Exercise 1.2. Let $\sim$ be an equivalence relation on a set S. We need to show that $\mathcal{P}_\sim$ is a set of non-empty disjoint sets whose union is S. Non-emptiness is easy: since $\sim$ is an equivalence relation, $a \sim a$ for all $a \in S$ and we conclude $a \in [a]_\sim$ by the definition of $[a]_\sim$.

Disjointness takes a bit more work. If $S = \varnothing$, then $P_\sim = \{\}$ is trivially disjoint. Similarly, if $S = \{a\}$, then $P = \{[a]_\sim\} = \{\{a\}\}$ is disjoint.

So, we now assume $|S| \geq 2$ and let $r, s \in S$ be distinct. For a contradiction, suppose that there exists $c \in S$ and distinct $[r]_\sim, [s]_\sim \in \mathcal{P}_\sim$ such that $c \in [r]_\sim$ and $c \in [s]_\sim$. Hence, $c \sim r$ and $c \sim s$. By the symmetry and transitivity of $\sim$, $r \sim s$ and $s \sim r$. But this means that for any $x \in [r]_\sim$ and any $y \in [s]_\sim$, we have $x \sim r$ from which it follows by transitivity that $x \sim s$ and, similarly, $y \sim s$, from which it follows that $y \sim r$. We conclude that $[r]_\sim = [s]_\sim$; by contradiction, $[r]_\sim, [s]_\sim$ must be disjoint.

Finally, suppose there exists $c \in S$ such that $c \notin \mathcal{P}_\sim$. But $[c]_\sim \in \bigcup_{x\in S}[x]_\sim$ and $c \in [c]_\sim$ so it must be the case that $\bigcup_{x\in S}[x]_\sim \subseteq S$. Now suppose that there exists $c \in \bigcup_{x\in S}[x]_\sim$ with $c \notin S$. There then exists a $r \in S$ such that $c \in [r]_\sim$. But

$$[r]_\sim = \{s \in S | s \sim r\}$$

so $c \in S$ and $S \subseteq \bigcup_{x\in S}[x]_\sim$. We conclude $S = \bigcup_{x\in S}[x]_\sim = \bigcup_{T\in P_\sim} T$ as required.

Exercise 1.3. Let $a, b \in S$. Define $\sim$ by

$$a \sim b \iff (\exists T \in \mathcal{P}_\sim)\ a, b \in T$$

All that's left is to show $\sim$ satisfies the required properties.

- reflexivity: Let $a \in S$. Since the union of the sets of $\mathcal{P}_\sim$ is equal to S, there exists $T \in \mathcal{P}_\sim$ with $a \in T$. Hence, $a \sim a$.
- symmetry: Let $a, b \in S$ and suppose $a \sim b$. Then $(\exists T \in \mathcal{P}_\sim)\ a, b \in T$ from which we conclude $b \sim a$.
- transitivity: Let $a, b, c \in S$ and suppose $a \sim b, b \sim c$. Then, $(\exists T \in \mathcal{P}_\sim)\ a, b \in T$ and $(\exists U \in \mathcal{P}_\sim)\ b, c \in U$. But since $\mathcal{P}_\sim$ consists of disjoint sets, $T = U$ since b is in both of them. Hence, $c \in T$ and we conclude $a \sim c$.