QGSS'22

Day 2 Introduction to Linear Algebra, Prerequisite Math, Circuit Composition

- Bloch Sphere
- Measurement → how to measure a photon?

$\alpha|0\rangle + \beta|1\rangle$

$$\left.\begin{aligned}\text{Prob. of } |0\rangle &= |\alpha|^2 \\ \text{prob. of } |1\rangle &= |\beta|^2\end{aligned}\right\} \quad |\alpha|^2 + |\beta|^2 = 1$$

$\alpha|0\rangle + \beta|1\rangle$

$$= \alpha\left(\frac{1}{\sqrt{2}}\left(|+\rangle + |-\rangle\right)\right) + \beta\left(\frac{1}{\sqrt{2}}\left(|+\rangle - |-\rangle\right)\right)$$

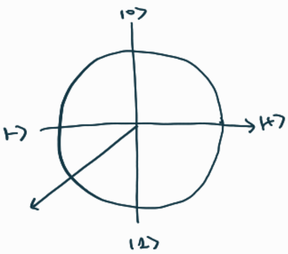
$$= \frac{1}{\sqrt{2}}(\alpha+\beta)|+\rangle + \frac{1}{\sqrt{2}}(\alpha-\beta)|-\rangle$$

prob of $|+\rangle = \frac{1}{2}|\alpha+\beta|^2$, prob of $|-\rangle = \frac{1}{2}|\alpha-\beta|^2$

$$\langle +|\left(\alpha|0\rangle + \beta|1\rangle\right) = \alpha\langle +|0\rangle + \beta\langle +|1\rangle$$

$$= \alpha\cdot\frac{1}{\sqrt{2}} + \beta\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}(\alpha+\beta)$$

- Quantum Gates



$|\psi\rangle \longrightarrow |\psi'\rangle$

$\begin{pmatrix} a \\ b \end{pmatrix} \longrightarrow \begin{pmatrix} a' \\ b' \end{pmatrix}$

→ what do these vectors represent in real world?

→ what's qubit in reality?

$$M = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix} \longrightarrow \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} m_1 a + m_2 b \\ m_3 a + m_4 b \end{pmatrix}$$

$M\begin{pmatrix} a \\ b \end{pmatrix} \longrightarrow \begin{pmatrix} a' \\ b' \end{pmatrix}$ "Linear Operators"

$$M|\psi\rangle = |\psi'\rangle$$

$$M\left(\frac{|\psi\rangle + |\phi\rangle}{\sqrt{2}}\right) = \frac{M|\psi\rangle + M|\phi\rangle}{\sqrt{2}}$$

"Unitarity"

$$U^\dagger U = 1 \qquad U(\theta, \phi, \lambda) = \begin{pmatrix} \cos\frac{\theta}{2} & e^{-i\lambda}\sin\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} & e^{i(\phi-\lambda)}\cos\frac{\theta}{2} \end{pmatrix}$$

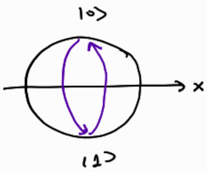
$$\langle\psi'|\psi'\rangle = \langle\psi|U^\dagger U|\psi\rangle = \langle\psi|\mathbb{1}|\psi\rangle = \langle\psi|\psi\rangle = 1$$

Single Qubit Gates

How these gates is constructed in real?

NOT

X Gate



180° about x

$\hat{X}|0\rangle = |1\rangle$

$\hat{X}|1\rangle = |0\rangle$

$\hat{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\hat{X}|+\rangle = \frac{1}{\sqrt{2}} X(|0\rangle + |1\rangle) = \frac{1}{\sqrt{2}}(X|0\rangle + X|1\rangle) = \frac{1}{\sqrt{2}}(|1\rangle + |0\rangle)$

$|+\rangle$: eigenvector

$+1$: eigenvalue

$\hat{X}|-\rangle = \frac{1}{\sqrt{2}} X(|0\rangle - |1\rangle) = -\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = -|-\rangle$

$|-\rangle$: eigen vector

-1 : eigen value

global phase

↳ does not affect measurement

Don't confuse

!

$\hat{X}\left(\underbrace{\frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)}_{|0\rangle}\right) = \underbrace{\frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)}_{|1\rangle}$

relative phase

Y Gate : 180° y $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ ± 1

Z Gate : 180° z $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ ± 1

Unitary Matrix eigenvector : $e^{i\phi}$

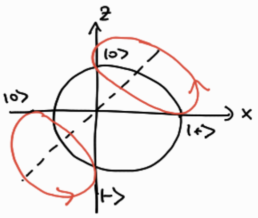
$U|\psi\rangle = a|\psi\rangle$

$(U|\psi\rangle)^{+} = \langle\psi|U^{+} = \langle\psi|a^{*}$

$\langle\psi|U^{+}U|\psi\rangle = aa^{*}\langle\psi|\psi\rangle = |a|^{2} = 1$

$a = e^{i\phi}$

Hadamard Gate



$\hat{H}|0\rangle = |+\rangle$

$\hat{H}|+\rangle = |0\rangle$

$\hat{H}|-\rangle = |1\rangle$

$\hat{H}|1\rangle = |-\rangle$

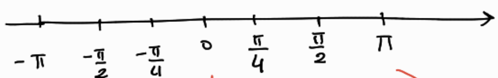
$\hat{H} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$H = H^{-1}$

$HH|\psi\rangle = 1|\psi\rangle = |\psi\rangle$

Phase Gate

$\begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix}$



$-\pi$: $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = Z$

$-\frac{\pi}{2}$: $\begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} = S^{+}$

$-\frac{\pi}{4}$: $\begin{pmatrix} 1 & 0 \\ 0 & e^{-i\pi/4} \end{pmatrix} = T^{+}$

0: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$

$\frac{\pi}{4}$: $\begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} = T$

$\frac{\pi}{2}$: $\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} = S$

π: $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = Z$

Bra-ket for gates

Outer product: $|a\rangle\langle b| = |a\rangle\langle b|$

$$|0\rangle\langle 0| = \begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}$$

$$|0\rangle\langle 1| = \begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix} = \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}$$

$$|1\rangle\langle 0| = \begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} = \begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}$$

$$|1\rangle\langle 1| = \begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix} = \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}$$

$$\alpha|0\rangle\langle 0| + \beta|0\rangle\langle 1| + \gamma|1\rangle\langle 0| + \delta|1\rangle\langle 1| = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$$

$$\alpha'|+\rangle\langle +| + \beta'|+\rangle\langle -| + \gamma'|-\rangle\langle +| + \delta'|-\rangle\langle -| \longrightarrow \begin{pmatrix}\alpha' & \beta'\\ \gamma' & \delta'\end{pmatrix}$$

Multiple Qubits

$$|a\rangle = \begin{pmatrix}a_1\\a_2\end{pmatrix} \qquad |b\rangle = \begin{pmatrix}b_1\\b_2\end{pmatrix}$$

Tensor product

$$|a\rangle \otimes |b\rangle = \begin{pmatrix}a_1\\a_2\end{pmatrix} \otimes \begin{pmatrix}b_1\\b_2\end{pmatrix}$$

$$= \begin{pmatrix} a_1\begin{pmatrix}b_1\\b_2\end{pmatrix} \\ a_2\begin{pmatrix}b_1\\b_2\end{pmatrix}\end{pmatrix} = \begin{pmatrix}a_1b_1\\a_1b_2\\a_2b_1\\a_2b_2\end{pmatrix}$$

$$(A \otimes B)(|a\rangle \otimes |b\rangle) = A|a\rangle \otimes B|b\rangle$$

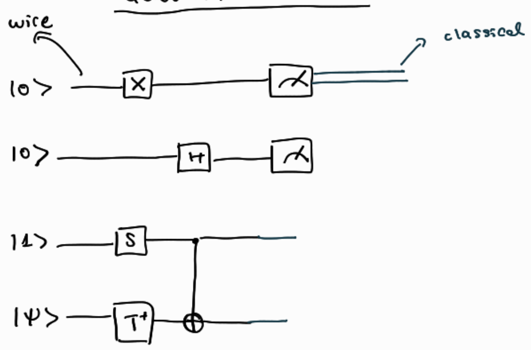
$$\begin{pmatrix}A_1 & A_2\\A_3 & A_4\end{pmatrix}\begin{pmatrix}B_1 & B_2\\B_3 & B_4\end{pmatrix} = \begin{pmatrix} A_1\begin{pmatrix}B_1 & B_2\\B_3 & B_4\end{pmatrix} & A_2\begin{pmatrix}B_1 & B_2\\B_3 & B_4\end{pmatrix} \\ A_3\begin{pmatrix}B_1 & B_2\\B_3 & B_4\end{pmatrix} & A_4\begin{pmatrix}B_1 & B_2\\B_3 & B_4\end{pmatrix}\end{pmatrix}$$

$$(\hat{X} \otimes \hat{Z})(|01\rangle) = -|11\rangle$$

$$(\hat{X} \otimes \hat{Z})(|0+\rangle) = |1-\rangle$$

2-qubit gates

Quantum Circuits



where is CX used?

C-NOT gate (CX)

Controlled - NOT

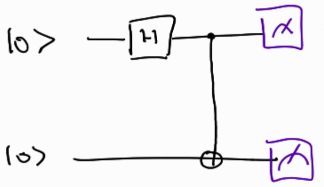
test → target

$CX\,|00\rangle = |00\rangle$

$CX\,|01\rangle = |01\rangle$

$CX\,|10\rangle = |11\rangle$

$CX\,|11\rangle = |10\rangle$



if it is $|0\rangle$

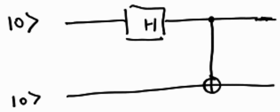
meaning the second qubit is $|0\rangle$ as well

$$|00\rangle \xrightarrow{H} \underbrace{(|0\rangle + |1\rangle)\,|0\rangle}_{|00\rangle + |10\rangle} \xrightarrow{CNOT} |00\rangle + |11\rangle$$

maximally entangled

Bell States

How to obtain Bell States in lab?



$|0\rangle + |1\rangle \rightarrow |00\rangle + |11\rangle$

$$|\Phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

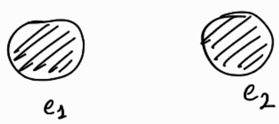
$$|\Phi^-\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}$$

$$\langle \Phi_i | \Phi_j \rangle = 0$$

$$|\Psi^+\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}$$

$$|\Psi^-\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

Pure and Mixed states



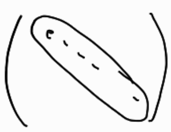
state vector

$$|\psi\rangle = |00\rangle + |11\rangle$$

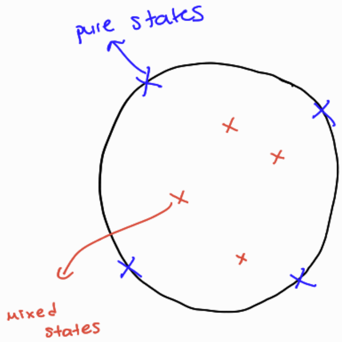
$$\varrho = |\psi\rangle\langle\psi| = (|00\rangle + |11\rangle)(\langle 00| + \langle 11|) = |00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11| = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

density matrix

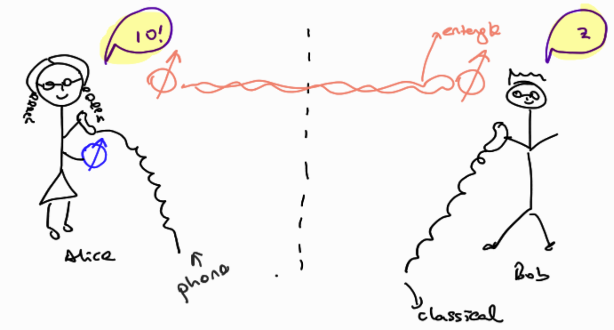
$$\varrho_1 = tr_2\, \varrho_{12}$$



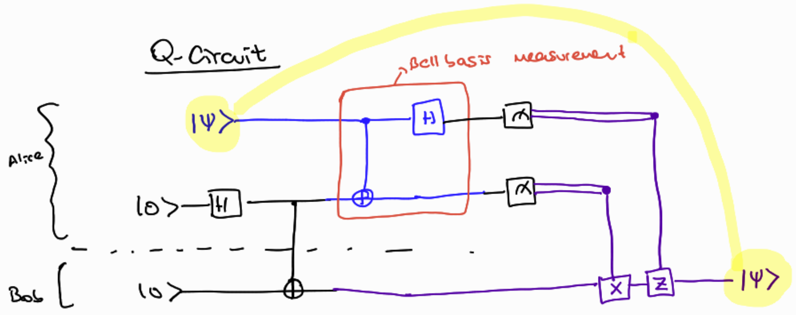
$$\sum_j \langle \psi_j | \varrho_{12} | \psi_j \rangle$$



Quantum teleportation



How to entangle two qubits far away?



$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

$|\psi\rangle|00\rangle \rightarrow (\alpha|0\rangle + \beta|1\rangle)(|00\rangle + |11\rangle) = \alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle$

CNOT $\rightarrow \alpha|000\rangle + \alpha|011\rangle + \beta|110\rangle + \beta|101\rangle$

H $\rightarrow \alpha|000\rangle + \alpha|100\rangle + \alpha|011\rangle + \alpha|111\rangle + \overbrace{\beta|010\rangle - \beta|110\rangle} + \overbrace{\beta|001\rangle - \beta|101\rangle}$

$|00\rangle\underbrace{(\alpha|0\rangle + \beta|1\rangle)}_{|\psi\rangle} + |10\rangle\underbrace{(\alpha|0\rangle - \beta|1\rangle)}_{Z|\psi\rangle} + |01\rangle\underbrace{(\alpha|1\rangle + \beta|0\rangle)}_{X|\psi\rangle} + |11\rangle\underbrace{(\alpha|1\rangle - \beta|0\rangle)}_{XZ|\psi\rangle}$

Questions for Day 2

→ What do state vectors represent in reality? Like polarization etc

→ What's qubit?

→ Gates

→ Bell States

→ Entanglement