Dimensional properties of self-similar and self-affine fractals

Zhou Feng

The Chinese University of Hong Kong

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Joint work with De-Jun Feng

What are fractals?

- Fractals: geometric objects with 'self-similarity' at arbitrarily small scales.
- Fractals are everywhere.

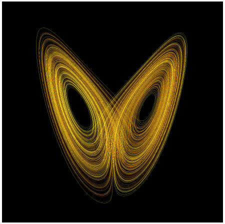
Romanesco broccoli

Islamic geometric patterns

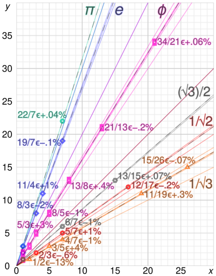
DJIA index

Table: Some fractals in nature, art and society.[1]

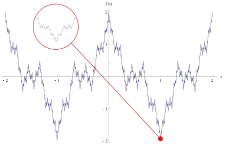
- Applications: art design, computer graphics, data compression, modeling real-world phenomenons...
- Theoretical research: dynamical systems, number theory, harmonic analysis, probability theory...



Lorenz attractor



Numbers defined by Diophantine properties



Graphs of Weierstrauss functions

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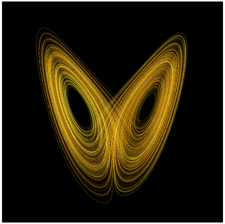

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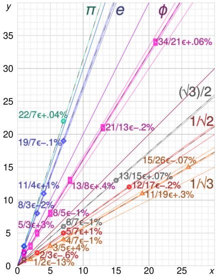

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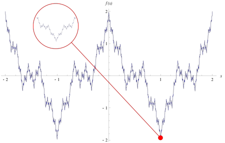

Graphs of Weierstrauss functions

It is important to understand the geometric properties of fractals both in practice and theory.

Dimension theory

- Fractal dimensions are the most important notions to study the 'size' of fractals in a rigorous way.
- A dimension is a number (usually not an integer) concerning how the fractal fills up space in small scales.

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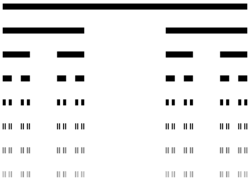
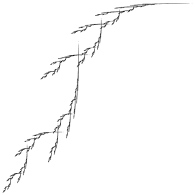
- Fractal dimensions are the most important notions to study the 'size' of fractals in a rigorous way.
- A dimension is a number (usually not an integer) concerning how the fractal fills up space in small scales.
- There are many different notions of dimensions including:
  - Hausdorff dimension $\dim_{\mathrm{H}}$;
  - box dimension $\dim_{\mathrm{B}}$.
- For the middle-third Cantor set C,

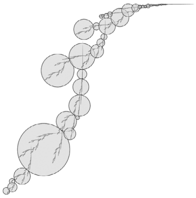


$$\dim_{\mathrm{H}} C = \dim_{\mathrm{B}} C = \frac{\log 2}{\log 3}.$$

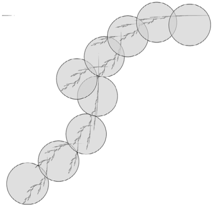
Hausdorff and box dimensions by covers:[2]



A self-affine set F



A cover $\mathcal{E}_r$ for $\dim_{\mathrm{H}} F$ by sets with diameters $\leq r$



A cover $\mathcal{E}_r$ for $\dim_{\mathrm{B}} F$ by sets with diameters $= r$

Hausdorff and box dimensions of a set F are respectively the infimum of the exponents s such that

$$\forall\, \varepsilon > 0,\ \exists\, r \to 0 \text{ and allowable cover } \mathcal{E}_r \text{ with } \sum_{E \in \mathcal{E}_r} |E|^s < \varepsilon,$$

where $|\cdot|$ denotes the diameter in Euclidean norm. Hence

$$0 \leq \dim_{\mathrm{H}} F \leq \dim_{\mathrm{B}} F \leq d \quad \text{for } F \subset \mathbb{R}^d.$$

[2]Figures by Fraser.

- By iterated function system (IFS) we mean a finite family of contracting affine maps

$$\mathcal{F} = \{f_j(x) = T_j x + a_j\}_{j=1}^m,$$

where T_j are invertible matrices on $\mathbb{R}^d$ with $\|T_j\| < 1$, and $a_j \in \mathbb{R}^d$.

- The self-affine set K is the unique nonempty compact set such that

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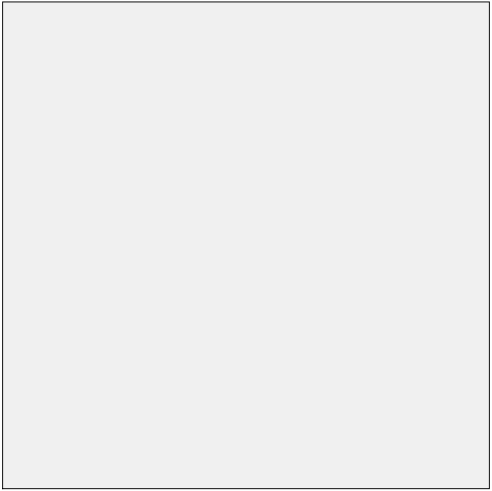
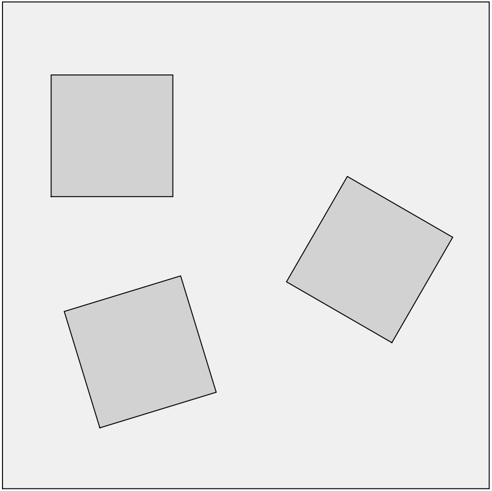
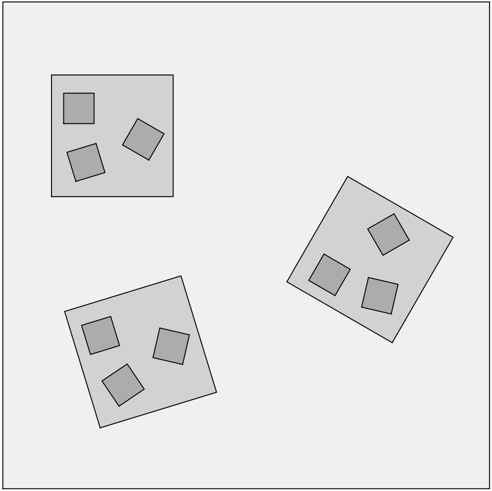
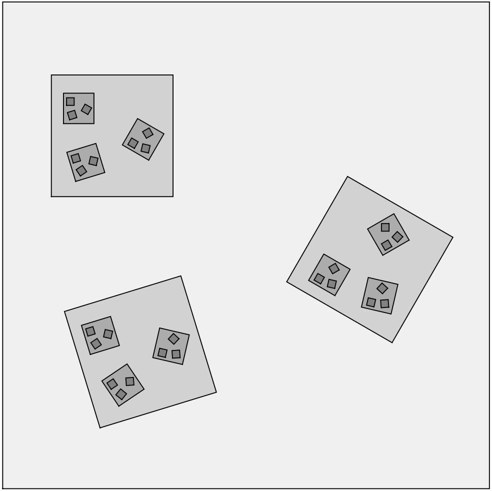
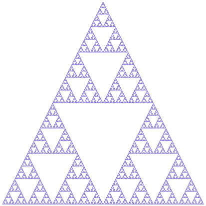
- If f_j are similarities, we call K a self-similar set and μ a self-similar measure.



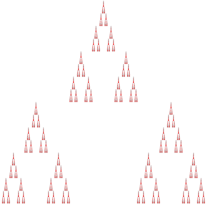




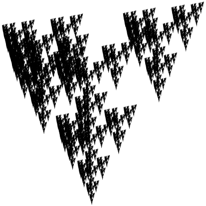




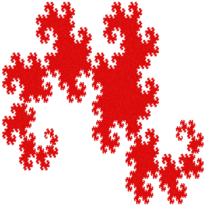
Sierpiński triangle



Bedford-McMullen carpet

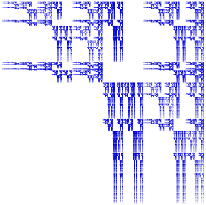


Typical self-affine set



Heighway dragon

Barnsley fern



Barański carpet

Table: Some examples of self-similar and self-affine sets[3].

[3]Figures by the scripts at GitHub repo zfengg/PlotIFS.jl under MIT License.

The goal of this talk

To briefly explain our results about the dimensional properties of self-similar and self-affine sets and measures.

Estimates on the dimension of self-similar measures with overlaps

For a self-similar measure μ, the dimension of μ is

$$\dim \mu = \lim_{r \to 0} \frac{\log \mu(B(x,r))}{\log r} \quad \text{for } \mu\text{-a.e. } x.$$

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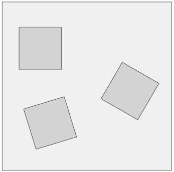
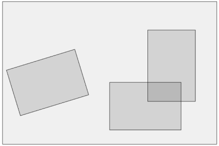


no overlaps



overlapping

- With sufficient separation conditions, $\dim \mu$ is relatively well understood. There is good deal of work in this direction.
- It is central and challenging to determine the dimension of overlapping self-similar measures.

Lower bounds on dim μ

- We developed tools from dynamical systems and presented a method to estimate the dimension of overlapping self-similar measures from below.

Lower bounds on $\dim \mu$

- We developed tools from dynamical systems and presented a method to estimate the dimension of overlapping self-similar measures from below.
- A uniform lower bound on the dimension of Bernoulli convolutions μ_β which are self-similar measures associated with

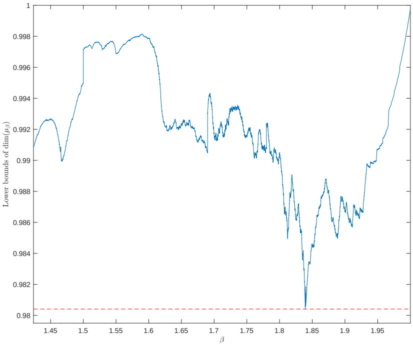
$$\{x/\beta \pm 1\},\ \beta \in (1,2) \quad \text{and} \quad (1/2, 1/2).$$

- Bernoulli convolutions have been studied since 1930s. (Erdős, Kahane, Garsia,...)

Theorem (Feng and F, 2022)

$\dim \mu_\beta \geq 0.98040856$ *for all* $\beta \in (1,2)$.

- (Hare and Sidorov, 2018) 0.82.
- (Kleptsyn, Pollicott, and Vytnova, 2022) 0.96399 through a different approach.



For $\beta_3 \approx 1.839$ called the tribonacci number $(x^3 - x^2 - x - 1 = 0)$,

$$\dim \mu_{\beta_3} = 0.98040931953 \pm 10^{-11}.$$

Recall our uniform lower bound: 0.98040856.

Conjecture (Feng and F, 2022)

$\dim \mu_{\beta_3} = \inf_{\beta \in (1,2)} \dim \mu_\beta.$

Upper bounds on $\dim \mu_\beta$ when β is Pisot

- When β is a Pisot number, e.g., the golden ratio $(\sqrt{5}+1)/2$,
- we establish a new relation between $\dim \mu_\beta$ and the entropy rate $h(\eta)$ of some equilibrium state η,

$$\dim \mu_\beta = \frac{h(\eta)}{\log \beta}.$$

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- The distribution of η is given by some products of matrices.
- Computable upper bounds on $\dim \mu_\beta$ in terms of the conditional entropies of η.

Example (Feng and F, 2022)

$\dim \mu_\alpha = 0.999995036 \pm 10^{-9}$, where $\alpha \approx 1.325$ is the largest root of $x^3 - x - 1 = 0$ (the smallest Pisot number called plastic constant).

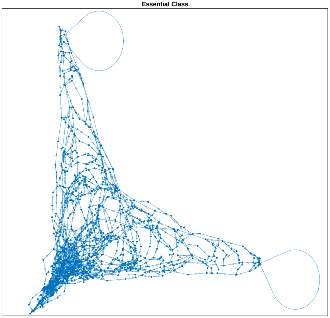

Figure: The directed graph for η in the previous example.

Dimension of homogeneous IFS with algebraic translations

- Let μ be a self-similar measure associated with

$$\mathcal{F} = \{f_j(x) = \lambda_j x + t_j\}_{j=1}^m \quad \text{and} \quad p = (p_j)_{j=1}^m.$$

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- A natural upper bound:

$$\dim \mu \leq \min \left\{1, \frac{H(p)}{\chi}\right\},$$

where $H(p)$ is the entropy and χ is the Lyapunov exponent.
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- The equality holds under sufficient separation conditions.
- We say $\mathcal{F}$ has exact overlaps if $f_{i_1} \circ \cdots \circ f_{i_n} = f_{j_1} \circ \cdots \circ f_{j_n}$ for some distinct words $i_1 \ldots i_n \neq j_1 \ldots j_n \in \{1, \ldots, m\}^n$.

Conjecture (exact overlaps conjecture)

If $\dim \mu < \min\{1, H(p)/\chi\}$*, then* $\mathcal{F}$ *has exact overlaps.*

A version for self-similar sets was due to (Simon, 1996).

Recall the IFS $\{\lambda_j x + t_j\}_{j=1}^m$.

Significant progress in recent years:

- (Hochman, 2014) all λ_j and t_j are algebraic numbers.
- (Rapaport, 2022) only λ_j are algebraic numbers.
- (Varjú, 2019) $m = 2$ and $\lambda_j = \lambda \in (0, 1)$. (Bernoulli convolutions)
- (Rapaport and Varjú, 2024) $\lambda_j = \lambda \in (0, 1)$ and t_j are rational numbers.

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Theorem (Feng and F, 2024)

Suppose $\lambda_j = \lambda \in (-1, 1) \setminus \{0\}$ and t_j are algebraic numbers. If $\mathcal{F}$ has no exact overlaps, then

$$\dim \mu = \min \left\{ 1, \frac{\sum_{j=1}^m p_j \log p_j}{\log|\lambda|} \right\}.$$

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The exact overlaps conjecture is still open!

Typical self-affine sets with nonempty interior

- Fix the linear part $\mathbf{T} = (T_1, \ldots, T_m)$ with $\|T_j\| < 1/2$.
- Consider self-affine sets $K^{\mathbf{a}}$ depending on translations $\mathbf{a} \in \mathbb{R}^{dm}$.
- In 1988, Falconer introduced the affinity dimension, $\dim_{\mathrm{AFF}} \mathbf{T}$.

Classical results

- (Falconer, 1988; Solomyak, 1998) For $\mathcal{L}^{dm}$-a.e. $\mathbf{a}$,

$$\dim_{\mathrm{H}} K^{\mathbf{a}} = \dim_{\mathrm{B}} K^{\mathbf{a}} = \min\left\{d, \dim_{\mathrm{AFF}} \mathbf{T}\right\}.$$

- (Jordan, Pollicott, and Simon, 2007) If $\dim_{\mathrm{AFF}} \mathbf{T} > d$, then $\mathcal{L}^d(K^{\mathbf{a}}) > 0$ for $\mathcal{L}^{dm}$-a.e. $\mathbf{a}$.

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Question

How about the interior of typical $K^{\mathbf{a}}$?

Although this seems a rather fundamental question, it has hardly been studied.

We introduce a quantity

$$\gamma(\mathbf{T}) = \inf \left\{ \gamma \geq 0 \colon \sup_{n \geq 1} \sum_{|I|=n} \alpha_d(T_I)^{\gamma} |\det T_I| \leq 1 \right\},$$

which only depends on $\mathbf{T}$.

Theorem (Feng and F, 2023)

If $\gamma(\mathbf{T}) > d$, then $K^{\mathbf{a}}$ has nonempty interior for almost all $\mathbf{a}$.

Fourier transform + Self-affine transversality.

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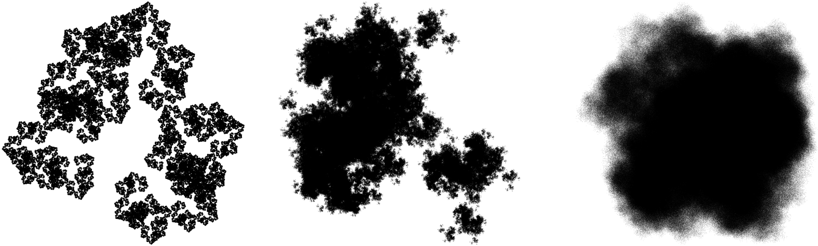
Fourier transform + Self-affine transversality.

Theorem (Feng and F, 2023)

Suppose $T_i T_j = T_j T_i$ for $1 \leq i, j \leq m$. If $\sum_{j=1}^{m} |\det T_j|^2 > 1$, then $K^{\mathbf{a}}$ has nonempty interior for almost all $\mathbf{a}$.

Symbolic representation + Steinhaus Theorem.



$\dim_{\mathrm{AFF}} \mathbf{T} < 2$ $\dim_{\mathrm{AFF}} \mathbf{T} > 2$ Our conditions satisfied

Table: Some numerical experiments and an open question.

- Intermediate dimensions were introduced by (Falconer, Fraser, and Kempton, 2020) to interpolate the Hausdorff and box dimensions.
- Impose $r^{1/\theta} \leq |E| \leq r$ $(0 \leq \theta \leq 1)$ on the allowable covers.
- Intermediate dimensions have seen interesting applications.

Intermediate dimensions of subsets of typical self-affine sets

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- Impose $r^{1/\theta} \leq |E| \leq r$ $(0 \leq \theta \leq 1)$ on the allowable covers.
- Intermediate dimensions have seen interesting applications.

Theorem (F, 2023)

Let $\pi^{\mathbf{a}} : \Sigma \to \mathbb{R}^d$ the natural coding and $E \subset \Sigma$. For $0 < \theta \leq 1$ and $\mathcal{L}$-a.e. $a \in \mathbb{R}^{md}$, the θ-intermediate dimensions of $\pi^{\mathbf{a}}(E) \subset K^{\mathbf{a}}$ are constants given by the capacity dimensions.

Results extended to the generalized Φ-intermediate dimensions introduced by (Banaji, 2023) in several other settings.

Replace $r^{1/\theta} \leq |E| \leq r$ with $\Phi(r) \leq |E| \leq r$.

Theorem (F, 2023)

Suppose $\lim_{r\to 0} r^{\varepsilon} \log \Phi(r) = 0$ *for all* $\varepsilon > 0$. *Then* *Φ-intermediate dimensions of*

- *almost all orthogonal projections of a set in Euclidean space;*
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These settings share a phenomenon called transversality.

Dimension coincidence for some affine-invariant sets

In specific nonconformal dynamical systems, invariant sets usually have distinct Hausdorff and box dimensions.

Theorem (F, 2024)

Let K be a compact subset of $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ invariant under the endomorphism $(x, y) \mapsto (nx, my) \mod 1$ $(n > m \geq 2)$. Suppose the symbolic coding of K satisfies weak specification. The following statements are equivalent.

- (A) $\dim_{\mathrm{H}} K = \dim_{\mathrm{B}} K$.
- (B) $0 < \mathcal{H}^{\varphi}(K) < \infty$ *for some gauge function* φ.
- (C) $\dim \mu = \dim_{\mathrm{H}} K$, *where* μ *is the measure of maximal entropy on* K.

When $d \geq 3$, we find examples in which (A) does not hold but (C) holds, which is a new phenomenon not appearing in the planar cases.

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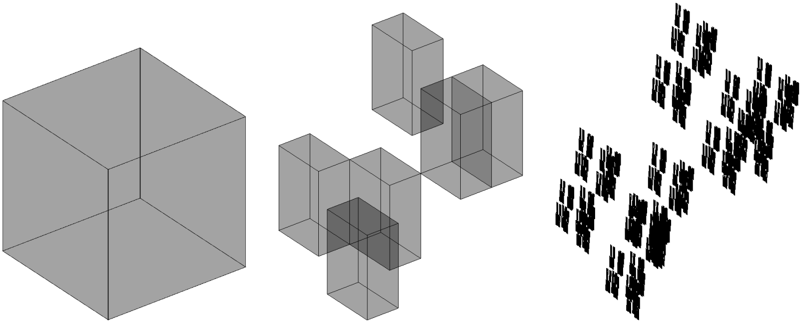
Theorem (F, 2024)

(A) and (B) are equivalent for Bedford-McMullen sponges.



Initial cube | First iteration | The self-affine set

Table: A Bedford-McMullen sponge[4].

[4]Figures generated by the scripts at GitHub repo zfengg/SelfAffine.

Conclusion

- Fractals are everywhere. In fractal geometry, we study the 'size' of fractals and how they behave under geometric operations.
- Self-similar and self-affine sets and measures are an important class of dynamically-driven fractals.
- We have investigated the dimensional properties of these fractals from different perspectives: numerically, theoretically, typically, and specifically.

Thank you for listening!

A fractal in the sky! (Helsinki, Jun 2023)