

Typical self-affine sets with non-empty interior

Zhou Feng

The Chinese University of Hong Kong

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Joint work with De-Jun Feng

- Let T_1, \dots, T_m be $d \times d$ invertible real matrices with $\|T_j\| < 1/2$ for $1 \leq j \leq m$. Write $\mathbf{T} := (T_1, \dots, T_m)$.
- For $\mathbf{a} = (a_1, \dots, a_m) \in \mathbb{R}^{dm}$, we consider the **affine iterated function system**,

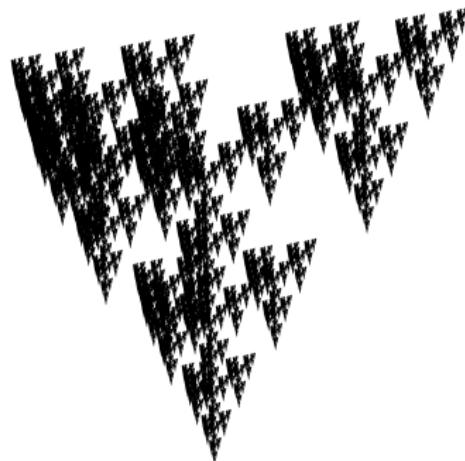
$$\{f_j^{\mathbf{a}}(x) = T_j x + a_j\}_{j=1}^m.$$

- It is well known that there is a unique non-empty compact set $K^{\mathbf{a}}$, called **self-affine set**, such that

$$K^{\mathbf{a}} = \bigcup_{j=1}^m f_j^{\mathbf{a}}(K^{\mathbf{a}}).$$

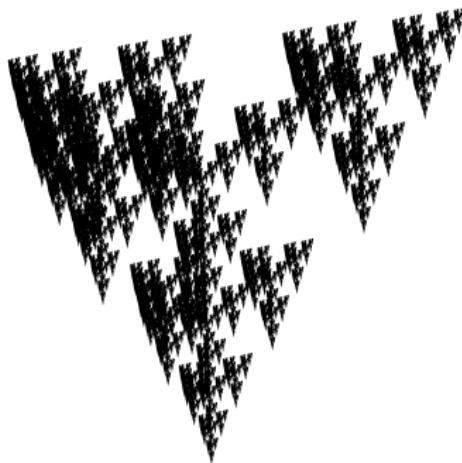
- In this talk, we fix the **linear part** \mathbf{T} and study $K^{\mathbf{a}}$ with the **translations** \mathbf{a} changing.

For example, some self-affine set K^a looks like



Research target

For example, some self-affine set $K^{\mathbf{a}}$ looks like



Goal

To provide some sufficient conditions on \mathbf{T} such that $K^{\mathbf{a}}$ has non-empty interior for \mathcal{L}^{dm} -a.e. (typical) \mathbf{a} .

Motivation

In 1988, Falconer introduced a quantity $\dim_{\text{AFF}} \mathbf{T}$ called **affinity dimension** which only depends on the **linear part** \mathbf{T} .

Classical results

- (Falconer, 1988; Solomyak, 1998) For \mathcal{L}^{dm} -a.e. \mathbf{a} ,

$$\dim_H K^\mathbf{a} = \dim_B K^\mathbf{a} = \min \{d, \dim_{\text{AFF}} \mathbf{T}\}.$$

- (Jordan, Pollicott, and Simon, 2007) If $\dim_{\text{AFF}} \mathbf{T} > d$, then $\mathcal{L}^d(K^\mathbf{a}) > 0$ for \mathcal{L}^{dm} -a.e. \mathbf{a} .

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Question

How about the **interior** of typical $K^\mathbf{a}$?

Although this seems a rather fundamental question, it has hardly been studied.

Main results: general case

To state the result, we define

$$\gamma(\mathbf{T}) = \inf \left\{ \gamma \geq 0 : \sup_{n \geq 1} \sum_{|I|=n} \alpha_d(T_I)^\gamma |\det T_I| \leq 1 \right\}.$$

where $\alpha_d(T)$ denotes the smallest singular value of a matrix T and $T_I = T_{i_1} \cdots T_{i_n}$ for $I = i_1 \dots i_n \in \{1, \dots, m\}^n$.

In short, $\gamma(\mathbf{T})$ is a quantity only depending on \mathbf{T} .

Theorem A

If $\gamma(\mathbf{T}) > d$, then $K^{\mathbf{a}}$ has non-empty interior for \mathcal{L}^{dm} -a.e. \mathbf{a} .

The idea for proving Theorem A

- By a classical result (see e.g. (Mattila, 2015)), it suffices to find measures $\mu^{\mathbf{a}}$ supported on $K^{\mathbf{a}}$ such that the Fourier transform $\widehat{\mu^{\mathbf{a}}}$ satisfies

$$\int_{B_\rho} \int_{\mathbb{R}^d} |\widehat{\mu^{\mathbf{a}}}(\xi)|^2 |\xi|^\gamma d\xi d\mathbf{a} < \infty \quad \text{for some } \gamma > d.$$

- By the transversality arguments of Falconer and Solomyak, and some key inequalities, the problem is reduced to finding some measure on $\{1, \dots, m\}^{\mathbb{N}}$ with enough regularity.
- The condition $\gamma(\mathbf{T}) > d$ is discovered and provides such a regular measure.

Theorem B

Suppose $T_i T_j = T_j T_i$ for $1 \leq i, j \leq m$. If $\sum_{j=1}^m |\det T_j|^2 > 1$, then K^a has non-empty interior for \mathcal{L}^{dm} -a.e. a .

Main results: commutative case

Theorem B

Suppose $T_i T_j = T_j T_i$ for $1 \leq i, j \leq m$. If $\sum_{j=1}^m |\det T_j|^2 > 1$, then K^a has non-empty interior for \mathcal{L}^{dm} -a.e. a .

For simplicity, we show [the idea of the proof](#) for the homogeneous case where

$$\mathbf{T} = (T, \dots, T).$$

- By symbolic expression,

$$K^a = E^a + TE^a,$$

where E^a is the self-affine set generated by $\{T^2 x + a_j\}_{j=1}^m$.

- Then $m \cdot |\det T|^2 > 1$ implies typically $\mathcal{L}^d(E^a) > 0$ (Jordan, Pollicott, and Simon, 2007).
- The proof is finished by [Steinhaus theorem](#).

An open question

By definition,

$$(1) \quad \gamma(\mathbf{T}) > d \iff \exists n \text{ such that } \sum_{|I|=n} \alpha_d(T_I)^d |\det T_I| > 1$$

$$(2) \quad \dim_{\text{AFF}} \mathbf{T} > 2d \iff \sum_{j=1}^m |\det T_j|^2 > 1$$

$$(3) \quad \dim_{\text{AFF}} \mathbf{T} > d \iff \sum_{j=1}^m |\det T_j| > 1.$$

Recall that (1) and (2) are respectively assumed in Theorems A and B. And (3) implies typically $\mathcal{L}^d(K^a) > 0$.

Note (1) \Rightarrow (2) \Rightarrow (3).

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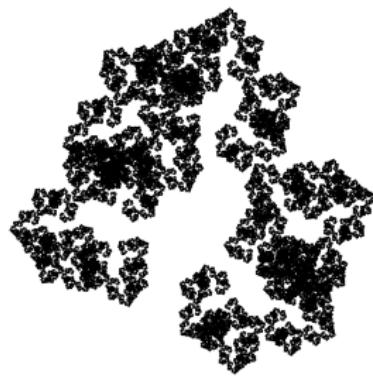
Recall that (1) and (2) are respectively assumed in Theorems A and B. And (3) implies typically $\mathcal{L}^d(K^a) > 0$.

Note (1) \Rightarrow (2) \Rightarrow (3).

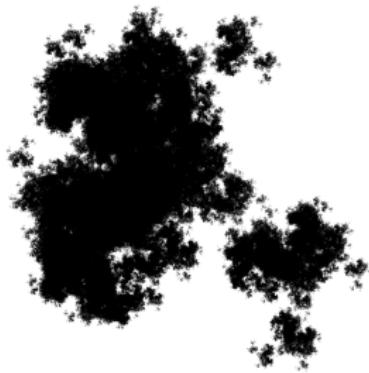
Open question

Does typical K^a have non-empty interior under the condition (3)?

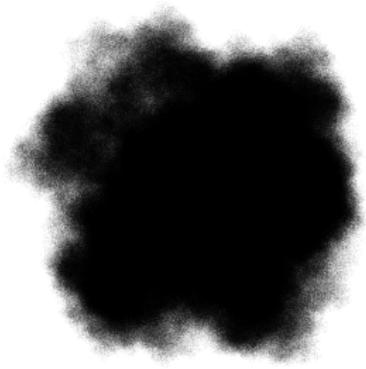
Some numerical experiments



$\dim_{\text{AFF}} \mathbf{T} < 2$



(3) satisfied



(1) & (2) satisfied

Thank you for listening!



Cheers for fractal!