Exercise 1.3.2.

a) Let $A = \{0\}$. Then $\inf A = \sup A = 0$.

b) Impossible. Let the set in question be $\{a_1, a_2, \ldots, a_n\}$ and suppose $a_s \geq a_i$ for all i (such an s must exist because the set is finite). Then a_s is the maximum of the set and hence the set has a supremum.

c) The set $\{\frac{1}{n} \mid n \in \mathbb{Z}\}$ does the trick.

Exercise 1.3.3.

a) Since A is bounded below, $B \neq \varnothing$. Furthermore, since A is nonempty, B must have an upper bound. By the Axiom of Completeness, $\sup B$ exists.

   Each $a \in A$ must be an upper bound for B since $b \leq a$. Hence, by part ii) of the definition, $\sup B \leq a$ for each a, establishing $\sup B$ as a lower bound of A. Part i) of the definition says that for all lower bounds $b \in B$, $b \leq \sup B$. So, we conclude that $\sup B = \inf A$.

b) Part a) shows that any nonempty set that is bounded below has an infimum.

Exercise 1.3.4.

a) By the Axiom of Completeness, $\sup A_i$ exists for each i. Let $s_1 = \sup A_2$ and $s_2 = \sup A_2$ and suppose $s_1 \geq s_2$. For each $a_1 \in A_1$ and $a_2 \in A_2$, $s_1 \geq a_1$ and $s_2 \geq a_2$. Hence, $s_1 \geq a_2$ and we conclude that s_1 is an upper bound of $A_1 \cup A_2$.

   Let b be an upper bound of $A_1 \cup A_2$. Then, $a_1 \leq b$ and $a_2 \leq b$. So, b is also an upper bound of A_1 and A_2. But $s_1 \leq b$ so $\sup A_1 \cup A_2 = s_1$.

   Similarly, if $s_2 \geq s_1$, we have $\sup A_1 \cup A_2 = s_2$. We conclude

$$\sup(A_1 \cup A_2) = \max(\sup A_1, \sup A_2)$$

   In general,

$$\sup\left(\bigcup_{k=1}^{n} A_k\right) = \max(\sup A_1, \ldots, \sup A_n)$$

   This can be proved by induction on k, with the base case ($k = 1$) being trivial to prove and the inductive step following directly from part a) (where you have to use the fact that $\max(\max(x_1, \ldots, x_n), x_{n+1}) = \max(x_1, \ldots, x_n, x_{n+1})$).

b) No, because an infinite union of bounded sets can result in an unbounded set, which has no supremum. For example,

$$\bigcup_{k=1}^{\infty} \{x \mid x \leq k\} = \mathbb{R}$$

Exercise 1.3.7.

Part i) of the definition is satisfied by assumption. For part ii), let b be an upper bound of A. Then, for all $a' \in A$, $a' \leq b$ and, crucially, $a \leq b$. Hence, $a = \sup A$.

Exercise 1.3.10.

a) By the Axiom of Completeness, $c = \sup A$ exists. Furthermore, since $a < b$ for all $a \in A, b \in B$, we have that $c \leq b$. But $a \leq c$ by definition, so we're done.

b) First, we need to construct disjoint nonempty sets A, B such that $A \cup B = \mathbb{R}$. Let $B = \{x \mid x > e \text{ for all } e \in E\}$. Since E is nonempty is bounded above by some b, B is nonempty (namely, it must contain $b+1$). Let $A = B^c = \{x \mid x \leq e \text{ for all } e \in E\}$.

   Now, for $a \in A, b \in B, e \in E$, we have $a \leq e$ and $b > e$. Hence, $a < b$ as required. By the Cut Property, there exists $c \in R$ such that $a \leq c$ and $c \leq b$. From the definition of A, this means that for all $e \in E$, $e \leq c$ such that c is an upper bound on E, satisfying part i) of the definition of the supremum. Additionally, suppose d is an upper bound on E. If $d \in A$, then $e \leq d$ and $d \notin B$, meaning that $d < b$. This requires that $d = c$: if $c < d$, then $a \not\leq c$ for all a. Otherwise, $d \in B$, from which it follows that $c \leq d$. We conclude that $c = \sup E$.

c) Let $A = \{x \in \mathbb{Q} \mid x < \sqrt{2}\}$ and $B = \{x \in \mathbb{Q} \mid x > \sqrt{2}\}$. Clearly, $A \cup B = \mathbb{Q}$ (since $\sqrt{2} \notin \mathbb{Q}$) and $a < b$ for all $a \in A$, $b \in B$. By the Cut Property, there exists $c \in \mathbb{Q}$ such that for all $a \in A$, $b \in B$, $a \leq c$ and $b \geq c$. Since $A \cup B = \mathbb{Q}$ and A and B are disjoint, either $c \in A$ or $c \in B$. If $c \in A$, then $1 < c < \sqrt{2}$ and

$$c < c + \frac{2 - c^2}{2} < \sqrt{2}$$

   But this means that there's a rational number (namely $c + (2 - c^2)/2$) in A larger than c, which is a contradiction. A similar argument can be made if $c \in B$.

Exercise 1.3.11.

a) True. Since A and B are nonempty and bounded, $\sup A$ and $\sup B$ exist. Since $A \subseteq B$, $a \leq \sup B$ for all $a \in A$, i.e., $\sup B$ is an upper bound for A. Additionally, $\sup A \leq b$ for all upper bounds b of A. Hence, $\sup A \leq \sup B$.

b) True. Since $\sup A$ and $\inf B$ exist, A and B must be nonempty and bounded. Suppose there does not exist $c \in \mathbb{R}$ such that $a < c < b$ for all $a \in A, b \in B$. Then, there exists $a' \in A$ and $b' \in B$ with $b' \leq a'$. (If $a' < b'$ then $a' < a' + (b' - a')/2 = (a' + b')/2 < b'$ so it must be the case that $b' \leq a'$.) But by assumption $a' \leq \sup A < \inf B \leq b'$, which is a contradiction.

c) False. Let $A = B = \varnothing$, then the assumption trivially holds but neither $\sup A$ nor $\inf B$ exists.

Exercise 1.4.1.

a) Let $a = p/q$ and $b = m/n$. Then $ab = (pm)/(qn)$. Since $\mathbb{Z}$ is closed under multiplication, $ab \in \mathbb{Q}$. Similar for $a + b$, given that $\mathbb{Z}$ is closed under addition.

b) Suppose $a + t \in \mathbb{Q}$. Then, there exists $p, q \in \mathbb{Z}$ such that $a + t = p/q$. Now, since $a = m/n$ for some $m, n\mathbb{Z}$, we have $t = p/q - m/n = (np - mq)/(qn)$. But this contradicts the irrationality of t, so $a + t \in \mathbb{I}$. Similar argument for $at \in \mathbb{I}$.

c) Nothing. By part (b), $1 - \sqrt{2}, \sqrt{2}/2 - 1, \sqrt{2}/2 + 1 \in \mathbb{I}$. But $(1 - \sqrt{(2)} + \sqrt{(2)} = 1 \in \mathbb{Q}$ and $(\sqrt{2}/2 - 1) + (\sqrt{2}/2 + 1) = \sqrt{2} \in \mathbb{I}$. Similarly, $\sqrt{2}\sqrt{2} \in \mathbb{Q}$ but $(\sqrt{2} + 1)\sqrt{2} = 2 + \sqrt{2} \in \mathbb{I}$. Hence, $\mathbb{I}$ is neither closed under addition nor multiplication.

Exercise 1.4.2. First, we show that s is an upper bound of A. Let $a' \in A$ and suppose $a' > s$, which implies $a' - s > 0$. By assumption, $s + 1/n \geq a'$ for all $n \in \mathbb{N}$. By the Archimedean Property, there exists $n_0 \in \mathbb{N}$ such that $1/n_0 < a' - s$. But this implies $s + 1/n_0 < a'$, which is a contradiction. Hence, $s \geq a$ for all $a \in A$.

Now, we want to show that s is a least upper bound. Let b be an upper bound for A. By assumption, $s - 1/n < b$ for all $n \in \mathbb{N}$ so $1/n > s - b$. Now, suppose $s > b$. Then $s - b > 0$ and so, by the Archimedean Property, there exists $n_0 \in \mathbb{N}$ with $1/n_0 < s - b$. But this is a contradiction, so $s \leq b$.

Exercise 1.4.3. Suppose there exists $x \in \cap_{n=1}^{\infty}(0, 1/n)$. Then, $0 < x < 1/n$ for all $n \in \mathbb{N}$. But, by the Archimedean Property, there exists $n_0 \in \mathbb{N}$ such that $1/n_0 < x$. This is a contradiction, so $\cap_{n=1}^{\infty}(0, 1/n) = \varnothing$.

Exercise 1.4.5. By Theorem 1.4.3, there exists $r \in \mathbb{Q}$ with $a - \sqrt{2} < r < b - \sqrt{2}$. Hence, $a < r + \sqrt{2} < b$. But, by Exercise 1.4.1 part (a), $r + \sqrt{2} \in \mathbb{I}$, so we're done.

Exercise 1.4.7. Choose $n_0 \in \mathbb{N}$ large enough so that

$$\frac{1}{n_0} < \frac{\alpha^2 - 2}{2\alpha}$$

Then

$$\alpha^2 - \frac{2\alpha}{n_0} > \alpha^2 - (\alpha^2 - 2) = 2.$$

But this contradicts the fact that α is a least upper bound of T. Hence, we conclude that $\alpha^2 = 2$.