Robust Clearing Price Mechanisms for Reserve Price Optimization
Supplementary Material

C OMITTED PROOFS FROM SECTION 3

C.1 Proof of Proposition 3.1

PROOF. Taking the derivative of $\mathbb{E}_v\left[\ell^c(p, \beta(v); \lambda)\right]$ w.r.t p, we have

$$\begin{aligned}
& \nabla_p \mathbb{E}_v[\ell^c(p, \beta(v); \lambda)] \\
= \;& \mathbb{E}_v\left[\nabla_p \ell^c(p, \beta(v); \lambda)\right] = \mathbb{E}_v\left[\lambda - \sum_{i=1}^{n} \mathbb{I}\{\beta_i(v_i) \geq p\}\right] \\
= \;& \lambda - \mathbb{E}_v\left[\sum_{i=1}^{n} \mathbb{I}\{\beta_i(v_i) \geq p\}\right] = \lambda - \sum_{i=1}^{n}(1 - D_i(\beta_i^{-1}(p))) \\
= \;& \sum_{i=1}^{n} D_i(\beta_i^{-1}(p)) - (n - \lambda)
\end{aligned}$$

Setting the gradient to be zero and by the first order condition, we complete the proof. □

C.2 Proof of Theorem 3.3

PROOF. We first rewrite the expected utility function $\mathbb{E}_{v_i}[\hat{u}_{i,2}(\alpha; v)]$ in the following way,

$$\begin{aligned}
& \mathbb{E}_v[\hat{u}_{i,2}(\alpha; v)] \\
= \;& \mathbb{E}_v\left[\mathbb{E}_z\left[\mathbb{I}\{v_i \geq \max_{j \neq i}\{v_j, p^*(\alpha) + z\}\} \cdot (v_i - \max_{j \neq i}\{v_j, p^*(\alpha) + z\})\right]\right] \\
= \;& \mathbb{E}_{v: v_i \geq m_i}\Big[\int_{m_i - p^*(\alpha)}^{v_i - p^*(\alpha)} (v_i - p^*(\alpha) - z) f(z) dz + \int_{-\infty}^{m_i - p^*(\alpha)} (v_i - m_i) f(z) dz\Big] \\
= \;& \mathbb{E}_{v: v_i \geq m_i}\left[(v_i - p^*(\alpha))(F(v_i - p^*(\alpha)) - F(m_i - p^*(\alpha))) - \int_{m_i - p^*(\alpha)}^{v_i - p^*(\alpha)} z f(z) dz + (v_i - m_i) F(m_i - p^*(\alpha))\right] \\
= \;& \mathbb{E}_{v: v_i \geq m_i}\left[(v_i - p^*(\alpha)) F(v_i - p^*(\alpha)) - (m_i - p^*(\alpha)) F(m_i - p^*(\alpha)) - \int_{m_i - p^*(\alpha)}^{v_i - p^*(\alpha)} z f(z) dz\right]
\end{aligned}$$

For notation simplicity, we denote

$$T(v_i, m_i, \alpha) = \left[(v_i - p^*(\alpha)) F(v_i - p^*(\alpha)) - (m_i - p^*(\alpha)) F(m_i - p^*(\alpha)) - \int_{m_i - p^*(\alpha)}^{v_i - p^*(\alpha)} z f(z) dz\right] \cdot g_i(m_i)$$

By Leibniz integral rule and integral by part, we compute the quantity

$$\lim_{\alpha \to 0} \frac{\mathbb{E}_v[\hat{u}_{i,2}(1 + \alpha; v)] - \mathbb{E}_v[\hat{u}_{i,2}(1 - \alpha; v)]}{2\alpha}$$

as follows,

$$
\begin{aligned}
& \lim_{\alpha\to 0}\frac{\mathbb{E}_v[\hat{u}_{i,2}(1+\alpha;v)]-\mathbb{E}_v[\hat{u}_{i,2}(1-\alpha;v)]}{2\alpha} \\
= & \left.\frac{\partial\mathbb{E}_{v_i}\left[\int_0^{v_i} T(v_i,m_i,\alpha)d\,m_i\right]}{\partial\alpha}\right|_{\alpha=1} \\
= & \mathbb{E}_{v_i}\left[v_i\cdot T(v_i,v_i,\alpha)+\int_0^{v_i}\frac{\partial T(v_i,m_i,\alpha)}{\partial\alpha}d\,m_i\Big|_{\alpha=1}\right] \\
= & \mathbb{E}_{v_i}\left[0+\int_0^{v_i}\left(F(v_i-p^*(\alpha))\cdot\frac{\partial[v_i-p^*(\alpha)]}{\partial\alpha}-F(m_i-p^*(\alpha))\cdot\frac{\partial[m_i-p^*(\alpha)]}{\partial\alpha}\right)g_i(m_i)d\,m_i\Big|_{\alpha=1}\right] \\
& \text{(By the fact that } T(v_i,v_i,\alpha)=0) \\
= & \mathbb{E}_{v_i}\left[0+\int_0^{\alpha v_i}\left(F(v_i-p^*(\alpha))\cdot\left(-\frac{\partial p^*(\alpha)}{\partial\alpha}\right)+F(m_i-p^*(\alpha))\cdot\frac{\partial p^*(\alpha)}{\partial\alpha}\right)g_i(m_i)d\,m_i\Big|_{\alpha=1}\right] \\
= & \mathbb{E}_{v_i}\left[\int_0^{v_i}\left[\left(F(m_i-p^*(1))-F(v_i-p^*(1))\right)\cdot\eta\right]g_i(m_i)d\,m_i\right]
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
& \mathtt{DIC}_i \\
= & \lim_{\alpha\to 0}\frac{\mathbb{E}_v[\hat{u}_{i,2}(1+\alpha;v)]-\mathbb{E}_v[\hat{u}_{i,2}(1-\alpha;v)]}{2\alpha\mathbb{E}_{v_i}[v_i x_i(v_i)]} \\
= & -\frac{\eta\cdot\mathbb{E}_{v_i}\left[\int_0^{v_i}(F(v_i-p^*(1))-F(m_i-p^*(1)))g_i(m_i)d\,m_i\right]}{\mathbb{E}_v[v_i x_{i,1}(v)]} \\
= & -\eta\cdot\frac{\mathbb{E}_{v_i}\left[(F(v_i-p^*(1))G_i(v_i)-\int_0^{v_i}F(m_i-p^*(1)))d\,G_i(m_i)\right]}{\mathbb{E}_v[v_i x_{i,1}(v)]} \\
= & -\eta\cdot\frac{\mathbb{E}_{v_i}\left[F(v_i-p^*(1))G_i(v_i)-F(m_i-p^*(1))G_i(m_i)\Big|_0^{v_i}\right]}{\mathbb{E}_v[v_i x_{i,1}(v)]}-\eta\cdot\frac{\mathbb{E}_{v_i}\left[\int_0^{v_i}G_i(m_i)f(m_i-p^*(1))d\,m_i\right]}{\mathbb{E}_v[v_i x_{i,1}(v)]} \\
& \text{(By intergral by part)} \\
= & -\eta\cdot\frac{\mathbb{E}_{v_i}\left[\int_0^{v_i}G_i(m_i)f(m_i-p^*(1))d\,m_i\right]}{\mathbb{E}_{v_i}[v_i\cdot G_i(v_i)]}
\end{aligned}
$$

where the last equality above holds because

$$x_{i,1}(v)=\mathbb{P}(m_i\le v_i)=G_i(v_i).$$

□

C.3 Proof of Proposition 3.4

PROOF. Given the definition of $p^*(\alpha)$ and Proposition 3.1, we have $D_i\left(\frac{p^*(\alpha)}{\alpha}\right)+\sum_{j\neq i}D_j(p^*(\alpha))=n-\lambda$. Taking gradient w.r.t α of the both sides in the above equation, we have

$$\frac{\partial p^*(\alpha)}{\partial\alpha}=\frac{p^*(\alpha)D_i'\left(\frac{p^*(\alpha)}{\alpha}\right)}{\alpha D_i'\left(\frac{p^*(\alpha)}{\alpha}\right)+\alpha^2\sum_{j\neq i}D_j'(p^*(\alpha))}.$$

Therefore, $\eta = \frac{p^*(1)D_i'(p^*(1))}{D_i'(p^*(1))+\sum_{j\neq i} D_j'(p^*(1))}$ □

D OMITTED PROOFS FROM SECTION 4

D.1 Proof of Proposition 4.1

PROOF.

$$\begin{aligned}
&\mathbb{E}_{v\sim D, z\sim F}[\ell^c(p,\beta(v)+z;\lambda)]\\
=\;& \sum_{i=1}^n \int_0^1 \int_{p-\beta_i(v)}^{\infty} (\beta_i(v)+z-p)dF(z)dD_i(v)+\lambda p\\
=\;& \sum_{i=1}^n \int_0^1 (\beta_i(v)-p)\cdot(1-F(p-\beta_i(v)))dD_i(v) + \sum_{i=1}^n \int_0^1 \int_{p-\beta_i(v)}^{\infty} zdF(z)dD_i(v)+\lambda p
\end{aligned}$$

Taking the partial gradient of the above formula w.r.t p, we have

$$\begin{aligned}
&\frac{\partial \mathbb{E}_{v,z}[\ell^c(p,\beta(v)+z;\lambda)]}{\partial p}\\
=\;& -\sum_{i=1}^n \int_0^1 (1-F(p-\beta_i(v)))dD_i(v)+\lambda\\
=\;& \sum_{i=1}^n \int_0^1 F(p-\beta_i(v))dD_i(v)-(n-\lambda)
\end{aligned}$$

If $\sum_{i=1}^n \mathbb{E}_{v_i}[F(-\beta_i(v_i))] \geq n-\lambda$, the reserve price is equal to 0. Otherwise, setting the gradient to be zero, we complete the proof. □

D.2 Proof of Theorem 4.3

PROOF. We first derive the expected utility of bidder i at stage 2,

$$\begin{aligned}
&\mathbb{E}_{v_i}[\hat{u}_{i,2}(\alpha;v)]\\
=\;& \mathbb{E}_v\left[\mathbb{I}\{v_i \geq \max_{j\neq i}\{v_j, r^*(\alpha)\}\}\cdot(v_i-\max_{j\neq i}\{v_j,r^*(\alpha)\})\right]\\
=\;& \mathbb{E}_{v:v_i\geq m_i, v_i\geq r^*(\alpha)}\left[(v_i-m_i)\cdot\mathbb{I}\{m_i\geq r^*(\alpha)\}+(v_i-r^*(\alpha))\cdot\mathbb{I}\{m_i\leq r^*(\alpha)\leq v_i\}\right]\\
=\;& \mathbb{E}_{v_i:v_i\geq r^*(\alpha)}\left[\int_{r^*(\alpha)}^{v_i}(v_i-m_i)g_i(m_i)dm_i\right]+\int_{r^*(\alpha)}\int_0^{r^*(\alpha)}(v_i-r^*(\alpha))g_i(m_i)dm_idD_i(v_i)\\
=\;& \mathbb{E}_{v_i:v_i\geq r^*(\alpha)}\left[\int_{r^*(\alpha)}^{v_i}(v_i-m_i)g_i(m_i)dm_i\right]+\int_{r^*(\alpha)}(v_i-r^*(\alpha))G_i(r^*(\alpha))dD_i(v_i)
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\lim_{\alpha\to 0}\frac{\mathbb{E}_v[\hat{u}_{i,2}(1+\alpha;v)]-\mathbb{E}_v[\hat{u}_{i,2}(1-\alpha;v)]}{2\alpha}\\
=\;& \frac{1}{2}\cdot\left(\frac{\partial\mathbb{E}_v[\hat{u}_{i,2}(\alpha;v)]}{\partial\alpha}\Big|_{\alpha=1^+}+\frac{\partial\mathbb{E}_v[\hat{u}_{i,2}(\alpha;v)]}{\partial\alpha}\Big|_{\alpha=1^-}\right)\\
=\;& -\mathbb{E}_{v_i:v_i\geq r^*(1)}\left[(v_i-r^*(1))g_i(r^*(1))\cdot\zeta\right]+\mathbb{E}_{v_i:v_i\geq r^*(1)}\left[-\eta G_i(r^*(1))+(v_i-r^*(1))g_i(r^*(1))\zeta\right]\\
=\;& -\zeta\mathbb{E}_{v_i:v_i\geq r^*(1)}\left[G_i(r^*(1))\right],
\end{aligned}$$

where $\zeta=\frac{1}{2}\left[\frac{\partial r^*(\alpha)}{\partial\alpha}\Big|_{\alpha=1^+}+\frac{\partial r^*(\alpha)}{\partial\alpha}\Big|_{\alpha=1^-}\right]$. Therefore, the IC-metric for bidder i is

$$\mathtt{DIC}_i = 1-\frac{\zeta\cdot G_i(r^*(1))\cdot(1-D_i(r^*(1)))}{\mathbb{E}_{v_i}[v_i\cdot G_i(v_i)]}$$

Then we can derive ζ in the following way, by Proposition 4.1, when $\sum_{i=1}^n \mathbb{E}_{v_i}[F(-v_i)] < n-\lambda$, we have

$$\sum_{i=1}^n \int_0^1 F(r^*(\alpha)-\alpha v_i)dD_i(v_i) = n-\lambda$$

Taking derivative with respect to α in the both sides, we have

$$\sum_{i=1}^{n} \int_0^1 f(r^*(\alpha) - \alpha v_i) \cdot \left(\frac{\partial r^*(\alpha)}{\partial \alpha} - v_i \right) dD_i(v_i) = 0$$

Thus, we get

$$\frac{\partial r^*(\alpha)}{\partial \alpha}\Big|_{\alpha=1} = \frac{\sum_{i=1}^n \int_0^1 v_i f(r^*(1) - v_i) dD_i(v_i)}{\sum_{i=1}^n \int_0^1 f(r^*(1) - v_i) dD_i(v_i)}$$

Then we characterize ζ following a case analysis:

- When $\sum_{i=1}^n \mathbb{E}_{v_i}[F(-v_i)] > n - \lambda$, there exists a $\delta > 0$, $r^*(\alpha) = 0, \forall \alpha \in [1-\delta, 1+\delta]$. Thus $\zeta = 0$.
- When $\sum_{i=1}^n \mathbb{E}_{v_i}[F(-v_i)] = n - \lambda$, the left derivative of $r^*(\alpha)$ at $\alpha = 1$ ($\alpha = 1^-$) is 0, and the right derivative of $r^*(\alpha)$ at $\alpha = 1$ ($\alpha = 1^+$) is

$$\frac{\sum_{i=1}^n \int_0^1 v_i f(r^*(1) - v_i) dD_i(v_i)}{\sum_{i=1}^n \int_0^1 f(r^*(1) - v_i) dD_i(v_i)}.$$

  Then $\zeta = \frac{\sum_{i=1}^n \int_0^1 v_i f(r^*(1) - v_i) dD_i(v_i)}{2\sum_{i=1}^n \int_0^1 f(r^*(1) - v_i) dD_i(v_i)}$.

Therefore, we complete the proof. □