Problem 2 (4 marks): Rigid-body transformations

Place your answers to the following, along with your work, in a pdf file named **matrix.pdf.**

Suppose you are given a 3D transformation Φ specified by a matrix M of the following form.

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} & t_1 \\ r_{21} & r_{22} & r_{23} & t_2 \\ r_{31} & r_{32} & r_{33} & t_3 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

where the upper 3 by 3 submatrix R of M is orthonormal, i.e., $R^T = R^{-1}$.

1. [1 mark] What is the inverse of M? Note that you should not use brute force or a package such as Maple or Matlab to answer this question.
2. [1 mark] Let V = P_2 - P_1 be a vector in 3-dimensional real (Euclidean) space, where P_1 and P_2 are points in that space. Is it the case that Φ is linear in 3D? In other words, does $\Phi(P_2 - P_1) = \Phi(P_2) - \Phi(P_1)$?
3. [2 marks] Prove that the transformation Φ preserves lengths, angles, and the area of triangles in 3D.

Problem 1.

$$\because \begin{pmatrix} r_{11} & r_{12} & r_{13} & t_1 \\ r_{21} & r_{22} & r_{23} & t_2 \\ r_{31} & r_{32} & r_{33} & t_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & t_1 \\ 0 & 1 & 0 & t_2 \\ 0 & 0 & 1 & t_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= A \cdot B$$

$$\therefore \begin{pmatrix} r_{11} & r_{12} & r_{13} & t_1 \\ r_{21} & r_{22} & r_{23} & t_2 \\ r_{31} & r_{32} & r_{33} & t_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} = B^{-1} \cdot A^{-1}$$

$$= \begin{pmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 1 & 0 & 0 & t_1 \\ 0 & 1 & 0 & t_2 \\ 0 & 0 & 1 & t_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1}$$

$\because R^T = R^{-1}$

$$\therefore \begin{pmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} r_{11} & r_{21} & r_{31} & 0 \\ r_{12} & r_{22} & r_{32} & 0 \\ r_{13} & r_{23} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\because \begin{pmatrix} 1 & 0 & 0 & t_1 \\ 0 & 1 & 0 & t_2 \\ 0 & 0 & 1 & t_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 & -t_1 \\ 0 & 1 & 0 & -t_2 \\ 0 & 0 & 1 & -t_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\therefore$ The result is:

$$\begin{pmatrix} r_{11} & r_{21} & r_{31} & 0 \\ r_{12} & r_{22} & r_{32} & 0 \\ r_{13} & r_{23} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & -t_1 \\ 0 & 1 & 0 & -t_2 \\ 0 & 0 & 1 & -t_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} r_{11} & r_{21} & r_{31} & -r_{11}\cdot t_1 - r_{21}\cdot t_2 - r_{31}\cdot t_3 \\ r_{12} & r_{22} & r_{32} & -r_{12}\cdot t_1 - r_{22}\cdot t_2 - r_{32}\cdot t_3 \\ r_{13} & r_{23} & r_{33} & -r_{13} t_1 - r_{23}\cdot t_2 - r_{33}\cdot t_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \text{Ans.}$$

Problem 2. Ans: ϕ is linear in 3D.

Prove: Let $P_1 = \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$ $P_2 = \begin{pmatrix} a \\ b \\ c \\ 1 \end{pmatrix}$

$$\therefore P_2 - P_1 = \begin{pmatrix} a-x \\ b-y \\ c-z \\ 0 \end{pmatrix}$$

$$\because \phi = \begin{pmatrix} r_{11} & r_{12} & r_{13} & t_1 \\ r_{21} & r_{22} & r_{23} & t_2 \\ r_{31} & r_{32} & r_{33} & t_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\therefore \phi(P_2 - P_1) = \begin{pmatrix} r_{11} & r_{12} & r_{13} & t_1 \\ r_{21} & r_{22} & r_{23} & t_2 \\ r_{31} & r_{32} & r_{33} & t_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a-x \\ b-y \\ c-z \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} r_{11}(a-x) + r_{12}(b-y) + r_{13}(c-z) \\ r_{21}(a-x) + r_{22}(b-y) + r_{23}(c-z) \\ r_{31}(a-x) + r_{32}(b-y) + r_{33}(c-z) \\ 0 \end{pmatrix} = LHS$$

$$\because \quad \phi(P_2) = \begin{pmatrix} r_{11} & r_{12} & r_{13} & t_1 \\ r_{21} & r_{22} & r_{23} & t_2 \\ r_{31} & r_{32} & r_{33} & t_3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} r_{11}\cdot a + r_{12}\cdot b + r_{13}\cdot c \\ r_{21}\cdot a + r_{22}\cdot b + r_{23}\cdot c \\ r_{31}\cdot a + r_{32}\cdot b + r_{33}\cdot c \\ 1 \end{pmatrix}$$

$$\phi(P_1) = \begin{pmatrix} r_{11} & r_{12} & r_{13} & t_1 \\ r_{21} & r_{22} & r_{23} & t_2 \\ r_{31} & r_{32} & r_{33} & t_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} r_{11}\cdot x + r_{12}\cdot y + r_{13}\cdot z \\ r_{21}\cdot x + r_{22}\cdot y + r_{23}\cdot z \\ r_{31}\cdot x + r_{32}\cdot y + r_{33}\cdot z \\ 0 \end{pmatrix}$$

$\therefore \quad RHS = \phi(P_2) - \phi(P_1)$

$$= \begin{pmatrix} r_{11}\cdot a + r_{12}\cdot b + r_{13}\cdot c \\ r_{21}\cdot a + r_{22}\cdot b + r_{23}\cdot c \\ r_{31}\cdot a + r_{32}\cdot b + r_{33}\cdot c \\ 0 \end{pmatrix} - \begin{pmatrix} r_{11}\cdot x + r_{12}\cdot y + r_{13}\cdot z \\ r_{21}\cdot x + r_{22}\cdot y + r_{23}\cdot z \\ r_{31}\cdot x + r_{32}\cdot y + r_{33}\cdot z \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} r_{11}(a-x) + r_{12}(b-y) + r_{13}(c-z) \\ r_{21}(a-x) + r_{22}(b-y) + r_{23}(c-z) \\ r_{31}(a-x) + r_{32}(b-y) + r_{33}(c-z) \\ 0 \end{pmatrix}$$

$$= LHS = \phi(P_2 - P_1)$$

$\therefore \quad \phi(P_2 - P_1) = \phi(P_2) - \phi(P_1)$

$\therefore \quad \phi$ is linear in 3D.

Problem 3

From Problem 2, we have known the ϕ is linear in 3D. And therefore, ϕ preserves the lengths between corresponding points. And also, the lengths of corresponding sides are equal.

Let pre-image triangle be $\triangle ABC$, the image one be $\triangle A'B'C'$. $\therefore AB = A'B'$, $AC = A'C'$, $BC = B'C'$.

$\Rightarrow \triangle ABC \cong \triangle A'B'C'$ (SSS)

So, we can make a conclusion that the image and the pre-image are congruent ($\cong$). According to the property of congruence, the two images are identical. And therefore, the corresponding angles are equal ($\angle A = \angle A'$, $\angle B = \angle B'$, $\angle C = \angle C'$). Also, the areas of two triangles are equal, given they are congruent.