Sequences, Series, and Multivariable Calculus (MATH 408D) – Techniques of Integration

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0.1 u-Substitution Review

u-Substitution is one of the most useful and prioritized tools when it comes to solving antiderivatives. The antiderivative power-rule doesn't help us evaluate integrals such as

$$\int 2x\sqrt{1+x^2}dx$$

> **Remark 0.1.** To find this integral, we can change the variable x into another variable to simplify the integration process.

Solution. Consider setting $1+x^2$ to another variable u

$$u = 1 + x^2$$

Taking the derivative of u with respect to x yields

$$\begin{aligned} u' = \frac{du}{dx} &= 2x \\ \frac{du}{2x} &= dx \end{aligned}$$

Substituting u and dx back into the original equation,

$$\int \frac{2x\sqrt{u}}{2x}du = \int \sqrt{u}du = \frac{2}{3}u^{3/2} + C$$

Finally, plugging in the original u,

$$= \boxed{\frac{2}{3}(1+x^2)^{3/2} + C}$$

□

> **Remark 0.2.** For the most part, u-substitution usually involves setting the "inside" function to u.

Theorem 0.3

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)dx = \int f(u)du$$

Example 0.4

$$\int \cos(5x)dx$$

Solution. Set up u-substitution terms

$$\begin{aligned} u &= 5x \\ \frac{du}{dx} &= 5 \\ \frac{du}{5} &= dx \end{aligned}$$

Substitute into original equation

$$\begin{aligned} \int \cos(5x)dx &= \frac{1}{5}\int \cos(u)du \\ &= \frac{1}{5}\sin(u) + C \\ &= \boxed{\frac{1}{5}\sin(5x) + C} \end{aligned}$$

□

Example 0.5

$$\int_0^2 2x(x^2+1)^3 dx$$

Solution. Set up u-substitution terms

$$\begin{aligned} u &= x^2 + 1 \\ \frac{du}{dx} &= 2x \\ \frac{du}{2x} &= dx \end{aligned}$$

Substitute into original equation

$$\begin{aligned} \int_0^2 2x(x^2+1)^3 dx &= \int_0^2 \frac{2x(u)^3}{2x} du \\ &= \int_0^2 u^3 du \\ &= \frac{1}{4}u^4\bigg|_0^2 \\ &= \frac{1}{4}(x^2+1)^4\bigg|_0^2 \end{aligned}$$

$$= \boxed{156}$$

> **Remark 0.6.** In this solution, I chose to substitute the $u = x^2 + 1$ equation back into into my solved integral instead of finding new limits of integration with resepect to u.

□

Example 0.7

$$\int_0^1 \frac{x}{1+x^2}dx$$

Solution.

$$\begin{aligned} u &= 1 + x^2 \\ \frac{du}{dx} &= 2x \\ dx &= \frac{du}{2x} \end{aligned}$$

□

$$\begin{aligned} \int_0^1 \frac{x}{1+x^2}dx &= \int_0^1 \frac{x}{u(2x)}du \\ &= \frac{1}{2}\int_0^2 \frac{1}{u}du \\ &= \frac{1}{2}\ln(|u|)\Big|_1^2 \\ &= \boxed{\frac{1}{2}\ln(2)} \end{aligned}$$

1 Integration by Parts

Integration by parts comes from the product rule from derivatives. Consider the product rule where $f = f(x)$ and $g = g(x)$

$$(fg)' = f'g + fg'$$

Integrate both sides of the equation

$$\begin{aligned}\int (fg)'dx &= \int f'gdx + \int fg'dx \\ fg &= \int f'gdx + \int fg'dx \\ \int fg'dx &= fg - \int f'gdx\end{aligned}$$

Theorem 1.1

If f and g are differentiable functions then,

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$$

or, equivalently,

$$\int udv = uv - \int vdu$$

where $u = f(x)$ and $v = g(x)$

1.1 Steps to consider

- Look for a product of two functions, e.g. $f(x)g(x)$
- Choose u and dv
- Choose u in order of **L.I.A.T.E** (logarithmic, inverse trig, algebraic, trig, exponential), choose u in that order

  > **Remark 1.2.** In practice, once integration by parts becomes more fluid, it's not necessary to consider the **L.I.A.T.E** rule every time.

- Differentiate u to find du
- Integrate dv to find v
- Plug into formula $\int udv = uv - \int vdu$ and solve

Example 1.3

$$\int xe^x dx$$

Solution. Find u and dv

$$u = x$$
$$dv = e^x dx$$

Find du

$$u = x$$
$$\frac{du}{dx} = 1$$
$$du = dx$$

Find v

$$dv = e^x dx$$
$$\int dv = \int e^x dx$$
$$v = e^x$$

Plug back into equation $\int udv = uv - \int vdu$

$$\int xe^x dx = xe^x - \int e^x dx$$
$$= \boxed{xe^x - e^x + C}$$

□

Example 1.4

$$\int \ln(x) dx$$

Solution.

$$u = \ln(x)$$
$$\frac{du}{dx} = \frac{1}{x}$$
$$du = \frac{1}{x} dx$$

$$dv = dx$$

$$\int dv = \int dx$$
$$v = x$$

Plug back into equation $\int udv = uv - \int vdu$

$$\int \ln(x)dx = x\ln(x) - \int \frac{x}{x}dx$$
$$= \boxed{x\ln(x) - x + C}$$

□

Example 1.5

$$\int x\sin(x)dx$$

Solution.

$$u = x$$
$$\frac{du}{dx} = 1$$
$$du = dx$$

$$dv = \sin(x)dx$$
$$\int dv = \int \sin(x)dx$$
$$v = -\cos(x)$$

Plug back into equation $\int udv = uv - \int vdu$

$$\int x\sin(x) = -x\cos(x) + \int \cos(x)dx$$
$$= \boxed{-x\cos(x) + \sin(x)}$$

□

Example 1.6

$$\int e^x\cos(x)dx$$

Solution.

$$u = \cos(x)$$

$$du = -\sin(x)dx$$

$$dv = e^x dx$$
$$v = e^x$$

$$\int e^x \cos(x)dx = e^x \cos(x) + \int e^x \sin(x)dx$$

$$u = \sin(x)$$
$$du = \cos(x)dx$$
$$dv = e^x dx$$
$$v = e^x$$

$$\begin{aligned}
\int e^x \cos(x)dx &= e^x \cos(x) + \int e^x \sin(x)dx \\
&= e^x \cos(x) + e^x \sin(x) - \int e^x \cos(x)dx \\
2\int e^x \cos(x)dx &= e^x \cos(x) + e^x \sin(x) \\
\int e^x \cos(x)dx &= \boxed{\frac{e^x \cos(x) + e^x \sin(x)}{2} + C}
\end{aligned}$$

□

Example 1.7

$$\int_0^1 \arctan(x)dx$$

2 Trigonometric Integration

2.1 Review of Trig Identities

$$\sin^2(x) + \cos^2(x) = 1 \tag{1}$$
$$\sec^2(x) - \tan^2(x) = 1 \tag{2}$$
$$\csc^2(x) - \cot^2(x) = 1 \tag{3}$$
(4)

$$\sin(2x) = 2\sin(x)\cos(x) \tag{5}$$

$$\cos(2x) = \cos^2(x) - \sin^2(x) \tag{6}$$
$$\sin^2 x = \frac{1 - \cos(2x)}{2} \tag{7}$$
$$\cos^2 x = \frac{1 \cos(2x)}{2} \tag{8}$$
$$\tag{9}$$

Example 2.1

$$\int \cos^5(x)dx$$

Solution. Plug in pythagorean identities:

$$\begin{aligned} \int \cos^5(x)dx &= \int \cos^4(x)\cos(x)dx \\ &= \int (1 - \sin^2(x))^2 \cos(x)dx \end{aligned}$$

Use u-substitution:

$$\begin{aligned} u &= \sin(x) \\ \frac{du}{dx} &= \cos(x) \\ du &= \cos(x)dx \end{aligned}$$

Solve:

$$\int \cos^5(x)dx = \int (1 - u^2)^2 du$$

Remark 2.2. Clever students have noticed that $u = 1 - \sin^2(x)$ is probably easier

□

2.2 Products of sine and cosine

Example 2.3

$$\int \sin^5(x)\cos^2(x)dx$$

Solution. Plug in pythagorean identities:

$$\begin{aligned}\int \sin^5(x)\cos^2(x)dx &= \int \sin^4(x)\cos^2(x)\sin(x)dx \\ &= \int (1-\cos^2(x))^2\cos^2(x)\sin(x)dx\end{aligned}$$

Use u-substitution:

$$\begin{aligned}u &= \cos(x) \\ \frac{du}{dx} &= -\sin(x) \\ du &= -\sin(x)dx\end{aligned}$$

Solve:

$$\int \sin^5(x)\cos^2(x)dx = -\int (1-u^2)^2u^2du$$

□

Example 2.4

$$\int \sin^2(x)\cos^2(x)dx$$

Solution. Plug in identities

$$\begin{aligned}\int \sin^2(x)\cos^2(x)dx &= \int \frac{1-\cos(2x)}{2}\times\frac{1+\cos(2x)}{2}dx \\ &= \frac{1}{4}\int 1-\cos^2(2x)dx \\ &= \frac{1}{4}\int \sin^2(2x)dx \\ &= \frac{1}{4}\int \frac{1-\cos(4x)}{2}dx \\ &= \frac{1}{8}\int 1-\cos(4x)dx\end{aligned}$$

□

2.3 Products of secant and tangent

Example 2.5

$$\int \sec^4 x dx$$

Solution.

$$\begin{aligned}
\int \sec^4 x dx &= \int \sec^2 x \sec^2 x dx \\
u &= \tan x \\
du &= \sec^2 x dx \\
&= \int (1+\tan^2 x)\sec^2 x dx \\
&= \int (1+u^2) du
\end{aligned}$$

□

Example 2.6

$$\int \tan^6 x \sec^4 x dx$$

Solution.

$$\begin{aligned}
\int \tan^6 x \sec^4 x dx &= \int \tan^6 x \sec^2 x \sec^2 x dx \\
&= \int \tan^6 x(1+\tan^2 x)\sec^2 x dx \\
u &= \tan x \\
du &= \sec^2 x dx \\
&= \int u^6(1+u^2) du
\end{aligned}$$

□

Example 2.7

$$\int \tan^5 x \sec^7 x dx$$

Solution.

$$\begin{aligned}
\int \tan^5 x \sec^7 x dx &= \int \tan^4 x \sec^6 x \tan x \sec x dx \\
&= \int (\sec^2 x - 1)^2 \sec^6 x \tan x \sec x dx \\
u &= \sec x \\
du &= \tan x \sec x dx \\
&= \int (u^2-1)^2 u^6 du
\end{aligned}$$

□

Example 2.8

$$\int \sec^2 x \tan^3 x dx$$

Solution.

$$\begin{aligned} u &= \tan x \\ du &= \sec^2 x dx \\ \int \sec^2 x \tan^3 x dx &= u^3 du \\ &= \frac{1}{4}u^4 + C \\ &= \boxed{\frac{1}{4}\tan^4 x + C} \end{aligned}$$

□

2.4 Proudct of sine and cosine with different angles

$$\begin{aligned} \sin A \cos B &= \frac{1}{2}(\sin(A+B) + \sin(A-B)) \\ \sin A \sin B &= \frac{1}{2}(\cos(A-B) - \cos(A+B)) \\ \cos A \cos B &= \frac{1}{2}(\cos(A-B) + \cos(A+B)) \end{aligned}$$

Example 2.9

$$\int \sin(3x)\cos(2x)dx$$

Solution.

$$\int \sin(3x)\cos(2x)dx = \frac{1}{2}\int(\sin(5x) + \sin x)dx$$

□