Lecture Notes 19: Complex Numbers

We have seen that the sinusoidal response of an LTI system (which obeys a linear differential equation with constant coefficients) can be calculated by simply plugging in that the input is $x = X\cos(\omega t)$, the output will be $y = Y\cos(\omega t + \phi)$, and then solving for ϕ and Y (or, equivalently, the magnitude gain $M = Y/X$). But is sure is painful. It's much easier if we use complex numbers.

We begin with the definition of the square root:

$$\sqrt{x} \triangleq \text{The number which, when squared, yields } x$$

By this definition, $\sqrt{-1}$ cannot be represented using ordinary **real** numbers, because both positive and negative numbers, when squared, yield only positive numbers. Therefore, we define an object called an **imaginary number** as follows:

$$\boxed{j = \sqrt{-1}}$$

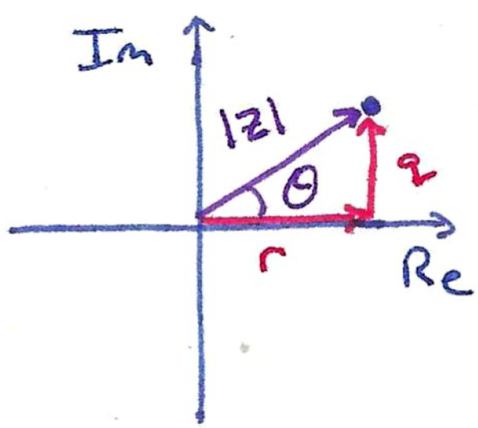
which implies that $j^2 = -1$ and $1/j = j/j^2 = -j$. (You may have seen the letter i used by heathens and barbarians for imaginary numbers. Civilized people who already use i extensively for current do not make such *faux pas*).

You can do all the same things with imaginary numbers that you can do with real numbers: add them together, multiply them together, and so forth. But *what about a real number plus an imaginary number*? Well, real plus imaginary is neither real nor imaginary, so we need yet another concept to deal with this: the **complex number**:

$$z = \underbrace{r}_{\text{Real Part } \mathrm{Re}\{z\}=r} + \underbrace{q}_{\text{Imaginary Part } \mathrm{Im}\{z\}=q} j$$

where we note that $\mathrm{Im}\{z\} = q$ is a real number.

A complex number, having two parts, cannot be represented on a number *line* as real or imaginary numbers can. We need a 2D **complex plane** to represent a complex number graphically:



The complex plane suggests multiple ways to represent complex numbers:

$$z = r + qj \quad \textbf{Rectangular Coordinates} \tag{1}$$
$$z = |z|\angle\theta \quad \textbf{Polar Coordinates} \tag{2}$$

where $|z| = \sqrt{r^2 + q^2}$ and $\theta = \tan^{-1}(q/r)$, which implies that $r = |z|\cos\theta$ and $q = |z|\sin\theta$. These two representations are usful in different contexts. For example,

- Adding and subtracting is easier in rectangular coordinates:

$$z_1 \pm z_2 = (r_1 \pm r_2) + (q_1 \pm q_2)j$$

- Multiplying and dividing is easier in polar coordinates (which we'll prove shortly):

$$z_1 z_2 = |z_1||z_2|\angle(\theta_1 + \theta_2)$$

$$\frac{z_1}{z_2} = \frac{|z_1|}{|z_2|}\angle(\theta_1 - \theta_2)$$

Another common operation we are used to from real numbers is exponentiation, e^x. What would this be for a complex exponent, e^z? Well, $e^z = e^{r+qj} = e^r e^{qj}$, where e^r is real exponentiation that we know how to do. But what is e^{qj}?

Recall that $y = e^x$ is defined as "the function which is its own derivative" and therefore, by the chain rule, $dy/dx = de^{ax}/dx = ae^{ax}$. I propose that the following answer fits this description for complex numbers:

$$\boxed{e^{qj} = \cos q + j \sin q}$$

Let us prove that this matches the definition by showing that $de^{qj}/dq = j \times e^{qj}$. Proof:

$$\frac{d}{dq}\left[\cos q + j\sin q\right] = -\sin q + j\cos q = j\left[\frac{-\sin q}{j} + \cos q\right] = j\left[\cos q + j\sin q\right]$$

The answer to "what is e^{qj}?" above is one of the most important equations in all of mathematics, **Euler's Formula**. My tip is to *not* try to intuitively relate complex exponentials to cosines and sines – this way lies madness. Instead, return to the fundamental definition of e^x (the function which is its own derivative) when you need reassurance that Euler's Formula is in fact true.

Using Euler's Formula, we can express a complex number z in polar coordinates by using elementary functions instead of the awkward $\angle$ notation:

$$z = |z|e^{j\theta} \quad \text{instead of} \quad |z|\angle\theta$$

This is useful because we can do operations on $e^{j\theta}$. For example, we can prove the multiplication and division formulas we say earlier:

$$z_1 z_2 = |z_1||z_2|e^{\theta_1 + \theta_2}$$

$$\frac{z_1}{z_2} = \frac{|z_1|}{|z_2|}e^{\theta_1 - \theta_2}$$

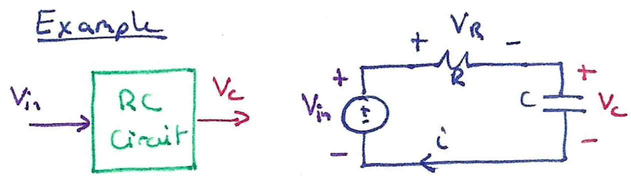
One final observations that is germane to our discussion of the sinusoidal response of LTI systems: $A\cos(\omega t + \phi)$ can be thought of as the real part of a complex exponential:

$$A\cos(\omega t + \phi) = \text{Re}\left\{Ae^{j(\omega t + \phi)}\right\}$$

The complex exponential also has an imaginary part, but because we're taking Re$\{\}$, it doesn't hurt that the imaginary part is there. This might seem like a silly transformation until we realize that *differentiating an exponential leads to massive simplifications compared to differentiating sines/cosines.* We take a lot of derivatives in differential equations.

Now, consider the RC circuit again: where we have already derived the differential equation,

$$RC\frac{dv_c}{dt} + v_c = v_i$$



Now, instead of plugging in $v_i = V_i \cos(\omega t)$ and $v_o = V_o \cos(\omega t + \phi)$, let us plug in

$$v_i = \mathrm{Re}\left\{V_i e^{j(\omega t)}\right\} \tag{3}$$

$$v_o = \mathrm{Re}\left\{V_o e^{j(\omega t+\phi)}\right\} \tag{4}$$

which are equivalent to the original cosines and where V_o and ϕ are still unknown. Pluggin in gives us

$$RC\frac{d}{dt}\mathrm{Re}\left\{V_o e^{j(\omega t+\phi)}\right\} + \mathrm{Re}\left\{V_o e^{j(\omega t+\phi)}\right\} = \mathrm{Re}\left\{V_i e^{j\omega t}\right\}$$

Now note that we can push/pull real constants and time derivatives in/out of the $\mathrm{Re}\{\}$ function, so $A\frac{d}{dt}\mathrm{Re}\{z\} = \mathrm{Re}\left\{A\frac{d}{dt}z\right\}$. Therefore,

$$\Rightarrow \mathrm{Re}\left\{j\omega RC V_o e^{j(\omega t+\phi)}\right\} + \mathrm{Re}\left\{V_o e^{j(\omega t+\phi)}\right\} = \mathrm{Re}\left\{V_i e^{j\omega t}\right\}$$

$$\Rightarrow \mathrm{Re}\left\{(V_o + j\omega RC V_o)e^{j\omega t}e^{j\phi}\right\} = \mathrm{Re}\left\{V_i e^{j\omega t}\right\}$$

Now let us leverage an identity that you will prove in homework:

Identity: If $\mathrm{Re}\left\{Ae^{j\omega t}\right\} = \mathrm{Re}\left\{Be^{j\omega t}\right\}$ where A, and B are complex constants, then A must equal B.

This identity allows us to effectively drop the $\mathrm{Re}\{\}$ from the equation. This exposes $e^{j\omega t}$ to be canceled on each side, leaving us with

$$\boxed{j\omega RC \underbrace{V_o e^{j\phi}}_{\text{Output Phasor } \vec{v}_o} + V_o e^{j\phi} = V_i = \underbrace{V_i e^{j0}}_{\text{Input Phasor } \tilde{v}_i}}$$

The "wave" part $e^{j\omega t}$ is gone and all that remains are **phasors** that capture the amplitude and phase of each wave. The fact that the input and output are both sine waves of the same frequency is now implied, not explicit. This fact dropping out of the equation is evidence that we have an LTI system for which an input sine wave *must* result in an output sine wave of the same frequency. The use of complex numbers naturally shows us that writing this fact down again and again is, in some sense, redundant.

We can now find the thing that people normally call the **transfer function**,

$$\text{Transfer Function} = \frac{\vec{v}_o}{\vec{v}_i}$$

which itself is a *complex* function of frequency. Interpreting the transfer function directly requires a lot more experience. However, we can quickly find the magnitude gain V_o/V_i and the phase difference ϕ by simply finding the magnitdue and angle of the transfer function:

$$\boxed{\text{Magnitude Gain} \quad M = \left|\frac{\vec{v}_o}{\vec{v}_i}\right| = \frac{|\vec{v}_o|}{|\vec{v}_i|}}$$

$$\boxed{\text{Phase Difference} \quad \phi = \angle\left(\frac{\vec{v}_o}{\vec{v}_i}\right)} = \angle\vec{v}_o - \angle\vec{v}_i$$

In the case of the RC circuit above, the transfer function is

$$\frac{\vec{v}_o}{\vec{v}_i} = \frac{1}{1 + j\omega RC}$$

$$M = \frac{\sqrt{1^2 + 0^2}}{\sqrt{1^2 + (\omega RC)^2}} = \frac{1}{\sqrt{1 + (\omega RC)^2}}$$

and

$$\angle\left(\frac{\vec{v}_o}{\vec{v}_i}\right) = \tan^{-1}\left(\frac{0}{1}\right) - \tan^{-1}\left(\frac{\omega RC}{1}\right) = -\tan^{-1}(\omega RC)$$

(Do these look familiar?)

With just a little practice, it is possible to skip the derivation and jump straight to the phasor equation by inspection. From there, it's simply algebra to find the complex transfer function, from which the gain and phase can be rapidly calculated. The entire process should only take a few lines.