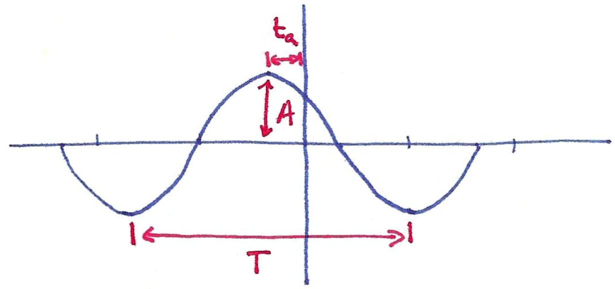
Lecture Notes 16: Sine Waves and Fourier Series

Consider the following signal that repeats in time (where we have only drawn ∼one cycle):



Using the language of time, we would say that this is a wave that repeats periodically with a period T. It "starts" at an advanced time t_a and it has an amplitude of A. We often translate expressions of time into expressions of frequency:

Period T	$\rightarrow$	Frequency $f = 1/T$
Length of Time	$\rightarrow$	Fraction of period or angle (2π rad/cycle or 360° per cycle)

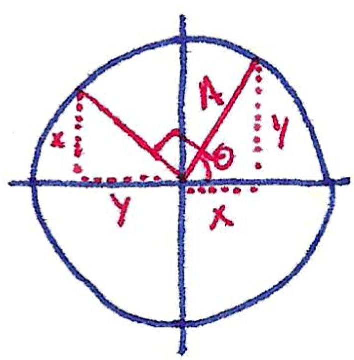
This conversion causes the advance time t_a to be expressed as a **phase angle** $\phi = 2\pi t_a/T$ (in radians). If we combine the concept of "frequency" with a desire to express time in terms of angles, we arrive at the concept of **angular frequency** $\omega = 2\pi f$ (in radians per second). Using this language, we usually express sinusoidal waves as:

$$\boxed{y = A\cos(\omega t + \phi)}$$

where $\phi > 0$ means the cosine appears to be shifted to the left, while $\phi < 0$ causes the cosine to appear to be shifted to the right.

Notice that the language of frequency has produced a wonderful shorthand when the function is periodic. We don't have to *say* that the function is periodic – it's *implied* by our using the language of frequency to describe it.

Recall the connection between sine waves and geometry:



For a line/vector/point that is a distance A and an angle θ from the $+x$-axis, the x-coordinate of that line/vector/point is $A\cos\theta$ and the y-axis coordinate is $A\sin\theta$. We also know that $\cos\theta = \sin(\theta + 90°)$ while $\sin\theta = \cos(\theta - 90°)$. Finally, the Pythagorean Theorem states that $x^2 + y^2 = A^2$, or

$$\boxed{\cos^2\theta + \sin^2\theta = 1}$$

A single sine wave can be expressed in **amplitude-phase form**:

$$y = A\cos(\omega t + \phi)$$

The *same* sine wave can also be expressed in **sine-cosine form**:

$$y = a\cos(\omega t) + b\sin(\omega t)$$

In words, we can describe a sine wave with an amplitude A and a phase ϕ, *or* we can describe the same wave with a cosine-part a and a sine-part b. You will prove in homework that $a = A\cos\phi$, $b = -A\sin\phi$, $A = \sqrt{a^2 + b^2}$, and $\phi = \tan^{-1}(-b/a)$.

This is all fine and good for signals that are actually close to sinusoidal, but many applications do not have sinusoidal signals. Digital signals are more like square waves; power converters tend to produce a lot of square waves; the wireless signals picked up by your phone are extremely distorted to the point of appearing to be random; and so forth. Does the language of frequencies have any usefulness in those areas?

The answer is yes, through an extremely powerful theorem known as **Fourier's Theorem**:

> Fourier's Theorem: Any periodic signal can be *equivalently* expressed as a sum of sine waves: a constant or "dc" part ($\omega = 0$), a part at the same frequency as the original signal, or "fundamental" (ω_0), and a part at each integer multiple of the fundamental ($n \times \omega_0$).

This theorem is remarkably insightful. When you and I look at a square wave, for example, we do not see a sine wave. But Fourier observed that if he added up multiple sine waves with the right amplitudes, frequencies, and relative phases, he could reconstruct a square wave (and other periodic signals).

Fourier's Theorem leads us to the concept of the **Fourier Series**, in which functions are represented as an infinite series of sine waves.

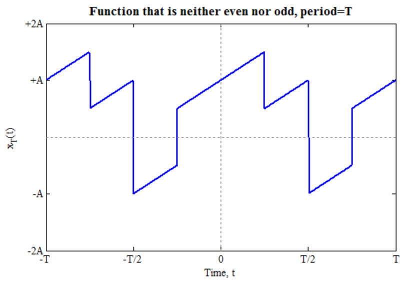
$$\boxed{\underbrace{f(t)}_{\textit{any}\,\text{periodic function}} = \underbrace{a_0}_{\text{dc part}} + \sum_{n=1}^{\infty} \underbrace{c_n\cos(n\omega_0 t + \phi_n)}_{\text{sine waves in amplitude}-\text{phase form}} = \underbrace{a_0}_{\text{dc part}} + \sum_{n=1}^{\infty} \underbrace{[a_n\cos(n\omega_0 t) + b_n\sin(n\omega_0 t)]}_{\text{sine waves in cosine}-\text{sine form}}}$$

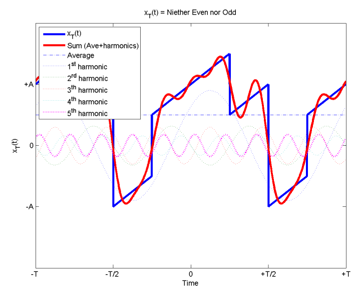
It is helpful to understand Fourier Series for the first time through an analogy to Taylor Series, which you already know.

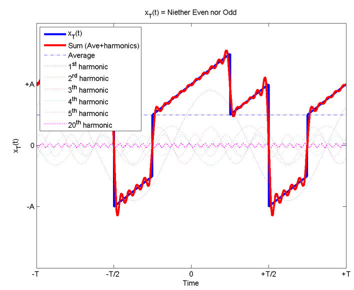
Taylor Series	Fourier Series
Exactly equal to the function if infinite terms are used	Exactly equal to the function if infinite terms are used
When truncated, becomes a worse approximation far away from the operating point	When truncated, becomes a worse approximation for fast transitions (like steps)
Using a single term results in a linear approximation	Using a single term results in a sinusoidal approximation

Proving Fourier's Theorem and figuring out how to solve for c_n and ϕ_n (or, equivalently, a_n and b_n is beyond the scope of this course. Nevertheless, hopefully I can persuade you that Fourier's Theorem is true through an example. One of the best examples I know can be found here `https://lpsa.swarthmore.edu/Fourier/Series/ExFS.html` :

This is a very badly behaved functions with discontinuities and no discernible symmetry. This ought to be a very difficult test of Fourier's Theorem. But with only 6 terms (dc plus 5 sine waves) in the middle plot, we can already see that the Fourier Series is beginning to approximate the function nicely. Using 20 sine waves







results in the plot on the right, where the resemblance is striking. Obviously $20 < \infty$, so even this is not a perfect approximation, but it's quite good.

Remember that we can't just add up sine waves willy-nilly – they have to have the right amplitudes and phases (or, equivalently, the right cosine parts and sine parts). In case you're curious, the sine waves that produce the plot above are:

	Sine-Cosine		Amplitude-Phase	
n	a_n	b_n	c_n	ϕ_n
0 (dc)	0.5	-	0.5	-
1	0.64	0.64	0.9	45°
2	0	-0.32	0.32	-90°
3	-0.21	0.21	0.30	135°
4	0	-0.16	0.16	-90°
5	0.13	0.13	0.18	45°
6	0	-0.11	0.11	-90°
7	-0.09	0.09	0.13	135°

This table translates into:

$$f(t) = 0.5 + 0.9\cos(1\omega_0 t + 45°) + 0.32\cos(2\omega_0 t - 90°) + \cdots$$

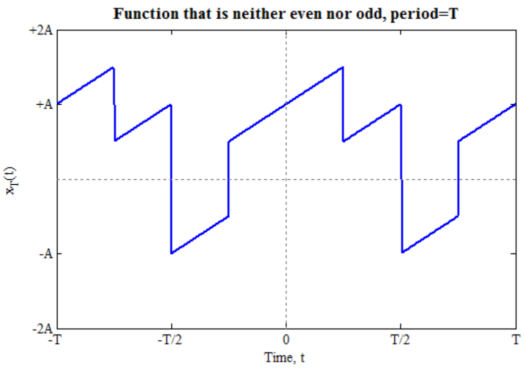
If a Fourier Series can work for this crazy function, hopefully you will trust that it will work for any function. If you need more evidence, see `https://en.wikipedia.org/wiki/Fourier_series`.

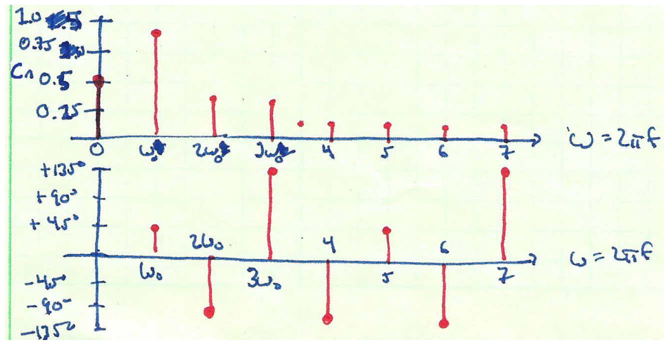
When we express a function as its Fourier Series and proceed to do operations on the individual sine waves, we are working in the **Frequency Domain**.

In the time domain, we plot the value of the function at each moment in time, producing familiar shapes like sine waves. In the frequency domain, we plot the magnitude and the phase of the function at each frequency. The function above is plotted in the time domain and the frequency domain below:

Some additional examples:

- Math Behind Music and Synthesis on Youtube at 7:47
  - Notice how the waves that are closer to a sign wave have strong fundamentals and weak harmonics. As the wave departs more from a sine wave, it includes more harmonic content. Discontinuities result in a large magnitude of the high-freuqency harmonics.
- Fourier Series Audio Demo





- Notice how the trumpet has many frequencies, producing a rich sound, while the oboe is primarily first and second harmonic, producing purer tone. Interestingly, both instruments produce stronger second harmonics than their fundamental!
- Further notice that we could truncate the Fourier Series after the first few terms without really losing much (especially with the oboe). This is the foundation of most **compression** algorithms, which attempt to throw away some information while keeping the actual experience (what an audio file sounds like or what an image looks like) as much the same as possible.

- Fourier Series on ComPADRE.org
  - Try with the oboe $\begin{bmatrix} 1.000 & 1.869 & 0.042 & 0.022 \end{bmatrix}$, and try eliminating the last two values. A file with just the first two values requires less storage and is easier to transmit than, say, transmitting the first thousand harmonics. If we can achieve this compression and if it sounds very much the same, then we've very much achieved the goal!
  - Try some different fundamental frequencies...