Exercise 1.2.i. We proceed by induction on $n-k$. For the base case $(n-k=1)$, there exists a bijective map $s : \mathbb{R}^{n-1} \to W$

$$(c_1, \ldots, c_{n-1}) \mapsto c_1 e_1 + \cdots + c_{n-1} e_{n-1}$$

Now, let $e_n \in V \setminus W$ and consider the map $t : W \to V$

$$(c_1, \ldots, c_n) \mapsto c_1 e_1 + \cdots + c_{n-1} e_{n-1} + c_n e_n$$

Suppose $\sum_{i=1}^{n} c_i e_i = 0$ and $c_n \neq 0$. Then $\sum_{i=1}^{n-1} c_i e_i = -c_n e_n$ since $e_n \neq 0$ because $0 \in W$. But this is impossible, because no linear combination of vectors in W can equal e_n. Hence, $c_n = 0$ and $\sum_{i=1}^{n-1} c_i e_i = 0$ (since s is injective), so we conclude that t is injective.

There exists a linear map $r : V \to W$ which is the identity if $v \in W$ and otherwise is 0. But this means that $\ker r = V \setminus W \cup \{0\}$ and hence $\dim V \setminus W = 1$. So, any vector in $V \setminus W$ is a basis of $V \setminus W$ (since $V \setminus W$ must have some basis consisting of a single vector and any vector is then just a linear scaling of that vector, i.e. any linear scaling of the basis vector is itself a basis vector.)

Now, let $v \in V$. If $v \in W$, then clearly there exists $(c_1, \ldots, c_n)$ such that $t(c_1, \ldots, c_n) = v$ (since since s spans W; $c_n = 0$ in this case). So, let $v \in V \setminus W$. Then, there exists c_n such that $v = c_n e_n$, i.e.,

$$v = t(0, \ldots 0, c_n)$$

and we conclude that t spans V and hence is a bijection.

For the inductive step, suppose the proposition is true for $n-k \leq K$ and suppose $n-k = K+1$. Choose any $e_{k+1} \in V \setminus W$ and consider the subspace $S = \{w + \lambda e_{k+1} \mid w \in W, \lambda \in \mathbb{R}\}$. By similar arguments to above, $\dim S = k+1$ and $e_1, \ldots, e_k, e_{k+1}$ forms a basis of S. So, we can invoke the inductive hypothesis and we're done.