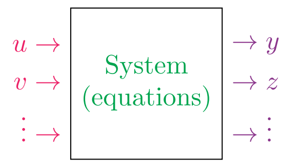
Lecture Notes 6: Superposition and Thevenin Equivalents

Recall that a system is linear if $a\vec{u}_1 + b\vec{u}_2 \to a\vec{y}_1 + b\vec{y}_2$ for any constants a,b. As a reminder, $\vec{u}$ is the vector of inputs, $\vec{u} = [u, v, \cdots]$ and $\vec{y}$ is the vector of outputs, $\vec{y} = [y, z, \cdots]$.



If we know that a system is linear, we can do something clever with the definition of linearity, which applies to any choice of $\vec{u}_1$, $\vec{u}_2$, etc.

Let $\vec{u}_1 = [u, 0, 0, 0, 0, \cdots]$. This means "apply input u alone with all other inputs set to 0." When you do this, it will result in a certain output $\vec{y}_1$.

Let $\vec{u}_2 = [0, v, 0, 0, 0, \cdots]$. This means "apply input v alone with all other inputs set to 0." When you do this, it will result in a certain output $\vec{y}_2$.

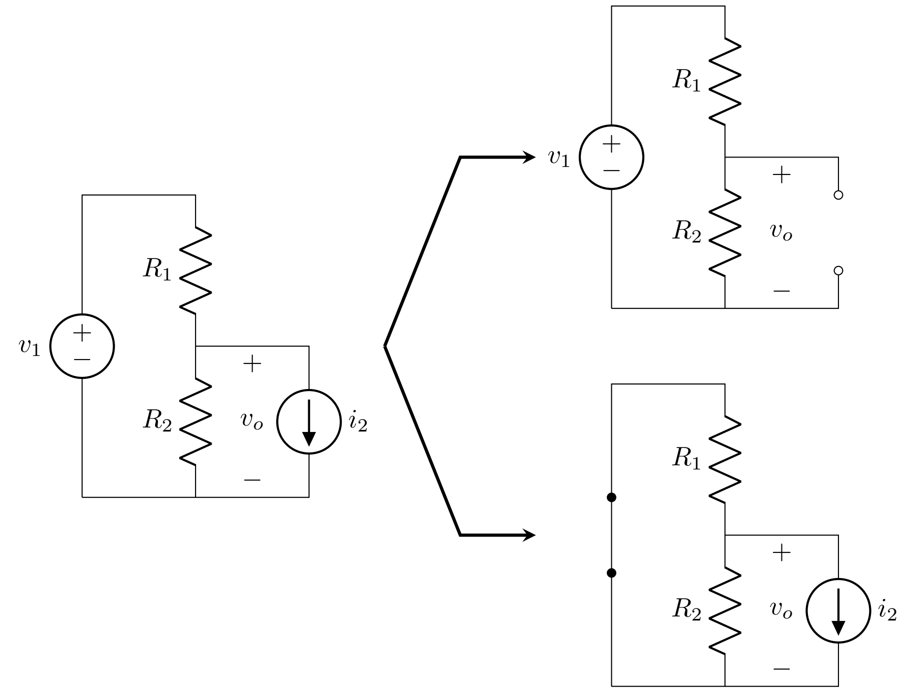
$\Rightarrow$ Linearity guarantees that the output $\vec{y}$ if all inputs are applied simultaneously will be

$$\vec{y}(\vec{u}) = \underbrace{\vec{y}\left(\begin{bmatrix} u \\ 0 \\ 0 \\ \vdots \end{bmatrix}\right)}_{\text{the output when u is applied alone}} + \underbrace{\vec{y}\left(\begin{bmatrix} 0 \\ v \\ 0 \\ \vdots \end{bmatrix}\right)}_{\text{the output when v is applied alone}} + \underbrace{\vec{y}\left(\begin{bmatrix} 0 \\ 0 \\ w \\ \vdots \end{bmatrix}\right)}_{\text{the output when w is applied alone}} + \cdots$$

This leads us to the technique of **superposition**.

Superposition is the notion that the output can be thought of as *independent* contributions from each of the inputs and **only applies to linear systems**. The superposition method/technique is an analysis strategy in which one calculates the response (output) to each input *individually*. Then, the true output (with all inputs applied simultaneously) is simply the sum of the individual responses.

Consider the example circuit below, which has two inputs v_1 and i_2. Let's say we're interested in the voltage across i_2 (which is also the voltage across R_2), v_o. Superposition tells us that we can apply *one source at a time* and solve for v_o *due to that source*. Then the "true" v_o is simply the sum of the responses due to each source.



$$\boxed{v_o = \frac{R_2}{R_1+R_2}v_1}$$

Contribution of v_1 alone with $i_2 = 0$ (open circuit)

$$\boxed{v_o = -\frac{R_1 R_2}{R_1+R_2}i_2}$$

Contribution of i_2 alone with $v_1 = 0$ (short circuit)

To apply v_1 alone, we need to set all other sources equal to zero. Setting i_2 equal to zero makes it a "zero amp current source" or, equivalently, "a component that always has 0 amps flowing through it" or, in other words, an open circuit. We solve for the output to find $(v_o \,\mathrm{due\,to\,v_1\,alone}) = R_2/(R_1 + R_2) \times v_1$.

To apply i_2 alone, we need to set all other sources equal to zero. Setting v_1 equal to zero makes it a "zero volt volage source" or, equivalently, "a component that always has 0 volts across it" or, in other words, a short circuit. We solve for the output to find $(v_o \,\mathrm{due\,to\,i_2\,alone}) = -R_1R_2/(R_1 + R_2) \times i_2$.

Now that we have found how each source contributes to the output, the "true" output is simply the sum of those responses:

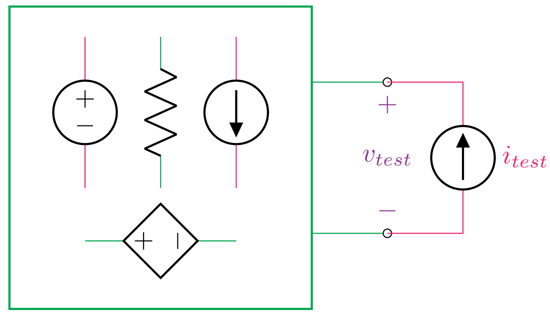
$$\boxed{v_o = \underbrace{\frac{R_2}{R_1+R_2}v_1}_{\mathrm{contribution\ from\ v_1}} + \underbrace{-\frac{R_1R_2}{R_1+R_2}i_2}_{\mathrm{contribution\ from\ i_2}}}$$

Another powerful consequence of linearity pertains to equivalent circuits of one-port networks. Consider a box that contains a linear system plus, possibly, independent sources. We might loosely call this a "linear system," though technically it's an "affine" system since it contains independent sources. Consider this linear system to only have one port with which it interacts with the outside world.

Let us see if we can find an equivalent circuit for this box without knowing what is inside it. As usual with equivalent circuits, apply a test current to the port and measure the resulting voltage. We don't know what's in the circuit, so we can't find the answer, but *if the circuit is linear* then we know what *form* the answer will take:

$$v_{test} = \underbrace{(av_1 + bv_2 + ci_3 + \cdots)}_{\mathrm{contributions\ from\ sources\ inside\ the\ box}} + \underbrace{R_o i_{test}}_{\mathrm{contribution\ from\ }i_{test}}$$

where the constant attached to i_{test} must have units of Ohms since it multiplies current to get voltage – hence, we have gone ahead and used the letter R.



"Linear" One-Port System

An external circuit doesn't distinguish between all the different contributions inside the box – it just knows that the stuff inside contributes something and the stuff outside contributes something. So let's collapse all of the terms coming from inside the box into a single number with units of volts:

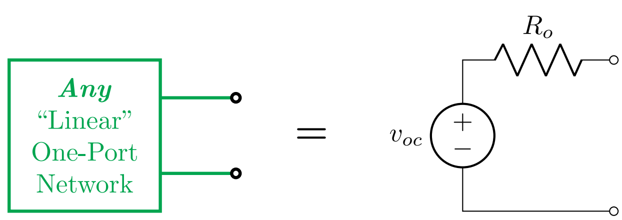
$$\boxed{v_{test} = v_{oc} + R_o i_{test}}$$

The voltage that is due to sources inside the box is called the **open-circuit voltage** because it is the voltage you would actually see at the port if i_{test} were zero (if the port were open-circuited).

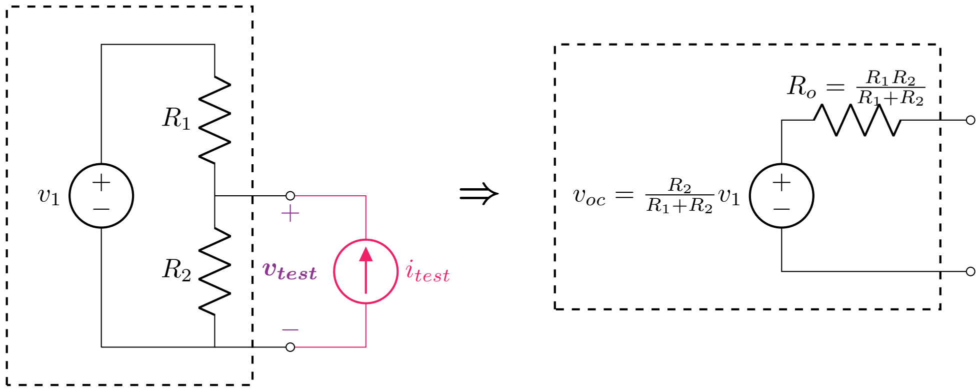
The constant attached to i_{test} is called the **output resistance** of the circuit. It relates how much the port voltage goes up and down as current is pumped across the port.

The fact that this will *always* be the case for a linear system is **Thevenin's Theorem**. For this reason, the open-circuit voltage is sometimes called the **Thevenin equivalent voltage** v_{th} and the output resistance is sometimes called the **Thevenin equivalent resistance** R_{th}.

We can express Thevenin's Theorem with an equivalent circuit. Since we know that the port voltage will be a contribution of a constant voltage *plus* a resistance-like term, this suggests a series connection of a voltage source and a resistor. Thus, another way to express Thevenin's Theorem is to say that *any* linear system will have an equivalent circuit of a voltage source in series with a resistor, called its **Thevenin Equivalent Circuit**.



Consider the example of a voltage divider. The "box" contains one voltage source and two resistors. To find the equivalent circuit, as always, apply a test current to the port and find the resulting voltage.

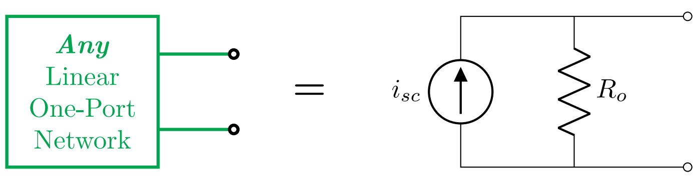


We already know the answer from the example we did for superposition, but now let us identify how the answer maps to the Thevenin Equivalent:

$$\boxed{v_{test} \quad = \quad \underbrace{\frac{R_2}{R_1+R_2}v_1}_{v_{oc}=\text{contribution from sources inside box}} \quad + \quad \underbrace{\frac{R_1R_2}{R_1+R_2}}_{R_o=\text{coefficient of contribution from } i_{test}} \quad \times\, i_{test}}$$

The Thevenin Equivalent Circuit is extremely useful. It allows you to collapse the essence of a circuit down to just two numbers. You can then forget about what's going on inside the circuit and focus on how that circuit interacts with other circuits. **Thevenin equivalent circuits are the key to developing complex systems, as they allow you to design small circuit modules and then unerstand (simply) how those modules will interact.**

Through a source transformation, we can show that any linear circuit can also be represented by a **Norton Equivalent Circuit**.



$$i_{test} = -i_{sc} + \frac{1}{R_o}v_{test}$$

The current that is due to sources inside the box is called the **short-circuit current** because it is the current you would actually see at the port if v_{test} were zero (if the port were short-circuited). This current is also called the **Norton Current**.

The constant attached to v_{test} is still called the **output resistance** of the circuit (though it shows up as the reciprocal). It relates how much the port current goes up and down as voltage is applied across the port. This term is less commonly called the Norton Equivalent Resistance. In fact, even when using the Norton equivalent circuit, the resistor is still called the Thevenin resistance.

You can solve for the Norton equivalent circuit by either first solving the Thevenin equivalent followed by a source transformation, or by directly applying a test voltage and measuring the resulting test current to find

$$i_{test} \quad = \quad \underbrace{\text{(something)}}_{\text{whatever you solve here is } -i_{sc}} \quad + \quad \underbrace{\text{(something else)}}_{\text{whatever you solve here is } 1/R_o} \quad \times v_{test}$$