CHAPTER TWO

Time Value of Money

Biggest Lottery Jackpots in U.S. History – $1.58 Billion Powerball The world's biggest lottery jackpot of $1.58 billion ($1,586,400,000 to be exact) on January 13, 2016, was a three-way split of $528 million each. The odds of winning the $1.58 billion jackpot were 1 in 292.2 million, which are probabilistically impossible odds to hope for, but there are always a few lucky people to claim such a record-shattering jackpot. The three winners were from California, Florida, and Tennessee, and they opted to take the $327.8 million lump sum ($983,505,233 split three ways) rather than receiving the $528 million sum in 30 annual installments.

How much do they take home?[1] First, they have to pay federal and state income taxes. For state income taxes, none will have to pay income taxes to their states. Some states don't have an

[1] *Source:* Mike Tarson and Christine Romans, "How Much Will the Powerball Jackpot Winners Get?" @ CNNMoney, January 14, 2016. (http://money.cnn.com/2016/01/14/news/powerball-winnings/).

income tax like Florida and Tennessee, yet other states have longtime tax exemptions for lottery winners like California.

Second, after paying 39.6% in federal income taxes on their prizes, each will take home about $197 million. So what should they do with that money? A couple of easy options to think of are as follows:

- If they are looking for the safest investment, they might purchase a 30-year U.S. government bond. With a 2.85% yield, each winner would get about $5.64 million a year to live on for the next 30 years, without touching the principal amount.
- If they are willing to take some risk, they could invest their money in the stock market. For example, with that money, they could have bought 1.9 million shares of Apple (AAPL) stock around $104 a share. A year later they could have increased their wealth by 34% as Apple shares went up by 34%.

If you were the winner of the aforementioned jackpot, you might well wonder why the value of the single-sum payment—$197 million paid immediately—is so much lower than the total value of the annuity payments—$17.6 (=$528 million/30) million received in 30 installments over 29 years (the first installment is paid immediately). Isn't receiving the annuity of $17.6 million overall a lot better than receiving just $197 million now? The answer to your question involves the principles we will discuss in this chapter, namely, the operation of interest and the time value of money.

The question we just posed provides a good starting point for this chapter. If it is better to receive a dollar today than it is to receive a dollar in 10 years, how do we quantify the difference? Our lottery example is complex. Instead of a choice between two single payments, the lottery winners were faced with a decision between a single payment now and a series of future payments. First, most people familiar with investments would tell you that receiving $197 million today is likely to prove a better deal than taking $17.6 million a year for 30 annual installments with the first payment immediately. In fact, based on the principles you will learn in this chapter, the real present value of the 29-year payment series— the value that you could receive today in the financial marketplace for the promise of $17.6 million a year for the next 29 years—can be shown to be considerably less than $197 million. And that is even before we consider the effects of inflation! The reason for this surprising result is the **time value of money**; the earlier a sum of money is received, the more it is worth because over time money can earn more money via interest.

In engineering economic analysis, the principles discussed in this chapter are regarded as the underpinnings of nearly all project investment analysis. It is imperative to understand these principles because we always need to account for the effect of interest operating on sums of cash over time. Fortunately, we have interest formulas that allow us to place different cash flows received at different times in the same time frame to make comparisons possible. As will become apparent, almost our entire study of engineering economic analysis is built on the principles introduced in this chapter.

2.1 Interest: The Cost of Money

Most of us have a general appreciation of the concept of interest. We know that money left in a savings account earns interest so that the balance over time is higher than the sum of the deposits. We know that borrowing to buy a car means repaying an amount over time, including the interest, and thus the amount paid is more than the amount borrowed. However, what may be unfamiliar to us is that, in the financial world, money itself is a commodity, and like other goods that are bought and sold, money costs money.

The cost of money is established and measured by an **interest rate**, a percentage that is periodically applied and added to an amount (or to varying amounts) of money over a specified length of time. When money is borrowed, the interest paid is the charge to the borrower for the use of the lender's property. When money is loaned or invested, the interest earned is the lender's gain for providing a good to another person. **Interest**, then, may be defined as the cost of having money available for use. In this section, we examine how interest operates in a free-market economy and establish a basis for understanding the more complex interest relationships that are presented later in the chapter.

2.1.1 The Time Value of Money

The time value of money seems like a sophisticated concept, yet it is one that you encounter every day. Should you buy something today or buy it later? Here is a simple example of how your buying behavior can have varying results: Pretend you have $100 and you want to buy a $100 refrigerator for your dorm room. (Assume that you are currently sharing a large refrigerator with your roommates in a common area.)

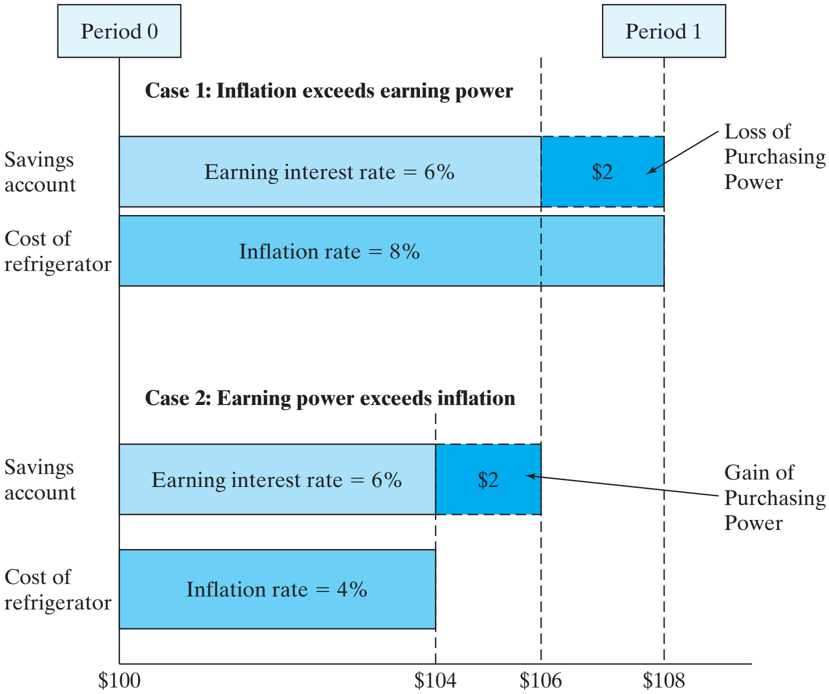


Figure 2.1 Gains achieved or losses incurred by delaying consumption.

- If you buy it now, you end up broke. But if you invest your money at 6% annual interest, then in a year you can still buy the refrigerator, and you will have $6 left over with no change in the price of the refrigerator. Clearly, you need to ask yourself whether the inconvenience of not having the refrigerator in your own room for a year can be compensated by the financial gain in the amount of $6.
- If the price of the refrigerator increases at an annual rate of 8% due to inflation, then you will not have enough money (you will be $2 short) to buy the refrigerator a year from now (Case 1 in Figure 2.1). In that case, you probably are better off buying the refrigerator now. If the inflation rate is running at only 4%, then you will have $2 left over if you buy the refrigerator a year from now (Case 2 in Figure 2.1).

Clearly, the rate at which you earn interest should be higher than the inflation rate in order to make any economic sense of the delayed purchase. In other words, in an inflationary economy, your purchasing power will continue to decrease as you further delay the purchase of the refrigerator. In order to make up for this future loss in purchasing power, the rate at which you earn interest must be sufficiently higher than the anticipated inflation rate. After all, time, like money, is a finite resource. There are only 24 hours in a day, so time has to be budgeted, too. What this example illustrates is that we must connect *earning power* and *purchasing power* to the concept of time.

The way interest operates reflects the fact that money has a time value. This is why amounts of interest depend on lengths of time; interest rates, for example, are typically given in terms of a percentage per year. We may define the principle of the time value of money as follows: The economic value of a sum depends on when the sum is received. Because money has both **earning power** and **purchasing power** over time (i.e., it can be put to work, earning more money for its owner), a dollar received today has a higher value than a dollar received at some future time. When we deal with large amounts of money, long periods of time, and high interest rates, a change in the value of a sum of money over time becomes extremely significant. For example, at a current annual interest rate of 10%, $1 million will earn $100,000 in interest in a year; thus, to wait a year to receive $1 million clearly involves a significant sacrifice. When deciding among alternative proposals, we must take into account the operation of interest and the time value of money in order to make valid comparisons of different amounts at various times.

When financial institutions quote lending or borrowing interest rates in the marketplace, those interest rates reflect the desired earning rate as well as any protection from loss in the future purchasing power of money because of inflation. Interest rates, adjusted for inflation, rise and fall to balance the amount saved with the amount borrowed, which affects the allocation of scarce resources between present and future uses.

Unless stated otherwise, we will assume that the interest rates used in this book reflect the **market interest rate**, which considers the earning power of money as well as the effect of inflation perceived in the marketplace. We will also assume that all cash flow transactions are given in terms of **actual dollars**, for which the effect of inflation, if any, is reflected in the amount.

2.1.2 Elements of Transactions Involving Interest

Many types of transactions involve interest (e.g., borrowing money, investing money, or purchasing machinery on credit), and certain elements are common to all of these types of transactions. These elements are:

1. The initial amount of money invested or borrowed in a transaction is called the **principal** (P).
2. The **interest rate** (i) measures the cost or price of money and is expressed as a percentage per period of time.
3. A period of time called the **interest period** (n) determines how frequently interest is calculated. (Note that, even though the length of time of an interest period can vary, interest rates are frequently quoted in terms of an annual percentage rate. We will discuss this potentially confusing aspect of interest in Chapter 3.)
4. A specified length of time marks the duration of the transaction and thereby establishes a certain **number of interest periods** (N).
5. **A plan for receipts or disbursements** (A_n) yields a particular cash flow pattern over a specified length of time. (For example, we might have a series of equal monthly payments that repay a loan.)
6. A **future amount of money** (F) results from the cumulative effects of the interest rate over a number of interest periods.

Example of an Interest Transaction

As an example of how the elements we have just defined are used in a particular situation, let us suppose that you apply for an education loan in the amount of $30,000 from a bank at a 9% annual interest rate. In addition, you pay a $300 loan origination fee[2] when the

[2] The loan origination fee covers the administrative costs of processing the loan. It is often expressed in points. One point is 1% of the loan amount. In our example, the $30,000 loan with a loan origination fee of one point would mean the borrower pays a $300 fee. This is equivalent to financing $29,700, but the payments are based on a $30,000 loan. Both payment plans are based on a rate of 9% interest.

TABLE 2.1 Repayment Plans Offered by the Lender

End of Year	Receipts	Payments	
		Plan 1	Plan 2
Year 0	$30,000	$300.00	$300.00
Year 1		$7,712.77	0
Year 2		$7,712.77	0
Year 3		$7,712.77	0
Year 4		$7,712.77	0
Year 5		$7,712.77	$46,158.72

loan commences. The bank offers two repayment plans, one with equal payments made at the end of every year for the next five years (installment plan) and the other with a single payment made after the loan period of five years (deferment plan). These payment plans are summarized in Table 2.1.

- In Plan 1, the principal amount, P, is $30,000 and the interest rate, i, is 9%. The interest period, n, is one year, and the duration of the transaction is five years, which means that there are five interest periods $(N = 5)$. It bears repeating that while one year is a common interest period, interest is frequently calculated at other intervals as well— monthly, quarterly, or semiannually, for instance. For this reason, we used the term **period** rather than **year** when we defined the preceding list of variables. The receipts and disbursements planned over the duration of this transaction yield a cash flow pattern of five equal payments, A, of $7,712.77 each, paid at year-end during years 1 through 5.
- Plan 2 has most of the elements of Plan 1 except that instead of five equal repayments, we have a grace period followed by a single future repayment (lump sum), F, of $46,158.72.

Cash Flow Diagrams

Problems involving the time value of money can be conveniently represented in graphic form with a **cash flow diagram** (Figure 2.2). Cash flow diagrams represent time by a horizontal line marked off with the number of interest periods specified. Arrows represent the cash flows over time at relevant periods. Upward arrows represent positive flows (receipts), and downward arrows represent negative flows (expenditures). Note, too, that the arrows actually represent **net cash flows**; two or more receipts or disbursements made at the same time are summed and shown as a net single arrow. For example, $30,000 received during the same period as a $300 payment is being made would be recorded as an upward arrow of $29,700. The lengths of the arrows can also suggest the relative values of particular cash flows.

Cash flow diagrams function in a manner similar to free-body diagrams or circuit diagrams, which most engineers frequently use. Cash flow diagrams give a convenient summary of all the important elements of a problem and serve as a reference point for determining whether the elements of a problem have been converted into their appropriate parameters. This book frequently employs this graphic tool, and you are strongly encouraged to develop the habit of using well-labeled cash flow diagrams as a means

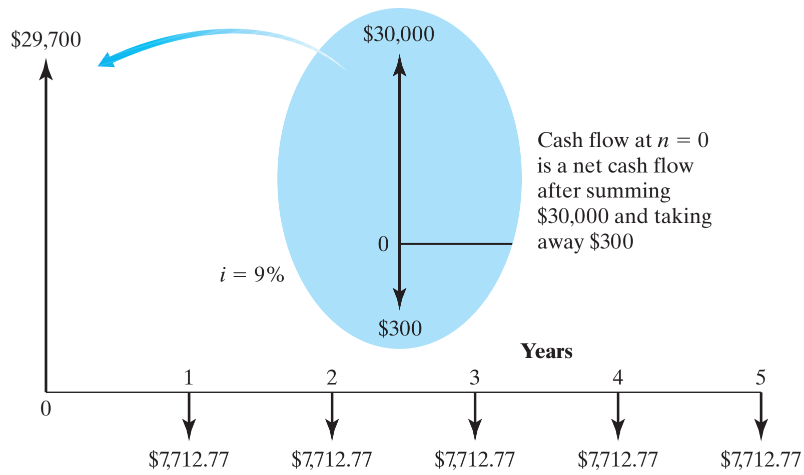


Figure 2.2 A cash flow diagram for Plan 1 of the loan repayment example.

to identify and summarize pertinent information in a cash flow problem. Similarly, a table such as Table 2.1 can help you organize information in another summary format.

End-of-Period Convention

In practice, cash flows can occur at the *beginning* or in the *middle* of an interest period or at practically any point in time. One of the simplifying assumptions we make in engineering economic analysis is the **end-of-period convention**, which is the practice of placing all cash flow transactions at the *end* of an interest period. This assumption relieves us of the responsibility of dealing with the effects of interest within an interest period, which would greatly complicate our calculations. Like many of the simplifying assumptions and estimates we make in modeling engineering economic problems, the end-of-period convention inevitably leads to some discrepancies between our model and real-world results.

Suppose, for example, that $100,000 is deposited during the first month of the year in an account with an interest period of one year and an interest rate of 10% per year. In such a case, if the deposit is withdrawn one month before the end of the year, the investor would experience a loss of $10,000— all of it interest! This results because, under the end-of-period convention, the $100,000 deposit made during the interest period is viewed as if it were made at the end of the year, as opposed to 11 months earlier. This example gives you a sense of why financial institutions choose interest periods that are less than one year, even though they usually quote their rate in terms of *annual percentage*. Armed with an understanding of the basic elements involved in interest problems, we can now begin to look at the details of calculating interest.

2.1.3 Methods of Calculating Interest

Money can be loaned and repaid in many ways, and similarly, money can earn interest in many different ways. Usually, however, at the end of each interest period, the interest earned on the principal amount is calculated according to a specified interest rate.

There are two computational schemes for calculating this earned interest yield: **simple interest** and **compound interest**.

Simple Interest

The first scheme considers interest earned on only the principal amount during each interest period. In other words, the interest earned during each interest period does not earn additional interest in the remaining periods, *even if you do not withdraw the earned interest*.

In general, for a deposit of P dollars at a simple interest rate of i for N periods, the total earned interest I would be

$$I = (iP)N. \tag{2.1}$$

The total amount available at the end of N periods, F, thus would be

$$F = P + I = P(1 + iN). \tag{2.2}$$

Simple interest is commonly used with add-on loans or bonds.

Compound Interest

Under a compound interest scheme, the interest earned in each period is calculated based on the total amount at the end of the previous period. This total amount includes the original principal plus the accumulated interest that has been left in the account. In this case, you are, in effect, increasing the deposit amount by the amount of interest earned. In general, if you deposited (invested) P dollars at an interest rate i, you would have $P + iP = P(1 + i)$ dollars at the end of one interest period. If the entire amount (principal and interest) were reinvested at the same rate i for next period, you would have at the end of the second period

$$\begin{aligned} P(1 + i) + i[P(1 + i)] &= P(1 + i)(1 + i) \\ &= P(1 + i)^2. \end{aligned}$$

Continuing, we see that the balance after period three is

$$P(1 + i)^2 + i[P(1 + i)^2] = P(1 + i)^3.$$

This interest-earning process repeats, and after N periods, the total accumulated value (balance) F will grow to

$$F = P(1 + i)^N. \tag{2.3}$$

Engineering economic analysis uses the compound interest scheme exclusively, as it is most frequently practiced in the real world.

EXAMPLE 2.1 Simple versus Compound Interest

Suppose you deposit $1,000 in a bank savings account that pays interest at a rate of 8% per year. Assume that you do not withdraw the interest earned at the end of each period (year) but instead let it accumulate. (1) How much would you have at the end of year 3 with simple interest? (2) How much would you have at the end of year 3 with compound interest?

DISSECTING THE PROBLEM

Given: $P = \$1{,}000$, $N = 3$ years, and $i = 8\%$ per year.
Find: F.

METHODOLOGY

Use Eqs. (2.2) and (2.3) to calculate the total amount accumulated under each computational scheme.

SOLUTION

(a) Simple interest: Using Eq. (2.2) we calculate F as

$$F = \$1{,}000\,[1 + (0.08)^3] = \$1{,}240.$$

Year by year, the interest accrues as shown:

End of Year	Beginning Balance	Interest Earned	Ending Balance
1	$1,000	$80	$1,080
2	$1,080	$80	$1,160
3	$1,160	$80	$1,240

(b) Compound interest: Applying Eq. (2.3) to our three-year, 8% case, we obtain

$$F = \$1{,}000\,(1 + 0.08)^3 = \$1{,}259.71.$$

The total interest earned is $259.71, which is $19.71 more than accumulated under the simple-interest method. We can keep track of the interest-accruing process more precisely as follows:

End of Year	Beginning Balance	Interest Earned	Ending Balance
1	$1,000.00	$80.00	$1,080.00
2	$1,080.00	$86.40	$1,166.40
3	$1,166.40	$93.31	$1,259.71

COMMENTS: At the end of the first year, you would have a total of $1,080 which consists of $1,000 in principal plus $80 in interest. In effect, at the beginning of the second year, you would be depositing $1,080, rather than $1,000. Thus, at the end of the second year, the interest earned would be $0.08(\$1{,}080) = \$86.40$ and the balance would be $\$1{,}080 + \$86.40 = \$1{,}166.40$. This is the equivalent amount you would be depositing at the beginning of the third year, and the interest earned for that period would be $0.08(\$1{,}166.40) = \$93.31$. With a beginning principal amount of $1,166.40 plus the $93.31 interest, the total balance would be $1,259.71 at the end of year 3.

2.2 Economic Equivalence

The observation that money has a time value leads us to an important question: If receiving $100 today is not the same as receiving $100 at any future point, how do we measure and compare various cash flows? How do we know, for example,

- Economic equivalence exists between cash flows that have the *same economic effect* and could therefore be traded for one another.
- Even though the amounts and timing of the cash flows may differ, the appropriate interest rate makes them equal.

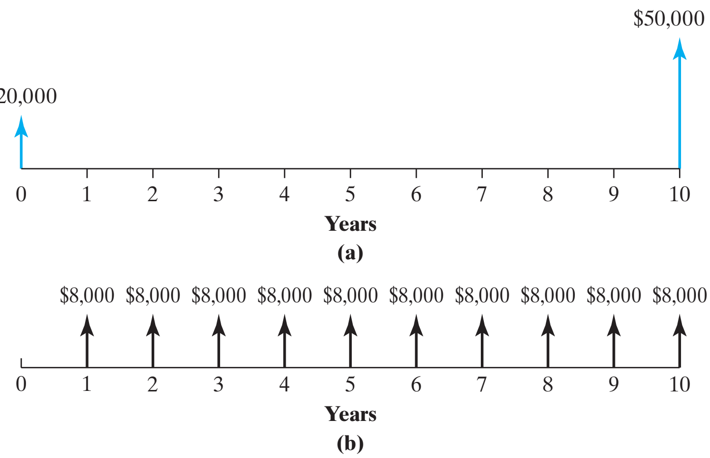


Figure 2.3 Which option would you prefer? (a) Two payments ($20,000 now and $50,000 at the end of 10 years) or (b) 10 equal annual receipts in the amount of $8,000 each.

whether we should prefer to have two payments, $20,000 today and $50,000 in 10 years from now, or $8,000 each year for the next 10 years? (See Figure 2.3.) In this section, we will describe the basic analytical techniques for making these comparisons. Then in Section 2.3, we will use these techniques to develop a series of formulas that can greatly simplify our calculations.

2.2.1 Definition and Simple Calculations

The central factor in deciding among alternative cash flows involves comparing their economic worth. This would be a simple matter if, in the comparison, we did not need to consider the time value of money. We could simply add up the individual payments within a cash flow, treating receipts as positive cash flows and payments (disbursements) as negative cash flows. Calculations for determining the economic effects of one or more cash flows are based on the concept of economic equivalence.

Economic equivalence exists between cash flows that have the same economic effect and could therefore be traded for one another in the financial marketplace (which we assume to exist). Economic equivalence refers to the fact that any cash flow—whether a single payment or a series of payments—can be converted to an *equivalent* cash flow at any point in time. The critical thinking on the present value of future cash flows is that the present sum is equivalent in value to the future cash flows because, if you had the present value today, you could transform it into the future cash flows simply by investing it at the market interest rate, also referred to as the **discount rate**. This process is shown in Figure 2.4.

The strict concept of equivalence may be extended to include the comparison of alternatives. For instance, we could compare the values of two proposals by finding the equivalent values of each at any common point in time. If financial proposals that appear to be quite different could turn out to have the same monetary value, then we can be *economically indifferent* in choosing between them. Likewise, in terms of economic effect, one would be an even exchange for the other, so there is no reason to prefer one over the other.

- If you deposit P dollars today for N periods at i, you will have F dollars at the end of period N.
- F dollars at the end of period N is equal to a single sum of P dollars now if your earning power is measured in terms of the interest rate i.

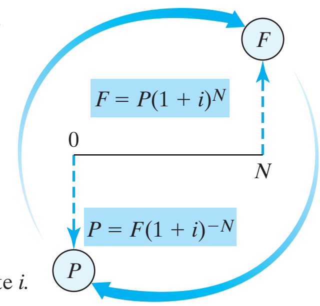


Figure 2.4 Using compound interest to establish economic equivalence.

A way to see the concepts of equivalence and economic indifference at work in the real world is to note the variety of payment plans offered by lending institutions for consumer loans. Recall Table 2.1, where we showed two different repayment plans for a loan of \$30,000 for five years at an annual interest rate of 9%. You will notice that the two plans require significantly different repayment patterns and different total amounts of repayment. However, because of the time value of money, these plans are equivalent—economically, the bank is indifferent to the consumer's choice of plan. We will now discuss how such equivalence relationships are established.

Equivalence Calculations: A Simple Example

Equivalence calculations can be viewed as an application of the compound-interest relationships we learned in Section 2.1. Suppose, for example, that we invest \$1,000 at 12% annual interest for five years. The formula developed for calculating compound interest, $F = P(1 + i)^N$ [Eq. (2.3)], expresses the equivalence between some present amount P and a future amount F for a given interest rate i and a number of interest periods, N. Therefore, at the end of the investment period, our sums grow to

$$\$1{,}000(1 + 0.12)^5 = \$1{,}762.34.$$

Thus, we can say that at 12% interest, \$1,000 received now is equivalent to \$1,762.34 received in five years, and we could trade \$1,000 now for the promise of receiving \$1,762.34 in five years. Example 2.2 further demonstrates the application of this basic technique.

EXAMPLE 2.2 Equivalence

Suppose you are offered the alternative of receiving either \$2,007 at the end of five years or \$1,500 today. There is no question that the \$2,007 will be paid in full (i.e., there's no risk of nonreceipt). Assuming that the money will not be needed in the next five years, you would deposit the \$1,500 in an account that pays i% interest. What value of i would make you indifferent to your choice between \$1,500 today and the promise of \$2,007 at the end of five years?

DISSECTING THE PROBLEM

Our job is to determine the present amount that is economically equivalent to $2,007 in five years, given the investment potential of i% per year. Note that the statement of the problem assumes that you would exercise the option of using the earning power of your money by depositing it. The "indifference" ascribed to you refers to economic indifference; that is, within a marketplace where i% is the applicable interest rate, you could trade one cash flow for the other.

Given: $F = \$2{,}007$, $N = 5$ years, $P = \$1{,}500$. See Figure 2.5a.
Find: i.

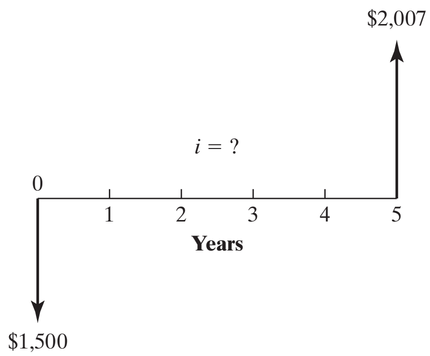


Figure 2.5a Cash flow diagram.

METHODOLOGY

Use Eq. (2.3), $F = P(1 + i)^N$ and solve for i.

SOLUTION

Using the expression of Eq. (2.3) we obtain

$$\$2{,}007 = \$1{,}500(1 + i)^5.$$

Solving for i yields:

$$i = \left(\frac{F}{P}\right)^{1/N} - 1 = \left(\frac{2{,}007}{1{,}500}\right)^{1/5} - 1$$
$$= 0.06 \ (\text{or } 6\%).$$

We can summarize the problem graphically as in Figure 2.5b.

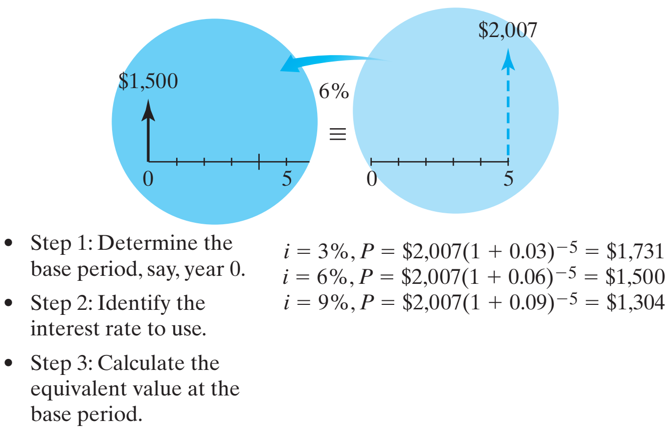


Figure 2.5b Equivalence calculations at varying interest rate.

COMMENTS: In this example, it is clear that if i is anything less than 6%, you would prefer the promise of \$2,007 in five years to \$1,500 today; if i is more than 6%, you would prefer \$1,500 now. As you may have already guessed, at a lower interest rate, P must be higher in order to be equivalent to the future amount. For example, at $i = 4\%$, $P = \$1{,}650$.

2.2.2 Equivalence Calculations Require a Common Time Basis for Comparison

Referring to Figure 2.3, how can we compare these two different cash flow series? Since we know how to calculate the equivalent value of a single cash flow, we may be able to convert each cash flow in the series to its equivalent value at a common base period. One aspect of this basis is the choice of a single point in time at which to make our calculations. In Example 2.2, if we had been given the magnitude of each cash flow and had been asked to determine whether the two were equivalent, we could have chosen any reference point and used the compound-interest formula to find the value of each cash flow at that point. As you can readily see, the choice of $n = 0$ or $n = 5$ would make our problem simpler, because we would need to make only one set of calculations: At 6% interest, either convert \$1,500 at time 0 to its equivalent value at time 5, or convert \$2,007 at time 5 to its equivalent value at time 0.

When selecting a point in time at which to compare the values of alternative cash flows, we commonly use either the present time, which yields what is called the **present worth** of the cash flows, or some point in the future, which yields their **future worth**. The choice of the point in time to use often depends on the circumstances surrounding a particular decision, or the choice may be made for convenience. For instance, if the present worth is known for the first two of three alternatives, simply calculating the present worth of the third will allow us to compare all three. For an illustration, consider Example 2.3.

EXAMPLE 2.3 Equivalence Calculations

Consider the cash flow series given in Figure 2.6. Compute the equivalent lump-sum amount at $n = 3$ at 10% annual interest.

DISSECTING THE PROBLEM	**Given:** The cash flows given in Figure 2.6, and $i = 10\%$ per year. **Find:** V_3 (or equivalent worth at $n = 3$).

METHODOLOGY

We find the equivalent worth at $n = 3$ in two steps. First, we find the future worth of each cash flow at $n = 3$ for all cash flows that occur before $n = 3$. Second, we find the present worth of each cash flow at $n = 3$ for all cash flows that occur after $n = 3$.

SOLUTION

- **Step 1:** Find the equivalent lump-sum payment of the first four payments at $n = 3$:

$$\$100(1 + 0.10)^3 + \$80(1 + 0.10)^2 + \$120(1 + 0.10)^1 + \$150 = \$511.90.$$

- **Step 2:** Find the equivalent lump-sum payment of the remaining two payments at $n = 3$:

$$\$200(1 + 0.10)^{-1} + \$100(1 + 0.10)^{-2} = \$264.46.$$

- **Step 3:** Find V_3, the total equivalent value:

$$V_3 = \$511.90 + \$264.46 = \$776.36.$$

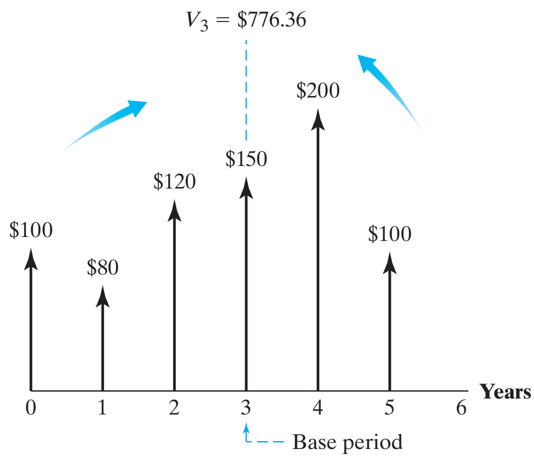


Figure 2.6 Equivalent worth-calculation at $n = 3$.

2.3 Interest Formulas for Single Cash Flows

We begin our coverage of interest formulas by considering the simplest of cash flows: single cash flows.

2.3.1 Compound-Amount Factor

Given a present sum P invested for N interest periods at interest rate i, what sum will have accumulated at the end of the N periods? You probably noticed right away that this description matches the case we first encountered in describing compound interest. To solve for F (the future sum), we use Eq. (2.3):

$$F = P(1 + i)^N.$$

Because of its origin in the compound-interest calculation, the factor $(1 + i)^N$ is known as the **compound-amount factor**. Like the concept of equivalence, this factor is one of the foundations of engineering economic analysis. Given this factor, all other important interest formulas can be derived.

The process of finding F is often called the **compounding process**. The cash flow transaction is illustrated in Figure 2.7. (Note the time-scale convention: The first period begins at $n = 0$ and ends at $n = 1$.) If a calculator is handy, it is easy enough to calculate $(1 + i)^N$ directly.

Interest Tables

Interest formulas such as the one developed in Eq. (2.3), $F = P(1 + i)^N$, allow us to substitute known values from a particular situation into the equation and solve for the unknown. Before the calculator was developed, solving these equations was very tedious. Imagine needing to solve by hand an equation with a large value of N, such as $F = \$20{,}000(1 + 0.12)^{15}$. More complex formulas required even more involved calculations. To simplify the process, tables of compound-interest factors were developed. These tables allow us to find the appropriate factor for a given interest rate and the number of interest periods. Even though many online financial calculators are now

- *Single-payment compound-amount factor* (growth factor)
- Given:
  $i = 10\%$
  $N = 8$ years
  $P = \$2{,}000$
- Find:
  $F = \$2{,}000(1 + 0.10)^8$
  $= \$2{,}000(F/P, 10\%, 8)$
  $= \$4{,}287.18$

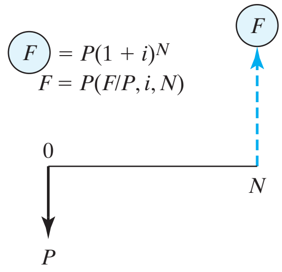


Figure 2.7 Compounding process: Find F, given P, i, and N.

readily available, it is still often convenient to use these tables, which are included in this text in Appendix B. Take some time now to become familiar with their arrangement. If you can, locate the compound-interest factor for the example just presented, in which we know P and, to find F, we need to know the factor by which to multiply \$20,000 when the interest rate i is 12% and the number of periods is 15:

$$F = \$20{,}000\underbrace{(1 + 0.12)^{15}}_{5.4736} = \$109{,}472.$$

Factor Notation

As we continue to develop interest formulas in the rest of this chapter, we will express the resulting compound-interest factors in a conventional notation that can be substituted into a formula to indicate precisely which table factor to use in solving an equation. In the preceding example, for instance, the formula derived as Eq. (2.3) is $F = P(1 + i)^N$. To specify how the interest tables are to be used, we may also express that factor in a functional notation as $(F/P, i, N)$, which is read as "Find F, given P, i, and N." This factor is known as the **single-payment compound-amount factor**. When we incorporate the table factor into the formula, the formula is expressed as follows:

$$F = P(1 + i)^N = P(F/P, i, N).$$

Thus, in the preceding example, where we had $F = \$20{,}000(1.12)^{15}$, we can now write $F = \$20{,}000(F/P, 12\%, 15)$. The table factor tells us to use the 12% interest table and find the factor in the F/P column for $N = 15$. Because using the interest tables is often the easiest way to solve an equation, this factor notation is included for each of the formulas derived in the upcoming sections.

EXAMPLE 2.4 Single Amounts: Find *F*, Given *P*, *i*, and *N*

If you had \$1,000 now and invested it at 7% interest compounded annually, how much would it be worth in eight years (Figure 2.8)?

DISSECTING THE PROBLEM	**Given:** $P = \$1{,}000$, $i = 7\%$ per year, and $N = 8$ years. **Find:** F.

METHODOLOGY	SOLUTION
Method 1: Using a Calculator You can simply use a calculator to evaluate the $(1 + i)^N$ term (financial calculators are preprogrammed to solve most future-value problems).	$F = \$1{,}000(1 + 0.07)^8 = \$1{,}718.19.$
Method 2: Using Compound-Interest Tables The interest tables can be used to locate the compound-amount factor for $i = 7\%$ and $N = 8$. The number you get can be substituted into the equation. Compound-interest tables are included in Appendix B of this book.	Using this method, we obtain $F = \$1{,}000(F/P, 7\%, 8) = \$1{,}000(1.7182) = \$1{,}718.20.$ This amount is essentially identical to the value obtained by direct evaluation of the single cash flow compound-amount factor. The slight deviation is due to rounding differences.
Method 3: Using a Computer Many financial software programs for solving compound-interest problems are available for use with personal computers. As summarized in Appendix D, many spreadsheet programs such as Excel also provide financial functions to evaluate various interest formulas.	With Excel, the future-worth calculation looks like the following: = FV(7%,8,0,–1000,0) = \$1,718.20.

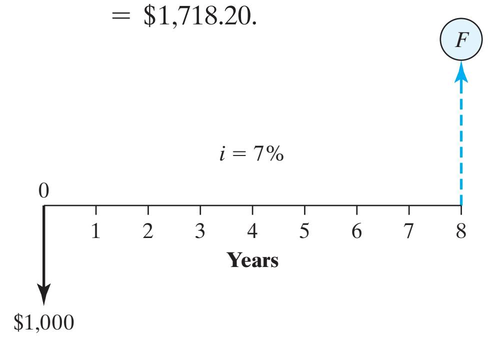


Figure 2.8 Cash flow diagram.

COMMENTS: A better way to take advantage of the powerful features of Excel is to develop a worksheet as shown in Table 2.2. Here, we are not calculating just the future worth of the single payment today. Instead, we can show in both tabular and graphical form how the deposit balances change over time. For example, the $1,000 deposit today will grow to $1,500.73 in six years. If you want to see how the cash balances change over time at a higher interest rate, say, 10%, you just change the interest rate in cell C6 and press the "ENTER" button.

2.3.2 Present-Worth Factor

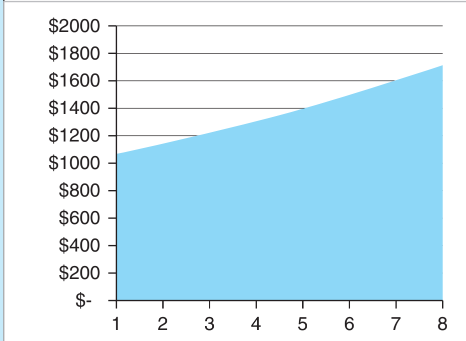
Finding the present worth of a future sum is simply the reverse of compounding and is known as the **discounting process**. (See Figure 2.9.) In Eq. (2.3), we can see that if we need to find a present sum P, given a future sum F, we simply solve for P:

$$P = F\left[\frac{1}{(1 + i)^N}\right] = F(P/F, i, N). \tag{2.4}$$

TABLE 2.2 **An Excel Worksheet to Illustrate How the Cash Balances Change over Time**

	A	B	C	D	E	F	G
1	Single Cash Flows						
2							
3	Inputs				Output		
4							
5	(*P*) Present Worth ($)		1000		(*F*) Future Worth ($)		1718.19
6	(*i*) Interest Rate (%)		7				
7	(*N*) Interest Periods		8		Cash Balance Over Time		
8							
9	Period (*n*)	Deposit (*P*)	Cash Balance				
10							
11	0	$ 1000	$ 1000.00				
12	1		$ 1070.00				
13	2		$ 1144.90				
14	3		$ 1225.04				
15	4		$ 1310.80				
16	5		$ 1402.55				
17	6		$ 1500.73				
18			1605.78				
19	8		$ 1718.19				
20							

= C18*(1+$C$6%)+B19



The factor $1/(1+i)^N$ is known as the **single-payment present-worth factor** and is designated $(P/F, i, N)$. Tables have been constructed for P/F factors and for various values of i and N. The interest rate i and the P/F factor are also referred to as the **discount rate** and the **discounting factor**, respectively.

- *Single-payment present-worth factor (discount factor)*
- Given:
  $i = 12\%$
  $N = 5$ years
  $F = \$1,000$
- Find:
  $P = \$1,000(1+0.12)^{-5}$
  $= \$1,000(P/F, 12\%, 5)$
  $= \$567.40$

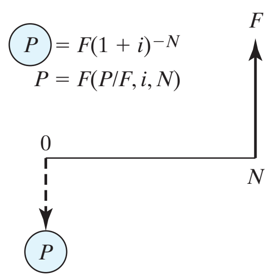


Figure 2.9 Discounting process: Find P, given F, i, and N.

EXAMPLE 2.5 Single Amounts: Find *P*, Given *F*, *i*, and *N*

A zero-coupon bond[3] is a popular variation on the bond theme for some investors. What should be the price of an eight-year zero-coupon bond with a face value of \$1,000 if similar, nonzero-coupon bonds are yielding 6% annual interest?

DISSECTING THE PROBLEM

As an investor of a zero-coupon bond, you do not receive any interest payments until the bond reaches maturity. When the bond matures, you will receive \$1,000 (the face value). In lieu of getting interest payments, you can buy the bond at a discount. The question is, "What should the price of the bond be in order to realize a 6% return on your investment?" (See Figure 2.10.)

Given: $F = \$1{,}000$, $i = 6\%$ per year, and $N = 8$ years.
Find: P.

METHODOLOGY

Using a calculator may be the best way to make this simple calculation. It is equivalent to finding the present value of the \$1,000 face value at 6% interest.

SOLUTION

Using a calculator, we obtain

$$P = \$1{,}000(1 + 0.06)^{-8} = \$1{,}000(0.6274) = \$627.40.$$

We can also use the interest tables to find that

$$P = \$1{,}000\overbrace{(P/F, 6\%, 8)}^{(0.6274)} = \$627.40.$$

Again, you could also use a financial calculator or computer to find the present worth. With Excel, the present-value calculation looks like the following:

$$= \text{PV}(6\%,8,0,1000,0)$$
$$= -\$\,627.40.$$

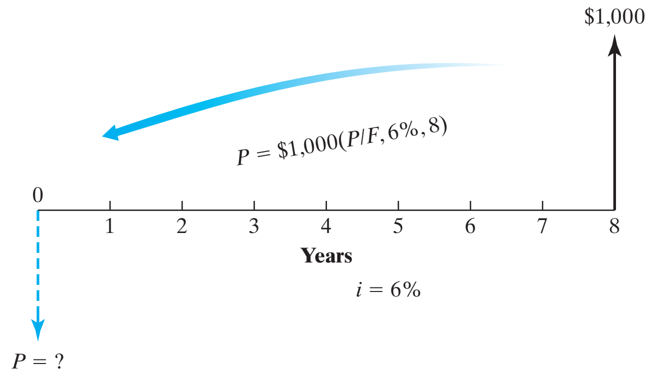


Figure 2.10 Cash flow diagram.

[3] Bonds are loans that investors make to corporations and governments. In Example 2.5, the \$1,000 of principal is the **face value** of the bond, the yearly interest payment is its **coupon**, and the length of the loan is the bond's **maturity**.

2.3.3 Solving for Time and Interest Rates

At this point, you should realize that the compounding and discounting processes are reciprocals of one another and that we have been dealing with one equation in two forms:

$$\text{Future-value form: } F = P(1 + i)^N$$

and

$$\text{Present-value form: } P = F(1 + i)^{-N}.$$

There are four variables in these equations: P, F, N, and i. If you know the values of any three, you can find the value of the fourth. Thus far, we have always given you the interest rate (i) and the number of years (N), plus either P or F. In many situations, though, you will need to solve for i or N, as we discuss next.

EXAMPLE 2.6 Solving for *i*

Suppose you buy a share of stock for \$10 and sell it for \$20; your profit is thus \$10. If that happens within a year, your rate of return is an impressive 100% ($\$10/\$10 = 1$). If it takes five years, what would be the annual rate of return on your investment? (See Figure 2.11.)

DISSECTING THE PROBLEM

Here, we know P, F, and N, but we do not know i, the interest rate you will earn on your investment. This type of rate of return calculation is straightforward, since you make only a one-time lump-sum investment.

Given: $P = \$10$, $F = \$20$, and $N = 5$ years.
Find: i.

METHODOLOGY

We start with the following relationship:

$$F = P(1 + i)^N.$$

We then substitute in the given values:

$$\$20 = \$10(1 + i)^5.$$

Next, we solve for i by one of two methods.

Method 1: Trial and Error
Go through a trial-and-error process in which you insert different values of i into the equation until you find a value that "works," in the sense that the right-hand side of the equation equals \$20.

SOLUTION

The solution value is $i = 14.87\%$. The trial-and-error procedure is extremely tedious and inefficient for most problems, so it is not widely practiced in the real world.

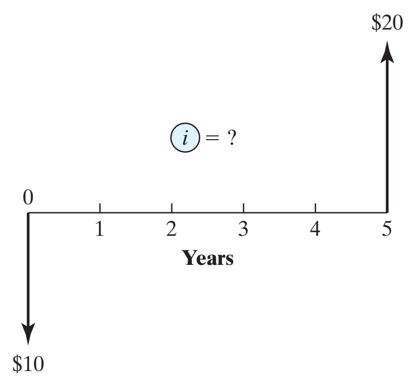


Figure 2.11 Cash flow diagram.

METHODOLOGY

Method 2: Interest Tables
You can solve the problem by using the interest tables in Appendix B.

SOLUTION

Start with the equation

$$\$20 = \$10(1 + i)^5,$$

which is equivalent to

$$2 = (1 + i)^5 = (F/P, i, 5).$$

Now look across the $N = 5$ row under the $(F/P, i, 5)$ column until you can locate the value of 2. This value is approximated in the 15% interest table at $(F/P, 15\%, 5) = 2.0114$, so the interest rate at which \$10 grows to \$20 over five years is very close to 15%. This procedure will be very tedious for fractional interest rates or when N is not a whole number, as you may have to approximate the solution by linear interpolation.

Method 3: Practical Approach
The most practical approach is to use either a financial calculator or an electronic spreadsheet such as Excel. A financial function such as RATE(N, 0, P, F) allows us to calculate an unknown interest rate.

The precise command statement would be as follows:

$$= \text{RATE}(5{,}0{,}-10{,}20) = 14.87\%.$$

Note that we always enter the present value (P) as a negative number in order to indicate a cash outflow in Excel.

EXAMPLE 2.7 Single Amounts: Find *N*, Given *P*, *F*, and *i*

You have just purchased 200 shares of a biotechnology stock at \$15 per share. You will sell the stock when its market price doubles. If you expect the stock price to increase 12% per year, how long do you expect to wait before selling the stock? (See Figure 2.12.)

DISSECTING THE PROBLEM

Given: $P = \$3{,}000$, $F = \$6{,}000$, and $i = 12\%$ per year.
Find: N (years).

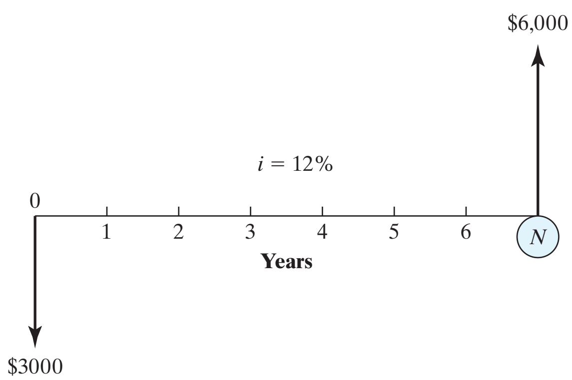


Figure 2.12 Cash flow diagram.

METHODOLOGY	SOLUTION
Method 1: Using a Calculator Using the single-payment compound-amount factor, we write $F = P(1+i)^N = P(F/P, i, N),$ which in this case is $\$6{,}000 = \$3{,}000(1+0.12)^N$ $= \$3{,}000(F/P, 12\%, N),$ or $2 = (1.12)^N = (F/P, 12\%, N).$	We start with $\log 2 = N \log 1.12.$ Solving for N gives $N = \dfrac{\log 2}{\log 1.12}$ $= 6.12 \approx 6$ years.
Method 2: Using Excel Again, we could use a computer spreadsheet program to find N.	Within Excel, the financial function NPER(i, 0, P, F) computes the number of compounding periods it will take an investment P to grow to a future value F, earning a fixed interest rate i per compounding period. In our example, the Excel command would look like this: = NPER(12%,0,−3000,6000). The calculated result is 6.1163 years.

COMMENTS: A very handy rule called the *Rule of 72* can determine approximately how long it will take for a sum of money to double. The rule states that to find the time it takes for the present sum of money to grow by a factor of two, we divide 72 by the interest rate. For our example, the interest rate is 12%. Therefore, the Rule of 72 indicates that it will take $\frac{72}{12} = 6$ years for a sum to double. This result is, in fact, relatively close to our exact solution.

2.4 Uneven-Payment Series

A common cash flow transaction involves a series of disbursements or receipts. Familiar examples of series payments are payment of installments on car loans and home mortgage payments, which typically involve identical sums to be paid at regular intervals. When there is no clear pattern over the series, we call the transaction an *uneven cash flow series.*

We can find the present worth of any uneven stream of payments by calculating the present worth of each individual payment and summing the results. Once the present worth is found, we can make other equivalence calculations. (For instance, future worth can be calculated by the interest factors developed in the previous section.)

EXAMPLE 2.8 Tuition Prepayment Option

The Tuition Prepayment Option (TPO) offered by many colleges provides savings by eliminating future tuition increases. When you enroll in the plan, you prepay all remaining undergraduate tuition and required fees at the rate in effect when you enter the plan. Tuition and fees (not including room and board) for the 2017–18 academic year are \$46,132 at Harvard University. Total undergraduate tuition for an entering freshman at this rate is \$184,528. Tuition, fees, room, and board normally increase each year, but it is difficult to predict by how much, since costs depend on future economic trends and institutional priorities. The following chart lists the tuition and required fee rates since 2013:

Academic Year	Tuition and Fees	Required Prepayment
2013–14	\$39,848	\$159,396
2014–15	\$41,406	\$165,624
2015–16	\$42,654	\$170,616
2016–17	\$44,356	\$177,424
2017–18	\$46,132	\$184,528

Suppose that you enrolled in the TPO for the academic year 2014–15. In 2018, looking back four years from the time of enrollment, knowing now exactly what the actual tuitions were, do you think your decision was justified in an economic sense to prepay "when money saved or invested was earning" at an interest rate of 4%?

DISSECTING THE PROBLEM

This problem is equivalent to asking what value of P would make you indifferent in your choice between P dollars today and the future expense stream of (\$41,406, \$42,654, \$44,356, \$46,132)

Given: Uneven cash flow in Figure 2.13; $i = 4\%$ per year.
Find: P.

METHODOLOGY

One way to deal with an uneven series of cash flows is to calculate the equivalent present value of each single cash flow and then sum the present values to find P. In other words, the cash flow is broken into four parts, as shown in Figure 2.13.

SOLUTION

Assuming that the tuition payment occurs at the beginning of each academic year, we sum the individual present values as follows:

$$\begin{aligned} P &= \$41{,}406 + \$42{,}654(P/F, 4\%, 1) + \$44{,}356(P/F, 4\%, 2) \\ &\quad + \$46{,}132(P/F, 4\%, 3) \\ &= \$164{,}439 < \$165{,}624 \end{aligned}$$

Since the equivalent present worth amount of the future tuition payments is less than the required prepayment amount at the beginning of the 2014–15 academic year, you would be better off paying tuition on a yearly basis.

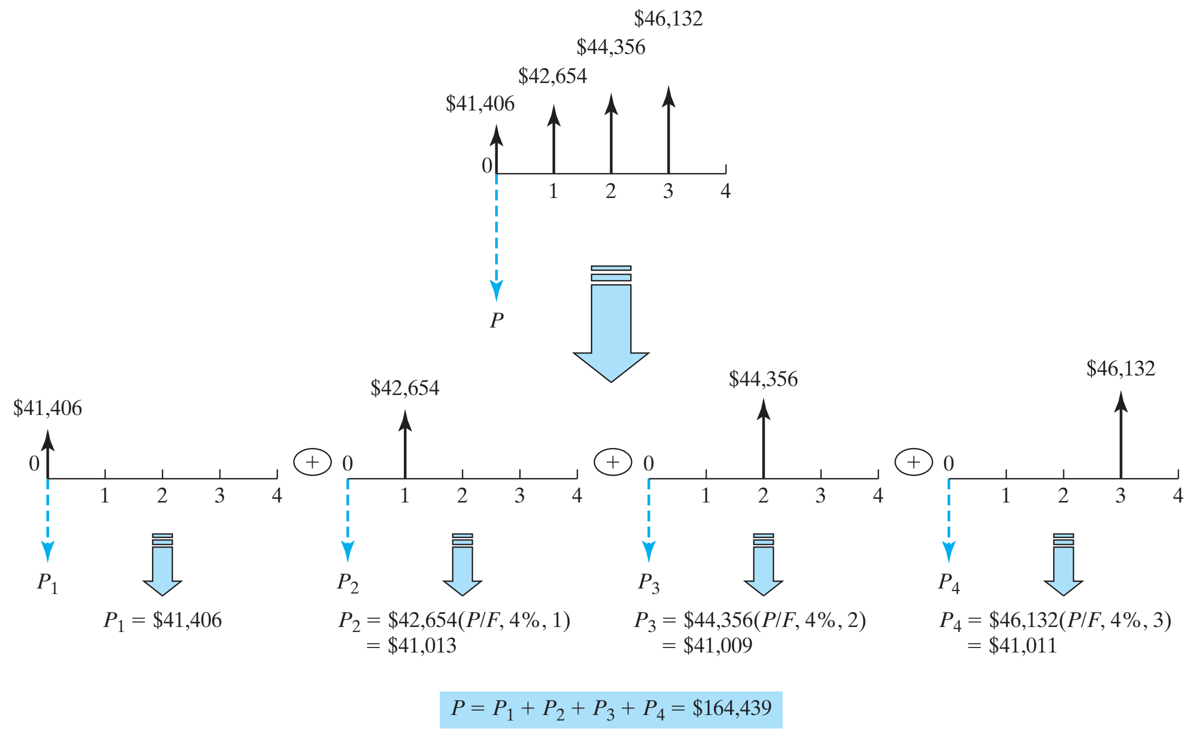


Figure 2.13 Decomposition of uneven cash flow series.

COMMENTS: Of course, you do not know in advance how much the future tuition would increase, so the difference between $164,439 and $165,624 (or $1,185) may be viewed as the risk premium you may choose to pay to lock in the future tuition at your freshman year amount.

2.5 Equal-Payment Series

As we learned in Example 2.8, we can always find the present worth of a stream of future cash flows by summing the present worth of each individual cash flow. However, if cash flow regularities are present within the stream, we may use some shortcuts, such as finding the present worth of an equal-payment (or a uniform) series. We often encounter transactions in which a uniform series of payments exists. Rental payments, bond interest payments, and commercial installment plans are based on uniform payment series. Our concern is to find the equivalent present worth (P) or future worth (F) of such a series, as illustrated in Figure 2.14.

2.5.1 Compound-Amount Factor: Find F, Given A, i, and N

Suppose we are interested in the future amount F of a fund to which we contribute A dollars each period and on which we earn interest at a rate of i per period. The contributions are made at the end of each of the N periods. These transactions are graphically illustrated in Figure 2.15. Looking at this diagram, we see that if an amount

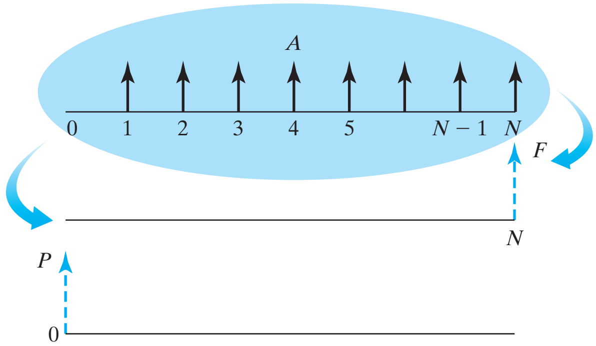


Figure 2.14 Equal-payment series: Find equivalent P or F.

A is invested at the end of each period for N periods, the total amount F that can be withdrawn at the end of N periods will be the sum of the compound amounts of the individual deposits.

As shown in Figure 2.15, the A dollars we put into the fund at the end of the first period will be worth $A(1 + i)^{N-1}$ at the end of N periods. The A dollars we put into the fund at the end of the second period will be worth $A(1 + i)^{N-2}$, and so forth. Finally, the last A dollars that we contribute at the end of the Nth period will be worth exactly A dollars at that time (no time to earn interest). This means we have a series in the form

$$F = A(1 + i)^{N-1} + A(1 + i)^{N-2} + \cdots + A(1 + i) + A,$$

or, expressed alternatively,

$$F = A + A(1 + i) + A(1 + i)^2 + \cdots + A(1 + i)^{N-1}. \tag{2.5}$$

Multiplying Eq. (2.5) by $(1 + i)$ results in

$$(1 + i)F = A(1 + i) + A(1 + i)^2 + \cdots + A(1 + i)^N. \tag{2.6}$$

Subtracting Eq. (2.5) from Eq. (2.6) to eliminate common terms gives us

$$F(1 + i) - F = -A + A(1 + i)^N.$$

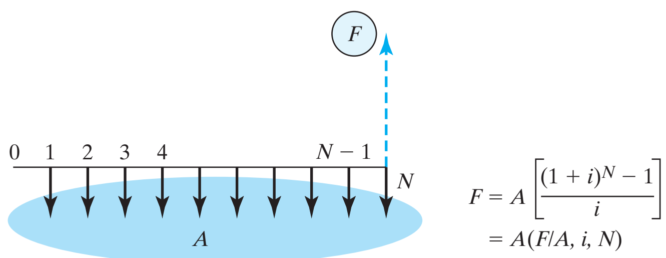


Figure 2.15 Cash flow diagram of the relationship between A and F.

Solving for F yields

$$F = A\left[\frac{(1+i)^N - 1}{i}\right] = A(F/A, i, N). \tag{2.7}$$

The bracketed term in Eq. (2.7) is called the **equal-payment-series compound-amount factor** (also known as the **uniform-series compound-amount factor)**; its factor notation is (F/A, i, N). This interest factor has been also calculated for various combinations of i and N in the tables in Appendix B.

EXAMPLE 2.9 Equal-Payment Series: Find *F*, Given *i*, *A*, and *N*

Suppose you make an annual contribution of \$5,000 to your savings account at the end of each year for five years. If your savings account earns 6% interest annually, how much can be withdrawn at the end of five years? (See Figure 2.16.)

DISSECTING THE PROBLEM

Given: $A = \$5{,}000$, $N = 5$ years, and $i = 6\%$ per year.
Find: F.

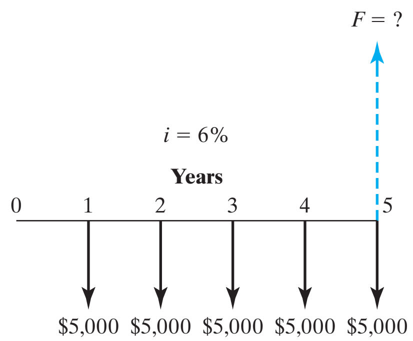


Figure 2.16 Cash flow diagram.

METHODOLOGY

Use the equal-payment-series compound-amount factor and Excel.

SOLUTION

Using the equal-payment-series compound-amount factor, we obtain

$$\begin{aligned} F &= \$5{,}000(F/A, 6\%, 5) \\ &= \$5{,}000(5.6371) \\ &= \$28{,}185.46. \end{aligned}$$

To obtain the future value of the annuity in Excel, we may use the following financial function:

= FV(6%,5,−5000,0).

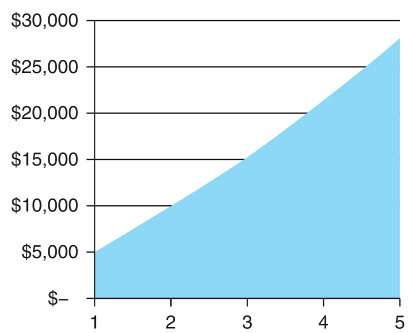
COMMENTS: We may be able to keep track of how the periodic balances grow in the savings account to reach the amount of \$28,185.46 at the end of year 5 as follows:

Year	1	2	3	4	5
Beginning Balance	\$0	\$5,000.00	\$10,300.00	\$15,918.00	\$21,873.08
Interest Earned (6%)	\$0	\$300.00	\$618.00	\$955.08	\$1,312.38
Deposit Made	\$5,000.00	\$5,000.00	\$5,000.00	\$5,000.00	\$5,000.00
Ending Balance	**\$5,000.00**	**\$10,300.00**	**\$15,918.00**	**\$21,873.08**	**\$28,185.46**

As shown in Table 2.3, an Excel worksheet can be easily created to determine the periodic cash balance over time.

TABLE 2.3 **An Excel Worksheet to Illustrate How the Cash Balance Changes over Time**

	A	B	C	D	E	F	G
1	Equal-Payment Cash Flows						
2					=FV($C$6%,$C$7,-$C$5)		
3	Inputs				Output		
4							
5	(*A*) Annuity ($)		5,000.00		(F) Future Worth ($)		28,185.46
6	(*i*) Interest Rate (%)		6				
7	(*N*) Interest Periods		5		Cash Balance Over Time		
8							
9	Period (*n*)	Deposit (*P*)	Cash Balance				
10							
11	0	$ –	$ –				
12	1	$ 5000	$ 5,000				
13	2	$ 5000	$ 10,300				
14	3	$ 5000	$ 15,918				
15	4	$ 5000	$ 21,873				
16	5	$ 5000	$ 28,185				
17							
18							
19		=C15*(1+$C$6%)+B16					
20							



EXAMPLE 2.10 Handling Time Shifts in an Equal-Payment Series

In Example 2.9, all five deposits were made at the *end* of each period— the first deposit being made at the end of the first period. Suppose that all deposits were made at the *beginning* of each period instead or commonly known as "**annuity due**." How would you compute the balance at the end of period 5?

DISSECTING THE PROBLEM

Compare Figure 2.17 with Figure 2.16. Each payment in Figure 2.17 has been shifted one year earlier; thus, each payment is compounded for one extra year.

Given: Cash flow diagram in Figure 2.17; $i = 6\%$ per year.
Find: F.

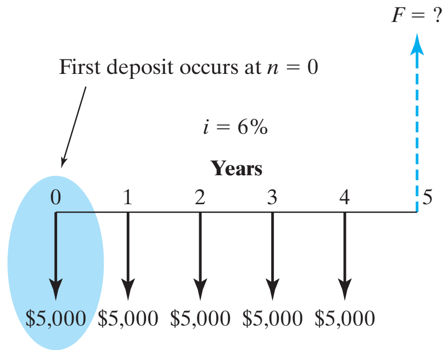


Figure 2.17 Cash flow diagram.

METHODOLOGY

Use the future value (FV) command in Excel. Note that with the end-of-year deposit, the ending balance F was \$28,185.46. With the beginning-of-year deposit, the same balance accumulates by the end of period 4. This balance can earn interest for one additional year.

SOLUTION

We can easily calculate the resulting balance as

$$F = \$28{,}185.46(1.06) = \$29{,}876.59.$$

Annuity due can be easily evaluated by the following financial function available in Excel:

$$= \text{FV}(6\%, 5, -5{,}000, 1).$$

COMMENTS: Another way to determine the ending balance is to compare the two cash flow patterns. By adding the \$5,000 deposit at period 0 to the original cash flow and subtracting the \$5,000 deposit at the end of period 5, we obtain the second cash flow. Therefore, we can find the ending balance by making an adjustment to the \$28,185.46:

$$F = \$28{,}185.46 + \$5{,}000(F/P, 6\%, 5) - \$5{,}000 = \$29{,}876.59.$$

2.5.2 Sinking-Fund Factor: Find A, Given F, i, and N

If we solve Eq. (2.7) for A, we obtain

$$A = F\left[\frac{i}{(1+i)^N - 1}\right] = F(A/F, i, N). \tag{2.8}$$

The term within the brackets is called the **equal-payment-series sinking-fund factor**, or just **sinking-fund factor**, and is referred to with the notation (A/F, i, N). A sinking fund is an interest-bearing account into which a fixed sum is deposited each interest period; it is commonly established for the purpose of replacing fixed assets.

EXAMPLE 2.11 College Savings Plan: Find *A*, Given *F*, *N*, and *i*

You want to set up a college savings plan for your daughter. She is currently 10 years old and will go to college at age 18. You assume that when she starts college, she will need at least \$100,000 in the bank. How much do you need to save each year in order to have the necessary funds if the current rate of interest is 7%? Assume that end-of-year deposits are made.

DISSECTING THE PROBLEM	**Given:** Cash flow diagram in Figure 2.18; $i = 7\%$ per year, and $N = 8$ years. **Find:** A.
METHODOLOGY *Method 1: Sinking-Fund Factor*	**SOLUTION** Using the sinking-fund factors, we obtain $A = \$100{,}000(A/F, 7\%, 8)$ $= \$9{,}746.78.$

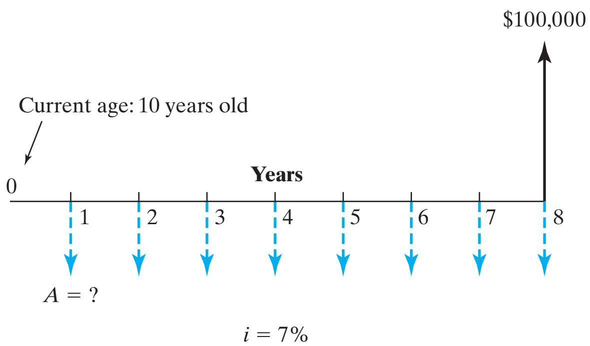


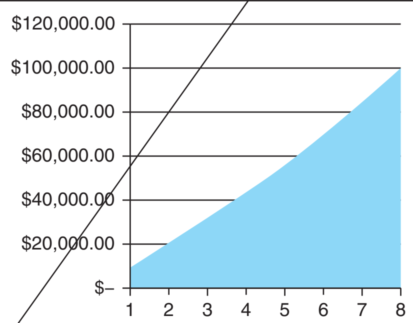
Figure 2.18 Cash flow diagram.

Method 2: Excel's PMT Function	Or, using the Excel's built-in financial function, we get the same result: $= \text{PMT}(7\%, 8, 0, 100000)$ $= -\$9{,}746.78.$

COMMENTS: With Excel, we can easily confirm the process of accumulating \$100,000 at the end of 8 years. As shown in Table 2.4, the cash balance at the end of year 1 will be just \$9,746.78, as the first deposit has no time to earn any interest. At the end of year 2, the cash balance consists of two contributions: The deposit made at the end of year 1 will grow to $\$9{,}746.78(1 + 0.07) = \$10{,}429.05$, and the second deposit, in the amount of \$9,746.78, resulting in a total of \$20,175.83. If the process continues for the remaining deposit periods, the final balance will be exactly \$100,000.

TABLE 2.4 An Excel Worksheet to Illustrate the Cash Balances over Time

	A	B	C	D	E	F	G
1	Equal-Payment Cash Flows (Sinking Fund)						
2							
3	Inputs				Output		
4							
5	(*F*) Future Worth		$ 100,000.00		(*A*) Annuity		$9,746.78
6	(*i*) Interest Rate		7%				
7	(*N*) Payment Periods		8		Cash Balance Over Time		
8							
9	Period (*n*)	Deposit (*P*)	Cash Balance				
10							
11	0	0	–				
12	1	$9,746.78	$ 9,746.78				
13	2	$9,746.78	$ 20,175.83				
14	3	$9,746.78	$ 31,334.91				
15	4	$9,746.78	$ 43,275.13				
16	5	$9,746.78	$ 56,051.17				
17	6	$9,746.78	$ 69,721.52				
18	7	$9,746.78	$ 84,348.81				
19	8	$9,746.78	$ 100,000.00				
20							



=PMT($C$6,$C$7,0,-$C$5,0)

2.5.3 Capital-Recovery Factor (Annuity Factor): Find *A*, Given *P*, *i*, and *N*

We can determine the amount of a periodic payment, A, if we know P, i, and N. Figure 2.19 illustrates this situation, where a typical loan transaction is described by either the lender's point of view or the borrower's point of view. To relate P to A, recall the relationship between P and F in Eq. (2.3): $F = P(1 + i)^N$. By replacing F in Eq. (2.8) by $P(1 + i)^N$, we get

$$A = P(1 + i)^N\left[\frac{i}{(1 + i)^N - 1}\right],$$

or

$$A = P\left[\frac{i(1 + i)^N}{(1 + i)^N - 1}\right] = P(A/P, i, N). \tag{2.9}$$

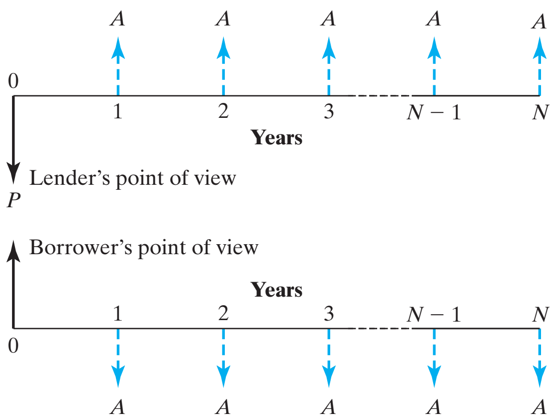


Figure 2.19 Cash flow diagram of the relationship between P and A.

Now we have an equation for determining the value of the series of end-of-period payments, A, when the present sum P is known. The portion within the brackets is called the **capital-recovery factor**, which is designated $(A/P, i, N)$. In finance, this A/P factor is referred to as the **annuity factor**. The annuity factor indicates a series of payments of a fixed, or constant, amount for a specified number of periods.

EXAMPLE 2.12 Paying Off an Educational Loan: Find A, Given P, i, and N

You borrowed $21,061.82 to finance the educational expenses for your senior year of college. The loan will be paid off over five years. The loan carries an interest rate of 6% per year and is to be repaid in equal annual installments over the next five years. Assume that the money was borrowed at the beginning of your senior year and that the first installment will be due a year later. Compute the amount of the annual installments (Figure 2.20).

DISSECTING THE PROBLEM

Given: $P = \$21{,}061.82$, $i = 6\%$ per year, and $N = 5$ years.
Find: A.

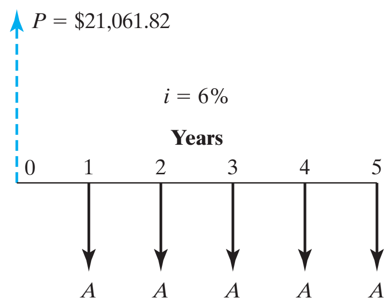


Figure 2.20 Cash flow diagram.

METHODOLOGY

Use the capital-recovery factor.

SOLUTION

Using the capital-recovery factor, we obtain

$$\begin{aligned} A &= \$21{,}061.82(A/P, 6\%, 5) \\ &= \$21{,}061.82(0.2374) \\ &= \$5{,}000. \end{aligned}$$

The table below illustrates how the \$5,000 annual repayment plan would retire the debt in five years. The Excel solution, with annuity function commands, is as follows:

$$\begin{aligned} &= \text{PMT}(i, N, P) \\ &= \text{PMT}(6\%{,}5{,}21061.82). \end{aligned}$$

The result of this formula is −\$5,000.

Year	1	2	3	4	5
Beginning Balance	\$21,061.82	\$17,325.53	\$13,365.06	\$9,166.96	\$4,716.98
Interest Charged (6%)	\$1,263.71	\$1,039.53	\$801.90	\$550.02	\$283.02
Payment Made	−\$5,000.00	−\$5,000.00	−\$5,000.00	−\$5,000.00	−\$5,000.00
Ending Balance	**\$17,325.53**	**\$13,365.06**	**\$9,166.96**	**\$4,716.98**	**\$0.00**

EXAMPLE 2.13 Deferred Loan Repayments

Suppose, in Example 2.12, that you want to negotiate with the bank to defer the first loan payment until the end of year 2. (But you still desire to pay off the loan by making five equal installments at 6% interest.) If the bank wishes to earn the same profit, what should be the new annual installment? (See Figure 2.21.)

DISSECTING THE PROBLEM

In deferring one year, the bank will add the interest accrued during the first year to the principal. In other words, you are effectively borrowing \$22,325.53 at the end of year 1, P'.

Given: P = \$21,061.82, i = 6% per year, and N = 5 years, but the first payment occurs at the end of year 2.
Find: A.

METHODOLOGY

Calculate equivalent worth of P'.

SOLUTION

$$\begin{aligned} P' &= \$21{,}061.82(F/P, 6\%, 1) \\ &= \$22{,}325.53. \end{aligned}$$

For the loan to be retired with five equal installments, the deferred equal annual payment, A', will be

$$\begin{aligned} A' &= \$22{,}325.53(A/P, 6\%, 5) \\ &= \$5{,}300. \end{aligned}$$

By deferring the first payment for one year, you need to make an additional $300 in payments each year.

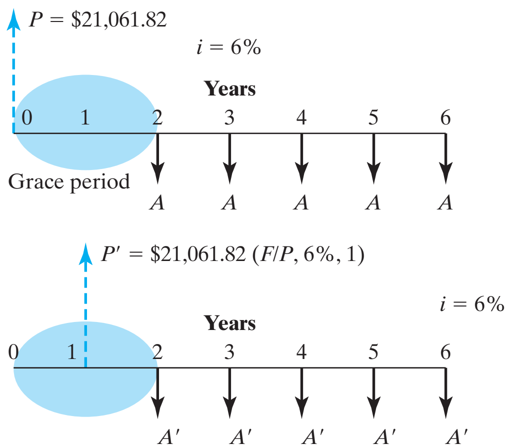


Figure 2.21 A deferred-loan cash flow diagram.

2.5.4 Present-Worth Factor: Find *P*, Given *A*, *i*, and *N*

What would you have to invest now in order to withdraw A dollars at the end of each of the next N periods? We now face just the opposite of the equal-payment capital-recovery factor situation: A is known, but P has to be determined. With the capital-recovery factor given in Eq. (2.9), solving for P gives us

$$P = A\left[\frac{(1+i)^N - 1}{i(1+i)^N}\right] = A(P/A, i, N). \tag{2.10}$$

The bracketed term is referred to as the **equal-payment-series present-worth factor** and is designated $(P/A, i, N)$.

EXAMPLE 2.14 Equal-Payment Series: Find *P*, Given *A*, *i*, and *N*

A truck driver from Georgia won the Powerball, a multistate lottery game similar to the one introduced in the chapter opening story. The winner could choose between a single lump sum of $116.5 million or a total of $195 million paid out over 20 annual installments (or $9.75 million per year and the first installment being paid out immediately). The truck driver opted for the lump sum. From a strictly economic standpoint, did he make the more lucrative choice?

DISSECTING THE PROBLEM

If the winner could invest his money at 8% annual interest, what is the lump-sum amount that would make him indifferent to each payment plan? (See Figure 2.22.)

Given: $i = 8\%$ per year, $A = \$9.75$ million, and $N = 19$ years.
Find: P.

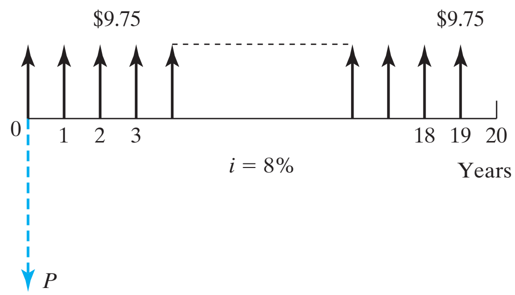


Figure 2.22 Cash flow diagram for the annual installment option.

METHODOLOGY

Basically you are finding the present worth of 19 installments of \$9.75 to be exactly "$P - \$9.75$."

SOLUTION

$$P = \$9.75 + \$9.75(P/A, 8\%, 19)$$

or

$$\begin{aligned} P - \$9.75 &= \$9.75(P/A, 8\%, 19) \\ &= \$103.39 \text{ million.} \\ &= \text{PV}\,(8\%, 20, 9.75, 1) = -\$103.39 \text{ million.} \end{aligned}$$

COMMENTS: Clearly, we can tell the winner that giving up \$9.75 million a year for 19 years to receive \$116.5 million today is a winning proposition if he can earn an 8% return on its investment. At this point, we may be interested in knowing the minimum rate of return at which accepting the \$116.5 million lump sum would make sense. Since we know that $P = \$116.5$ million, $N = 19$ and $A = \$9.75$ million, we solve for i.

$$\$116.5 = \$9.75 + \$9.75(P/A, i, 19)$$

$$(P/A, i, 19) = \frac{\$116.5 - \$9.75}{\$9.75} = 10.9487.$$

If you know the cash flows and the present value (or future value) of a cash flow stream, you can determine the interest rate. In this case, we are looking for the interest rate that causes the *P/A* factor to equal $(P/A, i, 19) = 10.9487$. Since we are dealing with an annuity due, we could proceed as follows:

- With a financial calculator, enter $N = 19$, $P = -106.75$, and $A = 9.75$, and then press the i key to find that $i = 6.2436\%$. For a typical financial calculator, the symbols such as PV, PMT, and FV are commonly adopted on its keypad. These symbols correspond to PV $= P$, PMT $= A$, and FV $= F$, respectively. Note that, just as with Excel, either P or A must be entered as a negative number to evaluate i correctly.

- If you want to use the compound-interest tables, look up the values of $(P/A, i, 19)$ that are closest to 10.9487. In the P/A column with $N = 19$ in Appendix B you will find that $(P/A, 6\%, 19) = 11.1581$ and $(P/A, 7\%, 19) = 10.3356$. Then, we may approximate i by a linear **interpolation** as shown in Table 2.5:

$$i \cong 7\% - (1\%)\frac{10.9487 - 10.3356}{11.1581 - 10.3356} = 7\% - (1\%)\frac{0.6131}{0.8225}$$
$$= 6.2546\%.$$

- In Excel, simply apply the following command to solve the unknown-interest-rate problem for an annuity:

$$= \text{RATE}(N,A,P,F,type,guess)$$
$$= \text{RATE}(19, 9.75, -106.75, 0, 0, 5\%)$$
$$= 6.2436\%.$$

As long as the lottery winner can find a financial investment that provides a rate of return higher than 6.2436%, his decision to go with the lump-sum payment option appears to be a sound one. Table 2.6 illustrates why the lump-sum amount (calculated at $116.5 million) deposited at an interest rate of 6.2436% is equivalent to receiving 20 installments of $9.75 million. In other words, if you deposit the lump-sum amount today in a bank account that pays 6.2436% annual interest, you will be able to withdraw $9.75 million annually over 19 years. At the end of that time, your bank account will show a zero balance.

TABLE 2.5 Finding Unknown *i* by a Linear Interpolation

	K	L	M	N	O	P
14						
15	**Interest Rate**	***(P/A, i, 19)***				
16						
17	6%	11.1581				
18						
19	*i*	10.9487				
20						
21	7%	10.3356				
22						
23	*a*	7% – *i*				
24	*b*	0.6131				
25	*c*	1%				
26	*d*	0.8225				
27						
28	*i*	6.2546%				
29						

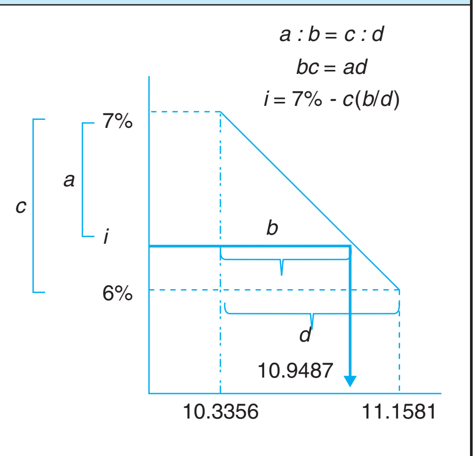
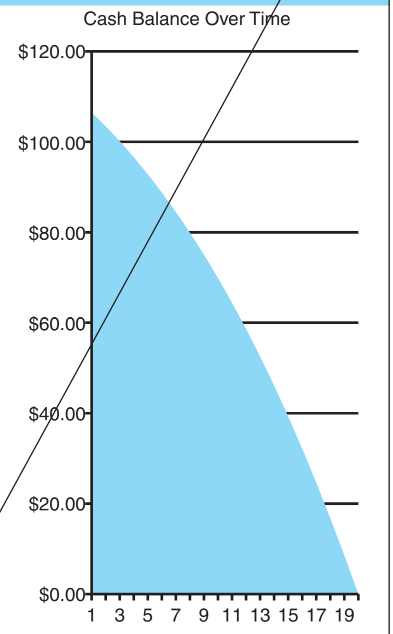


TABLE 2.6 **An Excel Worksheet to Illustrate the Process of Deleting $116.50 Million Initial Cash Balance over Time**

	A	B	C	D	E	F	G	H
1	Equal-Payment Cash Flows (Present Worth)							
2								
3	Inputs					Output		
4								
5	(*A*) Annuity			$ 9.75		(P) Present Worth		$116.50
6	(*i*) Interest Rate			6.2436%				
7	(*N*) Payment Period			19				
8								
9	Period (*n*)	Deposit	Withdrawal	Cash Balance				
10								
11	0	$116.50	($9.75)	$106.75				
12	1		($9.75)	$103.67				
13	2		($9.75)	$100.39				
14	3		($9.75)	$96.91				
15	4		($9.75)	$93.21				
16	5		($9.75)	$89.28				
17	6		($9.75)	$85.10				
18	7		($9.75)	$80.66				
19	8		($9.75)	$75.95				
20	9		($9.75)	$70.94				
21	10		($9.75)	$65.62				
22	11		($9.75)	$59.97				
23	12		($9.75)	$53.96				
24	13		($9.75)	$47.58				
25	14		($9.75)	$40.80				
26	15		($9.75)	$33.60				
27	16		($9.75)	$25.95				
28	17		($9.75)	$17.82				
29	18		($9.75)	$9.18				
30	19		($9.75)	$0.00				
31								



=PV($D$6,$D$7,-$D$5,0,0)+$D$5

EXAMPLE 2.15 Start Saving Money as Soon as Possible: Composite Series That Requires Both $(F/P, i, N)$ and $(F/A, i, N)$ Factors

Consider the following two savings plans that you think about starting at the age of 21:

- Option 1: Save $2,000 a year for 10 years. At the end of 10 years, make no further investments, but invest the amount accumulated at the end of every 10 years until you reach the age of 65. (Assume that the first deposit will be made when you are 22.)
- Option 2: Do nothing for the first 10 years. Start saving $2,000 a year every year thereafter until you reach the age of 65. (Assume that the first deposit will be made when you turn 32.)

If you were able to invest your money at 8% over the planning horizon, which plan would result in more money saved by the time you are 65? (See Figure 2.23.)

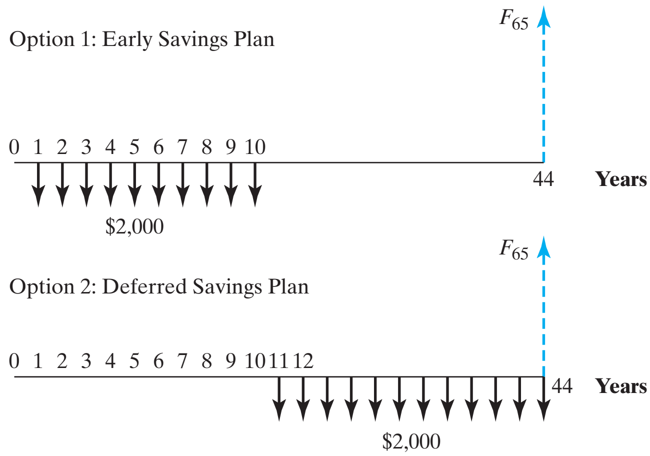


Figure 2.23 Cash flow diagram's for two different savings plans.

DISSECTING THE PROBLEM

Given: $i = 8\%$ per year, deposit scenarios shown in Figure 2.23.
Find: F when you are 65 for each savings plan.

METHODOLOGY

Option 1: Computing the Final Balance in Two Steps for Early Savings Plan - First find the balance at the end of year 10 using the F/A factor, then redeposit the entire balance until the end of year 44, using the F/P factor.

SOLUTION

First, compute the accumulated balance at the end of 10 years (when you are 31). Call this amount F_{31}.

$$F_{31} = \$2{,}000(F/A, 8\%, 10) = \$28{,}973.$$

Then use this amount to compute the result of reinvesting the entire balance for another 34 years. Call this final result F_{65}.

$$F_{65} = \$28{,}973(F/P, 8\%, 34) = \$396{,}646.$$

Option 2: Deferred Savings Plan - Just use the F/A factor to find the balance at the end of 44 years.

Since you have only 34 years to invest, the resulting balance will be

$$F_{65} = \$2{,}000(F/A, 8\%, 34) = \$317{,}253.$$

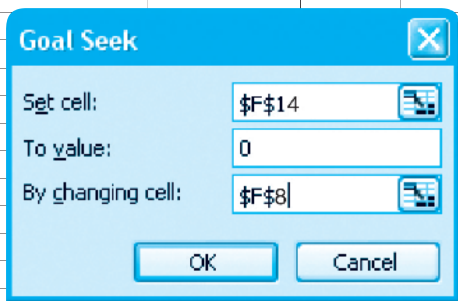
With the early savings plan, you will be able to save $79,393 more.

COMMENTS: In this example, the assumed interest rate is 8%. Certainly, we would be interested in knowing at what interest rate these two options would be equivalent. We can use Excel's **Goal Seek**[4] function to answer this question. As shown in Table 2.7, we enter the amount of deposits over 44 years in the second and third columns. Cells F10 and F12, respectively, display the future value of each option. Cell F14 contains the difference between the future values of the two options, or = F10 − F12. To begin using the Goal Seek function, first define Cell F14 as your *set cell*. Specify *set value* as 0, and set the *By changing cell* to be F8.

[4]Goal Seek is part of a suite of commands sometimes called *what-if analysis* tools in Microsoft Excel. When you know the desired result of a single *formula* but not the input value the formula needs to determine the result, you can use the Goal Seek feature by clicking **Goal Seek** on the **Tools** menu. When you are *goal seeking,* Excel varies the value in one specific cell until a formula that is dependent on that cell returns the result you want.

TABLE 2.7 Using the Goal Seek to Find the Break-Even Interest Rate to Make Two Options Equivalent

	A	B	C	D	E	F	G	H
1								
2	Year	Option 1	Option 2		Goal Seek Function Parameters			
3								
4	0							
5	1	-$2,000			=FV(F8,10,-2000)*(1+F8)^(34)			
6	2	-$2,000						
7	3	-$2,000						
8	4	-$2,000			Interest rate (%)	8%	By changing cell	
9	5	-$2,000						
10	6	-$2,000			FV of Option 1	$ 396,645.95		
11	7	-$2,000						
12	8	-$2,000			FV of Option 2	$317,253.34		
13	9	-$2,000						
14	10	-$2,000			Difference	$ 79,392.61	Set cell	
15	11		-$2,000					
16	12		-$2,000					
17	13		-$2,000		=FV(F8,34,-2000)		=$F$10-$F$12	
18	14		-$2,000					
19	15		-$2,000					
20	16		-$2,000					
21	17		-$2,000					
22	18		-$2,000					
23	19		-$2,000					
41	37		-$2,000					
42	38		-$2,000					
43	39		-$2,000					
44	40		-$2,000					
45	41		-$2,000					
46	42		-$2,000					
47	43		-$2,000					
48	44		-$2,000					
49								



Use the Goal Seek function to change the interest rate in Cell F8 incrementally until the value in Cell F14 equals 0. The break-even interest rate is 6.538%.

Note that in Table 2.7, rows 24–40 are hidden from display to reduce the size of the table. Clearly, it will be quite tedious if we resort on an analytical method. To do so, we may select the base period at $n = 10$ and find the equivalent value of a stream of payments at that base period:

$$\$2{,}000(F/A, i, 10) = \$2{,}000(P/A, i, 34)$$

and solve for unknown i by a linear interpolation method as shown in Table 2.5.

2.5.5 Present Value of Perpetuities

A perpetuity is a stream of cash flows that continues forever. A good example is a share of preferred stock that pays a fixed cash dividend each period (usually a quarter of a year) and never matures. An interesting feature of any perpetual annuity is that you cannot compute the future value of its cash flows because it is infinite. However, it has a well-defined present value. It appears counterintuitive that a series of cash flows that lasts forever can have a finite value today.

So what is the value of a perpetuity? We know how to calculate the present value for an equal-payment series with the finite stream as shown in Eq. (2.10). If we take a limit on this equation by letting $N \rightarrow \infty$, we can find the closed-form solution as follows:

$$P = \frac{A}{i}. \tag{2.11}$$

To illustrate, consider Example 2.16.

EXAMPLE 2.16 Present Value of Perpetuities: Find *P*, Given *A*, *i*, and *N*

Consider a perpetual stream of \$1,000 per year, as depicted in Figure 2.24. If the interest rate is 10% per year, how much is this perpetuity worth today?

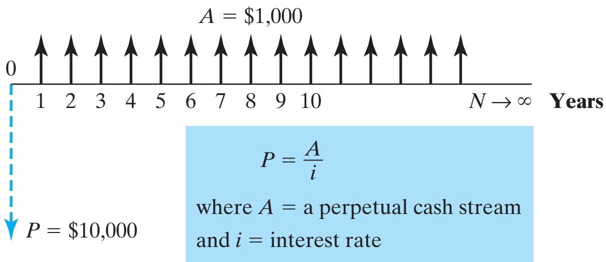


Figure 2.24 Present value of perpetual cash streams.

DISSECTING THE PROBLEM	
The question is basically: "How much do you need to deposit now in an account that pays 10% interest such that you will be able to withdraw \$1,000 a year forever?" (See Figure 2.24.)	**Given:** $i = 10\%$ per year, $A = \$1,000$, and $N = \infty$ years. **Find:** P.
METHODOLOGY	**SOLUTION**
Use Eq. (2.11).	$P = \frac{A}{i} = \frac{\$1,000}{0.10}$ $= \$10,000$

COMMENTS: If you put in \$10,000, then at the end of the first year, you would have \$11,000 in the account. You take out \$1,000, leaving \$10,000 for the next year. Clearly, if the interest rate stays at 10% per year, you will not eat into the principal, so you could continue to take out \$1,000 every year forever.

2.6 Dealing with Gradient Series

Engineers frequently encounter situations involving periodic payments that increase or decrease by a constant amount G or constant percentage (growth rate) from period to period. We can easily develop a series of interest formulas for this situation, but Excel will be a more practical tool to calculate equivalent values for these types of cash flows.

2.6.1 Handling Linear Gradient Series

Sometimes, cash flows will increase or decrease by a set amount, G, the gradient amount. This type of series is known as a **strict gradient series**, as seen in Figure 2.25. Note that each payment is $A_n = (n-1)G$. Note also that the series begins with a zero cash flow at the end of period 1. If $G > 0$, the series is referred to as an *increasing* gradient series. If $G < 0$, it is referred to as a *decreasing* gradient series.

Present-Worth Factor: Linear Gradient: Find *P*, Given *G*, *N*, and *i*

How much would you have to deposit now in order to withdraw the gradient amounts specified in Figure 2.26?

To find an expression for the present amount P, we apply the single-payment present-worth factor to each term of the series, obtaining

$$P = 0 + \frac{G}{(1+i)^2} + \frac{2G}{(1+i)^3} + \cdots + \frac{(N-1)G}{(1+i)^N},$$

or

$$P = \sum_{n=1}^{N} (n-1)G(1+i)^{-n}. \tag{2.12}$$

After some algebraic operations, we obtain

$$P = G\left[\frac{(1+i)^N - iN - 1}{i^2(1+i)^N}\right] = G(P/G, i, N). \tag{2.13}$$

The resulting factor in brackets is called the **gradient-series present-worth factor**[5] and is designated by the notation $(P/G, i, N)$.

[5] We can obtain an equal-payment series equivalent to the gradient series by multiplying Eq. (2.13) by Eq. (2.9). The resulting factor is referred to as the **gradient-to-equal payment series conversion factor** with designation of $(A/G, i, N)$, or $A = G\left[\frac{(1+i)^N - iN - 1}{i[(1+i)^N - 1]}\right]$.

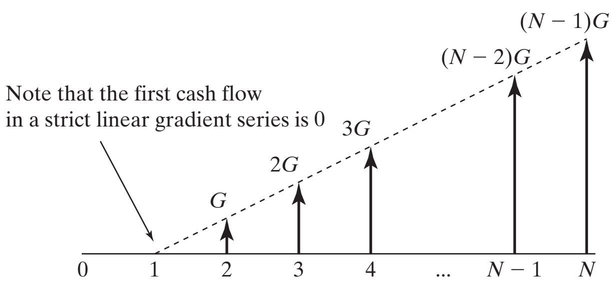


Figure 2.25 Cash flow diagram of a strict gradient series.

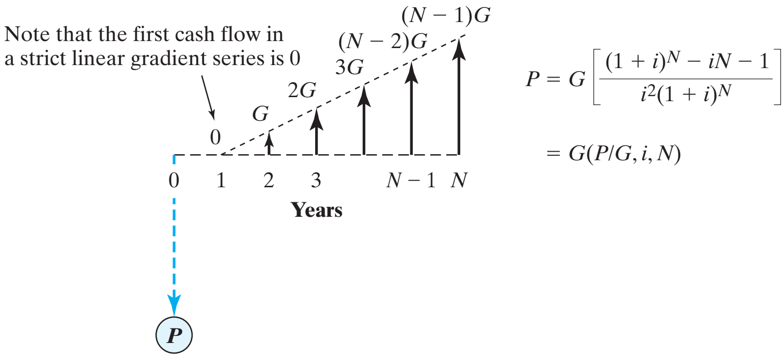


Figure 2.26 Cash flow diagram of a strict gradient series.

Linear Gradient Series as Composite Series

Unfortunately, the strict form of the increasing or decreasing gradient series does not correspond to the form that most engineering economic problems take. A typical problem involving a linear gradient series includes an initial payment during period 1 that increases by G during some number of interest periods, a situation illustrated in Figure 2.27. This configuration contrasts with the strict form illustrated in Figure 2.26, in which no payment is made during period 1 and the gradient is added to the previous payment beginning in period 2.

In order to use the strict gradient series to solve typical problems, we must view cash flows as shown in Figure 2.27 as a **composite series**, or a set of two cash flows, each corresponding to a form that we can recognize and easily solve: a uniform series of N payments of amount A_1 and a gradient series of increments of a constant amount G. The need to view cash flows that involve linear gradient series as composites of two series is very important for the solution of problems, as we shall now see.

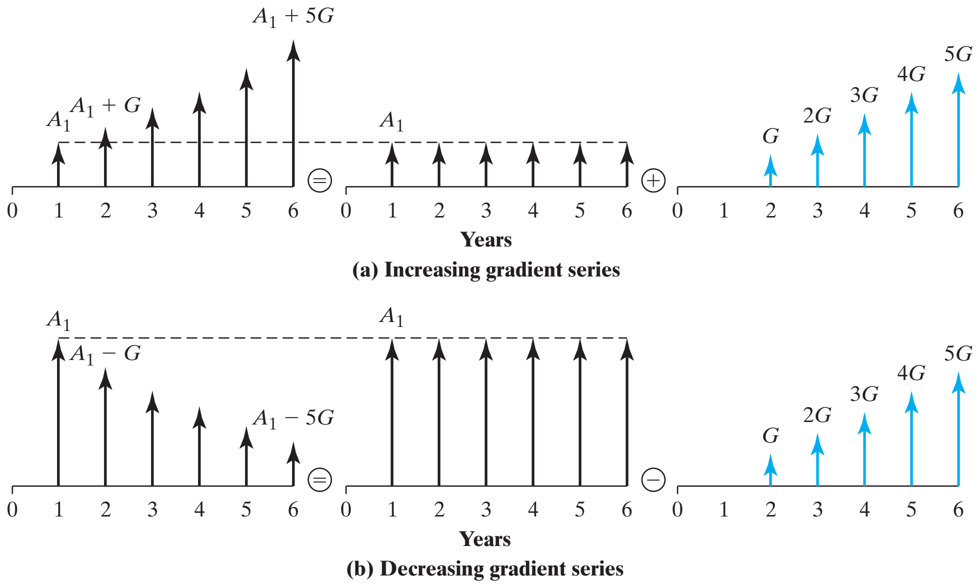


Figure 2.27 Two types of linear gradient series as composites of a uniform series of N payments of A_1 and a gradient series of increments of a constant amount G.

EXAMPLE 2.17 Creating a Graduated Loan Repayment with a Linear Gradient Series

You borrowed \$10,000 from a local bank with the agreement that you will pay back the loan according to a graduated payment plan. If your first payment is set at \$1,500, what would the remaining payment look like at a borrowing rate of 10% over five years?

DISSECTING THE PROBLEM

Basically, we are calculating the amount of gradient (G) such that the equivalent present worth of the gradient payment series will be exactly \$10,000 at an interest rate of 10%.

Given: $P = \$10{,}000$, $A_1 = \$1{,}500$, $N = 5$ years, and $i = 10\%$ per year. (See cash flow in Figure 2.28.)
Find: G.

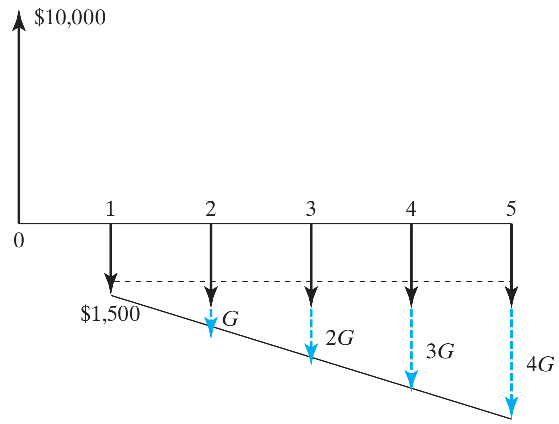


Figure 2.28 Cash flow diagram representing a graduated payment plan.

METHODOLOGY

Method 1: Calculate Present Value

SOLUTION

Since the loan payment series consists of two parts—(1) a \$1,500 equal-payment series and (2) a strict gradient series (unknown, yet to be determined)—we can calculate the present value of each series and equate them with \$10,000:

$$
\begin{aligned}
\$10{,}000 &= \$1{,}500(P/A, 10\%, 5) + G(P/G, 10\%, 5) \\
&= \$5{,}686.18 + 6.8618G \\
6.8618G &= \$4{,}313.82 \\
G &= \$628.67.
\end{aligned}
$$

Method 2: Use Excel's Goal Seek

Using the **Goal Seek** function in Excel, we could reproduce the same result, as in Table 2.8.

First, we designate cells B4 (interest rate) and B5 (borrowing amount) as input cells and cells E4 (gradient amount) and E5 (present worth of the repayment series) as output cells. In terms of repayment series, the equal-payment portion is entered in cells B12 through B16. The gradient portion is listed in cells C12 through C16. Cells D12 through D16 contain the repayment amount in each period over the life of the loan. Finally, cells E12 through E16 display the equivalent present worth of the loan repayment.

Second, to use the **Goal Seek** function, we will designate cell E4 as "By changing cell" and cell E5 as "Set cell" with its value at 10,000. The gradient amounts shown in cells C12 through C16 are obtained by varying the initial gradient amount in cell E4. The correct gradient amount (G) is determined at $628.67, which will cause the set cell value to be exactly $10,000.

TABLE 2.8 **An Excel Worksheet to Determine the Size of the Gradient Amount (Example 2.17)**

	A	B	C	D	E	F	G	H	I
1	**Example 2.16**								
2									
3	**Input:**			**Output:**					
4	Interest Rate (%)	10		Gradient (*G*)	$628.67	By changing cell			
5	Borrowing (*B*)	$10,000		Present Worth (*P*)	$10,000.00	Set cell			
6									
7									
8	Period		Repayment Series		Present Worth	=SUM(E12:E16)			
9	(*n*)	A1	G	Total					
10									
11	0								
12	1	$1,500.00	$0.00	$1,500.00	$1,363.64				
13	2	$1,500.00	$628.67	$2,128.67	$1,759.23		=D13*(1+$B$4)^(-A13)		
14	3	$1,500.00	$1,257.34	$2,757.34	$2,071.63				
15	4	$1,500.00	$1,886.01	$3,386.01	$2,312.69				
16	5	$1,500.00	$2,514.69	$4,014.69	$2,492.80				
17									
18									
19		=$E$4*(A16-1)		=B16+C16					
20									
21									

EXAMPLE 2.18 You Could Be the Next to Win $16 Million

Consider a jackpot winner of a SuperLotto Plus lottery in the State of California. The jackpot winner has 60 days to decide to take a one time lump-sum cash payment or 26 graduated annual installments. For an example, the prize money for the drawing on July 13, 2016, was $16 million! Before playing a SuperLotto Plus jackpot, you have a choice between getting the entire jackpot in 26 annual graduated payments or

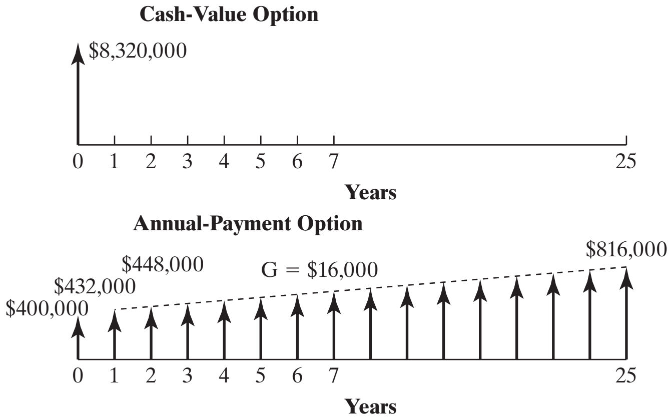


Figure 2.29 Cash flow diagram.

receiving one lump sum that will be less than the announced jackpot. (See Figure 2.29.) What would these choices come out to for an announced jackpot of $16 million?

- **Lump-sum cash-value option:** The winner would receive the present cash value of the announced jackpot in one lump sum. In this case, the winner would receive about 52%, or $8.32 million, in one lump sum (less tax withholdings). This cash value is based on average market costs determined by U.S. Treasury zero-coupon bonds with 5.35% annual yield.
- **Annual-payments option:** The winner would receive the jackpot in 26 graduated annual payments. In this case, the winner would receive $400,000 as the first payment (2.5% of the total jackpot amount) immediately. The second payment would be $432,000. Over the course of the next 25 years, these payments would gradually increase each year by $16,000 to a final payment of $816,000.

If the U.S. Treasury zero-coupon rate is reduced to 5% (instead of 5.35%) at the time of winning, what would be the equivalent lump-sum payment of the lottery?

DISSECTING THE PROBLEM

This problem is identical to asking what the equivalent present worth for this annual-payment series is at 5% interest.

Given: $A_0 = \$400{,}000$, $A_1 = \$432{,}000$, $G = \$16{,}000$ (from payment periods 2 to 25), $i = 5\%$ per year, and $N = 25$ years, as shown in Figure 2.29.
Find: P.

METHODOLOGY

Method 1: Compute the Equivalent Cash Value
Note that the annual payment series consists of a single payment at $n = 0$ and a linear gradient series.

SOLUTION

This method yields the following:

$$\begin{aligned} P &= \$400{,}000 + \$432{,}000(P/A, 5\%, 25) \\ &\quad + \$16{,}000(P/G, 5\%, 25) \\ &= \$8{,}636{,}224. \end{aligned}$$

The cash value now has increased from $8,320,000 to $8,636,224. In other words, if you check the “Cash Value” box on your lottery ticket and you win, you will receive the present cash value of the announced jackpot in one lump sum in the amount of $8,636,224.

Method 2: Excel Spreadsheet
Using Excel, we could reproduce the same result, as in Table 2.9.

In obtaining column C of the spreadsheet in Table 2.9, we could get the annual cash flow amount in each year by adding \$16,000 to the amount in the previous period. For example, Cell C14 is obtained by = C13 + \$C\$7. Then Cell C15 is obtained by = C14 + \$C\$7 and so on.

TABLE 2.9 **An Excel Worksheet to Calculate Equivalent Cash Value**

	A	B	C	D	E	F
1	**Example 2.18: Cash Value Calculation**					
2						
3						
4		**Winning Jackpot**	$ 16,000,000			
5		**Interest Rate (%)**	5.00%			
6		**Base Amount**	$ 432,000			
7		**Gradient Amount**	$ 16,000			
8						
9	**Payment**	**Annual Payment**	**Cash Flow Pattern**		**Discounting**	**Present**
10	**Period**	**before Taxes**	**Base**	**Graident**	**Factor (5%)**	**Cash Value**
11						
12	0	$ 400,000	$ 400,000		1.00000	$ 400,000
13	1	$ 432,000	$ 432,000		0.95238	$ 411,429
14	2	$ 448,000	$ 432,000	$ 16,000	0.90703	$ 406,349
15	3	$ 464,000	$ 432,000	$ 16,000	0.86384	$ 400,821
16	4	$ 480,000	$ 432,000	$ 16,000	0.82270	$ 394,897
17	5	$ 496,000	$ 432,000	$ 16,000	0.78353	$ 388,629
18	6	$ 512,000	$ 432,000	$ 16,000	0.74622	$ 382,062
19	7	$ 528,000	$ 432,000	$ 16,000	0.71068	$ 375,240
20	8	$ 544,000	$ 432,000	$ 16,000	0.67684	$ 368,201
21	9	$ 560,000	$ 432,000	$ 16,000	0.64461	$ 360,981
22	10	$ 576,000	$ 432,000	$ 16,000	0.61391	$ 353,614
23	11	$ 592,000	$ 432,000	$ 16,000	0.58468	$ 346,130
24	12	$ 608,000	$ 432,000	$ 16,000	0.55684	$ 338,557
25	13	$ 624,000	$ 432,000	$ 16,000	0.53032	$ 330,921
26	14	$ 640,000	$ 432,000	$ 16,000	0.50507	$ 323,243
27	15	$ 656,000	$ 432,000	$ 16,000	0.48102	$ 315,547
28	16	$ 672,000	$ 432,000	$ 16,000	0.45811	$ 307,851
29	17	$ 688,000	$ 432,000	$ 16,000	0.43630	$ 300,172
30	18	$ 704,000	$ 432,000	$ 16,000	0.41552	$ 292,527
31	19	$ 720,000	$ 432,000	$ 16,000	0.39573	$ 284,928
32	20	$ 736,000	$ 432,000	$ 16,000	0.37689	$ 277,391
33	21	$ 752,000	$ 432,000	$ 16,000	0.35894	$ 269,925
34	22	$ 768,000	$ 432,000	$ 16,000	0.34185	$ 262,541
35	23	$ 784,000	$ 432,000	$ 16,000	0.32557	$ 255,248
36	24	$ 800,000	$ 432,000	$ 16,000	0.31007	$ 248,054
37	25	$ 816,000	$ 432,000	$ 16,000	0.29530	$ 240,967
38						
39	**Total**	$ 16,000,000	**$ 11,200,000**	**$ 384,000**		**$ 8,636,224**
40						
41						
42						
43						
44						
45						

2.6.2 Handling Geometric Gradient Series

Another kind of gradient series is formed when the series in a cash flow is determined, not by some fixed amount like \$500 but by some *fixed rate* expressed as a percentage. Many engineering economic problems, particularly those relating to construction costs or maintenance costs, involve cash flows that increase or decrease over time by a constant percentage (**geometric**), a process that is called **compound growth**. Price changes caused by inflation are a good example of such a geometric series.

If we use g to designate the percentage change in a payment from one period to the next, the magnitude of the nth payment A_n is related to the first payment A_1 as follows:

$$A_n = A_1(1 + g)^{n-1}, n = 1, 2, \ldots, N. \tag{2.14}$$

The g can take either a positive or a negative sign, depending on the type of cash flow. If $g > 0$, the series will increase; if $g < 0$, the series will decrease. Figure 2.30 illustrates the cash flow diagram for this situation.

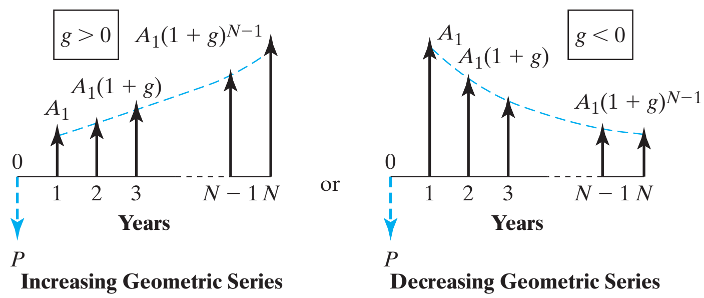
Present-Worth Factor: Find P, Given A_i, g, i, and N

Notice that the present worth P_n of any cash flow A_n at an interest rate i is

$$P_n = A_n(1 + i)^{-n} = A_1(1 + g)^{n-1}(1 + i)^{-n}. \tag{2.15}$$

To find an expression for the present amount for the entire series, P, we apply the **single-payment present-worth factor** to each term of the series:

$$P = \sum_{n=1}^{N} A_1(1 + g)^{n-1}(1 + i)^{-n}.$$



$$P = \begin{cases} A_1\left[\dfrac{1 - (1+g)^N(1+i)^{-N}}{i - g}\right], & \text{if } i \neq g; \\[2ex] A_1\dfrac{N}{(1+i)}, & \text{if } i = g \end{cases}$$

Figure 2.30 A geometrically increasing or decreasing gradient series.

The expression in Eq. (2.15) has the following closed expression:

$$P = \begin{cases} A_1\left[\dfrac{1-(1+g)^N(1+i)^{-N}}{i-g}\right] & \text{if } i \neq g, \\ A_1\left(\dfrac{N}{1+i}\right) & \text{if } i = g. \end{cases} \tag{2.16}$$

Or we can write

$$P = A_1(P/A_1, g, i, N).$$

The factor within brackets is called the **geometric-gradient-series present-worth factor** and is designated $(P/A_1, g, i, N)$. In the special case where $i = g$, Eq. (2.16) becomes $P = [A_1/(1+i)]N$.

There is an alternative way to derive the geometric-gradient-series present-worth factor. Bringing the constant term $A_1(1+g)^{-1}$ in Eq. (2.15) outside the summation yields

$$P = \frac{A_1}{(1+g)}\sum_{n=1}^{N}\left[\frac{1+g}{1+i}\right]^n = \frac{A_1}{(1+g)}\sum_{n=1}^{N}\frac{1}{\left[1+\dfrac{i-g}{1+g}\right]^n}. \tag{2.17}$$

If we define

$$g' = \frac{i-g}{1+g},$$

then we can rewrite P as follows:

$$P = \frac{A_1}{(1+g)}\sum_{n=1}^{N}(1+g')^{-n} = \frac{A_1}{(1+g)}(P/A, g', N). \tag{2.18}$$

We do not need another interest-factor table for this **geometric-gradient-series present-worth factor** as we can evaluate the factor with $(P/A, g', N)$. In the special case where $i = g$, Eq. (2.18) becomes $P = [A_1/(1+i)]N$, as $g' = 0$.

EXAMPLE 2.19 Required Cost-of-Living Adjustment Calculation

Suppose that your retirement benefits during your first year of retirement are \$50,000. Assume that this amount is just enough to meet your cost of living during the first year. However, your cost of living is expected to increase at an annual rate of 5% due to inflation. Suppose you do not expect to receive any cost-of-living adjustment in your retirement pension. Then, some of your future cost of living has to come from savings other than retirement pension. Determine the amount of shortcomings that must be funded from your other savings at the time of your retirement. Assume that your savings account earns 7% interest a year.

DISSECTING THE PROBLEM

Given: $A_1 = \$50{,}000$, $g = 5\%$, $i = 7\%$ per year, and $N = 25$ years, as shown in Figure 2.31.
Find: P.

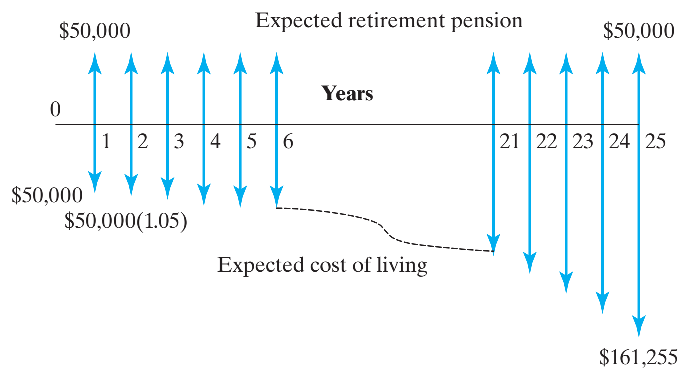


Figure 2.31 Cash flow diagram.

METHODOLOGY

Method 1: Calculate Present Worth

SOLUTION

- Find the equivalent amount of total benefits paid over 25 years:

$$\begin{aligned} P &= \$50{,}000(P/A, 7\%, 25) \\ &= \$582{,}679. \end{aligned}$$

- Find the equivalent amount of total cost of living with inflation. Using Eq. (2.16), the equivalent present worth of the cost of living is

$$\begin{aligned} P &= \$50{,}000(P/A_1, 5\%, 7\%, 25) \\ &= \$50{,}000\left[\frac{1-(1+0.05)^{25}(1+0.07)^{-25}}{0.07-0.05}\right] \\ &= \$50{,}000(18.8033) \\ &= \$940{,}167.22. \end{aligned}$$

Alternatively, to use Eq. (2.18), we need to find the value of g':

$$g' = \frac{0.07-0.05}{1+0.05} = 0.019048.$$

Then, using Eq. (2.18), we find P to be

$$\begin{aligned} P &= \frac{50{,}000}{1+0.05}(P/A, 1.9048\%, 25) \\ &= \$940{,}167. \end{aligned}$$

Method 2: Excel Worksheet

Or you can confirm the P value through an Excel worksheet as shown in Table 2.10. The required additional savings to meet the future increase in cost of living will be

$$\begin{aligned}\Delta P &= \$940{,}167 - \$582{,}679 \\ &= \$357{,}488.\end{aligned}$$

TABLE 2.10 An Excel Worksheet to Illustrate the Process of Calculating the Savings Required for Retirement

Geometric-Gradient Series (Present Worth)

Inputs

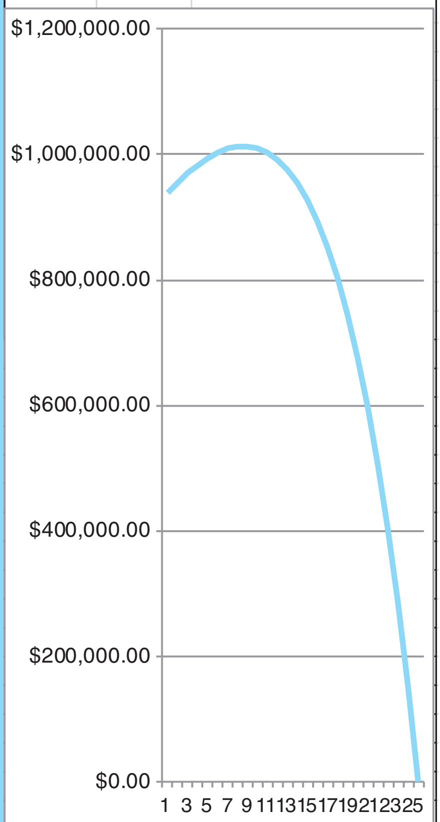
(A_1) Initial Amount	$ 50,000.00
(g) Growth Rate	5.0000%
(i) Interest Rate	7.0000%
(N) Payment Periods	25

Output

(P) Present Worth	$940,167.22

Period (n)	Withdrawals	Cash Balance
0		$940,167.22
1	$ 50,000.00	$ 955,978.92
2	$ 52,500.00	$ 970,397.45
3	$ 55,125.00	$ 983,200.27
4	$ 57,881.25	$ 994,143.04
5	$ 60,775.31	$ 1,002,957.74
6	$ 63,814.08	$ 1,009,350.70
7	$ 67,004.78	$ 1,013,000.47
8	$ 70,355.02	$ 1,013,555.48
9	$ 73,872.77	$ 1,010,631.59
10	$ 77,566.41	$ 1,003,809.39
11	$ 81,444.73	$ 992,631.32
12	$ 85,516.97	$ 976,598.54
13	$ 89,792.82	$ 955,167.63
14	$ 94,282.46	$ 927,746.90
15	$ 98,996.58	$ 893,692.61
16	$ 103,946.41	$ 852,304.68
17	$ 109,143.73	$ 802,822.28
18	$ 114,600.92	$ 744,418.92
19	$ 120,330.96	$ 676,197.28
20	$ 126,347.51	$ 597,183.58
21	$ 132,664.89	$ 506,321.55
22	$ 139,298.13	$ 402,465.93
23	$ 146,263.04	$ 284,375.51
24	$ 153,576.19	$ 150,705.60
25	$ 161,255.00	$ 0.00

Cash Balance Over Time



2.7 More on Equivalence Calculations

So far most of our equivalence calculations involve constant or systematic changes in cash flows. We calculate the equivalent present values or future values of these cash flows. However, many financial transactions contain several components of cash flows that do not exhibit an overall standard pattern that we have examined in earlier section. Consequently, it is necessary to expand our analysis to handle situations with mixed types of cash flows.

To illustrate, consider the cash flow stream shown in Figure 2.32. We want to compute the equivalent present worth for this mixed-payment series at an interest rate of 15%. Three different methods are presented.

Method 1: A "brute-force" approach is to multiply each payment by the appropriate $(P/F, 15\%, n)$ factors and then to sum these products to obtain the present worth of the cash flows, $543.72 in this case. Recall that this is exactly the same procedure we used to solve the category of problems called the uneven-payment series, which were described in Section 2.4. Figure 2.32 illustrates this computational method. Excel is the best tool for this type of calculation.

Method 2: We may group the cash flow components according to the type of cash flow pattern that they fit, such as the single payment, equal-payment series, and so forth as shown in Figure 2.33. Then the solution procedure involves the following steps:

- Group 1: Find the present worth of $50 due in year 1:

$$\$50(P/F, 15\%, 1) = \$43.48.$$

- Group 2: Find the equivalent worth of a $100 equal-payment series at year 1 (V_1), and then bring in this equivalent worth at year 0:

$$\underbrace{\$100(P/A, 15\%, 3)}_{V_1}(P/F, 15\%, 1) = \$198.54.$$

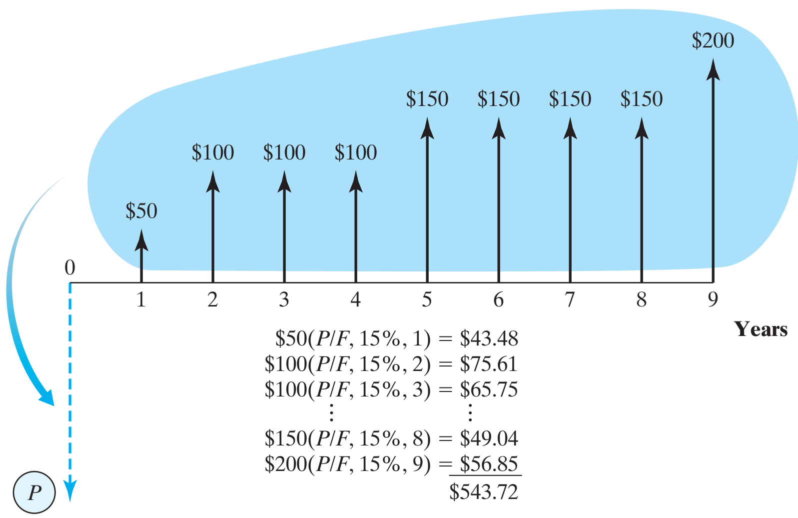


Figure 2.32 Method 1: A "brute-force" approach using P/F factors.

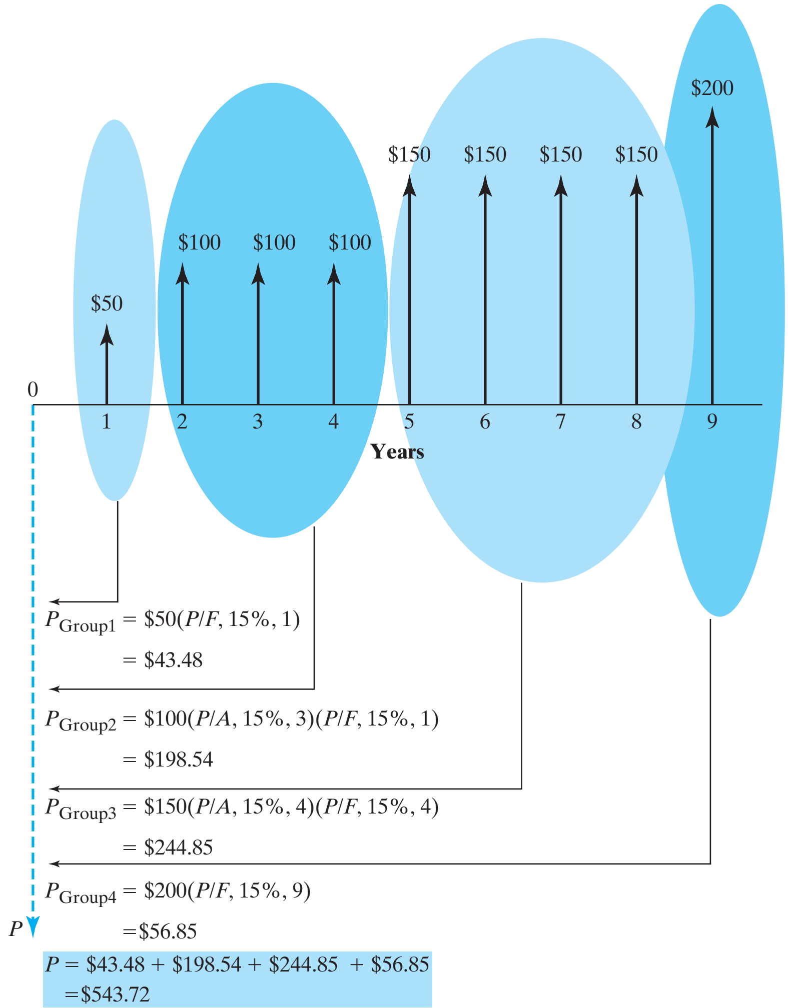


Figure 2.33 Method 2: Grouping approach using *P/F* and *P/A* factors.

- Group 3: Find the equivalent worth of a \$150 equal-payment series at year 4 (V_4), and then bring in this equivalent worth at year 0.

$$\underbrace{\$150(P/A, 15\%, 4)}_{V_4}(P/F, 15\%, 4) = \$244.85.$$

- Group 4: Find the equivalent present worth of the \$200 due in year 9:

$$\$200(P/F, 15\%, 9) = \$56.85.$$

- For the group total, sum the components:

$$\begin{aligned} P &= \$43.48 + \$198.54 + \$244.85 + \$56.85 \\ &= \$543.72. \end{aligned}$$

TABLE 2.11 **Method 3: Using Excel to Compute the Present Worth**

	A	B	C	D
1	**Composite Cash Flows**			
2				
3	**Input:**			
4	Interest Rate (%)	15%		=NPV($B$4,B12:B20)
5	**Output:**			
6	Present Worth (P)	$543.72		
7				
8	Period	Cash	Present	
9	(*n*)	Flow	Worth	
10				
11	0			
12	1	$50.00	$43.48	←=B12*(1+$B$4)^(-A12)
13	2	$100.00	$75.61	←=B13*(1+$B$4)^(-A13)
14	3	$100.00	$65.75	←=B14*(1+$B$4)^(-A14)
15	4	$100.00	$57.18	←=B15*(1+$B$4)^(-A15)
16	5	$150.00	$74.58	←=B16*(1+$B$4)^(-A16)
17	6	$150.00	$64.85	←=B17*(1+$B$4)^(-A17)
18	7	$150.00	$56.39	←=B18*(1+$B$4)^(-A18)
19	8	$150.00	$49.04	←=B19*(1+$B$4)^(-A19)
20	9	$200.00	$56.85	←=B20*(1+$B$4)^(-A20)
21				
22			$543.72	
23				=SUM(C12:C20)
24				
25				

A pictorial view of this computational process is also given in Figure 2.33. Either the brute-force method in Figure 2.32 or the method using both the $(P/A, i, n)$ and $(P/F, i, n)$ factors in Figure 2.33 can be used to solve problems of this type. Method 2 is much easier if the annuity component runs for many years, however. For example, this solution would be clearly superior for finding the equivalent present worth of a payment stream consisting of $50 in year 1, $200 in years 2 through 19, and $500 in year 20.

Method 3: With Excel, we may create a worksheet as shown in Table 2.11. In computing the present worth of the entire series, we may compute the equivalent present worth for each cash flow and sum them up. Or we could use the NPV function with the parameters, = NPV ($B$4, B12:B20), which will calculate the present worth of the entire series.

EXAMPLE 2.20 Retirement Planning: Composite Series That Requires Multiple Interest Factors

You want to supplement your retirement income through IRA contributions. You have 15 years left until retirement and you are going to make 15 equal annual deposits into your IRA until you retire with the first deposit being made at the end of year 1. You need to save enough so that you can make 10 annual withdrawals that will begin at the end of year 16. The first withdrawal will be $10,000, and each subsequent withdrawal will increase at a rate of 4% over the previous year's withdrawal in line with expected increase in cost-of-living. Your last withdrawal will be at the end of year 25. What is the amount of the equal annual deposit amount (C) for the first 15 years? Assume the interest rate is 8% compounded annually before and after you retire.

DISSECTING THE PROBLEM

Given: $i = 8\%$ per year, deposit and withdrawal series shown in Figure 2.34a.
Find: A

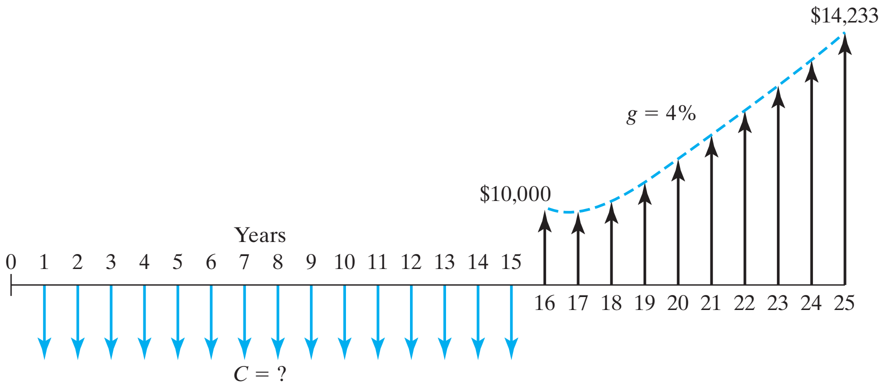


Figure 2.34a Establishing a retirement fund.

METHODOLOGY

Method 1: Establish the Economic Equivalence at Period 0

SOLUTION

First, compute the equivalent worth of the total deposit series at $n = 0$.

$$P_{\text{deposit}} = C(P/A, 8\%, 15) = 8.5595C.$$

Find the equivalent single lump-sum withdrawal now in two steps by finding the equivalent amount at $n = 15$ and then bring this value back to $n = 0$:

$$P_{\text{withdrawal}} = \overbrace{\$10{,}000\underbrace{(P/A_1, 4\%, 8\%, 10)}_{7.8590}}^{V_{15}}\underbrace{(P/F, 8\%, 15)}_{0.3152}$$

Since the two amounts are equivalent, by equating, we obtain C.

$$8.5595C = \$24{,}771.60$$
$$C = \$2{,}894.$$

Method 2: Establish the Economic Equivalence at $n = 15$.

First, compute the accumulated balance at the end of 15 years. Call this amount F_{15}.

$$F_{15} = C(F/A, 8\%, 15) = \$27.1521C.$$

Then find the equivalent lump-sum for the geometric withdrawal series at the end of retirement contribution, $n = 15$. Call this V_{15}:

$$V_{15} = \$10{,}000(P/A_1, 4\%, 8\%, 15) = \$78{,}590.$$

Since the two amounts must be the same, $F_{15} = V_{15}$, we obtain

$$27.1521C = \$78{,}590$$
$$C = \$2{,}894.$$

The computational steps are illustrated in Figure 2.34b.

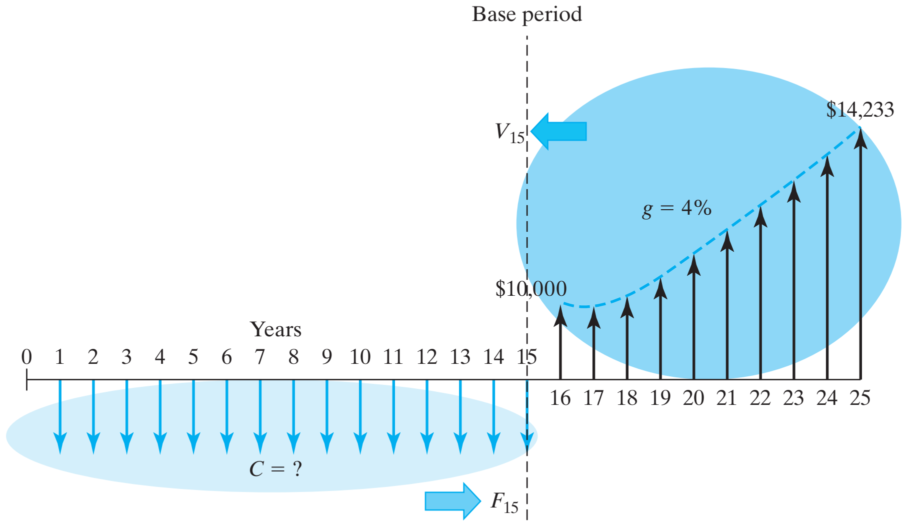


Figure 2.34b Establishing economic equivalence by selecting the base period at $n = 15$.

COMMENTS: In general, Method 2 is the more efficient way to obtain an equivalence solution to this type of decision problem as it requires fewer interest factors.

SUMMARY

- Money has a time value because it can earn more money over time. A number of terms involving the time value of money were introduced in this chapter:

  Interest *is the cost of money. More specifically, it is a cost to the borrower and an earning to the lender above and beyond the initial sum borrowed or loaned.*

  Interest rate *is a percentage periodically applied to a sum of money to determine the amount of interest to be added to that sum.*

  Simple interest *is the practice of charging an interest rate only to an initial sum.*

  Compound interest *is the practice of charging an interest rate to an initial sum and to any previously accumulated interest that has not been withdrawn from the initial sum. Compound interest is by far the most commonly used system in the real world.*

  Economic equivalence *exists between individual cash flows or between patterns of cash flows that have the same value. Even though the amounts and timing of the cash flows may differ, the appropriate interest rate makes them equal.*

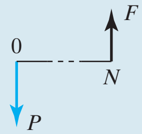
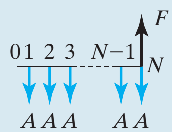
- The following compound-interest formula is perhaps the single most important equation in this text:

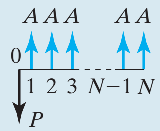
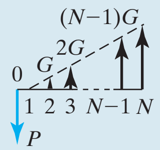
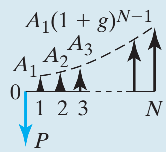
$$F = P(1 + i)^N.$$

In this formula, P is a present sum, i is the interest rate, N is the number of periods for which interest is compounded, and F is the resulting future sum. All other important interest formulas are derived from this one.

- **Cash flow diagrams** are visual representations of cash inflows and outflows along a time line. They are particularly useful for helping us detect which of the five patterns of cash flow a particular problem represents.
- The five patterns of cash flow are as follows:
  1. Single payment: A single present or future cash flow.
  2. Equal-payment series: A series of flows of equal amounts at regular intervals.
  3. Linear gradient series: A series of flows increasing or decreasing by a fixed amount at regular intervals. Excel is one of the most convenient tools to solve this type of cash flow series.
  4. Geometric-gradient series: A series of flows increasing or decreasing by a fixed percentage at regular intervals. Once again, this type of cash flow series is a good candidate for solution by using Excel.
  5. Uneven series: A series of flows exhibiting no overall pattern. However, patterns might be detected for portions of the series.
- **Cash flow patterns** are significant because they allow us to develop **interest formulas**, which streamline the solution of equivalence problems. Table 2.12 summarizes the interest formulas developed in this section and the cash flow situations in which they should be used. Recall that all the interest formulas developed in this section are applicable to situations only *where the interest (compounding) period is the same as the payment period* (e.g., annual compounding with annual payment). We also present some of Excel's useful financial commands in this table.

TABLE 2.12 **Summary of Compound-Interest Formulas**

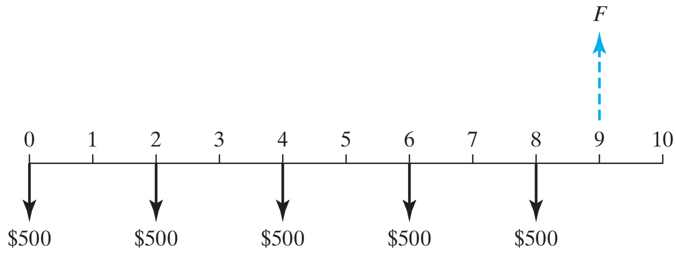
Flow Type	Factor Notation	Formula	Excel Command	Cash Flow Diagram
S I N	Compound Amount $(F/P, i, N)$	$F = P(1 + i)^N$	= FV (i%, N, 0, P)	
G L E	Present Worth $(P/F, i, N)$	$P = F(1 + i)^{-N}$	= PV (i%, N, 0, F)	
E Q U A L	Compound Amount $(F/A, i, N)$	$F = A\left[\frac{(1 + i)^N - 1}{i}\right]$	= FV (i%, N, A)	
P A Y M	Sinking Fund $(A/F, i, N)$	$A = F\left[\frac{i}{(1 + i)^N - 1}\right]$	= PMT (i%, N, 0, F)	

Flow Type	Factor Notation	Formula	Excel Command	Cash Flow Diagram
ENT SERIES	Present Worth $(P/A, i, N)$	$P = A\left[\frac{(1+i)^N - 1}{i(1+i)^N}\right]$	= PV (i%, N, A)	
	Capital Recovery $(A/P, i, N)$	$A = P\left[\frac{i(1+i)^N}{(1+i)^N - 1}\right]$	= PMT (i%, N, P)	
GRADIENT	Linear Gradient Present Worth $(P/G, i, N)$	$P = G\left[\frac{(1+i)^N - iN - 1}{i^2(1+i)^N}\right]$		
	Equal-Payment Conversion Factor $(A/G, i, N)$	$A = G\left[\frac{(1+i)^N - iN - 1}{i(1+i)^N - i}\right]$		
SERIES	Geometric-Gradient Present Worth $(P/A_1, g, i, N)$	$P = \begin{bmatrix} A_1\left[\frac{1-(1+g)^N(1+i)^{-N}}{i-g}\right] \\ A_1\left(\frac{N}{1+i}\right), (\text{if } i = g) \end{bmatrix}$		

SELF-TEST QUESTIONS

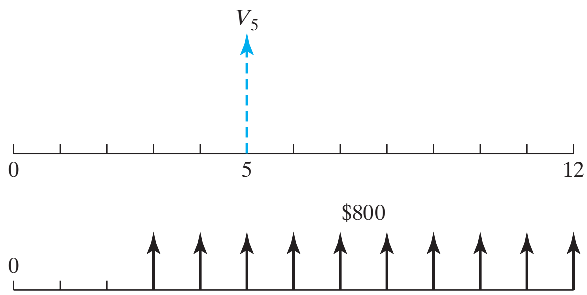
2s.1 You wish to have \$10,000 in an account 8 years from now. How much money must be deposited in the account now in order to have this amount if the account pays 10% compounded annually?
(a) \$3,855
(b) \$4,665
(c) \$5,403
(d) \$5,835

2s.2 Assume that \$500 is deposited today, two years from now, four years from now, six years from now, and eight years from now. At a 10% interest compounded annually, determine the future value at the end of year 9.

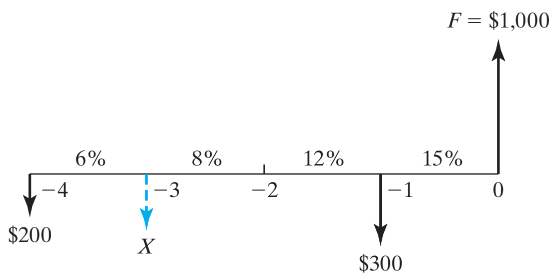


(a) $4,174
(b) $3,790
(c) $2,085
(d) $1,895

2s.3 What single payment at the end of year 5 is equivalent to an equal annual series of payments of $800 beginning at the end of year 3 and ending at the end of year 12? The interest rate is 8% compounded annually.
(a) $5,797
(b) $6,260
(c) $6,762
(d) $6,883

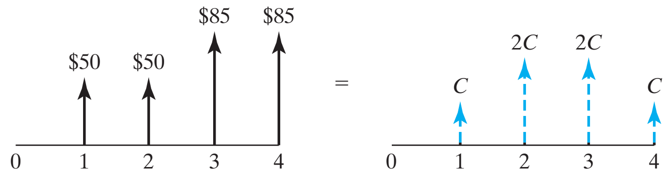


2s.4 Four years ago, you opened a mutual fund account and made three deposits ($200 four years ago, $X three years ago, and $300 a year ago) where you earned varying interest rates according to the following diagram. Today, your balance shows $1,000. Determine the amount of deposit that was made three years ago ($X). See the following figure.
(a) $215
(b) $237
(c) $244
(d) $259



2s.5 How much money should be deposited now in an account that pays 5% interest compounded annually in order to make ten equal annual withdrawals of $7,000?
(a) $54,052
(b) $55,402
(c) $58,916
(d) $60,594

2s.6 What value of C makes these two cash flows equivalent at an interest rate of 10%?

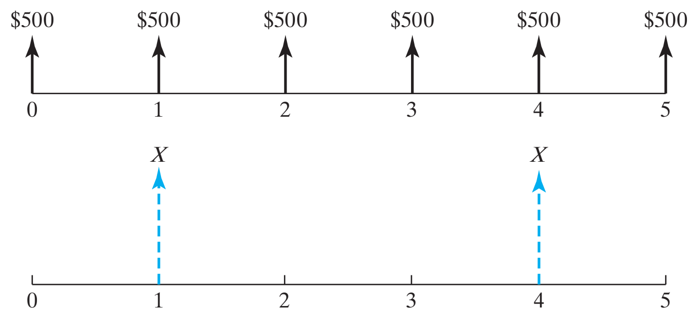


(a) $29.65
(b) $35.98
(c) $47.33
(d) $43.96

2s.7 Calculate the future worth of 12 annual $5,000 deposits in a savings account that earns 7% (compounded annually). Assume that all deposits are made at the *beginning* of each year.
(a) $126,005
(b) $111,529
(c) $95,703
(d) $92,037

2s.8 You borrow $34,000 from a bank to be repaid in three equal annual installments at 11% interest compounded annually. What is the portion of interest payment for the second annual payment?
(a) $2,621
(b) $2,511
(c) $1,980
(d) $1,521

2s.9 The following two cash flows are said to be economically equivalent at 10% interest. Determine the value of X for the second cash flow series.

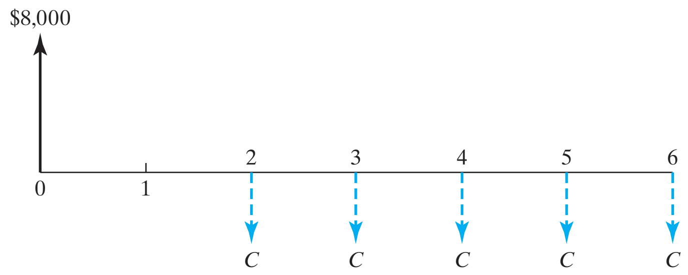


(a) $X = \$1,505$
(b) $X = \$1,500$

(c) $X = \$1{,}197$
(d) $X = \$1{,}192$

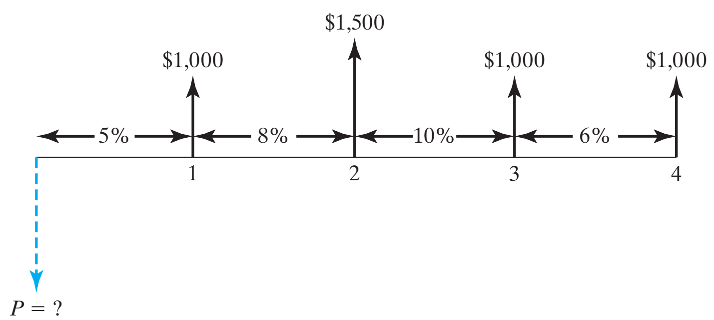
2s.10 What is the amount of five equal annual deposits that can provide five annual withdrawals, where a first withdrawal of \$1,000 is made at the end of year 6 and subsequent withdrawals increase at the rate of 10% year over the previous year's if the interest rate is 10% compounded annually?
(a) \$745
(b) \$789
(c) \$1,000
(d) \$1,563

2s.11 You borrowed \$8,000 to finance your educational expenses at the beginning of your junior year of college at an interest rate of 5% compounded annually. You are required to pay off the loan with five equal annual installments, but the first payment will be deferred until your graduation. Determine the value of C, the amount of annual payments.



(a) $C = \$950$
(b) $C = \$1{,}256$
(c) $C = \$1{,}421$
(d) $C = \$1{,}940$

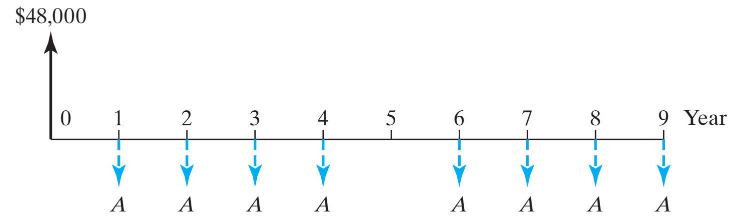
2s.12 Consider the following cash flow series at varying interest rates. What is the equivalent present worth of the cash flow series?



(a) $P = \$5{,}068$
(b) $P = \$4{,}442$
(c) $P = \$4{,}077$
(d) $P = \$3{,}833$

2s.13 At what annual interest rate will \$1,000 invested today be worth \$2,000 in 10 years?
(a) 6.5%
(b) 7.2%
(c) 9.3%
(d) 5.8%

2s.14 If you borrow \$48,000 at an interest rate of 12% compounded annually with the repayment schedule as shown, what is the amount A? (Note that there is a missing payment in year 5.)

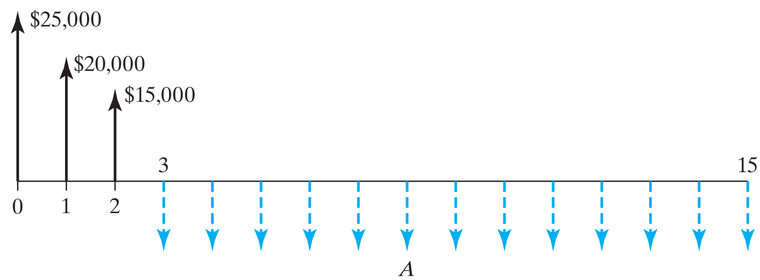


(a) $A = \$8{,}967$
(b) $A = \$10{,}082$
(c) $A = \$10{,}966$
(d) $A = \$12{,}820$

2s.15 You plan to make 20 annual deposits in a savings account that pays 4% interest compounded annually. If the first deposit—of \$5,000—is made at the end of the first year and each subsequent deposit is \$1,000 more than the previous one, the value of the account at the end of 20 years will be nearly:
(a) \$263,265
(b) \$268,120
(c) \$295,833
(d) \$393,338

2s.16 How long would it take an investment to *triple* if the interest rate is 8% compounded annually?
(a) 9 years
(b) 12 years
(c) 14 years
(d) 16 years

2s.17 If you borrow \$25,000 at an interest rate of 12% compounded annually with the following repayment schedule, what is the required amount A?

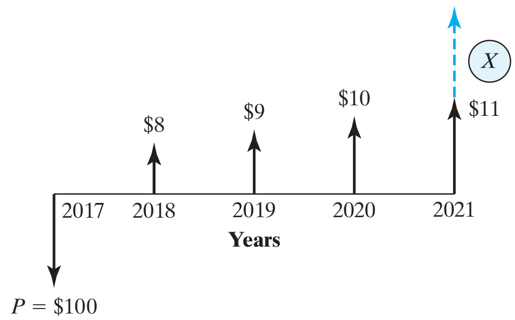


(a) $A = \$5{,}576$
(b) $A = \$8{,}883$
(c) $A = \$10{,}706$
(d) $A = \$12{,}014$

2s.18 If a sum of \$2,000 is deposited in a savings account at the beginning of each year for 10 years (a total number of deposits = 10) and the account draws interest at 9% compounded annually, the value of the account at the end of 10 years will be nearly:
(a) \$21,400
(b) \$33,121
(c) \$40,996
(d) \$43,865

2s.19 You are preparing to buy a vacation home five years from now. The home will cost \$125,000 at that time. You plan on saving three deposits at an interest rate of 10%:
Deposit 1: Deposit \$30,000 today.
Deposit 2: Deposit \$24,000 two years from now.
Deposit 3: Deposit \$$X$ three years from now.
How much do you need to deposit in year 3 to ensure that you have the necessary funds to buy the vacation home at the end of year 5?
(a) \$36,976
(b) \$42,586
(c) \$43,030
(d) \$44,115

2s.20 The accompanying diagram shows the anticipated cash dividends for Delta Electronics over the next four years. John is interested in buying some shares of this stock for a total of \$100 and will hold them for four years. If John's interest rate is known to be 6% compounded annually, what would be the desired (minimum) total selling price for the set of shares at the end of the fourth year?

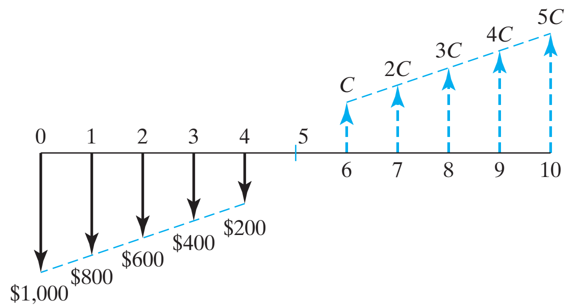


(a) \$93.67
(b) \$66.35
(c) \$86.13
(d) \$90.11

2s.21 You are planning to contribute \$6,500 a year to a mutual fund that earns an average of 8% per year. If you continue to contribute for the next 25 years, how much would you have in your account?
(a) \$412,725
(b) \$432,226

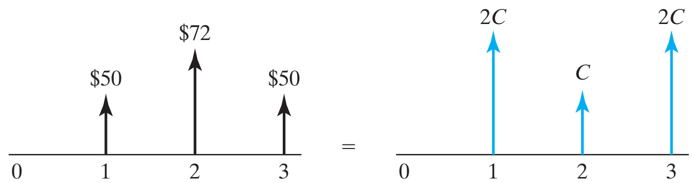
(c) \$457,881
(d) \$475,188

2s.22 Consider the cash flow series given in the accompanying table. What value of C makes the deposit series equivalent to the withdrawal series at an interest rate of 9% compounded annually?



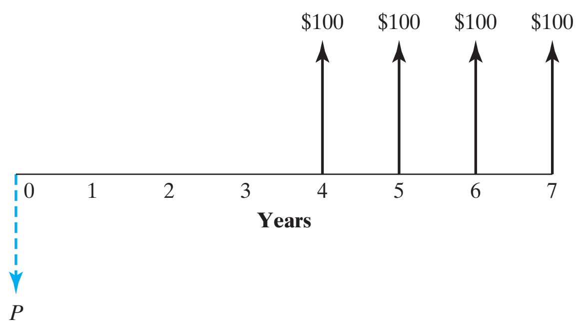
(a) \$334.50
(b) \$376.17
(c) \$390.15
(d) \$409.65

2s.23 What value of C makes the two cash flows equal? Assume $i = 10\%$.

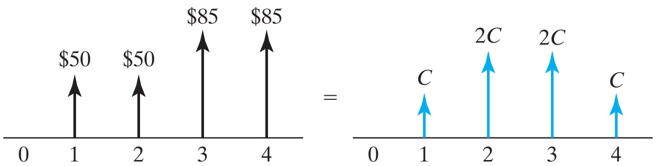


(a) \$34
(b) \$30
(c) \$43
(d) \$39

2s.24 In computing the equivalent present worth of the cash flow series at period 0, which of the following expressions is *incorrect*?
(a) $P = \$100(P/A, i, 4)(P/F, i, 4)$.
(b) $P = \$100(F/A, i, 4)(P/F, i, 7)$.
(c) $P = \$100(P/A, i, 7) - \$100(P/A, i, 3)$.
(d) $P = \$100[(P/F, i, 4) + (P/F, i, 5) + (P/F, i, 6) + (P/F, i, 7)]$.

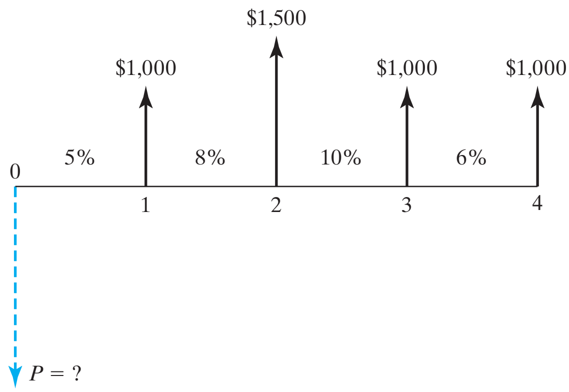


2s.25 State the value of C that makes the following two cash flow transactions economically equivalent at an interest rate of 10%:



(a) \$38.76
(b) \$40.38
(c) \$52.25
(d) \$43.96

2s.26 Consider the following cash flow series at varying interest rates. What is the equivalent present worth of the cash flow series?



(a) \$3,833
(b) \$2,987
(c) \$4,021
(d) \$3,985

PROBLEMS

Methods of Calculating Interest

2.1 What is the amount of interest earned on \$5,000 for eight years at 10% simple interest per year?

2.2 If you open a savings account that earns 7.5% simple interest per year, what is the minimum number of years you must wait to double your balance? Suppose you open another account that earns 7% interest compounded yearly. How many years will it take now to double your balance?

2.3 Compare the interest earned by \$10,000 for five years at 10% simple interest with that earned by the same amount for five years at 10% compounded annually.

2.4 You are considering investing \$4,500 at an interest rate of 8.5% compounded annually for five years or investing the \$4,500 at 9% per year simple interest for five years. Which option is better?

2.5 You are considering investing $1,000 at an interest rate of 6.5% compounded annually for five years or investing the $1,000 at 6.8% per year simple interest for five years. Which option is better?

2.6 You are about to borrow $15,000 from a bank at an interest rate of 8% compounded annually. You are required to make three equal annual repayments in the amount of $5,820.50 per year, with the first repayment occurring at the end of year 1. Show the interest payment and principal payment in each year.

The Concept of Equivalence

2.7 Suppose you have the alternative of receiving either $36,000 at the end of nine years or P dollars today. Currently, you have no need for money, so you could deposit the P dollars in a bank that pays 4% interest. What value of P would make you indifferent in your choice between P dollars today and the promise of $36,000 at the end of nine years?

2.8 Suppose that you obtain a personal loan from your uncle in the amount of $26,000 (now) to be repaid in three years to cover some of your college expenses. If your uncle usually earns 10% interest (annually) on his money, which is invested in various sources, what minimum lump-sum payment three years from now would make your uncle happy?

2.9 If you deposited $200 now $(n = 0)$ and $600 four years from now $(n = 4)$ in a savings account that pays 5% annual interest, how much would you have at the end of year 14?

Single Payments (Use of *F*/*P* or *P*/*F* Factors)

2.10 The average price of a new home is $260,000. If new home prices are increasing at a rate of 5% per year, how much will a new home cost in 7 years?

2.11 You are interested in buying a piece of real property that could be worth $400,000 in 8 years. Assuming that your money is worth 10%, how much would you be willing to pay for the property?

2.12 If the interest rate is 5.1%, what is the equivalent three-year discount rate?

2.13 What will be the amount accumulated by each of these present investments?
(a) $5,000 in 5 years at 7% compounded annually.
(b) $7,250 in 15 years at 9% compounded annually.
(c) $9,000 in 33 years at 6% compounded annually.
(d) $12,000 in 8 years at 5.5% compounded annually.

2.14 What is the present worth of these future payments?
(a) $25,500 eight years from now at 12% compounded annually.
(b) $58,000 twelve years from now at 4% compounded annually.
(c) $25,000 nine years from now at 6% compounded annually.
(d) $35,000 four years from now at 9% compounded annually.

2.15 For an interest rate of 12% compounded annually, determine the following:
(a) How much can be lent now if $15,000 will be repaid at the end of eight years?
(b) How much will be required in six years to repay a loan of $35,000 received now?

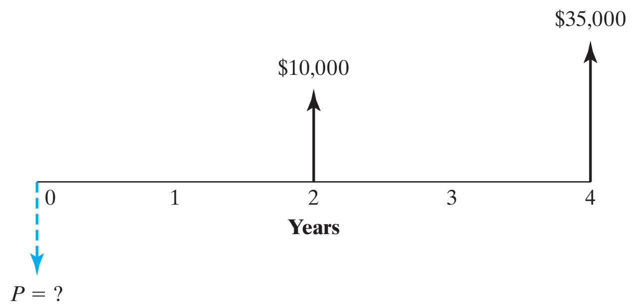
2.16 How many years will it take an investment to triple if the interest rate is 8% compounded annually?

2.17 How many years will it take to double your investment of $2,000 if it has an interest rate of 6% compounded annually?

2.18 You bought 300 shares of Facebook (FB) stock at $54,000 on November 9, 2017. Your intention is to keep the stock until it doubles in value. If you expect 8% annual growth for FB stock, how many years do you anticipate holding onto the stock? Compare your answer with the solution obtained by the Rule of 72 (discussed in Example 2.7).

2.19 In 1626 the Indians sold Manhattan Island to Peter Minuit of the Dutch West Company for $24. If they saved just $1 from the proceeds in a bank account that paid an 8% annual interest, how much would their descendents have in 2020?

2.20 If you want to withdraw $10,000 at the end of two years and $35,000 at the end of four years, how much should you deposit now into an account that pays 7% interest compounded annually? See the accompanying cash flow diagram.



2.21 You are interested in buying a piece of real estate property that could be worth $680,000 in 10 years. If your earning interest rate is 7%, how much would you be willing to pay for this property now?

2.22 John and Susan just opened savings accounts at two different banks. Each deposited $1,000. John's bank pays simple interest at an annual rate of 10%, whereas Susan's bank pays compound interest at an annual rate of 9.5%. No principal or interest will be taken out of the accounts for a period of five years. At the end of five years, whose balance will be higher and by how much (to the nearest dollar)?

Uneven-Payment Series

2.23 A project is expected to generate a cash flow of $2,000 in year 1, $800 in year 2, and $1,000 in year 3. At an interest rate of 10%, what is the maximum amount that you could invest in the project at year 0?

2.24 If you desire to withdraw the amounts given in Table P2.24 over the next five years from a savings account that earns 7% interest compounded annually, how much do you need to deposit now?

TABLE P2.24

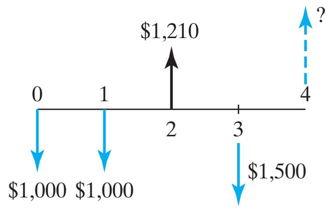
N	Amount
2	$15,000
3	$23,000
4	$36,000
5	$48,000

2.25 If \$2,000 is invested now, \$2,500 two years from now, and \$3,000 four years from now at an interest rate of 6% compounded annually, what will be the total amount in 10 years?

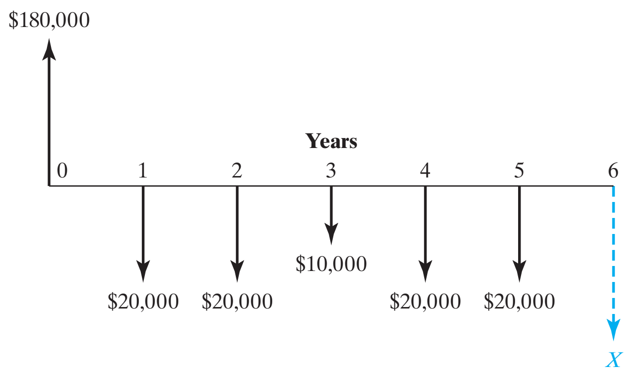
2.26 A local newspaper headline blared, "Bo Smith Signed for \$30 Million." A reading of the article revealed that on April 1, 2014, Bo Smith, the former record-breaking running back from Football University, signed a \$30 million package with the Dallas Rangers. The terms of the contract were \$3 million immediately, \$2.4 million per year for the first five years (with the first payment after one year) and \$3 million per year for the next five years (with the first payment at year 6). If Bo's interest rate is 8% per year, what would his contract be worth at the time he signs it?

2.27 How much invested now at 7% would be just sufficient to provide three payments, with the first payment in the amount of \$12,000 occurring three years hence, then \$8,500 five years hence, and finally \$3,000 eight years hence?

2.28 Consider the following sequence of deposits and withdrawals over a period of four years. If you earn 10% interest on your deposits, how much will you able to withdraw at the end of four years?



2.29 A company borrowed \$180,000 at an interest rate of 9% compounded annually over six years. The loan will be repaid in installments at the end of each year according to the accompanying repayment schedule. What will be the size of the last payment (X) that will pay off the loan?



2.30 You are considering purchasing a machine that is expected to produce the following cash flows: \$60,000 in year 1, \$77,000 in year 2, \$65,000 in year 3, \$57,000 in year 4, and \$45,000 in year 5. If your interest rate is 14%, what would be your maximum offer (purchase price) on this machine?

Equal-Payment Series

2.31 What would be the future worth of a series of equal year-end deposits of $4,500 for 10 years in a savings account that earns 3% annual interest if the following were true?

(a) All deposits are made at the *end* of each year?

(b) All deposits are made at the *beginning* of each year?

2.32 What is the future worth of the following series of payments?

(a) $8,000 at the end of each year for five years at 11.75% compounded annually.

(b) $2,000 at the end of each year for 12 years at 4.25% compounded annually.

(c) $7,000 at the end of each year for 20 years at 6.45% compounded annually.

(d) $4,000 at the end of each year for 12 years at 7.75% compounded annually.

2.33 What equal annual series of payments must be paid into a sinking fund to accumulate the following amounts?

(a) $54,000 in 8 years at 11% compounded annually.

(b) $24,000 in 20 years at 2% compounded annually.

(c) $30,000 in 4 years at 10% compounded annually.

(d) $415,000 in 25 years at 9% compounded annually.

2.34 Your company wants to set aside a fixed amount every year as a sinking fund to replace a piece of industrial equipment costing $185,000 at the end of six years from now. The sinking fund is expected to earn 6% interest. How much must your company deposit each year to meet this goal?

2.35 You want to save money from your business operation to replace a truck that has been used in delivery. The truck will be replaced after seven years from now and the replacement cost will be about $62,000. If you earn 5% interest on your savings, how much must you deposit at the end of each year to meet your needs?

2.36 Part of the income that a machine generates is put into a sinking fund to replace the machine when it wears out. If $2,600 is deposited annually at 8% interest, how many years must the machine be kept before a new machine costing $33,000 can be purchased?

2.37 A no-load (commission-free) mutual fund has grown at a rate of 14% compounded annually since its beginning. If it is anticipated that it will continue to grow at that rate, how much must be invested every year so that $200,000 will be accumulated at the end of twelve years?

2.38 You open a bank account, making a deposit of $1,000 now and deposits of $500 every other year. What is the total balance at the end of eight years from now if your deposits earn 5% interest compounded annually?

2.39 What equal annual payment series is required to repay the following present amounts?

(a) $15,000 in six years at 3.5% interest compounded annually.

(b) $7,500 in seven years at 7.5% interest compounded annually.

(c) $2,500 in five years at 5.25% interest compounded annually.

(d) $12,000 in 15 years at 6.25% interest compounded annually.

2.40 You have borrowed $20,000 at an interest rate of 10% compounded annually. Equal payments will be made over a three-year period with each payment made at the end of the corresponding year. What is the amount of the annual payment? What is the interest payment for the second year?

2.41 What is the present worth of the following series of payments?
(a) \$1,000 at the end of each year for eight years at 7.2% compounded annually.
(b) \$4,500 at the end of each year for 12 years at 9.5% compounded annually.
(c) \$1,900 at the end of each year for 13 years at 8.25% compounded annually.
(d) \$19,300 at the end of each year for eight years at 7.75% compounded annually.

2.42 From the interest tables in Appendix B, determine the values of the following factors by interpolation and compare your results with those obtained from evaluating the *A/P* and *P/A* interest formulas.
(a) The capital-recovery factor for 38 periods at 6.25% interest.
(b) The equal-payment series present-worth factor for 85 periods at 9.25% interest.

2.43 If \$1,600 is deposited in a savings account at the *beginning* of each year for 12 years and the account earns 8% interest compounded annually, what will be the balance on the account at the end of the 12 years (F)?

2.44 You have borrowed \$50,000 at an interest rate of 12%. Equal payments will be made over a three-year period. (The first payment will be made at the end of the first year.) What will the annual payment be, and what will the interest payment be for the second year?

2.45 You are considering buying a piece of industrial equipment to automate a part of your production process. This automation will save labor costs by as much as \$35,000 per year over 10 years. The equipment costs \$200,000. Should you purchase the equipment if your interest rate is 12%?

2.46 An investment costs \$5,620 and pays \$320 in perpetuity. What is the interest earned on this investment?

2.47 At an interest rate of 8%, what is the present value of an asset that produces \$1,000 a year in perpetuity?

Linear Gradient Series

2.48 An individual deposits an annual bonus into a savings account that pays 5% interest compounded annually. The size of the bonus increases by \$4,000 each year, and the initial bonus amount was \$25,000. Determine how much will be in the account immediately after the seventh deposit.

2.49 Five annual deposits in the amounts of \$16,000, \$14,000, \$12,000, \$10,000, and \$8,000, in that order, are made into a fund that pays interest at a rate of 11% compounded annually. Determine the amount in the fund immediately after the fifth deposit.

2.50 Compute the value of P in the accompanying cash flow diagram, assuming that $i = 8\%$.

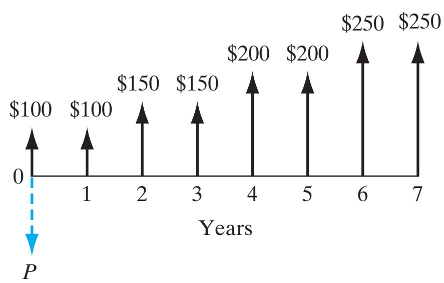


Figure P2.50

2.51 What is the equal-payment series for 15 years that is equivalent to a payment series of $30,000 at the end of the first year, decreasing by $2,000 each year over 15 years? Interest is 5% compounded annually.

2.52 How much do you have to deposit now in your savings account that earns a 6% annual interest if you want to withdraw the annual payment series in the figure below?

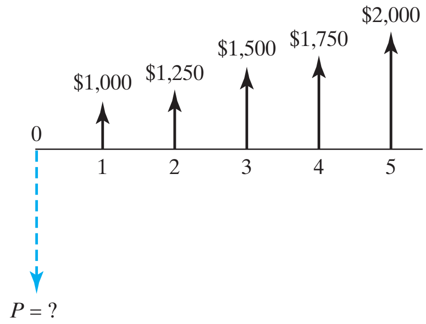


Figure P2.52

Geometric-Gradient Series

2.53 Joe's starting salary as a mechanical engineer is around $80,000. Joe is planning to place a total of 10% of his salary each year in the mutual fund. Joe expects a 5% salary increase each year for the next 30 years of employment. If the mutual fund will average 7% annual return over the course of his career, what can Joe expect at retirement?

2.54 Suppose that an oil well is expected to produce 100,000 barrels of oil during its first year in production. However, its subsequent production (yield) is expected to decrease by 10% over the previous year's production. The oil well has a proven reserve of 1,000,000 barrels.

(a) Suppose that the price of oil is expected to be $60 per barrel for the next several years. What would be the present worth of the anticipated revenue stream at an interest rate of 12% compounded annually over the next seven years?

(b) Suppose that the price of oil is expected to start at $60 per barrel during the first year, but to increase at the rate of 5% over the previous year's price. What would be the present worth of the anticipated revenue stream at an interest rate of 12% compounded annually over the next seven years?

(c) Consider part (b) again. After three years' production, you decide to sell the oil well. What would be a fair price?

2.55 A city engineer has estimated the annual toll revenues from a newly proposed highway construction over 20 years as follows:

$$A_n = (2{,}000{,}000)(n)(1.06)^{n-1}$$
$$n = 1, 2, \ldots, 20$$

To validate the bond, the engineer was asked to present the estimated total present value of toll revenue at an interest rate of 6%. Assuming annual compounding, find the present value of the estimated toll revenue.

2.56 What is the amount of 10 equal annual deposits that can provide five annual withdrawals? A first withdrawal of $15,000 is made at the end of year 11 and subsequent withdrawals increase at the rate of 8% per year over the previous year's withdrawal. Determine the amounts from the following rates.

(a) The interest rate is 9% compounded annually.

(b) The interest rate is 6% compounded annually.

2.57 You are planning to save $1 million for retirement over the next 30 years.

(a) If you are earning interest at the rate of 6% and you live 20 years after retirement, what annual level of living expenses will those savings support?

(b) Suppose your retirement living expenses will increase at an annual rate of 3% due to inflation. Determine the annual spending plan in line with your inflation.

Equivalence Calculations

2.58 Find the present worth of the cash receipts where $i = 12\%$ compounded annually with only four interest factors.

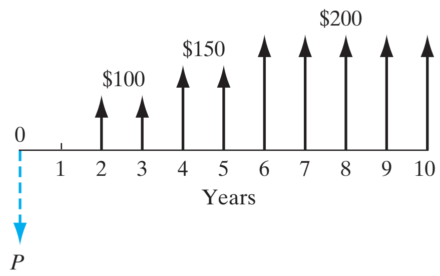


Figure P2.58

2.59 Find the equivalent present worth of the cash receipts where $i = 8\%$. In other words, how much do you have to deposit now (with the second deposit in the amount of $200 at the end of the first year) so that you will be able to withdraw $200 at the end of second year, $120 at the end of third year, and so forth if the bank pays you an 8% annual interest on your balance?

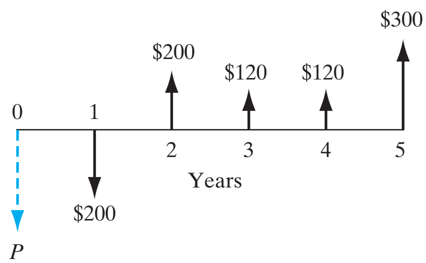


Figure P2.59

2.60 What value of A makes the two annual cash flows equivalent at 15% interest compounded annually?

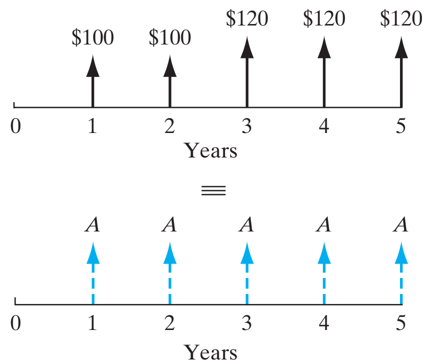


Figure P2.60

2.61 The two cash flow transactions shown in the accompanying cash flow diagram are said to be equivalent at 8% interest compounded annually. Find the unknown value of X that satisfies the equivalence.

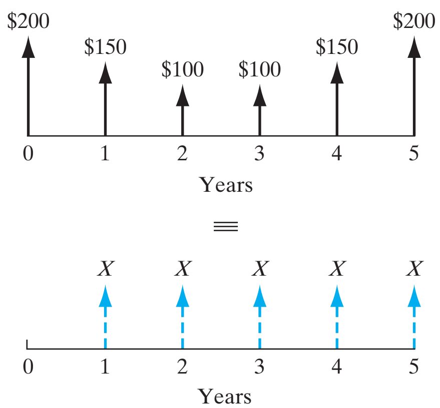


Figure P2.61

2.62 From the accompanying cash flow diagram, find the value of C that will establish the economic equivalence between the deposit series and the withdrawal series at an interest rate of 8% compounded annually.

(a) $1,335
(b) $862
(c) $1,283
(d) $828

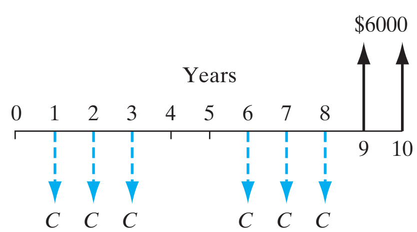


Figure P2.62

2.63 The following equation describes the conversion of a cash flow into an equivalent equal payment series with $N = 10$:

$$\begin{aligned} A = {} & [800 + 20(A/G, 6\%, 7)] \\ & \times (P/A, 6\%, 7)(A/P, 6\%, 10) \\ & + [300(F/A, 6\%, 3) - 500](A/F, 6\%, 10) \end{aligned}$$

Reconstruct the original cash flow diagram.

2.64 Consider the cash flow shown in the accompanying diagram. What value of C makes the inflow series equivalent to the outflow series at an interest rate of 10%?

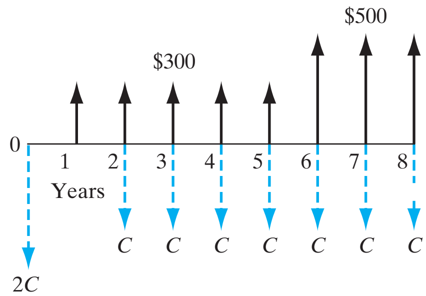


Figure P2.64

2.65 Henry Cisco is planning to make two deposits: \$25,000 now and \$30,000 at the end of year 6. He wants to withdraw C at the end of each year for the first six years and $(C + \$1{,}000)$ each year for the next six years. Determine the value of C if the deposits earn 10% interest compounded annually.

(a) \$7,711

(b) \$5,794

(c) \$6,934

(d) \$6,522

2.66 On the day his baby is born, a father decides to establish a savings account for the child's college education. Any money that is put into the account will earn an interest rate of 8% compounded annually. The father will make a series of annual deposits in equal amounts on each of his child's birthdays from the 1st through the 18th, so that the child can make four annual withdrawals from the account in the amount of \$30,000 on each birthday. Assuming that the first withdrawal will be made on the child's 18th birthday, which of the following equations are correctly used to calculate the required annual deposit?

(a) $A = (\$30{,}000 \times 4)/18$

(b) $A = \$30{,}000(F/A, 8\%, 4) \times (P/F, 8\%, 21)$
$(A/P, 8\%, 18)$

(c) $A = \$30{,}000(P/A, 8\%, 18) \times (F/P, 8\%, 21)$
$(A/F, 8\%, 4)$

(d) $A = [\$30{,}000(P/A, 8\%, 3) + \$30{,}000]$
$(A/F, 8\%, 18)$

(e) $A = \$30{,}000[(P/F, 8\%, 18) + (P/F, 8\%, 19)$
$+ (P/F, 8\%, 20) + (P/F, 8\%, 21)]$
$(A/P, 8\%, 18)$

2.67 Find the value of X so that the two cash flows shown in the diagram are equivalent for an interest rate of 8%.

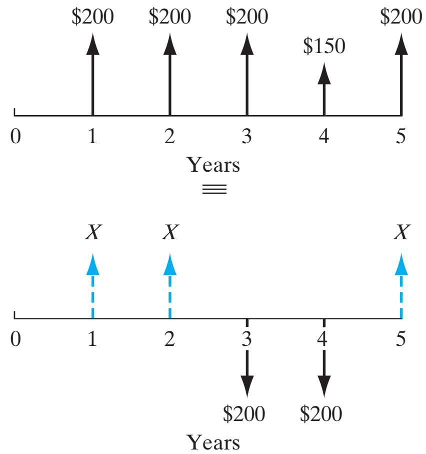


Figure P2.67

2.68 Consider the cash flow series given. In computing the equivalent worth at $n = 4$, which of the following equations is *incorrect*?

(a) $V_4 = [\$100(P/A, i, 6) - \$100(P/F, i, 4)]$
$(F/P, i, 4)$

(b) $V_4 = \$100(F/A, i, 3) + \$100(P/A, i, 2)$

(c) $V_4 = \$100(F/A, i, 4) - \$100 + \$100(P/A, i, 2)$

(d) $V_4 = [\$100(F/A, i, 6) - \$100(F/P, i, 2)]$
$(P/F, i, 2)$

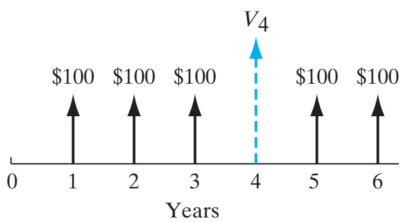


Figure P2.68

2.69 Consider the following cash flow.

TABLE P2.69

Year End	Payment
0	$500
1–5	$1,000

In computing F at the end of year 5 at an interest rate of 12%, which of the following equations is *incorrect*?

(a) $F = \$1{,}000(F/A, 12\%, 5) - \$500(F/P, 12\%, 5)$

(b) $F = \$500(F/A, 12\%, 6) + \$500(F/A, 12\%, 5)$

(c) $F = [\$500 + \$1{,}000(P/A, 12\%, 5)]$
$\times (F/P, 12\%, 5)$

(d) $F = [\$500(A/P, 12\%, 5) + \$1{,}000]$
$\times (F/A, 12\%, 5)$

Solving for an Unknown Interest Rate

2.70 It is said that a lump-sum amount of $50,000 at the end of five years is equivalent to an equal-payment series of $5,000 per year for 10 years, where the first payment occurs at the end of year 1. What earning interest is assumed in this calculation?

2.71 At what rate of interest compounded annually will an investment double in five years?

2.72 Determine the interest rate (i) that makes the pairs of cash flows shown economically equivalent.

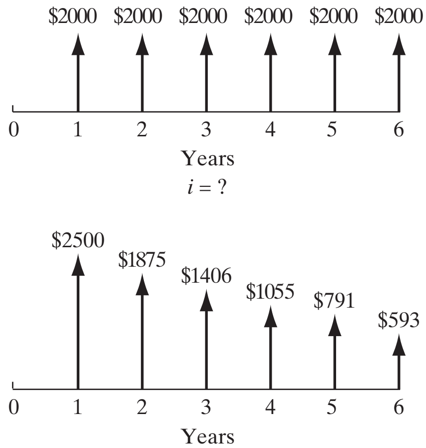


Figure P2.72

2.73 You have $15,000 available for investment in stock. You are looking for a growth stock whose value can grow to $40,000 over five years. What kind of growth rate are you looking for?

2.74 You may have already won $2 million! Just peel the game piece off the Instant Winner Sweepstakes ticket, and mail it to us along with your order for subscriptions to your two favorite magazines. As a grand-prize winner, you may choose between a $1 million cash prize paid immediately or $100,000 per year for 20 years—that's $2 million! Suppose that, instead of receiving one lump sum of $1 million, you decide to accept the 20 annual installments of $100,000. If you are like most jackpot winners, you will be tempted to spend your winnings to improve your lifestyle during the first several years. Only after you get this type of spending "out of your system" will you save later sums for investment purposes. Suppose that you are considering the following two options.

Option 1: You save your winnings for the first seven years and then spend every cent of the winnings in the remaining 13 years.

Option 2: You do the reverse, spending for seven years and then saving for 13 years.

If you can save winnings at 7% interest, how much would you have at the end of 20 years, and what interest rate on your savings will make these two options equivalent?

Short Case Studies with Excel

2.75 The state of Florida sold a total of $36.1 million worth of lottery tickets at $1 each during the first week of January 2018. As prize money, a total of $41 million will be distributed over the next 21 years ($1,952,381 at the *beginning* of each year). The distribution of the first-year prize money occurs now, and the remaining lottery proceeds are put into the state's educational reserve funds, which earn interest at the rate of 6% compounded annually. After the last prize distribution has been made (at the beginning of year 21), how much will be left in the reserve account?

2.76 A newspaper headline reads, "Millionaire Babies: How to Save Our Social Security System." It sounds a little wild, but the concept expressed in the title of this newspaper headline probably refers to an economic plan proposed by a member of Congress. Former Senator Bob Kerrey (D-Nebraska 1989–2001) proposed giving every newborn baby a $1,000 government savings account at birth, followed by five annual contributions of $500 each. (Kerrey offered this idea in a speech devoted to tackling Social Security reform.) If the funds are left untouched in an investment account, Kerrey says, then by the time each baby reaches age 65, his or her $3,500 contribution will have grown to $600,000 over the years, even at medium returns for a thrift-savings plan. At about 9.4% compounded annually, the balance would grow to be $1,005,132. (How would you calculate this number?) Since about 4 million babies are born each year, the proposal would cost the federal government $4 billion annually. About 90% of the total annual Social Security tax collections of more than $300 billion is used to pay current beneficiaries, making Social Security one of the largest federal programs in dollar expenditure. The remaining 10% is invested in interest-bearing government bonds that finance the day-to-day expenses of the federal government. Discuss the economics of Senator Bob Kerrey's Social Security savings plan.

2.77 Giancarlo Stanton's $325 million mega-contract with Marlins was known to be the largest contract in sports history. The 13-year payout schedule looks like the following:

2015: $6.5 million
2016: $9 million
2017: $14.5 million
2018: $25 million
2019: $26 million
2020: $26 million
2021: $29 million
2022: $29 million
2023: $32 million
2024: $32 million
2025: $32 million
2026: $29 million
2027: $25 million plus $10 million buyout option.

Note that the deal allows Stanton to opt out after the end of the 2020 World Series, but if he does, he leaves a substantial sum of money on the table by opting for free agency. In years 2015–20 he will earn $107 million of the $325 million deal, but in years 2021–26 where he would leave the Marlins, the salaries total $218 million would be foregone, or more than twice the sum of the early years in the contract.

(a) With the salary paid at the beginning of each season, what is the worth of his contract at an interest rate of 6%?

(b) Suppose he got another lucrative contract at the end of the 2020 season being a free agent valued at $210 million to be paid over 7 seasons ($30 million paid at the beginning of each season). Is it worth being a free agent to take this offer at 6% interest?