CS70 Midterm 1 Review

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February 14, 2014

Topics

- Propositions and Proof Techniques
- Induction
- Stable Marriage
- Modular Arithmetic

Propositions

- Definition:
  - A statement which is either true or false
- Propositions can be joined together into propositional forms.
  - Conjunction: $P \wedge Q$. True only when both P and Q are true.
  - Disjunction: $P \vee Q$. True when at least one of P and Q is true.
  - Negation: $\neg P$. True when P is false.
  - Implication: $P \Rightarrow Q$. This is the same as "if P, then Q."
    - Contrapositive $\neg Q \Rightarrow \neg P$. Logically equivalent to implication.
    - Converse $Q \Rightarrow P$. NOT logically equivalent to implication.

Quantifiers and Negation

- Use De Morgan's Law for negations.
  - $\neg(P \wedge Q) \Leftrightarrow (\neg P) \vee (\neg Q)$
  - $\neg(P \vee Q) \Leftrightarrow (\neg P) \wedge (\neg Q)$
- Example $\neg(\forall x \in \mathbb{N} \ \exists y \in \mathbb{N} \ P(x, y)) \Rightarrow (\exists x \in \mathbb{N} \ \forall y \in \mathbb{N} \ \neg P(x, y))$

Topics   Propositions and Proof Techniques   Induction   Stable Marriage   Modular Arithmetic   Public Key Cryptography

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Direct Proof

- Theory:
  - Direct Proof of $P \Rightarrow Q$: Assume P ... Therefore Q
  - This is often useful when you have an easy-to-expand expression as P (For example, in the exercise below)

- Exercise:
  Prove that, if m, n are odd integers, then mn is also an odd integer.

- Solution:

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- Exercise:
  Prove that, if m, n are odd integers, then mn is also an odd integer.

- Solution:

$$\text{We can write } m = 2k + 1, n = 2q + 1 \text{ for some } k, q \in \mathbb{Z}$$
$$(2k + 1)(2q + 1)$$
$$4kq + 2k + 2q + 1$$
$$2(2kq + k + q) + 1$$
$$2a + 1 \text{ for some } a \in \mathbb{Z}$$
$$\text{So } mn \text{ is, by definition, an odd number}$$

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Proof by Contraposition

- Theory:
  - Proof by Contraposition of $P \Rightarrow Q$: Assume $\neg Q$ ... Therefore $\neg P$
    So $\neg Q \Rightarrow \neg P \equiv P \Rightarrow Q$
  - This is often useful when you have an easy-to-expand expression as P (For example, in the exercise below)

- Exercise:
  Prove that, if x^2 is even, then x is even, for all $x \in \mathbb{Z}$

- Solution:

Proof by Contraposition

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  - Proof by Contraposition of $P \Rightarrow Q$: Assume $\neg Q$ ... Therefore $\neg P$
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- Exercise:
  Prove that, if x^2 is even, then x is even, for all $x \in \mathbb{Z}$

- Solution:

Suppose that x is not even.

Let $x = 2k + 1$ for $k \in \mathbb{Z}$

$$x^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$$

We proved that x^2 is an odd number, so we have completed

our proof by contrapositive

Proof by Contradiction

- Theory:
    - Proof by Contradiction of P:
      Assume $\neg P$
      . . .
      R
      . . .
      $\neg R$.
      Contradiction, therefore P
- Exercise:
  Prove that there is no greatest integer.
- Solution:

Proof by Contradiction

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  - Proof by Contradiction of P:
    Assume $\neg P$

    . . .
    R
    . . .
    $\neg R$.
    Contradiction, therefore P

- Exercise:
  Prove that there is no greatest integer.

- Solution: Assume that there is a greatest integer N. But we know that $N + 1$ is also an integer (since integers are closed under addition, by definition), and $N + 1 > N$ so N is not the greatest integer. Contradiction. Therefore there is no greatest integer.

Proofs by Induction

Prove by induction $\forall k \in \mathbb{N}, P(k)$, where $\sum\limits_{i=1}^{k} i = \frac{k(k+1)}{2}$

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Induction Hypothesis: Assume $P(m)$ for some m.

Topics
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Propositions and Proof Techniques
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Induction
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Stable Marriage
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Public Key Cryptography
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Induction Hypothesis: Assume $P(m)$ for some m.

Induction Step: Prove for $P(m+1)$.

$$
\begin{aligned}
P(m+1) &= P(m) + (m+1) \\
&= \frac{m(m+1)}{2} + (m+1) \\
&= \frac{m^2 + m + 2(m+1)}{2} \\
&= \frac{m^2 + 3m + 2}{2} \\
&= \frac{(m+1)(m+2)}{2}
\end{aligned}
$$

$$(1)$$

Cubed → Squared

Prove by induction $\forall k \in \mathbb{N}, P(k)$, where $\sum\limits_{i=1}^{k} i^3 = (\frac{k(k+1)}{2})^2$

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$$
\begin{aligned}
P(m+1) &= P(m) + (m+1)^3 \\
&= (\frac{m(m+1)}{2})^2 + (m+1)^3 \\
&= \frac{m^2(m+1)^2}{4} + \frac{4(m+1)(m+1)^2}{4} \\
&= \frac{(m^2 + 4m + 4)(m+1)^2}{4} \\
&= \frac{(m+2)^2(m+1)^2}{4} \\
&= (\frac{(m+1)(m+2)}{2})^2
\end{aligned}
$$

Divisible?

(Fa06 Papadimitriou and Vazirani #2) Prove by induction that for every odd positive integer n, $3^n + 4^n$ is divisible by 7.

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Induction Hypothesis: Assume $3^n + 4^n$, where n is some positive odd integer, is divisible by 7.

Induction Step: We want to prove that $3^{n+2} + 4^{n+2}$ is divisible by 7.

$3^{n+2} + 4^{n+2} = 9 * 3^n + 16 * 4^n$
We know that
$9 * 3^n + 16 * 4^n$ is divisible by 7 if
$9 * 3^n + 16 * 4^n \pmod 7 = 0$.
$9 * 3^n + 16 * 4^n \pmod 7 = 2 * 3^n + 2 * 4^n$
$= 2(3^n + 4^n) \pmod 7$
From the induction hypothesis, we know that $3^n + 4^n$ is divisible by 7, and therefore so is $2(3^n + 4^n)$.

Chocolate

(Vazirani Fa12 Midterm 1 #3b) You wish to break a standard mn Hershey chocolate bar into mn little squares to distribute to mn kids. In each step you can pick up exactly one piece of chocolate and break it along one of the horizontal or vertical lines etched into the bar. No stacking! Prove by induction that the minimum number of steps required to completely break the bar into mn little squares is $mn - 1$.

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We will use induction on $r = mn$; induction on the number of pieces in the bar.

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Base Case: The minimum number of breaks required is $g(r)$. $r = 1$. $m = 1$, $n = 1$. No breaks are necessary, and we observe that $g(r) = r - 1 = 0$.

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Induction Hypothesis: Assume $\forall s \leq k, g(s) = s - 1$. (Strong induction)

Chocolate Step

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Induction Step:
Consider $r = k + 1$. Since we have one big piece and $k + 1$ pieces to distribute, we need to begin by making 1 break. This results in two bars, one with $u < k + 1$ and another with $v < k + 1$ pieces, where $u + v = k + 1$. Applying the inductive hypothesis, it takes a minimum of $g(v) = v - 1$ breaks on the bar of size v, and it takes $g(u) = u - 1$ breaks on the bar of size u. The total minimum number of breaks is then
$g(k + 1) = 1 + (v - 1) + (u - 1) = (u + v) - 1 = (k + 1) - 1$. Therefore, for any bar of size mn, it takes a minimum of $mn - 1$ breaks.

Stable Marriage Question

Objective: Given that there are n men and n women, match them up (male to female pairing) in such a way that there are no *rogue couples*.

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Rogue couples: A man and a woman who prefer each other as opposed to their current partners. (BOTH sides must prefer each other to their current partners!)

Topics   Propositions and Proof Techniques   Induction   **Stable Marriage**   Modular Arithmetic   Public Key Cryptography
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Few things to note

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1. **TMA is male-optimal**: You can think of it this way - if all of the males prefer unique woman for their first choice, would the women have ANY say in the matter?
   Remember, male-optimal means that every male has the highest choice woman he can be hoped to paired with in ANY stable pairing.

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2. **Stable pairing that is both male- and female-optimal**: Is it possible to have both in a stable pairing? Yes. However, that also means that there is only 1 stable pairing possible, as that specific pairing needs to be both Male-optimal AND male-pessimal, female-optimal AND female-pessimal.

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3. **Runtime of TMA**: TMA has runtime of $O(n^2)$. You can think of it this way - even in the worst case situation, there will be at least 1 man who was rejected at each cycle. Since there are n men and each has n choices, there will be at longest n^2 number of iterations.

Applying the logic

Male Pref	1	2	3	4
A	E	F	G	H
B	G	F	E	H
C	E	G	H	F
D	F	E	G	H

If you are told that the male-optimal stable marriage pairing is {(A, E), (B, F), (C, G), (D, H)}, can you have a stable marriage pairing such that D is paired up with someone other than H?

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If you are told that the male-optimal stable marriage pairing is {(A, E), (B, F), (C, G), (D, H)}, can you have a stable marriage pairing such that D is paired up with someone other than H?

No, because if there was such a pairing, then that pairing would have formed a rogue couple in this set.

Go go go!

Now let's try a practice question.

What is the male- and female-optimal pairing in this set?

Male Pref	1	2	3	4
Jim	S	A	M	N
Matt	A	N	M	S
Valerian	N	A	M	S
Tychus	M	N	A	S

Female Pref	1	2	3	4
Sarah	J	M	V	T
Ariel	J	M	V	T
Mira	J	V	M	T
Nova	V	M	J	T

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Female Pref	1	2	3	4
Sarah	J	M	V	T
Ariel	J	M	V	T
Mira	J	V	M	T
Nova	V	M	J	T

The male-optimal pairings are: {(Sarah, Jim), (Ariel, Matt), (Nova, Valerian), (Mira, Tychus)}
The female-optimal pairings are: {(Sarah, Jim), (Ariel, Matt), (Nova, Valerian), (Mira, Tychus)}

Euclid's Extended Algorithm

Evaluate:

$$\text{extended-gcd}(37,10)$$

Show all recursive steps and return values. Use this information to provide a solution, if any to:

$$10x = 1 \bmod 37$$

```
algorithm extended-gcd(x,y):
   if y = 0 then return(x, 1, 0)
   else:
      (d, a, b) := extended-gcd(y, x mod y)
      return((d, b, a - (x div y) * b))
```

Euclid's Extended Algorithm

$$extended - gcd(37, 10) \quad -> \tag{2}$$
$$extended - gcd(10, 7) \quad -> \tag{3}$$
$$extended - gcd(7, 3) \quad -> \tag{4}$$
$$extended - gcd(3, 1) \quad -> \tag{5}$$
$$extended - gcd(1, 0) \quad -> \tag{6}$$

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$$extended - gcd(10, 7) \quad -> \tag{3}$$
$$extended - gcd(7, 3) \quad -> \quad returns(1, 1, -2) \tag{4}$$
$$extended - gcd(3, 1) \quad -> \quad returns(1, 0, 1) \tag{5}$$
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Euclid's Extended Algorithm

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Euclid's Extended Algorithm

$$extended - gcd(37, 10) \quad -> \quad returns(1, 3, -11) \tag{2}$$

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So a solution for $10x = 1 \bmod 37$ is $x = -11$, or equivalently, $x = 26 \bmod 37$.

Fermat's Little Theorem:

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$a^{p-1} = 1 \bmod p$ (p is a prime)

Calculate $2^{125} \bmod 127$. Hint: 127 is prime.

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$$(1)$$
$$(2)$$
$$(3)$$
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Fermat's Little Theorem

Calculate $2^{125} \bmod 127$. Hint: 127 is prime.

$$= \ 2^{-1} \cdot 2^{126} \bmod 127 \qquad (1)$$

$$(2)$$

$$(3)$$

$$(4)$$

$$(5)$$

Fermat's Little Theorem

Calculate $2^{125} \bmod 127$. Hint: 127 is prime.

$$= 2^{-1} \cdot 2^{126} \bmod 127 \tag{1}$$

$$= 2^{-1} \cdot 1 \bmod 127 \tag{2}$$

$$= 2^{-1} \bmod 127 \text{ (using extended euclid's)} \tag{3}$$

$$\tag{4}$$

$$\tag{5}$$

Fermat's Little Theorem

Calculate 2^{125} mod 127. Hint: 127 is prime.

$$= 2^{-1} \cdot 2^{126} \text{ mod } 127 \tag{1}$$
$$= 2^{-1} \cdot 1 \text{ mod } 127 \tag{2}$$
$$= 2^{-1} \text{ mod } 127 \text{ (using extended euclid's)} \tag{3}$$
$$= 64 \text{ mod } 127 \tag{4}$$
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Calculate $2^{125} \bmod 127$. Hint: 127 is prime.

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\begin{align}
&= 2^{-1} \cdot 2^{126} \bmod 127 \tag{1} \\
&= 2^{-1} \cdot 1 \bmod 127 \tag{2} \\
&= 2^{-1} \bmod 127 \text{ (using extended euclid's)} \tag{3} \\
&= 64 \bmod 127 \tag{4} \\
&= 2^{-1} = -63 = 64 \bmod 127 \tag{5}
\end{align}
$$

RSA Review

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- p, q : large prime numbers (approx. 512 bits each, $N = pq$)
- e : positive integer relatively prime to $(p-1)(q-1)$
- d : inverse of e (mod $(p-1)(q-1)$)

Topics   Propositions and Proof Techniques   Induction   Stable Marriage   Modular Arithmetic   Public Key Cryptography
○   ○○○○○   ○○○○○   ○○○○   ○○○○   ●
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RSA Review

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- e : positive integer relatively prime to $(p - 1)(q - 1)$
- d : inverse of e (mod $(p - 1)(q - 1)$)

Public key: (N, e)
Private key: d
Unencrypted message: x
Encrypted message: x^e (mod N)

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- d : inverse of e (mod $(p-1)(q-1)$)

Public key: (N, e)
Private key: d
Unencrypted message: $x = x^{ed}$ (mod N)
Encrypted message: x^e (mod N)

What's wrong with these RSA schemes?

- $p = 3$
- $q = 4$
- $e = 5$
- $d = 5$

1. p, q are too small.
2. ...

What's wrong with these RSA schemes?

- $p = 3$
- $q = 4$
- $e = 5$
- $d = 5$

1. p, q are too small.
2. ... q is not prime!

What's wrong with these RSA schemes?

- $p = 7$
- $q = 11$
- $e = 3$
- $d = 23$

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2. ...
3. ...

What's wrong with these RSA schemes?

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- $e = 3$
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1. p, q are too small.
2. ... e is not relatively prime to $(p-1)(q-1)$!
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What's wrong with these RSA schemes?

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- $q = 11$
- $e = 3$
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3. ... therefore, e has no inverse mod $(p-1)(q-1)$ and d does not exist!

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- $p = 11$
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- $e = 9$
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Topics
○

Propositions and Proof Techniques
○○○○○

Induction
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Stable Marriage
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Modular Arithmetic
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Public Key Cryptography
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   (Also, p and q are the same... Semi-related aside: taking the square root of a number does not take a lot of time, so it would not cost much for a hacker to check if $p == q == \sqrt{N}$. It also doesn't take a lot of time for you to generate another large prime - in other words, don't choose $p == q$)

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Great! Now we're ready to announce our public key (N, e) to the world!

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From here, we can just choose a message (x), encrypt the message ($x^e \pmod{N}$), and send the message to the person who knows d without having to worry about the message being intercepted. Let's say our message is $x = 2$, which means our encrypted message is $2^3 \pmod{15} = 8$.

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Thus, the original message was $x = 2$! We're done! RSA worked!

That's it!

Thanks for coming!
Good luck on your midterm!