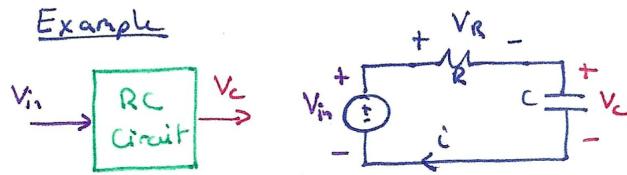


Lecture Notes 20: Rapid Transfer Functions and Impedance

Previously, we saw that replacing $A \cos(\omega t + \phi)$ with $\operatorname{Re}\{Ae^{j\omega t}e^{j\phi}\}$ simplified the solution to an LTI differential equation. We further noticed a pattern that allowed us to skip a lot of steps:

- The $\operatorname{Re}\{\cdot\}$ and $e^{j\omega t}$ are designed to be eliminated from every term
- An n 'th derivative always causes a multiplication by $\times(j\omega)^n$.

So the sinusoidal response of an RC circuit becomes almost a one-liner problem:



$$RC \frac{dv_c(t)}{dt} + v_c(t) = v_{in}(t)$$

$$RC(j\omega)V_c e^{j\phi} + V_c e^{j\phi} = V_{in} e^{j0}$$

or, in phasor notation,

$$RC(j\omega)\vec{v}_c + \vec{v}_c = \vec{v}_{in}$$

so the transfer function is,

$$\frac{\vec{v}_o}{\vec{v}_i} = \frac{V_o e^{j\phi}}{V_i e^{j0}} = \frac{1}{1 + j\omega RC}$$

Note that the transfer function is a complete answer to “what will the output be if the input is a sine wave,” but it is difficult to interpret because it is complex. We can extract real numbers that are easier to interpret as follows:

- The magnitude gain (the amplitude of the output relative to the input) is V_o/V_i . Observe that the magnitude of the transfer function is

$$\left| \frac{\vec{v}_o}{\vec{v}_i} \right| = \left| \frac{V_o e^{j\phi}}{V_i e^{j0}} \right| = \frac{|V_o e^{j\phi}|}{|V_i e^{j0}|} = \frac{V_o}{V_i} = \text{Magnitude Gain}$$

- The phase of the output relative to the input is $\phi - 0 = \phi$. Observe that the transfer function \vec{v}_o/\vec{v}_i is a complex number, and its angle is

$$\angle \frac{\vec{v}_o}{\vec{v}_i} = \angle \frac{V_o e^{j\phi}}{V_i e^{j0}} = \angle \frac{V_o}{V_i} e^{j(\phi-0)} = \phi$$

Therefore, we have simple ways to extract useful information from the transfer function:

The magnitude gain (output amplitude / input amplitude) is found by taking the complex magnitude of the transfer function, $|\vec{v}_o/\vec{v}_i|$.

The phase (output phase - input phase = output phase, since input phase = 0) is found by taking the angle of the transfer function, $\angle \vec{v}_o/\vec{v}_i$

Transfer functions usually appear as ratios of complex numbers, z_{num}/z_{den} , so it is important to recognize that the magnitude of a ratio is equal to the ratio of the magnitudes,

$$\left| \frac{z_{num}}{z_{den}} \right| = \frac{|z_{num}|}{|z_{den}|}$$

so the magnitude of the transfer function for the RC circuit is

$$M = \left| \frac{\vec{v}_o}{\vec{v}_i} \right| = \frac{|1|}{|1 + j\omega RC|} = \frac{1}{\sqrt{1 + (\omega RC)^2}}$$

Likewise, it is important to recognize that the angle of a ratio is equal to the difference of the angles,

$$\angle \frac{z_{num}}{z_{den}} = \angle z_{num} - \angle z_{den}$$

so the phase of the transfer function for the RC circuit is

$$\phi = \angle \frac{\vec{v}_o}{\vec{v}_i} = \angle \frac{1}{1 + j\omega RC} = \angle 1 - \angle(1 + j\omega RC) = \tan^{-1} \left(\frac{0}{1} \right) - \tan^{-1} \left(\frac{\omega RC}{1} \right) = -\tan^{-1}(\omega RC)$$

With a little practice, the following process becomes very quick:

- Use KVL/KCL/Component Laws (or variants like node analysis) to set up a system of equations and reduce it to the differential equation for a circuit
- Skip most of the hard derivation and simply replace the input and output with their phasors and any derivatives d^n/dt^n with $(j\omega)^n$
- Solve for the transfer function
- Extract magnitude and phase from the transfer function

Now we have considered the transfer function as a way to solve a differential equation for an entire circuit. But consider that an individual component's equation might also be a differential equation, e.g.,

$$i_c = C \frac{dv_c}{dt}$$

Well, could we apply the concept of phasors and transfer functions to this? If we consider i_c as the input and v_c as the output, then

$$I_c e^{j0} = C(j\omega) V_c e^{j\phi}$$

This is interesting: in the time domain, the capacitor $i-v$ equation has a derivative in it, but in the frequency domain, the capacitor current is proportional to its voltage. The voltage-to-current ratio in the frequency domain is

$$\frac{\vec{v}_c}{\vec{i}_c} = \frac{V_c e^{j\phi}}{I_c e^{j0}} = \frac{1}{j\omega C} = -j \frac{1}{\omega C}$$

The fact that the capacitor equation *in the frequency domain* looks like a resistor equation, $v \propto i$, suggests that we can treat capacitors like resistors in the frequency domain. The constant of proportionality is just complex.

This more expansive concept of a complex “resistance” is called **impedance** and is usually written with the letter Z .

Using this language, we might write

$$\vec{v}_c = Z_c \vec{i}_c \quad Z_c = \frac{1}{j\omega C} = -j \frac{1}{\omega C}$$

For a resistor,

$$\vec{v}_R = Z_R \vec{i}_R \quad Z_R = R$$

One final component, which you have not seen before, is called an **inductor**. Despite knowing nothing about the inductor, your knowledge of the frequency domain prepares you to understand its properties. The inductor is the dual of the capacitor: in the time domain, its $i - v$ equation is:

$$v_L = L \frac{di_L}{dt}$$

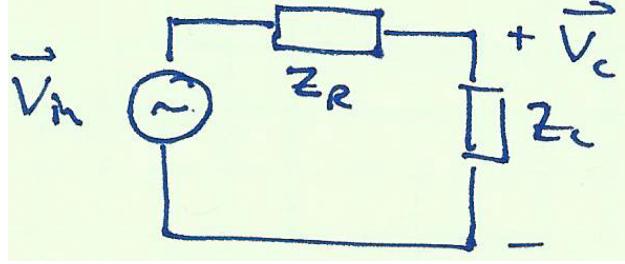
where the constant of proportionality, L , is its **inductance**. In the frequency domain, this becomes

$$\vec{v}_L = Z_L \vec{i}_L \quad Z_L = j\omega L$$

We have observed that resistors, capacitors, and inductors all have the same fundamental equation in the frequency domain, $v \propto i$, where the constant of proportionality has a different character depending on the component (real versus imaginary, positive versus negative). This suggests that the sinusoidal response of an LTI circuit can be solved like resistor circuits:

All LTI circuits are solved just like resistor circuits, because all LTI $i - v$ relationships becomes $\vec{v} = (\text{const}) \times \vec{i}$ in the phasor domain. The only catch is that the constant(s) may be imaginary or even complex.

By converting to the frequency domain (phasor domain) from the beginning, solving the RC circuit becomes *even easier!*



This is just a voltage divider, but with impedances instead of resistances. With this knowledge, the solution is a one-liner:

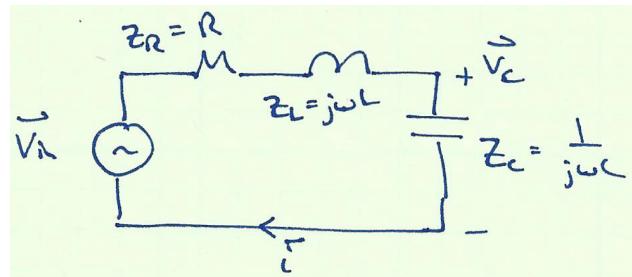
$$\vec{v}_c = \frac{Z_C}{Z_C + Z_R} \vec{v}_{in} = \frac{1/j\omega C}{1/j\omega C + R} \vec{v}_{in} = \frac{1}{1 + j\omega RC} \vec{v}_{in}$$

Impedances add in series and combine in parallel *just like resistors*, which gives us rapid circuit solving superpowers. For example, consider a series RLC circuit: We can say right away that

$$\vec{i} = \frac{\vec{v}_{in}}{Z_{tot}} = \frac{1}{R + \frac{1}{j\omega C} + j\omega L} \vec{v}_{in} = \frac{j\omega C}{(1 - LC\omega^2) - j\omega RC} \vec{v}_{in}$$

Then, if we're interested in solving for the voltage across the capacitor (for example),

$$\vec{v}_c = Z_C \vec{i}_c = \frac{1}{j\omega C} \times \frac{j\omega C}{(1 - LC\omega^2) - j\omega RC} \vec{v}_{in} = \frac{1}{(1 - LC\omega^2) - j\omega RC} \vec{v}_{in}$$



Then, if we ever want to “translate” the answers back into the time domain, we need only take the magnitude $|x|$ and angle $\angle x$ of the corresponding phasor.

One final bit of terminology. Just as we often talk of the dual of resistance R (voltage/current), which is conductance G (current/voltage), we also often speak of the dual of impedance Z (phasor voltage / phasor current), which is called **admittance** Y (phasor current / phasor voltage).