Introduction to Electrical Engineering (ECE 302H) – Homework 3

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Problem 3.1. Consider the circuit below which is composed of resistors (which you know) and a new component called a comparator. Comparators are quite easy to understand – they simply compare the voltages at their inputs. If the positive input is greater than the negative input, then the comparator outputs a digital "1". If the negative input is greater, then the comparator outputs a digital "0". Comparators do not draw any current from their inputs, i.e. they do not disturb the circuit they are measuring. If the outputs of the comparators are (from top down) [0001] then we interpret the digital output D as "D=1"; if the output of the comparators is [0011] then we interpret the digital output D as "2", and so forth. This is "thermometer code" because the 1's at the output rise like mercury in an old thermometer.

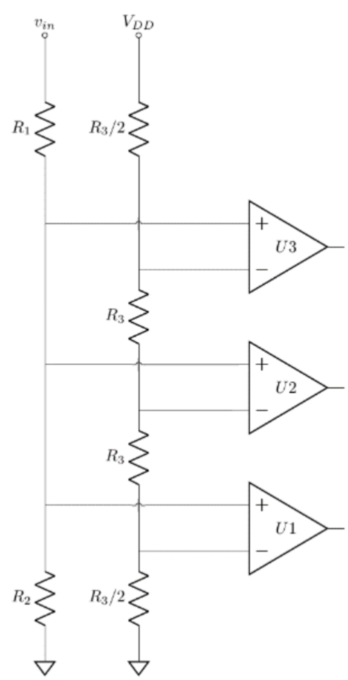
a) Calculate the voltage at the positive input to each comparator assuming vin is set by a voltage source. Call it vin,low (note that this node is not connected to the other resistor string; the wire lines "hop over" the other resistor string). You expect the maximum value of vin to be 19.8 V and each pin of the comparator is rated to handle up to 3.3 V; what must the ratio R1 / R2 be so that you can measure the full input voltage range while using the full pin rating? Please use that value for the rest of the problem.

b) Calculate the voltage at the negative input to each comparator assuming VDD is set by a voltage source. Call them Vref1, Vref2, and Vref3. Let VDD = 3.3V.

c) Sketch a graph of the digital output D ("1", "2", etc.) as a function of vin. How many different levels can D represent? How many bits would you need to represent this number of levels in binary code (i.e., how many bits of resolution are there)?

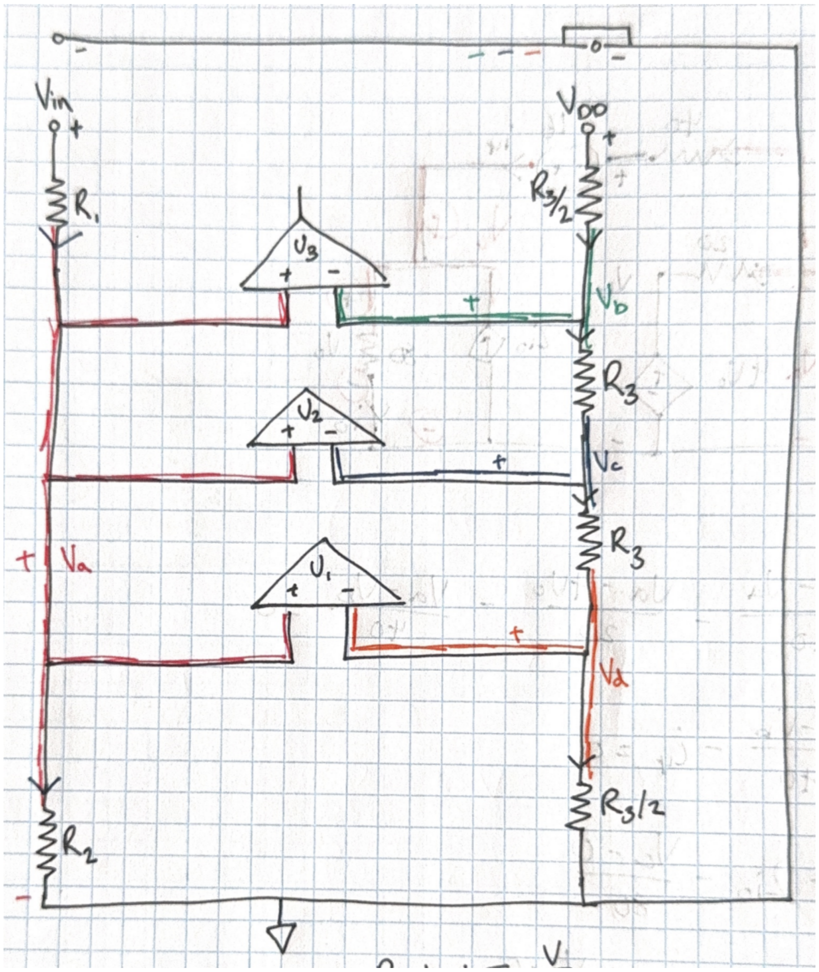
d) How many more resistors and comparators would you need to add one more bit of precision?

e) Discrete resistors are typically rated to dissipate about 1/4 W of power each. Using this rating and the ratio you found in (a), calculate the minimum allowable values for R1,R2, and R3.



Solution. □

a) Redraw circuit and use node analysis to solve for V_a:



$$
\begin{aligned}
\sum I = I_{R_1} - I_{R_2} &= 0 \\
\frac{V_{in} - V_a}{R_1} - \frac{V_a - 0}{R_2} &= 0 \\
\frac{V_{in} - V_a}{R_1} &= \frac{V_a}{R_2} \\
\frac{V_{in}}{R_1} &= \frac{V_a}{R_2} + \frac{V_a}{R_1} \\
&= V_a\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \\
&= V_a\left(\frac{R_1 + R_2}{R_1 R_2}\right) \\
V_a &= \boxed{\frac{V_{in} R_2}{R_1 + R_2}}
\end{aligned}
$$

Plug in 3.3 V for V_a and 19.8 V for V_{in}:

$$\begin{aligned} V_a &= \frac{V_{in}R_2}{R_1+R_2} \\ 3.3 &= \frac{19.8R_2}{R_1+R_2} \\ \frac{3.3}{19.8} &= \frac{R_2}{R_1+R_2} \\ \frac{19.8}{3.3} &= \frac{R_1+R_2}{R_2} \\ &= \frac{R_1}{R_2}+1 \\ \frac{R_1}{R_2} &= \frac{19.8}{3.3}-1 \\ &= \boxed{5} \end{aligned}$$

b) Use node analysis on node B to set up KCL equation:

$$\sum I = \frac{V_{DD}-V_b}{R_3/2} - \frac{V_b-V_c}{R_3} = 0$$

Use node analysis on node C to set up KCL equation:

$$\sum I = \frac{V_b-V_c}{R_3} - \frac{V_c-V_d}{R_3} = 0$$

Use node analysis on node D to set up KCL equation:

$$\sum I = \frac{V_c-V_d}{R_3} - \frac{V_d-0}{R_3/2} = 0$$

> **Remark.** Notice now that we have a scenario with 3 unknown variables: V_b, V_c, and V_d and 3 equations. That means that we can solve for each unkown variable

$$\begin{aligned} \frac{2V_{DD}}{R_3} - \frac{2V_b}{R_3} - \frac{V_b}{R_3} + \frac{V_c}{R_3} &= 0 \quad \Rightarrow \quad V_c = 3V_b - 2V_{DD} \\ \frac{V_b}{R_3} - \frac{V_c}{R_3} - \frac{V_c}{R_3} + \frac{V_d}{R_3} &= 0 \quad \Rightarrow \quad V_b = 2V_c - V_d \\ \frac{V_c}{R_3} - \frac{V_d}{R_3} - \frac{2V_d}{R_3} &= 0 \quad \Rightarrow \quad V_c = 3V_d \end{aligned}$$

Solve system of equations:

$$\begin{aligned} V_c &= 3V_b - 2V_{DD} \\ &= 3(2V_c - V_d) - 2V_{DD} \\ &= 6V_c - 3V_d - 2V_{DD} \\ -5V_c &= -3V_d - 2V_{DD} \end{aligned}$$

$$\begin{aligned} V_c &= 3V_d \\ 5V_c &= 15V_d \end{aligned}$$

$$\begin{cases} -5V_c = -3V_d - 2V_{DD} \\ 5V_c = 15V_d \end{cases} \Rightarrow \quad 0 = 12V_d - 2V_{DD} \quad \Rightarrow \quad V_d = \frac{1}{6}V_{DD}$$

$$\begin{aligned} V_c &= 3V_d \\ &= 3(\frac{1}{6}V_{DD}) \\ &= \frac{1}{2}V_{DD} \end{aligned}$$

$$\begin{aligned} V_b &= 2V_c - V_d \\ &= 2(\frac{1}{2}V_{DD}) - \frac{1}{6}V_{DD} \\ &= \frac{5}{6}V_{DD} \end{aligned}$$

Plug in $V_{DD} = 3.3\,\text{V}$ and solve for reference voltages:

$$\begin{aligned} V_d = V_{ref1} &= \frac{1}{6}(3.3) = \boxed{0.55\,\text{V}} \\ V_c = V_{ref2} &= \frac{1}{2}(3.3) = \boxed{1.65\,\text{V}} \\ V_b = V_{ref3} &= \frac{5}{6}(3.3) = \boxed{2.75\,\text{V}} \end{aligned}$$

c) This problem asks us to find digital output D as a function of V_{in}. The issue is that D is a non-continuous piece-wise of V_a. But this is okay because V_a is a function of V_{in}, so in the end, we are really graphing $D(V_a(V_{in}))$.

> **Remark.** I just wanted to make that clear.

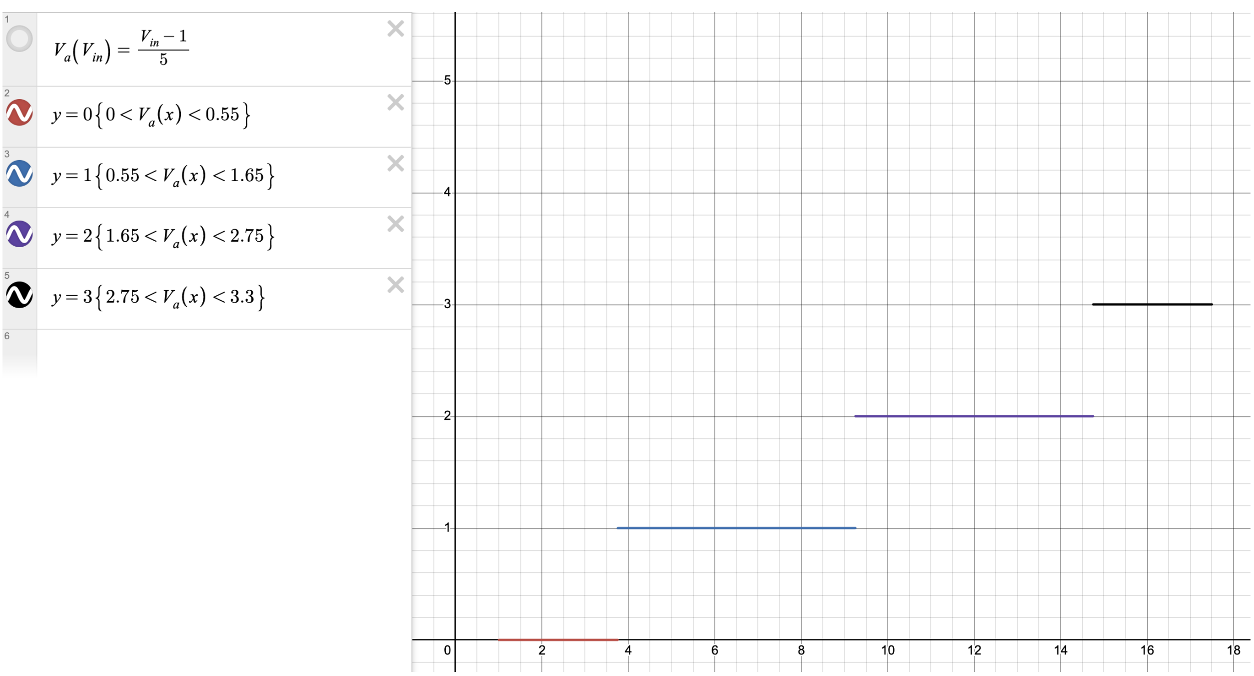
Consider what D would be given a particular V_a:

$$\begin{aligned}
D = 0&: \quad 0 < V_a < 0.55 \\
D = 1&: \quad 0.55 < V_a < 1.65 \\
D = 2&: \quad 1.65 < V_a < 2.75 \\
D = 3&: \quad 2.75 < V_a < 3.3
\end{aligned}$$

Rewrite V_a in the form of $V_a(V_{in})$:

$$\begin{aligned}
V_a &= \frac{V_{in} R_2}{R_1 + R_2} \\
\frac{1}{V_a} &= \frac{1}{V_{in}} \frac{R_1 + R_2}{R_2} \\
V_{in} &= V_a \frac{R_1}{R_2} + 1 \\
&= 5V_a + 1 \\
V_a &= \frac{1}{5} V_{in} - \frac{1}{5}
\end{aligned}$$

Graph $D(V_a)$:



There are two ways to interpret how many bits in binary you need to add.

The first representation is similar to the way the problem statements represents levels where 4 levels is represented with: 000, 001, 011, 111. And each next consecutive level is represented with one extra bit. 5 levels would be 0000, 0001, 0011, 0111, 1111. 6 levels: 00000, 00001, 00011, 00111, 01111, 11111. The formula for how many bits are needed to represent n levels would be $n_{\text{bits}} = n_{\text{levels}} - 1$. So in this case, you would need 3 bits to represent 4 levels.

The second representation is based off of how many distinct numbers each bit of binary could represent. 4 levels can be represented by 2 bits of binary: 00, 01, 10, 11. 3 bits of binary can represent 2^3 levels. n bits of binary can represent 2^n levels.

d) You would only need 1 more resistor and 1 more comparator to add another level of precision.

e) We know that $P_{\text{resistor}} = \frac{V^2}{R}$. Write equation for P_1:

$$\begin{aligned} P_1 = \frac{1}{4} &= \frac{(19.8 - 3.3)^2}{R_1} \\ R_1 &= 4(19.8 - 3.3)^2 \\ R_1 &= \boxed{1089\,\Omega} \end{aligned}$$

Apply the ratio R_1/R_2:

$$\begin{aligned} \frac{R_1}{R_2} &= 5 \\ R_2 &= \frac{R_1}{5} \\ R_2 &= \frac{1089}{5} \\ R_2 &= \boxed{217.8\,\Omega} \end{aligned}$$

Write equation for P_3:

$$\begin{aligned} P_3 = \frac{1}{4} &= \frac{(1.65 - 0.55)^2}{R_3} \\ R_3 &= 4(1.65 - 0.55)^2 \\ R_3 &= \boxed{4.84\,\Omega} \end{aligned}$$

Problem 3.2. In radio-frequency (rf) systems, it is common to have a high-power source of energy and a sensitive detector or load that can't absorb very much power. In these cases, you can use an attenuator to reduce the amount of power absorbed by the load by a known amount. Let the energy source be modeled by a voltage vin with a resistor Rs in series; let the load be modeled by a resistor RL. Refer to the Figure.

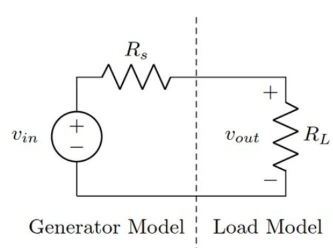
a) Calculate the power delivered to the load when no attenuator is used.

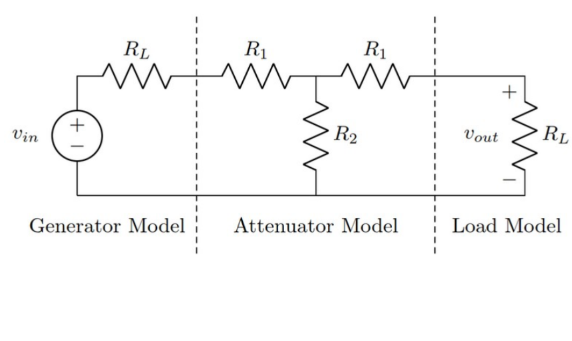
b) Consider the attenuator in the Figure, which consists of a T-network of two resistors of values R1 and one resistor of value R2. Calculate what the new network "looks like" from the perspective of the generator. In other words, calculate the equivalent resistance Rnet of everything to the right of the generator.

c) Generators of rf energy often behave best when they are loaded with the same resistance as the generator's internal resistance (i.e., when RL =RS), but when we add the attenuator we risk changing the resistance that the source "sees" (i.e., Rnet). Let RL = Rs = 50 ohm. Normally our design task would be to find values for R1 and R2 that give a target attenuation while making the load+attenuator resistance look like 50 ohm. This is pretty messy to calculate in the general case so let's do a specific case instead. Suppose that R2 = 25.975 ohm. Calculate the required value of R1 to make the equivalent resistance of the load+attenuator combination equal to 50 ohm.

d) Calculate the attenuation of the resulting attenuator circuit, i.e., the ratio of power delivered to the actual load when the attenuator is inserted in the circuit divided by the power delivered to the load with no attenuator.

e) Simulate the circuit you designed in LTspice. What measurements would you take, and what calculations would you make based on those measurements, to evaluate if your circuit does what you want it to do? Submit a screenshot of your LTSpice graphs and circuit and, separately, any calculations you need to do to prove that the circuit works. Be sure to label any nodes whose voltage you will be measuring so that the graph is self-explanatory. (Currents and powers don't need to be labeled since they're named after their component)





Solution. □

a) Recall that $P = V \times I$ and that V in a resistor is IR. Thus, power through a resistor can be represented as $P_{\text{resistor}} = I^2R$.

$$\begin{aligned}
\sum V &= V_{in} - V_S - V_L = 0 \\
&= V_{in} - I_S R_S - I_L R_L = 0 \\
&= V_{in} - I_L R_S - I_L R_L = 0 \\
I_L R_L + I_L R_S &= V_{in} \\
I_L &= \frac{V_{in}}{R_L + R_S}
\end{aligned}$$

$$\begin{aligned}
P_L &= I_L^2 R_L \\
&= \boxed{\frac{V_{in}^2 R_L}{(R_L + R_S)^2}}
\end{aligned}$$

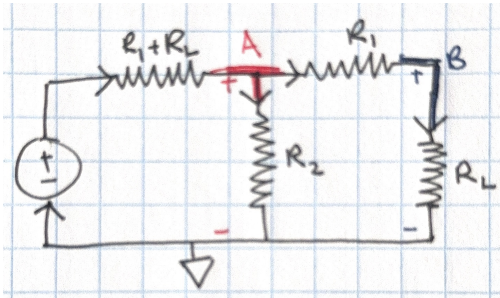
b) For this problem we can just use equivalent resistances to solve for R_{eq}.

$$\begin{aligned}
R_{eq1} &= R_1 + R_L \\
\frac{1}{R_{eq2}} &= \frac{1}{R_2} + \frac{1}{R_{eq1}} \\
&= \frac{1}{R_2} + \frac{1}{R_1 + R_L} \\
&= \frac{R_1 + R_2 + R_L}{R_2(R_1 + R_L)} \\
R_{eq2} &= \frac{R_2(R_1 + R_L)}{R_1 + R_2 + R_L} \\
R_{eq} &= R_1 + R_{eq2} \\
&= \boxed{R_1 + \frac{R_2(R_1 + R_L)}{R_1 + R_2 + R_L}}
\end{aligned}$$

c) Plug in values of R_2, R_L, and R_{eq}:

$$\begin{aligned}
R_{eq} &= R_1 + \frac{R_2(R_1 + R_L)}{R_1 + R_2 + R_L} \\
50 &= R_1 + \frac{25.975(R_1 + 50)}{R_1 + 25.975 + 50} \\
R_1 &= \boxed{30.37\,\Omega}
\end{aligned}$$

d) Label circuit and perform node analysis.



Node A:

$$\begin{aligned}\sum I = \frac{V_{in} - V_a}{R_1 + R_L} - \frac{V_a - V_b}{R_1} - \frac{V_a - 0}{R_2} &= 0\\ \frac{V_{in} - V_a}{30.37 + 50} - \frac{V_a - V_b}{30.37} - \frac{V_a}{25.975} &= 0\\ \frac{V_{in} - V_a}{80.37} - \frac{V_a - V_b}{30.37} - \frac{V_a}{25.975} &= 0\end{aligned}$$

Node B:

$$\begin{aligned}\sum I = \frac{V_a - V_b}{R_1} - \frac{V_b - 0}{R_L} &= 0\\ \frac{V_a - V_b}{30.37} - \frac{V_b}{50} &= 0\end{aligned}$$

> **Remark.** Notice now that we have a 2 equation 2 unknown: V_a and V_b system of equations that we can solve using algebra.

$$\begin{aligned}\frac{V_a - V_b}{30.37} - \frac{V_b}{50} &= 0\\ \frac{V_a}{30.37} - \frac{V_b}{30.37} - \frac{V_b}{50} &= 0\\ V_a &= V_b + \frac{30.37V_b}{50}\\ &= V_b + 0.61V_b\\ &= 1.61V_b\end{aligned}$$

$$\frac{V_{in} - V_a}{80.37} - \frac{V_a - V_b}{30.37} - \frac{V_a}{25.975} = 0$$

$$\begin{aligned}
\frac{V_{in} - 1.61V_b}{80.37} - \frac{1.61V_b - V_b}{30.37} - \frac{1.61V_b}{25.975} &= 0 \\
\frac{V_{in}}{80.37} - 0.02V_b - 0.02V_b - 0.06V_b &= 0 \\
\frac{V_{in}}{80.37} &= 0.1V_b \\
0.124V_{in} &= V_b \\
V_b &\approx \frac{1}{8}V_{in}
\end{aligned}$$

$$\begin{aligned}
V_a &= 1.61V_b \\
&= 1.61(0.125V_{in}) \\
&= 0.201V_{in} \\
&\approx \frac{1}{5}V_{in}
\end{aligned}$$

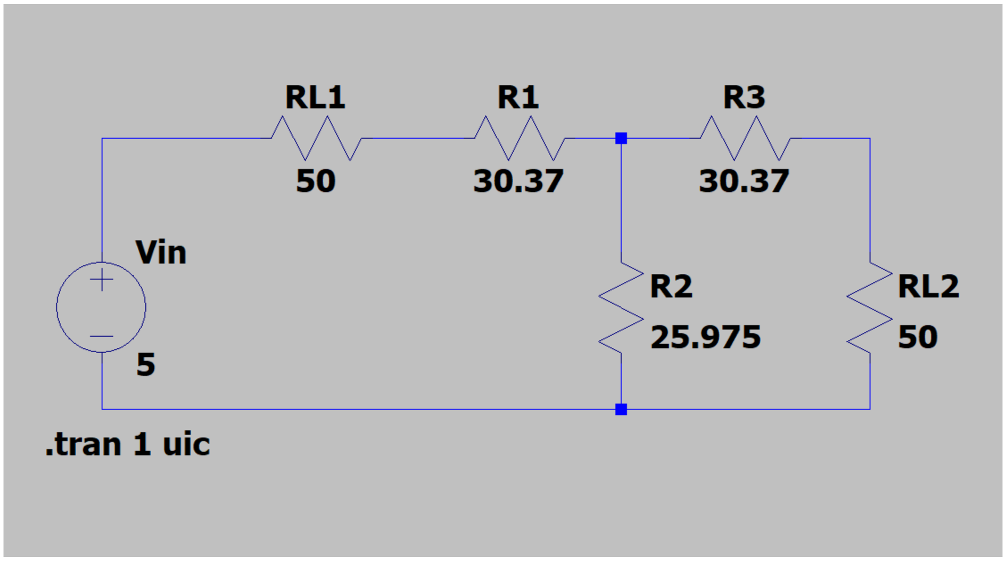
$$\begin{aligned}
I_L &= \frac{V_b}{R_L} \\
&= \frac{V_b}{50} \\
&= \frac{V_{in}}{8(50)} \\
&= \frac{V_{in}}{400}
\end{aligned}$$

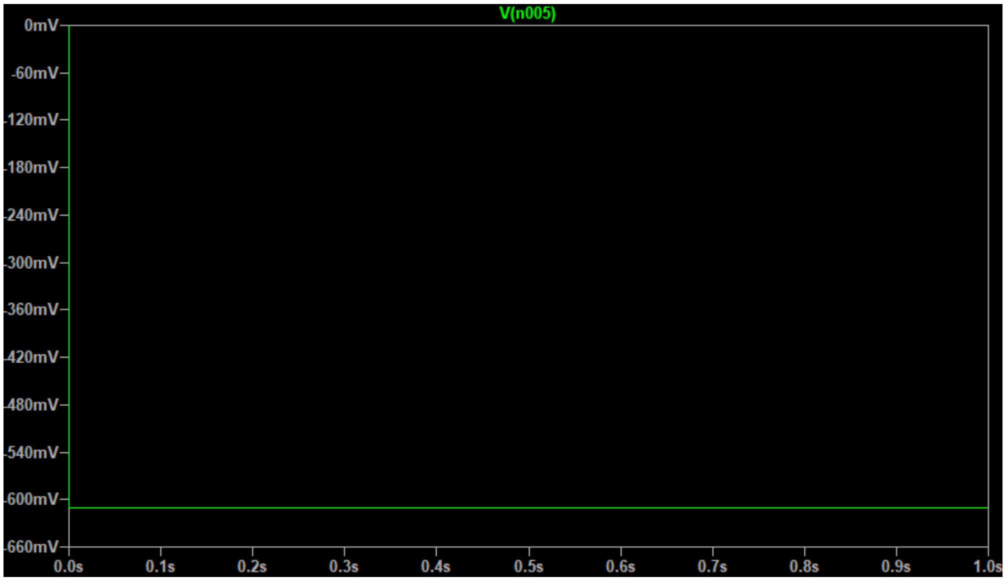
$$\begin{aligned}
P_{\text{attenuated}} &= I_L^2 R_L \\
&= \left(\frac{V_{in}}{400}\right)^2 50 \\
&= \frac{V_{in}^2}{3200}
\end{aligned}$$

$$\begin{aligned}
P_{\text{without}} &= \frac{V_{in}^2 50}{(50+50)^2} \\
&= \frac{V_{in}^2}{200}
\end{aligned}$$

$$\begin{aligned}
\frac{P_{\text{attenuated}}}{P_{\text{without}}} &= \frac{V_{in}^2}{3200}\frac{200}{V_{in}^2} \\
&\approx \boxed{0.063}
\end{aligned}$$

e) Create attenuator circuit on LTSpice and plug in values for resistors.





From the data in the graph:

$$V_{in} = 5\,\text{V}$$
$$V_{out} = -610\,\text{mV}$$

From the expected equation:

$$V_b = V_{out} \approx \frac{1}{8} V_{in}$$

$$V_{out} \approx \frac{1}{8}(5)$$
$$V_{out} \approx 625\,\mathrm{mV}$$

We can see that the expected and experimental data are reflective in their magnitudes. The difference is caused by the approximation of $V_{out} \approx \frac{1}{8}V_{in}$.

Problem 3.3. Use the definition of linearity to show whether the following SISO functions are linear:

a) $f(x) = mx$, where m is a constant

b) $f(x) = mx$, where $m = 1/x^2$

c) $f(x) = mx + b$, where m and b are constants

d) $f(x) = e^x$

Solution. □

a)

$$f(x) = mx$$
$$f(u_1) = mu_1$$
$$f(u_2) = mu_2$$

$$\begin{aligned} f(x) &= mx \\ f(au_1 + bu_2) &= m(au_1 + bu_2) \\ f(au_1 + bu_2) &= a(mu_1) + b(mu_2) \\ f(au_1 + bu_2) &= af(u_1) + bf(u_2) \end{aligned}$$

$\boxed{\text{Linear :D}}$

b)

$$f(x) = \frac{1}{x^2}x$$
$$f(x) = \frac{1}{x}$$
$$f(u_1) = \frac{1}{u_1}$$
$$f(u_2) = \frac{1}{u_2}$$

$$\begin{aligned} f(x) &= \frac{1}{x} \\ f(au_1 + bu_2) &= \frac{1}{au_1 + bu_2} \\ af(u_1) + bf(u_2) &= \frac{a}{u_1} + \frac{b}{u_2} \\ \frac{1}{au_1 + bu_2} &\neq \frac{a}{u_1} + \frac{b}{u_2} \end{aligned}$$

$\boxed{\text{Not Linear :(}}$

c)

$$\begin{aligned} f(x) &= mx + b \\ f(u_1) &= mu_1 + b \\ f(u_2) &= mu_2 + b \end{aligned}$$

$$\begin{aligned} f(x) &= mx + b \\ f(au_1 + cu_2) &= m(au_1 + cu_2) + b \\ f(au_1 + cu_2) &= amu_1 + cmu_2 + b \\ af(u_1) + cf(u_2) &= amu_1 + ab + cmu_2 + cb \\ amu_1 + cmu_2 + b &\neq amu_1 + ab + cmu_2 + cb \end{aligned}$$

$\boxed{\text{Not Linear :(}}$

d)

$$\begin{aligned} f(x) &= e^x \\ f(u_1) &= e^{u_1} \\ f(u_2) &= e^{u_2} \end{aligned}$$

$$\begin{aligned} f(x) &= e^x \\ f(au_1 + bu_2) &= e^{au_1 + bu_2} \\ af(u_1) + bf(u_2) &= ae^{u_1} + be^{u_2} \\ e^{au_1 + bu_2} &\neq ae^{u_1} + be^{u_2} \end{aligned}$$

$\boxed{\text{Not Linear :(}}$

Problem 3.4. In a previous homework, we analyzed a MOSFET-based circuit. We said that the MOSFET could be modeled by a current source of value gmvgs, where gm was called the transconductance and vgs was the voltage between the MOSFET gate and source. This model is actually a simplification from a more complete model for the current that flows from drain to source, given by

$$i = \frac{1}{2}\mu_n C_{ox}\frac{W}{L}\left(v_{gs} - V_{TH}\right)^2\left(1 + \lambda v_{ds}\right)$$

where micro n is the mobility of electrons, Cox is the gate capacitance per unit area, Vth is called the "threshold voltage" of the MOSFET, and W and L are the MOSFET width and length. The parameter lambda represents channel-length modulation (if lambda = 0, then the term 1 + lambda vds falls out of the expression). We'll learn more about all of these parameters later in the course.

a) Please identify two features of the above equation that make it nonlinear. (Hint – identify which terms are circuit variables and which ones are constants)

b) We have surprisingly few tools for dealing with nonlinear systems. The most common thing we can do is "linearize" the system, which means that we need to find a linear equation that is approximately equal to the real equation for small deviations away from an operating point. This is also called "small-signal modeling." Replace the circuit quantities i, vgs, and vds with an operating point value plus a small deviation away from that operating point. Thus i → I + delta i, vgs → Vgs + delta vgs, and vds → Vds + delta vds. Show that the resulting equation reduces to the following:

$$\Delta i = \left(\mu_n C_{ox}\frac{W}{L}\left(1 + \lambda V_{ds}\right)\left(V_{gs} - V_{TH}\right)\right)\Delta v_{gs} + \left(\frac{1}{2}\mu_n C_{ox}\frac{W}{L}\left(V_{gs} - V_{TH}\right)^2\lambda\right)\Delta v_{ds}$$

Be sure to not confuse the operating point values (indicated by capital letter) from the small perturbation values (indicated by delta). You will need to cancel an object on the left hand side with an equal object on the right hand side. Further, you will need to make an approximation that any product of small quantities can be approximated as zero (a small number times a small number is a really small number). So, any terms in your answer with two or more delta quantities multiplied together can be approximated as zero. These are the nonlinear terms – approximating them as zero is the "linearization" part of this process.

c) Using sentences and a circuit diagram, show how this equation can be converted to a dependent current source in parallel with a resistor. What is the current source voltage-to-current constant gm? What is the resistance value ro? You can see now that the previous homework lied to you just a little bit. The model was correct, but it only models the behavior of the deviations of the variables from their operating points (the previous homework also assumed lambda=0). Still, that's often all we care about, so the result is useful.

Solution. □

a) Given the list of constants that are defined to us. We can simplify down the equation to represent constants together.

$$i = C_1(v_{gs} - C_2)^2(1 + \lambda v_{ds})$$

(a) The first giveaway is the $(v_{gs} + C_2)^2$ term. In a linear equation the dependent variables have to be in the first order, but when $(v_{gs} + C_2)^2$ is expanded, there will be a dependent variable in the second order, violating the rule of linearity.

(b) The next giveaway is that when this equation is fully expanded, there will be a $+C$ term when the equation is fully expanded. In the previous problem, we saw that $f(x) = mx + b$ is non linear because of the added constant term, so we can conclude that the $+C$ term here will also violate linearity.

b)

$$\begin{aligned}
i &= C_1(v_{gs} - C_2)^2(1 + \lambda v_{ds}) \\
I + \Delta i &= C_1(V_{gs} + \Delta v_{gs} - C_2)^2(1 + \lambda(V_{ds} + \Delta v_{ds})) \\
I + \Delta i &= C_1(V_{gs} + \Delta v_{gs} - C_2)^2(1 + \lambda V_{ds} + \lambda \Delta v_{ds}) \\
I + \Delta i &= C_1((V_{gs} - C_2)^2 + 2\Delta v_{gs}(V_{gs} - C^2) + (\Delta v_{gs})^2)(1 + \lambda V_{ds} + \lambda \Delta v_{ds}) \\
I + \Delta i &= C_1((V_{gs} - C_2)^2 + 2\Delta v_{gs}(V_{gs} - C^2) + 0)(1 + \lambda V_{ds} + \lambda \Delta v_{ds}) \\
I + \Delta i &= C_1((V_{gs} - C_2)^2 + 2\Delta v_{gs}(V_{gs} - C^2))(1 + \lambda V_{ds} + \lambda \Delta v_{ds}) \\
I + \Delta i &= i + C_1 2\Delta v_{gs}(V_{gs} - C^2)(1 + \lambda V_{ds}) + C_1((V_{gs} - C_2)^2 \lambda \Delta v_{ds} + 0) \\
I + \Delta i &= i + C_1 2\Delta v_{gs}(V_{gs} - C^2)(1 + \lambda V_{ds}) + C_1(V_{gs} - C_2)^2 \lambda \Delta v_{ds} \\
\Delta i &= C_1 2\Delta v_{gs}(V_{gs} - C^2)(1 + \lambda V_{ds}) + C_1(V_{gs} - C_2)^2 \lambda \Delta v_{ds} \\
\Delta i &= C_1 2\Delta v_{gs}(V_{gs} - V_{TH})(1 + \lambda V_{ds}) + C_1(V_{gs} - V_{TH})^2 \lambda \Delta v_{ds} \\
\Delta i &= \mu_n C_{ox}\frac{W}{L}\Delta v_{gs}(V_{gs} - V_{TH})(1 + \lambda V_{ds}) + \frac{1}{2}\mu_n C_{ox}\frac{W}{L}(V_{gs} - V_{TH})^2 \lambda \Delta v_{ds}
\end{aligned}$$

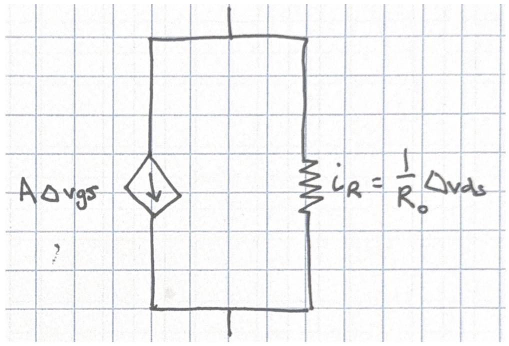
c) The equation given to use here can be represented in the form of

$$\Delta i = A\Delta v_{gs} + B\Delta v_{ds}$$

where A and B are constants.

In this equation we see the summation of two currents which happens when a circuit is in parallel, thus agreeing with the problem statement.

Consider this circuit:



In this scenario, $g_m = A$ and $R_0 = \frac{1}{B}$.

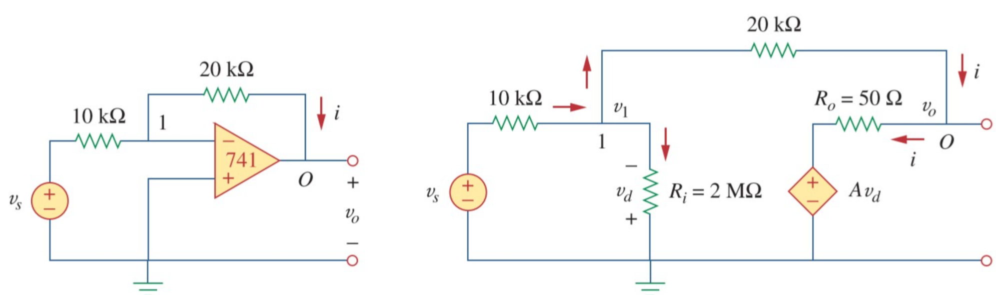
$$A = \mu_n C_{ox} \frac{W}{L}(1 + \lambda V_{ds})(V_{gs} - V_{TH})$$
$$B = \frac{1}{2}\mu_n C_{ox} \frac{W}{L}(V_{gs} - V_{TH})^2 \lambda$$

$$g_m = \boxed{\mu_n C_{ox} \frac{W}{L}(1 + \lambda V_{ds})(V_{gs} - V_{TH})}$$
$$R_0 = \boxed{\left(\frac{1}{2}\mu_n C_{ox} \frac{W}{L}(V_{gs} - V_{TH})^2 \lambda\right)^{-1}}$$

Problem 3.5. We most commonly encounter dependent voltage sources when we use circuit elements called operational amplifiers or "op-amps." An ideal op-amp is nothing more than a voltage-dependent voltage source; nonideal op-amps also have some input resistance Ri and output resistance Ro. Therefore, even though we have not yet covered op-amps explicitly in class, you are nevertheless prepared to analyze op-amp circuits. A circuit involving an op amp is shown in the figure, with its equivalent circuit model. For all parts, let the parameter A be a variable and then take A→infinity in your final answers.

a) Using the equivalent circuit model, analyze the circuit for an ideal op-amp, i.e. let Ri=infinity (open circuit) and let Ro =0 (short circuit). What is the output voltage for a given input voltage? What is the voltage at the negative input of the op-amp, v1?

b) Using the equivalent circuit model, analyze the circuit again for a non-ideal op-amp (Ri=2 M ohm, Ro =50 ohm as shown), calculating vo and v1.



Solution. □

a) The first thing we need to realize is that if $R_i = \infty$, that means that no current will flow from node 1 through R_i into GND.

The second thing we need to realize is that if $R_0 = 0$, that means that node 0 is directly connected to the dependent voltage source Av_d. Thus, $v_0 = Av_d$.

The third thing we need to recognize from the circuit is that $v_1 = -v_d$. This relationship can be seen where v_d is labeled by R_i.

Node 1 KCL:

$$\begin{aligned}\sum I = \frac{v_s - v_1}{10000} - \frac{v_1 - v_0}{20000} &= 0\\ 2v_s - 2v_1 - v_1 + v_0 &= 0\\ 2v_s - 3v_1 + v_0 &= 0\\ v_0 &= 3v_1 - 2v_s\end{aligned}$$

$$\begin{aligned}v_0 &= Av_d\\ v_0 &= -Av_1\end{aligned}$$

$$v_1 = -\frac{v_0}{A}$$

$$\begin{aligned}
v_0 &= 3v_1 - 2v_s \\
v_0 &= -\frac{3v_0}{A} - 2v_s \\
v_0 &= -\frac{3v_0}{\infty} - 2v_s \\
v_0 &= \boxed{-2v_s}
\end{aligned}$$

$$\begin{aligned}
v_1 &= -\frac{v_0}{A} \\
v_1 &= -\frac{v_0}{\infty} \\
v_1 &= \boxed{0}
\end{aligned}$$

b) Node 1 KCL:

$$\begin{aligned}
\sum I = \frac{v_s - v_1}{10000} - \frac{v_1 - 0}{2000000} - \frac{v_1 - v_0}{20000} &= 0 \\
200v_s - 200v_1 - v_1 - 100v_1 + 100v_0 &= 0 \\
200v_s - 301v_1 + 100v_0 &= 0
\end{aligned}$$

Node 0 KCL:

$$\begin{aligned}
\sum I = \frac{v_1 - v_0}{20000} - \frac{v_0 - Av_d}{50} &= 0 \\
\frac{v_1 - v_0}{20000} - \frac{v_0 + Av_1}{50} &= 0 \\
v_1 - v_0 - 400v_0 - 400Av_1 &= 0 \\
v_1 - 401v_0 - 400Av_1 &= 0
\end{aligned}$$

> **Remark.** Notice that now it is a 2 equation 2 unknown system of equations.

$$\begin{aligned}
200v_s - 301v_1 + 100v_0 = 0 \\
v_1 = \frac{200v_s + 100v_0}{301}
\end{aligned}$$

$$v_1 - 401v_0 - 400Av_1 = 0$$

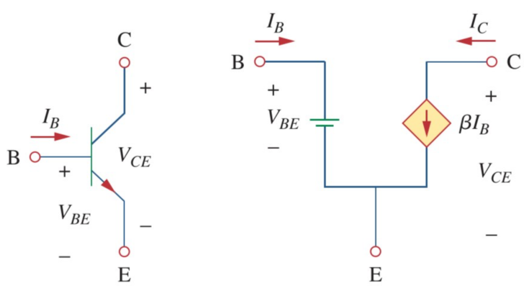
$$
\begin{aligned}
v_1(1-400A) &= 401v_0 \\
\frac{200v_s+100v_0}{301}(1-400A) &= 401v_0 \\
200v_s+100v_0-80000Av_s-40000Av_0 &= 120701v_0 \\
200v_s+100v_0-80000Av_s-40000Av_0-120701v_0 &= 0 \\
200v_s-80000Av_s-v_0(40000A+120701) &= 0 \\
200v_s-80000Av_s &= v_0(40000A+120701) \\
v_0 &= \frac{200v_s-80000Av_s}{40000A+120701} \\
v_0 &= \frac{200v_s}{40000A+120701}-\frac{80000Av_s}{40000A+120701} \\
v_0 &= \frac{200v_s}{40000A+120701}-\frac{80000v_s}{40000+\frac{120701}{A}} \\
v_0 &= \frac{200v_s}{40000\infty+120701}-\frac{80000v_s}{40000+\frac{120701}{\infty}} \\
v_0 &= 0-\frac{80000v_s}{40000} \\
v_0 &= \boxed{-2v_s}
\end{aligned}
$$

$$
\begin{aligned}
200v_s-301v_1+100v_0 &= 0 \\
v_0 &= \frac{301v_1-200v_s}{100}
\end{aligned}
$$

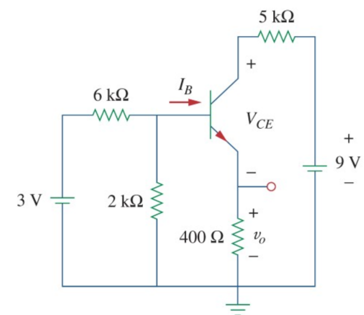
$$
\begin{aligned}
v_1-401v_0-400Av_1 &= 0 \\
v_1-\frac{120701v_1-80200v_s}{100}-400Av_1 &= 0 \\
100v_1-120701v_1-80200v_s-40000Av_1 &= 0 \\
-120601v_1-40000Av_1-80200v_s &= 0 \\
v_1(-120601-40000A) &= 80200v_s \\
v_1 &= \frac{-80200v_s}{120601+40000A} \\
v_1 &= \frac{-80200v_s}{120601+40000\infty} \\
v_1 &= \boxed{0}
\end{aligned}
$$

Problem 3.6. The first transistor was not a MOSFET, but rather a Bipolar Junction Transistor (BJT, or bipolar transistor). The BJT has three terminals like the MOSFET, but they are called the Collector (instead of Drain), the Emitter (instead of Source), and Base (instead of Gate). BJTs have largely been supplanted by MOSFETs in modern times, but still find use in applications that demand very high speed or high current.

A BJT is shown in the figure below along with its equivalent circuit when in its active mode, which consists of a fixed voltage source VBE and a current-dependent current source. The parameters VBE and beta are properties of the transistor. Use Node Analysis to solve for IB and vo in the circuit in the figure for a BJT with beta=200 and VBE =0.7 V.

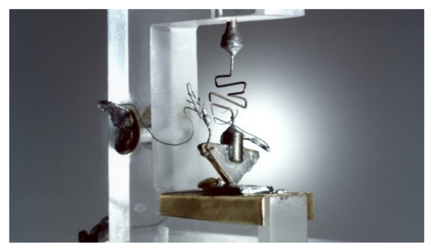


The bipolar transistor circuit diagram and its equivalent circuit model



The circuit to be solved

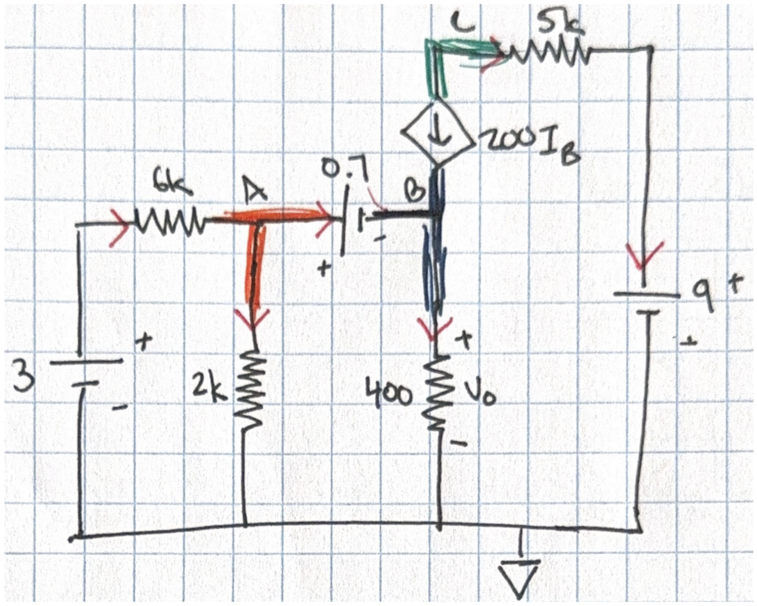
The developers of the first working transistor, a BJT: John Bardeen, Willliam Shockley, and Walter Brattain



The first working transistor was created at Bell Labs in 1947 out of Germanium

Solution. □

Redraw circuit:



Node A KCL:

$$\sum I = \frac{3 - V_a}{6000} - \frac{V_a - 0}{2000} - I_B = 0$$

Node B KCL:

$$\sum I = I_B + 200 I_B - \frac{V_b - 0}{400} = 0$$

Node C KCL:

$$\sum I = -200 I_B - \frac{V_c - 9}{5000} = 0$$

Floating voltage source:

$$\begin{aligned} V_a - V_0 &= 0.7 \\ V_a &= V_0 + 0.7 \end{aligned}$$

Voltage difference equivalents:

$$V_b = V_0$$

Remark. 5 equation 5 unkown :D

$$\begin{aligned}
\sum I = \frac{3 - V_a}{6000} - \frac{V_a - 0}{2000} - I_B &= 0 \\
\frac{3 - V_0 - 0.7}{6000} - \frac{V_0 + 0.7 - 0}{2000} - I_B &= 0 \\
\frac{2.3 - V_0}{6000} - \frac{V_0 + 0.7}{2000} - I_B &= 0 \\
\frac{2.3 - V_0}{6000} - \frac{3V_0 + 2.1}{6000} - I_B &= 0 \\
\frac{0.2 - 4V_0}{6000} - I_B &= 0 \\
\frac{0.2}{6000} - \frac{V_0}{1500} - I_B &= 0
\end{aligned}$$

$$\begin{aligned}
I_B + 200I_B - \frac{V_b - 0}{400} &= 0 \\
201I_B - \frac{V_0}{400} &= 0 \\
V_0 &= 80400I_B
\end{aligned}$$

$$\begin{aligned}
\frac{0.2}{6000} - \frac{V_0}{1500} - I_B &= 0 \\
\frac{0.2}{6000} - \frac{80400I_B}{1500} - I_B &= 0 \\
\frac{0.2}{6000} - 54.6I_B &= 0 \\
I_B &= 0.000000611 \\
I_B &= \boxed{0.611\,\mu\text{A}}
\end{aligned}$$

$$\begin{aligned}
V_0 &= 80400I_B \\
V_0 &= 80400(0.000000611) \\
V_0 &= 0.0491 \\
V_0 &= \boxed{0.0491\,\text{V}}
\end{aligned}$$