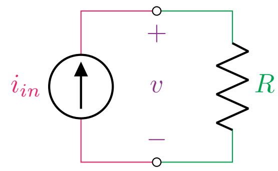
Lecture Notes 5: Linearity

Mathematically, a **system** is a set of equations that relate **inputs** to **outputs**. In circuits, independent sources are inputs, all other voltages/currents are possible outputs, and the component values are called system parameters.

Input (u) → System (equations) → Output (y)



During this and future courses, you will come to appreciate that we have powerful tools to analyze a special class of systems called **linear** systems and comparatively few tools to analyzer nonlinear systems.

A linear system is one whose equation(s) look like this:

$$c_y y = c_u u$$

$$\underbrace{c_y}_{\text{System Parameter}} \times \underbrace{y}_{\text{Output}} = \underbrace{c_u}_{\text{System Parameter}} \times \underbrace{u}_{\text{Input}}$$

i.e., every equation contains terms with only one input or output per term, where each input or output is only raised to the first power (not quadratic, log, exponential, etc.), and multiplied by a constant. Later, we will see that derivatives and integrals are also linear.

Formally, linearity is defined as follows:

> Let an output y be a function of an input u, which may be written $y = f(u)$ or $u \to y$. The equation $y = f(u)$ is linear if, for any constants a and b,
>
> $$f(au_1 + bu_2) = af(u_1) + bf(u_2)$$

Example: $c_u u = c_y y$ or $y = (c_u/c_y)u$

$$\begin{aligned} f(au_1 + bu_2) &= \frac{c_u}{c_y}(au_1 + bu_2) \\ &= a\frac{c_u}{c_y}u_1 + b\frac{c_u}{c_y}u_2 \\ &= af(u_1) + bf(u_1) \end{aligned}$$

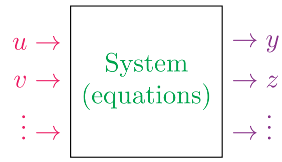
This example is linear.

Another example: $y = cu^2$

$$\begin{aligned} f(au_1 + bu_2) &= c(au_1 + bu_2)^2 \\ &= ca^2u_1^2 + cb^2u_2^2 + cabu_1u_2 \\ &\neq a(cu_1^2) + b(cu_2)^2 \end{aligned}$$

This example is not linear.

A system with a single input and single output **(SISO)** has only one equation. A system with multiple inputs and outputs **(MIMO)** has multiple equations:



where we expect N equations for N outputs. Fewer equations leaves the problem unsolvable. More equations implies that some of the equations are linearly dependent (contain redundant information) – this is fine. An example where one might obtain too many equations is by doing KCL at every node instead of every-node-but-one.

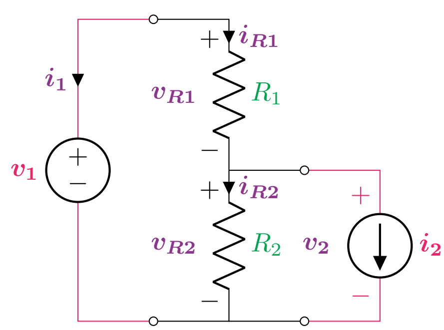
This can be converted into matrix form. The inputs and outputs are expressed as vectors:

$$\vec{u} = \begin{bmatrix} u \\ v \\ \vdots \end{bmatrix} \qquad \vec{y} = \begin{bmatrix} y \\ z \\ \vdots \end{bmatrix}$$

Linearity is now defined as follows:

> A set of inputs $\vec{u}_1$ yields a set of outputs $\vec{y}_1$
> A different set of inputs $\vec{u}_2$ yields a set of outputs $\vec{y}_2$
> A system is linear if, for any $\vec{u}_1$, $\vec{u}_2$ and for any constants a, b,
>
> $$\boxed{a\vec{u}_1 + b\vec{u}_2 \to a\vec{y}_1 + b\vec{y}_2}$$

This applies directly to circuits, where the **input(s) are independent sources** and the **output(s) are whatever voltages or currents we are interested in**. The **component values are the "system parameters."**



Is this circuit (system) linear? Consider its equations:

KVL:	$v_1 - v_{R1} - v_{R2}$	$=$	0	Since KVL takes the form $\sum v = 0$,
	$v_{R2} - v_2$	$=$	0	KVL equations are always linear
KCL:	$i_1 + i_{R2}$	$=$	0	Since KCL takes the form $\sum i = 0$,
	$i_{R1} - i_{R2} - i_2$	$=$	0	KCL equations are always linear
Component Laws:	v_{R1}	$=$	$R_1 i_{R1}$	Resistors have linear component laws!
	v_{R2}	$=$	$R_2 i_{R2}$	(Not all components do)

Since all circuits consist of KVL equations, KCL equations, and component laws, and since KVL and KCL are always linear, we can conclude that

> $\Rightarrow$ Whether a circuit is linear depends solely on whether the components are linear.

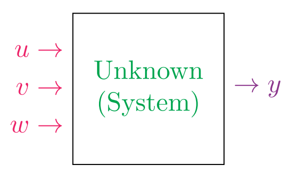
This circuit can be solved as a matrix:

$$\underbrace{\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 1 & -R_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -R_2 & 0 & 0 \end{bmatrix}}_{\text{System Parameters}} \underbrace{\begin{bmatrix} v_{R1} \\ i_{R1} \\ v_{R2} \\ i_{R2} \\ i_1 \\ v_2 \end{bmatrix}}_{\text{Outputs}} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}}_{\text{System Parameters}} \underbrace{\begin{bmatrix} v_1 \\ i_2 \end{bmatrix}}_{\text{Inputs}}$$

For now, it is important that you be able to intuitively recognize when an equation is linear or not and that you be able to formally prove whether an equation or system is linear or not. As you go throughout your career, you will discover just how central linearity is to most of engineering. In this course alone, you'll learn many tools that only apply to linear systems:

- Superposition - only applies to linear systems
- Thevenin and Norton Equivalent Circuits – only exist for linear systems
- Feedback – most feedback and control theory only applies to linear systems
- Small Signal Analysis – the most common way to analyze non-linear systems ... by devising a linear approximation for the system!
- Transfer Functions – only applies to linear systems
- Impedance – a generalization of resistance that only applies to linear systems

If you know that a system is linear, then you know what *form* its equations will take. This makes **system identification** much easier.



If the system above is linear, then its equation must be

$$y = \underbrace{c_u}_{\text{Unknown}} u + \underbrace{c_y}_{\text{system}} v + \underbrace{c_w}_{\text{parameters}} w$$

Through experiments, you could gather data about the system. For example, suppose you ran three experiments by inputting different values of u, v, and w and then recorded the output y as follows:

	u	v	w	y
Experiment 1	-50	2	1	10
Experiment 2	+50	2	1	-10
Experiment 3	0	4	1	20

$$\Rightarrow \begin{bmatrix} -50 & 2 & 1 \\ 50 & 2 & 1 \\ 0 & 4 & 1 \end{bmatrix} \begin{bmatrix} c_u \\ c_v \\ c_w \end{bmatrix} = \begin{bmatrix} 10 \\ -10 \\ 20 \end{bmatrix}$$

By solving this system of equations, we can discover c_u, c_v, c_w based on a few experiments. Once we know the system parameters, we can predict any output from any input. This is only possible because we knew what form $y(u, v, w)$ took and it is particularly accessible when that form is linear.