Advanced Continuum Modelling
Magnetohydrodynamics (part 2)

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Outline

Outline

1. One-dimensional magnetohydrodynamics
2. Numerical considerations
3. Resistive MHD
4. A resistive model for lightning

Conservation law form

- Last lecture we introduced the conservation law form of the equations of MHD

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot \left[\rho \mathbf{v} \otimes \mathbf{v} + \left(p + \frac{1}{2} B^2 \right) \mathbf{I} - \mathbf{B} \otimes \mathbf{B} \right] = 0$$

$$\frac{\partial U}{\partial t} + \nabla \cdot \left[\left(U + p + \frac{1}{2} B^2 \right) \mathbf{v} - (\mathbf{v} \cdot \mathbf{B}) \mathbf{B} \right] = 0$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \cdot (\mathbf{B} \otimes \mathbf{v} - \mathbf{v} \otimes \mathbf{B}) = 0$$

- It is clear that this is written in standard conservation form

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot \mathbf{f}(\mathbf{u}) = 0$$

What happens in one dimension?

- The basis of numerical methods so far has been to consider one-dimensional equations, derivatives and Riemann problems - this will not change

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{f}(\mathbf{u})}{\partial x} + \frac{\partial \mathbf{g}(\mathbf{u})}{\partial y} + \frac{\partial \mathbf{h}(\mathbf{u})}{\partial z} = 0$$

- For MHD, writing the evolution equations in 1D is not as simple as removing all y- and z-components from the equations
- The 1D equations do still have to satisfy

$$\frac{\partial \mathbf{g}}{\partial y} = \frac{\partial \mathbf{h}}{\partial z} = 0$$

- But we **cannot** assume $B_y = B_z = 0$ and $v_y = v_z = 0$

Why don't y- and z-terms vanish?

- Consider the magnetic field evolution equation,

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \cdot (\mathbf{B} \otimes \mathbf{v} - \mathbf{v} \otimes \mathbf{B}) = 0$$

- For B_y, this becomes

$$\frac{\partial B_y}{\partial t} + \frac{\partial}{\partial x}(B_y v_x - v_y B_x) = 0$$

- A non-zero y-component of the magnetic field will evolve, and this will feed back into the momentum and energy equations
- However, the x-derivative of the magnetic field does lead to a vanishing equation

$$\frac{\partial B_x}{\partial t} + \frac{\partial}{\partial x}(B_x v_x - v_x B_x) = \frac{\partial B_x}{\partial t} = 0$$

- In 1D, B_x is **always** constant in time (though not necessarily zero)

Velocity evolution in one dimension

- Assuming that B_x does not vanish entirely (e.g. if a 1D Riemann solver is being used in a 2D simulation), then this also affects the velocity equations

$$\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot \left[\rho \mathbf{v} \otimes \mathbf{v} + \left(p + \frac{1}{2} B^2 \right) \mathbf{I} - \mathbf{B} \otimes \mathbf{B} \right] = 0$$

- Consider the y-component of the velocity

$$\frac{\partial \rho v_y}{\partial t} + \frac{\partial}{\partial x} \left(\rho v_x v_y - B_x B_y \right) = 0$$

- Even if v_y is initially zero, the magnetic field evolution can give it a value
- Therefore, only one equation vanishes moving from 3D to 1D MHD - the B_x evolution equation

The one-dimensional equations

- If we explicitly write out the 1D ideal MHD equations, we get a system of seven equations
- This has implications for solving Riemann problems for ideal MHD
- As we have seven equations, do we expect seven different waves for MHD?

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho v_x}{\partial x} = 0$$

$$\frac{\partial \rho v_x}{\partial t} + \frac{\partial}{\partial x}\left(\rho v_x^2 + p + \frac{1}{2}B^2 - B_x B_x\right) = 0$$

$$\frac{\partial \rho v_y}{\partial t} + \frac{\partial}{\partial x}\left(\rho v_x v_y - B_x B_y\right) = 0$$

$$\frac{\partial \rho v_z}{\partial t} + \frac{\partial}{\partial x}\left(\rho v_x v_z - B_x B_z\right) = 0$$

$$\frac{\partial U}{\partial t} + \frac{\partial}{\partial x}\left[\left(U + p + \frac{1}{2}B^2\right) v_x - \left(\mathbf{v} \cdot \mathbf{B}\right) B_x\right] = 0$$

$$\frac{\partial B_y}{\partial t} + \frac{\partial}{\partial x}\left(B_y v_x - B_x v_y\right) = 0$$

$$\frac{\partial B_z}{\partial t} + \frac{\partial}{\partial x}\left(B_z v_x - B_x v_z\right) = 0$$

Primitive variable form

- We can investigate the number of expected waves whilst also verifying that we actually have a hyperbolic system of equations
- Whilst we could work out the Jacobian matrix for the conserved system, and then compute its eigenvalues, this would not be fun
- As with the Euler equations, we are better off using the primitive variable form
- Like density, the magnetic field is its own primitive variable, the calculation is then fairly straightforward
- For example, the B_y component can be written

$$\frac{\partial B_y}{\partial t} + v_x \frac{\partial B_y}{\partial x} + B_y \frac{\partial v_x}{\partial x} - v_y \frac{\partial B_x}{\partial x} - B_x \frac{\partial v_y}{\partial x} = 0$$

- We've removed the B_x equation - what do we do about the derivative of this quantity?

Primitive variable form

- Recall that Maxwell's equations must obey the constraint $\nabla \cdot \mathbf{B} = 0$
- Since all y- and z-derivatives vanish in 1D, this gives

$$\nabla \cdot \mathbf{B} = \frac{\partial B_x}{\partial x} = 0$$

- This term simply vanishes from the primitive variable form, giving

$$\frac{\partial}{\partial t}\begin{pmatrix} \rho \\ v_x \\ v_y \\ v_z \\ p \\ B_y \\ B_z \end{pmatrix} + \begin{pmatrix} v_x & \rho & 0 & 0 & 0 & 0 & 0 \\ 0 & v_x & 0 & 0 & 1/\rho & B_y/\rho & B_z/\rho \\ 0 & 0 & v_x & 0 & 0 & -B_x/\rho & 0 \\ 0 & 0 & 0 & v_x & 0 & 0 & -B_x/\rho \\ 0 & \rho c_s^2 & 0 & 0 & v_x & 0 & 0 \\ 0 & B_y & -B_x & 0 & 0 & v_x & 0 \\ 0 & B_z & 0 & -B_x & 0 & 0 & v_x \end{pmatrix} \frac{\partial}{\partial x}\begin{pmatrix} \rho \\ v_x \\ v_y \\ v_z \\ p \\ B_y \\ B_z \end{pmatrix} = 0$$

- Once again, the Eigenvalues are a lot easier to compute, however, easier is a relative term; in this case it is still a very long calculation

MHD wave speeds

- If we did persist with this calculation, we would find seven distinct, real eigenvalues,

$$v_x, \quad v_x \pm c_{sl}, \quad v_x \pm c_a, \quad v_x \pm c_f$$

- Since they are all real, we do indeed have **a hyperbolic system**
- And we can expect up to seven waves in an MHD solution, governed by the velocity, v_x, the Alfvén speed c_a, and the slow and fast magneto-acoustic speeds, c_{sl} and c_f

$$c_a = \frac{|B_x|}{\sqrt{\rho}}, \qquad c_{f,sl} = \sqrt{\frac{1}{2}\left[c_s^2 + c_a^2 \pm \sqrt{\left(c_s^2 + c_a^2\right)^2 - 4\frac{c_s^2 B_x^2}{\rho}}\right]}$$

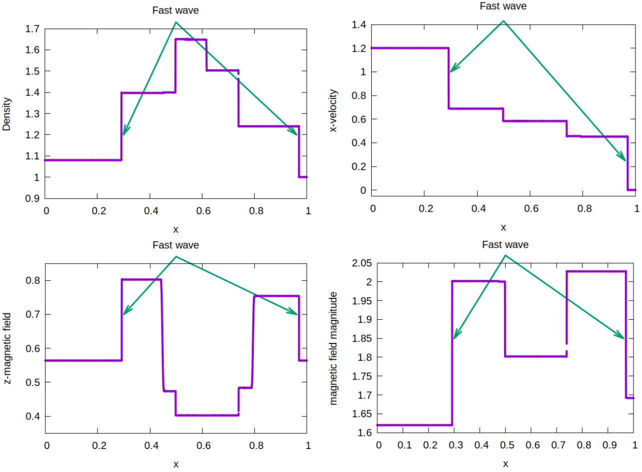
- For these equations, c_s, the speed of sound must not be confused with the slow wave speed c_{sl}

Waves in MHD

- With seven unique eigenvalues, we have seven waves, two of each of the fast, Alfvén and slow waves, and one contact discontinuity
- We can still have shock waves, rarefactions and contact discontinuities, but the presence of the magnetic field allows for other waves
- Note that across contact discontinuities, magnetic field, like pressure and velocity, is constant
- **Tangential discontinuities** occur when velocity and magnetic field are parallel to the wave; here, total pressure and normal velocity are continuous, but other variables jump
- These waves occur in limiting cases for the slow wave
- **Rotational discontinuities** have constant normal velocity and density and magnetic field **magnitude** but the magnetic field components can jump
- Alfvén waves are always rotational discontinuities, and it is important to note that there is mass transfer across this discontinuity

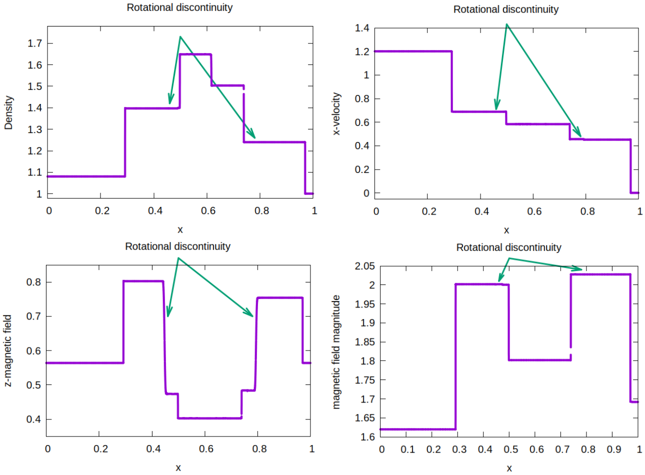
The seven waves in MHD

- The fast wave can be a shock or a rarefaction



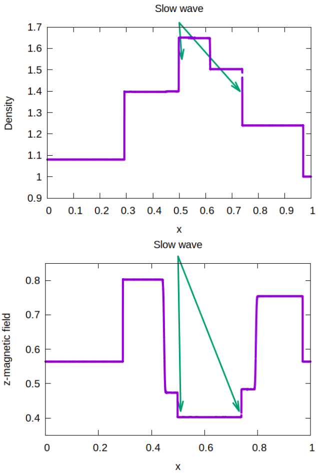
The seven waves in MHD

- Alfvén waves only have magnetic field jumps (but not in magnitude)



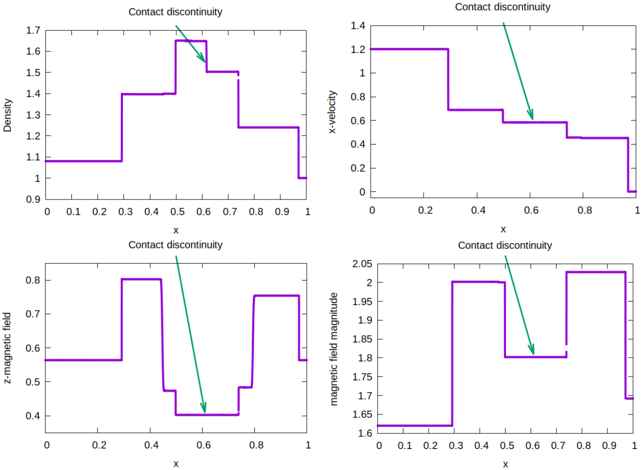
The seven waves in MHD

- The slow wave can be a shock, rarefaction or tangential discontinuity



The seven waves in MHD

- Only density (and energy) jump in a contact discontinuity



Degenerate wave cases

$$c_a = \frac{|B_x|}{\sqrt{\rho}}, \qquad c_{f,sl} = \sqrt{\frac{1}{2}\left[c_s^2 + c_a^2 \pm \sqrt{\left(c_s^2 + c_a^2\right)^2 - 4\frac{c_s^2 B_x^2}{\rho}}\right]}$$

- Note that if $\mathbf{B} = 0$, then $c_a = 0$ and $c_{sl} = 0$, whilst $c_f = c_s$, hence, as expected, we recover the wave speeds (and behaviour) for the standard Euler equations
- If $B_x = 0$, then we **still** have $c_a = 0$, $c_{sl} = 0$ and $c_f^2 = c_s^2$
- In this case, the Alfvén and slow waves are again degenerate, and this corresponds to the y- and z-momentum equations 'vanishing' from the system
- It is clear from the equations that $B_x = 0$ means there is no evolution of v_y possible any more (but B_y and B_z can still evolve through their evolution equations)

$$\frac{\partial \rho v_y}{\partial t} + \frac{\partial}{\partial x}\left(\rho v_x v_y - B_x B_y\right) = 0$$

Outline

Centred schemes in MHD

- Now we have the MHD equations in conservation form, we can begin to apply numerical methods to solve them
- A SLIC (or FORCE) scheme will work - only the number of variables and definition of fluxes needs to change, as well as the calculation of the time step
- For SLIC, the choice of variable for limiting is important; total energy will not always be a good choice, since this doesn't jump across Alfvén waves
- The time step is governed by the fastest moving wave, which, in MHD is (appropriately) the fast wave
- The wave speeds are, in fact, always ordered

$$v_x - c_f \leq v_x - c_a \leq v_x - c_{sl} \leq v_x \leq v_x + c_{sl} \leq v_x + c_a \leq v_x + c_f$$

- Therefore, a maximum wave speed of $a_{\max} = |v_x| + c_f$ is appropriate, or $a_{\max} = |\mathbf{v}| + c_f$ in multiple dimensions

The Riemann problem for MHD

- As we have seen, Riemann problem-based Godunov-type methods can often achieve better capturing of the waves than centred schemes
- Although it is possible to use an exact solver, which models all seven waves, this is generally accepted to be too computationally expensive (and solutions may not always be unique)
- Instead, approximate Riemann solvers are a standard technique, they are based on those for the familiar solvers for the Euler equations (HLL, HLLC etc.)
- They still consider a reduced number of waves in the solution, e.g. HLLC still has a contact discontinuity, a left-moving wave and a right-moving wave

HLL for MHD

- We originally considered the HLL approximate Riemann solver for the Euler equations
- However, when we defined it, we did not need to use anything specific to the Euler equations, we simply expressed it for general conserved quantities, $\mathbf{u}$ and $\mathbf{f}(\mathbf{u})$
- As a result, the method can be used (almost) unchanged for MHD
- We still consider the solution between two states, the left state, $\mathbf{u}_L$ and the right state, $\mathbf{u}_R$, where there is a single intermediate state, $\mathbf{u}^*$
- The intermediate state is assumed to obey the Rankine-Hugoniot conditions

$$\mathbf{f}^* - \mathbf{f}_{L/R} = S_{L/R}\left(\mathbf{u}^* - \mathbf{u}_{L/R}\right)$$

- The only 'difference' between MHD and Euler equations is the wave speed estimates (or, more accurately, the constraints on accurate choices of these speeds)

HLL for MHD

- We can take the maximum and minimum wave speeds (S_R and S_L) to be

$$S_L = \min\left(v_{x,L}, v_{x,R}\right) - \max\left(c_{f,L}, c_{f,R}\right), \quad S_R = \max\left(v_{x,L}, v_{x,R}\right) + \max\left(c_{f,L}, c_{f,R}\right)$$

- In other words, simply use the fast wave speed in place of the sound speed
- The HLL intermediate state is still given by

$$\mathbf{u}^{\mathrm{HLL}} = \frac{S_R \mathbf{u}_R - S_L \mathbf{u}_L + \mathbf{f}_L - \mathbf{f}_R}{S_R - S_L}$$

- And the HLL approximate flux is still given by

$$\mathbf{f}^{\mathrm{HLL}} = \frac{S_R \mathbf{f}_L - S_L \mathbf{f}_R + S_L S_R \left(\mathbf{u}_R - \mathbf{u}_L\right)}{S_R - S_L}$$

HLLC for MHD

- The HLLC approximate Riemann solver offered a compromise between accuracy (sharper contact discontinuities) and simplicity (iterative solvers are not required)
- We still assume a three-wave solution between the left state, $\mathbf{u}_L$ and the right state, $\mathbf{u}_R$, with intermediate variables $\mathbf{u}_L^*$ and $\mathbf{u}_R^*$, a contact discontinuity between them
- The general form for HLLC is unchanged from the Euler equations

$$\mathbf{f}^{\mathrm{HLLC}} = \begin{cases} \mathbf{f}_L & 0 \leq S_L \\ \mathbf{f}_L + S_L\left(\mathbf{u}_L^* - \mathbf{u}_L\right) & S_L \leq 0 \leq v_x^* \\ \mathbf{f}_R + S_R\left(\mathbf{u}_R^* - \mathbf{u}_R\right) & v_x^* \leq 0 \leq S_R \\ \mathbf{f}_R & S_R \leq 0 \end{cases}$$

- The HLLC solver we describe is found in Li, ‘An HLLC Riemann Solver for Magnetohydrodynamics’ (2005)

Total pressure

$$\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot \left[\rho \mathbf{v} \otimes \mathbf{v} + \left(p + \frac{1}{2} B^2 \right) \mathbf{I} - \mathbf{B} \otimes \mathbf{B} \right] = 0$$

- If we consider the evolution equation for momentum, we see that the term $(p + \frac{1}{2} B^2)$ only acts in the normal direction
- This normal forcing was how we crudely justified including pressure in our original derivation of the Euler equations
- For MHD, we can consider two contributions which act like a pressure, the **hydrodynamic pressure**, p, and the **magnetic pressure**, $\frac{1}{2} B^2$
- Together, these are sometimes known as the **total pressure**, p_T (or, unfortunately, p as well)
- Because Li uses this for defining the HLLC scheme, we shall do so too

HLLC for MHD - intermediate velocity

- The HLLC scheme again considers the two intermediate states to be governed by the Rankine-Hugoniot jump conditions

$$\mathbf{f}_{L/R}^{\mathrm{HLLC}} - \mathbf{f}_{L/R} = S_{L/R}\left(\mathbf{u}_{L/R}^{\mathrm{HLLC}} - \mathbf{u}_{L/R}\right)$$

- If we consider the intermediate wave between the star states to be a contact discontinuity, moving with wave speed $S_M = v_{x,L}^* = v_{x,R}^* = S^*$, then coupled with our estimates for the left and right wave speeds, this system can be solved exactly
- Based on the original HLLC method, the intermediate wave speed is given by

$$S^* = \frac{\rho_R v_{x,R}\left(S_R - v_{x,R}\right) - \rho_L v_{x,L}\left(S_L - v_{x,L}\right) + p_{T,L} - p_{T,R} - B_{x,L}^2 + B_{x,R}^2}{\rho_R\left(S_R - v_{x,R}\right) - \rho_L\left(S_L - v_{x,L}\right)}$$

HLLC for MHD - intermediate states

- Although the rest of the star states can be computed from the Rankine-Hugoniot conditions, unfortunately this gives us a solution which is **inconsistent** with the integral form of the MHD equations
- Such a method is therefore guaranteed to fail
- The method of Li modifies this approach to maintain consistency, and in doing so, still retains many of the standard HLLC properties
- For example, pressure is computed from the Rankine-Hugoniot conditions

$$p_T^{\mathrm{HLLC}} = \rho \left(S_K - v_{x,K}\right)\left(S^* - v_{x,K}\right) + p_T - B_{x,K}^2 + \left(B_{x,K}^{\mathrm{HLLC}}\right)^2$$

- For all definitions, we consider the intermediate states, $K = [L, R]$
- This satisfies $p_{T,L}^{\mathrm{HLLC}} = p_{T,R}^{\mathrm{HLLC}}$ only if (using the definition for S^* can show this)

$$B_{x,L}^{\mathrm{HLLC}} = B_{x,R}^{\mathrm{HLLC}}$$

HLLC for MHD - intermediate states

- Several other relationships from the Rankine-Hugoniot conditions can still be used in MHD:

$$\rho_K^{\text{HLLC}} = \rho_K \frac{S_K - v_{x,K}}{S_K - S^*}, \qquad (\rho v_x)_K^{\text{HLLC}} = \rho_K^{\text{HLLC}} S^*$$

$$\left(\rho v_{(y,z)}\right)_K^{\text{HLLC}} = \rho_K v_{(y,z),K} \frac{S_K - v_{x,K}}{S_K - S^*} - \frac{B_{x,K}^{\text{HLLC}} B_{(y,z),K}^{\text{HLLC}} - B_{x,K} B_{(y,z),K}}{S_K - S^*}$$

$$U_K^{\text{HLLC}} = U_K \frac{S_K - v_{x,K}}{S_K - S^*} + \frac{p_T^{\text{HLLC}} S^* - p_{T,K} v_{x,K} - \left[B_x^{\text{HLLC}} \left(\mathbf{B} \cdot \mathbf{v}\right)^{\text{HLLC}} - B_{x,K} \left(\mathbf{B} \cdot \mathbf{v}\right)\right]}{S_L - S^*}$$

HLLC for MHD - intermediate magnetic fields

- The relationships so far have removed the unknown fluid quantities from the intermediate states, but not the magnetic fields
- Although these can be derived from the Rankine-Hugoniot, it is these definitions that are not consistent with the underlying equations - this uses the consistency condition (see Toro)

$$\frac{S^* - S_L}{S_R - S_L}\mathbf{u}_L^{\mathrm{HLLC}} + \frac{S_R - S^*}{S_R - S_L}\mathbf{u}_R^{\mathrm{HLLC}} = \frac{S_R\mathbf{u}_R - S_L\mathbf{u}_L - (\mathbf{f}_R - \mathbf{f}_L)}{S_R - S_L} = \mathbf{u}^{\mathrm{HLL}}$$

- It can be shown that the HLLC definitions for intermediate momentum only satisfy this condition if

$$B_{x,L}^{\mathrm{HLLC}} B_{y,L}^{\mathrm{HLLC}} = B_{x,R}^{\mathrm{HLLC}} B_{y,R}^{\mathrm{HLLC}}, \qquad B_{x,L}^{\mathrm{HLLC}} B_{z,L}^{\mathrm{HLLC}} = B_{x,R}^{\mathrm{HLLC}} B_{z,R}^{\mathrm{HLLC}}$$

- This tells us that magnetic field components do not jump across the contact discontinuity - the simplest values to give these components, which also obey the consistency condition, are

$$B_y^{\mathrm{HLLC}} = B_y^{\mathrm{HLL}} \qquad B_z^{\mathrm{HLLC}} = B_z^{\mathrm{HLL}}$$

HLLC for MHD - completing the solver

- In order to obtain a condition on B_x, we consider the consistency condition for the energy equation, which holds only if

$$B_{x,L}^{\mathrm{HLLC}} \left(\mathbf{B}\cdot\mathbf{v}\right)_L^{\mathrm{HLLC}} = B_{x,R}^{\mathrm{HLLC}} \left(\mathbf{B}\cdot\mathbf{v}\right)_R^{\mathrm{HLLC}}$$

- Since tangential velocities jump across the interface, we cannot use the intermediate velocity relationships to satisfy the consistency condition, instead we define

$$\left(\mathbf{B}\cdot\mathbf{v}\right)^{\mathrm{HLLC}} = \mathbf{B}^{\mathrm{HLL}}\cdot\mathbf{v}^{\mathrm{HLL}}$$

- Finally, we need an expression for B_x^{HLLC} (this is only needed for more than 1D) - following the fact that this component is constant across the contact discontinuity, we set

$$B_x^{\mathrm{HLLC}} = B_x^{\mathrm{HLL}}$$

HLLC for MHD - completing the solver

- Now all intermediate states have been identified, the inter-cell fluxes are computed using the standard HLLC method

$$\mathbf{f}^{\mathrm{HLLC}} = \begin{cases} \mathbf{f}_L & 0 \leq S_L \\ \mathbf{f}_L + S_L \left(\mathbf{u}_L^{\mathrm{HLLC}} - \mathbf{u}_L \right) & S_L \leq 0 \leq S^* \\ \mathbf{f}_R + S_R \left(\mathbf{u}_R^{\mathrm{HLLC}} - \mathbf{u}_R \right) & S^* \leq 0 \leq S_R \\ \mathbf{f}_R & S_R \leq 0 \end{cases}$$

The divergence constraint

- The full form of the ideal MHD equations is the four evolution equations plus the divergence constraint

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot \left[\rho \mathbf{v} \otimes \mathbf{v} + \left(p + \frac{1}{2}B^2\right)\mathbf{I} - \mathbf{B} \otimes \mathbf{B}\right] = 0$$

$$\frac{\partial U}{\partial t} + \nabla \cdot \left[\left(U + p + \frac{1}{2}B^2\right)\mathbf{v} - (\mathbf{v} \cdot \mathbf{B})\,\mathbf{B}\right] = 0$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \cdot (\mathbf{B} \otimes \mathbf{v} - \mathbf{v} \otimes \mathbf{B}) = 0$$

$$\nabla \cdot \mathbf{B} = 0$$

- Mathematically, provided the initial data satisfies the divergence constraint, then the solution to these equations continues to do so
- Unfortunately, numerically, this is not the case - numerical errors mean it is unlikely that divergence completely vanishes, and this can lead to growing instabilities and conservation errors

Avoiding diverging fields

- Ensuring divergence-free simulations is essential for accurate results, and there is more than one approach, see, for example, Vides *et al.* 'Divergence-free MHD simulations with the Heracles code', (2013)
- Two key methods from this paper are divergence cleaning and constrained transport
- Divergence cleaning can be built directly upon a naive implementation of the underlying equations, but introduces an additional evolution equation
- Constrained transport requires a staggered grid approach, for which magnetic field components are evolved at a different location to the fluid variables, and thus must be built into the numerical method
- For simplicity of numerical implementation, we consider only the divergence cleaning approach

Divergence cleaning

- The idea behind this technique is that we can 'evolve away' any errors that appear in the divergence constraint
- This is done through the introduction of a new variable, ψ, which appears in the evolution of the magnetic field, and also is evolved itself

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \cdot (\mathbf{B} \otimes \mathbf{v} - \mathbf{v} \otimes \mathbf{B}) + \nabla \psi = 0$$

$$\mathcal{D}(\psi) + \nabla \cdot \mathbf{B} = 0$$

- If the operator $\mathcal{D}$ is chosen correctly, then it can be shown that divergence errors dissipate and propagate away
- Following Dedner *et al.* 'Hyperbolic Divergence Cleaning for the MHD Equations' (2002), we use

$$\mathcal{D}(\psi) = \frac{1}{c_h^2} \frac{\partial \psi}{\partial t} + \frac{1}{c_p^2} \psi$$

- This definition uses a hyperbolic wave speed, c_h, and a parabolic damping term, c_p

Divergence cleaning MHD equations

- We now have an additional hyperbolic equation to solve

$$\frac{\partial \psi}{\partial t} + c_h^2 \nabla \cdot \mathbf{B} = -\frac{c_h^2}{c_p^2}\psi$$

- Numerically, this is fairly simple, since, for a 1D Riemann-solver based technique, this equation, plus the evolution of B_x decouple from the complete system (see Vides *et al.*)

$$\frac{\partial \psi}{\partial t} + c_h^2 \frac{\partial B_x}{\partial x} = -\frac{c_h^2}{c_p^2}\psi, \qquad \frac{\partial B_x}{\partial t} + \frac{\partial \psi}{\partial x} = 0$$

- The value for c_h is the speed of propagation of the divergence errors - in order to maintain stability of the system, whilst keeping propagation as fast as possible, we pick the maximum possible wave speed

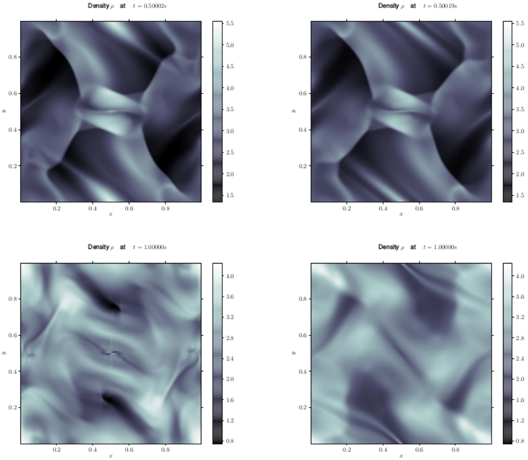
$$c_h = \max\left(|v_x| + c_{f,x}, |v_y| + c_{f,y}, |v_z| + c_{f,z}\right)$$

- The value for c_p is not so obvious, and was, in fact, determined by numerical experiment - Dedner *et al.* give

$$\frac{c_p^2}{c_h} = 0.18$$

What does divergence cleaning do?

- We show images taken from Vides *et al.*, the left hand side shows no limitation on divergence, and the right hand side shows divergence cleaning
- It is clear that the effects of the violation of $\nabla \cdot \mathbf{B} = 0$ take a while to build up - bottom plots have run for twice as long as the top plots
- It is clear that error builds up if there is no divergence constraint treatment



Outline

Limits of the ideal approximation

- Ideal MHD is a valuable starting point for understanding and modelling plasma
- Many astrophysical applications happen on large enough scales that this approximation is valid, and the behaviour in the **centre** of a fusion reactor can also be treated this way
- Mathematically, all types of MHD have the same conserved variables in the ideal limit, our solution shows all the possible wave types, and we can use the same numerical techniques
- However, most terrestrial applications, such as **full** simulations of a fusion reactor, lightning, and many other astrophysical ones, cannot neglect resistivity
- Whilst incorporating these effects into the equations is straightforward, mathematically, there may be better treatments of the system

Beyond ideal MHD

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot \left[\rho \mathbf{v} \otimes \mathbf{v} + \left(p + \frac{1}{2} \frac{B^2}{\mu_0} \right) \mathbf{I} - \frac{1}{\mu_0} \mathbf{B} \otimes \mathbf{B} \right] = 0$$

$$\frac{\partial U}{\partial t} + \nabla \cdot \left[\left(U + p + \frac{1}{2} \frac{B^2}{\mu_0} \right) \mathbf{v} - \frac{1}{\mu_0} (\mathbf{v} \cdot \mathbf{B}) \mathbf{B} \right] = 0$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \cdot (\mathbf{B} \otimes \mathbf{v} - \mathbf{v} \otimes \mathbf{B}) = 0$$

$$\mathbf{J} = \mu_0^{-1} \nabla \times \mathbf{B}$$

$$\mathbf{E} = -\mathbf{v} \times \mathbf{B}$$

$$\nabla \cdot \mathbf{B} = 0$$

- To obtain the resistive MHD equations, we could start our derivation, combining Maxwell's equations and the Euler equations into a single conservative system again
- But that is a lot of effort, and we can fairly easily 'correct' our system instead
- The changes are that electric field is now given by

$$\mathbf{E} = \eta \mathbf{J} - \mathbf{v} \times \mathbf{B}$$

and there was a source term for the internal energy equation, $\eta \mathbf{J} \cdot \mathbf{J}$

Resistive MHD - magnetic field

$$\mathbf{E} = \eta \mathbf{J} - \mathbf{v} \times \mathbf{B}$$

- The electric field was used in deriving the conservation law for the magnetic field, and it is fairly simple to see how things change
- Recall that the magnetic field equation can be written

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = \frac{\partial \mathbf{B}}{\partial t} + \nabla \times (\eta \mathbf{J}) - \nabla \times (\mathbf{v} \times \mathbf{B}) = 0$$

- This gives the same conservation equation, but now we include source terms

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \cdot (\mathbf{B} \otimes \mathbf{v} - \mathbf{v} \otimes \mathbf{B}) = -\nabla \times (\eta \mathbf{J})$$

Resistive MHD - total energy

- When deriving the energy equation for ideal MHD, we included a contribution for the magnetic component of the total energy through

$$\mathbf{B} \cdot \left(\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} \right) = 0$$

- This gives an additional term in the energy equation
- Before obtaining the source term, we need to consider one thing; in unscaled units,

$$U = \rho \varepsilon + \frac{1}{2} \rho v^2 + \frac{1}{2\mu_0} B^2$$

- So when we consider resistivity, in unscaled units, we actually want to include a component

$$\frac{1}{\mu_0} \mathbf{B} \cdot \left(\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} \right) = -\frac{1}{\mu_0} \mathbf{B} \cdot \nabla \times (\eta \mathbf{J})$$

Resistive MHD - total energy

$$\frac{1}{\mu_0}\mathbf{B}\cdot\left(\frac{\partial \mathbf{B}}{\partial t} + \nabla\times\mathbf{E}\right) = -\frac{1}{\mu_0}\mathbf{B}\cdot\nabla\times(\eta\mathbf{J})$$

- A second part of the source term comes from the resistive contribution to the internal energy evolution equation

$$\frac{\partial \rho e}{\partial t} + \nabla\cdot(\rho e\mathbf{v}) + p\nabla\cdot\mathbf{v} = \eta\mathbf{J}\cdot\mathbf{J}$$

- From these two contributions, we obtain a source term

$$s_U(\mathbf{u}) = \eta\mathbf{J}\cdot\mathbf{J} - \frac{1}{\mu_0}\mathbf{B}\cdot[\nabla\times(\eta\mathbf{J})]$$

Resistive MHD - total energy

- Using $\mathbf{J} = \mu_0^{-1}\nabla \times \mathbf{B}$, we can write this source contribution as

$$\eta\mathbf{J}\cdot\mathbf{J} - \frac{1}{\mu_0}\mathbf{B}\cdot\left[\nabla\times(\eta\mathbf{J})\right] = \frac{1}{\mu_0}\nabla\cdot(\mathbf{B}\times\eta\mathbf{J})$$

- This then gives us a single source term for our conservation of energy equation

$$\frac{\partial U}{\partial t} + \nabla\cdot\left[\left(U + p + \frac{1}{2}\frac{B^2}{\mu_0}\right)\mathbf{v} - \frac{1}{\mu_0}\left(\mathbf{v}\cdot\mathbf{B}\right)\mathbf{B}\right] = \frac{1}{\mu_0}\nabla\cdot(\mathbf{B}\times\eta\mathbf{J})$$

- Although this appears to be a divergence term, it is not of the form $\mathbf{f}\left(\mathbf{u}\right)$, because current density is related to the derivative of the magnetic field
- This then conceals a second order derivative in this term

The full resistive model

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot \left[\rho \mathbf{v} \otimes \mathbf{v} + \left(p + \frac{1}{2} \frac{B^2}{\mu_0} \right) \mathbf{I} - \frac{1}{\mu_0} \mathbf{B} \otimes \mathbf{B} \right] = 0$$

$$\frac{\partial U}{\partial t} + \nabla \cdot \left[\left(U + p + \frac{1}{2} \frac{B^2}{\mu_0} \right) \mathbf{v} - \frac{1}{\mu_0} (\mathbf{v} \cdot \mathbf{B}) \mathbf{B} \right] = \frac{1}{\mu_0} \nabla \cdot (\mathbf{B} \times \eta \mathbf{J})$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \cdot (\mathbf{B} \otimes \mathbf{v} - \mathbf{v} \otimes \mathbf{B}) = -\nabla \times (\eta \mathbf{J})$$

- We now see that resistive MHD is "just" ideal MHD with source terms
- Unfortunately, these source terms contain second derivatives of a conserved variable - recall the implications this has for the time step
- Sometimes, additional physical approximations let us avoid these problems (lightning), sometimes they don't (fusion)

Further complications for fusion

- For the rest of the lecture, we are going to focus on the specific application of lightning, rather than fusion
- Partly, this is because we have discussed (briefly) how to deal with source terms such as those in resistive MHD
- However, the resistive model presented here is still not the one used in many fusion applications
- A plasma contains ions and electrons, both of which have very different mass, but the formulations so far have assumed they behave in the same way (have the same velocity)
- Models exist which relax this assumption, **two-fluid** approximations, and this type of model is not covered until Introduction to Computational Multiphysics

Outline

Is this the formulation for lightning?

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

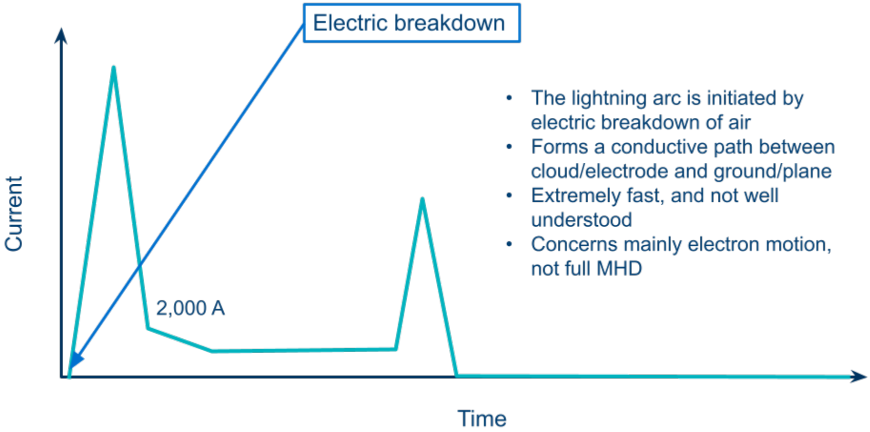
$$\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot \left[\rho \mathbf{v} \otimes \mathbf{v} + \left(p + \frac{1}{2}\frac{B^2}{\mu_0}\right)\mathbf{I} - \frac{1}{\mu_0}\mathbf{B} \otimes \mathbf{B}\right] = 0$$

$$\frac{\partial U}{\partial t} + \nabla \cdot \left[\left(U + p + \frac{1}{2}\frac{B^2}{\mu_0}\right)\mathbf{v} - \frac{1}{\mu_0}(\mathbf{v} \cdot \mathbf{B})\,\mathbf{B}\right] = \frac{1}{\mu_0}\nabla \cdot (\mathbf{B} \times \eta \mathbf{J})$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \cdot (\mathbf{B} \otimes \mathbf{v} - \mathbf{v} \otimes \mathbf{B}) = -\nabla \times (\eta \mathbf{J})$$

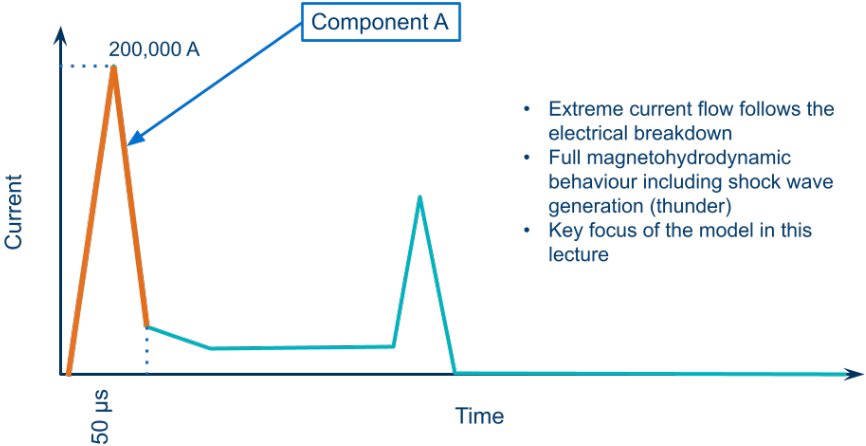
- Lightning is a complex, dynamic process, and happens over a range of time and length scales
- So before answering this question, we consider what happens during a lightning strike

The progress of a lightning strike



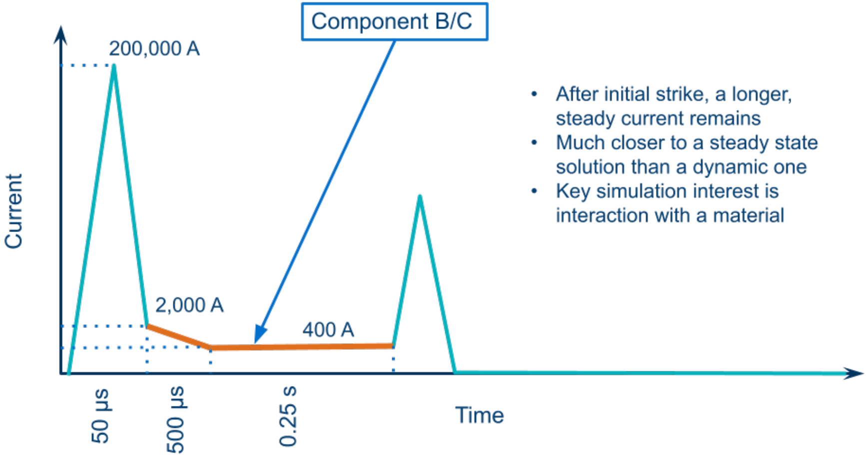
- The lightning arc is initiated by electric breakdown of air
- Forms a conductive path between cloud/electrode and ground/plane
- Extremely fast, and not well understood
- Concerns mainly electron motion, not full MHD

The progress of a lightning strike



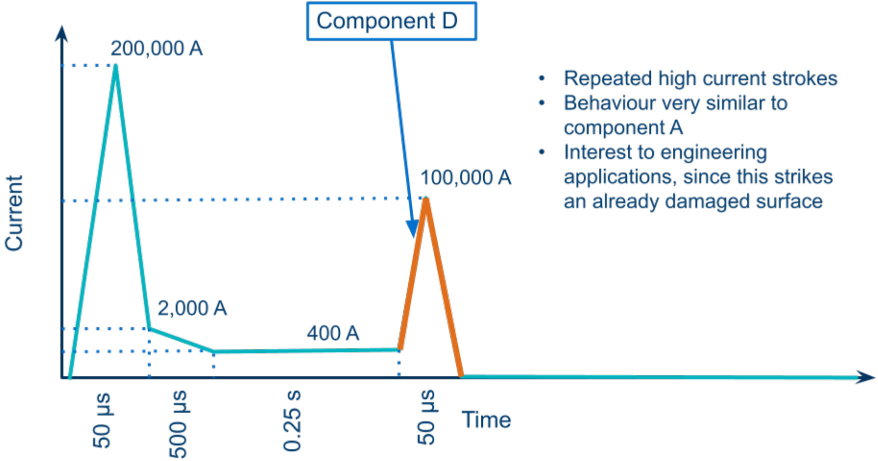
- Extreme current flow follows the electrical breakdown
- Full magnetohydrodynamic behaviour including shock wave generation (thunder)
- Key focus of the model in this lecture

The progress of a lightning strike



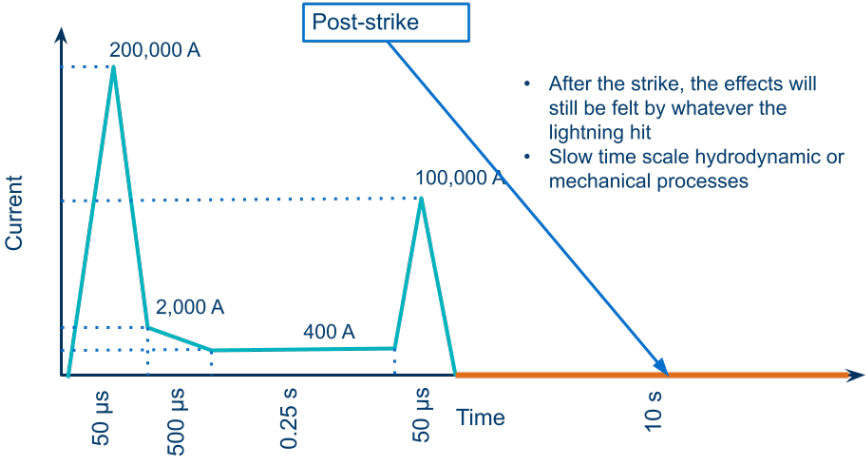
- After initial strike, a longer, steady current remains
- Much closer to a steady state solution than a dynamic one
- Key simulation interest is interaction with a material

The progress of a lightning strike



- Repeated high current strokes
- Behaviour very similar to component A
- Interest to engineering applications, since this strikes an already damaged surface

The progress of a lightning strike



- After the strike, the effects will still be felt by whatever the lightning hit
- Slow time scale hydrodynamic or mechanical processes

The challenges of lightning



- Electric current, I, is related to current density, $\mathbf{J}$, for a given volume, V, we have

$$I = \int_S \mathbf{J} \cdot \mathbf{n} \mathrm{d}S$$

- The source of this current is typically not modelled, for reasons complexity and physical understanding
- Current flow is due to the flow of electrons, which move **quickly** ($\sim 0.3c$) - attempts to include the evolution could lead to very small time steps, which is the issue we are trying to avoid

How can we solve for a current-induced plasma?

- This doesn't mean we can't solve for current, Chen *et al.* (2014) suggest a 1D wave equation formulation

$$\frac{\partial^2 I_{\mathrm{in}}}{\partial t^2} - \frac{0.3c^2}{\left(V^*\right)^2}\frac{\partial^2 I_{\mathrm{in}}}{\partial z^2} = 0$$

- Once this is solved, the magnetic field has to be recomputed based on the current distribution
- We could solve this equation in 1D at least, the time step restriction could be overcome, but current profiles, in both lightning and experimental representations, are **not** 1D
- However, this formulation does tell us one thing - the evolution of the magnetic field is driven by the current, not the fluid velocity
- We can use this to create an **electrostatic** representation of the electric and magnetic fields, removing time derivatives of these quantities

The electrostatic approach for current-induced plasma

- Effectively, we are saying that the magnetic field is evolving much faster than the hydrodynamics of the system
- If we consider the magnetic field evolution, then the fluid variables are close to static
- This would be a steady-state solution where all that matters is our boundary conditions
- However, if we consider the fluid variables, as they evolve, it appears that the magnetic field nearly instantaneously 'relaxes' to match their evolution
- This would imply the magnetic field is a background field for the hydrodynamic system
- We shall consider this a bit more quantitatively too once we have some equations

Lightning MHD equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\rho \left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right] + \nabla p - \mathbf{J} \times \mathbf{B} = 0$$

$$\frac{\partial \rho \varepsilon}{\partial t} + \nabla \cdot (\rho \varepsilon \mathbf{v}) + p \nabla \cdot \mathbf{v} = \eta \mathbf{J} \cdot \mathbf{J}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0$$

$$\mathbf{J} = \mu_0^{-1} \nabla \times \mathbf{B}$$

$$\mathbf{E} = \eta \mathbf{J} - \mathbf{v} \times \mathbf{B}$$

$$\nabla \cdot \mathbf{B} = 0$$

- We return to our system of hydrodynamic equations and Maxwell's equations
- Under the electrostatic approach, the magnetic field evolution equation can simply be ignored
- We'll worry about how we compute it later
- But for now, we consider what this means for the hydrodynamic variables; the magnetic terms can only appear as **source terms**

MHD for lightning

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v} + p\mathbf{I}) = \mathbf{J} \times \mathbf{B}$$

$$\frac{\partial \hat{U}}{\partial t} + \nabla \cdot \left[\left(\hat{U} + p \right) \mathbf{v} \right] = \mathbf{v} \cdot (\mathbf{J} \times \mathbf{B}) + \eta \mathbf{J} \cdot \mathbf{J}$$

- Obviously the conservation of mass still exists, as previously
- We reached the form of the conservation of momentum when deriving the MHD equations in conservative form
- And we also know that we reach the conservation of energy equation by combining the evolution equation for internal energy with the dot product of velocity and the conservation of momentum
- So the source terms follow this pattern
- Note that $\hat{U} = \rho\varepsilon + \frac{1}{2}\rho v^2$

MHD for lightning

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v} + p\mathbf{I}) = \mathbf{J} \times \mathbf{B}$$

$$\frac{\partial \hat{U}}{\partial t} + \nabla \cdot \left[\left(\hat{U} + p \right) \mathbf{v} \right] = \mathbf{v} \cdot (\mathbf{J} \times \mathbf{B}) + \eta \mathbf{J} \cdot \mathbf{J}$$

- The MHD formulation for lightning has a few clear advantages:
  1. The equations have reduced to a system which looks like the Euler equations with some source terms
  2. These source terms don't depend on the derivative of a conserved variable; we do not suffer the same time step issues
- Unfortunately, we are not free of time step issues, but a linear reduction now exists (not a $\propto \Delta x^2$ reduction)

Effects of this system

- We started with a system of equations which had a seven-wave solution
- And now we have a three-wave solution - can consider how this is justified
- We can, by considering the wave speeds for the lightning system

$$c_a = \frac{|B_x|}{\sqrt{\mu_0 \rho}}, \qquad c_{f,sl} = \sqrt{\frac{1}{2}\left[c_s^2 + c_a^2 \pm \sqrt{\left(c_s^2 + c_a^2\right)^2 - 4\frac{c_s^2 B_x^2}{\mu_0 \rho}}\right]}$$

- Note we have dimensionalised the Alfvén wave speed
- In order to justify this, we are saying that $c_a \ll c_s$
- Within a lightning strike, we have $\rho \sim 0.1\,\mathrm{kg/m^3}$ and $p \sim 10^6\,\mathrm{Pa}$, giving, even under crude models, $c_s \sim 10^4 \mathrm{m/s}$
- For $|\mathbf{B}| < 0.1\,\mathrm{T}$ we could justify this, but lighting can reach up to 10 T close to the arc

Justifying the model

- The original justification of this approach can be found in Chemartin et al. "Three dimensional simulation of a DC free burning arc. Application to lightning physics", (2009)
- They do it by justifying how they can neglect time derivatives of the magnetic field (to achieve the same result)
- Considering cases of around 100 A (not 100,000 A!)
- The full application to a high-current lightning strike is therefore a little risky
- Fortunately, Chemartin and co-authors (and us in the LSC) have shown it does produce good results
- But why? We shall return to this after we have worked out how to compute the magnetic field

Computing the current density

- In order to compute magnetic field, we also need current density, the two are related through Ampere's law
$$\mu_0 \mathbf{J} = \nabla \times \mathbf{B}$$
- Previously, we knew the magnetic field, so worked the other way to compute current density
- This was one of the simplification in deriving the ideal MHD equations, we were able to 'remove' the charge continuity equation
$$\frac{\partial \tau}{\partial t} + \nabla \cdot \mathbf{J} = 0$$
- We can now use this equation to compute the current density
- Our assumption that the current propagation is much faster than the evolution of the plasma means we can neglect the time derivatives for charge density, hence we don't need to actually work out what τ is
$$\nabla \cdot \mathbf{J} = 0$$

From current density to electric potential

$$\nabla \cdot \mathbf{J} = 0$$

- This is now a steady state problem; one of the key ways to solve this is to convert it to an elliptic problem
- We do this through the original evolution equation for magnetic field

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0 \qquad \rightarrow \qquad \nabla \times \mathbf{E} = 0$$

- This implies that $\mathbf{E}$ is a **conservative field**, i.e. that it must be the gradient of a scalar quantity
- This scalar is termed the electric potential, where

$$\mathbf{E} = \nabla \phi$$

From current density to electric potential

- Current density is then related to the electric field through Ohm's law

$$\mathbf{J} = \sigma\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right)$$

- Our electrostatic approach means we assume $\mathbf{v} \times \mathbf{B} \sim 0$ - is this valid?
- Current density can reach $\mathcal{O}\left(10^8\right)$ A/m^2, whilst extreme values for σ, $\mathbf{v}$ and $\mathbf{B}$ are $\mathcal{O}\left(10^3\right)$ S/m, $\mathcal{O}\left(1000\right)$ m/s and $\mathcal{O}\left(10\right)$ T respectively, giving a total maximum magnitude of $\mathcal{O}\left(10^7\right)$ A/m^2
- Though potentially comparable in magnitude, in the evolution of a plasma arc, maximum values for velocity and conductivity do not coincide, and as a result, we reduce the magnitude of $\sigma\left(\mathbf{v} \times \mathbf{B}\right)$ by 1-2 orders of magnitude
- As a result, the lightning plasma literature is able to consider Ohm's law of the form $\mathbf{J} = \sigma\mathbf{E}$
- To obtain the current density, we the have to solve an elliptic equation for electric potential

$$\nabla \cdot \sigma \nabla \phi = 0$$

Computing the magnetic field

- Once we know the current density, we compute the magnetic field by solving Ampere's law
$$\mu_0 \mathbf{J} = \nabla \times \mathbf{B}$$
- Again, this is a steady state problem that we would like to express as an elliptic equation
- Because the magnetic field is divergence-free, we can define a **magnetic vector potential**
$$\nabla \cdot \mathbf{B} = \nabla \cdot (\nabla \times \mathbf{A}) = 0$$
- This allows us to write Ampere's law as
$$\mu_0 \mathbf{J} = \nabla \times (\nabla \times \mathbf{A}) = -\nabla^2 \mathbf{A} + \nabla (\nabla \cdot \mathbf{A})$$
- The definition of the magnetic vector potential is not unique
- Additional curl-free components could be added whilst still satisfying $\mathbf{B} = \nabla \times \mathbf{A}$

Computing the magnetic field

$$\mu_0 \mathbf{J} = \nabla \times (\nabla \times \mathbf{A}) = -\nabla^2 \mathbf{A} + \nabla (\nabla \cdot \mathbf{A})$$

- In order to obtain a unique magnetic vector potential, we make a **gauge choice**
- In doing so, we define something about $\nabla \cdot \mathbf{A}$, constraining the curl-free components
- It is standard to pick **Lorenz gauge**,

$$\frac{1}{c^2}\frac{\partial \phi}{\partial t} + \nabla \cdot \mathbf{A} = 0$$

- Since we drop time derivatives, we arrive at an elliptic equation for $\mathbf{A}$

$$\mu_0 \mathbf{J} = -\nabla^2 \mathbf{A}$$

Dealing with elliptic terms

- We have not encountered elliptic equations yet in the compressible courses, and, so far, we have tried our best to avoid them
- We will need a different set of numerical methods to solve this equation (Numerical Methods for Incompressible Fluid Dynamics)
- And although implicit solvers are less efficient (for a given time step) than explicit ones, they are still more efficient than evolving at the magnetic field's time step
- Additionally, code platforms, such as AMReX, offer a means to solve hyperbolic and elliptic (and parabolic) problems on the same Cartesian mesh

The full system

- The full system of equations for a current-driven plasma is:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v} + p\mathbf{I}) = \mathbf{J} \times \mathbf{B}$$

$$\frac{\partial \hat{U}}{\partial t} + \nabla \cdot \left[\left(\hat{U} + p \right) \mathbf{v} \right] = \mathbf{v} \cdot (\mathbf{J} \times \mathbf{B}) + \eta J^2$$

$$\nabla \cdot \sigma \nabla \phi = 0$$

$$\mu_0 \mathbf{J} = -\nabla^2 \mathbf{A}$$

- This is coupled with the definitions

$$\hat{U} = \rho \epsilon + \frac{1}{2} \rho v^2, \quad \mathbf{J} = \sigma \nabla \phi, \quad \mathbf{B} = \nabla \times \mathbf{A}$$

- (Almost) all we need to close the system is an equation of state

Cylindrical symmetry

- Before we do this, we return to the outstanding issue of how can the lightning MHD system be justified

$$c_a = \frac{|B_x|}{\sqrt{\mu_0 \rho}}, \qquad c_{f,sl} = \sqrt{\frac{1}{2}\left[c_s^2 + c_a^2 \pm \sqrt{\left(c_s^2 + c_a^2\right)^2 - 4\frac{c_s^2 B_x^2}{\mu_0 \rho}}\right]}$$

- Lightning strikes are largely cylindrically symmetric, especially if we focus on a region close to an attachment site
- We can use this geometric argument to resolve the wave speed issue
- In cylindrical symmetry, our θ derivatives all vanish, this means:

$$\mathbf{J} = \sigma \nabla \phi = J_r \mathbf{e}_r + J_z \mathbf{e}_z,$$

$$\mu_0 \mathbf{J} = -\nabla^2 \mathbf{A} \implies \mathbf{A} = A_r \mathbf{e}_r + A_z \mathbf{e}_z,$$

$$\mathbf{B} = \nabla \times \mathbf{A} = B_\theta \mathbf{e}_\theta$$

Cylindrical symmetry

$$c_a = \frac{|B_x|}{\sqrt{\mu_0 \rho}}, \qquad c_{f,sl} = \sqrt{\frac{1}{2}\left[c_s^2 + c_a^2 \pm \sqrt{\left(c_s^2 + c_a^2\right)^2 - 4\frac{c_s^2 B_x^2}{\mu_0 \rho}}\right]}$$

$$\mathbf{B} = \nabla \times \mathbf{A} = B_\theta \mathbf{e}_\theta$$

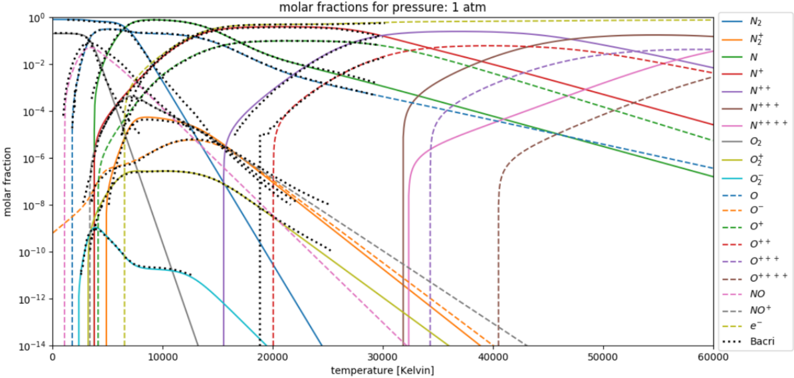
- Note that in the derivation of the fast wave speed, it is only the magnetic field **in the direction of the wave**
- Although we derived this in Cartesian coordinates, because we can write cylindrical coordinates as "Cartesian plus source terms", this does not change
- In other words, in the r- and z-directions (the only ones which can support waves), the magnetic field vanishes
- Therefore the information propagation speeds are either e.g. v_r or $v_r \pm c_s$
- Therefore the lightning MHD model can still capture the correct wave behaviour

Properties of a plasma

- The equation of state for a completely ionised plasma is simple, $p = (\gamma - 1)\,\rho\epsilon$, $\gamma = 5/3$
- However, for lightning (and similar applications) this is not applicable, our equation of state has to cover both the partially ionised plasma arc and the surrounding unionised air (for which $\gamma = 1.4$)
- The transition from air to an ionised material is not simple either
- Temperature rises allow for dissociation of certain species, e.g. nitrogen or oxygen, which can then combine to produce new species, nitric oxide, and their presence completely changes the properties of the material
- Further temperature rises actually start ionising the material too
- To make things more challenging, whilst many gasses and mixtures at ambient conditions can be described by $\gamma = 1.4$, the composition can greatly affect the higher-temperature behaviour

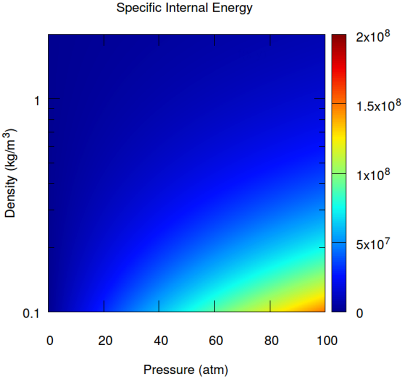
Plasma composition

- Fortunately, the requirement to know the properties and composition of plasma goes beyond numerical simulations - for example, NASA need to know how air mixes with high-temperature rocket fuel

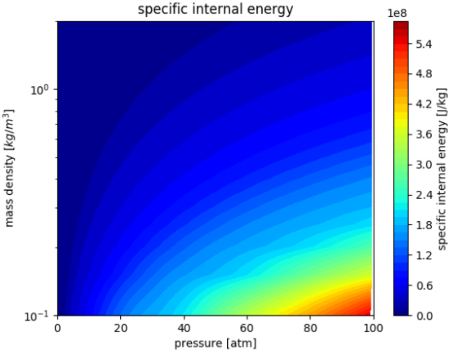


Source: Frederik Träuble

- Ionisation of air results in a substantial change in thermodynamic properties, whilst we still have a monotonic relationship between ρ, p and ϵ, it does not follow a simple curve



Ideal Gas, $\gamma = 5/3$



Ionising mixture

Properties of species change

- The internal energy and pressure depend on the temperature and the chemical composition of the plasma
- One technique to model this is to consider the specific internal energy and pressure as a sum over individual specific internal energies for each of the plasma components, treating each as an ideal gas

$$\epsilon = \sum_{i=1}^{n_s} \frac{\rho_i}{\rho} c_{v,i}\left(T\right) T + \sum_{i=1}^{n_s} \frac{\rho_i}{\rho} \Delta f_i^0 \quad p = \sum_{i=1}^{n_s} \rho_i \frac{R}{m_i} T$$

- The number of species, n_s, includes atoms, ions, molecules and free electrons, all formed in at appropriate temperatures and pressures
- The specific heat is no longer a constant, and that we have included a heat of formation term, Δf_i^0 to account for the creation of new species
- Though we have an explicit form for ϵ, this is not easily invertible, root-finding techniques are required for a full solution
- Additionally, even this is a simplification - specific heat is not a function of temperature alone, but of e.g. pressure and temperature

How to deal with complicated mixtures

- For an air plasma, an accurate, yet explicit, description of its properties is available based on curve fitting to chemical equilibrium calculations (d'Angola *et al.* (2008) use temperature and pressure to describe the plasma)
- Inverting a series of complex equation of state definitions is possible, but very time consuming, made more complicated by the vast differences in scale of the fitting parameters (many orders of magnitude)
- Instead, this data can be used to constructed the tabulated EoS that we hinted existed a few lectures ago

Do we need any more source terms?

- So far, we have described our system in the absence of (non-magnetic) source terms

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v} + p\mathbf{I}) = \mathbf{J} \times \mathbf{B} + \mathbf{S}$$

$$\frac{\partial \hat{U}}{\partial t} + \nabla \cdot \left[\left(\hat{U} + p \right) \mathbf{v} \right] = \mathbf{v} \cdot (\mathbf{J} \times \mathbf{B}) + \eta J^2 + S_T$$

$$\nabla \cdot \sigma \nabla \phi = 0$$

$$\mu_0 \mathbf{J} = -\nabla^2 \mathbf{A}$$

- The momentum equation allows for body forces, such as gravity - though under the timescales and length scales typically considered for plasma modelling, we can (as usual) set $\mathbf{S} = 0$
- The energy equation can include terms for thermal effects, which may include diffusion, radiation, convection and conduction - whether these are necessary also depends on timescale

Thermal source terms

- We can neglect conduction and convection, and for many cases we can neglect diffusion too, but radiation cannot be ignored
- The temperature at the centre of a plasma arc ($> 30,000\,K$) is high enough that there is substantial energy loss due to radiative emission
- Radiative models are complex - they depend on the interactions between the species present, across a range of frequencies, and include angular and gradient dependencies
- One of the simplest (without substantial accuracy loss) methods is to use the net emission coefficient, ϵ_N

$$S_r = 4\pi\epsilon_N = 4\pi \int_{\nu_1}^{\nu_2} \frac{2hc^2\nu^3}{\exp\left(hc\nu/k_B T\right) - 1} K_\nu \exp\left(-K_\nu R_p\right) \mathrm{d}\nu$$

- Even this requires calculation absorption and emission properties of a range of possible molecular interactions
- However, once computed, the net emission coefficient can be stored for a range of pressure-temperature pairs, i.e. it is incorporated into the equation of state

What do we need to model lightning

- To summarise, we have three evolution equations, for which we are familiar with the underlying numerical methods

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v} + p\mathbf{I}) = \mathbf{J} \times \mathbf{B}$$

$$\frac{\partial \hat{U}}{\partial t} + \nabla \cdot \left[\left(\hat{U} + p \right) \mathbf{v} \right] = \mathbf{v} \cdot (\mathbf{J} \times \mathbf{B}) + \eta J^2 - S_r$$

- Note that because we consider **radiative losses**, we have a minus sign on the source term
- We also need to solve two elliptic equations - many resources are available

$$\nabla \cdot \sigma \nabla \phi = 0$$

$$\mu_0 \mathbf{J} = -\nabla^2 \mathbf{A}$$

- And we need a tabulated equation of state, which includes the radiative effects (we have one of these in within the LSC should you need it)