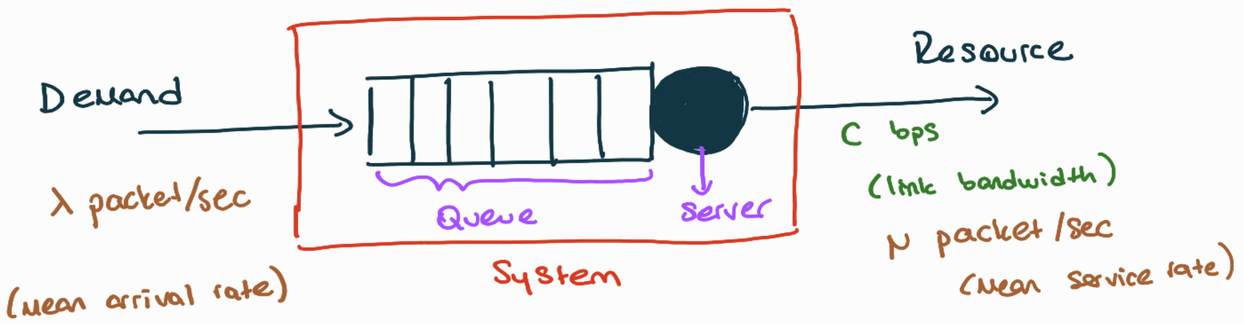
QUEUING ANALYSIS

Reference: Ece Güran Schmidt EE444 Slides



Demand > Resource → Queuing delay occurs

Service Time = $S = \dfrac{P \text{ bits}}{C \text{ bits/sec}}$

t_a : the last bit of the packet arrives at the node

t_s : the first bit starts to get service

t_d : the last bit of the packet leaves the node

$T_S = t_d - t_a$ (Time in the system = queue + server)

$t_d = t_s + S$

$T_Q = t_s - t_a$

queuing delay

Kendall's Notation: A / B / m / K / M

- A: arrival statistics
- B: service statistics
- m: # of servers
- K: Buffer size
- M: Customer Population

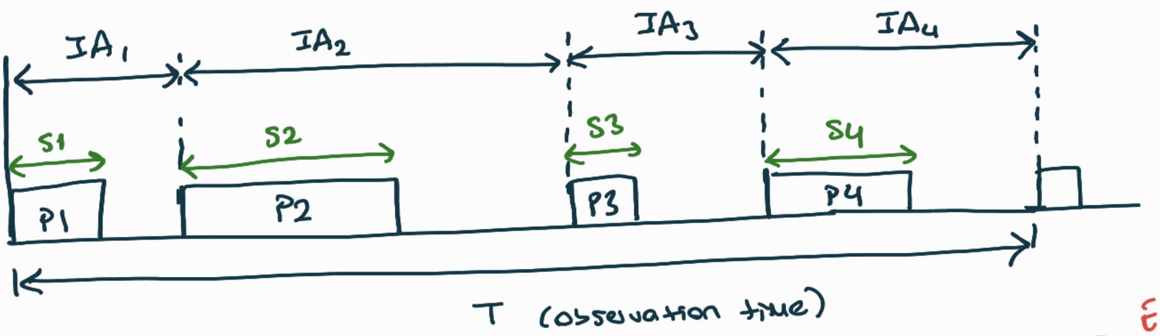
(K, M: generally not specified)

M: exponential

D: deterministic

G: general

Utilization (ρ) $= \dfrac{B \text{ sec} \rightarrow \text{service is busy}}{T \text{ sec} \rightarrow \text{observation time}}$ $= E[N_{server}] = ?$



$$T = \sum_{i=1}^{4} IA_i \qquad \rho = \frac{\sum_{i=1}^{4} S_i}{\sum_{i=1}^{4} IA_i} = \frac{\left[\sum_{i=1}^{4} S_i\right]/4}{\left[\sum_{i=1}^{4} IA_i\right]/4} = \frac{E[S]}{E[IA]} = \frac{\lambda}{\mu}$$

$E[S] = \frac{1}{\mu}$

$E[IA] = \frac{1}{\lambda}$

Throughput (X) $= \dfrac{F \text{ packets}}{T \text{ sec}} = \dfrac{F \text{ packets}}{B \text{ sec}} \cdot \dfrac{B \text{ sec}}{T \text{ sec}} = \boxed{\mu \cdot \rho = \lambda = X}$

what goes in goes out

Little's law

$$E[N_s] = E[T_s] \cdot \lambda$$

system

$$E[N_Q] = E[T_Q] \cdot \lambda$$

$$E[N_{server}] = E[T_{server}] \, \lambda$$

The next state depends only on the present state and present arrivals/departures

This is called "Markovian Property".

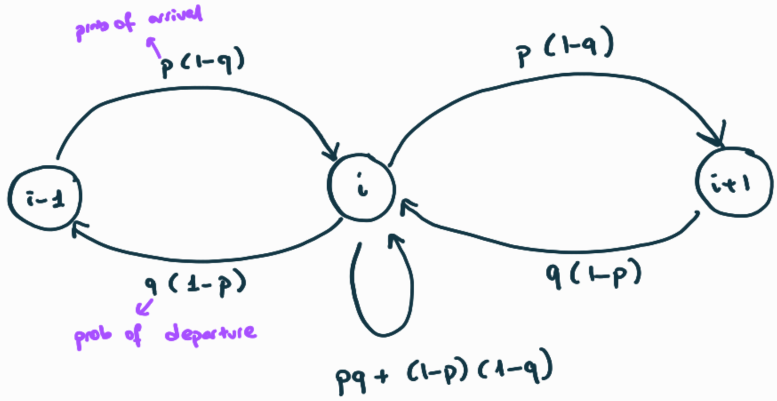
$E[N_s]$: The expected (average) # of packets in the system at steady state

π_i : The steady state probability that the system is in state i

$$E[N_s] = \sum_{i=0}^{\infty} i \, \pi_i \qquad \sum_{i=0}^{\infty} \pi_i = 1$$

Q1) How to model the discrete time system?

Discrete time Markov Chain



$$\pi_i = \pi_i[pq + (1-p)(1-q)] + \pi_{i-1}\, p(1-q) + \pi_{i+1}\, q(1-p)$$

Balance Equation:

$$\pi_i[(1-p)q + (1-q)p] = \pi_{i-1}[p(1-q)] + \pi_{i+1}[q(1-p)]$$

Get the real-life continous time model:

For time step δ, prob. of packet arrival is p

$$\lambda = \frac{p}{\delta} \qquad \mu = \frac{q}{\delta}$$

packet arrival rate

packet departure rate

$$\pi_i[q - pq + p - pq] = \pi_{i-1}[p - pq] + \pi_{i+1}[q - qp]$$

$$\pi_i[\mu\delta - \cancel{\mu\lambda\delta^2}_{0} + \lambda\delta - \cancel{\mu\lambda\delta^2}_{0}] = \pi_{i-1}[\lambda\delta - \cancel{\lambda\mu\delta^2}_{0}] + \pi_{i+1}[\mu\delta - \cancel{\mu\lambda\delta^2}_{0}] \qquad \begin{matrix}(\delta \to 0)\\ \delta^2 \to 0\end{matrix}$$

$$\pi_i(\mu+\lambda)\cancel{\delta} = \pi_{i-1}\lambda\cancel{\delta} + \pi_{i+1}\mu\cancel{\delta}$$

$$\boxed{\pi_i(\mu+\lambda) = \pi_{i-1}(\lambda) + \pi_{i+1}(\mu)} \Rightarrow \text{Continous Time Balance Equation}$$

Continous Time Markov Chain

$\pi_0 \lambda = \pi_1 \mu$

$\pi_1 = \pi_0 \frac{\lambda}{\mu}$

$\pi_1 (\lambda+\mu) = \pi_2 \mu + \pi_0 \lambda$

$\pi_2 \mu = \pi_0 \frac{\lambda}{\mu} (\lambda+\mu) - \pi_0 \lambda$

$\pi_2 \mu = \pi_0 \left[\frac{\lambda^2}{\mu} + \cancel{\lambda} - \cancel{\lambda} \right]$

$\pi_2 = \pi_0 \left[\frac{\lambda^2}{\mu^2} \right]$

$$\boxed{\pi_i = \pi_0 \left(\frac{\lambda}{\mu}\right)^i = \pi_0 \, \underbrace{\rho}_{\text{utilization}}{}^i}$$

$$\sum \pi_i = 1 \Rightarrow \underbrace{\pi_0}_{1-\rho} \underbrace{\sum_{i=0}^{\infty} \rho^i}_{\frac{1}{1-\rho}} = 1$$

system is empty

$$\boxed{\begin{aligned} \pi_0 &= 1-\rho \\ \pi_i &= (1-\rho)\, \rho^i \end{aligned}}$$

✡✡

Probability that the system is not empty $= 1 - \pi_0 = \rho$

Expected # of packets in the system:

$$E[N_S] = \sum_{i=1}^{\infty} i \cdot \pi_i = \frac{\rho}{1-\rho}$$

Expected # of packets in the queue

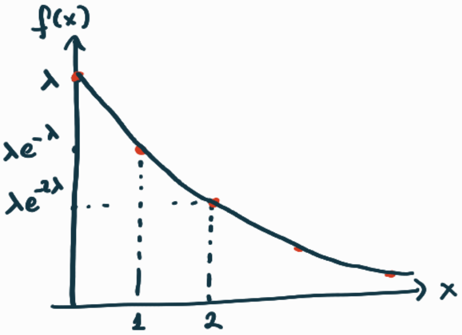
$$E[N_Q] = E[N_S] - E[N_{server}] = \frac{\rho}{1-\rho} - \rho = \frac{\rho^2}{1-\rho}$$

This was an M/M/1 Queue!

* Interarrival times and service times are exponentially distributed

(Memoryless)

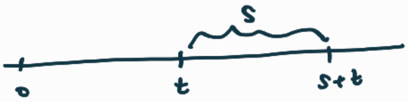
Exponential Distribution



$$E[x] = \int_{-\infty}^{\infty} x f(x)\,dx = \frac{1}{\lambda}$$

$$P(X > s+t \mid X > t) = P(X > s)$$

memoryless



* Interarrivals are exponentially distributed $\iff$ Arrivals is a Poisson process

Poisson Process

A sequence of events in time

$N(t)$ = # of total events that occur by time t

Probability of k events on a time interval t is:

$$\frac{e^{-\lambda t}(\lambda t)^k}{k!}$$

rate of Poisson process

Expected # of arrivals on a interval t

$$E[N(t)] = \lambda t$$

* For a given single queue, if the arrivals are poisson with λ, departure is also a poisson process with λ.

Question 1

Assume that the arrival rate is increased from λ to k

by a factor k

At the same time, transmission capacity is increased by a factor k

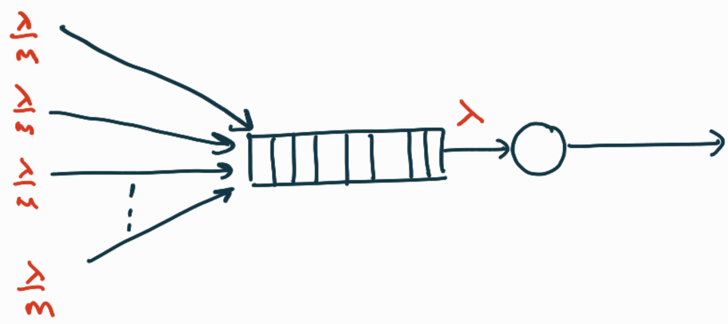
$$\rho_{new} = \frac{\lambda k}{\mu k} = \frac{\lambda}{\mu} = \rho_{old}$$

$$N_{new} = \frac{\rho}{1-\rho}$$

$$T_{new} = \frac{1}{k\mu - k\lambda} = \frac{1}{k} T_{old}$$

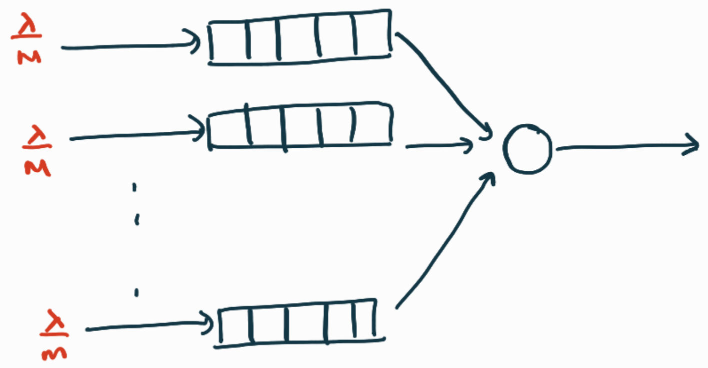
Multiplexing Traffic

Statistical multiplexing



$$T = \frac{1}{\mu - \lambda}$$

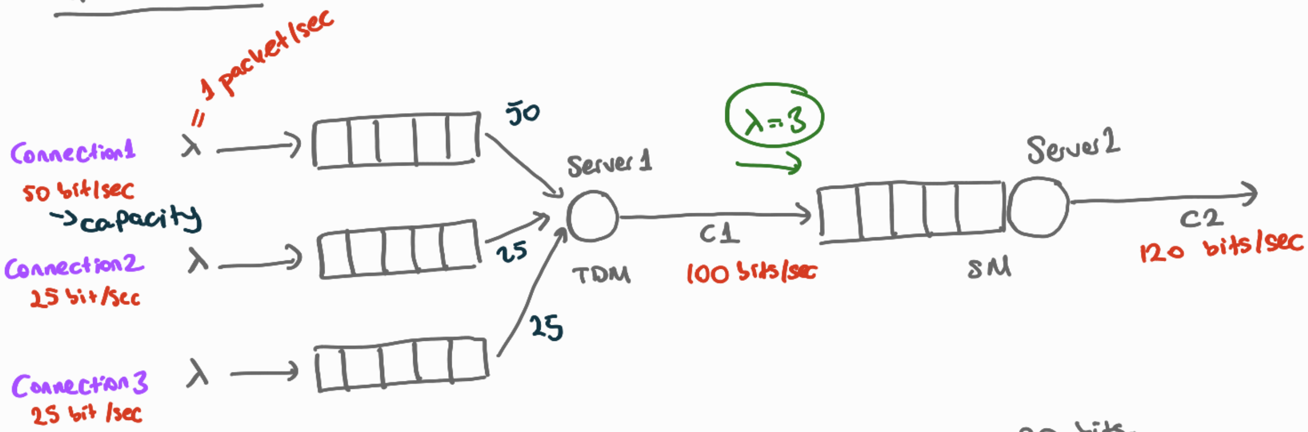
Static Allocation (TDM / FDM)



$$T = \frac{1}{\mu/m - \lambda/m} = \frac{M}{\mu - \lambda}$$

* All sessions behave like seperate queuing system

Question 2



Packet sizes are exponentially distributed with an avg. size 20 bits.

a) What is the expected number of packets in the total system.

$$E[S_{c_1}] = \frac{P}{C} = \frac{20\text{bits}}{50\text{bits}} = \frac{2}{5} = \frac{1}{\mu_1} \Rightarrow \mu_1 = 2.5 \Longrightarrow \rho_1 = \frac{\lambda_i}{\mu_1} = \frac{1}{2.5} = 0.4$$

$$E[S_{c_2}] = E[S_{c_3}] = \frac{P}{C} = \frac{20\text{bits}}{25\text{sec}} = \frac{1}{\mu_2} \Rightarrow \mu_2 = \mu_3 = 1.25 \Rightarrow \rho_2 = \rho_3 = \frac{1}{1.25} = 0.8$$

$$E[N_{server_1}] = \underbrace{(0.4) \times 0.5}_{0.2} + \underbrace{(0.8) \times 0.25}_{0.2} + \underbrace{0.8 \times (0.25)}_{0.2} = \boxed{0.6}$$

$$E[NQ_1] = \frac{\rho_1^2}{1-\rho_1} = \frac{(0.4)^2}{1-0.4} = \frac{0.16}{0.6}$$

$$E[NQ_2] = \frac{\rho_2^2}{1-\rho_2} = \frac{(0.8)^2}{1-0.8} = \frac{0.64}{0.2} = E[NQ_3]$$

$$E[N_{S_1}] = E[NQ_1] + E[NQ_2] + E[NQ_3] + E[N_{server}] = \frac{0.16}{0.6} + \frac{0.64}{0.2} \times 2 + 0.6$$

$E[NQ] = \frac{\rho^2}{1-\rho}$ $\quad E[S] = \frac{1}{\mu} = \frac{P}{C} \Rightarrow \mu_2 = \frac{C}{P} = \frac{120 \text{ bits/sec}}{20 \text{ bits}} = 6$, $\lambda = 3 \Rightarrow \rho = 0.5$

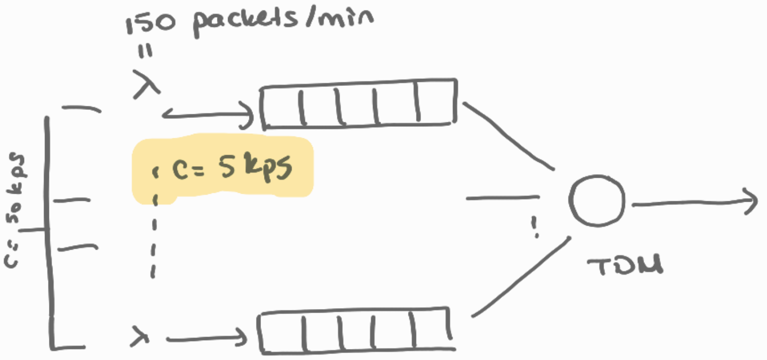
$$\left.\begin{aligned} E[NQ] &= \frac{(0.5)^2}{1-0.5} = 0.5 \\ E[N_{server_2}] &= \rho = 0.5 \end{aligned}\right\} 1 = E[NS_2]$$

b) Expected end to end delay of a packet from connection 1 in the total system

$$E[T_S] = E[T_{S\,TDM}] + E[T_{S\,SM}]$$

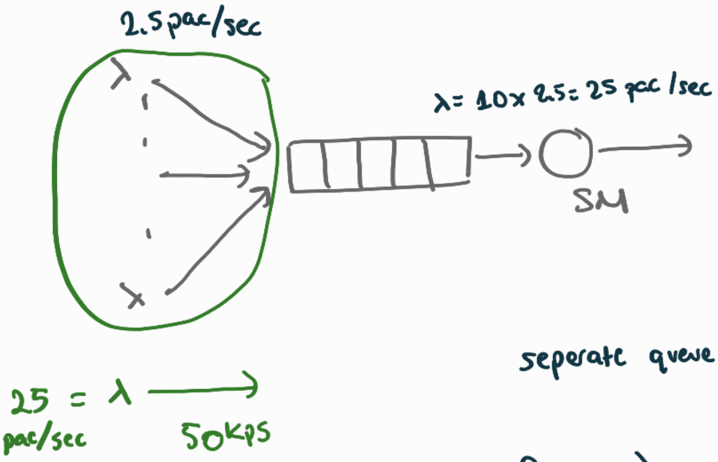
$$= \frac{1}{\mu_1 - \lambda_1} + \frac{1}{\mu_2 - \lambda_2} = \frac{1}{2.5 - 1} + \frac{1}{(6-3)}$$

Question 3



Packet lengths are exponentially distributed with mean 1000 bits.

a) Find the average # of packets in the queue and the avg delay per packet when i) TDM ii) SM



a.i) $E[NQ] = \frac{\rho^2}{1-\rho}$

$\rho = \frac{\lambda}{\mu}$ $\quad \lambda = \frac{150 \text{ packets}}{60 \text{ sec}} = 2.5$ packet/sec

seperate queue $\rightarrow E[S_i] = \frac{1}{\mu} = \frac{P}{C} = \frac{1\cancel{000} \text{ bits}}{5 \times 1\cancel{000} \text{ bits/sec}} = 0.2 = \frac{2}{10} = \frac{1}{\mu_i}$

$\mu_i = 5$

$\rho_i = \frac{\lambda}{\mu_i} = \frac{2.5}{5} = 0.5$

$E[NQ_i] = \frac{\rho_i^2}{1-\rho_i} = \frac{0.5 \times \cancel{0.5}}{\cancel{0.5}} = 0.5 \rightarrow$ per queue

$E[NQ] = 10 \times E[NQ_i] = 10 \times 0.5 = 5 \rightarrow$ all queues

Average delay $E[T_S] = \frac{1}{\mu - \lambda} = \frac{1}{5 - 2.5} = \frac{1}{2.5} = \frac{4}{10} = 0.4$ per packet a.ii) $E[NQ] = \frac{\rho^2}{1-\rho}$

SM

$\rho = \frac{\lambda}{N}$ $\lambda = 10 \times 2.5 = 25$ pac/sec

$E[S] = \frac{1}{N} = \frac{P}{C} = \frac{1000}{\underset{\downarrow 50kps}{50 \times 1000}} = \frac{1}{50} \Rightarrow N = 50$

$\rho = \frac{25}{50} = 0.5$

$E[NQ] = \frac{0.5^2}{1-0.5} = 0.5 \rightarrow$ single queue

$E[TS] = \frac{1}{N-\lambda} = \frac{1}{50-25} = \frac{1}{25} \rightarrow$ per packet

b) If 5 of the session generate traffic with rate of 250 packets/min and other 5 with 50 packets/min, find the average # of packets in the queue and the average delay per packet when i) TDM and ii) SM is used

i) TDM 5 packets $\rightarrow \lambda = \frac{25\cancel{0} \text{ packets}}{6\cancel{0} \text{ sec}} = \frac{25}{6}$ pac/sec

$N_i = 5$ $\rho_i = \frac{\lambda}{N_i} = \frac{25/6}{5} = \frac{5}{6}$ per packet

5 packets $\rightarrow \lambda = \frac{5\cancel{0}}{6\cancel{0}}$ $\rightarrow \rho_i = \frac{5/6}{5} = \frac{1}{6}$ per packet

$E[NQ_1] = \frac{\rho_i^2}{1-\rho} = \frac{(5/6) \times (5/6)}{1 - 5/6} = \frac{5/6 \times 5/\cancel{6}}{1/\cancel{6}} = \frac{25}{6}$ per queue

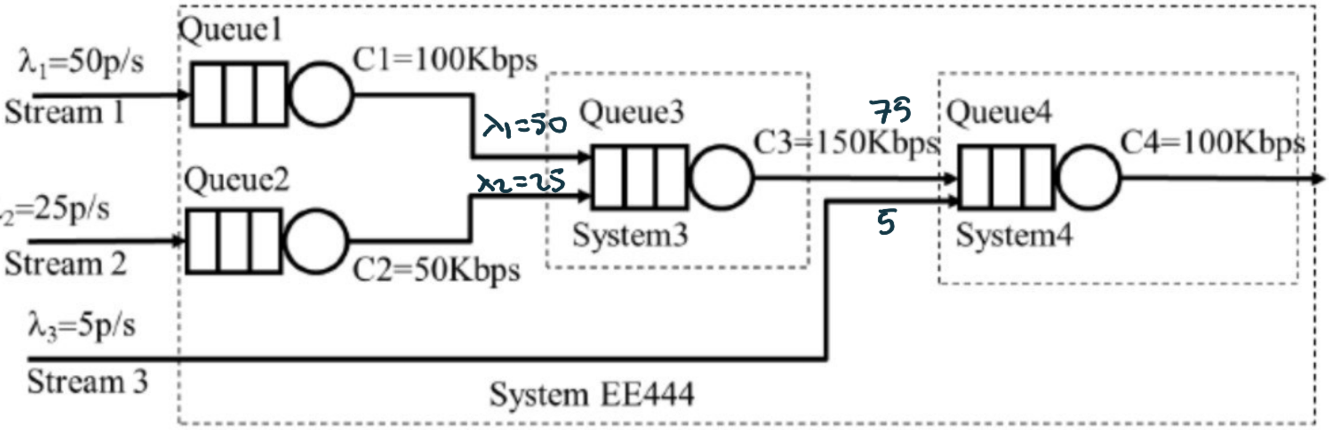
$E[NQ_2] = \frac{\rho_i^2}{1-\rho_i} = \frac{(1/6) \times (1/\cancel{6})}{5/\cancel{6}} = \frac{1}{30}$ per queue

$\boxed{E[NQ] = 5 \times \left(\frac{25}{6} + \frac{1}{30}\right)}$

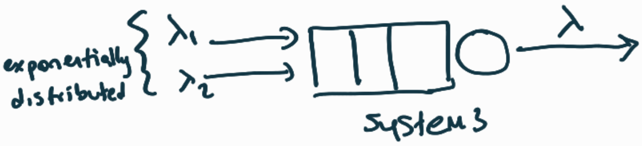
ii) SM $\rightarrow \frac{\rho^2}{1-\rho} = E[NQ] = \frac{0.5^2}{0.5} = 0.5$

$\rho = \frac{\lambda}{\underset{\downarrow 50kbps}{N}}$ $\lambda = 5 \times \left(\frac{25}{6}\right) + 5 \times \left(\frac{5}{6}\right) = \cancel{6} \times \frac{25}{\cancel{6}} = 25$ pac/sec $\rho = \frac{25}{50} = 0.5$

Q4 In the queuing system below, Stream 1, Stream 2 and Stream 3 are Poisson arrivals. The packet sizes for Stream 1 and Stream 2 are exponentially distributed with an average of 1000 bits/packet. Stream 3 has **fixed** size packets with a size of 1000bits. **Given information:** The average number of packets in Queue 4 is 3. All queues have infinite capacity. No packets are dropped.



a) what kind of queuing does System3 have in X/X/X notation?



$\lambda_1 + \lambda_2 = \lambda$ is also exponential distributed

M/M/1

b) what is the probability that System3 server is busy $= \rho_3$ at a given time?

$E[NQ_1] = \frac{\rho_1^2}{1-\rho_1}$ $\quad \rho_1 = \frac{\lambda_1}{\mu_1}$ $\quad E[S_1] = \frac{1}{\mu_1} = \frac{P_1}{C_1} = \frac{1\ k\ \text{bits}}{100\ \text{kbits/sec}} \Rightarrow \mu_1 = 100$ $\quad \rho = \frac{\lambda_1}{\mu_1} = \frac{50}{100} = 0.5$ $\quad E[NQ_1] = \frac{0.5^2}{0.5} = 0.5$

$E[NQ_2] = \frac{\rho_2}{1-\rho_2}$ $\quad \rho_2 = \frac{\lambda_2}{\mu_2}$ $\quad E[S_2] = \frac{1}{\mu_2} = \frac{P_2}{C_2} = \frac{1\ k\text{bits}}{50\ k\text{bit/sec}} \Rightarrow \mu_2 = 50$ $\quad \rho = \frac{\lambda_2}{\mu_2} = \frac{25}{50} = 0.5$ $\quad E[NQ_2] = 0.5$

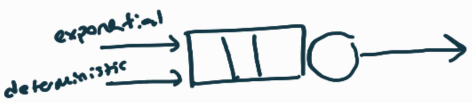
$E[N_{server}] = \rho_3 = \frac{\lambda_3}{\mu_3}$ $\quad \lambda_3 = \lambda_1 + \lambda_2 = 75\ p/s$ $\quad E[S_3] = \frac{1}{\mu_3} = \frac{P_3}{C_3} = \frac{1\ k\text{bits}}{150\ \text{kps}} \Rightarrow \mu_3 = 150$ $\quad \rho_3 = \frac{\lambda_3}{\mu_3} = \frac{75}{150} = 0.5$ $\quad E[N_{server}] = 0.5$

$E[N_S] = E[NQ_1] + E[NQ_2] + E[N_{server}] = 3 \times 0.5 = 1.5$ X (No need) The question asks only $\boxed{\rho_3 = 0.5}$

c) what is the average delay of any given packet in System 3?

$E[T_S] = \frac{1}{\mu_3 - \lambda_3} = \frac{1}{150-75} = \frac{1}{75}$ per packet

d) what kind of queuing does System4 have in X/X/X notation?



M/G/1

e) what is the average delay of packets on Stream1 in SystemEE444?

$E[T_S] = \frac{1}{\mu_1 - \lambda_1} + \frac{1}{\mu_3 - \lambda_3} + E[T_{S4}]$

$\mu_1 = 100$, $\lambda_1 = 50$, $\mu_3 = 150$, $\lambda_3 = 75$, $E[T_{S4}] = \frac{3.8}{80}$

$E[N_{S4}] = \underbrace{E[NQ_4]}_{3} + \underbrace{E[N_{server\,4}]}_{} = 3.8$

$\rho = \frac{\lambda_4}{\mu_4} = \frac{80}{100} = 0.8$

Little's Law:

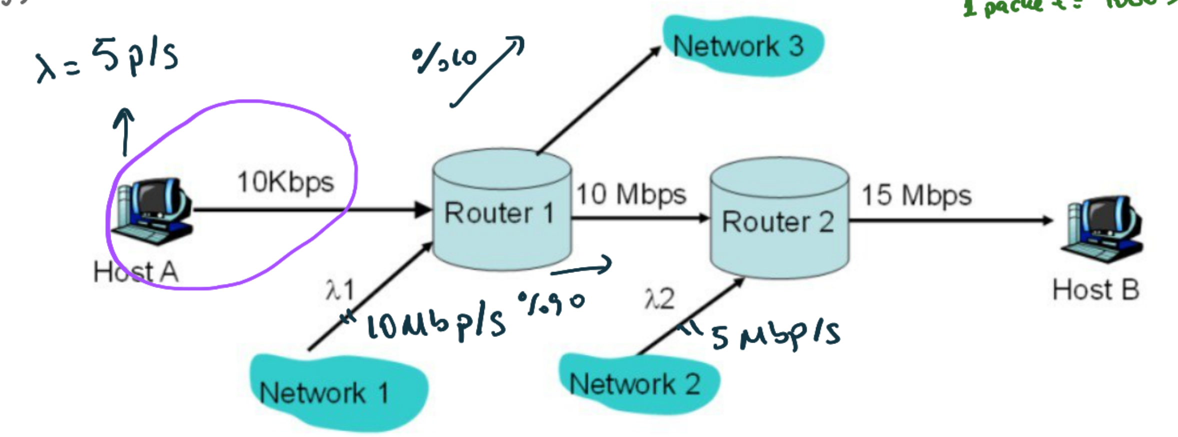
$\underbrace{E[N_{S4}]}_{3.8} = \underbrace{E[T_{S4}]}_{\frac{3.8}{80}} \underbrace{\lambda_4}_{80}$

$\mu_4 = \frac{C_4}{P_4} = \frac{100\ \cancel{k}\text{bit/sec}}{1000\ \text{bit/p}} = 100$

Q5)

$\lambda_1 = 10Mbps = 10 \times 10^6$ bit/sec $= \frac{10 \times \cancel{10^6}^{10^3}}{\cancel{10^3}} = 10^4 = 10kp/s$

1 packet = 1000 bit

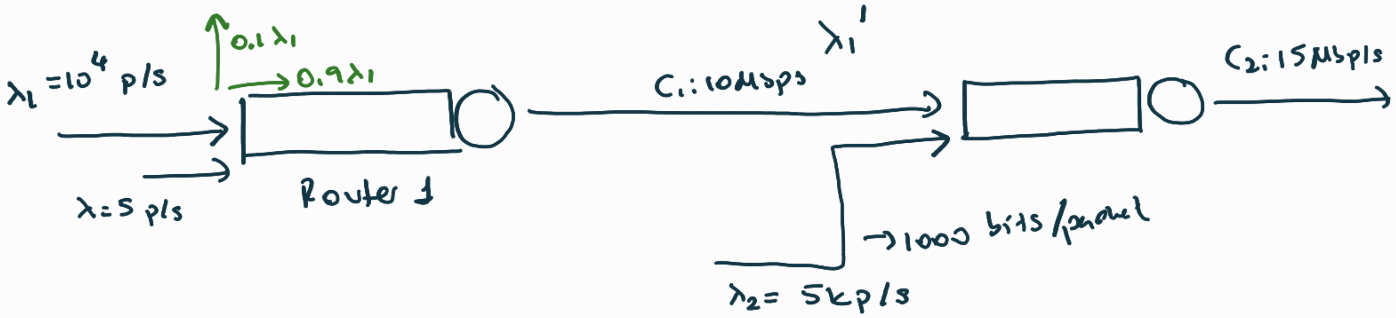


For all packets in this network, the packet sizes are exponentially distributed with an average size of 1000 bits/packet. All packet generation processes are Poisson processes. Ignore all propagation delays.

Host A generates packets with a rate of 5 packets/sec (↑too small) and sends them to Host B over Router 1 and Router 2. The total traffic that arrives from Network 1 at Router 1 has a rate of λ1=10Mbps. 10% of the packets that arrive from Network 1 are routed to Network 3 while the rest of them go on to Router 2. The total traffic that arrives from Network 2 at Router 2 has a rate of λ2=5Mbps.

What is the total average delay from Host A to Host B. Clearly specify all your assumptions and the properties that you use.

$\lambda_1 >> \lambda_A \quad 10Mbps >> 5p/s$



$$E[T_{R1}] = \frac{1}{\mu_{R1} - \lambda_{R1}} = \frac{1}{10k - 9k} = \frac{1}{1k}$$

$$E[S_{R1}] = \frac{1}{\mu_{R1}} = \frac{P_{R1}}{C_{R1}} = \frac{10\cancel{00} \text{ bits}}{10 \times \cancel{10^6} \text{ bit/sec}} \Rightarrow \mu_1 = 10k$$

$$\lambda_{R1} = 0.9 \times \lambda_1 + \cancel{5p/s} \simeq 9k\,p/s$$

$$E[T_{R2}] = \frac{1}{\mu_{R2} - \lambda_{R2}} = \frac{1}{15k - 14k} = \frac{1}{1k}$$

$$E[S_{R2}] = \frac{1}{\mu_{R2}} = \frac{P_{R2}}{C_{R2}} = \frac{10\cancel{00} \text{ bits}}{15 \times \cancel{10^6}^{10^3} \text{ bits/sec}} \Rightarrow \mu_2 = 15k\,p/s$$

$$\lambda_{R2} = 0.9 \times \lambda_1 + \lambda_2 + \cancel{5p/s} = 9kp/s + 5kp/s = 14kp/s$$

$$E[T_S] = E[T_{R1}] + E[T_{R2}] = \frac{1}{500} = 2\,\mu sec$$

Total average delay: 102 μsec

* Transmission delay: $\frac{1k \text{ bits}}{10k \text{ bit/sec}} = 0.1 sec = 100\,\mu sec$

$\frac{10\cancel{00} \text{ bits}}{10 \times \cancel{10^6}^{10^3} \text{ bit/sec}} = \frac{1}{10^4} sec = 0.1\,msec$ too small

Finite Buffer Case:

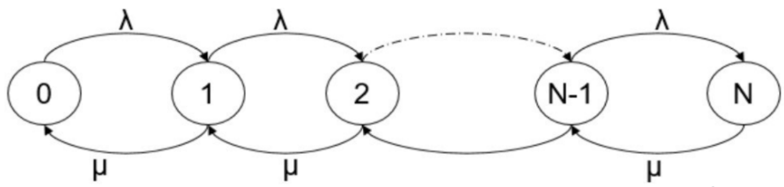
Single queue with G buffer space $\Rightarrow \lambda \neq X$

$\lambda(1 - P(G \text{ jobs in the system})) \rightarrow$ Completion Rate

$E[N_S] = \lambda(1 - P(G \text{ jobs in the system}))\, E[T_S]$

Q6

- Repeat the analysis in class for M/M/1/N (Finite buffer with N buffers)
- Write the balance equations: rate in rate out
- Find the limiting probabilities: π_i
- Find π_0 using $\Sigma\pi_i=1$ (FINITE SUM)
- Probability of a packet loss: Probability of a packet arriving and finding the system with N packets: π_N $\rightarrow \pi_N = \left(\frac{\lambda}{\mu}\right)^N \pi_0$



$$\sum_{i=1}^{N} \pi_i = 1 = \sum_{i=1}^{N} \left(\frac{\lambda}{\mu}\right)^i \underbrace{\pi_0}_{1-\rho} = \pi_0 \underbrace{\sum_{i=1}^{N} \left(\frac{\lambda}{\mu}\right)^i}$$

finite sum $= \sum_{n=0}^{N} a^N = \frac{(1-a^{N+1})}{1-a}$

$$1 = \pi_0 \frac{(1-(\lambda/\mu)^{N+1})}{1-\frac{\lambda}{\mu}}$$

$$\boxed{\pi_0 = \frac{1-(\lambda/\mu)}{1-(\lambda/\mu)^{N+1}}}$$