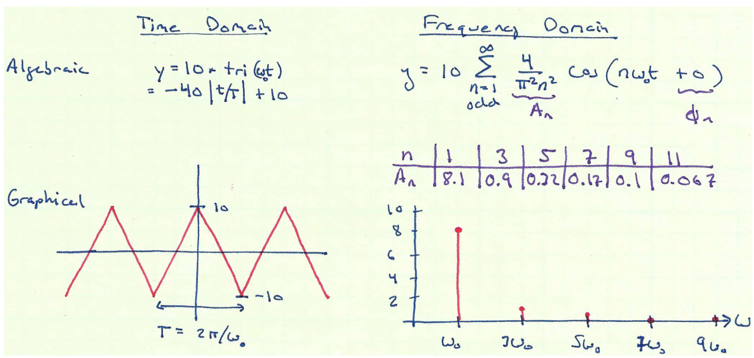
Lecture Notes 17: Log Plots

In the frequency domain (and other times in engineering), it is common to want to see a large **dynamic range** in numbers,

$$\text{Dynamic Range} = \frac{\text{Maximum Number In Set}}{\text{Minimum Number In Set}}$$

For example, a data set that only includes the numbers 7, 10, 4, 8, and 12 has very low dynamic range. However, a data set that includes 0.0003, 10, and 84,692 has very large dynamic range.

Traditional “linear” plots are ill-suited to analyzing data with large dynamic ranges. For example, consider a triangle wave in the frequency domain:

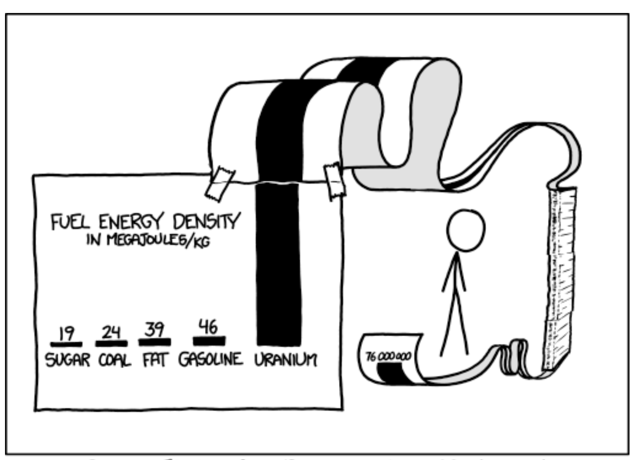


Notice how the data for the amplitudes of the sine waves in the frequency domain has high dynamic range. The largest amplitude is 8.1, while the smallest number (so far) is 0.067. When these numbers are plotted on a linear plot, you run into one of the following problems:

1. If you plot the largest number with a reasonable height, the lowest numbers are indistinguishable from zero.

2. If you plot the smallest number with a reasonable height, the largest numbers are off the chart

The only way to plot this properly with a linear plot would be to have an *extremely* tall graph. This is an absurd solution humorously captured by XKCD:

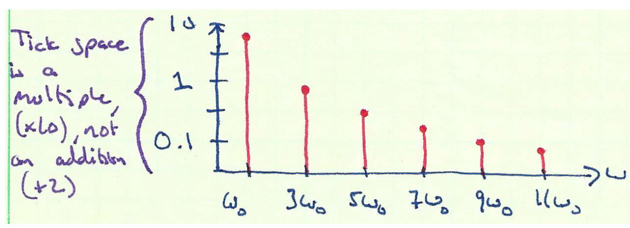


SCIENCE TIP: LOG SCALES ARE FOR QUITTERS WHO CAN'T FIND ENOUGH PAPER TO MAKE THEIR POINT *PROPERLY*.

The right solution is to use a **log(arithmic) scale** to "compress" data with large dynamic ranges to fit in a smaller space.

> On a log scale, tick marks are assigned by multiplying the last tick mark by a constant. For example, you could evenly-spaced tick marks at 1, 10, 100, etc.
>
> [This stands in contrast to a linear plot, where tick marks are assigned by adding a constant to the last tick mark. For example, you could have evenly spaced tick marks at 1, 3, 5, 7, etc.]

A **log(arithmic) plot** is one that makes use of a log scale on one or both axes. Consider the triangle wave amplitude data again:



Here, the horizontal axis (ω) is plotted on a linear scale as usual, but the vertical axis (amplitude) is plotted on a log scale. This is called a **semilog-y** plot. Notice how all of the data points are legible now. In fact, this plot even goes out to the 11th harmonic and is still legible (by contrast, the linear plot was already illegible at about the 5th harmonic).

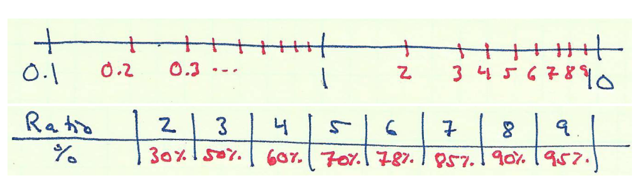
The tradeoff with log plots is that you lose a little bit of visual precision. For example, is the highest data point equal to 6, 8, or 10? It's hard to see. We're quite confident that it's less than 10 and more than 5, but a little unclear beyond that. However, in exchange, we can intuitively read the very small numbers with similar precision as the large numbers. This is very important because it's these very small numbers that make a triangle wave a triangle wave.

The key to the log scale is that tick spacing is multiplicative, whereas tick spacing is additive on a linear scale. In linear scale, the spacing is arbitrary – whether your ticks go $1,2,3,4,\cdots$ or $2,4,6,8,\cdots$ is entirely up to you. With log scales, the spacing is also arbitrary – whether your ticks go $2,4,8,16\cdots$ or $1,10,100,1000,\cdots$ is entirely up to you. Nevertheless, there are more common spacings than others. By far the most common is factors of 10, or **decades**. Another common spacing is factors of 2, or **octaves**.

Minor ticks on a log scale based on factors of 10 are drawn according to the following formula:

$$\boxed{\text{Minor Tick \% Between Major Ticks} = \log_{10}\left(\frac{\text{Number Being Plotted}}{\text{Lower Major Tick}}\right)}$$

For example: on a log scale with major tick marks at 0.1, 1, 10, etc., the number 0.32 should be plotted at $\log_{10}(0.32/0.1) = 50.5\%$ of the way from the 0.1 tick mark to the 1 tick mark. This causes the minor tick marks to not be evenly spaced, but rather to compress together as the next major tick mark is approached, shown both graphically and numerically below:



Some of these are worth remembering. **Most engineers know that halfway between tick marks on a log scale is very close to a factor of three**, for example.

Note that there is no "origin" or "zero" on a log scale. As you move down in numbers, they simply get smaller, $0.1 \to 0.01 \to 0.001 \to \cdots$. As a corollary to this, there is no such thing as an "intercept" on a log scale. We draw the vertical and horizontal lines to help us visually, but the line for the x-axis does not represent $y = 0$ and the line for the y-axis does not represent $x = 0$.

There are three main ways to draw and think about log scales:

1. You are plotting y on a log scale. The tick marks are actually labeled as 0.1, 1, 10, etc, but the number you are plotting is y itself.

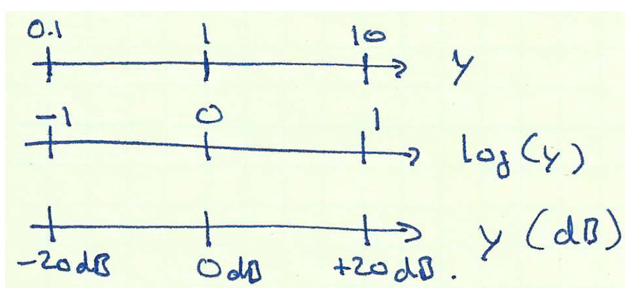
2. You are plotting $\log(y)$ on a linear scale. The tick marks are actually labeld as -1, 0, 1, etc. (a linear scale), but the number you are plotting is not y itself but rather $\log(y)$. The number by the tick is therefore the exponent related to y (e.g., "2" really means 10^2)

3. You can plot y in **decibels**, which are defined as

$$\boxed{\text{Number in dB} = 20\log_{10}(\text{Number})}$$

   for voltages and currents (for power, the definition is $10\log(\#)$ – we won't bother with that here).

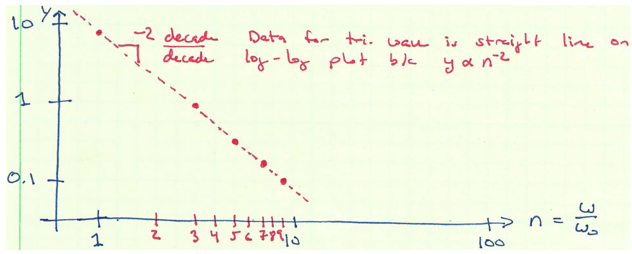
These three representations are *exactly the same*, but you need to be fluent in all three of them because all three are used in practice



We have one important point to make for dB (or for plotting $\log(y)$). Since dB$= 20\log(\#)$, it is essential that # be dimensionless. Therefore, dB is properly used exclusively for *ratios*, such as the output voltage of an amplifier divided by the input voltage v_o/v_i. If you are interested in an actual number, not a ratio, and you still want to use dB, you can plot the number relative to an arbitrary reference. For example, 100V cannot be expressed in dB, but 100V/1V can be (40 dB). In these cases, you need to communicate to your reader what the arbitrary reference is, which is usually done by appending the reference to dB. So we would say that 100V = 40 dBV ("dB-volts"). If we had used 1 μV as the reference, we would say 100V = 160 dBμV (8 orders of magnitude or 8 decades).

Recall that a log plot can plot just the vertical axis on a log scale (semilog-y), just the horizontal axis on a log scale (**semilog-x**), or both axes on a log scale (**log-log**, also often just called "log"). Something very interesting happens when we plot the data for the amplitudes of the sine waves that make up the triangle wave on a log-log plot:

The data appear to be on a straight line! Is this just happenstance? No! It turns out there are very useful patterns with log plots:



- Monomials $y = ax^n$ appear as straight lines on log-log plots. The proof is found by taking the log of both sides, which gives $\log(y) = \log(a) + n\log(x)$, which becomes a straight line when plotting $\log(y)$ vs $\log(x)$. In this case, the slope of the line (in decades per decade) is equal to n.
- Exponentials $y = ae^x$ appear as straight lines on semilog-y plots. In this case, the slope of the line is the base of the exponential (often e, but if it weren't, this is how you would know).
- Logarithms $y = a\log(x)$ appear as straight lines on semilog-x plots. In this case, the slope of the line is the base of the logarithm.