CS419 Digital Image and Video Analysis
Assignment 1

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Question 1

a) Convolution operation is commutative:

Considering the following formula:

$$(f * g)(x) = \int_{-\infty}^{+\infty} f(y)g(x-y)\,dy$$

Apply change of variables by taking $z = x - y$

$$\frac{dz}{dy} = \frac{d(x-y)}{dy} \rightarrow dz = -dy$$

Lower limit:

$$y = -\infty \rightarrow z = t - (-\infty) = +\infty$$

Upper limit:

$$y = +\infty \rightarrow z = t - (+\infty) = -\infty$$

Substitute the new values into the equation:

$$(f * g)(x) = \int_{+\infty}^{-\infty} f(x-z)g(z)\ -dz$$

Move the negative sign outside of the integral:

$$= -\int_{+\infty}^{-\infty} f(x-z)g(z)\,dz$$

Interchange the limits to remove the negative sign:

$$= \int_{-\infty}^{+\infty} f(x-z)g(z)\,dz$$

Change the order of f and g:

$$= -\int_{+\infty}^{\infty} g(z)f(x-z)\,dz$$

Now as z and y are both variables independent of x, I can replace z with y.

$$= -\int_{+\infty}^{\infty} g(y)f(y-z)\,dz$$

This is equal to the expression $(g * f)(x)$, hence the operation is commutative.

b) Cross-correlation is not commutative:

$$(f \star g)(x) = \int_{-\infty}^{+\infty} f(y)g(x+y)\,dy$$

Apply change of variables by inputting $z = x + y$ which implies $y = z - x$.

$$\frac{dz}{dy} = \frac{d(x+y)}{dy} \rightarrow dz = dy$$

Lower limit:

$$y = -\infty \rightarrow z = t + (-\infty) = -\infty$$

Upper limit:

$$y = +\infty \rightarrow z = t + (+\infty) = +\infty$$

Hence the limits do not change. Plugging the new values the result is:

$$(f \star g)(x) = \int_{-\infty}^{+\infty} f(z-x)g(z)\,dz$$

Change the order of f with g and replace y to be z as they are both variables independent of x.

$$\int_{-\infty}^{+\infty} g(y)f(y-x)\,dy = (g \star f)(-x)$$

This is equal to the expression $(g \star f)(-x)$, hence the operation is not commutative.

c) Convolution is associative:

Start by expanding the expression of $g(x) * h(x)$

$$f(x) * (g(x) * h(x)) = f(x) * \int_{-\infty}^{+\infty} h(y)g(x-y)\,dy \tag{1}$$

Now since commutativity is proven in section a, expand f(x) in order to calculate (g(x) * h(x)) * f(x).

$$= \int_{-\infty}^{+\infty} f(z) \int_{-\infty}^{+\infty} h(y)g(x-y-z)\,dy\,dz \tag{2}$$

Change the order of integration

$$= \int_{-\infty}^{+\infty} h(y) \int_{-\infty}^{+\infty} f(z)g(x-y-z)\,dz\;\;dy \tag{3}$$

Combine $(f * g)$ which is equal to $\int_{-\infty}^{+\infty} f(z)g(x-z)\,dz(x)$

$$= \int_{-\infty}^{+\infty} h(y)(f * g)(x-y)\,dy \tag{4}$$

This simply shows the expansion of this equation:

$$= ((f * g) * h)(x) \tag{5}$$

Hence the associativity holds.

Question 2

To solve this question, I will calculate $\frac{\partial^2 f}{\partial y^2}$ and $\frac{\partial^2 f}{\partial x^2}$ separately. Then I will combine them to get the final result.

a) Calculating the second order partial derivative with respect to y:

Starting with expressing the partial derivative with respect to y, in terms of x' and y' using the chain rule:

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial x'}\frac{\partial x'}{\partial y} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial y}$$

Replacing the x' and y' according to the given equations:

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial x'}\frac{xcos(\theta) - ysin(\theta)}{\partial y} + \frac{\partial f}{\partial y'}\frac{xsin(\theta) + ycos(\theta)}{\partial y}$$

Calculating the derivatives with respect to y:

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial x'}(-sin(\theta)) + \frac{\partial f}{\partial y'}cos(\theta)$$

Now apply take the second order derivative:

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}(\frac{\partial f}{\partial x'}(-sin(\theta)) + \frac{\partial f}{\partial y'}cos(\theta)) = \frac{\partial}{\partial y}\frac{\partial f}{\partial x'}(-sin(\theta)) + \frac{\partial}{\partial y}\frac{\partial f}{\partial y'}cos(\theta)$$

Swapping partial derivatives as now $\frac{\partial f}{\partial y}$ is a known equation.

So we write $\frac{\partial}{\partial y}\frac{\partial f}{\partial x'}$ as $\frac{\partial}{\partial x'}\frac{\partial f}{\partial y}$ and similarly, write $\frac{\partial}{\partial y}\frac{\partial f}{\partial y'}$ as $\frac{\partial}{\partial y'}\frac{\partial f}{\partial y}$.

Now in order to calculate $\frac{\partial^2 f}{\partial y^2}$ we need the values of $\frac{\partial}{\partial y'}\frac{\partial f}{\partial y}$ and $\frac{\partial}{\partial x'}\frac{\partial f}{\partial y}$

From the previous equation we know that:

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial x'}(-sin(\theta)) + \frac{\partial f}{\partial y'}cos(\theta)$$

Substitute this into the equation to calculate the unknowns:

$$\frac{\partial}{\partial x'}\frac{\partial f}{\partial y} = \frac{\partial}{\partial x'}(\frac{\partial f}{\partial x'}(-sin(\theta)) + \frac{\partial f}{\partial y'}cos(\theta)) = \frac{\partial^2 f}{\partial x'^2}(-sin(\theta)) + \frac{\partial^2 f}{\partial x'\partial y'}cos(\theta)$$

Similarly:

$$\frac{\partial}{\partial y'}\frac{\partial f}{\partial y} = \frac{\partial}{\partial y'}(\frac{\partial f}{\partial x'}(-sin(\theta)) + \frac{\partial f}{\partial y'}cos(\theta)) = \frac{\partial^2 f}{\partial x'\partial y'}(-sin(\theta)) + \frac{\partial^2 f}{\partial y'^2}cos(\theta)$$

Now plug these values into the second order derivative equation which gives us:

$$\frac{\partial^2 f}{\partial y^2} = (\frac{\partial^2 f}{\partial x'^2}(-sin(\theta)) + \frac{\partial^2 f}{\partial x'\partial y'}cos(\theta))(-sin(\theta)) + (\frac{\partial^2 f}{\partial x'\partial y'}(-sin(\theta)) + \frac{\partial^2 f}{\partial y'^2}cos(\theta))cos(\theta)$$

After doing the multiplications this is the result:

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial x'^2}sin^2(\theta) + \frac{\partial^2 f}{\partial x'\partial y'}(-2sin(\theta)cos(\theta)) + \frac{\partial^2 f}{\partial y'^2}cos^2(\theta)$$

b) Calculating the second order partial derivative with respect to x:

Starting with expressing the partial derivative with respect to x, in terms of x' and y' using the chain rule:

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial x}$$

Replacing the x' and y' according to the given equations:

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial x'}\frac{xcos(\theta) - ysin(\theta)}{\partial x} + \frac{\partial f}{\partial y'}\frac{xsin(\theta) + ycos(\theta)}{\partial x}$$

Taking the derivatives with respect to x, we get:

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial x'}cos(\theta) + \frac{\partial f}{\partial y'}sin(\theta)$$

Applying the second order derivative:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}(\frac{\partial f}{\partial x'}cos(\theta) + \frac{\partial f}{\partial y'}sin(\theta)) = \frac{\partial}{\partial x}\frac{\partial f}{\partial x'}cos(\theta) + \frac{\partial}{\partial x}\frac{\partial f}{\partial y'}sin(\theta)$$

Swapping partial derivatives as now $\frac{\partial f}{\partial x}$ is a known equation.

So we write $\frac{\partial}{\partial x}\frac{\partial f}{\partial x'}$ as $\frac{\partial}{\partial x'}\frac{\partial f}{\partial x}$ and similarly, write $\frac{\partial}{\partial x}\frac{\partial f}{\partial y'}$ as $\frac{\partial}{\partial y'}\frac{\partial f}{\partial x}$.

Now in order to calculate $\frac{\partial^2 f}{\partial x^2}$ we need the values of $\frac{\partial}{\partial x'}\frac{\partial f}{\partial x}$ and $\frac{\partial}{\partial y'}\frac{\partial f}{\partial x}$

From the previous equation we know that:

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial x'}cos(\theta) + \frac{\partial f}{\partial y'}sin(\theta)$$

Substitute this into the equation to calculate the unknowns:

$$\frac{\partial}{\partial x'}\frac{\partial f}{\partial x} = \frac{\partial}{\partial x'}(\frac{\partial f}{\partial x'}cos(\theta) + \frac{\partial f}{\partial y'}sin(\theta)) = \frac{\partial^2 f}{\partial x'^2}cos(\theta) + \frac{\partial^2 f}{\partial y'\partial x'}sin(\theta)$$

Similarly:

$$\frac{\partial}{\partial y'}\frac{\partial f}{\partial x} = \frac{\partial}{\partial y'}(\frac{\partial f}{\partial x'}cos(\theta) + \frac{\partial f}{\partial y'}sin(\theta)) = \frac{\partial^2 f}{\partial x'\partial y'}cos(\theta) + \frac{\partial^2 f}{\partial y'^2}sin(\theta)$$

Now plug these values into the second order derivative equation which gives us:

$$\frac{\partial^2 f}{\partial x^2} = (\frac{\partial^2 f}{\partial x'^2}cos(\theta) + \frac{\partial^2 f}{\partial y'\partial x'}sin(\theta))cos(\theta) + (\frac{\partial^2 f}{\partial x'\partial y'}cos(\theta) + \frac{\partial^2 f}{\partial y'^2}sin(\theta))sin(\theta)$$

After doing the multiplications this is the result:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial x'^2}cos^2(\theta) + \frac{\partial^2 f}{\partial y'\partial x'}2sin(\theta)cos(\theta) + \frac{\partial^2 f}{\partial y'^2}sin^2(\theta)$$

c) Combining the results to reach the final conclusion:

Adding the results together from section a and b, we get the following equation:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial x'^2}(sin^2(\theta) + cos^2(\theta)) + \frac{\partial^2 f}{\partial y'^2}(sin^2(\theta) + cos^2(\theta))$$

For all degrees in between, it it known that $sin^2(\theta) + cos^2(\theta)$ is equal to 1. So replace all those occurrences with 1 in order to get the final equation below:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial x'^2} + \frac{\partial^2 f}{\partial y'^2}$$

Hence Laplacian operator is rotation invariant.

Question 3

For the given equation:

$$h(x, y) = 3f(x, y) + 2f(x-1, y) + 2f(x+1, y) - 17f(x, y-1) + 99f(x, y+1)$$

To prove linearity, it needs to hold 2 conditions: Additivity and Homogenity

a) Additivity:

Additivity claims that h(f_2 + f_2) = h(f_1) + h(f_2) for 2 arbitrary functions. To start with, let f_1 and f_2 be 2 different functions.

Inputting f_1 into the filter:

$$h_1(x, y) = 3f_1(x, y) + 2f_1(x-1, y) + 2f_1(x+1, y) - 17f_1(x, y-1) + 99f_1(x, y+1)$$

Inputting f_2 into the filter:

$$h_2(x, y) = 3f_2(x, y) + 2f_2(x-1, y) + 2f_2(x+1, y) - 17f_2(x, y-1) + 99f_2(x, y+1)$$

Inputting $f_1 + f_2$ into the filter:

$$h_3(x, y) = 3(f_2(x, y) + f_1(x, y)) + 2(f_2(x-1, y) + f_1(x-1, y)) + 2(f_2(x+1, y)$$

$$+f_1(x+1, y)) - 17(f_2(x, y-1) + f_1(x, y-1)) + 99(f_2(x, y+1) + f_1(x, y+1))$$

Expanding the equation:

$$h_3(x, y) = 3f_2(x, y) + 3f_1(x, y) + 2f_2(x-1, y) + 2f_1(x-1, y) + 2f_2(x+1, y)$$

$$+2f_1(x+1, y) - 17f_2(x, y-1) - 17f_1(x, y-1) + 99f_2(x, y+1) + 99f_1(x, y+1)$$

Rearranging the arguments:

$$h_3(x, y) = 3f_2(x, y) + 2f_2(x-1, y) + 2f_2(x+1, y) - 17f_2(x, y-1) + 99f_2(x, y+1)$$

$$+3f_1(x, y) + 2f_1(x-1, y) + 2f_1(x+1, y) - 17f_1(x, y-1) + 99f_1(x, y+1)$$

This is equivalent to $h_3(x, y) = h_1(x, y) + h_2(x, y)$, hence additivity holds.

b) Homogeneity:

Homogeneity claims that h(cf) = ch(f).
To start with, let c be a constant.
Giving the input cf(x,y) into the function:

$$h(x,y) = 3(cf(x,y)) + 2(cf(x-1,y)) + 2(cf(x+1,y)) - 17f(c(x,y-1)) + 99(cf(x,y+1))$$

Expanding the equation:

$$h(x,y) = 3cf(x,y) + 2cf(x-1,y) + 2cf(x+1,y) - 17cf(x,y-1) + 99cf(x,y+1))$$

Factoring out the common c:

$$h(x,y) = c(3f(x,y) + 2f(x-1,y) + 2f(x+1,y) - 17f(x,y-1) + 99f(x,y+1)))$$

This is equivalent to ch(x,y) homogeneity also holds.
So, the filter is indeed linear.

c) Convolution Mask:

In order to extract the convolution mask, take f(x,y) as the center point and create a 3x3 matrix. The filter can be interpreted as below:

$$h(x,y) = 3f(x,y) + 2f(x-1,y) + 2f(x+1,y) - 17f(x,y-1) + 99f(x,y+1) +$$
$$0f(x-1,y-1) + 0f(x+1,y-1) + 0f(x-1,y+1) + 0f(x+1,y+1)$$

The function now includes all the neighbouring cells coefficients to fill a 3x3 matrix. Filling all the coefficients relative to the central one, the following matrix is obtained.

$$\begin{bmatrix} 0 & -17 & 0 \\ 2 & 3 & 2 \\ 0 & 99 & 0 \end{bmatrix}$$