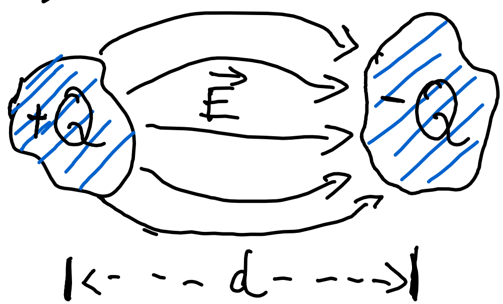
Quantization of electrical circuits

Goal: given an electrical circuit composed of inductors and capacitors, find the system Hamiltonian.

N.B. Here no Josephson junction (for now) and no resistors.

Lumped element representation:

- consider two metallic islands charged with $\pm Q$



$\vec{E}$ electric field

d characteristic lenght of the circuit

In a static configuration charges arrange on the surface of the conductor and the electric field vanishes inside the metal.

The time to reach a static charge configuration $\boxed{\tau \sim d/c}$, with c the velocity of the light.

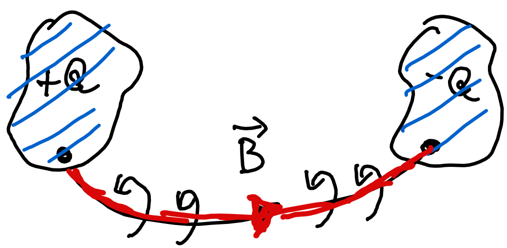
If $Q(t)$ changes slowly compared to τ, the electric field follows quasi-instantaneously.

The energy required to move one unit charge from one to the other island is path-independent and given by the voltage V. $V = Q/C$ — C is the capacitance; it depends on geometry and dielectric medium.

$\Rightarrow$ Total energy stored in the electric field

$$E_{el} = \int_0^Q dQ'\, V(Q') = \int_0^Q dQ' \frac{Q'}{C} = \frac{Q^2}{2C}$$

• Now let's consider an additional wire connecting the 2 islands



$I = -\dot{Q}$

$\Phi_B = L I$ → current

magnetic flux ← Φ_B

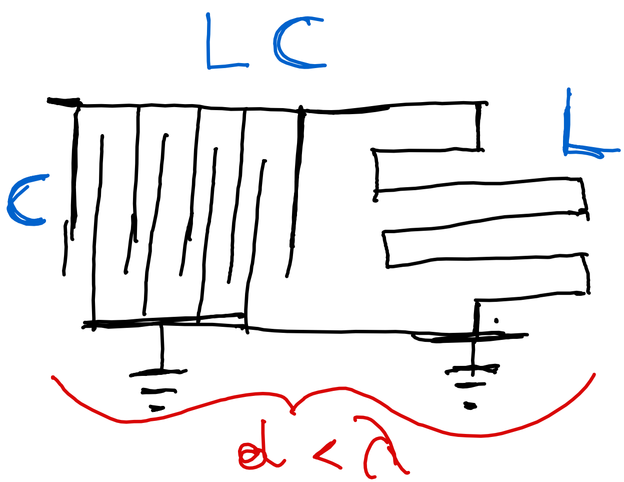
L ↓ Inductance (geometry-dependent)

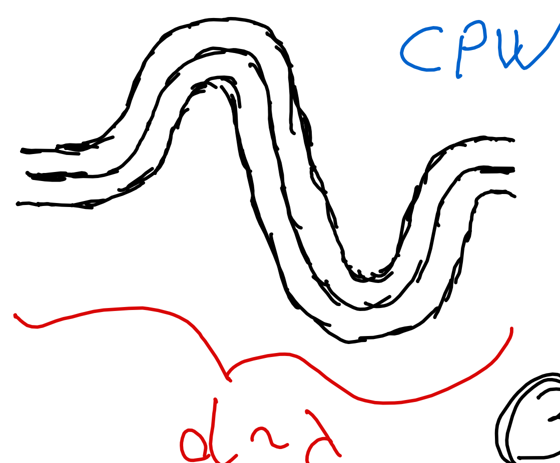
Energy stored in magnetic field $\vec{B}$:

$$E_{mag} = \int_0^I dI'\, \Phi(I') = \frac{1}{2} L I^2 = \Phi^2/2L$$

$$\Phi_B(t) = \int_0^t V(t')\,dt'$$

There are two types of on-chip electronic components: lumped elements and distributed elements.
The two types are distinguished by how large they are by comparison with the wavelength of microwave radiation at the relevant frequency: For lumped elements, the size "d" of the components is much smaller than the wavelength, $d \ll \lambda$; whereas for distributed elements, the size of the component is roughly the same size as λ or bigger: $d \gtrsim \lambda$. For GHz photons $\Rightarrow \lambda \sim$ cm





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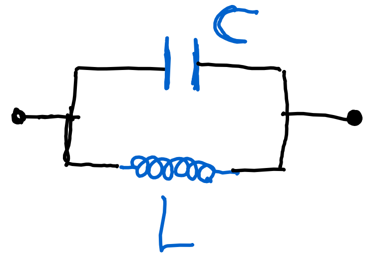
The low frequency dynamics of this system are fully captured by two effective parameters L, C.

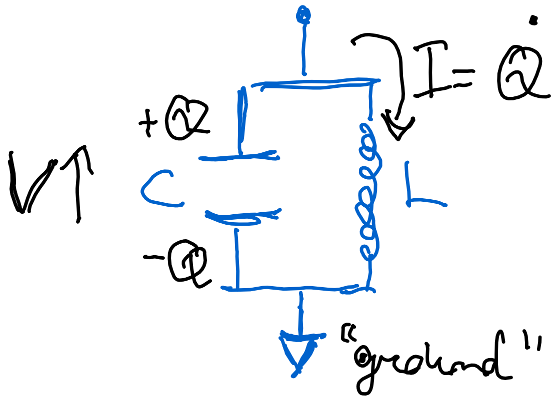
The finite dimention of the system "el" can be neglected.

The system is represented by a LUMPED ELEMENT model



Quantization of the LC resonator

Consider



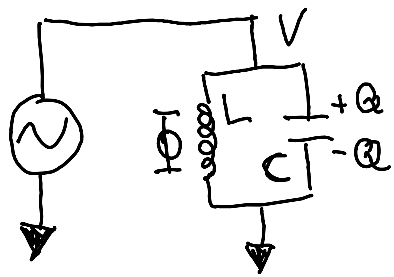
Electric field : $Q^2/2C$

Magnetic " : $\Phi^2/2L$

Faraday's law of induction $\dot{\Phi} = V$

$$\Rightarrow \ddot{\Phi} = \dot{V} = \frac{\dot{Q}}{C} = -\frac{I}{C} = -\frac{\Phi}{LC} = -\omega_0^2 \Phi \quad , \quad \omega_0 = \sqrt{\frac{1}{LC}}$$

Electronic Harmonic Oscillator



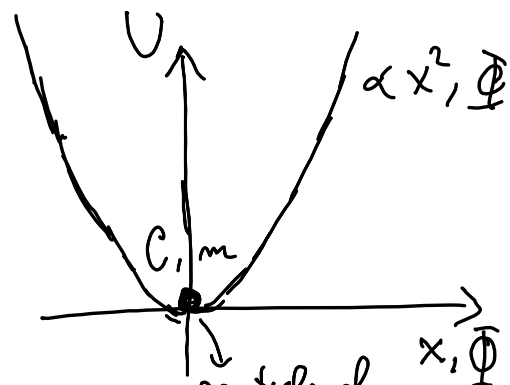
- charge on capacitor
$$Q = CV$$
- flux in the inductor
$$\Phi = LI$$
- voltage across oscillator
$$V = \frac{Q}{C} = -L\dot{I} = -\dot{\Phi}$$

Hamiltonian

$$H = \frac{CV^2}{2} + \frac{LI^2}{2} = \frac{Q^2}{2C} + \frac{\Phi^2}{2L}$$

electrostatic energy

magnetic energy



- Compare to mechanical harmonic oscillator
$$H = \frac{p^2}{2m} + \frac{Kx^2}{2}$$
kinetic K + potential U

Characteristic quantities

mechanical	electronic
position x	flux Φ
momentum p	charge Q
mass m	capacitance C
spring constant K	inverse inductance $1/L$
resonance freq. $\omega = \sqrt{\frac{k}{m}}$	$\omega = \sqrt{\frac{1}{LC}}$

Harmonic oscillator

conjugate variables

$$\frac{\partial H}{\partial \Phi} = \frac{\Phi}{L} = I = \dot{Q}$$

$$\frac{\partial H}{\partial Q} = \frac{Q}{C} = V = -L\dot{I} = -\ddot{\Phi}$$

$$\boxed{\begin{array}{l} x, p \text{ are canonical} \\ \Phi, Q \text{ variables} \end{array}}$$

- quantum mechanical operators

$$\hat{x} = x \qquad \hat{\Phi} = \Phi$$

$$\hat{p} = -i\hbar \frac{\partial}{\partial x} \qquad \hat{Q} = -i\hbar \frac{\partial}{\partial \Phi}$$

- commutation relation

$$[\hat{x}, \hat{p}] = i\hbar \qquad [\hat{\Phi}, \hat{Q}] = i\hbar \Leftrightarrow \left[2\pi \frac{\hat{\Phi}}{\Phi_0}, \frac{Q}{2e}\right] = [\hat{\delta}, \hat{N}] = i$$

Hamiltonian operator

- using conjugate variables Q, Φ

$$\hat{H} = \frac{\hat{Q}^2}{2C} + \frac{\hat{\Phi}^2}{2L} = -\frac{\hbar^2}{2C}\frac{\partial^2}{\partial\Phi^2} + \frac{1}{2L}\Phi^2$$

- using creation and annihilation operators

$$\hat{a}^+ = \frac{1}{\sqrt{2\hbar Z_c}}\left(Z_c \hat{Q}^+ - i\hat{\Phi}^+\right) \quad \text{creation operators}$$

$$\hat{a} = \frac{1}{\sqrt{2\hbar Z_c}}\left(Z_c \hat{Q} + i\hat{\Phi}\right) \quad \text{annihilation}$$

with $Z_c = \sqrt{L/C}$ impedance of oscillator

$$\hat{H} = \hbar\omega\left(\hat{a}^+\hat{a} + \frac{1}{2}\right)$$

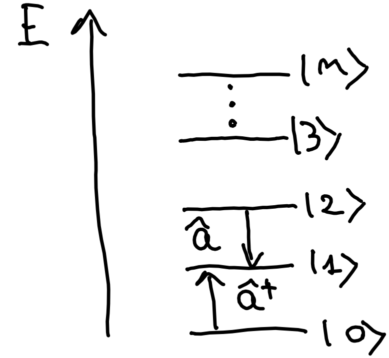
$a^+a = n$ number operators

- Properties of $\hat{a}^+$ and $\hat{a}$

$$\hat{a}^+|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

$$\hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle$$

$$\hat{a}^+\hat{a}\,|n\rangle = n\,|n\rangle$$



Energy Spectrum

with $|n\rangle$ number (Fock) state of harmonic oscillator

- relation to $\hat{Q}$ and $\hat{\Phi}$

$$\hat{Q} = \sqrt{\frac{\hbar}{2Z_c}}\,(\hat{a}^+ + \hat{a}) \longrightarrow$$ related to electric field stored on capacitor

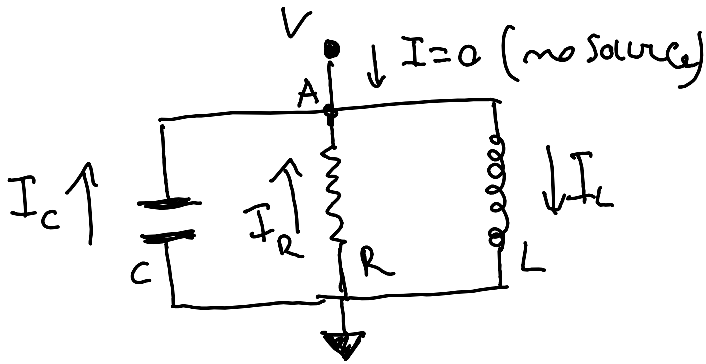
$$\hat{\Phi} = i\sqrt{\frac{\hbar Z_c}{2}}\,(\hat{a}^+ - \hat{a}) \longrightarrow$$ relate to magnetic field stored in inductor

or $$\hat{V} = \sqrt{\frac{\hbar\omega}{2C}}\,(\hat{a}^+ + \hat{a})$$

$$\hat{I} = i\sqrt{\frac{\hbar\omega}{2L}}\,(\hat{a}^+ - \hat{a})$$

with $\omega = \frac{1}{\sqrt{LC}}$ and $V = \frac{Q}{C}$ and $I = \Phi/L$

Dissipation in the Harmonic Oscillator



with

- current through resistor
  $I_R = V/R$
- displacement current
  $I_C = \dot{Q}_C = C\dot{V}$
- voltage across inductor
  $V = -L\dot{I}_L$

- Kirchoff law at point A

$$I_L = I_R + I_C + I$$

$$I_L - C\dot{V} - V/R = 0 \quad \text{(same voltage at A)}$$

$$I_L - C(-L\ddot{I}_L) + \frac{L}{R}\dot{I}_L = \frac{1}{LC} I_L + \ddot{I}_L + \frac{1}{RC}\dot{I}_L = 0$$

differential equation for current through inductor

- Solution $I_L(t) = I_L(0)\, e^{\lambda t}$ with $\lambda_{1,2} = \frac{1}{2LC}\left(-\frac{L}{R} \pm \sqrt{\left(\frac{L}{R}\right)^2 - 4LC}\right)$

Energy Decay Rate

- underdamped oscillator ($4LC \gg L/R$)

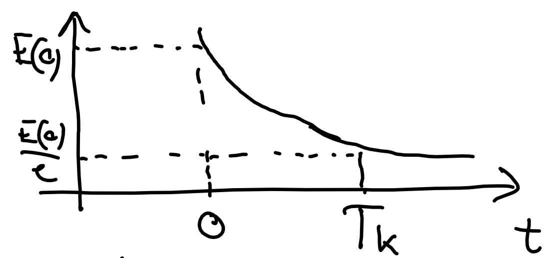
$$\lambda_{1,2} = -\frac{1}{2RC} \pm i\frac{1}{\sqrt{LC}} = -\alpha \pm i\omega$$

with $\alpha = \frac{1}{2RC} = \frac{1}{\tau}$ amplitude decay constant

$\tau = 2RC$ " " time

$\omega = 1/\sqrt{LC}$ oscillator frequency

- energy decay rate



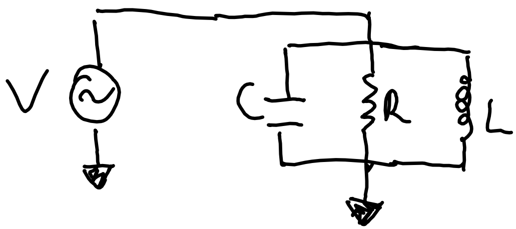
$$E \propto \frac{1}{2} L I_L^2 \propto e^{-\frac{1}{RC}t}$$

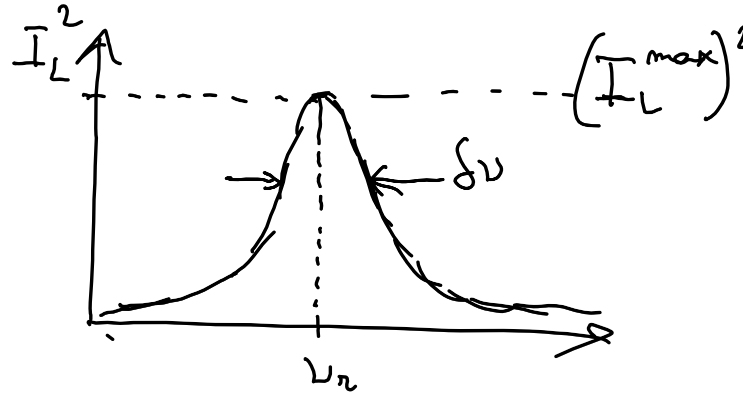
with $k = \frac{1}{RC}$ energy decay rate

$T_k = RC$ " " time

Spectral Response of Damped Harmonic Oscillator

- driven damped oscillator



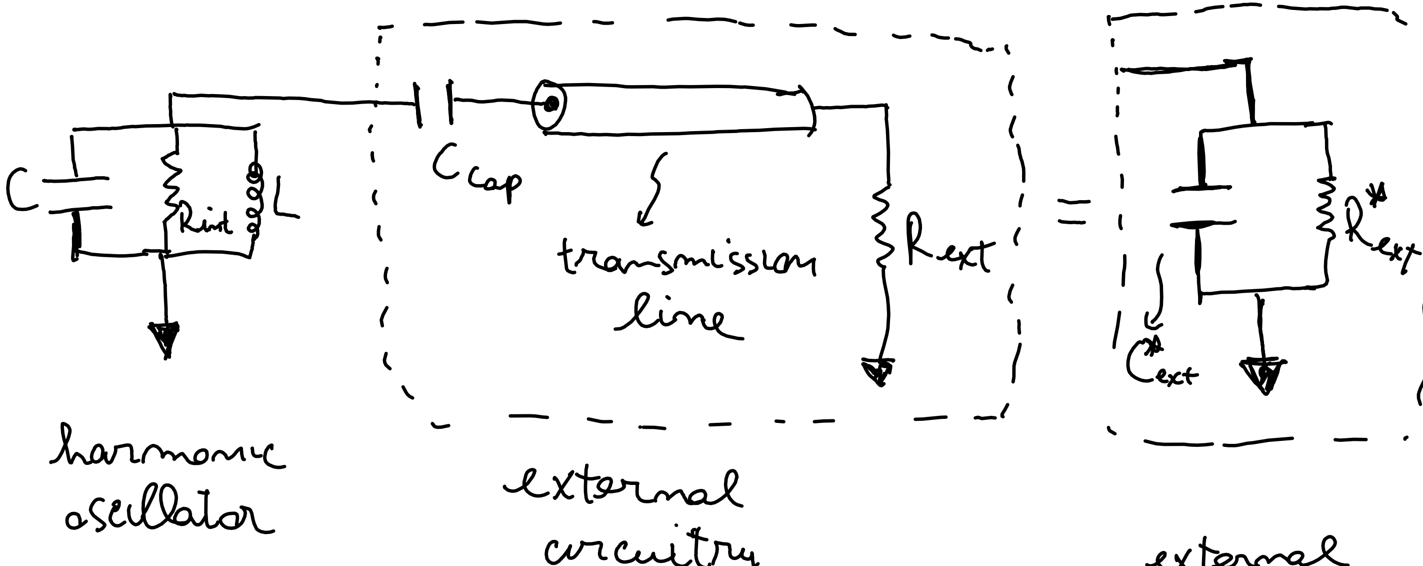


Lorentian line shape

$$I_L^2(\nu) = (I_L^{max})^2 = \frac{\delta\nu/\pi}{(\nu-\nu_r)^2+\delta\nu^2}$$

with $\delta\nu$: full width of line at half max

Internal and External Dissipation



external contribution to energy decay

- total effective resistance $\frac{1}{R_{tot}} = \frac{1}{R_{int}} + \frac{1}{R^*_{ext}}$

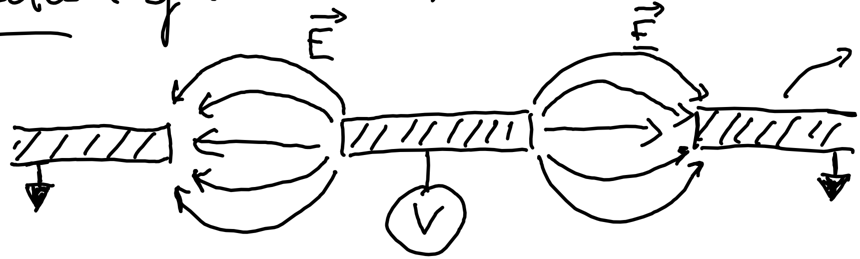
frequency shift due to external circuit

- total effective capacitance $C_{tot} = C_{int} + C^*_{ext}$

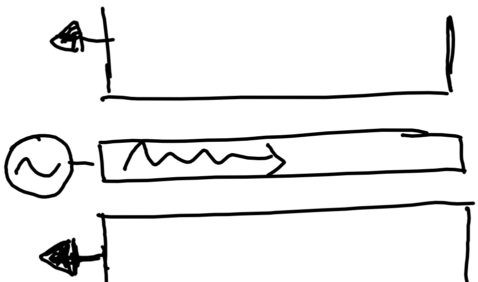
- energy decay time of combined system $T_K = R_{tot}\, C_{tot}$

Transmission line resonator

Field distrubution of the transmission line



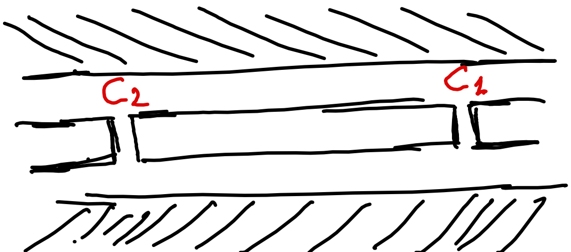
Wave propagation in a transmission line



$$E = E_0 e^{i(kx - \omega t)}$$

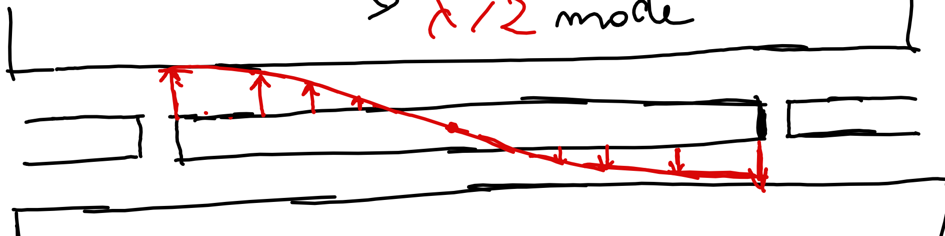
$$\omega = c_{eff} k = \frac{c}{\sqrt{\varepsilon}} \frac{2\pi}{\lambda}$$

Electric field at the boundary



Capacitors (C_i) as mirrors: defining nodes for the current, so antinodes (maximum) for the voltage.

Fundamental mode → $\lambda/2$ mode

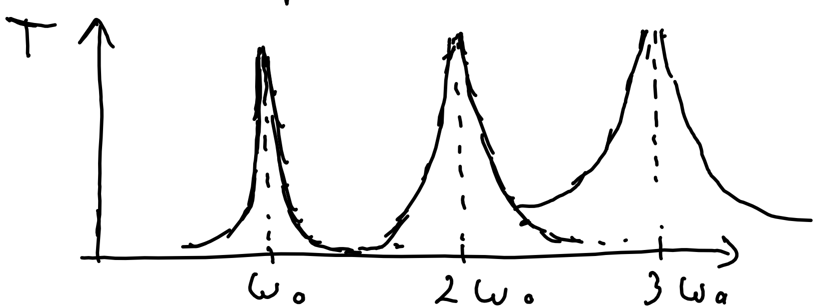


Higher harmonics: $2\frac{\lambda}{2}$; $3\frac{\lambda}{2}$; $4\frac{\lambda}{2}$; ... ; $n\frac{\lambda}{2}$

Resonance frequency?

$$\left.\begin{aligned} \omega &= c_{eff}\,\frac{2\pi}{\lambda} \\ l &= \frac{\lambda}{2}\,n \end{aligned}\right\} \quad \omega = c_{eff}\,\frac{2\pi}{2l} = \frac{\pi\, c_{eff}}{l}\,n$$

Excitation spectrum:



"Harmonic oscillator"

General procedure to find Hamiltonian

1. Set up Lagrange function

$$\mathcal{L} = T - V = \frac{1}{2} C \dot{\Phi}^2 - \frac{1}{2L} \Phi^2$$

"kinetic" "potential"

2. Legendre transformation

conjugate variable $\quad Q = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}} = C\dot{\Phi}$

3. Hamiltonian function

$$H = \dot{\Phi} Q - \mathcal{L} = \frac{Q^2}{C} - \frac{Q^2}{2C} - \left(-\frac{\Phi^2}{2L}\right) = \frac{Q^2}{2C} + \frac{\Phi^2}{2L}$$

4. Quantize $\quad Q \longrightarrow \hat{Q} \;;\; \Phi \longrightarrow \hat{\Phi}$ with $[\hat{\Phi}, \hat{Q}] = i\hbar$

5. Express $\hat{H}$ in terms of annihilation and creation operators

$$\hat{\Phi} = \left[\frac{\hbar \omega_0 L}{2}\right]^{\frac{1}{2}} (a + a^+) \;;\; \hat{Q} = \left[\frac{\hbar \omega_0 C}{2}\right]^{\frac{1}{2}} i (a - a^+)$$

to obtain $\hat{H} = \hbar \omega_0 a^+ a + \text{Const}$; $[a, a^+] = 1$

This procedure is generally applicable to more complicated circuits.

- **Properties of Quantum Harmonic Oscillator:**

The ground state satisfies $a|0\rangle = 0$

Eigenstates of $a^+ a = \hat{n}$ are called "Fock states"

$$a^+ a |n\rangle = n |n\rangle \quad , \quad n \in \{0, 1, 2, \ldots\}$$

The number n corresponds to the number of elementary excitations, i.e. photons at frequency ω_0.

Applying a (a^+) to state $|n\rangle$ raises (lowers) n by 1 :

$$a^+ |n\rangle = \sqrt{n+1}\, |n+1\rangle \quad ; \quad a|n\rangle = \sqrt{n}\, |n-1\rangle$$

which follows from the commutation relation.

$$a^+ a \underbrace{\left(a^+ |n\rangle \right)}_{\sim |n+1\rangle} = a^+ (1 + a^+ a) |n\rangle = \underbrace{(n+1)\, a^+ |n\rangle}_{\sim |n+1\rangle}$$

The zero-point fluctuations (zpf) of $\hat{\Phi}$ and $\hat{Q}$ of the ground state are

$$\langle 0 | \hat{\Phi}^2 | 0 \rangle = \Phi_{zpf}^2 \quad ; \quad \langle 0 | \hat{Q}^2 | 0 \rangle = Q_{zpf}^2$$

- An important class of states are "coherent states" defined:

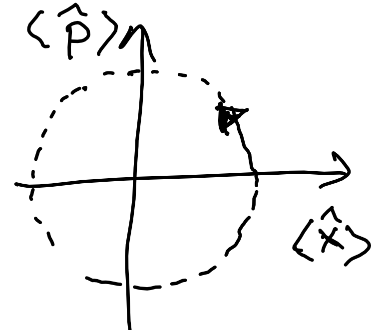
$$a |\alpha\rangle = \alpha |\alpha\rangle \quad , \quad \alpha \in \mathbb{C}$$

They are important because they have dynamics closely resembling the behaviour of a classical harmonic oscillator.

$$e^{-i \frac{\hat{H}}{\hbar} t} |\alpha_0\rangle = |\alpha(t)\rangle = |\alpha_0 e^{-i\omega_0 t}\rangle$$

$$\langle \alpha(t) | \hat{\Phi} | \alpha(t) \rangle = 2\, \Phi_{zpf}\, \alpha_0 \cos(\omega_0 t)$$

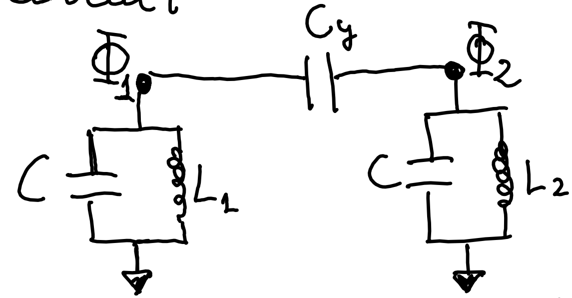
$$\langle \alpha(t) | \hat{Q} | \alpha(t) \rangle = 2\, Q_{zpf}\, \alpha_0 \sin(\omega_0 t)$$



$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \quad \left(\text{in the Fock state representation} \right)$$

Quantization of coupled resonator systems

Consider the electric circuit



with Lagrange function

$$\mathcal{L} = \frac{1}{2} C \dot{\Phi}_1^2 + \frac{1}{2} C \dot{\Phi}_2^2 + \frac{1}{2} C_g \left(\dot{\Phi}_2 - \dot{\Phi}_1 \right)^2 - \frac{\Phi_1^2}{2L_1} - \frac{\Phi_2^2}{2L_2}$$

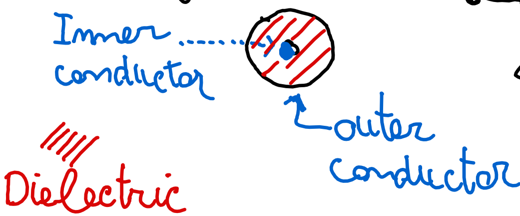
Applying the canonical transformation ....

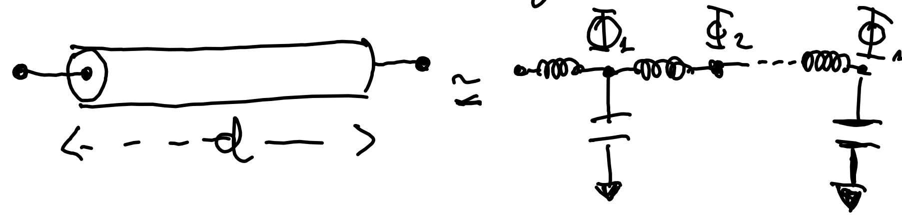
$$H = \hbar \omega_1 a^+ a + \hbar \omega_2 b^+ b + \hbar g (a - a^+)(b - b^+)$$

[see exercise]

Transmission line (distributed) resonators

A finite length transmission line, e.g. a coaxial waveguide





Can be represented by a series of inductances and capacitors (to ground).

In the continuum limit $n \to \infty$ the flux variables Φ_i become a position-dependent field $\Phi(x)$ and the Lagrange function is

$$\mathcal{L} = \int_0^d dx \left\{ \frac{c}{2} \dot{\Phi}(x)^2 - \frac{1}{2\ell} \left(\partial_x \Phi(x) \right)^2 \right\}$$

with c and ℓ are capacitance and inductance per unit length

Taking the boundary condition of vanishing current at the two open ends, we can express $\Phi(x)$ in terms of the normal modes

$$\Phi(x) = \sum_{n=1}^{\infty} \phi_n \cos(k_n x) \quad , \quad k_n = n\frac{\pi}{d}$$

resulting in

$$\mathcal{L} = \frac{1}{2}\sum_n \left(C\dot{\phi}_n^2 - \frac{1}{L_n}\phi_n^2 \right) \quad \text{with} \quad C = \frac{c\,d}{2} \quad L_n = \frac{2\,\ell\, d}{\pi^2 n^2}$$

Introducing $q_n = \dfrac{\partial \mathcal{L}}{\partial \dot{\phi}_n}$ we obtain

$$H = \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{q_n^2}{C} + \frac{\phi_n^2}{L_n}\right) = \quad \text{sum of harmonic oscillators}$$

$$= \sum_{n=1}^{\infty} \omega_n a_n^{+} a_n \qquad \text{with } \omega_n = \frac{1}{\sqrt{L_n C}} = n\frac{\pi v}{d}$$

and $v = \dfrac{1}{\sqrt{\ell c}}$ "phase velocity"