Exercise 1.2. Let \sim be an equivalence relation on a set S. We need to show that \mathcal{P}_{\sim} is a set of non-empty disjoint sets whose union is S. Non-emptiness is easy: since \sim is an equivalence relation, $a \sim a$ for all $a \in S$ and we conclude $a \in [a]_{\sim}$ by the definition of $[a]_{\sim}$.

Disjointness takes a bit more work. If $S = \emptyset$, then $P_{\sim} = \{\}$ is trivially disjoint. Similarly, if $S = \{a\}$, then $P = \{[a]_{\sim}\} = \{\{a\}\}$ is disjoint.

So, we now assume $|S| \geq 2$ and let $r,s \in S$ be distinct. For a contradiction, suppose that there exists $c \in S$ and distinct $[r]_{\sim}$, $[s]_{\sim} \in \mathcal{P}_{\sim}$ such that $c \in [r]_{\sim}$ and $c \in [s]_{\sim}$. Hence, $c \sim r$ and $c \sim s$. By the symmetry and transitivity of \sim , $r \sim s$ and $s \sim r$. But this means that for any $x \in [r]_{\sim}$ and any $y \in [s]_{\sim}$, we have $x \sim r$ from which it follows by transitivity that $x \sim s$ and, similarly, $y \sim s$, from which it follows that $y \sim r$. We conclude that $[r]_{\sim} = [s]_{\sim}$; by contradiction, $[r]_{\sim}$, $[s]_{\sim}$ must be disjoint.

Finally, suppose there exists $c \in S$ such that $c \notin \mathcal{P}_{\sim}$. But $[c]_{\sim} \in \bigcup_{x \in S} [x]_{\sim}$ and $c \in [c]_{\sim}$ so it must be the case that $\bigcup_{x \in S} [x]_{\sim} \subseteq S$. Now suppose that there exists $c \in \bigcup_{x \in S} [x]_{\sim}$ with $c \notin S$. There then exists a $r \in S$ such that $c \in [r]_{\sim}$. But

$$[r]_{\sim} = \{ s \in S | s \sim r \}$$

so $c \in S$ and $S \subseteq \bigcup_{x \in S} [x]_{\sim}$. We conclude $S = \bigcup_{x \in S} [x]_{\sim} = \bigcup_{T \in P_{\sim}} T$ as required.

Exercise 1.3. Let $a, b \in S$. Define \sim by

$$a \sim b \iff (\exists T \in \mathcal{P}_{\sim}) \ a, b \in T$$

All that's left is to show \sim satisfies the required properties.

- reflexivity: Let $a \in S$. Since the union of the sets of \mathcal{P}_{\sim} is equal to S, there exists $T \in \mathcal{P}_{\sim}$ with $a \in T$. Hence, $a \sim a$.
- symmetry: Let $a,b \in S$ and suppose $a \sim b$. Then $(\exists T \in \mathcal{P}_{\sim})$ $a,b \in T$ from which we conclude $b \sim a$.
- transitivity: Let $a,b,c \in S$ and suppose $a \sim b,b \sim c$. Then, $(\exists T \in \mathcal{P}_{\sim})$ $a,b \in T$ and $(\exists U \in \mathcal{P}_{\sim})$ $b,c \in U$. But since \mathcal{P}_{\sim} consists of disjoint sets, T = U since b is in both of them. Hence, $c \in T$ and we conclude $a \sim c$.

Exercise 1.5. Let $a, b \in \mathbb{N}$. Define a R b by

$$a R b \iff |a - b| \le 1$$

R is clearly reflexive: for all $a \in \mathbb{N}$, $|a-a|=0 \le 1$. R is symmetric since |x|=|-x| for any x. However, R is *not* transitive: let distinct $a,b,c \in \mathbb{N}$ and a R b and b R c. Then, |a-b|=1 (since $a \ne b$) and similarly |b-c|=1. Hence, either a-b=1 or b-a=1. In the former case, b-c=1 (since $c \ne a$) from which we conclude that a-c=2 and in the later case, c-b=1, and we have c-a=2 again. Hence, |a-c|=2 and thus a R c does not hold.

Exercise 1.6. Showing \sim satisfies the required properties:

- reflexivity: For all $a \in \mathbb{R}$, $a a = 0 \in \mathbb{Z}$.
- symmetry: Let $a, b \in \mathbb{R}$. If $b a \in \mathbb{Z}$, clearly $-(b a) = a b \in \mathbb{Z}$.
- transitivity: Let $a,b,c \in \mathbb{R}$ and suppose $a \sim b$ and $b \sim c$. Then $b-a \in \mathbb{Z}$ and $c-b \in \mathbb{Z}$. Since \mathbb{Z} is closed under addition, $c-a=(c-b)+(b-a)\in \mathbb{Z}$.
 - \sim relates $a, b \in \mathbb{R}$ if they have the same post-decimal representation.

Exercise 2.1. A bijection $f: S \to S$ can be constructed iteratively as follows. Set D = T = S. Choose $s_1 \in D$, $s_1' \in T$ and set $f(s_1) = s_1'$. Next, choose $s_2 \in D \setminus \{s_1\}$, $s_2' \in T \setminus \{s_1'\}$ and set $f(s_2) = s_2'$. Next, choose $s_3 \in D \setminus \{s_1, s_2\}$, $s_3' \in T \setminus \{s_1', s_2'\}$ and set $f(s_3) = s_3'$. Continue in this manner for n steps (until $D = T = \emptyset$).

Clearly, f is a bijection. If $s,s' \in S$, then $f(s) = f(s') \Rightarrow s = s'$ by construction. Similarly, since the algorithm continues until $T = \emptyset$, f must also be surjective. The algorithm allows for n! different constructions of f. There only question that remains is if there exists a bijection $g: S \to S$ which cannot be constructed in such a fashion. But g can be constructed in such a fashion: for each $s \in S$, simply choose the target of f as g(s). This is always possible because g is a bijection: for each $s' \in T$, there is an $s \in D$ such that g(s) uniquely maps to it. But such a construction means that f = g, which is a contradiction.

We conclude that there are exactly n! bijections from S to S.