Models with constant elasticity demand and homogeneous firms

- These notes draw on Chapter 3 of the notes by Allen and Arkolakis for reference.
- **What causes division of labor and trade?** In models with homogeneous firms, there is no comparative advantage based on productivity. Instead, these models assume distinct variety of goods produced in different regions and use a CES utility function to depict consumers' love of variety, which induces the division of labor and trades.

1. Basic settings

- A compact set S of countries.
- The origin country and destination country are denoted as i and j, respectively.
- Total spending of country j is X_j.
- The population of country j is L_j.
- Each consumer has a single labor unit that is inelastically supplied.
- Homogeneous firms.
- Ice-berg trade cost: d_{ij} $(i, j \in S)$, which is always assumed that $d_{ij} \geq 1$ $(i \neq j)$ and $d_{ii} = 1$. Triangle inequality: $d_{ij} \cdot d_{jk} \geq d_{ik}$.
- Allow different market types: perfect competition, Bertrand competition, monopolistic competition, etc.

2. Preliminary: CES demand

- Consumers gain utility via consuming various goods $u \in \Omega$. Suppose that the utility function for country j's consumers is as follows:

$$U_j = \left(\sum_{u \in \Omega} a_{ij}(u)^{\frac{1}{\sigma}} q_{ij}(u)^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}, \tag{1}$$

  where $\sigma \geq 0$ is the elasticity of substitution. Here i denotes different exporters.
- The maximization problem:

$$\max_{\{q_{ij}(u)\}_{u \in \Omega}} \left(\sum_{u \in \Omega} a_{ij}(u)^{\frac{1}{\sigma}} q_{ij}(u)^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}, \quad \text{s.t.} \quad \sum_{u \in \Omega} q_{ij}(u) p_{ij}(u) \leq X_j.$$

  The corresponding first-order condition (FOC) is $\mathrm{MRS}_{u,u'} = p_{ij}(u)/p_{ij}(u')$ for any two different goods u and u'.
- Substitute the FOC in the budget constraint function to yield the CES demand function:

$$q_{ij}(u) = a_{ij} \frac{p_{ij}(u)^{-\sigma}}{P_j^{1-\sigma}} X_j, \tag{2}$$

  where $P_j = \left(\sum_{u' \in \Omega} a_{ij}(u') p_{ij}(u')^{1-\sigma} \right)^{\frac{1}{1-\sigma}}$ is known as the Dixit-Stigliz price index. Substituting the demand function into (1) yields the indirect utility function: $V_j = X_j / P_j$.

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- Denote the value of good u traded from i to j as $X_{ij}(u)$. From (2), we have:

$$X_{ij}(u) = p_{ij}(u)q_{ij}(u) = a_{ij}\left(\frac{p_{ij}(u)}{P_j}\right)^{1-\sigma} X_j. \tag{3}$$

3. Armington model: perfect competition

- Paper: Anderson, J. E. (1979). A Theoretical Foundation for the Gravity Equation. *The American Economic Review*, *69*(1), 106–116.
- Armington model is the first theoretical foundation of the gravity equation.

3.1. Additional settings

- Each country produced a distinct variety of goods. Therefore, we can forget about index u and also use exporter index i to denote goods.
- Perfect competition.
- Productivity: labor input only. Each worker produces Z_i units of her/his country's good.
- The wage is w_i and thus the domestic price of the good is $p_i = w_i/Z_i$.
- The sale price of the good traded from country i to country j is:

$$p_{ij} = d_{ij}\frac{w_i}{Z_i}. \tag{4}$$

3.2. Gravity

- Substitute (4) in (3) to yield:

$$X_{ij} = a_{ij}d_{ij}^{1-\sigma}\left(\frac{w_i}{Z_i}\right)^{1-\sigma}\frac{X_j}{P_j^{1-\sigma}}. \tag{5}$$

- Define Y_i as the total income of country i. The total income equals to the total share:

$$Y_i = \sum_{j=1}^{N} X_{ij} = \left(\frac{w_i}{Z_i}\right)^{1-\sigma}\sum_{j=1}^{N} a_{ij}d_{ij}^{1-\sigma}\frac{X_j}{P_j^{1-\sigma}}.$$

  from which we can get:

$$\left(\frac{w_i}{Z_i}\right)^{1-\sigma} = \frac{Y_i}{\sum_j a_{ij}d_{ij}^{1-\sigma}\frac{X_j}{P_j^{1-\sigma}}}. \tag{6}$$

- We define $\Pi_i^{1-\sigma} \equiv \sum_j a_{ij}d_{ij}^{1-\sigma}X_jP_j^{\sigma-1}$ to simplify (6) and replacing this expression in (5) to yield:

$$X_{ij} = a_{ij}d_{ij}^{1-\sigma}\frac{Y_i}{\Pi_i^{1-\sigma}}\frac{X_j}{P_j^{1-\sigma}}. \tag{7}$$

  This structural gravity equation implies that bilateral trade flows are determined by five factors: bilateral trade resistance $a_{ij}d_{ij}^{1-\sigma}$, gross outputs of the exporter country Y_i and the importer country X_j (under equilibrium, total expenditure = total output), outward multilateral resistance (OMR) $\Pi_i^{-1-\sigma}$ and inward multilateral resistance (IMR) $P_j^{1-\sigma}$ (these two terms are named "multilateral" since they involve all countries).

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- Notice that this structural gravity equation can be represented by a commonly used empirical gravity equation:

$$X_{ij} = K_{ij}\gamma_i\delta_j,$$

where K_{ij} denotes bilateral trade resistance, and γ_i and δ_j refer to exporter and importer fixed effects, respectively. However, an advantage of the structural model is that it provides structural interpretations for estimates from the reduced-form specification.

3.3. Welfare

- Define the bilateral trade share as $\pi_{ij} \equiv X_{ij}/X_j$, we have:

$$\pi_{ij} = a_{ij}\left(\frac{p_{ij}}{P_j}\right)^{1-\sigma} = a_{ij}\left(\frac{d_{ij}}{Z_i}\right)^{1-\sigma}\left(\frac{w_i}{P_j}\right)^{1-\sigma}. \tag{8}$$

- By choosing $i = j$, Equation (8) can be transformed as:

$$\frac{w_j}{P_j} = \pi_{jj}^{\frac{1}{1-\sigma}} a_{jj}^{\frac{1}{\sigma-1}} Z_j. \tag{9}$$

- As the indirect utility function of country j equals to the real income (i.e., $V_j = X_j/P_j$), Equation (9) implies the relationship between welfare and bilateral trade share. Assume $\sigma > 1$ (which means goods produced by different countries are of high substitutability), every country benefits from the transition from autarky to trade.

4. Krugman model: monopolistic competition and increasing returns to scale

- Paper: Krugman, P. (1980). Scale Economies, Product Differentiation, and the Pattern of Trade. *American Economic Review*, *70*(5), 950–959.

4.1. Additional settings

- Armington model assumes that each country produces *a single* unique good, but Krugman model dispenses this assumption by introducing firms into the model.
- Krugman model assumes that *each firm produces a unique variety of goods* and consumers would like at least a little bit of every variety (CES demand).
- To take into account increasing returns to scale, the model assumes that a firm has to incur a fixed entry cost $w_i f_i^e$ in order to produce. The number of firms N_i in country i is determined in equilibrium.

4.2. Consumers

- In this model, we change discrete goods to a continuum of goods and rewrite the CES utility function as:

$$U_j = \left(\sum_{i\in S}\int_{u\in\Omega_i} q_{ij}(u)^{\frac{\sigma-1}{\sigma}}\mathrm{d}u\right)^{\frac{\sigma}{\sigma-1}}. \tag{10}$$

- Solving the maximization problem to yield the demand function and expenditure function of good u traded from i to j:

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$$q_{ij}(u) = \frac{p_{ij}^{-\sigma}}{P_j^{1-\sigma}} X_j, \tag{11}$$

and

$$X_{ij}(u) = \left(\frac{p_{ij}}{P_j}\right)^{1-\sigma} X_j, \tag{12}$$

where the price index $P_j \equiv \left(\sum_{i \in S} \int_{u \in \Omega_i} p_{ij}(u)^{1-\sigma} \mathrm{d}u\right)^{\frac{1}{1-\sigma}}$.

- The bilateral trade flow is then:

$$X_{ij} = \int_{u \in \Omega_i} X_{ij}(u) \mathrm{d}u = \frac{X_j}{P_j^{1-\sigma}} \int_{u \in \Omega_i} p_{ij}^{1-\sigma}(u) \mathrm{d}u. \tag{13}$$

4.3. Firms

- Assume all firms in country i have the same productivity Z_i. This is why Krugman model is classified as a homogeneous firms model.
- As each firm produces a unique variety of goods, we can use u to represent a variety as well a firm. As each firm produces a different variety of good, it has monopolistic power and is a price setter. Accordingly, the optimization problem faced by a firm u from country i is:

$$\max_{\{p_{ij}(u)\}_{j \in S}} \sum_{j \in S} \left(p_{ij}(u) q_{ij}(u) - w_i \frac{d_{ij} q_{ij}(u)}{Z_i}\right) - w_i f_i^e, \quad \text{s.t.} \quad q_{ij}(u) = \frac{p_{ij}^{-\sigma}}{P_j^{1-\sigma}} X_j.$$

- Because the production decision is independent across destinations, we can easily solve the optimal price:

$$p_{ij}(u) = \frac{\sigma}{\sigma - 1} \cdot \frac{w_i d_{ij}}{Z_i} \tag{15}$$

  It is noteworthy that the second term at the right hand side is the equilibrium price in the perfect competition case, thus (15) provides the markups in the monopolistic case.

4.4. Gravity

- Substitute (15) in (13) to yield:

$$X_{ij} = \left(\frac{\sigma}{\sigma - 1}\right)^{1-\sigma} d_{ij}^{1-\sigma} \left(\frac{w_i}{Z_i}\right)^{1-\sigma} \frac{X_j}{P_j^{1-\sigma}} \int_{u \in \Omega_i} \mathrm{d}u.$$

- Define $N_i \equiv \int_{u \in \Omega_i} \mathrm{d}u$, which represents the number of firms producing in country i. The above equation can be rewritten as:

$$X_{ij} = \left(\frac{\sigma}{\sigma - 1}\right)^{1-\sigma} d_{ij}^{1-\sigma} \left(\frac{w_i}{Z_i}\right)^{1-\sigma} \frac{X_j}{P_j^{1-\sigma}} N_i. \tag{16}$$

- Using the same method as in the Armington model, Equation (16) again provides a structural gravity equation:

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$$X_{ij} = d_{ij}^{1-\sigma} \frac{Y_i}{\Pi_i^{1-\sigma}} \frac{X_j}{P_j^{1-\sigma}}, \tag{17}$$

where $\Pi_i^{1-\sigma} \equiv (\frac{\sigma}{\sigma-1})^{1-\sigma} N_i \sum_j d_{ij}^{1-\sigma} \frac{X_j}{P_j^{1-\sigma}}$. Notice that (17) is very similar to (7), and the only difference is the composition of two multilateral resistance terms.

4.5. Welfare

- Define the bilateral trade share as $\pi_{ij} \equiv X_{ij}/X_j$, we have:

$$\pi_{ij} = \int_{u \in \Omega_i} \left(\frac{p_{ij}(u)}{P_j} \right)^{1-\sigma} \mathrm{d}u.$$

- Substituting (15) in the above equation and letting $i = j$ yields

$$\frac{w_j}{P_j} = \left(\frac{\sigma - 1}{\sigma} \right) z_j N_j^{\frac{1}{\sigma-1}} \pi_{jj}^{\frac{1}{1-\sigma}}. \tag{18}$$

  This equation implies that the real wage increases with higher productivity, more firm entries, and a lower local trade share.
- Note, however, that unlike the Armington model, firms are making positive profits, so that the real wage no longer captures the total welfare of a location.
- Additionally, we have not explicitly solved the firm entry decision yet.

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