Lecture Notes 8: Duality

You may have noticed a certain symmetry in circuit analysis:

- KVL looks like KCL but with voltage instead of current and with loops instead of nodes.
- Parallel (objects with the same voltage by virtue of KVL) is analogous to Series (objects with the same current by virtue of KCL)
- Through source transformations and Thevenin/Norton equivalents, we get the sense that voltage sources in series with resistors have some connection to current sources in parallel with resistors...

These are legitimate observations and examples of **duality**: if the roles of voltage and current are exchanged, a new circuit is made that has dual properties.

Example: If two components share the same **current** by virtue of **KCL**, then they are in **series**.
Example: If two components share the same **voltage** by virtue of **KVL**, then they are in **parallel**.

If we didn't already know about "parallel," we could have invented it by applying duality to "series."

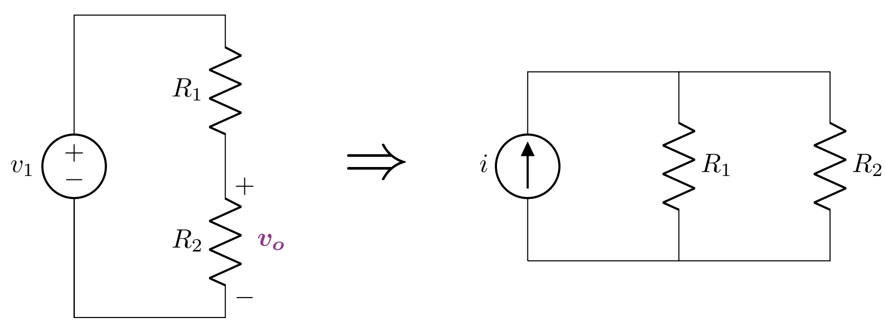
Use the following to find the dual of sentences describing circuits:

Voltage	$\iff$	Current
Across	$\iff$	Through
Short	$\iff$	Open
Parallel	$\iff$	Series
Mesh	$\iff$	Node
Outer Loop	$\iff$	Ground Node

As an example, consider the ever-useful voltage divider. In sentences, we might describe it (and its dual) as follows:

A **voltage** divider consists of a **voltage** source in **series** with two resistors. The output is the **voltage across** the second resistor.
A **current** divider consists of a **current** source in **parallel** with two resistors. The output is the **current through** the second resistor.

If we draw this dual circuit, we get the following:



But wait! In the example, the solutions to both circuits are:

$$v_o = \frac{R_2}{R_1 + R_2} v_1 \qquad i_o = \frac{R_1}{R_1 + R_2} i_1$$

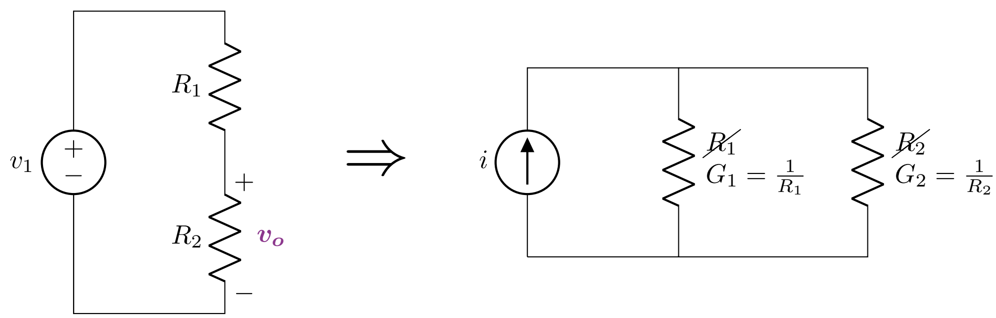
These equations look similar, but they aren't quite duals (if we just switched v's for i's, we wouldn't get the same thing). What have we forgotton? $\Rightarrow$ We need to take the dual of the resistor!

A **resistor** is a linear **voltage**-**current** relationship where **voltage** is **current** times a constant with units of **volts**/**amps** (**Ohms Ω**), or **resistance** R.

A **conductance** is a linear **current-voltage** relationship where **current** is **voltage** times a constant with units of **amps/volts** (**Siemens S**) or **conductance** G.

Because **conductances** behave just like resistors ($v = Ri$ vs $i = Gv$), we still draw them like resistors. When we take the dual of a circuit, the conductances have the same value as the resistors, but with units of Siemens instead of Ohms. This is the same as saying that the dual of a resistor R is a resistor of value $1/R$.

The correct dual of the voltage divider is then:



Now the solutions are

$$v_o = \frac{R_2}{R_1 + R_2} v_1 \qquad\qquad i_o = \frac{1/G_1}{1/G_1 + 1/G_2} i_1 = \frac{G_2}{G_1 + G_2} i_1$$

And now we have a proper dual.

Any statement we might make about a circuit also has a dual:

Circuit 1: As the **resistances** are increased, less **current flows**.
Circuit 2: As the **conductances** are increased, less **voltage develops**.

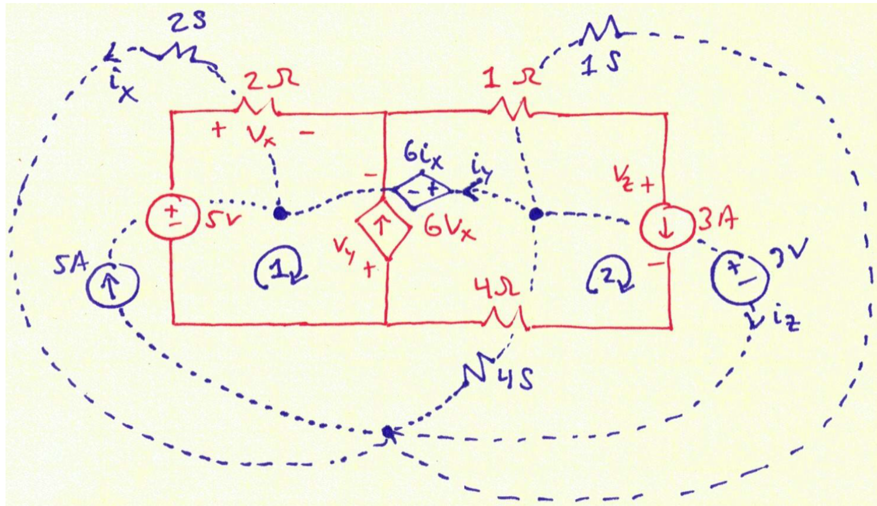
Duals are quite useful. For example,

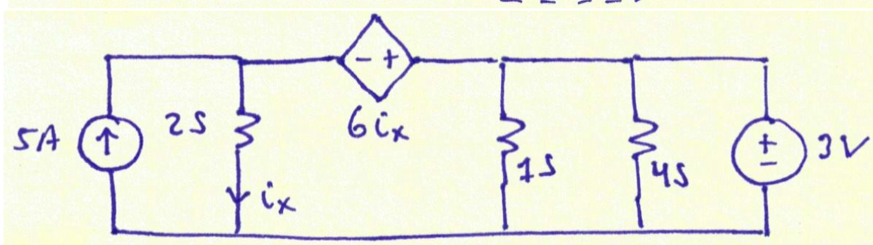
Resistors in **series** are simply $\boldsymbol{R_1} + \boldsymbol{R_2} + \cdots$ but **parallel** is not so easy.
Conductances in **parallel** are simply $\boldsymbol{G_1} + \boldsymbol{G_2} + \cdots$ but **series** is not so easy.

To systematically find a dual circuit, draw a dot inside each mesh and one on the outside. Thus, each mesh in the original circuit corresponds to a node in the dual circuit. Then, connect every dot to every other dot by passing through the components of the original circuit, replacing the component with its dual. To keep polarity straight, label the voltage across each component in the original circuit. If a component's reference voltage counts as positive when doing KVL clockwise around a given mesh (a positive charge would gain energy as it passes clockwise through that component), then the dual's reference current will go into the dual node associated with that mesh. If you obeyed the PSC (or not) for any component in the original circuit, continue to obey the PSC (or not) in the dual.

You can check your work by seeing how changing a value in the original circuit would affect some other value, and then checking that the dual does the same:

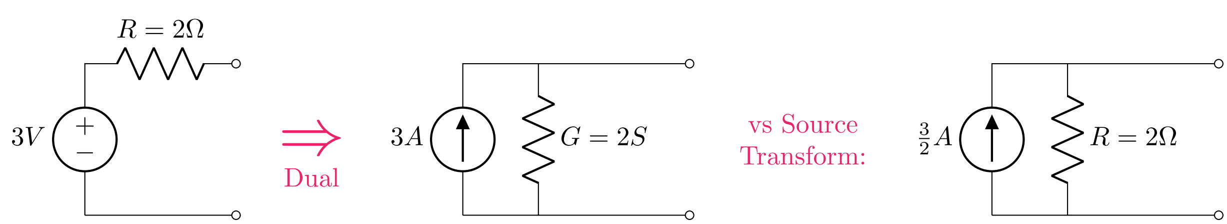
- Original: Increasing 5V would increase v_x $\Rightarrow$ Dual: Increasing 5A would increase i_x ✓





- Original: Increasing $6v_x$ would decrease v_x $\Rightarrow$ Dual: Increasing $6i_x$ would decrease i_x ✓
- Original: Increasing 3A would increase v_x $\Rightarrow$ Dual: Increasing 3V would increase i_x ✓

Note that sometimes circuits look like duals but aren't exactly:



A source transform is not a dual, as it has *equivalent* properties, not dual properties. Thevenin and Norton circuits are also not exactly duals for the same reason.

But they're close! Even if you don't want the dual exactly, duality can still provide powerful inspiration. Source transforms and Thevenin/Norton circuits are dual *topologies*, but not with dual-valued components. The dual topology alone (but with equivalent, not dual behavior) is a useful thing to have, as we've seen.

We'll see more duals later on, such as capacitors ($i = Cdv/dt$) and inductors ($v = Ldi/dt$).