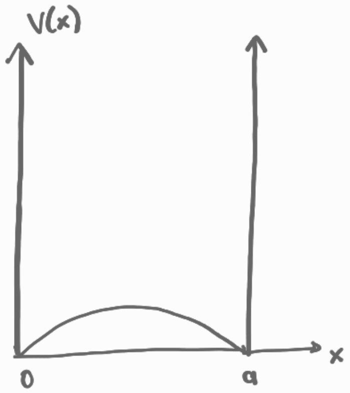
RECITATIONS

Question 1) Consider a particle in an infinite potential with width a. Bottom of the well is changed to have a shape



$$\lambda H_I = \mathcal{E} \sin\left(\frac{\pi x}{a}\right)$$

a) Show that 1st order correction to energy is given by $\lambda E_n^{(1)} = \frac{2}{\pi} \mathcal{E} \left(\frac{4n^2}{4n^2-1}\right)$

$$\lambda E_n^{(1)} = \langle \phi_n | \lambda H_I | \phi_n \rangle = \int_0^a \phi_n^*(x) \lambda H_I \phi_n(x)\, dx$$

For infinite potential well $\phi_n = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)$

$$\lambda E_n^{(1)} = \mathcal{E} \frac{2}{a} \int_0^a \sin^2\left(\frac{n\pi x}{a}\right) \sin\left(\frac{\pi x}{a}\right) dx$$

$$= \frac{2}{a} \mathcal{E} \frac{a}{2\pi} \left(\frac{\cancel{\cos(2n\pi)}^{1} + 8n^2 \cancel{-1}}{4n^2-1} \right)$$

Since $\cos(2n\pi) = 1$

$$\boxed{\lambda E_n^{(1)} = \frac{2\mathcal{E}}{\pi} \left(\frac{4n^2}{4n^2-1}\right)}$$

b) Write down ground state energy of full system corrected up to 1st order in perturbation

$$E_1 \simeq E_1^{(0)} + \lambda E_1^{(1)} = E_1^{(0)} + \frac{8\mathcal{E}}{3\pi} = E_1^{(0)}\left(1 + \frac{8}{3\pi}\frac{\mathcal{E}}{E_1^{(0)}}\right)$$

c) What smallness assumption is appropriate to $\mathcal{E}$ so that perturbation theory is applicable here?

$$\frac{\mathcal{E}}{E_1^{(0)}} \ll 1$$

d) Show that 1^{st} order correction to wave function.

$$\psi_n^{(1)} = \frac{4\varepsilon}{\pi E_1^{(0)}} \sum_{m\neq n} \frac{(1+\cos(n\pi)\cos(m\pi))}{((m-n)^2-1)((m+n)^2-1)} \cdot \frac{mn}{(n^2-m^2)}$$

$$\psi_n^{(1)} = \sum_{m\neq n} \lambda c_{nm}^{(1)} \phi_m = \sum_{m\neq n} \frac{\langle\phi_m|\lambda H_I|\phi_n\rangle}{(E_n^{(0)} - E_m^{(0)})} \phi_m$$

$$\langle\phi_m|\lambda H_I|\phi_n\rangle = \int_0^a \phi_m^*(x)\,\lambda H_I\,\phi_n(x)\,dx$$

$$= \frac{2\varepsilon}{a}\int_0^a \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{\pi x}{a}\right)dx$$

$$= \frac{2\varepsilon}{a}\,\frac{a\,2mn}{\pi}\,\frac{(1+\cos(n\pi)\cos(m\pi))}{((m-n)^2-1)((m+n)^2-1)}$$

$$E_n^{(0)} - E_m^{(0)} = E_1^{(0)}(n^2-m^2)$$

$$\psi_n^{(1)} = \frac{4\varepsilon}{\pi E_1^{(0)}} \sum_{m\neq n} \frac{(1+\cos(n\pi)\cos(m\pi))}{((m-n)^2-1)((m+n)^2-1)} \cdot \frac{mn}{(n^2-m^2)}\phi_m$$

e) Write down $\boxed{\text{ground state wavefunction}}$ of full system corrected up to 1^{st} order in perturbation keeping only the first non zero terms in the expansion.

$$\psi_1^{(1)} = \frac{4\varepsilon}{\pi E_1^{(0)}} \sum_{m\neq n\,(1)} \frac{(1+\overset{-1}{\cos\pi})\cos(m\pi)}{((m-1)^2-1)((m+1)^2-1)} \cdot \frac{m}{(1^2-m^2)}\phi_m$$

when m is even $\cos(m\pi) = 1 \Rightarrow \psi_1^{(1)} = 0$

So $m = 3, 5, 7, \dots \Rightarrow \cos(m\pi) = -1$

Take first two terms $m=3$ and $m=5$

$$\psi_1^{(1)} = \frac{4\varepsilon}{\pi E_1^{(0)}} \frac{2}{((3-1)^2-1)((3+1)^2-1)} \cdot \frac{3}{(1^2-3^2)}\phi_3 + \frac{4\varepsilon}{\pi E_1^{(0)}} \frac{2}{((5-1)^2-1)((5+1)^2-1)} \frac{5}{(1^2-5^2)}\phi_5$$

$$\psi_1^{(1)} = -\frac{\varepsilon}{15\pi E_1^{(0)}}\left(\phi_3 + \frac{1}{21}\phi_5\right)$$

Ground state wavefunction of full system corrected up to 1^{st} order in perturbation:

$$\psi_1 \simeq \phi_1 + \psi_1^{(1)} = \phi_1 + \psi_1^{(1)} = \phi_1 - \frac{\varepsilon}{15\pi E_1^{(0)}}\left(\phi_3 + \frac{1}{21}\phi_5\right)$$

Question 2

Suppose we put a delta function bump in the center of infinite square well:

$$H' = \alpha \delta\left(x - \frac{a}{2}\right)$$

a) Find first order correction to the allowed energies. Explain why energies not perturbed for even n.

$$\psi_n^{o}(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)$$

$$E_n^{(1)} = \langle \psi_n^{o} | H' | \psi_n^{o} \rangle = \frac{2\alpha}{a} \int_0^a \sin^2\left(\frac{n\pi x}{a}\right) \delta\left(x - \frac{a}{2}\right) dx$$

$$E_n^{(1)} = \frac{2\alpha}{a} \sin^2\left(\frac{n\pi}{2}\right) = \begin{cases} 0 & \text{, if } n \text{ is even} \\ \frac{2\alpha}{a} & \text{, if } n \text{ is odd} \end{cases}$$

For even n the wave function is zero at the location of pertur. @ ($x = a/2$)

$\Rightarrow$ No effect of bump

b) Find first three nonzero terms in the expansion of the correction to the ground state $\psi_1^{(1)}$.

$$\psi_1^{(1)} = \sum_{m \neq 1} \frac{\langle \psi_m^{o} | H' | \psi_1^{o} \rangle}{(E_1^{o} - E_m^{o})} \psi_m^{o}$$

$$\langle \psi_m^{o} | H' | \psi_1^{o} \rangle = \frac{2\alpha}{a} \int \sin\left(\frac{m\pi x}{a}\right) \delta\left(x - \frac{a}{2}\right) \sin\left(\frac{\pi x}{a}\right) dx$$

$$= \frac{2\alpha}{a} \sin\left(\frac{m\pi}{2}\right) \underbrace{\sin\left(\frac{\pi}{2}\right)}_{1} = \frac{2\alpha}{a} \sin\left(\frac{m\pi}{2}\right) = 0 \text{ for } m \text{ even}$$

First nonzero terms will be for $m = 3, 5, 7$

$$E_1^{o} - E_m^{o} = (1 - m^2) E_1^{o} = \frac{\pi^2 \hbar^2}{2ma^2} (1 - m^2)$$

$$\psi_1^{1} \simeq \sum_{m = 3,5,7} \frac{2\alpha}{a} \frac{2m a^2}{\pi^2 \hbar^2 (1 - m^2)} \sin\left(\frac{m\pi}{2}\right) \phi_m$$

$$\psi_1^1 = \frac{m\alpha}{\pi^2\hbar^2}\sqrt{\frac{a}{2}}\left(\sin\left(\frac{3\pi}{a}x\right) - \frac{1}{3}\sin\left(\frac{5\pi}{a}x\right) + \frac{1}{6}\sin\left(\frac{7\pi x}{a}\right)\right)$$

Question 3

Consider a charged particle in 1-d harmonic oscillator potential. Suppose we turn on a weak electric field (E), so that the potential energy shifted by $H' = -qEx$

a) Show that there is no first order change in energy levels and calculate second order correction.

$$E_n^{(1)} = \langle \psi_n^0 | H' | \psi_n^0 \rangle = -qE \langle n | x | n \rangle = 0$$

Second order correction $E_n^2 = \sum_{m\neq n} \frac{|\langle \psi_m^0 | H' | \psi_n^0 \rangle|^2}{E_n^0 - E_m^0}$

$$E_n^{(2)} = q^2 E^2 \sum_{m\neq n} \frac{\langle m | x | n \rangle^2}{(n-m)\hbar\omega}$$

$$\boxed{\text{Use } x = \sqrt{\frac{\hbar}{2m\omega}}\,(a_+ + a_-)}$$

$$E_n^{(2)} = \frac{(qE)^2}{\hbar\omega}\frac{\hbar}{2m\omega}\sum_{m\neq n}\frac{(\sqrt{n+1}\,\delta_{m,n+1} + \sqrt{n}\,\delta_{m,n-1})^2}{(n-m)}$$

$$= \frac{(qE)^2}{2m\omega^2}\sum_{m\neq n}\frac{(n+1)\delta_{m,n+1} + n\,\delta_{m,n-1}}{(n-m)}$$

$$= \frac{(qE)^2}{2m\omega^2}\left(\frac{(n+1)}{n-(n+1)} + \frac{n}{n-(n-1)}\right) = -\frac{(qE)^2}{2m\omega^2}$$

b) Schrödinger eqn. can be solved directly in this case, by a change of $x' = x - (qE/m\omega^2)$

Find exact energies and show that it is consistent with perturbation theory approximation.

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \left(\frac{1}{2}m\omega^2x^2 - qEx\right)\psi = E\psi$$

By using $x = x' + \frac{qE}{m\omega^2}$

$$\left(\frac{1}{2}m\omega^2x^2 - qEx\right) = \frac{1}{2}m\omega^2\left(x' + \frac{qE}{m\omega^2}\right)^2 - qE\left(x' + \left(\frac{qE}{m\omega^2}\right)\right)$$

$$= \frac{1}{2}m\omega^2x'^2 - \frac{1}{2}\frac{(qE)^2}{m\omega^2}$$

Schrödinger eqn. becomes:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx'^2} + \frac{1}{2}m\omega^2x'^2\psi = \left(E + \frac{1}{2}\frac{(qE)^2}{m\omega^2}\right)\psi$$

$$\Rightarrow E_n = \left(n + \frac{1}{2}\right)\hbar\omega - \frac{1}{2}\frac{(qE)^2}{m\omega^2}$$