Moran structures

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Abstract

A growing collection of basic facts on Moran structures.

1 Introduction

Roughly speaking, *Moran structures* are the geometric realizations of symbolic *trees*. A *Moran set* is a lim sup set generated from a Moran structure. *Moran measure*, concentrating on a Moran set, often come as a weak-$*$ limit of some sequence of measures along a Moran structure. The procedure of constructing Moran sets or Moran measures are called *Moran constructions*.

Since its original introduction in [7], Moran structure has been extensively generalized and analyzed. They have applications in various fields including fractal geometry, dynamic systems, number theory and etc.

1.1 What is a Moran structure?

For $n \in \mathbb{N}$, define $\Sigma_n \coloneqq \{i_1 \dots i_n \colon i_k \in \mathbb{Z}, 1 \le k \le n\}$. Each element in Σ_n is called a n-length word. We denote the empty word by $\varnothing$ and let $\Sigma_0 = \{\varnothing\}$. Set $|I| = n$ if $I \in \Sigma_n$. Define $\Sigma^* = \bigcup_{n \ge 0} \Sigma_n$.

For $I, J \in \Sigma^*$, define the concatenated word

$$IJ \coloneqq \begin{cases} i_1 \dots i_n j_1 \dots j_m & \text{if } I = i_1 \dots i_n, J = j_1 \dots j_m \in \Sigma^* \setminus \{\varnothing\}; \\ J & \text{if } I = \varnothing; \\ I & \text{if } J = \varnothing. \end{cases}$$

There is a natural partial order on Σ^*. For $I, J \in \Sigma^*$, we call I a *prefix* of J, denoted by $I \prec J$, if there exists some $\omega \in \Sigma^*$ such that $J = I\omega$.

Let $\mathcal{T} \subset \Sigma^*$. For $I \in \mathcal{T}$, define $S(I) \coloneqq \{J \in \mathcal{T} \colon I \prec J \text{ and } |J| = |I| + 1\}$.

Definition 1.1. We call $\mathcal{T}$ a *tree* if it satisfies

(i) $\varnothing \in \mathcal{T}$;

(ii) $S(I)$ is finite for each $I \in \mathcal{T}$.

(iii) $i_1 \dots i_{n-1} \in \mathcal{T}$ when $i_1 \dots i_n \in \mathcal{T}$.

Let X be a compact metric space.

Definition 1.2. A map $A\colon \mathcal{T} \to 2^X$ is called a (generalized) *Moran structure* in X if $A(J) \subset A(I)$ whenever $I \prec J$.

Denote $\mathcal{T}_n = \mathcal{T} \cap \Sigma_n$ for each $n \in \mathbb{Z}_{\geq 0}$. Define $E \coloneqq \bigcap_{n=0}^{\infty} \bigcup_{I \in \mathcal{T}_n} A(I)$. We call E a *Moran set* whenever $E \neq \emptyset$.

Remark 1.3. Sometimes the term *net structure* is also used to describe *Moran structure*. There is slight difference between *net* and *tree*. In general, we can find a *root*, $\varnothing$, in a tree while a net can expand endless. For example, the collection of all dyadic cubes on $\mathbb{R}$ is more like a net while the dyadic cubes restricted to $[0, 1]$ seem to constitute a tree. In the symbolic world, we may call the set of two-sided words $\bigcup_{n\in\mathbb{N}} \prod_{i=-n}^{n} \{0, 1\}$ a net while the set of one-sided words $\bigcup_{n\in\mathbb{N}} \prod_{i=0}^{n} \{0, 1\}$ a tree.

1.2 FAQ

- When and where do Moran structures appear?
- Why are they useful?
- When and how can we apply Moran construction?

2 Dimension theory

In this section, we focus on the dimensional properties about Moran sets [4, 5]. An example is also presented to compute dimensions of random covering sets via constructing Moran measures with low energies [6].

2.1 Homogeneous Moran sets

Fix any bounded closed interval Δ (usually $\Delta = [0, 1]$) with length δ. Let $(m_k)_{k=1}^{\infty} \in \mathbb{N}_{\geq 2}^{\mathbb{N}}$ and $(r_k)_{k=1}^{\infty} \in (0, 1)^{\mathbb{N}}$ such that $m_k r_k \leq 1$ for all $k \in \mathbb{N}$. For each $k \in \mathbb{N}$, let $0 \leq t_1^{(k)} < \cdots < t_{m_k}^{(k)} < 1$ be a set of k-order translations such that $t_{i+1}^{(k)} - t_i^{(k)} \geq r_k$ for $1 \leq i \leq m_k - 1$.

Consider the tree $\mathcal{T} = \{\varnothing\} \cup \bigcup_{k=1} \prod_{i=1}^{k} \{1, \dots, m_i\}$. Let $A\colon \mathcal{T} \to 2^{\Delta}$ to be the Moran structure such that

(i) $A(\varnothing) = \Delta$;

(ii) For $k \in \mathbb{N}$, $I \in \mathcal{T}_{k-1}$ and $1 \leq i \leq m_k$, define $A(Ii)$ to be the closed interval with length $r_k|A(I)|$ and the distance between left endpoints of $A(I)$ and $A(Ii)$ is $t_i^{(k)}|A(I)|$.

As the translations change, we define *homogeneous Moran sets* $\mathcal{M}(\Delta, (m_k), (r_k))$ as the collection of all the Moran sets generated by any such Moran structure A, i.e., for each $E \in \mathcal{M}(\Delta, (m_k), (r_k))$, there is a family of translations $\{(t_i^{(k)})_{i=1}^{m_k}\}_{k\in\mathbb{N}}$ and the corresponding Moran structure A such that

$$E = \bigcap_{n=0}^{\infty} \bigcup_{I\in\mathcal{T}_n} A(I).$$

Moreover, we have some special homogeneous Moran sets

Partial Cantor sets $C^*(\Delta, (m_k), (r_k))$ if $t_i^{(k)} = (i-1)r_k$ for $1 \le i \le m_k$ and $k \in \mathbb{N}$.

Symmetric Cantor sets $C(\Delta, (m_k), (r_k))$ if $t_i^{(k)} = (i-1)\dfrac{1-r_k}{m_k-1}$ for $1 \le i \le m_k$ and $k \in \mathbb{N}$.

When the root interval $\Delta = [0, 1]$, we omit the notation Δ. The following quantities are tight with the dimensions of homogeneous Moran sets. Define

$$\alpha = \liminf_{k\to\infty} \frac{\log m_1\cdots m_k}{-\log r_1\cdots r_k} \qquad \alpha^* = \liminf_{k\to\infty} \frac{\log m_1\cdots m_k}{-\log r_1\cdots r_k - \log r_{k+1}m_{k+1}}$$

$$\beta = \limsup_{k\to\infty} \frac{\log m_1\cdots m_k + \log m_{k+1}}{-\log r_1\cdots r_k + \log m_{k+1}} \quad \beta^* = \limsup_{k\to\infty} \frac{\log m_1\cdots m_k}{-\log r_1\cdots r_k}.$$

Note that $0 \le \alpha^* \le \alpha \le \beta^* \le \beta \le 1$.

Remark 2.1. If $\inf_k r_k > 0$, thus $\sup_k -\log(r_{k+1}m_{k+1}) < \infty$, thus $\alpha^* = \alpha = \beta^* = \beta$.

In [4], a net measure, comparable to Hausdorff measure, is constructed on every symmetric Cantor set C, resulting in a Hausdorff dimension formula of C. A packing dimension formula of C is proved in [5]. By symbolic representation, we know every partial Cantor set is a symmetric Cantor set. Hence the following result is a direct corollary of Theorem 2.5.

Theorem 2.2 (Dimensions of homogeneous Cantor sets). *Let $C = C((m_k), (r_k))$ and $C^* = C^*((m_k), (r_k))$. Then*

$$\dim_H C = \underline{\dim}_B C = \alpha \text{ and } \dim_P C = \overline{\dim}_B C = \beta$$

and

$$\dim_H C^* = \underline{\dim}_B C^* = \alpha^* \text{ and } \dim_P C^* = \overline{\dim}_B C^* = \beta^*.$$

Theorem 2.3 (Extreme value property). *Let $C = C((m_k), (r_k))$ and $C^* = C^*((m_k), (r_k))$. For any $E \in \mathcal{M}((m_k), (r_k))$, we have*

$$\dim_H C^* \le \dim_H E \le \dim_H C \le \dim_P C^* \le \dim_P E \le \dim_P C.$$

Proof. The major obstruction is to obtain the lower bounds. This comes with the density estimates of the measures, which often follows from the careful estimates of the intersecting number of fundamental intervals via geometrical observations.

(i) (Hausdorff)

(ii) (Packing)

□

Theorem 2.4 (Intermediate value property)**.** *For any $0 \le \alpha^* \le s \le \alpha \le \beta^* \le t \le \beta \le 1$, there exists $(m_k), (r_k)$ and $E, F \in \mathcal{M}((m_k),(r_k))$ such that*

$$\dim_H C^* = \alpha^*,\ \dim_H E = s,\ \dim_H C = \alpha$$
$$\dim_P C^* = \beta^*,\ \dim_P F = t,\ \dim_P C = \beta.$$

Proof. Existence of subset E with finite positive Hausdorff measure and F with finite positive packing measure; σ-stability of $\dim_H$ and $\dim_P$; Adjust Moran construction based on E and F (move the empty-intersection intervals to the leftmost to build partial Cantor sets with minimal dimensions). □

We present a slight non-essential generalization of [4, Theorem 1].

Theorem 2.5. *Let $E \in \mathcal{M}((m_k),(r_k))$ associated with translations $\{(t_i^{(k)})_{i=1}^{m_k}\}_{k\in\mathbb{N}}$. Denote $\ell_{\min}^{(k)}$ the minimal gap and $\ell_{\max}^{(k)}$ the maximal gap between the k-order intervals in the same $(k-1)$-order interval. Suppose there exists $M > 0$ such that $\ell_{\max}^{(k)}/\ell_{\min}^{(k)} \le M$ and $t_{m_k}^{(k)} - t_1^{(k)} + r_k \ge 1/M$ for all k. Then there exists $c > 0$ such that*

$$c \liminf_{k\to\infty} \prod_{i=1}^{k} m_i r_i^s \le \mathcal{H}^s(E) \le \liminf_{k\to\infty} \prod_{i=1}^{k} m_i r_i^s \tag{2.1}$$

where $0 \le s \le 1$. Take $c = 1/(48M^3)$ is sufficient.

Proof. The strategy is to construct a net measure comparable to Hausdorff measure. For $0 \le s \le 1$, denote $\|\mathcal{B}\|_s = \sum_{B\in\mathcal{B}} |B|^s$ and $\|\mathcal{B}\|_\infty = \sup_{B\in\mathcal{B}} |B|$ for each collection $\mathcal{B}$ of sets.

By compactness and $0 \le s \le 1$, we focus on the covering $\mathcal{B} = (B_i)$ consisting of finitely many closed disjoint intervals with endpoints in E and $B_i^\circ \cap E \ne \emptyset$. By $\mathscr{B}$ we denote the collection of all such coverings.

(i) ($\mathscr{B} \implies \mathscr{G}$) Define a collection of coverings as

$$\mathscr{G} = \left\{ (G_i)_{i=1}^N \colon G_i = \bigcup_{j\in\Lambda} A(Ij) \text{ for some } k \in \mathbb{N}, I \in \mathcal{T}_{k-1}, \Lambda \subset \{1,\dots,m_k\} \right\}.$$

Let $\mathcal{B} \in \mathscr{B}$. Fix any $B \in \mathcal{B}$. Let k be the minimal order of intervals that B° can contain. Then B° intersects at most two $(k-1)$-order intervals.

- Suppose B° intersects two intervals $A(I_1)$ and $A(I_2)$ with $I_1, I_2 \in \mathcal{T}_{k-1}$. Let α be the right endpoint of $A(I_1 m_k)$ and β be the left endpoint of $A(I_2 1)$. Set $B_1 = B \cap (-\infty, \alpha]$ and $B_2 = B \cap [\beta, \infty)$. Hence $|B|^s \ge (|B_1|^s + |B_2|^s)/2$. For $i = 1, 2$, define
$$G(B_i) = \bigcup_{J\in\mathcal{T}_k,\ A(J)\cap B_i \ne \emptyset} A(J).$$

Then $G(B_i) \in \mathscr{G}$ and $|G(B_i)| \le |B_i| + 2\prod_{i=1}^k r_i$. Without loss of generality assume B_1 contains a k-order interval, thus

$$|G(B_1)|^s \le (|B_1| + 2\prod_{i=1}^k r_i)^s \le 3^s |B_1|^s.$$

If B_2 also contains a k-order interval, then

$$|B|^s \ge \frac{1}{2}(|B_1|^s + |B_2|^s) \ge \frac{3^{-s}}{2}(|G(B_1)|^s + |G(B_2)|^s).$$

If B_2 does not contains any k-order interval. Being in E, the right endpoint of B_2 should lie in some k-order interval $A(I_2 a)$ contained in $A(I_2)$. Hence we must have $a = 1$ since otherwise $A(I_2 1)$ will be contained in B_2. Thus $|G(B_2)| \le \prod_{i=1}^k r_i \le |G(B_1)|^s$ and

$$|B|^s \ge \frac{1}{2}(|B_1|^s + |B_2|^s) \ge \frac{3^{-s}}{2}|G(B_1)|^s \ge \frac{3^{-s}}{4}(|G(B_1)|^s + |G(B_2)|^s).$$

- When B° intersects only one $(k-1)$-order interval, the same arguments gives

$$|B|^s \ge 3^{-s}|G(B)|^s.$$

Together, by replacing each $B \in \mathcal{B}$ with $G(B_1), G(B_2)$ or $G(B)$, there exists some $\mathcal{G} \in \mathscr{G}$ such that $\|\mathcal{G}\|_\infty \le 6\|\mathcal{B}\|_\infty$ and

$$\|\mathcal{B}\|_s \ge \frac{1}{12}\|\mathcal{G}\|_s. \tag{2.2}$$

(ii) ($\mathscr{G} \implies \mathscr{W}$) Let $k \in \mathbb{N}$ and $1 \le n < m_k$. Write $m_k = qn + r$ for some $0 \le r < n$ and $q \in \mathbb{N}$. For each $I \in \mathcal{T}_{k-1}$, define

$$\mathcal{W}_{k,n}(I) = \Big\{ \operatorname{co}(A(Ia) \cup A(Ib)) : j \in \{0, \dots, q\}, a = jn+1 \le m_k, b = \min\{a+n, m_k\} \Big\}$$

where $\operatorname{co}(F)$ denotes the convex hull of F. Then set

$$\mathcal{W}_{k,n} = \bigcup_{I \in \mathcal{T}_{k-1}} \mathcal{W}_{k,n}(I).$$

Let $\mathcal{G} = (G_i)_{i=1}^N \in \mathscr{G}$ for some $N \in \mathbb{N}$. Let k_1 and k_2 be the minimal and the maximal order of intervals contained in the some G_i.

Put

$$\begin{aligned} D_1 &= \{G \in \mathcal{G} \colon G \text{ is a union of } k_1\text{-order intervals}\} \\ D_2 &= \{A(I) \colon I \in \mathcal{T}_{k_1} \text{ and } A(I) \nsubseteq G \text{ for all } G \in \mathcal{G}\} \\ D &= D_1 \cup D_2. \end{aligned}$$

Then D is a covering of $\bigcup_{I \in \mathcal{T}_{k_1}} A(I)$. For $p \in D$, define

$$h(p) = \begin{cases} \dfrac{|p|^s}{\#p(k_1)} & \text{if } p \in D_1 \\[2ex] \|p \cap \mathcal{G}\|_s & \text{if } p \in D_2 \end{cases}$$

where $p(k_1)$ is the number of k_1-order intervals contained in p. If $G \notin D_1$, then by the definition of $\mathscr{G}$, we have $G \subset A(I)$ for some $I \in \mathcal{T}_{k-1}$. Since D is a finite set, suppose $h(p)$ attains minimum at $p = p_0$. Then

$$\begin{aligned}\|\mathcal{G}\|_s &= \sum_{p\in D_1} |p|^s + \sum_{p\in D_2} \|p\cap\mathcal{G}\|_s \\ &= \sum_{p\in D_1} \#p(k_1)h(p) + \sum_{p\in D_2} h(p) \\ &\geq \sum_{p\in D_1} \#p(k_1)h(p_0) + \sum_{p\in D_2} h(p_0) \\ &= m_1\cdots m_{k_1}h(p_0).\end{aligned}$$

(a) When $p_0 \in D_1$. Note that

$$\|\mathcal{G}\|_s \geq m_1\cdots m_{k_1-1}(m_{k_1}h(p_0)) = m_1\cdots m_{k_1-1}(\frac{m_{k_1}}{\#p_0(k_1)}|p_0|^s).$$

Write $n = \#p_0(k_1)$ and $m_{k_1} = qn + r$ for some $0 \leq r < n$ and $q \in \mathbb{N}$. Fix any $J \in \mathcal{T}_{k-1}$. Then

$$2\frac{m_{k_1}}{\#p_0(k_1)}|p_0|^s = 2\frac{qn+r}{n}|p_0|^s \geq (q+1)|p_0|^s \geq \frac{\|\mathcal{W}_{k_1,\#p_0(k_1)}(J)\|_s}{M^s}.$$

Hence

$$\|\mathcal{G}\|_s \geq \frac{1}{2M}\|\mathcal{W}_{k_1,\#p_0(k_1)}(J)\|_s. \tag{2.3}$$

(b) When $p_0 \in D_2$. Define

$$\mathcal{G}^* = \{p_0\cap\mathcal{G} + t\colon t = \min A(I) - \min p_0,\ I \in \mathcal{T}_{k_1}\}.$$

Then $\mathcal{G}^* \in \mathscr{G}$ and the minimal order of $\mathcal{G}^*$ is at least $k_1 + 1$. Moreover,

$$\|\mathcal{G}\|_s \geq m_1\cdots m_{k_1}h(p_0) = \|\mathcal{G}^*\|_s. \tag{2.4}$$

Since (b) increases the minimal order at least 1 while k_2 is finite, by (2.3) and (2.4) we have

$$\|\mathcal{G}\|_s \geq \frac{1}{2M}\|\mathcal{W}_{k^*,n}\|_s \tag{2.5}$$

for some $k_1 \leq k^* \leq k_2$ and $1 \leq n < m_{k^*}$.

(iii) ($\mathcal{W} \implies \mathcal{F}$) For $k \in \mathbb{N}$, define $\mathcal{F}_k = (A(I))_{I\in\mathcal{T}_k}$. Fix $k \in \mathbb{N}$ and let $1 \leq n < m_k$.

- If $n = 1$, then $\|\mathcal{W}_{k,n}\|_s = \|\mathcal{F}_k\|_s$.
- Suppose $n \geq 2$. Fix any $I \in \mathcal{T}_{k-1}$. Let $\mathcal{L}$ be the collection of gaps between intervals in $\mathcal{W}_{k,n}(I)$. Then $\#\mathcal{L} \leq q \leq \#\mathcal{W}_{k,n}(I) \leq q + 1$. For each $L \in \mathcal{L}$ and W being one of the first q intervals in $\mathcal{W}_{k,n}(I)$, we have $|L|^s \leq M|W|^s$. By the concavity of $y = x^s, 0 \leq s \leq 1$,

$$((1/M)|A(I)|)^s \leq \sum_{W\in\mathcal{W}_{k,n}(I)} |W|^s + \sum_{L\in\mathcal{L}}|L|^s \leq (M+1)\|\mathcal{W}_{k,n}(I)\|_s.$$

Hence $\|\mathcal{W}_{k,n}(I)\|_s \geq |A(I)|^s/(M(M+1))$ and

$$\|\mathcal{W}_{k,n}\|_s \geq \frac{1}{M(M+1)}\|\mathcal{F}_{k-1}\|_s.$$

Together, for all $1 \le n < m_k$, there exists some $k' \in \{k, k-1\}$ such that

$$\|\mathcal{W}_{k,n}\|_s \ge \frac{1}{2M^2} \|\mathcal{F}_{k'}\|_s. \tag{2.6}$$

Finally, it follows from (2.2), (2.5) and (2.6) that for each $\mathcal{B} \in \mathscr{B}$ with $\|\mathcal{B}\|_\infty \le \delta$, there exists $k \in \mathbb{N}$ such that

$$\|\mathcal{B}\|_s \ge \frac{1}{48M^3} \|\mathcal{F}_k\|_s.$$

Since $\mathcal{B}$ is arbitrary and let $\delta \to 0$, we have $\mathcal{H}^s(E) \ge \frac{1}{48M^3} \liminf_{k\to\infty} \prod_{i=1}^k m_i r_i^s$. Since each $\mathcal{F}_k$ is a covering of E, the other inequality of (2.1) also holds. □

2.2 Sets defined by digit restrictions

Some sets defined by b-ary expansions are homogeneous Moran sets satisfying the condition in Remark 2.1. Hence Theorem 2.2 and Theorem 2.3 lead to the following examples.

For $S \subset \mathbb{N}$, denote the lower and upper densities of S by $\underline{d}(S)$ and $\overline{d}(S)$ respectively. Let $|E|$ be the cardinality of any finite set E.

Example 2.6 ([1, Example 1.3.2])**.** Let $S \subset \mathbb{N}$. Define

$$A_S = \left\{ \sum_{n\in S} x_n 2^{-n} \colon x_n \in \{0,1\} \right\}.$$

Then

$$\dim_H A_S = \underline{\dim}_B A_S = \underline{d}(S) \quad \text{and} \quad \dim_P A_S = \overline{\dim}_B A_S = \overline{d}(S).$$

Example 2.7 ([1, Exercise 1.38])**.** Fix any $b \in \mathbb{N}_{\ge 2}$. Let $E, F \subset \{0, \dots, b-1\}$ and $S \subset \mathbb{N}$. Denote by (x_n) the b-ary expansion for each $x \in [0,1]$. Define

$$A_{E,F,S} = \left\{ \sum_{n=1}^{\infty} x_n b^{-n} \colon x_n \in \begin{Bmatrix} E & n \in S \\ F & n \notin S \end{Bmatrix} \right\}.$$

Then

$$\dim_H A_{E,F,S} = \begin{cases} \frac{\underline{d}(S)\log(|E|/|F|)+\log|F|}{\log b} & \text{if } |E| \ge |F|; \\ \frac{\overline{d}(S)\log(|E|/|F|)+\log|F|}{\log b} & \text{if } |E| < |F|. \end{cases}$$

Example 2.8 ([1, Exercise 1.39])**.** Fix any $b \in \mathbb{N}_{\ge 2}$. Let $(s_n) \in \{0, \dots, b-1\}^{\mathbb{N}}$ and $E_i \subset \{0, \dots, b-1\}$ for $0 \le i \le b-1$. Then

$$\dim_H \left\{ \sum_{n=1}^{\infty} x_n b^{-n} \colon x_n \in E_{s_n} \right\} = \liminf_{n\to\infty} \frac{\sum_{i=1}^n \log|E_{s_i}|}{n \log b}.$$

2.3 Random covering sets

3 Multifractal analysis

We record some examples to show how we can apply Moran structures during multifractal analysis in [2, 3].

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