Exercise 1.2. Let $\sim$ be an equivalence relation on a set S. We need to show that $\mathcal{P}_\sim$ is a set of non-empty disjoint sets whose union is S. Non-emptiness is easy: since $\sim$ is an equivalence relation, $a \sim a$ for all $a \in S$ and we conclude $a \in [a]_\sim$ by the definition of $[a]_\sim$.

Disjointness takes a bit more work. If $S = \varnothing$, then $P_\sim = \{\}$ is trivially disjoint. Similarly, if $S = \{a\}$, then $P = \{[a]_\sim\} = \{\{a\}\}$ is disjoint.

So, we now assume $|S| \geq 2$ and let $r, s \in S$ be distinct. For a contradiction, suppose that there exists $c \in S$ and distinct $[r]_\sim, [s]_\sim \in \mathcal{P}_\sim$ such that $c \in [r]_\sim$ and $c \in [s]_\sim$. Hence, $c \sim r$ and $c \sim s$. By the symmetry and transitivity of $\sim$, $r \sim s$ and $s \sim r$. But this means that for any $x \in [r]_\sim$ and any $y \in [s]_\sim$, we have $x \sim r$ from which it follows by transitivity that $x \sim s$ and, similarly, $y \sim s$, from which it follows that $y \sim r$. We conclude that $[r]_\sim = [s]_\sim$; by contradiction, $[r]_\sim, [s]_\sim$ must be disjoint.

Finally, suppose there exists $c \in S$ such that $c \notin \mathcal{P}_\sim$. But $[c]_\sim \in \bigcup_{x \in S} [x]_\sim$ and $c \in [c]_\sim$ so it must be the case that $\bigcup_{x \in S} [x]_\sim \subseteq S$. Now suppose that there exists $c \in \bigcup_{x \in S} [x]_\sim$ with $c \notin S$. There then exists a $r \in S$ such that $c \in [r]_\sim$. But

$$[r]_\sim = \{s \in S | s \sim r\}$$

so $c \in S$ and $S \subseteq \bigcup_{x \in S} [x]_\sim$. We conclude $S = \bigcup_{x \in S} [x]_\sim = \bigcup_{T \in P_\sim} T$ as required.

Exercise 1.3. Let $a, b \in S$. Define $\sim$ by

$$a \sim b \iff (\exists T \in \mathcal{P}_\sim)\ a, b \in T$$

All that's left is to show $\sim$ satisfies the required properties.

- reflexivity: Let $a \in S$. Since the union of the sets of $\mathcal{P}_\sim$ is equal to S, there exists $T \in \mathcal{P}_\sim$ with $a \in T$. Hence, $a \sim a$.
- symmetry: Let $a, b \in S$ and suppose $a \sim b$. Then $(\exists T \in \mathcal{P}_\sim)\ a, b \in T$ from which we conclude $b \sim a$.
- transitivity: Let $a, b, c \in S$ and suppose $a \sim b, b \sim c$. Then, $(\exists T \in \mathcal{P}_\sim)\ a, b \in T$ and $(\exists U \in \mathcal{P}_\sim)\ b, c \in U$. But since $\mathcal{P}_\sim$ consists of disjoint sets, $T = U$ since b is in both of them. Hence, $c \in T$ and we conclude $a \sim c$.

Exercise 1.5. Let $a, b \in \mathbb{N}$. Define $a\ R\ b$ by

$$a\ R\ b \iff |a - b| \leq 1$$

R is clearly reflexive: for all $a \in \mathbb{N}$, $|a - a| = 0 \leq 1$. R is symmetric since $|x| = |-x|$ for any x. However, R is *not* transitive: let distinct $a, b, c \in \mathbb{N}$ and $a\ R\ b$ and $b\ R\ c$. Then, $|a - b| = 1$ (since $a \neq b$) and similarly $|b - c| = 1$. Hence, either $a - b = 1$ or $b - a = 1$. In the former case, $b - c = 1$ (since $c \neq a$) from which we conclude that $a - c = 2$ and in the later case, $c - b = 1$, and we have $c - a = 2$ again. Hence, $|a - c| = 2$ and thus $a\ R\ c$ does not hold.

Exercise 1.6. Showing $\sim$ satisfies the required properties:

- reflexivity: For all $a \in \mathbb{R}$, $a - a = 0 \in \mathbb{Z}$.
- symmetry: Let $a, b \in \mathbb{R}$. If $b - a \in \mathbb{Z}$, clearly $-(b - a) = a - b \in \mathbb{Z}$.
- transitivity: Let $a, b, c \in \mathbb{R}$ and suppose $a \sim b$ and $b \sim c$. Then $b - a \in \mathbb{Z}$ and $c - b \in \mathbb{Z}$. Since $\mathbb{Z}$ is closed under addition, $c - a = (c - b) + (b - a) \in \mathbb{Z}$.

  $\sim$ relates $a, b \in \mathbb{R}$ if they have the same post-decimal representation.

Exercise 2.1. A bijection $f : S \to S$ can be constructed iteratively as follows. Set $D = T = S$. Choose $s_1 \in D, s_1' \in T$ and set $f(s_1) = s_1'$. Next, choose $s_2 \in D \setminus \{s_1\}, s_2' \in T \setminus \{s_1'\}$ and set $f(s_2) = s_2'$. Next, choose $s_3 \in D \setminus \{s_1, s_2\}, s_3' \in T \setminus \{s_1', s_2'\}$ and set $f(s_3) = s_3'$. Continue in this manner for n steps.

Clearly, f is a bijection. If $s, s' \in S$, then $f(s) = f(s') \Rightarrow s = s'$ by construction. Since the algorithm continues for n steps and assigns f a unique output for each step, f must also be surjective. The algorithm allows for $n!$ different constructions of f: at iteration i, we have $(n - i + 1)$ possible choices to choose as the output of f on the given input. The only question that remains is if there exists a bijection $g : S \to S$ which cannot be constructed in such a fashion. But g *can* be constructed in such a fashion: for each $s \in S$, simply choose the target of f as $g(s)$. This is always possible because g is a bijection: for each $s' \in T$, there is an $s \in D$ such that $g(s)$ uniquely maps to it. (And hence if $s \in D \setminus \{s_1, \dots, s_m\}$ after m iterations, $g(s) \in T \setminus \{s_1', \dots, s_m'\}$.) But such a construction means that $f = g$, which is a contradiction.

We conclude that there are exactly $n!$ bijections from S to S.

Exercise 2.2.

($\Longrightarrow$) Suppose $f : A \to B$ has a right-inverse, then there exists a function $g : B \to A$ such that $f \circ g = \mathrm{id}_B$. If we choose some $b \in B$, we have

$$(f \circ g)(b) = f(g(b)) = \mathrm{id}_B(b) = b$$

So, for any $b \in B$, we have $g(b) \in A$ with $f(g(b)) = b$ which shows f is indeed surjective.

($\Longleftarrow$) Now suppose f is surjective. We will construct a function $g : B \to A$ such that $f \circ g = \mathrm{id}_B$. Define an equivalence relation $\sim$ by

$$a \sim b \Longleftrightarrow f(a) = f(b)$$

$\sim$ is clearly an equivalence relation (indeed, this is immediate by the fact that $=$ is an equivalence relation). By Exercise 1.2, $\mathcal{P}_\sim$ is a partition of A. Define g as

$$g(b) = \text{choose any } c \in [a]_\sim \text{ with } f(a) = b$$

such an a is guaranteed to exist by the surjectivity of $g(b)$ and the ability to make such a choice is guaranteed by the axiom of choice. The only thing left is to show that g is indeed a right-inverse of f. To do so, let $b \in B$ and suppose $g(b) = c$ for some $c \in A$. Then,

$$f(g(b)) = f(c) = b$$

by the construction of g.

Exercise 2.3. Let $f : A \to B$ be a bijection with an inverse $f^{-1} : B \to A$. By definition,

$$f \circ f^{-1} = \mathrm{id}_B \qquad f^{-1} \circ f = \mathrm{id}_A$$

but this means that f must also be an of f^{-1}. By Corollary 2.2, f^{-1} is a bijection. Now, also consider bijection $g : B \to C$. By the associativity of function composition,

$$\begin{aligned}(g \circ f) \circ (f^{-1} \circ g^{-1}) &= g \circ (f \circ (f^{-1}) \circ g^{-1} \\ &= g \circ \mathrm{id}_B \circ g^{-1} \\ &= g \circ g^{-1} \\ &= \mathrm{id}_C\end{aligned}$$

and

$$\begin{aligned}f^{-1} \circ g^{-1} \circ g \circ f &= f^{-1} \circ (g^{-1} \circ g) \circ f \\ &= f^{-1} \circ \mathrm{id}_B \circ f \\ &= f^{-1} \circ f \\ &= \mathrm{id}_A\end{aligned}$$

By Corollarly 2.2 once again, $g \circ f$ is a bijection with inverse $f^{-1} \circ g^{-1}$.

Exercise 2.4. Consider some set of sets S.

- reflexivity: for all $A \in S$, we have $A \cong A$ (with $\mathrm{id}_A : A \to A$ as the bijection).
- symmetry: Consider $A, B \in S$ and suppose $A \cong B$ which means there exists a bijection $f : A \to B$. By Exercise 2.3, $f^{-1} : B \to A$ is also a bijection and hence $B \cong A$.
- transitivity: Let $A, B, C, \in S$ and suppose $A \cong B$ and $B \cong C$, meaning there exists bijections $f : A \to B$ and $g : B \to C$. By Exercise 2.3, $g \circ f : A \to C$ is a bijection, and hence $A \cong C$.

Exercise 2.5. A function $f : A \to B$ is an *epimorphism* if for all sets Z and all functions $g : Z \to B$, there exists $\alpha : Z \to A$ such that $f \circ \alpha = g$. The thing to prove is now that a function f is surjective if and only if it is an epimorphism.

($\Longrightarrow$) Suppose f is surjective. Define $\alpha : Z \to A$ as

$$\alpha(z) = a \text{ where } f(a) = g(z)$$

such an a is always guaranteed to exist by the surjectivity of f. Now, consider an arbitrary $z \in Z$. We have,

$$(f \circ \alpha)(z) = f(\alpha(z)) = g(z)$$

as required.

($\Longleftarrow$) Suppose f is an epimorphism and let $g = \mathrm{id}_B$. Since f is an epimorphism, there exists a function $\alpha : B \to A$ such that $f \circ \alpha = \mathrm{id}_B$. But this means that α is a right-inverse of f, so by Proposition 2.1, f is surjective.