Exercise 2.2.4.

a) $(-1, 1, -1, 1, -1, \dot{)}$

b) Impossibru. An infinite number of ones requires a "long-term" behavior where the sequence features 1. If the sequence doesn't converge to 1, it also has to feature other numbers–but then the sequence is oscillating between 1s and these other numbers and hence must be divergent or these other numbers must get so close to 1 that the sequence converges to 1.

c) $(1, 2, 2, 3, 3, 3, 4, 4, 4, 4, \dots)$

Exercise 2.2.5.

a) $\lim a_n = 0$. Let $\varepsilon > 0$. Choose $n \in \mathbb{N}$ such that $n \geq 5/\varepsilon + 1$. Then,

$$|a_n - 0| = a_n \leq \left[\left[\frac{5}{(5/\varepsilon)+1}\right]\right] \leq \frac{5}{(5/\varepsilon)+1} < \frac{5}{5/\varepsilon} = \varepsilon$$

as required.

b) $\lim a_n = 1$. Choose $N \in \mathbb{N}$ such that $N > 12/(3\varepsilon - 1)$ and let $n \geq \mathbb{N}$. Then,

$$\begin{aligned}
\left|a_n - \frac{4}{3}\right| &< \left|\left[\left[\frac{12 + 4\,(12/(3\varepsilon-1))}{3\,(12/(3\varepsilon-1))}\right]\right] - 1\right| \\
&\leq \frac{12 + 4\,(12/(3\varepsilon-1))}{3\,(12/(3\varepsilon-1))} - 1 \\
&= \frac{12 + (12/(3\varepsilon-1))}{3\,(12/(3\varepsilon-1))} \\
&= \frac{3\varepsilon}{3} \\
&= \epsilon
\end{aligned}$$

as required.

Exercise 2.2.6. Suppose $\lim a_n = a$ and also that $\lim a_n = b$ with $a \neq b$. Since $a \neq b$, there exists $\delta > 0$ such that $|a - b| = \delta$. Now, by Definition 2.2.3, for every $\epsilon > 0$, it follows that $|a_n - a| < \epsilon$ and $|a_m - b| < \epsilon$ when $n \geq N$ and $m \geq M$ for some M, N. Choose $\epsilon = \delta/4$ and set M and N appropriately. Now, let $R = \max\{M, N\}$ and set $r \geq R$. Then, $|a_r - a| < \delta/4$ and $|a_r - b| < \delta/4$. By the triangle inequality,

$$|a - b| \leq |a_r - a| + |a_r - b| < \delta/2 = \frac{|a-b|}{2}$$

which is nonsense because $a - b \neq 0$. By contradiction, $a = b$.

Exercise 2.2.7.

a) Only frequently since $(-1)^n = -1$ for all odd n.

b) Eventually implies frequently.

c) A sequence (a_n) converges to a if, given any ε-neighborhood $V_\varepsilon(a)$ of a, (a_n) is eventually in $V_\varepsilon(a)$.

d) No, the sequence $(-2)^n$ contains an infinite number of 2s but is not eventually in the interval (1.9, 2.1). It is, however, frequently in (1.9, 2.1). Indeed, any sequence containing an infinite number of 2s must be frequently in (1.9, 2.1). If this were not the case, there would be some $N \in \mathbb{N}$ such that for all $n \geq N$, $a_n \neq 2$. But then there would be at most N 2s in the sequence.