Introduction to Electrical Engineering (ECE 302H) – Homework 8

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Problem 8.1. Log Scale

a) You read that a function of frequency "rolls off at 20 dB per decade." Describe what this means.

b) Convert the ratio 40:1 to dB

c) Convert the ratio 3710:1 to dB

d) Convert 65 dB to a ratio

e) Convert 33 dB to a ratio

f) How many volts is 65 dB μV? How many volts is 33 dB mV?

g) A paper reports that an input signal has been attenuated by 60 dB. Has the signal become bigger or smaller, and by what factor?

h) On a log plot, you want to plot a number between X and $10X$ (e.g., between 10 and 100). If the number you want to plot is $2.5X$, where will it appear on the plot (i.e., what percentage of the distance from the X tick mark to $10X$ tick mark). Does it matter what X is?

i) On a linear plot, a data point halfway between tick marks is the "arithmetic mean" of the tick mark values (e.g., 2.5 is the average of 2 and 3 and it appears halfway between the tick mark for 2 and the tick mark for 3). Explain why a data point halfway between two tick marks on a log plot represents the "geometric mean" of the tick mark values, i.e. $\sqrt{Tick_1 \times Tick_2}$.

Solution.   □

a) The statement of a frequency "rolling off at $20\,\mathrm{dB}$ per decade" means that the function decreases by $20\,\mathrm{dB}$ when the frequency increases by a factor of 10.

b)

$$20 \times \log 40$$

$$\boxed{32.041\,\mathrm{dB}}$$

c)

$$20 \times \log 3710$$

$$\boxed{71.387\,\mathrm{dB}}$$

d)

$$10^{\frac{65}{20}}$$

$$\boxed{1778 : 1}$$

e)

$$10^{\frac{33}{20}}$$

$$\boxed{45 : 1}$$

f)

$$65\,\mathrm{dB\,\mu V} = 10^{\frac{65}{20}}\,\mathrm{\mu V}$$
$$65\,\mathrm{dB\,\mu V} = 1778\,\mathrm{\mu V}$$
$$\boxed{65\,\mathrm{dB\,\mu V} = 1.778\,\mathrm{mV}}$$

$$33\,\mathrm{dB\,mV} = 10^{\frac{33}{20}}\,\mathrm{mV}$$
$$\boxed{33\,\mathrm{dB\,mV} = 44.7\,\mathrm{mV}}$$

g) The signal would become $\boxed{\text{smaller}}$ by a factor of,

$$\boxed{10^{\frac{-60}{20}} = 0.001}$$

h)

$$\mathrm{tick\%} = \log \frac{2.5}{1} \times 100\%$$

$$\boxed{\mathrm{tick\%} = 39.79\%}$$

It does not depend on X.

i) Consider the two tick marks T_1 and T_2, where $T_1 < T_2$ the value x 50% between the two ticks is,

$$0.5 = \frac{\log \frac{x}{T_1}}{\log \frac{T_2}{T_1}}$$

$$0.5 \log \frac{T_2}{T_1} = \log \frac{x}{T_1}$$

$$\log \left(\frac{T_2}{T_1} \right)^{0.5} = \log \frac{x}{T_1}$$

$$\sqrt{\frac{T_2}{T_1}} = \frac{x}{T_1}$$

$$\boxed{x = \sqrt{T_1 T_2}}$$

Problem 8.2. Sine Wave Technology

a) For any function $f(x)$, we know that $f(x - a)$ represents a horizontal translation of the graph. If $a > 0$, does the graph translate left or right?

b) Consider a sine wave $A\cos(\omega t)$. What does the function $A\cos(\omega t + \phi)$ look like if $\phi > 0$? If $\phi < 0$? Sketch both scenarios relative to the base sine wave.

c) When $\phi > 0$, do we say that the new wave lags or leads the original? How about when $\phi < 0$? In two or three sentences, explain why this terminology makes sense.

d) It's often easier to measure the size of a sine wave from "peak to peak," meaning from its highest point to its lowest point. How does the peak-to-peak size of the sine wave relate to the amplitude A?

e) If we look at a sine wave, we can't measure ω directly. The easiest thing we can measure is the period T. How would we calculate ω from T?

f) Explain the difference between frequency f and angular frequency ω? (Not just the formulas)
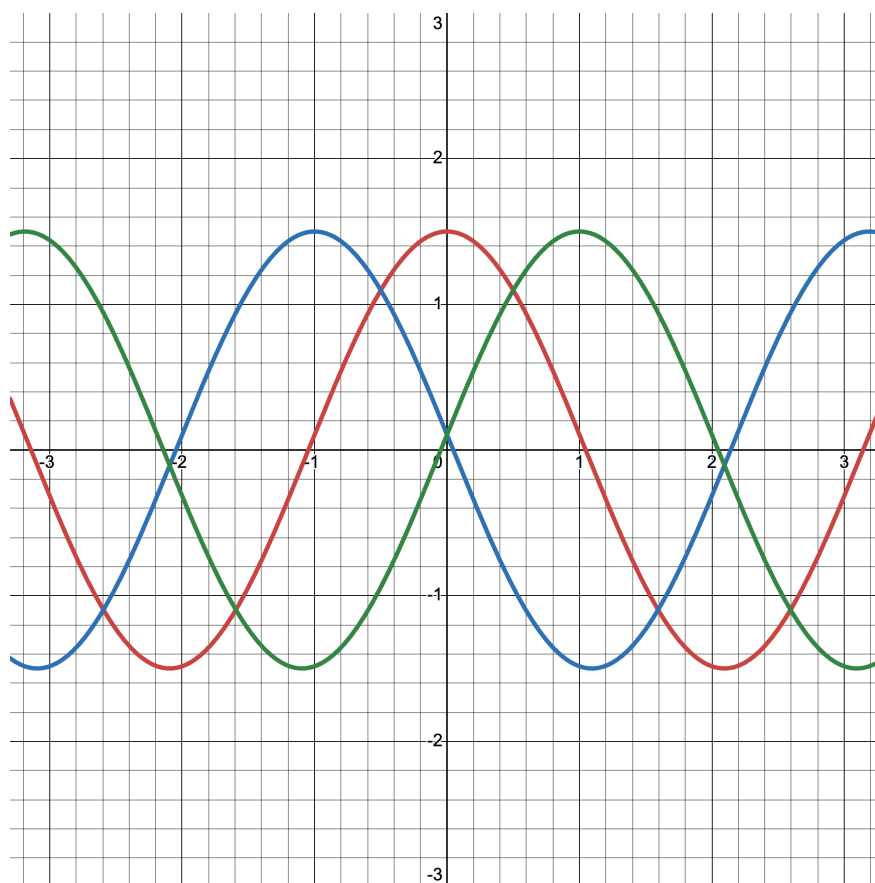
Solution. □

a) The translation $f(x - a)$, where $a > 0$, translates the graph to the $\boxed{\text{right}}$ by a units.

b) Based on the answer to the previous question, we can conclude that for a function $f(t) = A\cos(\omega t + \phi)$,

$$\phi < 0 \quad \Rightarrow \quad \text{right shift}$$
$$\phi > 0 \quad \Rightarrow \quad \text{left shift}$$

$$A = 1.5$$
$$\omega = 1.5$$
$$\textcolor{red}{\phi = 0}$$
$$\textcolor{green}{\phi = -1}$$
$$\textcolor{blue}{\phi = 1}$$

c) When $\phi < 0$, the wave can be closely modeled by the green line on the previous graph where $\phi = -1$. As you can see, the green line lags behind the base red line. Conversely, $\phi > 0$, which can be modeled by the blue line $\phi = 1$, leads the base red line. This explanation makes sense because relative to time (x-axis), the blue line experiences changes before red line, which experiences changes before the green line.

d) The highest point to its lowest point on a sine wave (peak-to-trough) is double its amplitude. $\boxed{\text{peak-to-trough} = 2A}$

e)

$$\boxed{\omega = \frac{2\pi}{T}}$$

f) Frequency measures the amount of cycles that something happened each second, while angular frequency captures a full cycle of that same something within the period of a circle (2πrad think sin and cos). That's what defines the relationship between the two,

$$\omega = 2\pi f$$

Problem 8.3. Combining Sine Waves

When two sine waves of the same frequency are added together, the result is a single sine wave of the same frequency, even if the original sine waves had different phases. Let us derive the amplitude and phase of the new sine wave. In particular, consider adding together two waves, $a = A\sin(\omega t)$ and $b = B\sin(\omega t + \delta)$. The result is a new sine wave $c = C\sin(\omega t + \phi)$, where the amplitude C and phase ϕ are unknown.

a) We contend that $a + b = c$. In particular, this must be true for any time t. Plug in $t = 0$ and solve for $\sin(\phi)$. Your answer will include knowns and unknowns, but it's a good constraint that your final answer must obey.

b) Now set $t = \pi/2\omega$ and solve for $\cos(\phi)$, again in terms of knowns and unknowns.

c) Use the fact that $\cos^2(x) + \sin^2(x) = 1$ to solve for C only in terms of knowns. This is your final answer for C. What trigonometric law does your answer remind you of?

d) Plug your answer for C back into part (a) or (b) and solve for ϕ only in terms of knowns. This is your final answer for ϕ.

Note – later in the course, we will learn an easier way to add sine waves using complex numbers.

Solution.   □

   a)

$$a + b = c$$
$$A\sin(\omega t) + B\sin(\omega t + \delta) = C\sin(\omega t + \phi)$$
$$A\sin(0) + B\sin(\delta) = C\sin(\phi)$$
$$\boxed{\sin(\phi) = \frac{B\sin(\delta)}{C}}$$

   b)

$$A\sin(\omega t) + B\sin(\omega t + \delta) = C\sin(\omega t + \phi)$$
$$A\sin(\omega \times \frac{\pi}{2\omega}) + B\sin(\omega \times \frac{\pi}{2\omega} + \delta) = C\sin(\omega \times \frac{\pi}{2\omega} + \phi)$$
$$A\sin(\frac{\pi}{2}) + B\sin(\frac{\pi}{2} + \delta) = C\sin(\frac{\pi}{2} + \phi)$$
$$A + B\cos(\delta) = C\cos(\phi)$$
$$\boxed{\cos(\phi) = \frac{A + B\cos(\delta)}{C}}$$

   c)

$$\sin^2(\phi) + \cos^2(\phi) = 1$$
$$\left(\frac{B\sin(\delta)}{C}\right)^2 + \left(\frac{A + B\cos(\delta)}{C}\right)^2 = 1$$
$$B^2\sin^2(\delta) + A^2 + 2AB\cos(\delta) + B^2\cos^2(\delta) = C^2$$
$$A^2 + B^2(\sin^2(\delta) + \cos^2(\delta)) + 2AB\cos(\delta) = C^2$$
$$A^2 + B^2 + 2AB\cos(\delta) = C^2$$
$$\boxed{C = \sqrt{A^2 + B^2 + 2AB\cos(\delta)}}$$

This equation looks a lot like the law of cosines.

   d)

$$\sin(\phi) = \frac{B\sin(\delta)}{C}$$
$$\sin(\phi) = \frac{B\sin(\delta)}{\sqrt{A^2 + B^2 + 2AB\cos(\delta)}}$$
$$\boxed{\phi = \arcsin\left(\frac{B\sin(\delta)}{\sqrt{A^2 + B^2 + 2AB\cos(\delta)}}\right)}$$

However, there is an easier way to solve for ϕ. Notice the two equations for $\sin(\phi)$ and $\cos(\phi)$,

$$\sin(\phi) = \frac{B\sin(\delta)}{C}$$

$$\cos(\phi) = \frac{A + B\cos(\delta)}{C}$$

$$\tan(\phi) = \frac{\sin(\phi)}{\cos(\phi)} = \frac{B\sin(\delta)}{A + B\cos(\delta)}$$

$$\boxed{\phi = \arctan\left(\frac{B\sin(\delta)}{A + B\cos(\delta)}\right)}$$

Problem 8.4. Mixing Sine Waves

One function that is commonly done (purposefully or not) to sine waves is to multiply them together. This is known as "mixing."

a) Use the identity $\cos(\theta_1 \pm \theta_2) = \cos\theta_1 \cos\theta_2 \pm \sin\theta_1 \sin\theta_2$ to prove that $\cos\theta_1 + \cos\theta_2 = 2 \cos\left(\frac{\theta_1 - \theta_2}{2}\right) \cos\left(\frac{\theta_1 + \theta_2}{2}\right)$.

b) When you listen to a speaker generate a sine wave, it sounds like a pure tone. If you add two identical sine waves together, it still sounds like a pure tone. However, if you add two sine waves together at slightly different frequencies, you hear a single tone play that slowly gets louder and quieter. Use the above identity to explain this concept. If the sine waves are at frequencies f_1 and f_2 which are close to each other, what tone do you expect to hear and how long do you expect it to take for the tone to go from loud to quiet and back to loud again?

c) In part (b), we explored what happens when you add two sine waves of different frequencies and we gained intuition by thinking of the sum instead as a product of two sine waves. In rf systems, we also do the opposite: we multiply two sine waves together and interpret the result as a sum of sine waves. The most common rf receiver architecture is called a "superheterodyne" structure. It works by multiplying ("mixing") the incoming rf signal at a frequency f_{rf} by another sine wave at frequency f_{LO} (LO stands for local oscillator; it can be lower or higher than f_{rf}). The difference between f_{rf} and f_{LO} is called the "intermediate frequency," f_{IF}. Suppose that you want to listen to KUT 90.5 which transmits at 90.5 MHz, and you multiply that signal with a local oscillator of the same magnitude at 40.25 MHz. On a plot versus frequency, sketch the rf signal, the local oscillator signal, and the frequencies of the signal after mixing, each as impulses whose height corresponds to the magnitude of the sine wave. Label the intermediate frequency and every other frequency both symbolically and numerically. Make sure your magnitudes are roughly to scale.

Solution.   □

a)

$$2\cos\left(\frac{\theta_1 - \theta_2}{2}\right)\cos\left(\frac{\theta_1 + \theta_2}{2}\right)$$

$$2\left(\cos\left(\frac{\theta_1}{2}\right)\cos\left(\frac{\theta_2}{2}\right) + \sin\left(\frac{\theta_1}{2}\right)\sin\left(\frac{\theta_2}{2}\right)\right)\left(\cos\left(\frac{\theta_1}{2}\right)\cos\left(\frac{\theta_2}{2}\right) - \sin\left(\frac{\theta_1}{2}\right)\sin\left(\frac{\theta_2}{2}\right)\right)$$

$$2\left(\left(\cos\left(\frac{\theta_1}{2}\right)\cos\left(\frac{\theta_2}{2}\right)\right)^2 + \left(\sin\left(\frac{\theta_1}{2}\right)\sin\left(\frac{\theta_2}{2}\right)\right)^2\right)$$

$$2\left(\cos^2\left(\frac{\theta_1}{2}\right)\cos^2\left(\frac{\theta_2}{2}\right) + \sin^2\left(\frac{\theta_1}{2}\right)\sin^2\left(\frac{\theta_2}{2}\right)\right)$$

$$2\left(\frac{1+\cos\theta_1}{2}\frac{1+\cos\theta_2}{2} - \frac{1-\cos\theta_1}{2}\frac{1-\cos\theta_2}{2}\right)$$

$$\frac{1}{2}(1+\cos\theta_1)(1+\cos\theta_2) - \frac{1}{2}(1-\cos\theta_1)(1-\cos\theta_2)$$

$$\frac{1}{2}(1+\cos\theta_1+\cos\theta_2+\cos\theta_1\cos\theta_2) - \frac{1}{2}(1-\cos\theta_1-\cos\theta_2+\cos\theta_1\cos\theta_2)$$

$$\frac{1}{2}\cos\theta_1 + \frac{1}{2}\cos\theta_2 + \frac{1}{2}\cos\theta_1 + \frac{1}{2}\cos\theta_2$$

$$\cos\theta_1 + \cos\theta_2$$

b)

$$\theta_1 = 2\pi f_1 t$$
$$\theta_2 = 2\pi f_2 t$$

For frequencies close togehter, $f_2 \to f_1 \quad \Rightarrow \quad f_2 = f_1$

$$\theta_2 = 2\pi f_1 t$$
$$\theta_1 = \theta2$$

$$2\cos\left(\frac{\theta_1 - \theta_2}{2}\right)\cos\left(\frac{\theta_1 + \theta_2}{2}\right)$$

$$2\cos(0)\cos\left(\frac{4\pi f_1 t}{2}\right)$$

$$2\cos(2\pi f_1 t)$$

Remark. Notice that the amplitude is constant, 2.

For frequencies far apart, $f_2 \neq f_1$,

$$\theta_1 = 2\pi f_1 t$$
$$\theta_2 = 2\pi f_2 t$$

$$2\cos\left(\frac{\theta_1 - \theta_2}{2}\right)\cos\left(\frac{\theta_1 + \theta_2}{2}\right)$$
$$2\cos\left(\frac{2\pi f_1 t - 2\pi f_2 t}{2}\right)\cos\left(\frac{2\pi f_1 t + 2\pi f_2 t}{2}\right)$$
$$2\cos(\pi t(f_1 - f_2))\cos(\pi t(f_1 + f_2))$$

If we think about $\cos(\pi t(f_1 + f_2))$ as the "base" wave, the amplitude of this wave can be modeled by $2\cos(\pi t(f_1 - f_2))$, since $A \times \cos(\omega t + \phi)$.

Notice that if $f_1 - f_2 = 0 \Rightarrow f_1 = f_2$, we would get the previous case with an amplitude of $2\cos(\pi t(f_1 - f_1)) = 2$

Now to find the period of this wave, AKA the time the it takes for it to go from loud to quiet and get back loud again, conceptually this would just be the time difference between consequtive max amplitudes.

$$2\cos(\pi t(f_1 - f_2)) = 2$$
$$\pi t(f_1 - f_2) = k\pi$$
$$t = \frac{k}{f_1 - f_2}, \quad \text{for } f_1 \neq f_2$$

$$T = t_{n+1} - t_n$$
$$T = \frac{n+1}{f_1 - f_2} - \frac{n}{f_1 - f_2}$$
$$\boxed{T = \frac{1}{f_1 - f_2}}$$

c)

$$b = 2\pi f_{rf} t$$
$$a = 2\pi f_{LO} t$$

$$\cos(a)\cos(b) = \cos\left(\frac{\theta_1 - \theta_2}{2}\right)\cos\left(\frac{\theta_1 + \theta_2}{2}\right)$$
$$a = \frac{\theta_1 - \theta_2}{2}$$

$$b = \frac{\theta_1 + \theta_2}{2}$$

$$\theta_1 = 2a + \theta_2$$
$$\frac{2a + \theta_2 + \theta_2}{2} = b$$
$$a + \theta_2 = b$$
$$\theta_2 = b - a = 2\pi t(f_{rf} - f_{LO})$$
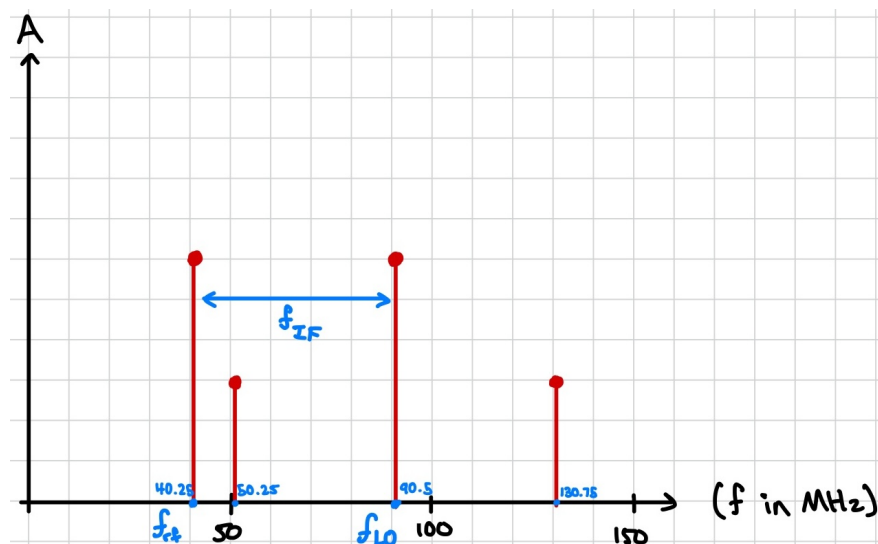$$\theta_1 = a + b = 2\pi t(f_{rf} + f_{LO})$$

$$\cos\left(\frac{\theta_1 - \theta_2}{2}\right)\cos\left(\frac{\theta_1 + \theta_2}{2}\right) = \frac{1}{2}\cos\theta_1 + \frac{1}{2}\cos\theta_2$$

$$= \boxed{\frac{1}{2}\cos(2\pi t(f_{rf} + f_{LO})) + \frac{1}{2}\cos(2\pi t(f_{LO} - f_{rf}))}$$
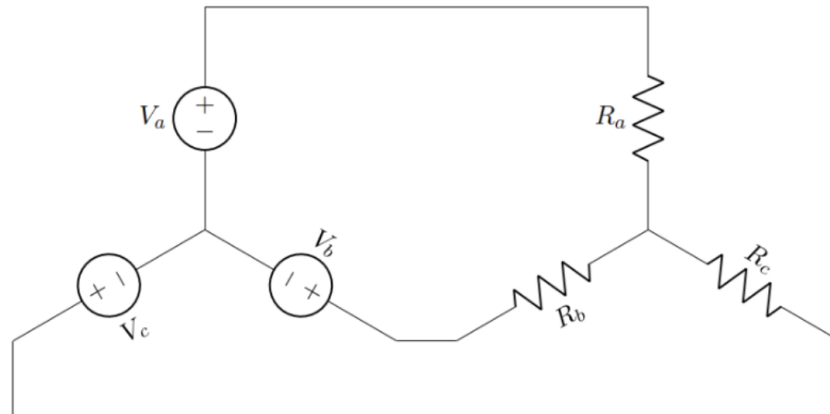
$$f_{rf} = 90.5\,\text{MHz}$$
$$f_{LO} = 40.25\,\text{MHz}$$
$$f_{IF} = 50.25\,\text{MHz}$$

Problem 8.5. Balanced 3-Phase Systems

In power transmission and distribution systems, it is common to use "three-phase" power with the source and load divided into three parts. Usually the voltages and resistances are balanced, meaning $v_a(t) + v_b(t) + v_c(t) = 0$ and $R_a = R_b = R_c$.

a) If the central node of the source (also known as the "neutral" voltage) is at ground, what will the central node of the load be? Use node analysis to prove it.

b) One way to ensure balancing $(v_a(t) + v_b(t) + v_c(t) = 0)$ is to have each voltage be a sine wave, each phase-shifted by a third of a cycle, i.e. $v_a(t) = V \cos(\omega t)$, $v_b(t) = V \cos\left(\omega t + \frac{2\pi}{3}\right)$, and $v_c(t) = V \cos\left(\omega t - \frac{2\pi}{3}\right)$. Show that their sum is always zero. Hint: recall the following trigonometric identities:

$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha)$$
$$\sin(-\alpha) = -\sin(\alpha)$$
$$\cos(-\alpha) = \cos(\alpha)$$

Solution.   □

a) Node analysis at the load node (V_L),

$$\frac{V_a - V_L}{R_a} + \frac{V_b - V_L}{R_b} + \frac{V_c - V_L}{R_c} = 0$$

$$\frac{V_a}{R_a} + \frac{V_b}{R_b} + \frac{V_c}{R_c} = \frac{V_L}{R_a} + \frac{V_L}{R_b} + \frac{V_L}{R_c}$$

$$\frac{V_a}{R_a} + \frac{V_b}{R_b} + \frac{V_c}{R_c} = V_L \left(\frac{1}{R_a} + \frac{1}{R_b} + \frac{1}{R_c} \right)$$

$$\boxed{V_L = \left(\frac{\frac{V_a}{R_a} + \frac{V_b}{R_b} + \frac{V_c}{R_c}}{\frac{1}{R_a} + \frac{1}{R_b} + \frac{1}{R_c}} \right)}$$

Balanced case,

$$V_L = \left(\frac{\frac{V_a}{R_a} + \frac{V_b}{R_b} + \frac{V_c}{R_c}}{\frac{1}{R_a} + \frac{1}{R_b} + \frac{1}{R_c}} \right)$$

$$V_L = \left(\frac{\frac{V_a}{R} + \frac{V_b}{R} + \frac{V_c}{R}}{\frac{1}{R} + \frac{1}{R} + \frac{1}{R}} \right)$$

$$V_L = \left(\frac{V_a + V_b + V_c}{3} \right)$$

$$V_L = \left(\frac{0}{3} \right)$$

$$\boxed{V_L = 0}$$

b)

$$v_a(t) + v_b(t) + v_c(t)$$

$$V \cos(\omega t) + V \cos\left(\omega t + \frac{2\pi}{3} \right) + V \cos\left(\omega t - \frac{2\pi}{3} \right)$$

$$V \cos(\omega t) + V \cos(\omega t) \cos \frac{2\pi}{3} - V \sin(\omega t) \sin \frac{2\pi}{3} + V \cos(\omega t) \cos \frac{2\pi}{3} + V \sin(\omega t) \sin \frac{2\pi}{3}$$

$$V \cos(\omega t) + V \cos(\omega t) \cos \frac{2\pi}{3} + V \cos(\omega t) \cos \frac{2\pi}{3}$$

$$V \cos(\omega t) \left(1 + 2 \cos \frac{2\pi}{3} \right)$$

$$V \cos(\omega t) \left(1 + 2 \left(-\frac{1}{2} \right) \right)$$

$$V \cos(\omega t)(0)$$

$$\boxed{V \cos(\omega t)(0) = 0}$$