

## Homework 10

## ECE 302H: Introduction to Electrical Engineering

**Problem 10.1 – Poles and Zeros (1h)**

Transfer functions of linear systems always look like  $T = \frac{\text{polynomial of } \omega}{\text{other polynomial of } \omega}$ .

If those polynomials are factored, then the transfer function looks like

$$T = A \frac{\left(1 + \frac{j\omega}{\omega_{z1}}\right) \left(1 + \frac{j\omega}{\omega_{z2}}\right) \left(1 + \frac{j\omega}{\omega_{z3}}\right) \dots}{\left(1 + \frac{j\omega}{\omega_{p1}}\right) \left(1 + \frac{j\omega}{\omega_{p2}}\right) \left(1 + \frac{j\omega}{\omega_{p3}}\right) \dots},$$

where  $A$  is a constant scaling factor. The known frequency elements  $\omega_z$  (they come from factoring the polynomial in the numerator) are called zeros, while the elements labeled  $\omega_p$  are called poles. Note that poles and zeros can be real, imaginary, or complex, and can have positive or negative values.

(Throughout this problem, be careful to distinguish the complex-valued  $T$  from the real-valued  $\Re T \vee \Im T$  and  $\angle T$ )

- a) What are the poles and zeros of  $T = 3 \frac{\left(1 - \frac{j\omega}{100}\right) \left(1 - \frac{j\omega}{1,000}\right)}{\left(1 - \frac{j\omega}{10}\right) \left(1 - \frac{j\omega}{10,000}\right)}$ , assuming that the numerical

frequencies have units of rad/sec? Are they real, imaginary, or complex? Are they positive or negative?

- b) The scaling factor  $A$  is often called the “dc gain.” Explain why by letting  $\omega \rightarrow 0$ . Are there any values of  $\omega_z$  or  $\omega_p$  that would ruin this interpretation?
- c) Sketch the magnitude of  $T$  on a log-log plot (y axis in dB) as follows:
- Start from low frequency. What is  $T$  approximately equal to? What is  $\Re T \vee \Im T$  approximately equal to in this region? At what frequency does this approximation break down? Sketch  $|T|$  up until this frequency.
  - Above the frequency you just found, what is  $T$  approximately equal to? What should the slope of  $|T|$  be in this region? At what frequency does this approximation break down? Continue sketching  $|T|$ , starting where you left off and ending at the frequency you just found.
  - Continue increasing frequency, identifying the slope of  $\Re T \vee \Im T$  in each region until you have a complete graph
- d) As frequency increases, what do you notice about the slope of  $\Re T \vee \Im T$  each time you hit a pole? Each time you hit a zero?
- e) Sketch the phase of  $T$  on a semilog-x plot in a similar way:
- Start from low frequency. What is  $T$  approximately equal to? What is the phase of  $T$  in this region? At what frequency does this approximation break down? Sketch the phase of  $T$  up until this frequency.
  - Above the frequency you just found, what is  $T$  approximately equal to? What is the phase of  $T$  in this region? At what frequency does this approximation break down?

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Sketch the phase of T in this frequency region. Roughly sketch a smooth connection from the phase in the last region to the phase in this region.

- Continue increasing frequency, identifying the phase of T in each region until you have a complete graph
- f) As frequency increases, what do you notice about the phase of T each time you hit a pole? Each time you hit a zero?
  - g) Suppose A were changed from 3 to 30. Describe in words how the magnitude and phase plots would change.

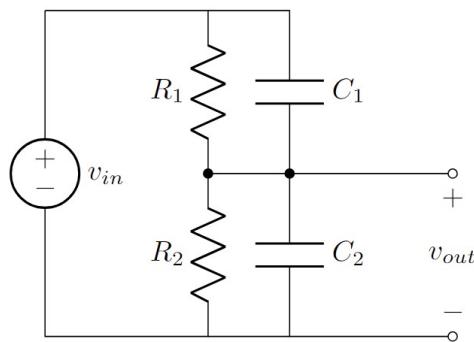
**Problem 10.2 – How Oscilloscope Probes Work and the “Doublet” Problem (1h)**

An oscilloscope probe can be modeled as one resistor  $R_1$  in parallel with a capacitor  $C_1$ . This parallel combination appears in series with the scope itself, which is modeled as another resistor  $R_2$  in parallel with a capacitor  $C_2$ . The bottom of  $R_2$  and  $C_2$  is ground, which connects back to the device under test, completing the circuit. The midpoint voltage between  $R_1$  and  $R_2$  is what the oscilloscope itself measures.

- a) Use circuit analysis to determine the differential equation that relates the input (the voltage of the device under test) and the output (the midpoint voltage) in the time domain.
- b) Let the input be a cosine wave,  $v_i(t) = V_i \cos \omega t$ . What form must the output  $v_{out}(t)$  take, and which parameters of  $v_{out}(t)$  are known or unknown at this stage?
- c) Write the input and output voltages as the real part of complex exponentials
- d) Plug your answers into the differential equation; perform any derivatives; reduce the problem to  $\Re[Ae^{j\omega t}] = \Re[Be^{j\omega t}]$  and use the theorem we learned in class to conclude that  $A=B$ . In the resulting equation, identify the input voltage phasor  $\bar{V}_i$  and the output voltage phasor  $\bar{V}_{out}$ .
- e) Solve for the complex transfer function and calculate its magnitude and phase as functions of  $\omega$
- f) Prove that the magnitude and phase of the transfer function are independent of frequency if  $\frac{R_1}{R_2} = \frac{C_2}{C_1}$ . (We call this “pole-zero cancellation”)
- g) Suppose you have a square wave voltage source applied to the circuit input. If the above condition is met, what do you expect the output (midpoint) voltage to look like? Why? (Hint: How does the transfer function affect the different frequency components of a square wave?)
- h) Let  $R_1 = 90.9 \text{ k}\Omega$ ,  $R_2 = 9.42 \text{ k}\Omega$ ,  $C_2 = 145 \text{ pF}$ ,  $C_1 = 15 \text{ pF}$  and simulate the circuit in LTSpice. Let the voltage source be a PULSE from 0V to 10V with rise and fall times of 1ns at a frequency of 10 kHz and 50% duty cycle (also set its AC amplitude to 1V for ac analysis simulations). Run transient simulation from 0 to 300 us and an ac simulation from 10 Hz to 10 MHz (on the ac simulation, adjust the magnitude scale to be from -40 dB to 0 dB and the phase scale to be from -90 degree to +90 degree). Include both plots.
- i) Repeat (h) for  $C_1 = 20 \text{ pF}$  and  $C_1 = 10 \text{ pF}$ . Include the plots and describe what you see in the time domain simulation. (In the transfer function, the close-but-imperfect cancellation of poles and zeros is called a “doublet”)
- j) Suppose  $C_1$  is adjustable (e.g., with a screwdriver), describe how you could properly tune an oscilloscope probe.

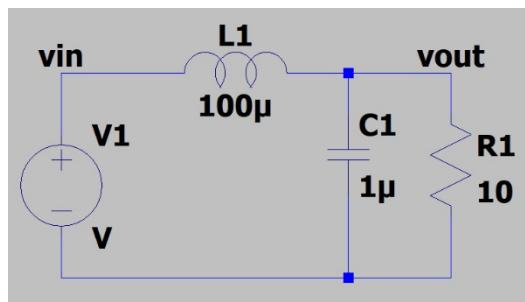
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## Problem 10.3 – Inductors and RLC Circuits (1h)

An inductor is a coil of wire, sometimes wrapped around a magnetic core. Its component law is the dual of the capacitor's, namely  $v_L = L \frac{di_L}{dt}$ . Imagine an LCR circuit like the one below (which happens to be the same as the circuit in your final project):

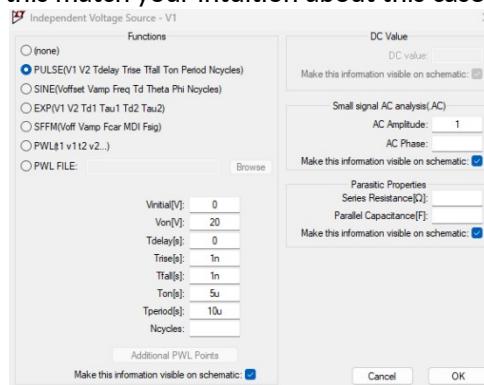


- Find the differential equation that relates the input voltage to the output voltage.
- If the input is a sine wave, we needn't go through the whole rigmarole of writing it as the real part of a complex exponential, plugging it in, etc. We've done this enough times to know what will happen. Instead, look directly at the differential equation and replace  $v_i$  and  $v_{out}$  with their phasors, recognizing that  $\frac{d}{dt}v_{out}$  will turn into  $j\omega \vec{v}_{out}$  and  $\frac{d}{dt}v_i$  will turn into  $j\omega \vec{v}_i$  in the frequency domain.
- Solve for the complex transfer function of the circuit, namely  $\frac{\vec{v}_{out}}{\vec{v}_i}$ . Calculate the magnitude and phase as a function of frequency  $\omega$ .
- On a log-log plot (Bode Plot), what is the slope of the magnitude at low frequencies (far below the LC resonance,  $\omega_r = 1/\sqrt{LC}$ ) and high frequencies (far above resonance)?
- Create this circuit in LTSpice. Right click on the voltage source, choose "advanced," and set the AC amplitude to 1. Then, run a simulation but, in the dialog box, change from "Transient" to "AC Analysis." Select "Decade" with 100 points per decade with a start frequency of 100 Hz and a stop frequency of 1 MHz. Submit the plot of Vout (since Vin is a cosine wave with a magnitude of 1 and a phase of 0, then plotting Vout is the same as plotting Vout/Vin).

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- f) A square wave, like all waves in the frequency domain, consists of a dc component, a fundamental component at the square wave frequency, and a sequence of harmonics at twice, thrice, and so forth of the fundamental frequency, of generally decreasing magnitude. If a square wave of frequency much less than resonance is applied to the above circuit, what do you expect the output to approximately look like? If a square wave of frequency much greater than resonance is applied to the above circuit, what do you expect the output to approximately look like?
- g) Verify your answer in LTSpice by right-clicking on the voltage source -> "Pulse" and assigning the following parameters to create a 100 Hz square wave that goes from 0V to 20V. Change the simulation type back to Transient and run the simulation for 50 ms. Submit the time-domain plot of Vin and Vout. Does this match your intuition about this case?



- h) Further verify your intuition from the frequency domain by adjusting the voltage source to have a frequency of 100 kHz. Run the simulation for 300us but only saving the last 150us. Submit the time-domain plot of Vin and Vout. Does this match your intuition about this case?
- i) For the 100 kHz case, right-click on the plot and select view -> FFT (Fast Fourier Transform, i.e. the frequency-domain representation of the signals). Make sure both Vin and Vout are selected and click ok. Make sure Vin and Vout are selected for visibility and then click ok again. Submit the plot.
- j) Finally, note that the dc value of the output is 10V in the 100 kHz case because the duty cycle of the input, i.e. the fraction of time that it is 20V instead of 0V, is 50%. If the duty cycle were 25%, what would you expect the dc value of Vout to be?

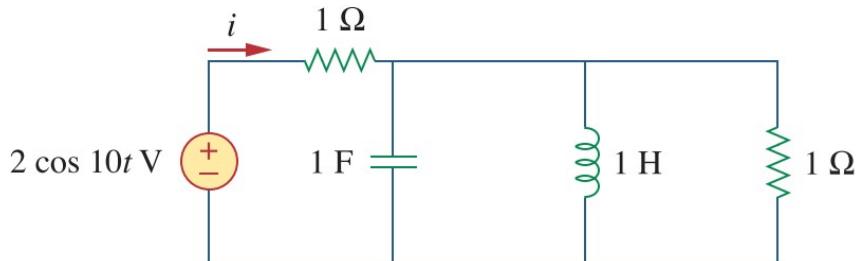
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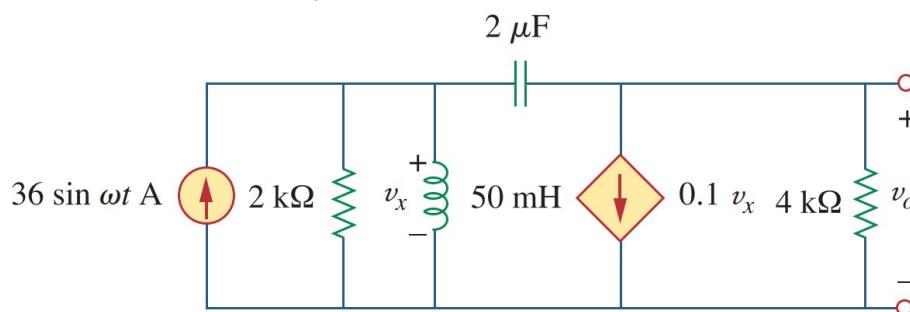
**Problem 10.4 – Rapid AC Solutions (1h)**

Having multiple times gone through the process of writing out the differential equations in the time domain and then converting to the frequency domain, we are now prepared to do direct ac analysis in the frequency domain. Treating all resistors, inductors, and capacitors as *impedances* and using all the same circuit analysis tools that we used for resistors, answer the following questions:

- a) Find the total *impedance* to the right of the voltage source and use your answer to solve for  $i$ .



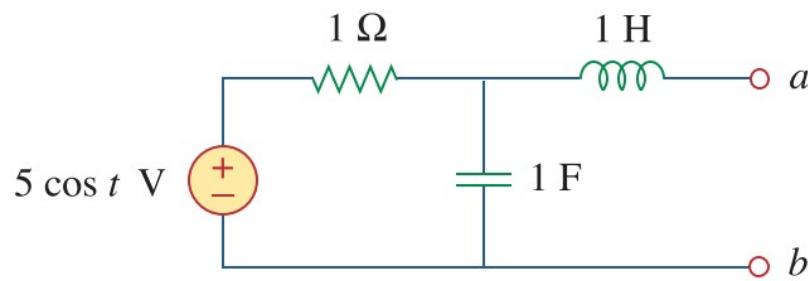
- b) Use node analysis to solve for  $v_o$  if  $\omega = 2 \text{ krad/s}$



- c) Find the Thevenin equivalent for this circuit:

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**Problem 10.5 – Extra Credit: Degree Plan (1h)**

Choose the tech core that you are most likely to choose and craft an accelerated degree plan for that tech core. You can use whatever criteria you want to craft your plan – including early and deep concentration, intentional hybridization, maintaining the opportunity to pivot, and so forth – but your plan must not be the same as the 4-year plan put forward by the ECE department. You can leave the gen-ed classes unspecified, but you do have to say which semesters they will occur in. Choose the elective ECE classes specifically, though. Write a short 1-2 paragraphs about the key features of your degree plan and your reasoning for some of the key choices.

You are in no way committing to that tech core or the accelerated degree plan, but going through the exercise will help you think creatively about your degree plan.

Only complete plans with sensible rationale will be awarded credit.