Introduction to Electrical Engineering (ECE 302H) – Homework 6

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Problem 6.1. The Mass Action Law and Massively Large Numbers

There is a very important relationship in semiconductors that relates the number of free electrons and the number of holes known as the Mass Action Law, which you will derive below. In addition, we deal with numbers between 10^{10} and 10^{22} when talking about electron and atom concentrations in semiconductors. These numbers are pretty wild to think about and can become fairly abstract. This problem will help you think about these numbers.

a) In semiconductors, mobile electrons and holes are constantly being generated. They are also constantly undergoing "recombination," meaning that an electron fills a hole and both "particles" become immobilized. Let the rate of generation be G (measured in $\#/cm^3$ per second) and let the rate of recombination be R (with the same units). In equilibrium, what must the relationship between R and G be? Explain why.

b) The rate of generation G is a function of temperature, $G(T)$. Explain why this makes sense.

c) The rate of recombination is proportional to the product of the mobile electron concentration n and the mobile hole concentration p. Explain why this makes sense.

d) By the above, the product np must be equal to a quantity that only depends on temperature. This quantity is called n_i^2 where n_i is called the "intrinsic concentration" and, at 300 K, is approximately equal to $10^{10}/cm^3$. This yields the **mass-action law** $np = n_i^2(T)$ What is $n_i^2(T)$ at 300 K, and what will n and p be for a piece of pure silicon?

e) The silicon lattice cell is a cube with 8 silicon atoms at the corners, 6 silicon atoms on the "faces" of the cube, and 4 silicon atoms wholly enclosed in the cube. The side length is 5.43 angstroms. What is the concentration of silicon atoms N_s in crystalline silicon? (Hint – atoms on the edges and faces are shared between multiple lattice cells) To appreciate the relative magnitudes of N_s and n_i, write them out long-form (not in scientific notation).

f) If you had a crystal with one silicon atom for every person on earth, how many mobile electrons would there be, rounded to the nearest whole number? (Current population of earth is about 8 billion people)

g) Suppose that a piece of silicon is doped with a concentration of donors N_d. There are now three kinds of uncompensated charges in the silicon – free electrons, free holes, and stationary donor atoms. At equilibrium, the material must be electrically neutral. Write an equation representing this statement using the three mentioned concentrations.

h) Using the mass action law, calculate the concentration of electrons n as a function of n_i and N_d. Show that, for $N_d \gg n_i$, $n \approx N_d$. What would p be in this approximation?

i) The maximum dopant concentration in silicon without substantially changing its crystal structure and electronic properties is about $10^{19}/cm^3$. If you had a crystal with one silicon atom for every person on earth, doped at the maximum rate given, how many free electrons would there be? (Current population of earth is about 8 billion people) This is approximately the size of what US metropolitan area? (visit the Wikipedia page for Metropolitan Statistical Area for a table)

Solution. □

a) In equilibrium, $R = G$ because the rate of generation and rate of recombination would have to be equal.

b) The rate of generation G is a function of temperature $G(T)$ bceause the temperature is the property that vibrates the electrons and allow them to move into the lattice.

c) The rate of recombination being proportional to the product of the mobile electron concentration n and the mobile hole concentration p makes sense because an increase in either will lead to an increase in the rate of recombination.

d)

$$\begin{aligned} n_i(300) &= 10^{10}/\text{cm}^3 \\ n_i^2(300) &= 10^{10^2}/\text{cm}^{3^2} \\ n_i^2(300) &= \boxed{10^{20}/\text{cm}^6} \end{aligned}$$

In pure silicon $n = p$.

$$\begin{aligned} n &= p \\ np &= n_i^2(T) \\ np_{300} &= n_i^2(300) \\ np_{300} &= 10^{20}/\text{cm}^6 \\ n &= \boxed{10^{10}/\text{cm}^3} \\ p &= \boxed{10^{10}/\text{cm}^3} \end{aligned}$$

e) The main factor that we need to consider for this problem are the shared edges and faces between distinct silicon lattice cells within the overall structure. We know from the strcuture of stacking cubes that each corner is shared by 8 cells while each face is shared by 2 cells. That means that on average, each cell only constitutes 8/8 silicon atoms at the corner, and 6/2 atoms for each face.

$$\begin{aligned} Si_{\text{atoms per cell}} &= \frac{8}{8} + \frac{6}{2} + 4 \\ Si_{\text{atoms per cell}} &= 8 \end{aligned}$$

An angstrom is 10^{-8}cm. That means that the concentration of silicon atoms N_s in #/cm^2 is,

$$\begin{aligned} N_s &= \frac{8Si}{5.43^3\text{angstrom}^3} \times \frac{1\text{angstrom}^3}{10^{-8^3}\text{cm}^3} \\ N_s &= 5.0 \times 10^{22} Si/\text{cm}^3 \\ N_s &= \boxed{50,000,000,000,000,000,000,000 \quad Si/\text{cm}^3} \end{aligned}$$

f)

$$
\begin{aligned}
n_{\text{mobile electrons}} &= \frac{8\times 10^{9} Si}{1} \times \frac{1\text{cm}^3}{5.0\times 10^{22} Si} \times \frac{10^{10}}{\text{cm}^3} \\
n_{\text{mobile electrons}} &= 4.828\times 10^{-5} \\
n_{\text{mobile electrons}} &= \boxed{1.6\times 10^{-3}}
\end{aligned}
$$

> **Remark.** With how small this number is and the fact that you can't have 1.6×10^{-3} of an electron, the effective number of mobile electrons is zero.

g)

$$\boxed{\text{free holes} + \text{donor atoms} - \text{free electrons} = 0}$$

$$\boxed{p + N_d - n = 0}$$

h)

$$
\begin{aligned}
np &= n_i^2 \\
p &= n - N_d \\
n(n - N_d) &= n_i^2 \\
n^2 - nN_d - n_i^2 &= 0
\end{aligned}
$$

Quadratic formula and solve for positive n solution,

$$n = \frac{N_d + \sqrt{N_d^2 + 4n_i^2}}{2}$$

Notice that because $N_d \gg n_i$, $N_d^2 + 4n_i^2 = N_d^2$,

$$
\begin{aligned}
n &\approx \frac{N_d + \sqrt{N_d^2}}{2} \\
n &\approx \frac{2N_d}{2} \\
n &\approx N_d
\end{aligned}
$$

Considering what p would be,

$$
\begin{aligned}
np &= n_i^2 \\
p &= \frac{n_i^2}{n} \\
p &\approx \frac{n_i^2}{N_d}
\end{aligned}
$$

i)

$$
\begin{aligned}
n_{\text{mobile electrons}} &= \frac{8 \times 10^9 Si}{1} \times \frac{1\text{cm}^3}{5.0 \times 10^{22} Si} \times \frac{10^{19}}{\text{cm}^3} \\
n_{\text{mobile electrons}} &= 1.6 \times 10^6
\end{aligned}
$$

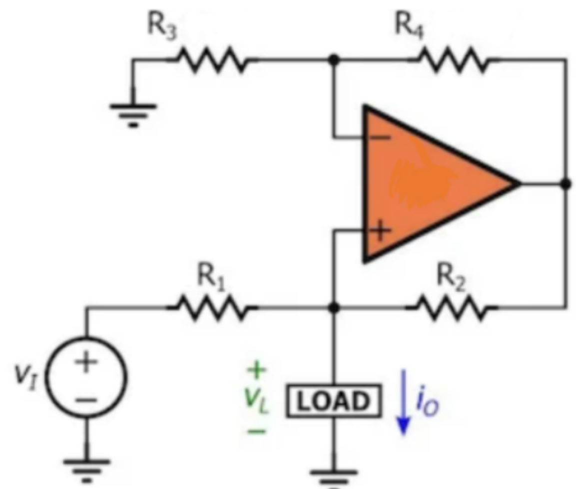
By the 2024 census, Milwaukee–Waukesha, WI comes in the closes with 1.574×10^6 population.

Problem 6.2. An Op Amp Current Source

a) Let the load be a voltage source of value v_L. Use node analysis to calculate the voltage at the negative op amp input v_- , the voltage at the positive op amp input v_+, the voltage at the output of the op amp v_o, and the load current i_o. Assume an ideal op amp with infinite gain.

b) In order for the op-amp circuit to act like a current source with respect to the load, what must its output resistance be? What relationship between the resistors makes this circuit behave as a current source?

c) When (b) is satisfied, what is the value of the current source?



Solution. □

a) Circuit equivalents,

$$v_L = v_+ = v_-$$

Node analysis at v_-,

$$\begin{aligned}
\frac{v_o - v_L}{R_4} - \frac{v_L}{R_3} &= 0 \\
R_3 v_o - R_3 v_L &= R_4 v_L \\
R_3 v_o &= (R_3 + R_4) v_L \\
v_o &= \frac{R_3 + R_4}{R_3} v_L
\end{aligned}$$

Node analysis at v_+,

$$\begin{aligned}
\frac{v_I - v_L}{R_1} - i_o - \frac{v_L - v_o}{R_2} &= 0 \\
R_2 v_I - R_2 v_L - R_1 R_2 i_o - R_1 v_L + R_1 v_o &= 0 \\
R_2 v_I - R_2 v_L - R_1 R_2 i_o - R_1 v_L + \frac{R_1 R_3 + R_1 R_4}{R_3} v_L &= 0 \\
\frac{R_1 R_3 + R_1 R_4 - R_1 R_3 - R_2 R_3}{R_2} v_L + R_2 v_I &= R_1 R_2 i_o \\
\frac{R_1 R_4 - R_2 R_3}{R_1 R_2^2} v_L + \frac{1}{R_1} v_I &= i_o
\end{aligned}$$

$$\begin{aligned}
v_- &= v_L \\
v_+ &= v_L \\
v_o &= \frac{R_3 + R_4}{R_3} v_L \\
i_o &= \boxed{\frac{R_1 R_4 - R_2 R_3}{R_1 R_2^2} v_L + \frac{1}{R_1} v_I}
\end{aligned}$$

b) For the circuit to act like a current source with respsect to the load, we want to set the v_L term to 0.

$$\begin{aligned}
\frac{R_1 R_4 - R_2 R_3}{R_1 R_2^2} &= 0 \\
R_1 R_4 - R_2 R_3 &= 0 \\
&\boxed{R_1 R_4 = R_2 R_3}
\end{aligned}$$

c)

$$\boxed{i_o = \frac{1}{R_1} v_I}$$

Problem 6.3. Sheet resistance

In integrated circuits, it is common to use thin sheets of resistive material to build resistors. Current flows laterally through the sheet. In this case, a common metric for the resistor deposition process is called the "sheet resistance" R_S.

a) Sheet resistance is expressed in units of "ohms per square," i.e., the amount of resistance a square sheet would have. The size of the square is not specified. Explain why it is intellectually defensible to do this based on how resistance relates to geometry.

b) Calculate the sheet resistance for a film of thickness t and resistivity ρ.

c) In terms of R_S, what is the resistance of a film that is twice as long as it is wide? Twice as wide as it is long?

Solution. □

a) We know that $R = \rho\frac{l}{A}$ for a material of length l and perpendicular cross sectional area A. Because current flows laterally through the sheet, the length of the sheet l stays the same. The remaining side of the sheet, lets say W makes up part of the area A. Let's call the other part of A a the thickness t.

$$R_S = \rho\frac{l}{Wt}$$

In a square $l = W$,

$$R_S = \rho\frac{l}{Wt}$$
$$R_S = \frac{\rho}{t}$$

As we can see, R_S is independent of the size of the square. So the measurement "ohms per square" makes sense.

b) From the previous part,

$$R_S = \boxed{\frac{\rho}{t}}$$

c)

$$R = \rho\frac{l}{Wt}$$
$$R = R_S\frac{l}{W}$$

Twice as long as it is wide,

$$R = R_S\frac{l}{W}$$
$$R = R_S\frac{2W}{W}$$
$$R = \boxed{2R_S}$$

Twice as wide as it is long,

$$R = R_S\frac{l}{W}$$

$$R = R_S \frac{l}{2l}$$

$$R = \boxed{\frac{1}{2} R_S}$$

Problem 6.4. Eletrical speed

We envision electricity as a fast process, but how fast do electrons actually move?

a) The conductivity of copper is about $5.9 \times 10^7 S/m$, due almost entirely to mobile electrons. There is about one free electron per atom, of which there are about 8.5×10^{28} per cubic meter. The maximum current density that one would normally allow in a copper wire is about $500 A/cm^2$. What is the maximum electron velocity in copper under these conditions?

b) Given your answer, explain how it is possible that a light bulb turns on almost instantly even when it is several meters away from the switch.

Solution. □

a)

$$V = \frac{500\,\mathrm{C}}{\mathrm{s} \times \mathrm{cm}^2} \times \frac{\mathrm{m}^3}{8.5 \times 10^{28} e} \times \frac{6.24 \times 10^{18} e}{1\,\mathrm{C}} \times \frac{100^2 \mathrm{cm}^2}{1 \mathrm{m}^2}$$
$$V = \boxed{3.671 \times 10^{-4} \mathrm{m/s}}$$

b) Although the velocity is very small, the light bulb turns on near instantly because there already electrons throughout the entire volume of wire. That means that there are already electrons right next to the light bulb, and when the switch is flicked on, those electrons will flow into the bulb.

Problem 6.5. Skin effect

At high frequencies, current does not flow over the entire cross-sectional area of a conductor, but rather only in a surface layer of thickness δ, where δ is known as the skin depth.

a) Calculate the ordinary resistance of a cylindrical conductor of radius b, length l, and conductivity σ.

b) Calculate the effective resistance of the cylindrical conductor when operated at high frequency such that the skin depth is δ.

c) Imagine an AWG 22 copper wire ($\sigma = 5.96 \times 10^7 S/m$), such as the ones we use in the lab. The skin depth is given by $\delta = \sqrt{2\rho/\omega\mu_0}$ where $\omega = 2\pi f$ is the angular frequency of the current and $\mu_0 = 4\pi \times 10^{-7} H/m$ is a constant of the universe known as the permeability of free space. At what frequency f is the skin depth equal to the wire radius? Below this frequency, current is well modeled as flowing evenly over the cross-sectional area of the conductor, as we're used to doing and as we calculated in part (a). Above this frequency, the conductor is "skin depth limited" and one must use the result from part (b).

d) For the wire in part (c), what is the ratio of effective resistance to ordinary resistance at 100 kHz, 1 MHz, and 10 MHz?

Solution. □

a)

$$R = \rho \frac{l}{A_{\text{cross section}}}$$
$$\rho = \frac{1}{\sigma}$$
$$R = \frac{l}{\sigma \pi b^2}$$

b)

$$A = \pi b^2 - \pi(b-\delta)^2$$
$$A = \pi b^2 - \pi b^2 + 2\pi b\delta - \pi\delta^2$$
$$A = 2\pi b\delta - \pi\delta^2$$

$$R = \frac{l}{\sigma A}$$
$$R = \frac{l}{\sigma 2\pi b\delta - \sigma\pi\delta^2}$$

c)

$$\delta = \sqrt{\frac{2\rho}{\omega\mu_0}}$$
$$\delta = \sqrt{\frac{2\rho}{2\pi f\mu_0}}$$
$$\delta = \sqrt{\frac{\rho}{\pi f\mu_0}}$$
$$\delta = \sqrt{\frac{1}{\sigma\pi f\mu_0}}$$
$$b = \sqrt{\frac{1}{\sigma\pi f\mu_0}}$$
$$b^2 = \frac{1}{\sigma\pi f\mu_0}$$
$$f = \frac{1}{\sigma\pi b^2\mu_0}$$
$$f = \frac{1}{(5.96\times 10^7)(\pi)(0.322\times 10^{-3})^2(4\pi\times 10^{-7})}$$
$$f = 40.990\,\text{kHz}$$

d) First calculate the ordinary resistance of an AWG 22 wire (0.322 mm radius),

> **Remark.** Because we are taking a ratio of resistances, I will treat each length l as 1 m, because it a constant variable.

$$R_0 = \frac{l}{\sigma\pi b^2}$$
$$R_0 = \frac{1}{(5.96 \times 10^7)(\pi)(0.322 \times 10^{-3})^2}$$
$$R_0 = 51.510\,\text{m}\Omega$$

At 100 kHz,

$$\delta = \sqrt{\frac{1}{\sigma\pi f\mu_0}}$$
$$\delta = \sqrt{\frac{1}{(5.96 \times 10^7)(\pi)(100 \times 10^3)(4\pi \times 10^{-7})}}$$
$$\delta = 2.062 \times 10^{-4}$$

$$R = \frac{1}{\sigma 2\pi b\delta - \sigma\pi\delta^2}$$
$$R = \frac{1}{\sigma\pi\delta(2b - \delta)}$$
$$R = \frac{1}{(5.96 \times 10^7)(\pi)(2.062 \times 10^{-4})(2(0.322 \times 10^{-3}) - 2.062 \times 10^{-4})}$$
$$R = 59.162\,\text{m}\Omega$$

$$\frac{R}{R_0} = 1.149$$

At 1 MHz,

$$\delta = \sqrt{\frac{1}{\sigma\pi f\mu_0}}$$
$$\delta = \sqrt{\frac{1}{(5.96 \times 10^7)(\pi)(1 \times 10^6)(4\pi \times 10^{-7})}}$$
$$\delta = 6.519 \times 10^{-5}$$

$$R = \frac{1}{\sigma\pi\delta(2b - \delta)}$$

$$R = \frac{1}{(5.96 \times 10^7)(\pi)(6.519 \times 10^{-5})(2(0.322 \times 10^{-3}) - 6.519 \times 10^{-5})}$$
$$R = 141.543\,\mathrm{m\Omega}$$

$$\frac{R}{R_0} = 2.748$$

At 10 MHz,

$$\delta = \sqrt{\frac{1}{\sigma \pi f \mu_0}}$$
$$\delta = \sqrt{\frac{1}{(5.96 \times 10^7)(\pi)(10 \times 10^6)(4\pi \times 10^{-7})}}$$
$$\delta = 2.062 \times 10^{-5}$$

$$R = \frac{1}{\sigma \pi \delta (2b - \delta)}$$
$$R = \frac{1}{(5.96 \times 10^7)(\pi)(2.062 \times 10^{-5})(2(0.322 \times 10^{-3}) - 2.062 \times 10^{-5})}$$
$$R = 415.492\,\mathrm{m\Omega}$$

$$\frac{R}{R_0} = 8.066$$

Problem 6.6. Semiconductor Exercises

a) For silicon doped with acceptors (like Boron) at a rate of $N_A = 2 \times 10^{18}/\text{cm}^3$, find the hole and electron concentrations.

b) For silicon doped with donors (like Phosphorus) at a rate of N_d (#/cm^3), what must N_d be if the hole concentration is below the intrinsic level by a factor of 10^8?

c) Find the end-to-end resistance of a bar that is 15 µm long, 3 µm wide, and 1 µm thick, made of the following materials:

   i. Intrinsic silicon
   ii. Silicon doped with donors (like Phosphorus) with $N_d = 5 \times 10^{16}/\text{cm}^3$
   iii. Silicon doped with donors (like Phosphorus) with $N_d = 5 \times 10^{18}/\text{cm}^3$
   iv. Silicon doped with acceptors (like Boron) with $N_a = 5 \times 10^{16}/\text{cm}^3$
   v. Aluminum with resistivity $2.8\,\mu\Omega \cdot \text{cm}$

   Assume that the mobility of electrons in silicon is $\mu_n = 1200\text{cm}^2/Vs$ and the mobility of holes is $\mu_p = \mu_n/3 = 400\text{cm}^2/Vs$. Be sure to take into account the contributions of both electrons and holes.

Solution. □

a) Because the silicon is doped, we can ignore the concentration of electrons and holes as a result of thermal generation,

$$\begin{aligned}
P &\approx \boxed{2 \times 10^{18}/\text{cm}^3} \\
n \times P &= 10^{20}/\text{cm}^6 \\
n &\approx \frac{10^{20}/\text{cm}^6}{10^{18}/\text{cm}^3} \\
n &\approx \boxed{50/\text{cm}^3}
\end{aligned}$$

b)

$$\begin{aligned}
P &= \frac{10^{10}/\text{cm}^3}{10^8} \\
P &= 100\,\text{cm}^3 \\
n \times P &= 10^{20}/\text{cm}^6 \\
n &= \frac{10^{20}/\text{cm}^6}{100/\text{cm}^3} \\
n &= \boxed{10^{18}\,\text{cm}^3}
\end{aligned}$$

c) i. We want σ in terms of S/m. First convert mobility and concentration to be in terms of m,

$$\begin{aligned}
n_i &= \frac{10^{10}}{\text{cm}^3} \times \frac{100^3\text{cm}^3}{\text{m}^3} \\
n_i &= 10^{16}\,\text{m}^{-3}
\end{aligned}$$

$$\begin{aligned}
\mu_n &= \frac{1200\text{cm}^2}{\text{V s}} \times \frac{\text{m}^2}{100^2\text{cm}^2} \\
\mu_n &= 0.12\,\text{m}^2\,\text{V}^{-1}\,\text{s}^{-1}
\end{aligned}$$

$$\mu_h = 0.04\,\text{m}^2\,\text{V}^{-1}\,\text{s}^{-1}$$

$$\begin{aligned}
\sigma &= q(n\mu_n + p\mu_p) \\
\sigma &= 1.602 \times 10^{-19}(10^{16}(0.12) + 10^{16}(0.04)) \\
\sigma &= 2.563 \times 10^{-4}\,\text{S}\,\text{m}^{-1}
\end{aligned}$$

$$\begin{aligned}
R &= \frac{1}{\sigma} \times \frac{l}{A} \\
R &= \frac{1}{2.563 \times 10^{-4}} \times \frac{15 \times 10^{-6}}{(3 \times 10^{-6})(10^{-6})} \\
R &= \boxed{19.508\,\mathrm{G\Omega}}
\end{aligned}$$

ii.

$$\begin{aligned}
N_d = n &= \frac{5 \times 10^{16}}{\mathrm{cm}^3} \times \frac{100^3\mathrm{cm}^3}{1\,\mathrm{m}^3} \\
n &= 5 \times 10^{22}\,\mathrm{m}^{-3}
\end{aligned}$$

$$\begin{aligned}
np &= 10^{20}\,\mathrm{cm}^{-6} \\
p &= \frac{10^{20}\,\mathrm{cm}^{-6}}{5 \times 10^{16}\,\mathrm{cm}^{-3}} \times \frac{100^3\mathrm{cm}^3}{1\,\mathrm{m}^3} \\
p &= 2 \times 10^{9}\,\mathrm{m}^{-3}
\end{aligned}$$

$$\begin{aligned}
\sigma &= q(n\mu_n + p\mu_p) \\
\sigma &= 1.602 \times 10^{-19}((5 \times 10^{22})(0.12) + (2 \times 10^{9})(0.04)) \\
\sigma &= 961.2\,\mathrm{S\,m^{-1}}
\end{aligned}$$

$$\begin{aligned}
R &= \frac{1}{\sigma} \times \frac{l}{A} \\
R &= \frac{1}{961.2} \times \frac{15 \times 10^{-6}}{(3 \times 10^{-6})(10^{-6})} \\
R &= \boxed{5.201\,\mathrm{k\Omega}}
\end{aligned}$$

iii.

$$\begin{aligned}
N_d = n &= \frac{5 \times 10^{18}}{\mathrm{cm}^3} \times \frac{100^3\mathrm{cm}^3}{1\,\mathrm{m}^3} \\
n &= 5 \times 10^{24}\,\mathrm{m}^{-3}
\end{aligned}$$

$$\begin{aligned}
np &= 10^{20}\,\mathrm{cm}^{-6} \\
p &= \frac{10^{20}\,\mathrm{cm}^{-6}}{5 \times 10^{18}\,\mathrm{cm}^{-3}} \times \frac{100^3\mathrm{cm}^3}{1\,\mathrm{m}^3} \\
p &= 2 \times 10^{7}\,\mathrm{m}^{-3}
\end{aligned}$$

$$\sigma = q(n\mu_n + p\mu_p)$$
$$\sigma = 1.602 \times 10^{-19}((5 \times 10^{24})(0.12) + (2 \times 10^{7})(0.04))$$
$$\sigma = 9.612 \times 10^{4}\,\mathrm{S\,m^{-1}}$$

$$R = \frac{1}{\sigma} \times \frac{l}{A}$$
$$R = \frac{1}{9.612 \times 10^{4}} \times \frac{15 \times 10^{-6}}{(3 \times 10^{-6})(10^{-6})}$$
$$R = \boxed{52.018\,\Omega}$$

iv.

$$N_a = p = 5 \times 10^{16}\,\mathrm{cm^{-3}}$$
$$p = \frac{5 \times 10^{16}}{\mathrm{cm^3}} \times \frac{100^3\mathrm{cm^3}}{1\,\mathrm{m^3}}$$
$$p = 5 \times 10^{22}\,\mathrm{m^{-3}}$$

$$np = 10^{20}\,\mathrm{cm^{-6}}$$
$$n = \frac{10^{20}\,\mathrm{cm^{-6}}}{5 \times 10^{16}\,\mathrm{cm^{-3}}} \times \frac{100^3\mathrm{cm^3}}{1\,\mathrm{m^3}}$$
$$n = 2 \times 10^{9}\,\mathrm{m^{-3}}$$

$$\sigma = q(n\mu_n + p\mu_p)$$
$$\sigma = 1.602 \times 10^{-19}((2 \times 10^{9})(0.12) + (5 \times 10^{22})(0.04))$$
$$\sigma = 320.4\,\mathrm{S\,m^{-1}}$$

$$R = \frac{1}{\sigma} \times \frac{l}{A}$$
$$R = \frac{1}{320.4} \times \frac{15 \times 10^{-6}}{(3 \times 10^{-6})(10^{-6})}$$
$$R = \boxed{15.605\,\mathrm{k\Omega}}$$

v.

$$\rho = 2.8\,\mathrm{\mu\Omega\,cm}$$
$$\rho = 2.8\,\mathrm{\mu\Omega\,cm} \times \frac{1\,\Omega}{10^{6}\,\mathrm{\mu\Omega}} \times \frac{1\,\mathrm{m}}{100\,\mathrm{cm}}$$

$$\rho = 2.8 \times 10^{-8}\,\Omega\,\mathrm{m}$$

$$R = \rho \times \frac{l}{A}$$

$$R = 2.8 \times 10^{-8} \times \frac{15 \times 10^{-6}}{(3 \times 10^{-6})(10^{-6})}$$

$$R = \boxed{0.14\,\Omega}$$

Problem 6.7. Parallel plate capacitor

A parallel plate capacitor has a dielectric of relative permittivity 3.9 and a thickness of 30 nm.

a) Find the capacitance per unit area

b) If a voltage of 2 V is applied and the dimensions of the rectangular plates are 180 nm $\times$ 2 µm, calculate the capacitance.

Solution. □

a)

$$\begin{aligned} C &= \frac{\epsilon A}{d} \\ \frac{C}{A} &= \frac{\epsilon_r \epsilon_0}{d} \\ \frac{C}{A} &= \frac{3.9 \times 8.854 \times 10^{-12}}{30 \times 10^{-9}} \\ \frac{C}{A} &= \boxed{1.151 \times 10^{-3}\,\mathrm{F/m^2}} \end{aligned}$$

b)

$$\begin{aligned} C &= 180 \times 10^{-9}\,\mathrm{m} \times 2 \times 10^{-6}\,\mathrm{m} \times 1.151 \times 10^{-3}\,\mathrm{F/m^2} \\ C &= \boxed{4.144 \times 10^{-16}\,\mathrm{F}} \end{aligned}$$

Problem 6.8. Capacitor with two dielectrics

A parallel plate capacitor has a dielectric that is composed of two layers. The first layer is 30 nm thick and has a relative permittivity of 3.9, while the second layer has a thickness of 50 nm and a relative permittivity of 18.

a) Draw a box on which to apply Gauss' law. The box's top should be embedded in the metal of the top electrode; the box's bottom should be embedded in the first dielectric layer. Use Gauss' law to calculate the electric field E_1 in the first dielectric as a function of the charge per unit area on the top plate.

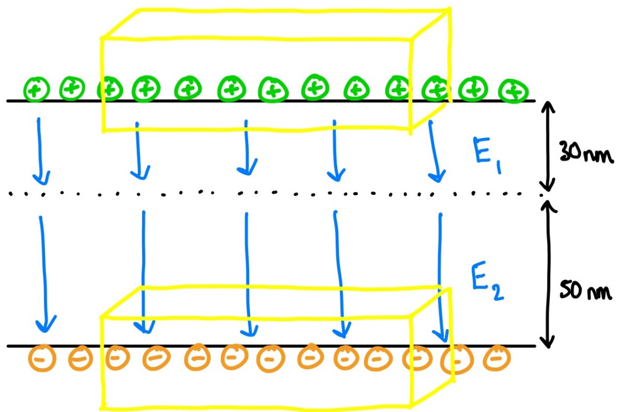
b) Draw another box on which to apply Gauss' law, but this time let the bottom of the box be embedded in the second dielectric layer. Use Gauss' law to calculate the electric field E_2 in the second dielectric as a function of the charge per unit area on the top plate.

c) Which dielectric will have the larger electric field?

d) Integrate $\int E \cdot dl$ from the top electrode to the bottom electrode to calculate the voltage across the capacitor as a function of E_1 and E_2.

e) Find the capacitance per unit area.

Solution. □



a)

$$\oint E\epsilon \cdot dA = Q_{\text{enclosed}}$$
$$E_1\epsilon_1\epsilon_0 A = Q_{\text{enclosed}}$$
$$E_1 = \frac{1}{\epsilon_1\epsilon_0} \times \frac{Q_{\text{enclosed}}}{A}$$
$$E_1 = \boxed{\frac{1}{3.9\epsilon_0} \times \frac{Q_{\text{enclosed}}}{A}}$$

b)

$$\oint E\epsilon \cdot dA = Q_{\text{enclosed}}$$
$$E_2\epsilon_2\epsilon_0 A = Q_{\text{enclosed}}$$
$$E_2 = \frac{1}{\epsilon_2\epsilon_0} \times \frac{Q_{\text{enclosed}}}{A}$$
$$E_2 = \boxed{\frac{1}{18\epsilon_0} \times \frac{Q_{\text{enclosed}}}{A}}$$

c) First dielectric will have a larger field because the relative permittivity is lower,

$$E \propto \frac{1}{\epsilon}$$

d)

$$V = \boxed{E_1 \times 30\,\text{nm} + E_2 \times 50\,\text{nm}}$$

e)

$$
\begin{aligned}
Q &= CV \\
C &= \frac{Q}{V} \\
C &= \frac{Q}{E_1 \times 30\,\mathrm{nm} + E_2 \times 50\,\mathrm{nm}} \\
C &= \frac{Q}{\frac{1}{3.9\epsilon_0} \times \frac{Q}{A} \times 30 \times 10^{-9} + \frac{1}{18\epsilon_0} \times \frac{Q}{A} \times 50 \times 10^{-9}} \\
C &= \frac{Q}{\frac{Q}{A}\left(\frac{30\times10^{-9}}{3.9\times8.854\times10^{-12}} + \frac{50\times10^{-9}}{18\times8.854\times10^{-12}}\right)} \\
\frac{C}{A} &= \frac{1}{\frac{30\times10^{-9}}{3.9\times8.854\times10^{-12}} + \frac{50\times10^{-9}}{18\times8.854\times10^{-12}}} \\
\frac{C}{A} &= \boxed{8.456 \times 10^{-4}\,\mathrm{F/m^2}}
\end{aligned}
$$