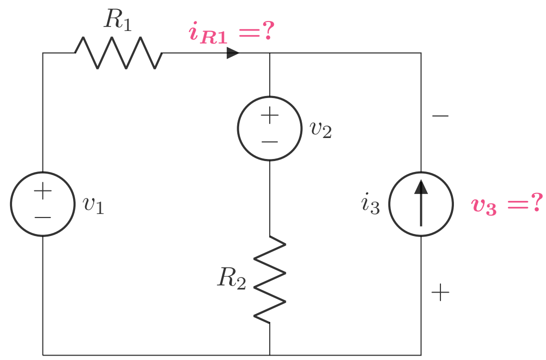
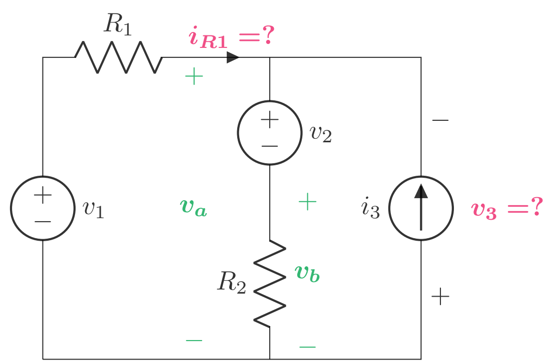
Lecture Notes 4: Node Analysis



Consider the circuit above. We are totally equipped to solve this circuit. It has 5 components and therefore 10 unknowns (7 if we were to call v_1, v_2, i_3 "known"), and we expect to get 2 KVL, 3 KCL, and 5 component law equations for a total of 10. The rest is just math.

Still, even this relatively simple circuit results in many equations with many unknowns and is a mess to solve.

Now consider that if we only knew *some* information, the rest of the problem would be trivial. For example, suppose we knew v_a and v_b somehow:

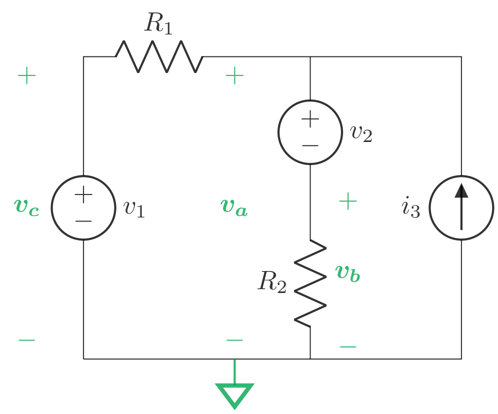


If we knew v_a and v_b, anything else we wanted to know would be trivial to solve for. For example, $i_{R1} = v_{R1}/R_1 = (v_1 - v_a)/R_1$, or $v_3 = -v_a$.

★ Is there a systematic way to solve for a smaller number of salient variables, from which anything else is easily calculated? Yes, the Node Voltage Method or Node Analysis!

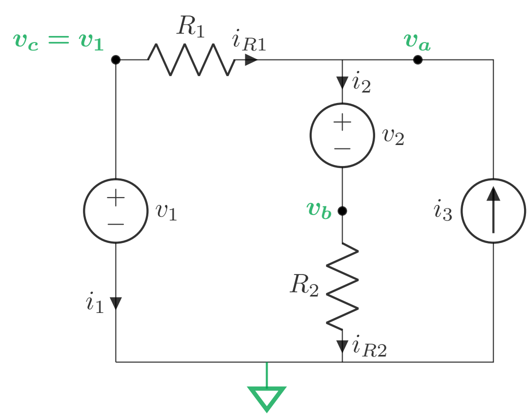
Node Analysis invites us to solve for the **node voltages**, meaning the voltages of each node relative to an arbitrarily-chosen **ground** which counts as zero volts. If we know all of the node voltages, simple subtraction (KVL) can tell us the voltage across any component, and then the current through every component can be determined from that component's $i - v$ relationship. Here are the steps:

Step 1: Arbitrarily assign a reference node or "ground." Define the node voltages as the voltage between each other node and the node you chose as ground. (Remember that there is no such thing as the voltage "at" a location, only the voltage between locations. Therefore, your choice of node to call "0 volts" or "ground" really is arbitrary).



Any voltage source connected to ground causes its corresponding node voltage to be automatically known. In our example, $\boxed{v_c = v_1}$.

Step 2: Write KCL at every *remaining* node except ground. Be sure to have labeled the reference directions for the currents as always!



In our example, we get two equations:

$$i_{R1} - i_2 + i_3 = 0 \tag{11}$$
$$i_2 - i_{R2} = 0 \tag{12}$$

You could simplify any series connections at this stage by saying that they have the same current (which is the same as incorporating $i_2 = i_{R2}$ into the other equations). We'll continue with the equations written explicitly, but in practice one would probably only have one KCL equation for this circuit.

Step 3: Express resistor currents in terms of node voltages (e.g., $i_{R1} \rightarrow (v_1 - v_a)/R_1$). Be sure that the subtraction matches the direction of the current label. Current source currents are known (e.g., i_3). Leave currents through non-grounded voltage sources as unknowns (e.g., i_2), but supplement the list of equations with the **floating** voltage source's component law (e.g., $v_a - v_b = v_2$). This causes our list of equations to become the following:

$$\frac{v_1 - v_a}{R_1} - i_1 + i_3 = 0 \tag{13}$$

$$i_2 - \frac{v_b - 0}{R_2} = 0 \tag{14}$$

$$v_a - v_b = 0 \tag{15}$$

where v_1, v_2, and i_3 are known, v_a, v_b, and i_2 are unknown node voltages and currents through floating voltage sources.

Step 4: Solve the reduced system of equations (now 3 equations and 3 unknowns. It could have been 10 if we had used a brute-force solution, and it could be 2 if we use a single variable to describe current through series components).

$$\frac{v_1 - v_a}{R_1} - \frac{v_b - 0}{R_2} + i_3 = 0 \Rightarrow \frac{v_1}{R_1} - \frac{v_a}{R_1} - \frac{v_a}{R_2} + \frac{v_2}{R_2} + i_3 = 0$$

$$\Rightarrow \boxed{v_a = \frac{\frac{v_1}{R_1} + \frac{v_2}{R_2} + i_3}{\frac{1}{R_1} + \frac{1}{R_2}}}$$

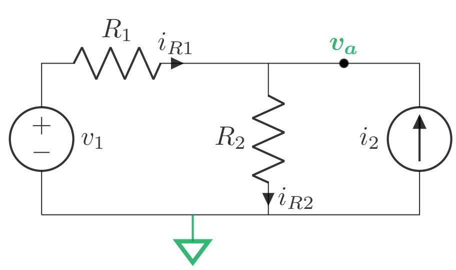
$$\Rightarrow \boxed{v_b = v_a - v_2} \quad \mathrm{v_a\ now\ known}$$

$$\Rightarrow \boxed{i_2 = \frac{v_b}{R_2}} \quad \mathrm{v_b\ now\ known}$$

Step 5: If you need to solve for something that is not a node voltage or a current through a floating voltage source, you now have plenty of information to solve for it easily. For example, if you needed to find i_{R1}, do node voltage analysis first to get v_a and v_b, *then* solve $i_{R1} = (v_1 - v_a)/R_1$.

Having gone through the entire process, reflect on the fact that ground-referenced voltage sources make life especially easy and non-ground-referenced (floating) voltage sources make life especially hard. This implies that *the best choice of ground is usually the one with the most voltage sources attached to it.*

Node analysis usually lets you solve circuits much faster than by brute force and is the foundation of most circuit simulators. Consider how quickly we can do the example below:



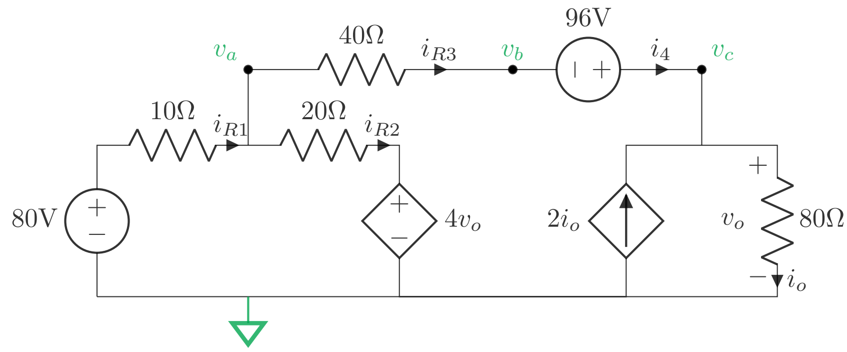
With experience, you can get a single equation and a single unknown by inspection:

$$\frac{v_1 - v_a}{R_1} - \frac{v_a}{2} + i_2 = 0$$

$$\Rightarrow \boxed{v_a = \frac{\frac{v_1}{R_1} + i_2}{\frac{1}{R_1} + \frac{1}{R_2}}}$$

Once v_a is known, anything else can be easily computed. What might have been an 8-equation problem can be done in one or two lines!

Consider another example from Alexander and Sadiku 3.30:



Step 1: Choose the bottom node as ground since it connects to 2 voltage sources. Define the node voltages relative to ground.

Step 2 and 3 can often be performed together by writing KCL and expressing resistor currents using node voltages.

$$\text{KCL node A} \quad \frac{80}{v_a}10 - \frac{v_a - 4v_o}{20} - \frac{v_a - v_b}{40} = 0 \tag{16}$$

$$\text{KCL node B} \quad \frac{v_a - v_b}{40} - i_4 = 0 \tag{17}$$

$$\text{KCL node O} \quad i_4 + 2i_o - \frac{v_o}{80} = 0 \tag{18}$$

Step 3: Supplement with the voltage-source law for any floating voltage sources ($v_o - v_b = 96$) and express any equations we might need to handle the dependent sources (in this case, $v_o = 80i_o$).

Step 4: Solve the system of 5 equations and 5 unknowns (instead of 16!). In node analysis, it is often helpful to multiply through to get rid of all the fractions. Merging the second two equations and multiplying through gives:

$$\frac{v_a - v_b}{40} + \frac{2v_o}{80} - \frac{v_o}{80} = 0 \Rightarrow \boxed{2v_a - 2v_b + v_o = 0}$$

Incorporating $v_o = v_b + 96$ gives

$$\boxed{2v_a - v_b + 96 = 0}$$

Simplifying the first equation gives

$$320 - 4v_a - 2v_a + 8v_o - v_a + v_b = 0 \Rightarrow 1088 - 7v_a + 9v_b = 0$$

Plugging in for v_b gives

$$1088 - 7v_a + 18v_a + 864 = 0 \Rightarrow \boxed{v_a = -177.455}$$

$$\Rightarrow \boxed{v_b = 2v_a + 96 = -258.91}$$

$$\Rightarrow \boxed{v_o = v_b + 96 = -162.91}$$

We can check our answer by calculating

$$i_{R1} = 25.75,\ i_{R2} = 23.708,\ i_{R3} = 2.036 \quad \sum i = 0 \text{ (KCL holds at node A)}$$

$$i_o = -2.036,\ i_4 = 2.036 \quad \sum i = 0 \ \text{(KCL holds at node O)}$$

It's also not a bad idea to check your answer using a circuit simulator such as LTSpice!