Introduction to Electrical Engineering (ECE 302H) – Homework 9

Dawson Zhang | Instructor: Dr. Hanson

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Problem 9.1. Python Tutorial

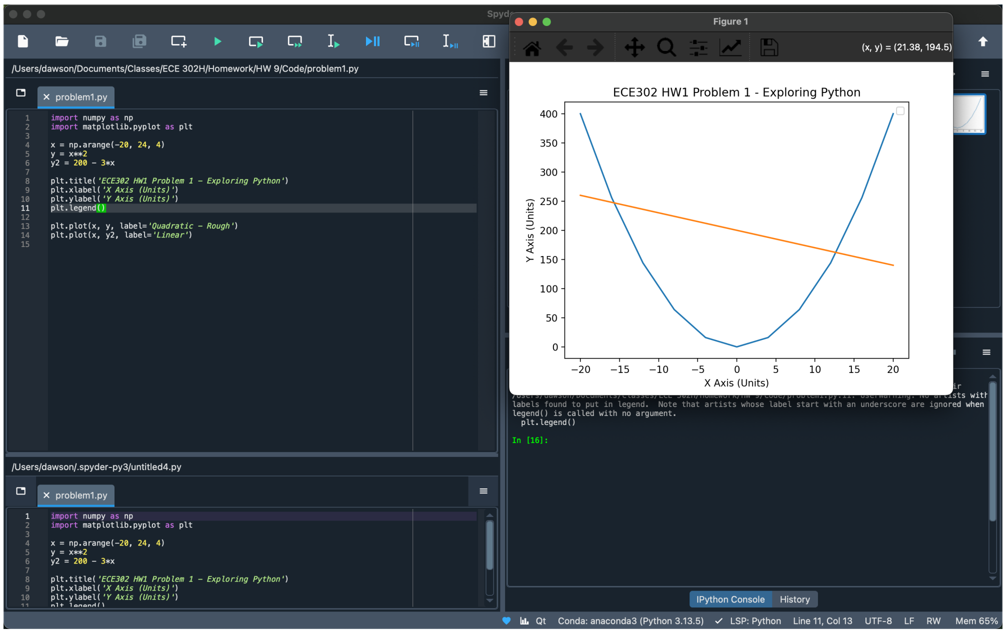
> **Remark.** Check HW document for full problem

Solution. □

i) Code:

```
np.arange(-20, 24, 4)
```

q) Picture for code and final graph:



Problem 9.2. Fourier Series Plotting

Consider the sawtooth wave. One cycle of this wave looks like a straight line $f(\omega t) = \omega t$ from $-\pi$ to π. This cycle repeats in the positive and negative ω directions forever and is therefore periodic. The Fourier series for this wave is given by

$$f(\omega t) = 2 \sum \frac{(-1)^{n+1}}{n} \sin(n\omega t)$$

Do the following in a single Python script and submit both your script and the final plots. Be sure to include a title, legend, x label, and y label. Note that you may not have been explicitly shown how to do each and every step here – a big part of learning to program is learning to research the commands for the tasks you want to perform.

a) Plot the sawtooth wave as a function of ωt from -3π to 3π.

b) Create a vector of ωt and use it to calculate a vector that represents the fundamental of the Fourier series. Plot it on top of the original sawtooth wave in a different color and style.

c) Calculate the sum of the first 3 harmonics of the sawtooth wave and plot it on top of the previous two plots in a different color and style.

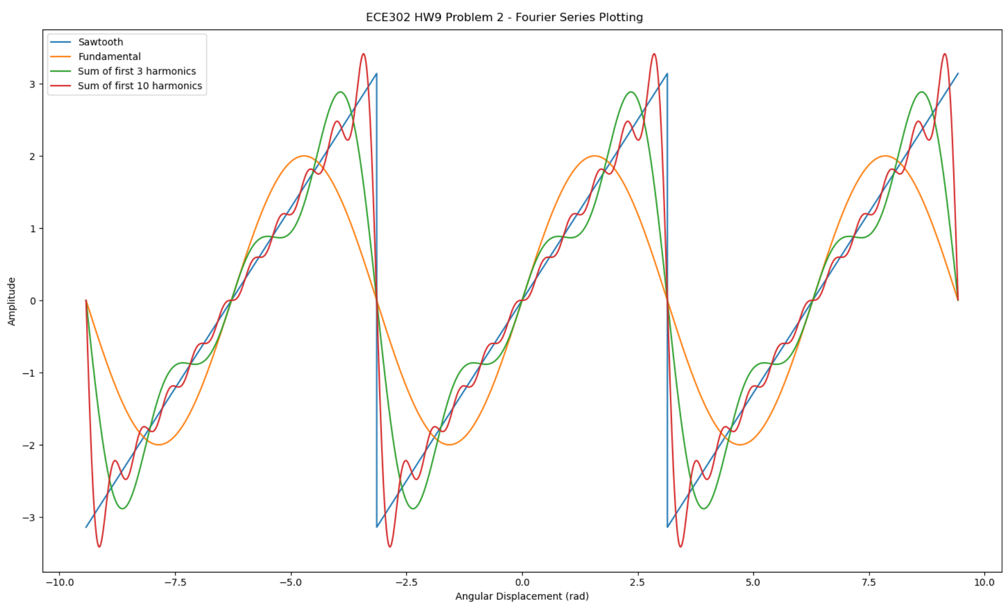
d) Do the same again for the sum of the first 10 harmonics.

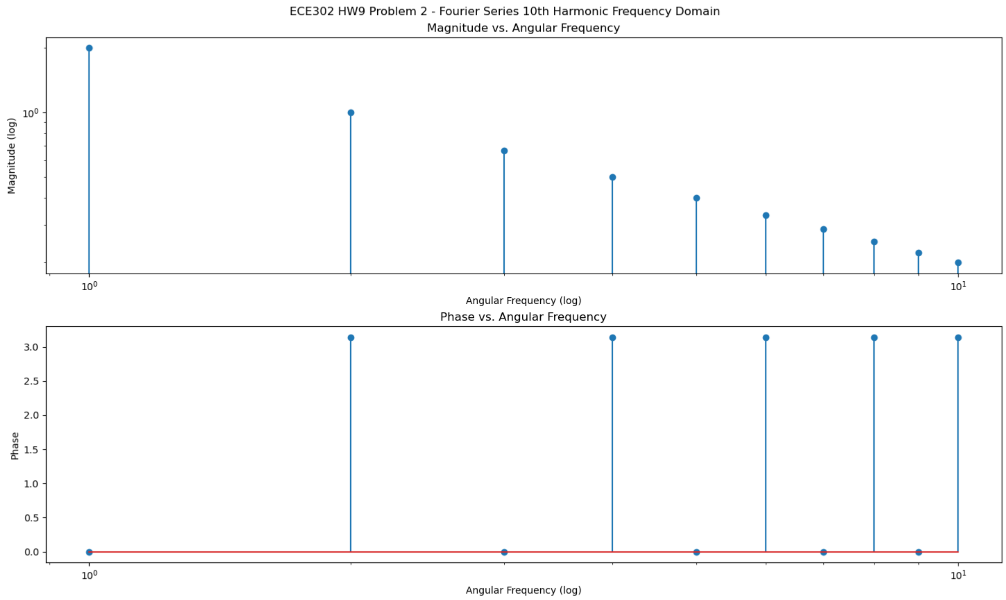
e) On a separate plot with two subplots, use the stem command (if you imported matplotlib as plt, then the stem plot command is plt.stem) to plot the magnitude and phase of the first 10 harmonics of the Fourier series of the sawtooth wave. Plot the magnitude on a log-log plot and the phase on a semilog-x plot.

Solution. □

```
#!/usr/bin/env python3
# -*- coding: utf-8 -*-
"""
Created on Wed Nov 5 19:47:47 2025

@author: dawson
"""

import numpy as np
import matplotlib.pyplot as plt

INFINITY = 1000
STEP = 0.001

x = np.arange(-3 * np.pi, 3 * np.pi, STEP)

y = ((x + np.pi) % (2 * np.pi)) - np.pi

y2 = 2 * (-1) ** (1 + 1) * (1 / 1) * np.sin(1 * x)

y3 = 0
for n in range(1, 4):
    y3 += 2 * (-1) ** (n + 1) * (1 / n) * np.sin(n * x)

y4 = 0
for n in range(1, 11):
    y4 += 2 * (-1) ** (n + 1) * (1 / n) * np.sin(n * x)

fig, axes = plt.subplots(1, constrained_layout=True)

fig.suptitle('ECE302 HW9 Problem 2 - Fourier Series Plotting')

axes.plot(x, y, label='Sawtooth')
axes.plot(x, y2, label='Fundamental')
axes.plot(x, y3, label='Sum of first 3 harmonics')
axes.plot(x, y4, label='Sum of first 10 harmonics')
axes.set_xlabel('Angular Displacement (rad)')
axes.set_ylabel('Amplitude')
axes.legend()


x = np.arange(1, 11)
y = np.abs(2 * (-1 ** (x + 1)) * (1 / x))
y2 = []

for n in range(1, 11):
    if n % 2 == 0:
        y2.append(np.pi)
    else:
        y2.append(0)
```

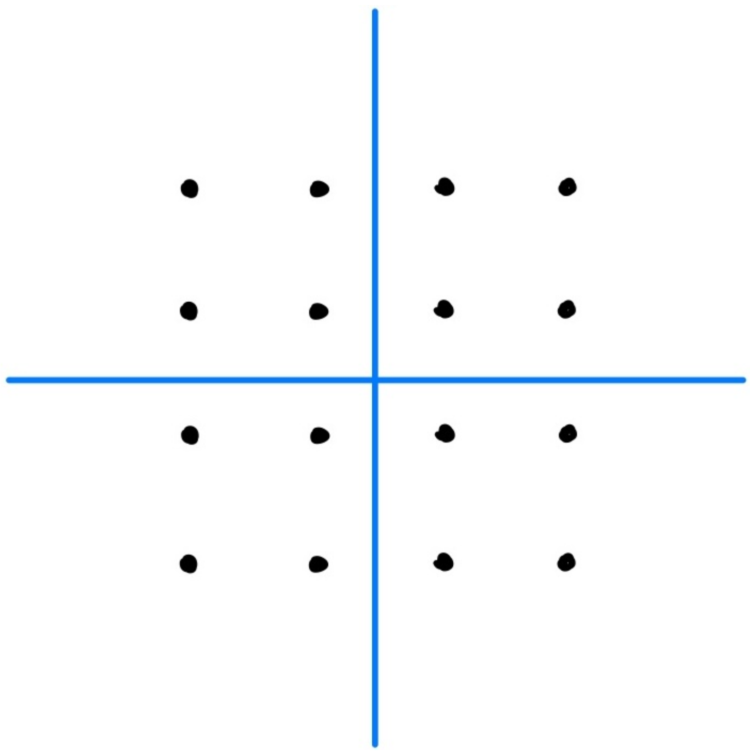
```python

fig, axes = plt.subplots(2, constrained_layout=True)

fig.suptitle('ECE302 HW9 Problem 2 - Fourier Series 10th Harmonic Frequency Domain')

axes[0].stem(x, y)
axes[0].set_xscale('log')
axes[0].set_yscale('log')
axes[0].set_title('Magnitude vs. Angular Frequency')
axes[0].set_xlabel('Angular Frequency (log)')
axes[0].set_ylabel('Magnitude (log)')

axes[1].stem(x, y2)
axes[1].set_xscale('log')
axes[1].set_title('Phase vs. Angular Frequency')
axes[1].set_xlabel('Angular Frequency (log)')
axes[1].set_ylabel('Phase')
```




ECE302 HW9 Problem 2 - Fourier Series 10th Harmonic Frequency Domain
Magnitude vs. Angular Frequency
Magnitude (log)
10^0
Angular Frequency (log)
10^0
10^1
Phase vs. Angular Frequency
Phase
3.0
2.5
2.0
1.5
1.0
0.5
0.0
Angular Frequency (log)

Problem 9.3. Quadrature Amplitude Modulation for Wireless Communication

Wireless communication systems send signals using cosine waves at high frequencies (anywhere from MHz to several GHz). These cosine waves are modulated, or changed slightly over time, to represent different digital values. Older modulation schemes would modulate frequency (FM) or amplitude (AM) to convey information. Modern systems modulate both the amplitude and the phase to convey more information in a single wave. As we've learned in class, a cosine wave with an amplitude and a phase can also be represented as a cosine wave plus a sine wave at the same frequency, each with their own amplitude but no phase. Wireless communications people usually think about their wave in the cosine sine form. The cosine part is called the "in-phase" component, and the sine part is called the "quadrature" component. Modulating both parts is called "Quadrature Amplitude Modulation" (QAM).



a) There are two variables that can be adjusted, the cosine amplitude and the sine amplitude (equivalent to the amplitude and phase of a single wave). These two variables can be plotted on a graph such as the Figure. Redraw the graph and label the x-axis "I" to represent the in-phase (cosine) amplitude. Label the y-axis "Q" to represent the quadrature (sin) amplitude.

b) The Figure represents a 16-QAM system. Each dot represents a digital value. How many bits are needed to represent all of the possible x coordinates? How many bits are needed to represent all of the possible y coordinates? Therefore, how many bits of information can be contained in a single incoming wave?

c) One way to assign digital values to the dots is to use "Gray Coding." Gray coding is not standard binary counting – it's an alternative in which increasing the number by one only flips a single bit from 1 to 0 or 0 to 1. Look up Gray Code in Wikipedia and create a table that shows the numbers 0 through 15, their Gray code representation, and also their ordinary binary code representation.

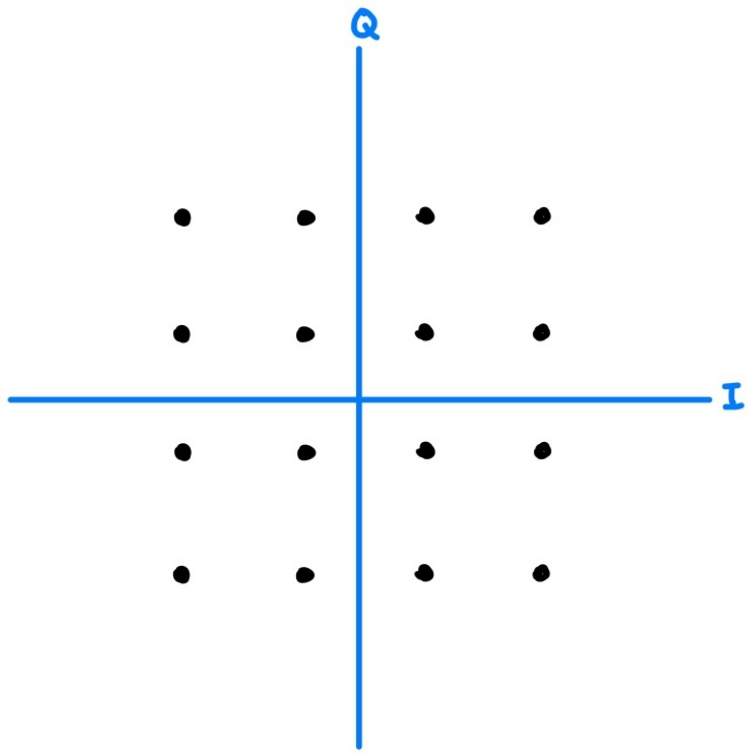
d) Use Gray coding to assign digital values to the dots in your figure. Use the two MSBs to represent the in-phase coordinate from most-negative to most-positive. Use the two LSBs to represent the quadrature component from most-negative to most-positive. A system that receives a cosine wave with a certain amplitude and phase will decompose that wave into its cosine (in-phase) and sine (quadrature) parts, look at their amplitudes, and find the closest dot on the graph – the system will then interpret the incoming cosine wave as the digital value assigned to that dot.

e) Each dot has an I and Q amplitude of either 1 or 3, so they are evenly spaced. Calculate the amplitude-phase representation of the [1011], [1111], [1010], and [1110] dots. Do you notice any correlation between the amplitude and phase of your answers and the locations of the dots on the graph?

f) If you wanted to have 64 dots instead of 16 without making the graph any larger (meaning, without having to design a circuit that can deal with any larger voltages), how many times more accurate would the transmitter need to be in creating I and Q components? How many times faster could your bitrate be?

Solution. □

a) Labeled Axis:



b) $\boxed{\text{2 bits}}$ for 4 unique x-coordinate

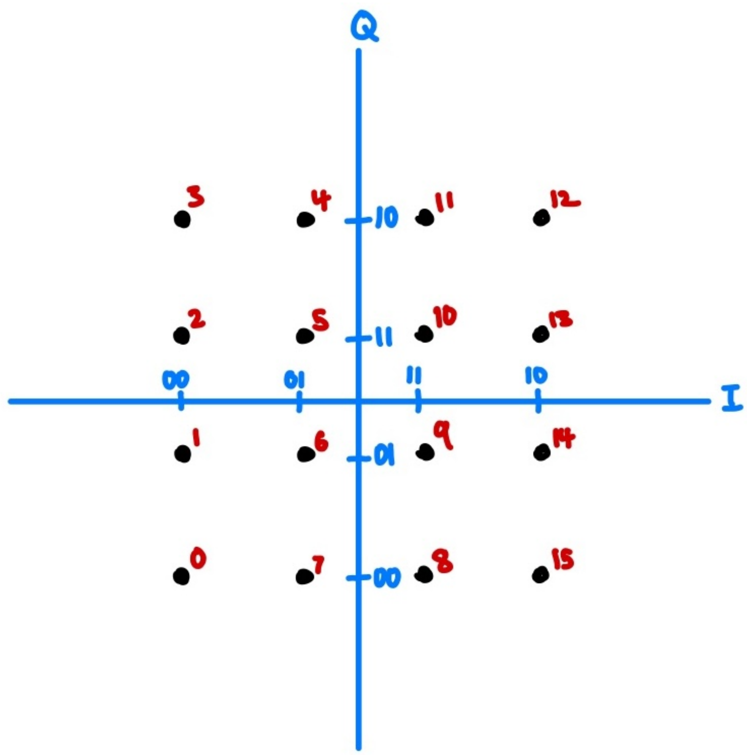
$\boxed{\text{2 bits}}$ for 4 unique y-coordinate

$\boxed{\text{4 bits}}$ of information $2 + 2 = 4$

c) Table:

Decimal	**Binary**	**Gray**
0	0000	0000
1	0001	0001
2	0010	0011
3	0011	0010
4	0100	0110
5	0101	0111
6	0110	0101
7	0111	0100
8	1000	1100
9	1001	1101
10	1010	1111
11	1011	1110
12	1100	1010
13	1101	1011
14	1110	1001
15	1111	1000

d) Labeled Ticks and Points:



e) [1011]: (3, 1). $\boxed{A = \sqrt{10}}$, $\boxed{\phi = \arctan(\frac{1}{3}) = 18.435°}$

[1111]: (1, 1). $\boxed{A = \sqrt{2}}$, $\boxed{\phi = \arctan(\frac{1}{1}) = 45°}$

[1010]: (3, 3). $\boxed{A = 3\sqrt{2}}$, $\boxed{\phi = \arctan(\frac{3}{3}) = 45°}$

[1110]: (1, 3). $\boxed{A = \sqrt{10}}$, $\boxed{\phi = \arctan(\frac{3}{1}) = 71.565°}$

All of these four points reside in the first quadrant.

f) The transmitter would need to be $\boxed{\text{2 times more}}$ accurate in producing in each component (I and Q).

In total, the bitrate would be $\boxed{\text{4 times faster}}$ since $2 \times 2 = 4$.

Problem 9.4. Complex Arithmetic Boot Camp

Complex numbers can be expressed as a real part plus an imaginary part, $z = r + qj$, where $j = \sqrt{-1}$. This is called rectangular form. They can also be expressed as a complex exponential, $z = Me^{j\phi}$. This is called polar form. In polar form, $|M|$ is called the magnitude, i.e. the distance from (0,0) to z on the complex plane, and ϕ is called the angle or the argument, i.e. the angle from the real (+x) axis to z on the complex plane. It is easier to add/subtract complex numbers in rectangular form, while it is easier to multiply/divide complex numbers in polar form.

a) Express $z = 3 + 4j$ as a complex exponential (polar form) and plot it on the complex plane

b) Express $z = -6 + 8j$ as a complex exponential (polar form) and plot it on the complex plane

c) Express $z = -j - 4j$ as a complex exponential (polar form) and plot it on the complex plane

d) Express $z = 2j$ as a complex exponential (polar form) and plot it on the complex plane

e) Express $z = \sqrt{-1 + j}$ as a complex exponential (polar form) and plot it on the complex plane

f) Express $z = 2e^{j\pi/6}$ in rectangular form and plot in on the complex plane

g) Express $z = -3e^{-j\pi/4}$ in rectangular form and plot in on the complex plane

h) Express $z = \sqrt{3}e^{j3\pi/4}$ in rectangular form and plot in on the complex plane

i) Express $z = -j^3$ in rectangular form

j) Express $z = -j^{-4}$ in rectangular form

k) Express $z = (2 + j)^2$ in rectangular form and polar form

l) Express $z = (3 - 2j)^3$ in rectangular form and polar form

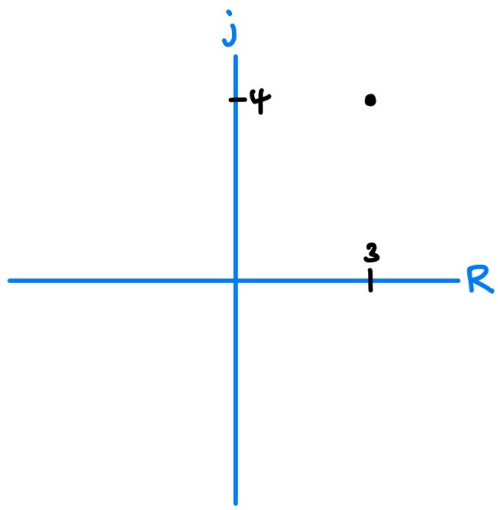
m) Calculate the magnitude and phase of $T = \frac{1+3j}{2-7j}$

n) Calculate the magnitude and phase of $T = \frac{1}{3+4j}$

Solution. □

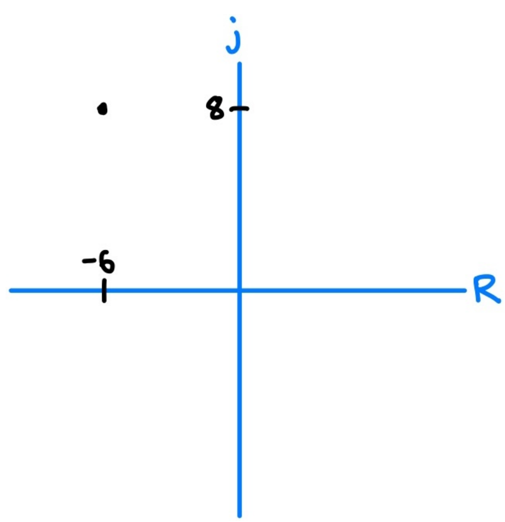
a)

$$z = 3 + 4j$$
$$z = \sqrt{3^2 + 4^2} e^{j \arctan(4/3)}$$
$$\boxed{z = 5e^{j0.927}}$$



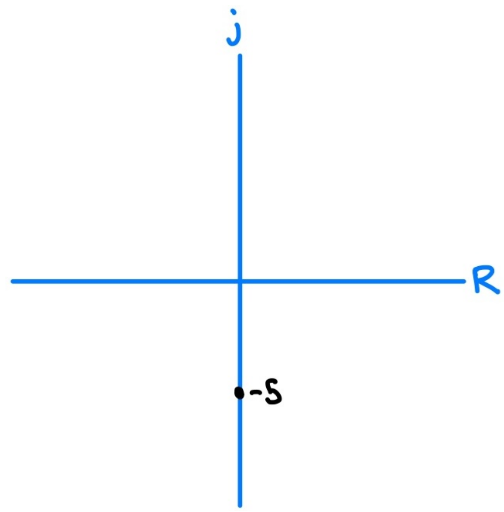
b)

$$z = -6 + 8j$$
$$z = \sqrt{6^2 + 8^2} e^{j \arctan(8/-6)}$$
$$\boxed{z = 10e^{j(\pi - 0.927)}}$$



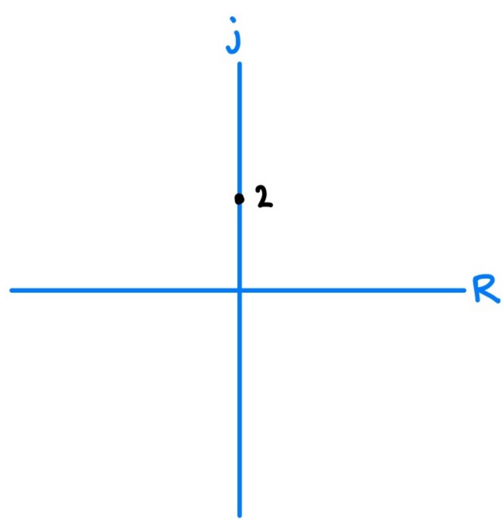
c)

$$z = -j - 4j$$
$$z = \sqrt{5^2} e^{j \arctan(-5/0)}$$
$$\boxed{z = 5e^{j3\pi/2}}$$



d)

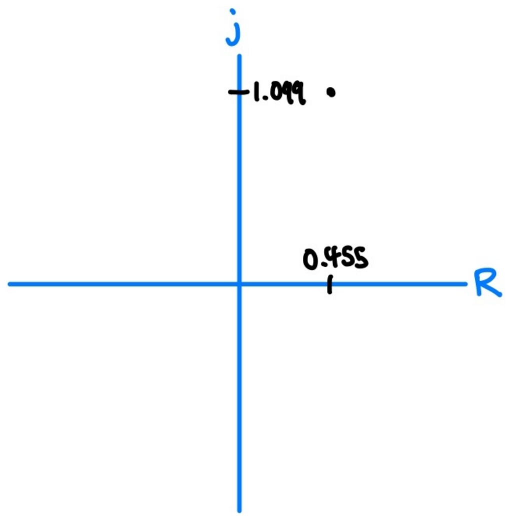
$$z = 2j$$
$$z = \sqrt{2^2} e^{j \arctan(2/0)}$$
$$\boxed{z = 2e^{j\pi/2}}$$



e)

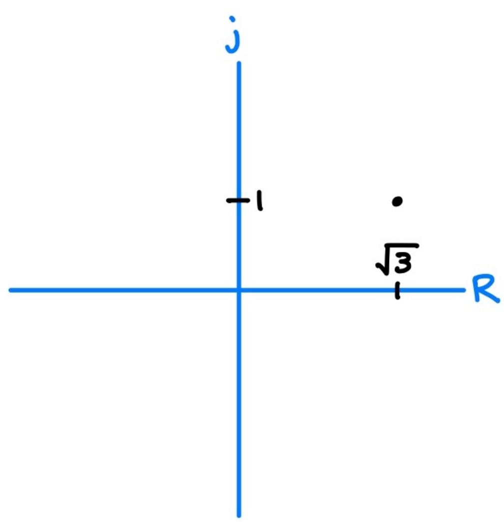
$$z = \sqrt{-1 + j}$$

$$z = \sqrt{\sqrt{1^2 + 1^2} e^{j \arctan(1/-1)}}$$
$$z = \sqrt{\sqrt{2} e^{j3\pi/4}}$$
$$\boxed{z = 2^{1/4} e^{j3\pi/8}}$$



f)

$$z = 2e^{j\pi/6}$$
$$z = 2\cos(\pi/6) + 2\sin(\pi/6)j$$
$$\boxed{z = \sqrt{3} + j}$$

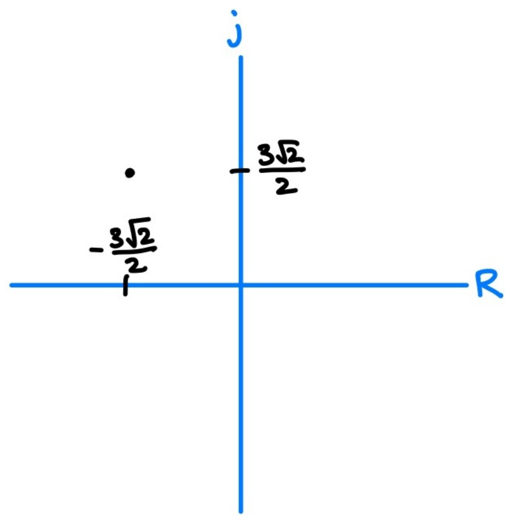


g)

$$z = -3e^{-j\pi/4}$$

$$z = -3\cos(-\pi/4) - 3\sin(-\pi/4)j$$

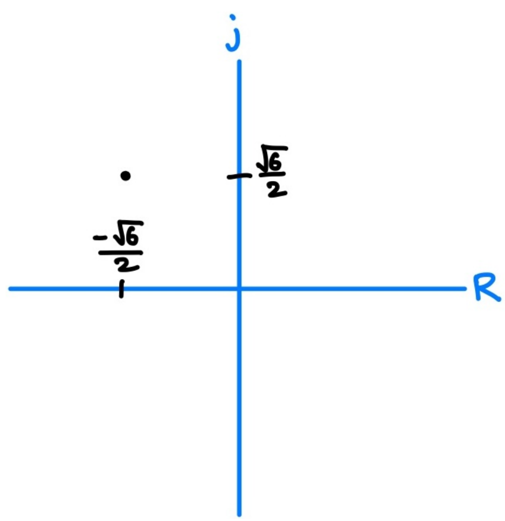
$$\boxed{z = -\frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}j}$$



h)

$$z = \sqrt{3}e^{j3\pi/4}$$
$$z = \sqrt{3}\cos(3\pi/4) + \sqrt{3}\sin(3\pi/4)j$$

$$\boxed{z = -\frac{\sqrt{6}}{2} + \frac{\sqrt{6}}{2}j}$$



i)

$$z = -j^3$$

$$z = -(-j)$$
$$\boxed{z = j}$$

j)

$$z = -j^{-4}$$
$$z = -\frac{1}{j^4}$$
$$\boxed{z = -1}$$

k)

$$z = (2 + j)^2$$
$$z = j^2 + 4j + 4$$
$$\boxed{z = 3 + 4j}$$
$$z = \sqrt{3^2 + 4^2} e^{j \arctan(4/3)}$$
$$\boxed{z = 5e^{j0.927}}$$

l)

$$z = (3 - 2j)^3$$
$$z = (\sqrt{3^2 + 2^2} e^{j \arctan(-2/3)})^3$$
$$z = (\sqrt{13} e^{j5.695})^3$$
$$\boxed{z = 13^{3/2} e^{j4.519}}$$
$$z = 13^{3/2} \cos(4.519) + 13^{3/2} \sin(4.519) j$$
$$\boxed{z = -9 - 46j}$$

m)

$$T = \frac{1 + 3j}{2 - 7j}$$
$$T = \frac{\sqrt{1^2 + 3^2} e^{j \arctan(3/1)}}{\sqrt{2^2 + 7^2} e^{j \arctan(-7/2)}}$$
$$T = \frac{\sqrt{10} e^{j1.249}}{\sqrt{53} e^{j4.991}}$$
$$T = \sqrt{\frac{10}{53}} e^{j2.541}$$
$$\boxed{|M| = \sqrt{\frac{10}{53}}}$$
$$\boxed{\phi = 2.541\,\text{rad}}$$

n)

$$T = \frac{1}{3+4j}$$
$$T = (\sqrt{3^2+4^2}e^{j\arctan(4/3)})^{-1}$$
$$T = (5e^{j0.927})^{-1}$$
$$T = \frac{1}{5}e^{j5.356}$$
$$\boxed{|M| = \frac{1}{5}}$$
$$\boxed{\phi = 5.356\,\text{rad}}$$