^- + \sigma_1^- \otimes \sigma_2^+)|\uparrow\rangle_1 \otimes |\downarrow\rangle_2$$

$$= \underbrace{\cancel{\sigma_1^+|\uparrow\rangle}}_{0} \otimes \underbrace{\cancel{\sigma_2^-|\downarrow\rangle_2}}_{0} + \underbrace{\sigma_1^-|\uparrow\rangle}_{|\downarrow\rangle} \otimes \underbrace{\sigma_2^+|\downarrow\rangle}_{|\uparrow\rangle} = |\downarrow\rangle_1 \otimes |\uparrow\rangle_2$$

$$|\Psi\rangle = |\uparrow\rangle_1 \otimes |\uparrow\rangle_2$$

$$(\sigma_1^+ \otimes \sigma_2^- + \sigma_1^- \otimes \sigma_2^+)|\uparrow\rangle_1 \otimes |\uparrow\rangle_2$$

$$= \underbrace{\cancel{\sigma_1^+|\uparrow\rangle}}_{0} \otimes \sigma_2^-|\uparrow\rangle_2 + \sigma_1^-|\uparrow\rangle \otimes \underbrace{\cancel{\sigma_2^+|\uparrow\rangle}}_{0} = 0$$

Dirac Notation

$$\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = |\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|$$

$$\sigma_1^z \otimes \sigma_2^z = \left(|\uparrow\rangle_1\langle\uparrow|_1 - |\downarrow\rangle_1\langle\downarrow|_1\right) \otimes \left(|\uparrow\rangle_2\langle\uparrow|_2 - |\downarrow\rangle_2\langle\downarrow|_2\right)$$

$$= |\uparrow\rangle_1\langle\uparrow|_1 \otimes |\uparrow\rangle_2\langle\uparrow|_2 + \ldots$$

$$= \left(|\uparrow\rangle_1 \otimes |\uparrow\rangle_2\right)\left(\langle\uparrow|_1 \otimes \langle\uparrow|_2\right) \leftarrow \cdots$$

→ 4x4 Matrices

$(1\ 0\ 0\ 0)$ $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$$= |\uparrow\uparrow\rangle\langle\uparrow\uparrow| + |\downarrow\downarrow\rangle\langle\downarrow\downarrow| - |\uparrow\downarrow\rangle\langle\uparrow\downarrow| - |\downarrow\uparrow\rangle\langle\downarrow\uparrow|$$

$$\begin{array}{c c} \begin{matrix} \uparrow\uparrow & \uparrow\downarrow & \downarrow\uparrow & \downarrow\downarrow \end{matrix} & \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} & \begin{matrix} \uparrow\uparrow \\ \uparrow\downarrow \\ \downarrow\uparrow \\ \downarrow\downarrow \end{matrix} \end{array}$$

Exchange interaction term $\sigma_1^+ \otimes \sigma_2^- + \sigma_1^- \otimes \sigma_2^+$

$$\sigma^+ = \frac{1}{2}(\sigma_x + i\sigma_y) = \begin{array}{cc} \begin{matrix} \uparrow & \downarrow \end{matrix} & \\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{matrix} \uparrow \\ \downarrow \end{matrix} \end{array} = |\uparrow\rangle\langle\downarrow|$$

$$\sigma^- = \frac{1}{2}(\sigma_x - i\sigma_y) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = |\downarrow\rangle\langle\uparrow|$$

$$\sigma_1^+ \otimes \sigma_2^- + \sigma_1^- \otimes \sigma_2^+ = |\uparrow\rangle_1\langle\downarrow|_1 \otimes |\downarrow\rangle_2\langle\uparrow|_2 + |\downarrow\rangle_1\langle\uparrow|_1 \otimes |\uparrow\rangle_2\langle\uparrow|_2$$

$$= |\uparrow\downarrow\rangle\langle\downarrow\uparrow| + |\downarrow\uparrow\rangle\langle\uparrow\downarrow|$$

$$\underset{\text{array form}}{=} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

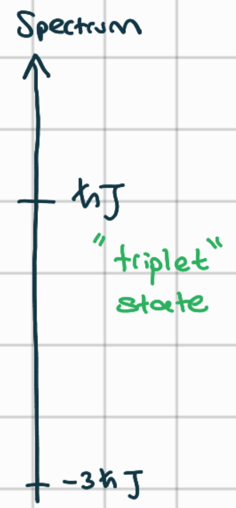
$$H = \hbar J \left\{ \sigma_1^z \otimes \sigma_2^z + 2\left[\sigma_1^+ \otimes \sigma_2^- + \sigma_1^- \otimes \sigma_2 \right] \right\}$$

$$\underset{\text{array form}}{=} \hbar J \begin{pmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & \boxed{1} \end{pmatrix}$$ Heisenberg Hamiltonian

Eigenvalues and Eigenvectors

$$|\uparrow\uparrow\rangle, \quad |\downarrow\downarrow\rangle, \quad \overbrace{\frac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{\sqrt{2}}}^{|\Phi_+\rangle}, \quad \overbrace{\frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}}}^{|\Phi_-\rangle}$$

$$\underbrace{+\hbar J \qquad +\hbar J \qquad +\hbar J}_{\text{degenerate}} \qquad -3\hbar J$$



Ground State
"single state"

$$\hbar J \left\{ \sigma_1^z \otimes \sigma_2^z + 2\left(\sigma_1^+ \otimes \sigma_2^- + \sigma_1^- \otimes \sigma_2^+ \right) \right\} |\uparrow\uparrow\rangle$$

$$= \hbar J \left(\sigma_1^z |\uparrow\rangle \otimes \sigma_2^z |\uparrow\rangle \right) = \hbar J \underbrace{|\downarrow\rangle \otimes |\downarrow\rangle}_{|\uparrow\uparrow\rangle}$$

$$\underset{\text{eigen basis}}{U_t} = \begin{pmatrix} e^{-i\hbar J} & & & 0 \\ & e^{-i\hbar J} & & \\ & & e^{-i\hbar J} & \\ 0 & & & e^{+3i\hbar J} \end{pmatrix}$$

$$\hbar J \left\{ \sigma_1^z \otimes \sigma_2^z + 2\left(\sigma_1^+ \otimes \sigma_2^- + \sigma_1^- \otimes \sigma_2^+ \right) \right\} \left(\frac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{\sqrt{2}} \right)$$

$$-\hbar J \left(\frac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{\sqrt{2}} \right) + 2\hbar J \left(\frac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{\sqrt{2}} \right)$$

Spectral decomposition of H

$$H = \hbar J \; |\uparrow\uparrow\rangle\langle\uparrow\uparrow| + \hbar J \, |\downarrow\downarrow\rangle\langle\downarrow\downarrow| + \hbar J \, |\Phi_+\rangle\langle\Phi_+| - 3\hbar J \, |\Phi^-\rangle\langle\Phi^-|$$

$$\underset{\text{comp. basis}}{\begin{pmatrix} 1 & & & \\ & -1 & 2 & \\ & 2 & -1 & \\ & & & 1 \end{pmatrix}} \rightsquigarrow \underset{\text{eigen-basis}}{\begin{pmatrix} \hbar J & & & \\ & \hbar J & & 0 \\ & 0 & \hbar J & \\ & & & 3\hbar J \end{pmatrix}}$$

$$U_{Hes} = e^{\frac{-itH}{\hbar}} = e^{itJ} |\uparrow\uparrow\rangle\langle\uparrow\uparrow| + e^{itJ} |\downarrow\downarrow\rangle\langle\downarrow\downarrow| + e^{itJ} |\Phi_+\rangle\langle\Phi_+| + e^{-3itJ} |\Phi_-\rangle\langle\Phi_-|$$

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Unitary Time Evolution

independent of time

$$i\hbar \frac{d}{dt} U_t = H U_t$$

$$|\Psi_t\rangle = \underbrace{U_t}_{4\times4 \text{ matrix}} |\Psi_{t=0}\rangle$$

$$U_t = e^{\frac{-it}{\hbar}H}$$

Dirac Notation

$$U_t = e^{itJ} |\uparrow\uparrow\rangle\langle\uparrow\uparrow| + e^{itJ} |\downarrow\downarrow\rangle\langle\downarrow\downarrow| + e^{itJ} |\Phi_+\rangle\langle\Phi_+| + e^{-3itJ} |\Phi_-\rangle\langle\Phi_-|$$

$$\underset{\text{comp. basis}}{U_t} = \begin{pmatrix} x & & & 0 \\ & x & x & \\ & x & x & \\ 0 & & & x \end{pmatrix}$$

The SWAP Gate

Unitary operation defined as

$$\underbrace{\text{SWAP}}_{4\times4 \text{ matrix}} |x,y\rangle = |y,x\rangle$$

$$|x\rangle\otimes|y\rangle \qquad |y\rangle\otimes|x\rangle$$

$x = \uparrow$ or $\downarrow$

$y = \uparrow$ or $\downarrow$

* Eigen vectors of SWAP are same as eigenvectors of $H_{\text{Heis.}}$

$$\text{SWAP}|\uparrow\uparrow\rangle = 1.|\uparrow\uparrow\rangle$$

$$\text{SWAP}|\downarrow\downarrow\rangle = 1.|\downarrow\downarrow\rangle$$

$$\text{SWAP}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) = 1.(|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle)$$

$$\text{SWAP}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) = -1.(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$$

$$\text{SWAP} = 1.|\uparrow\uparrow\rangle\langle\uparrow\uparrow| + 1.|\downarrow\downarrow\rangle\langle\downarrow\downarrow| + 1.\frac{(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)}{\sqrt{2}}\frac{(\langle\uparrow\downarrow| + \langle\downarrow\uparrow|)}{\sqrt{2}}$$

$$-1.\frac{(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)}{\sqrt{2}}\frac{(\langle\uparrow\downarrow| - \langle\downarrow\uparrow|)}{\sqrt{2}}$$

$$U_t = e^{-itJ}|\uparrow\uparrow\rangle\langle\uparrow\uparrow| + e^{-itJ}|\downarrow\downarrow\rangle\langle\downarrow\downarrow| + e^{-itJ}|\Phi_+\rangle\langle\Phi_+| + e^{+3itJ}|\Phi_-\rangle\langle\Phi_-|$$

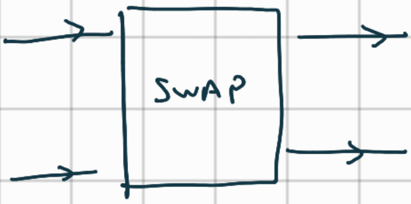
$$e^{-i\pi/4} \qquad t = \frac{\pi}{4J}$$

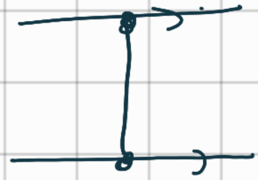
$$-e^{-i\frac{\pi}{4}} = e^{i\pi - \frac{\pi}{4}} = e^{i\frac{3\pi}{4}}$$

$$\boxed{e^{\frac{-it}{\hbar}H_{heis}} \underset{\substack{\downarrow \\ \text{for } t=\frac{\pi}{4J}}}{=} e^{-i\pi/4}\ \mathrm{SWAP}}$$

$$e^{\frac{-i\pi}{4\hbar J}H_{heis}}|\psi_0\rangle = \underbrace{e^{\frac{-i\pi}{4}}}_{\substack{\downarrow \\ \text{global phase}}}(\mathrm{SWAP}\,|\psi_0\rangle)$$

Circuit Representation





CNOT gate uses an anisotropic Heis. Hamiltonian

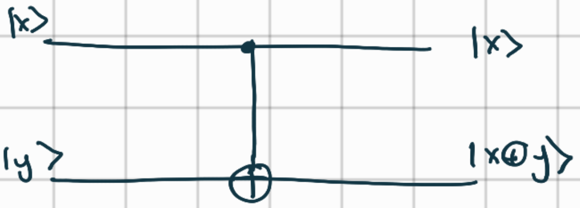
$$H_{anisotropic} = \hbar J\, \sigma_1^z \otimes \sigma_2^z$$

$$\underset{\text{comp basis}}{=} \begin{pmatrix} \hbar J & & & 0 \\ & -\hbar J & & \\ & & -\hbar J & \\ 0 & & & \hbar J \end{pmatrix}$$

$$U_t^{ani} = e^{\frac{-it}{H}H_{aniso}} = \begin{pmatrix} e^{-itJ} & & & 0 \\ & e^{+itJ} & & \\ & & e^{+itJ} & \\ 0 & & & e^{-itJ} \end{pmatrix}$$

By definition: $\quad CNOT\,|x\rangle \otimes |y\rangle = |x\rangle \otimes |\underbrace{x \oplus y}_{\text{mod}\,2}\rangle$ here $x = 0, 1$

$y = 0, 1$



$$CNOT = \begin{pmatrix} 1 & 0 & & \\ 0 & 1 & & 0 \\ & & 0 & 1 \\ 0 & & 1 & 0 \end{pmatrix} = \underbrace{|0\rangle\langle 0|}_{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}} \otimes \underbrace{\mathbb{I}}_{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} + \underbrace{|1\rangle\langle 1|}_{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}} \otimes X$$

$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$