P9120- Homework # 2

Assigned: October 10th, 2024 Due at 9pm EST on October 26, 2024

Maximum points that you can score in this Homework is 20.
Please include R or python code you used to complete this homework.

1. (5 points) Let $X \in \mathbb{R}^p$ be the vector of input variables and $Y \in \{-1, 1\}$ be the binary outcome. Consider classification rule $G(X) = \text{sign}(f(X))$, where f is a real-valued function on $\mathbb{R}^p$. For loss functions $L(y, f(\mathbf{x}))$ given in (a)-(d) below, let $f^* = \arg\min_f EL(Y, f(X))$, where the expectation is taken over the joint distribution of X and Y. Show that

   (a) (Binomial Deviance loss) If $L(y, f(\mathbf{x})) = \log[1+\exp(-yf(\mathbf{x}))]$, then $f^*(\mathbf{x}) = \log \frac{Pr(Y=1|X=\mathbf{x})}{Pr(Y=-1|X=\mathbf{x})}$.

   (b) (Hinge loss) If $L(y, f(\mathbf{x})) = [1-yf(\mathbf{x})]_+$, then $f^*(\mathbf{x}) = \text{sign}[Pr(Y = 1|X = \mathbf{x}) - \frac{1}{2}]$.

   (c) (quadratic loss) If $L(y, f(\mathbf{x})) = [y - f(\mathbf{x})]^2 = [1 - yf(\mathbf{x})]^2$, then $f^*(\mathbf{x}) = 2Pr(Y = 1|X = \mathbf{x}) - 1$.

   (d) (Exponential loss) If $L(y, f(\mathbf{x})) = \exp[-yf(\mathbf{x})]$, then $f^*(\mathbf{x}) = \frac{1}{2} \log \frac{Pr(Y=1|X=\mathbf{x})}{Pr(Y=-1|X=\mathbf{x})}$.

2. (5 points) In this problem, we review *Lagrangian duality* discussed in Lecture 4 with a simple example. Consider the optimization problem.

$$\min_{x \in \mathbb{R}} x^2 - x + 1 \text{ subject to } x \geq 1. \tag{1}$$

   (a) Plot the objective function $f(x) = x^2 - x + 1$ and indicate on your plot the feasible set (i.e. the set of x that satisfies the constraint). From this plot, determine the minimizer x^* and the value of objective function at the minimizer $f(x^*)$.

   (b) Write down the Lagrangian primal function $L(x, \lambda)$, where λ is the Lagrangian multiplier. Derive and plot the Lagrangian dual function $L_D(\lambda)$.

   (c) State and solve the Lagrangian dual problem (by hand). Denote the maximizer by λ^* and the corresponding Lagrangian dual function by $L_D(\lambda^*)$. Does strong duality hold? (i,.e. does $L_D(\lambda^*) = f^(x^*)$?)

   (d) Based on part (b) and part (c), we can solve the optimization problem (1) by choosing x that minimizes $L(x, \lambda^*)$. Verify that the minimizer of $L(x, \lambda^*)$ is indeed x^*.

3. (5 points) Python Lab 1 exercise. (P9120_Lab1_Exercise.ipynb)

4. (5 points) Python Lab 2 exercise. (P9120_Lab2_Exercise.ipynb)