Deutsch Problem

* Let f be a bit-valued function of 1 bit

$x = 0, 1 \qquad f(x) \in \{0,1\}$

$$f : \{0,1\} \to \{0,1\}$$

x	$f_1(x)$
0	0
1	0

x	f_2
0	1
1	1

x	f_3
0	0
1	1

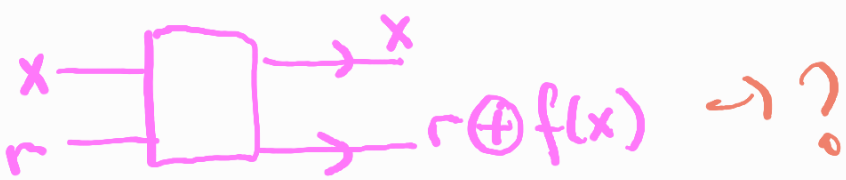
x	f_4
0	1
1	0

f_1 and f_2 are constant functions

f_3: Identity

f_4: NOT

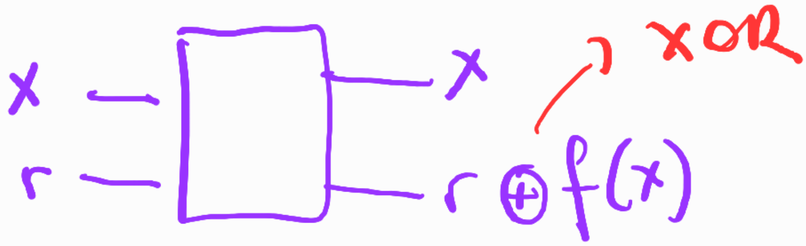
Deutsch's problem: A black box computes an unknown function f. Can you decide if f is constant or not by evaluating it only once?



Classically: No

Question: $f(0) \stackrel{?}{=} f(1)$

A quantum computer can solve this problem in one evaluation of f.



$$|xr\rangle \longrightarrow U|xr\rangle = |x, r \oplus f(x)\rangle$$

if input state is

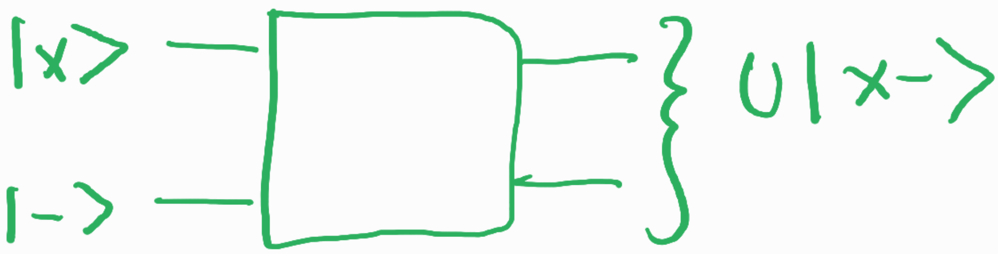
$$|\psi\rangle = \sum c_{xr} |xr\rangle$$ → superposition state

then final state is

$$U|\psi\rangle = \sum c_{xr} |x, r \oplus f(x)\rangle$$

* 1^{st} put the R qubit in state

$$|-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$



$$U|x-\rangle = U\left(\frac{|x0\rangle - |x1\rangle}{\sqrt{2}}\right)$$

$$= \frac{|x, f(x)\rangle - |x, 1 \oplus f(x)\rangle}{\sqrt{2}}$$

$$\begin{pmatrix} \frac{|x0\rangle - |x1\rangle}{\sqrt{2}} = |x-\rangle \text{ if } f(x) = 0 \\ \frac{|x1\rangle - |x0\rangle}{\sqrt{2}} = -|x-\rangle \text{ if } f(x) = 1 \end{pmatrix}$$

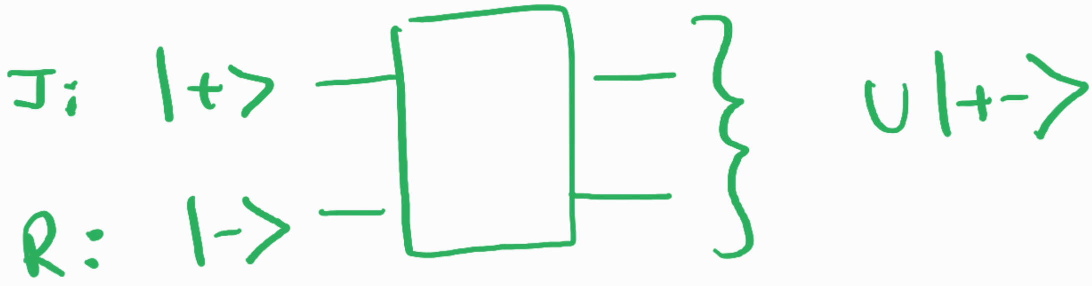
$$U|x-\rangle = (-1)^{f(x)} |x-\rangle$$

If the R qubit is in $|-\rangle$ state, then function evaluation just changes the overall phase factor of the state

physics is independent of the overall phase factor

If input qubit I is in $|+\rangle$ state

result " R " " $|-\rangle$ state



$$U|+-\rangle = U\,\frac{|0-\rangle + |1-\rangle}{\sqrt{2}}$$

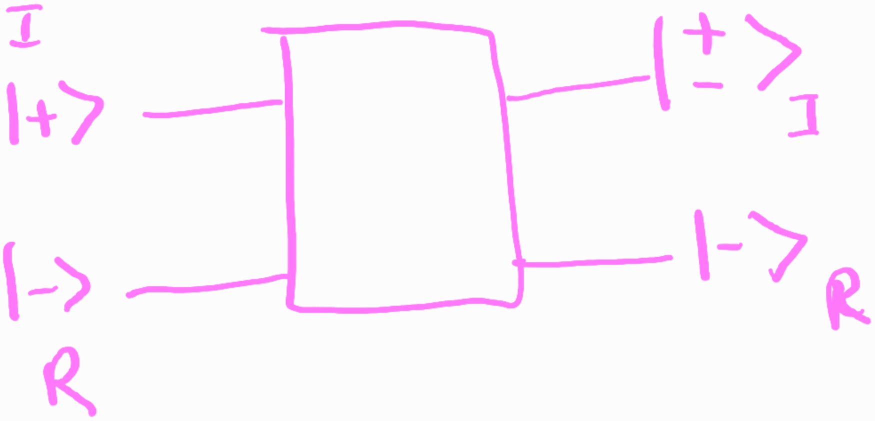
$$= \frac{(-1)^{f(0)}|0-\rangle + (-1)^{f(1)}|1-\rangle}{\sqrt{2}}$$

$$= (-1)^{f(0)}\left(\frac{|0-\rangle + (-1)^{f(1)-f(0)}|1-\rangle}{\sqrt{2}}\right)$$

$$= (-1)^{f(0)}\underbrace{\left(\frac{|0\rangle + (-1)^{f(1)-f(0)}|1\rangle}{\sqrt{2}}\right)}_{I}\otimes\underbrace{|-\rangle}_{R}$$

overall phase factor (not important)

$$= \begin{cases} (-1)^{f(0)} |+\rangle & \text{if } f(0)=f(1) \\ (-1)^{f(0)} |-\rangle & \text{if } f(0)\neq f(1) \end{cases}$$



Deutsch Algorithm

1. Prepare I in $|+\rangle$, R in $|-\rangle$ state
2. Run the chip once (evaluate function f once)
3. Measure I in $\{|+\rangle, |-\rangle\}$ basis

Conclusion: if result $= |+\rangle$ then f is const

if result $= -$ then f is non-const

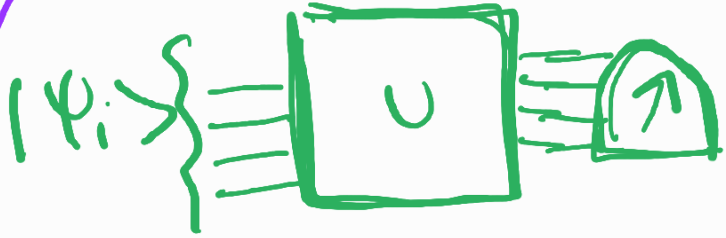
In a way, Deutsch algorithm computes $f(0) - f(1)$ without computing either $f(0)$ or $f(1)$. ☆☆

☆ Stages of quantum computation:

① initial preperation

② unitary transformation

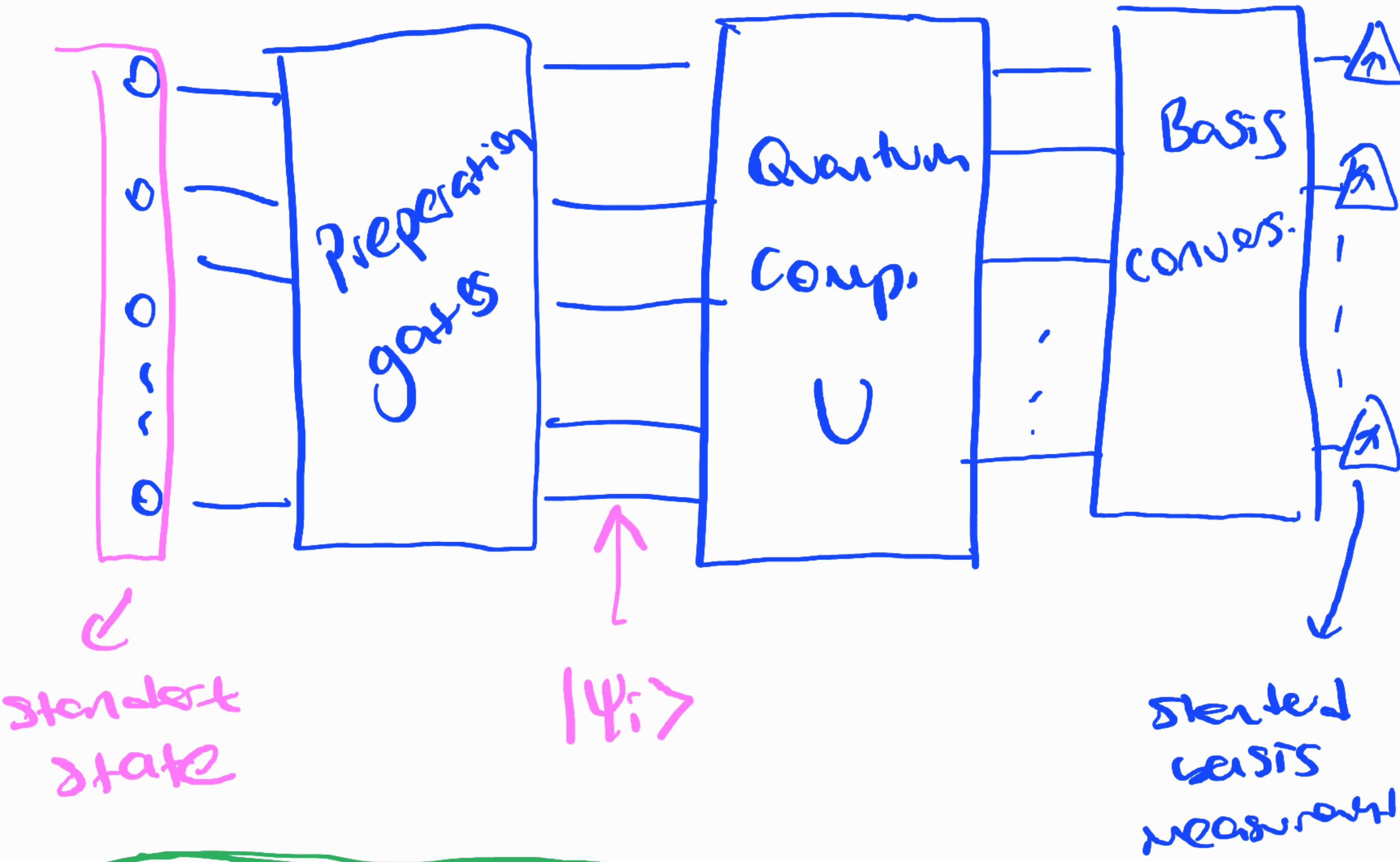
③ Measurement



obtain this by combining several elementary gates

prepare the qubits in some standard state and use some elementary gates to transform it to $|\psi_i\rangle$

→ Doing some elementary gates, transform your measurement to a measurement in standard basis.



Simple gates

① NOT gate or X gate

$$|x\rangle -\boxed{X}- |1 \oplus x\rangle$$

$$|0\rangle \longrightarrow |1\rangle$$
$$|1\rangle \longrightarrow |0\rangle$$

Unitary transform: is $\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = X$

② Hadamard gate:

$$|0\rangle - \boxed{H} - |+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$
$$|1\rangle - \boxed{H} - |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

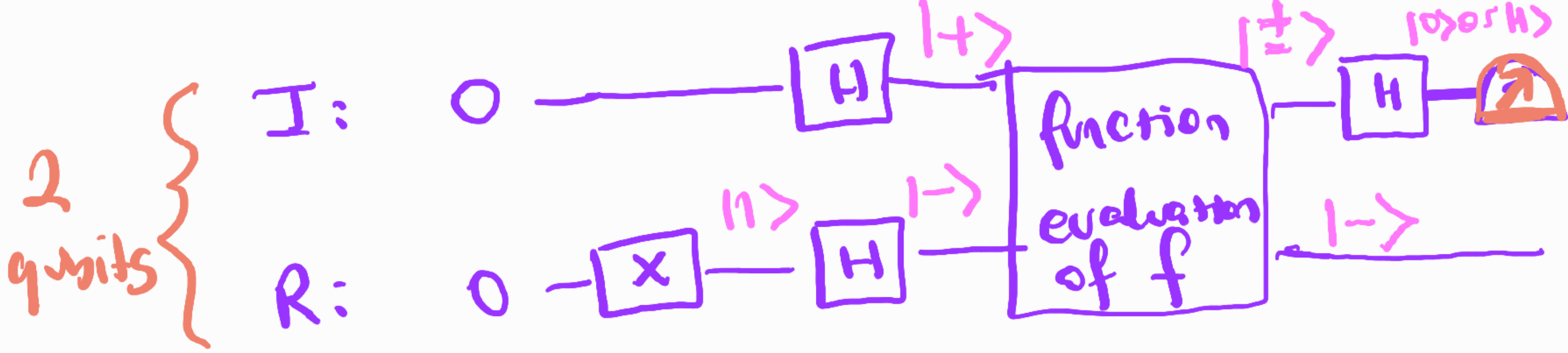
$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad H|x\rangle = \frac{1}{\sqrt{2}} \sum_{y=0}^{1} (-)^{xy} |y\rangle$$

★ Check: $H^2 = I$

$$H|+\rangle = |0\rangle$$
$$H|-\rangle = |1\rangle$$

Deutsch algorithm in quantum circuit notation

$\longrightarrow$ time



if result = 0 then f is constant

if result = 1 then f is non-constant

Deutsch - Josza Problem

f is a bit valued function of n-bits

$$f : \{ \underbrace{00\ldots0,\ 00\ldots1,\ \ldots,\ 11\ldots1}_{x = 0, 1, \ldots, 2^n - 1} \} \mapsto \{0,1\}$$

$$f(x) = 0 \text{ or } 1$$

★ Def: We say f is constant if (f(x) = 0 for all x) or (f(x) = 1 for all x)

★ Def: We say that f is balanced if the number of x values for which

$f(x)=0$ is equal to # of x' values for which $f(x')=1$

$$|\{x : f(x)=0\}| = |\{x' : f(x')=1\}|$$

$n=1$

x	f
0	1
1	0

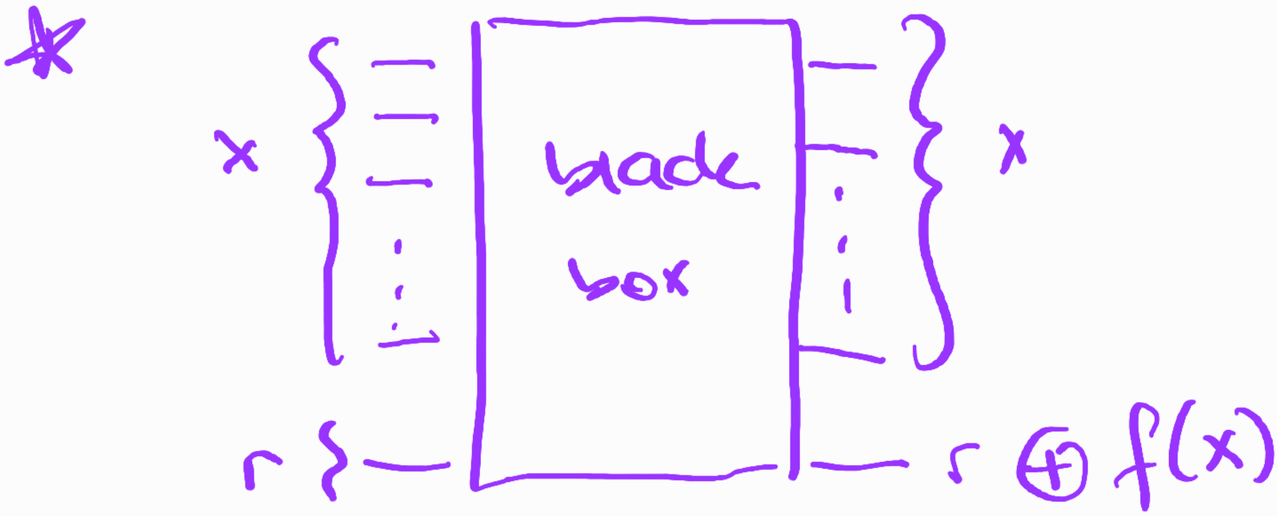
NOT

x	f'
0	0
1	1

$n=2$

x	g
00	0
01	1
10	1
11	0

balanced functions



A chip computes an unknown function f(x). You know that f is either constant or balanced, but you don't know which.

Deutsch-Josza problem:

By evaluating the function only once (by using the chip only once) can you decide if f is constant or balanced?

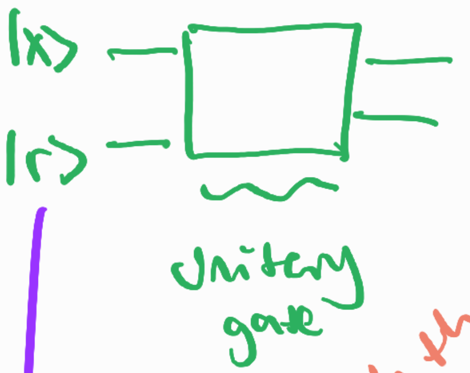
* Classically: You need to evalute f at least twice (best case), or sometimes $\frac{1}{2} 2^n + 1$ times → worst case

$n=3$

x	f(x)
000	0
001	0
010	0
011	0
100	?

→ 1 → balanced ?
→ 0 → constant ?

Quantum Computer requires only one evaluation of $f(x)$



$|xr\rangle$ = input state

$U|\underbrace{x}_{n\text{ bits}}r\rangle = |x, r \oplus f(x)\rangle$ output state

watch the notation!

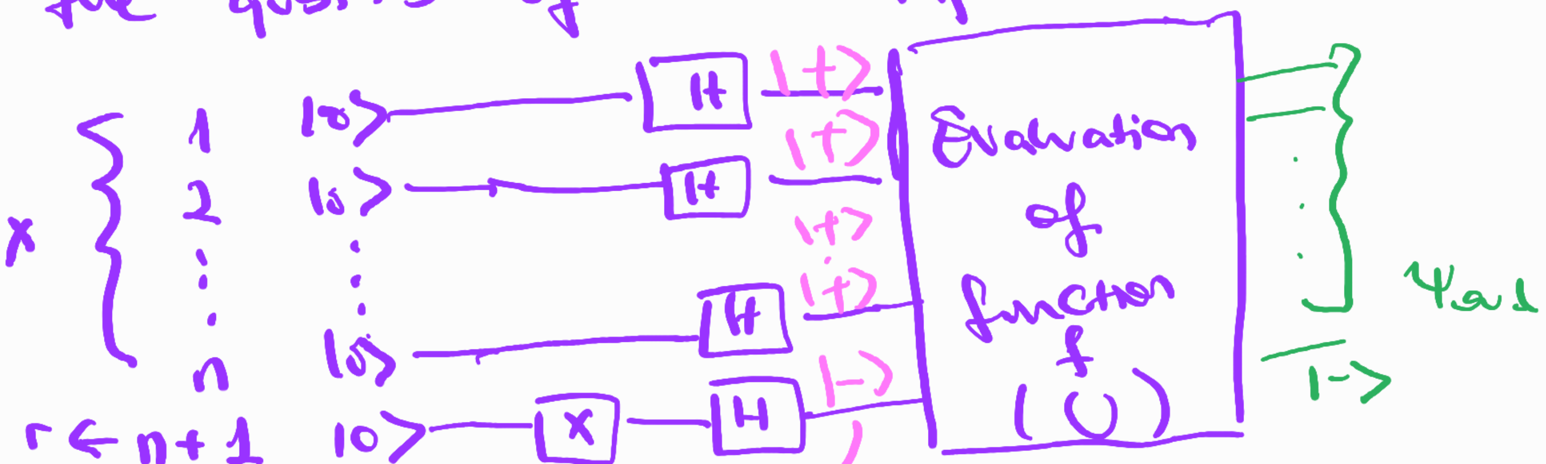
$$|\underbrace{x_1 x_2 \dots x_n r}_{n+1 \text{ bits}}\rangle = |x_1\rangle \otimes \dots \otimes |x_n\rangle \otimes |r\rangle$$

$$|-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

$$U|x,-\rangle = (-)^{f(x)} |x,-\rangle$$

If the register R is in the $|-\rangle$ state, then function evaluation changes the phase of terms.

* Apply a Hadamard gate to each of the qubits of the input I.



$$|\Psi_{in}\rangle = |++ + \cdots + -\rangle$$

$$|\Psi_{in}\rangle = |+ + \cdots + -\rangle = \left(\bigotimes \frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)_I \otimes |-\rangle_R$$

$$= \frac{1}{2^{n/2}}\left(|0\cdots0\rangle + |0\cdots1\rangle + \cdots\cdots + |1\cdots1\rangle\right) \otimes |-\rangle_R$$

$$= \underbrace{\frac{1}{2^{n/2}}\left(\sum_x |x\rangle_I\right)}_{} \otimes |-\rangle_R$$

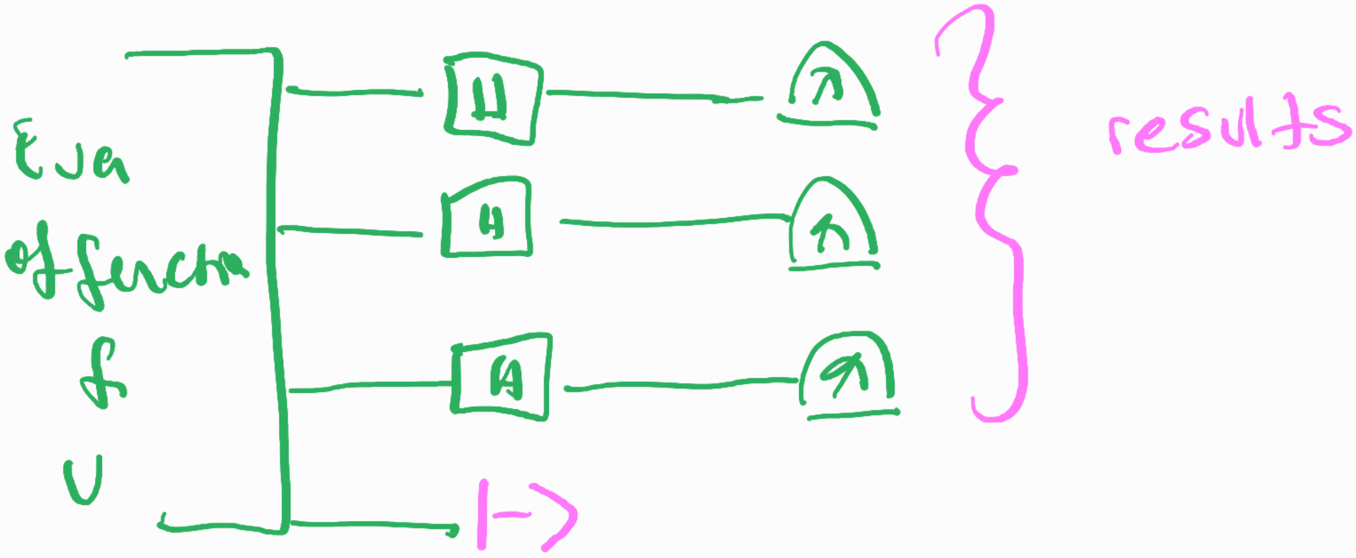
The input memory holds all numbers in a superposition state.

$$|\Psi_{out}\rangle = U\,|\Psi_{in}\rangle$$

$$= \frac{1}{2^{n/2}} U \sum_x |x\rangle_I \otimes |-\rangle_R$$

$$= \frac{1}{2^{n/2}} \sum_x (-)^{f(x)} |x\rangle \otimes |-\rangle_R$$

$$= \left(\frac{1}{2^{n/2}} \sum_x (-)^{f(x)} |x\rangle\right) \otimes |-\rangle_R$$



Register qubit (no need to measure)

$$(H^{\otimes n}) \otimes \mathbb{1}_R \, |\Psi_{out}\rangle = ?$$

Hadamard gate on a single qubit

$|y\rangle$ $y=0$ or $y=1$

$H|y\rangle = ?$ $\quad H|0\rangle = |+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$

$H|1\rangle = |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$

$$\boxed{H|y\rangle = \frac{1}{\sqrt{2}} \sum_z (-)^{yz} |z\rangle} \; ?$$

$$(H^{\otimes n}) \, |y_1 y_2 \cdots y_n\rangle = (H|y_1\rangle) \otimes (H|y_2\rangle) \otimes \cdots$$

$$= \left(\frac{1}{\sqrt{2}} \sum_{z_1} (-)^{y_1 z_1} |z_1\rangle\right) \otimes \left(\frac{1}{\sqrt{2}} \sum_{z_2} (-)^{y_2 z_2} |z_2\rangle\right) \otimes \cdots$$

$$= \frac{1}{2^{n/2}} \sum_{z} (-)^{y_1 z_1 + y_2 z_2 + \cdots y_n z_n} |z_1 \cdots z_n\rangle$$

Modulo 2 inner product of two n-bit sequences $y = y_1 \cdots\cdots y_n$

$z = z_1 \cdots\cdots z_n$

$$\boxed{\langle y|z\rangle = y_1 z_1 + \cdots + y_n z_n \pmod 2}$$

$$\boxed{H^{\otimes n} |y\rangle = \frac{1}{2^{n/2}} \sum_{z} (-)^{\langle y|z\rangle} |z\rangle}$$

<u>Ex</u> $\quad H^{\otimes n} |00\cdots 0\rangle = \frac{1}{2^{n/2}} \sum_{z} (-)^{\langle 0|z\rangle} |z\rangle$

$$= \frac{1}{2^{n/2}} \sum_{z} |z\rangle$$

$$\star \; |\psi_{out}\rangle = \frac{1}{2^{n/2}} \left(\sum_x (-)^{f(x)} |x\rangle \right) \otimes |-\rangle_R$$

$$(H^{\otimes n} \otimes 1_R) \, |\psi_{out}\rangle = \frac{1}{2^{n/2}} \left[\sum_x (-)^{f(x)} H^{\otimes n} |x\rangle \right) \otimes |-\rangle_R$$

$$= \frac{1}{2^n} \left(\sum_{x,z} (-)^{f(x) + \langle x|z\rangle} |z\rangle \right) \otimes |-\rangle_R$$

$$= \left[\sum_z |z\rangle \underbrace{\left(\frac{\sum_x (-)^{f(x)+\langle x|z\rangle}}{2^n} \right)} \right] \otimes |-\rangle_R$$

a_z = amplitude of z

$P_z = |a_z|^2$ = probability of obtaining the result z in the final measurement

$z = (000\ldots0)$

$P_{00\ldots0} = \,?$

$$a_{00\ldots0} = \frac{1}{2^n} \sum (-)^{f(x)}$$

$$a_{0\ldots0} = \begin{cases} \pm 1 & \text{if } f(x) = \text{const} \\ 0 & \text{if } f \text{ is balanced} \end{cases}$$

$$P_{0\ldots0} = \begin{cases} 1 & \text{if } f \text{ is constant} \\ 0 & \text{if } f \text{ is balanced} \end{cases}$$