Exercise 2.2.4.

a) $(-1, 1, -1, 1, -1, \dot{)}$

b) Impossibru. An infinite number of ones requires a "long-term" behavior where the sequence features 1. If the sequence doesn't converge to 1, it also has to feature other numbers–but then the sequence is oscillating between 1s and these other numbers and hence must be divergent or these other numbers must get so close to 1 that the sequence converges to 1.

c) $(1, 2, 2, 3, 3, 3, 4, 4, 4, 4, \dots)$

Exercise 2.2.5.

a) $\lim a_n = 0$. Let $\varepsilon > 0$. Choose $n \in \mathbb{N}$ such that $n \geq 5/\varepsilon + 1$. Then,

$$|a_n - 0| = a_n \leq \left[\!\left[\frac{5}{(5/\varepsilon) + 1}\right]\!\right] \leq \frac{5}{(5/\varepsilon) + 1} < \frac{5}{5/\varepsilon} = \varepsilon$$

as required.

b) $\lim a_n = 1$. Choose $N \in \mathbb{N}$ such that $N > 12/(3\varepsilon - 1)$ and let $n \geq \mathbb{N}$. Then,

$$\begin{aligned}
\left|a_n - \frac{4}{3}\right| &< \left|\left[\!\left[\frac{12 + 4\,(12/(3\varepsilon - 1))}{3\,(12/(3\varepsilon - 1))}\right]\!\right] - 1\right| \\
&\leq \frac{12 + 4\,(12/(3\varepsilon - 1))}{3\,(12/(3\varepsilon - 1))} - 1 \\
&= \frac{12 + (12/(3\varepsilon - 1))}{3\,(12/(3\varepsilon - 1))} \\
&= \frac{3\varepsilon}{3} \\
&= \epsilon
\end{aligned}$$

as required.

Exercise 2.2.6. Suppose $\lim a_n = a$ and also that $\lim a_n = b$ with $a \neq b$. Since $a \neq b$, there exists $\delta > 0$ such that $|a - b| = \delta$. Now, by Definition 2.2.3, for every $\epsilon > 0$, it follows that $|a_n - a| < \epsilon$ and $|a_m - b| < \epsilon$ when $n \geq N$ and $m \geq M$ for some M, N. Choose $\epsilon = \delta/4$ and set M and N appropriately. Now, let $R = \max\{M, N\}$ and set $r \geq R$. Then, $|a_r - a| < \delta/4$ and $|a_r - b| < \delta/4$. By the triangle inequality,

$$|a - b| \leq |a_r - a| + |a_r - b| < \delta/2 = \frac{|a - b|}{2}$$

which is nonsense because $a - b \neq 0$. By contradiction, $a = b$.

Exercise 2.2.7.

a) Only frequently since $(-1)^n = -1$ for all odd n.

b) Eventually implies frequently.

c) A sequence (a_n) converges to a if, given any ε-neighborhood $V_\varepsilon(a)$ of a, (a_n) is eventually in $V_\varepsilon(a)$.

d) No, the sequence $(-2)^n$ contains an infinite number of 2s but is not eventually in the interval (1.9, 2.1). It is, however, frequently in (1.9, 2.1). Indeed, any sequence containing an infinite number of 2s must be frequently in (1.9, 2.1). If this were not the case, there would be some $N \in \mathbb{N}$ such that for all $n \geq N$, $a_n \neq 2$. But then there would be at most N 2s in the sequence.

Exercise 2.3.1.

a) By the Algebraic Limit Theorem,

$$0 = \lim(x_n) = \lim(\sqrt{x_n}\sqrt{x_n}) = \lim(\sqrt{x})\lim(\sqrt{x})$$

so, $\lim(\sqrt{x}) = 0$.

b) Follows by the same argument in part a).

Exercise 2.3.3. By the Order Limit Theorem, $\lim y_n \leq \lim z_n = l$. Also, $l = \lim x_n \leq \lim y_n$. So $l \leq \lim y_n \leq l$ and we conclude $\lim y_n = l$.

Exercise 2.3.5. Suppose (z_n) is convergent, i.e. $\lim z_n = z$. Then, for all $\varepsilon > 0$ there's an $N \in \mathbb{N}$ such that for all $n \geq N$, $|z_n - z| < \varepsilon$. If n is odd, this is the same as $|x_{((n+1)/2)} - z| < \varepsilon$. If n is even, this is the same as $|y_{n/2} - z| < \varepsilon$. Define $n_x = 2n + 1$ and $n_y = 2n$. Clearly, $n_x \geq 2N + 1$ and $n_y \geq 2N$. Additionally, $|x_{n_x} - z| < \varepsilon$ and $|y_{n_y} - z| < \varepsilon$. We conclude that $\lim x_n = \lim y_n = \lim z_n = z$.

Now suppose that $\lim x_n = \lim y_n = z$. Let $\varepsilon > 0$. Then, there exists $N_x, N_y \in \mathbb{N}$ such that for all $n_x \geq N_x$ and $n_y \geq N_y$, $|x_{n_x} - z| < \varepsilon$ and $|y_{n_y} - z| < \varepsilon$. Set $N = \max\{2N_x, 2N_y\}$. Choose $n \geq N$. Clearly, $n \geq 2n_x - 1$ and $n \geq 2n_y$. If n is odd, then $|z_n - z| = |x_{(n+1)/2} - z|$. But $(n+1)/2 \geq n_x$ so $|x_{(n+1)/2} - z| < \varepsilon$. Similarly, if n is even, $|z_n - z| = |y_{n/2} - z|\varepsilon$. Hence, $|z_n - z| < \varepsilon$ for all $n \geq N$.

Exercise 2.3.7.

a) Let $(x_n) = (n)$ and let $(y_n) = (-n)$. Then, $(x_n + y_n) = (n + (-n)) = (0)$, which obviously converges.

b) Impossible. By the Algebraic Limit Theorem, $\lim(y_n) = \lim(y_n + x_n - x_n) = \lim(y_n + x_n) - \lim(x_n)$.

c) Let $(b_n) = (1/n)$. By Exercise 2.3.6, $\lim(1/n) = 0$.

d) Suppose $(a_n - b_n)$ is bounded. Then, there exists $M > 0$ such that $|a_n - b_n| \leq M$ for all $n \in \mathbb{N}$. Similarly, by Theorem 2.3.2, there eixsts $B > 0$ such that $|b_n| \leq B$ for all $n \in \mathbb{N}$. Since (a_n) is unbounded, for any $K \in \mathbb{R}$, there exists $n_0 \in \mathbb{N}$ such that $|a_{n_0}| > K$. So, choose $n_1 \in \mathbb{N}$ such that $|a_{n_1}| > M + B$. Then, $|a_{n_1} - b_{n_1}| > M + B - b_{n_1} > M$, which is a contradiction.

e) Let $(a_n) = (0)$ and $(b_n) = (-1)^n$. Clearly $(a_n b_n) = (0)$ converges, but (b_n) does not.

Exercise 2.3.8.

a) $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m$. By the Algebraic Limit Theorem,

$$\begin{aligned}\lim(p(x_n)) &= \lim(a_0) + \lim(a_1 x_n) + \lim(a_2 x_n^2) + \cdots + \lim(a_n x_n^m) \\ &= \lim(a_0) + \lim(a_1)\lim(x_n) + \lim(a_2)\lim(x_n)^2 + \cdots + \lim(a_m)\lim(x_n)^m \\ &= a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m \\ &= p(x)\end{aligned}$$

b) Let $f(x) = [[x]]$ and $(x_n) = (1.5)$. Clearly, $\lim f(x_n) = 1$ and $\lim(x_n) = 1.5$.

Exercise 2.3.11.

a) Let $\varepsilon > 0$ and $\lim x_n = x$. We need to find an $N > 0$ such that for all $n \geq N$,

$$\begin{aligned}|y_n - x| &= \left|\frac{x_1 + x_2 + \cdots + x_n}{n} - x\right| \\ &= \left|\frac{x_1 + x_2 + \cdots + x_n - nx}{n}\right| \\ &= \frac{1}{n}\left|(x_1 - x) + (x_2 - x) + \cdots + (x_n - x)\right| \\ &\leq \frac{1}{n}\left(|x_1 - x| + |x_2 - x| + \cdots + |x_n - x|\right) < \varepsilon\end{aligned}$$

Since (x_n) converges, there is an $M > 0$ such that $|x_n - x| < \varepsilon/2$ for all $n > M$. Hence, the above becomes

$$\begin{aligned}&\frac{1}{n}\left(|x_1 - x| + |x_2 - x| + \cdots + |x_n - x|\right) \\ &= \frac{1}{n}\left(|x_1 - x| + |x_2 - x| + \cdots + |x_{M-1} - x|\right) + \frac{1}{n}\left(|x_M - x| + \cdots + |x_n - n|\right) \\ &< \frac{1}{n}\left(|x_1 - x| + |x_2 - x| + \cdots + |x_{M-1} - x|\right) + \frac{\varepsilon}{2}\end{aligned}$$

Now $(|x_1 - x| + \cdots + |x_{M-1} - x|)$ is finite, so we can choose some $R > 0$ large enough such that—with the $1/n$ factor—it's less than $\varepsilon/2$ for all $n \geq R$. Namely,

$$R = \left[\left[\frac{2(|x_1 - x| + \cdots + |x_{M-1} - x|)}{\varepsilon}\right]\right] + 1$$

We choose $N = \max\{R, M\}$ and then have

$$\begin{aligned}|y_n - x| &< \frac{1}{n}\left(|x_1 - x| + |x_2 - x| + \cdots + |x_{M-1} - x|\right) + \frac{\varepsilon}{2} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon\end{aligned}$$

for all $n \geq N$, as required.

b) Consider $(x_n) = (-1)^n$. Then

$$y_n = \frac{(-1) + 1 + (-1) + \cdots + (-1)^n}{n} = \begin{cases} 0 & n \text{ even} \\ -1/n & n \text{ odd} \end{cases}$$

Clearly, (y_n) converges to 0 even though (x_n) does not converge.

Exercise 2.3.12.

a) True, follows immediately by part (iii) for the Order Limit Theorem.

b) True. Suppose $a \in (0,1)$. Then, $|a_n - a| > 0$ for all n since $a_n \notin (0,1)$. But then we can choose $\varepsilon = \text{argmin}_n(|a_n - a|/2)$ and have $|a_n - a| > \epsilon$ for all n, contradicting the existence of a.

c) False. The sequence where the nth term consists of the best decimal approximation of $\sqrt{2}$ to n places clearly converges to $\sqrt{2}$.

Exercise 2.4.1.

a) We'll use induction to show that $x_n \geq x_{n+1}$ for all $n \in \mathbb{N}$.

Base ($n = 1$): Clearly, $3 \geq 1/(4-3) = 1$.

Inductive step: Assume $x_n \geq x_{n+1}$ for all $n \leq K$. Then,

$$\begin{aligned} x_K &\leq x_{K+1} \\ 4 - x_K &\geq 4 - x_{K+1} \\ \frac{1}{4 - X_K} &\leq \frac{1}{4 - x_{K+1}} \\ x_{K+1} &\leq x_{K+2} \end{aligned}$$

as required. Hence, (x_n) is decreasing. Since $3 \geq x_n$ for all $n \in \mathbb{N}$, we have that $x_n \geq 0$ and conclude that $|x_n| \leq 3$. By Theorem 2.4.2, (x_n) converges.

b) The sequence (x_{n+1}) is just (x_n) without the first term–clearly, the converge to the same limit.

c) We have,

$$\begin{aligned} \lim x_{n+1} &= \lim \left(\frac{1}{4 - x_n} \right) \\ &= \frac{1}{4 - \lim x_n} && \text{(by the Algebraic Limit Theorem)} \\ &= \frac{1}{4 - \lim x_{n+1}} \end{aligned}$$

Hence, $\lim x_n(4 - \lim x_n) = 1$ or $\lim x_n{}^2 - 4 \lim x_n + 1 = 0$. The roots are $\frac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}$. Since $x_{n+1} < x_n$ and $x_1 = 3$, we have $\lim x_n = 2 - \sqrt{3}$.

Exercise 2.4.2.

a) The argument assumes that the limit exists in the first place (and it does not).

b) Yes, because the limit exists since it is monotone and bounded above (by 3).

Exercise 2.4.3.

a) We have

$$\begin{aligned}
x_n &= \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\cdots\sqrt{2+\sqrt{2}}}}}} \\
&< \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\cdots\sqrt{2+2}}}}} \quad \text{(The final } \sqrt{2} \text{ was replaced with 2)} \\
&= \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\cdots 2}}}} \\
&= \sqrt{2+\sqrt{2+\sqrt{2+2}}} \\
&= 2
\end{aligned}$$

Hence, 2 is an upper bound for the sequence. Additionally, clearly $x_{n+1} > x_n$ for all n, so the sequence is monotone and by Theorem 2.4.2, the sequence must converge and $\lim x_n$ exists. Now, $x_{n+1} = \sqrt{2+x_n}$, So,

$$\begin{aligned}
\lim x_{n+1}^2 &= \lim(2+x_n) \\
\lim x_n^2 &= 2 + \lim x_n \\
&\text{(By the ALT and since } \lim x_{n+1} = \lim x_n) \\
\lim x_n^2 - \lim x_n - 2 &= 0 \\
(\lim x_n + 1)(\lim x_n - 2) &= 0
\end{aligned}$$

Hence, $\lim x_n = 2$.

b) The n-th term in the sequence contains n (nested) square roots and assume $n > 3$. Then,

$$x_n = \sqrt{2\sqrt{2\sqrt{2}\cdots}} = \prod_{k=1}^{n} 2^{1/2^k} = 2^{\sum_{k=1}^{n} 1/2^k}$$

But $\sum_{k=1}^{\infty} 1/2^k = 1$, so that means that the sequence $(1/2^k)$ must be bounded, and, correspondingly that x_n must be bounded. Hence, $\lim x_n$ must exist. Namely, $\lim x_n = 2$.

Exercise 2.4.4.

a) Suppose that $\mathbb{N}$ is bounded from above and consider the sequence (n). Clearly, (n) is monotone so by the Monotone Congergence Theorem, $\lim n$ exists and we set $\lim n = \alpha$. Now, by the Algebraic Limit Theorem, $\lim(n+1) = \lim n + 1 = \alpha + 1$. But $\lim n = \lim(n+1)$, hence $\alpha = \alpha + 1$, which is a contradiction. We conclude that $\mathbb{N}$ is unbounded from above.

b) In the proof of Theorem 1.4.1, we consider the sequence (a_n), which is clearly monotone and bounded above by any b_n. Hence the limit $\lim a_n = \alpha$ must exist.

   To complete the proof, we need to show that for any convergent increasing sequence (x_n), $x_n \leq \lim x_n$ for all n. Suppose there exists an n_0 such that

$\lim x_n < x_{n_0}$. Since the sequence is increasing, this require that for all n, $\lim x_n < x_{n+n_0}$. But $\lim x_n = \lim x_{n+n_0}$ and, by the Order Limit Theorem, $\lim x_n < \lim x_{n+n_0} = \lim_x$, which is a contradiction.

So, we now have that $a_n \leq \alpha \leq b_n$ for all n, as required, and the rest of the proof follows just as it did in the AoC version.