Entanglement and Teleportation

Entanglement :

$$|\psi\rangle_{AB} = |\phi\rangle_A \otimes |\chi\rangle_B$$ ← product state

entangled if it is not a product state

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

For two qubits A, B

$$|\psi\rangle_{AB} = \sum_{i,j=0} c_{ij} |i\rangle_A \otimes |j\rangle_B$$

$$C = \begin{bmatrix} c_{00} & c_{01} \\ c_{10} & c_{11} \end{bmatrix}$$

$$\boxed{\begin{array}{l} |\psi\rangle \text{ is entangled} \iff \det C \neq 0 \\ |\psi\rangle \text{ is product st} \iff \det C = 0 \end{array}}$$

★ let A and B be any systems

$\mathcal{H}_A$ = Hilbert space of A = n-dim.

$\mathcal{H}_B$ = Hilbert space of B = m-dim

Let $|\psi\rangle_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$

$|\psi\rangle_{AB}$ = a state of the composite sys AB.

→ Let $|\alpha_1\rangle, \dots, |\alpha_n\rangle$ be an orthon. basis (o.n.b) of H_A

$|\beta_1\rangle, \dots |\beta_m\rangle$ " " of $\mathcal{H}_B$

Expand

$$|\psi\rangle = \sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij} |\alpha_i\rangle \otimes |\beta_j\rangle$$

$$C = \begin{bmatrix} c_{11} & \cdots & c_{1m} \\ \vdots & & \\ c_{n1} & \cdots & c_{nm} \end{bmatrix}$$ is $n \times m$ matrix

Singular value decomposition of C

$$C = U D V$$

$n \times n$ unitary

$n \times m$ diagonal

$m \times m$ unitary

$$|\psi\rangle = \sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij} \, |\alpha_i\rangle \otimes |\beta_j\rangle$$

$$= \sum_{i,j,k,l} U_{ik} D_{kl} V_{lj} \, |\alpha_i\rangle \otimes |\beta_j\rangle$$

- Define $|\tilde{\alpha}_k\rangle = \sum_i U_{ik} |\alpha_i\rangle$

  $|\tilde{\alpha}_1\rangle, \ldots |\tilde{\alpha}_n\rangle$ a new orb for $\mathcal{H}_A$

- " $|\tilde{\beta}_l\rangle = \sum_j V_{lj} |\beta_j\rangle$

  $|\tilde{\beta}_1\rangle, \ldots |\tilde{\beta}_m\rangle$ is a new orb for $\mathcal{H}_B$

$$|\psi\rangle = \sum_{k,l} \underbrace{D_{k,l}}_{d_k \delta_{kl}} |\tilde{\alpha}_k\rangle \otimes |\tilde{\beta}_l\rangle$$

$$|\psi\rangle = \sum_k d_k |\tilde{\alpha}_k\rangle \otimes |\tilde{\beta}_k\rangle$$

$\Uparrow$ Schmidt decomposition $\Downarrow$

For any state $|\psi\rangle_{AB}$ in $\mathcal{H}_{AB}$, we can find orthonormal basis $\mathcal{H}_A$ and $\mathcal{H}_B$ such that

$$|\psi\rangle_{AB} = \sum_{k=1}^{p} c_k |\alpha_k\rangle_A \otimes |\beta_k\rangle_B$$

$$\|\psi\|^2 = c_1^2 + c_2^2 + \cdots + c_p^2 = 1$$

p is called the Schmidt rank of $|\psi\rangle$

c_k (or c_k^2) are called Schmidt numbers of $|\psi\rangle$

$|\psi\rangle$ is product $\Leftrightarrow$ $p = \text{rank} = 1$

$|\psi\rangle$ is entangled $\Leftrightarrow$ $p \geq 2$

we say $|\psi\rangle$ is <u>maximally entangled</u> if

$$c_1 = c_2 = \dots = c_p = \frac{1}{\sqrt{p}}$$

For 2 qubits

$$\boxed{\begin{aligned} |\Phi_+\rangle_{AB} &= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \\ |\Phi_-\rangle_{AB} &= \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \\ |\Psi_+\rangle_{AB} &= \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \\ |\Psi_-\rangle_{AB} &= \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \end{aligned}}$$

All are maximally entangled states (rank = 2)

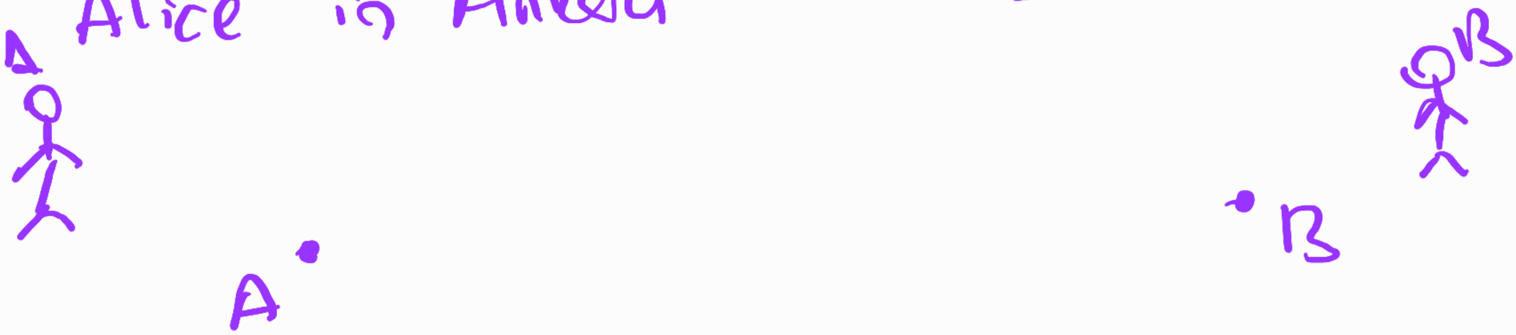
$\{ |\Phi_+\rangle, |\Phi_-\rangle, |\Psi_+\rangle, |\Psi_-\rangle \}$ forms a basis of the composite Hilbert space $\mathcal{H}_{AB}$.

Bell basis:

$$|\Phi_{\pm}\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle \pm |11\rangle\right) \quad \text{etc.}$$

Alice in Ankara

Bob in Bursa



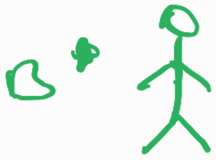
Suppose A and B interacted in the past & put into the $|\Phi_+\rangle_{AB}$ state.

After that they are seperated & A is sent to Bob

Reminder: if Alice does a measurement on A, then state of AB collapses

→ B state also collapses.

measure



collapse

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

* if Alice does σ_z measurement

prob $= 1/2$ $|00\rangle_{AB}$

prob $= \frac{1}{2}$ $|11\rangle_{AB}$

after the measurement, the particles are no longer entangled

* If Alice does σ_x measurement

$$\underbrace{\begin{aligned} \mathbb{P}_+ &= \left(|+\rangle\langle+|\right)_A \otimes \mathbb{1}_B \\ \mathbb{P}_- &= \left(|-\rangle\langle-|\right)_B \otimes \mathbb{1}_B \end{aligned}}_{\text{projection op.}}$$

$$|\Phi_+\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) \rightarrow |P_+|\Phi_+\rangle$$

$$= \frac{1}{\sqrt{2}}\left(|+\rangle\underbrace{\langle+|0\rangle}\otimes|0\rangle + |+\rangle\underbrace{\langle+|1\rangle}\otimes|1\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(|+\rangle\otimes|0\rangle + |+\rangle\otimes|1\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(|++\rangle\right)$$

$$P_-|\Phi_-\rangle = \ldots\ldots = \frac{1}{\sqrt{2}}|--\rangle$$

If Alice measures σ_x $\xrightarrow{prob = 1/2}$ $|+ + \rangle_{AB}$

$\xrightarrow{prob = 1/2}$ $|- - \rangle_{AB}$

★ Teleportation of an unknown quantum state:

- Alice has a qubit C in an unknown state $|\psi\rangle_C = a|0\rangle_C + b|1\rangle_C$

• Alice wants to transfer this state to one of Bob's particles.

Solution: Send particle C to Bob by using a quantum channel.

What if such a quantum channel is not available?

* If there is no prior entanglement between particles of Alice & particles of Bob Then this is an impossible task. (This violates the no-cloning theorem)

* But if there are particles with prior entanglement

$|\psi\rangle_C = a|0\rangle_C + b|1\rangle_C$

$|\Phi_+\rangle_{AB} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ } has to be maximally entanglement

some quantum channel must have been used in the past to distribute this

teleportation protocol

$\underbrace{|\Phi_+\rangle_{AB} \otimes |\psi\rangle_C}_{\text{before}} \rightarrow \underbrace{|garbage\rangle_{AC} \otimes |\psi\rangle_B}_{\text{after}}$

1. Alice does a Bell-basis measurement on CA

2. Alice sends the $\boxed{\text{2-bit}}$ classical info about the outcome of the experiment to Bob by using a classical communication channel.

③ Bob does a special unitary transformation on his particle B

↘ depends on the info received from Alice

$$\boxed{I, \sigma_x, \sigma_y, \sigma_z}$$

④ Voila!

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① Bell Basis
$$\begin{cases} |\Phi_{\pm}\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle) \\ |\Psi_{\pm}\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle) \end{cases}$$

or Alice measures an observable X whose eigenvectors are Bell basis vectors

$$X = \lambda_1 |\Phi_+\rangle\langle\Phi_+| + \lambda_2 | \rangle\langle | + \ldots$$

State of CAB $\boxed{\text{before the meas}}$

$$|\psi_{①}\rangle_{CAB} = |\psi_C\rangle \otimes |\Phi_+\rangle_{AB}$$

$$= (a|0\rangle + b|1\rangle)_C \otimes \left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right)_{AB}$$

$$|\psi_{①}\rangle_{CAB} = \frac{a|000\rangle + b|100\rangle + a|011\rangle + b|111\rangle}{\sqrt{2}}$$

$$= \frac{a}{\sqrt{2}}\left(\frac{|\Phi_+\rangle + |\Phi_-\rangle}{\sqrt{2}}\right)_{CA} \otimes (|0\rangle)_B + \frac{b}{\sqrt{2}}\, \frac{|\Psi_+\rangle - |\Psi_-\rangle}{\sqrt{2}} \otimes |0\rangle$$

$$+ \frac{a}{\sqrt{2}}\, \frac{|\Psi_+\rangle + |\Psi_-\rangle}{\sqrt{2}} \otimes |1\rangle + \frac{b}{\sqrt{2}}\, \frac{|\Phi_+\rangle - |\Phi_-\rangle}{\sqrt{2}} \otimes |1\rangle$$

$$= \frac{1}{2}\Big(\underbrace{|\Phi_+\rangle_{CA} \otimes (a|0\rangle + b|1\rangle)_B} + \underbrace{|\Phi_-\rangle_{CA} \otimes (a|0\rangle - b|1\rangle)_B}$$

$$\underbrace{+ \underbrace{|\Psi_+\rangle_{CA} \otimes (b|0\rangle + a|1\rangle)_B} + \underbrace{|\Psi_-\rangle_{CA} \otimes (a|1\rangle - b|0\rangle)_B}}$$

before measurement, Note: C is unentangled from AB

A and B are entangled with each other

Measurements

Collapse occurs to one of these 4 terms.

The state of CAB after measurement

$$\bullet\ |\psi_{②}\rangle = |\Phi_+\rangle \otimes \boxed{(a|0\rangle + b|1\rangle)}^{|\psi\rangle_B} \text{ if } \rightarrow$$

→ outcome is Φ_+ (prob $= \frac{1}{4}$)

(Now, B is unentangled due to the meas.)

- $|\psi_2\rangle = |\Phi_-\rangle \otimes (a|0\rangle + b|1\rangle)$ if outcome is Φ_- (prob $= \frac{1}{4}$)

- $|\psi_2\rangle = |\Psi_+\rangle \otimes (a|1\rangle + b|0\rangle) \rightarrow |\Psi_+\rangle$ 1/4

- $|\psi_2\rangle = |\Psi_-\rangle \otimes (a|1\rangle - b|0\rangle) \rightarrow |\Psi_-\rangle$ 1/4

② Alice conveys the 2bit info about the outcome of the meas.

③ Bob applies one of 4 possible unitary transformations to B

I: do nothing

$X = \sigma_x$: a rotation of spin around x axis by 180°

$Y = \sigma_y$: " " " y "

$Z = \sigma_z$: " " " z "

if it is a spin 1/2 particle

State of B after step ①	Unitary Bob should apply	Final state of B after Bob's unitary
$(a\|0\rangle + b\|1\rangle)$	I	$a\|0\rangle + b\|1\rangle = \|\psi\rangle$
$(a\|0\rangle - b\|1\rangle)$	σ_z	$a\|0\rangle + b\|1\rangle = \|\psi\rangle$
$(a\|1\rangle + b\|0\rangle)$	σ_x	$a\|0\rangle + b\|1\rangle = \|\psi\rangle$
$(a\|1\rangle - b\|0\rangle)$	σ_y	$-ia\|0\rangle - ib\|1\rangle = -i\|\psi\rangle$

$$\begin{pmatrix} \sigma_y|0\rangle = i|1\rangle \\ \sigma_y|1\rangle = -i|0\rangle \end{pmatrix}$$

overall phase factor

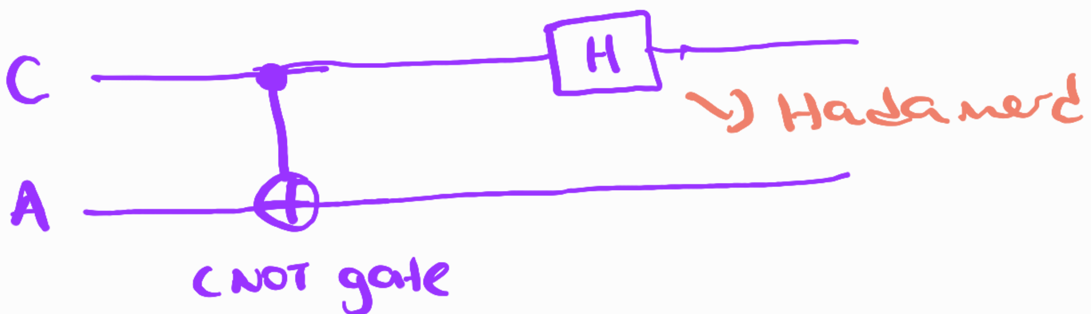
can not be measured

Done! :)

Questions

How can Alice do Bell Basis Measurement on CA?

Ans: 1st apply a unitary to particles so that same unitary converts Bell basis to a product basis.



CNOT: controlled NOT = controlled X

if C has state 0: do nothing

if " " " 1: apply NOT gate to A.

state of CA before	state of CA after
00	00
01	01
10 $\rightarrow$	11
11 $\rightarrow$	10

$$|\Phi_+\rangle_{CA} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$\downarrow \text{CNOT}$$

$$\frac{1}{\sqrt{2}}(|00\rangle + |10\rangle) = |+0\rangle$$

$$\downarrow H$$

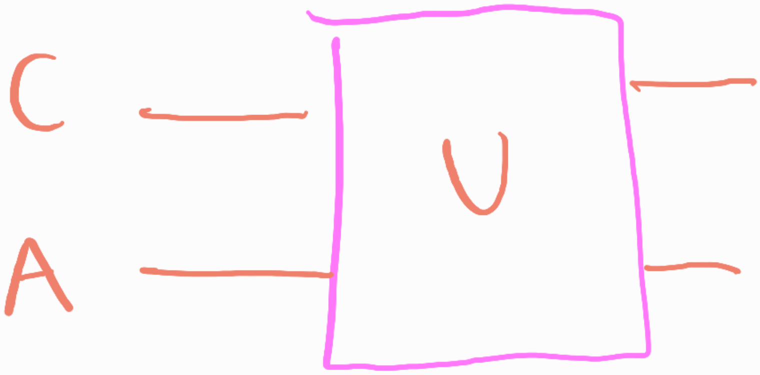
$$|00\rangle$$

$$|\Phi_-\rangle = \;.\;.\;.\;. \quad |10\rangle$$

$$|\Psi_+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \longrightarrow |01\rangle$$

$$|\Psi_-\rangle = \;.\;.\;.\;.\;. \longrightarrow |11\rangle$$

(Measurement of CA on Bell basis) = (Applying unitary U on CA and doing standard basis measurement)



Note: Alice measurement on CA must destroy the state $|\psi\rangle$ from her particles.

No-Cloning Theorem

before

after