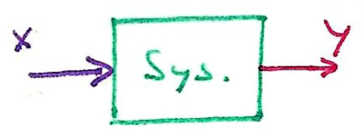
Lecture Notes 18: Transfer Functions

Physical systems obey differential equations. When the equations are linear (for example, with an output y and input x), they take the form

$$\cdots a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = \cdots d\frac{d^2x}{dt^2} + e\frac{dx}{dt} + fx$$

where $a, b, c, d, e, f, \cdots$ are system parameters. When the system parameters are constant in time, we have a **Linear Time Invariant (LTI)** system.



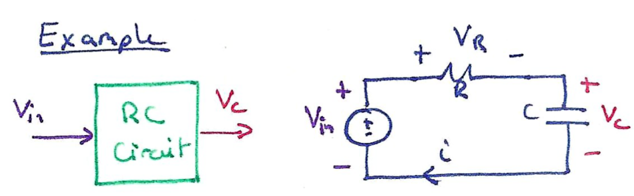
LTI systems have a very interesting property if the input (or **excitation**) is sinusoidal:

> If an LTI system is excited with a sine wave, $x(t) = X\cos(\omega t)$, then the output *must* be a sine wave at the same frequency, $y(t) = Y\cos(\omega t + \phi)$.

The proof of this can be seen in the differential equation. Plugging in $x(t) = X\cos(\omega t)$, the right hand side will reduce to $(\text{const})\cos(\omega t) + (\text{const})\sin(\omega t)$ which is just a sine wave in cosine-sine form. The only way for the left hand side to reduce to a sine wave (after all of the derivatives, multiplication by constants, and additions), is if the solution $y(t)$ is also sinusoidal. In addition, differentiation, multiplication by constants, and addition do not change the frequency of sine waves, so $y(t)$ must be a sine wave *at the same frequency* as $x(t)$.

This fact makes the **sinusoidal response** of a system relatively easy to calculate. And, through Fourier's Theorem, we know that *any* periodic function can be represented by a sum of sine waves, so the sinusoidal response of a system applies quite generally. Finally, if $x(t)$ is not sinusoidal, the principle of superposition tells us that we can calculate the response of the system to each of the sine waves that makes up $x(t)$ *individually* and then add the answers up. They key to all of this, of course, is the ability to calculate the sinusoidal response to the system.

Consider the example of an RC circuit, our first circuit with a capacitor and therefore our first differential equation:



First, we need to derive the differential equations, which we do by applying KVL, KCL, and the component laws (or, equivalently, node voltage analysis) as usual. Doing so tells us that

$$v_{in} - v_R - v_C = 0 \rightarrow v_in - iR - v_c = 0$$

The capacitor component law is $i_c = C\,dv_c/dt$, so

$$\boxed{RC\frac{dv_c}{dt} + v_c = v_{in}}$$

This is the input-output differential equation in the desired form.

Now, we wish to calculate the sinusoidal response. Therefore, plug in $x(t) = v_{in}(t) = V_{in}\cos(\omega t)$. Through the powerful statement we made above, we know a priori that the output will have the form $y(t) = v_c(t) = V_c\cos(\omega t + \phi)$. We just don't know what V_C and ϕ are yet. Plugging *both* of these into the differential equation yields:

$$\boxed{-RC\omega V_C \sin(\omega t + \phi) + V_C\cos(\omega t + \phi) = V_{in}\cos(\omega t)}$$

We usually don't solve for the output magnitude per se. Rather, we calculate the **magnitude gain** of the system, $M = Y/X$ which, in our example, is $M = V_C/V_{in}$,

$$M = \frac{V_C}{V_{in}} = \frac{\cos(\omega t)}{cos(\omega t + \phi) - RC\omega\sin(\omega t + \phi)}$$

This isn't a solution yet, because M depends on ϕ which is still unknown. This equation may even seem impossible to solve because we only have one equation and two unknowns. However, we can cleverly use the fact that this equation must be true *at all time* to come up with two independent equations.

Consider, for example, $t = 0$. The equation for M must be true at $t = 0$, so

$$t = 0 \quad \rightarrow \quad M = \frac{1}{\cos(\phi) - RC\omega\sin(\phi)}$$

Another time that simplifies the equation a lot is $t = -\phi/\omega$, which yields

$$t = -\frac{\phi}{\omega} \quad \rightarrow \quad M = -\cos(-\phi)$$

Now take these two interdependent equations with two unknowns (M and ϕ) and solve:

$$M = \cos(\phi) \qquad \sin(\phi) = \sqrt{1 - \cos^2\phi} = \sqrt{1 - M^2}$$

and therefore,

$$\begin{aligned} M &= \frac{1}{M - RC\omega\sqrt{1 - M^2}} \\ -RC\omega\sqrt{1 - M^2} &= \frac{(1 - M^2)}{M} \\ +R^2C^2\omega^2(1 - M^2) &= \frac{(1 - M^2)^2}{M^2} \\ R^2C^2\omega^2 &= \frac{(1 - M^2)}{M^2} \end{aligned}$$

and finally,

$$\boxed{M = \frac{1}{\sqrt{1 + (RC\omega)^2}}}$$

Let's take a moment to practice sketching this function by hand. Clearly, the behavior of the function changes a lot for $\omega \ll 1/RC$ versus $\omega \gg 1/RC$:

- For $\omega \ll 1/RC$, the function reduces to

  $$M \approx 1 = 0\,\mathrm{dB}$$

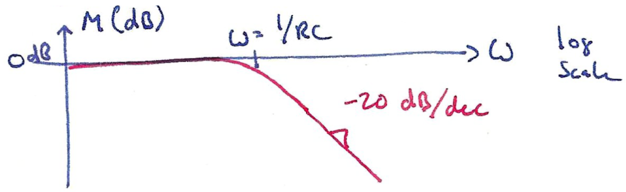
  which ought to be flat.

- For $\omega \gg 1/RC$, the function reduces to

$$M \approx \frac{1}{\sqrt{(RC\omega)^2}} = \frac{1}{RC\omega}$$

- To help us connect these two regions, at $\omega = 1/RC \Rightarrow M = 1/\sqrt{2} \approx -3\text{dB}$

This looks like a $1/\omega$ function. We know what this looks like on a linear plot, but recall that $1/\omega = \omega^-1$ is a *monomial* and so it should look like a straight line with a slope of -1 decade-per-decade = -20 dB/decade on a log-log plot. This gives us two **asymptotes** and we can hand-sketch the connection between them:



Great! This already tells us a bit about how the RC circuit will behave. At low frequencies, the output sine wave is essentially the same magnitude as the input sine wave. This makes sense: at low frequencies, the capacitor allows very little current to flow ($i = Cdv/dt$ is very small at low frequencies), so there's very little voltage drop across the resistor, so the capacitor voltage is effectively the same as the input voltage. At high frequencies, the capacitor voltage becomes smaller and smaller. This also makes sense: at high frequencies, the capacitor permits a lot of current to flow; the amount of current the actually does flow is more limited by the resistor than by the capacitor. As the frequency increases, this current results in less and less voltage across the capacitor.

Many systems have similar behavior, and we classify them as **low-pass filters** because they allow low-frequencies to "pass" through to the output while they "filter out" high frequencies.

We're not done yet, though! We still need to calculate ϕ (the phase difference between $v_c(t)$ and $v_{in}(t)$):

$$\phi = -\cos^{-1}\left(\frac{1}{\sqrt{1 + (RC\omega)^2}}\right)$$

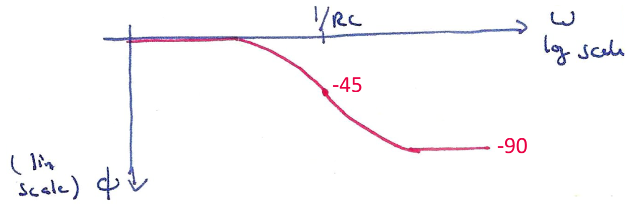
This is correct, but is usually expressed differently. We can arrive at the usual expression through an obscure trigonometric identity that says $\cos^{-1}(1/\sqrt{1 + x^2}) = \tan^{-1}(x)$. This gives us:

$$\boxed{\phi = -\tan^{-1}(RC\omega)}$$

Let's again practice examining this by hand before we reach for our calculators. Again, this function behaves differently for $\omega \ll 1/RC$ and $\omega \gg 1/RC$:

- For $\omega \ll 1/RC \Rightarrow \phi \approx 0$, and ought to be flat.
- For $\omega \gg 1/RC \Rightarrow \phi \approx -90°$, and ought to be flat.
- To help us connect these regions, at $\omega = 1/RC \Rightarrow \phi = -45°$.

There's no need to plot phase on a log-log plot. The phase does have a large dynamic range, but we don't much care about the distinction between 0° and 1°, for example. Phase is usually plotted on a semilog-x plot:



Together, $M(\omega)$ and $\phi(\omega)$ define how an input sine wave is transformed in magnitude and phase into an output sine wave (of the same frequency). M and ϕ are known as the magnitude and phase of the **transfer function**.

Example: At $\omega = 0.5/RC$, $M = 0.894$ and $\phi = -0.46\text{rad} = -26.6°$. This means that an input wave at $0.5/RC$ will produce an output wave that is 0.894 times as large as the input and will lag the input by $26.6°$.

Often, the transfer function simplifies greatly when ω is far from certain key frequencies, such as $1/RC$ in the example. Because of this, these key frequencies are given special names. In **first-order systems** (those with only a single derivative of the output in the original differential equation), the reciprocal of the key frequency is known as the **time constant** of the system, τ ($= RC$ in the example).

We have been very successful, *but* it was a lot of work to get here by doing everything in terms of cosines and sines.

There is a much easier way to solve these problems. In fact, the RC circuit's transfer function can be found in about two lines. The catch is that doing it the "easy way" requires the use of complex/imaginary numbers. Therefore, our next task is to review/learn complex numbers. We will then return to this problem and see how much easier life can be if we first learn some more sophisticated mathematics.