Exercise 1.3.2.

a) Let $A = \{0\}$. Then $\inf A = \sup A = 0$.

b) Impossible. Let the set in question be $\{a_1, a_2, \ldots, a_n\}$ and suppose $a_s \geq a_i$ for all i (such an s must exist because the set is finite). Then a_s is the maximum of the set and hence the set has a supremum.

c) The set $\{\frac{1}{n} \mid n \in \mathbb{Z}\}$ does the trick.

Exercise 1.3.3.

a) Since A is bounded below, $B \neq \varnothing$. Furthermore, since A is nonempty, B must have an upper bound. By the Axiom of Completeness, $\sup B$ exists.

Each $a \in A$ must be an upper bound for B since $b \leq a$. Hence, by part ii) of the definition, $\sup B \leq a$ for each a, establishing $\sup B$ as a lower bound of A. Part i) of the definition says that for all lower bounds $b \in B$, $b \leq \sup B$. So, we conclude that $\sup B = \inf A$.

b) Part a) shows that any nonempty set that is bounded below has an infimum.

Exercise 1.3.4.

a) By the Axiom of Completeness, $\sup A_i$ exists for each i. Let $s_1 = \sup A_2$ and $s_2 = \sup A_2$ and suppose $s_1 \geq s_2$. For each $a_1 \in A_1$ and $a_2 \in A_2$, $s_1 \geq a_1$ and $s_2 \geq a_2$. Hence, $s_1 \geq a_2$ and we conclude that s_1 is an upper bound of $A_1 \cup A_2$.

Let b be an upper bound of $A_1 \cup A_2$. Then, $a_1 \leq b$ and $a_2 \leq b$. So, b is also an upper bound of A_1 and A_2. But $s_1 \leq b$ so $\sup A_1 \cup A_2 = s_1$.

Similarly, if $s_2 \geq s_1$, we have $\sup A_1 \cup A_2 = s_2$. We conclude

$$\sup(A_1 \cup A_2) = \max(\sup A_1, \sup A_2)$$

In general,

$$\sup\left(\bigcup_{k=1}^{n} A_k\right) = \max(\sup A_1, \ldots, \sup A_n)$$

This can be proved by induction on k, with the base case ($k = 1$) being trivial to prove and the inductive step following directly from part a) (where you have to use the fact that $\max(\max(x_1, \ldots, x_n), x_{n+1}) = \max(x_1, \ldots, x_n, x_{n+1})$).

b) No, because an infinite union of bounded sets can result in an unbounded set, which has no supremum. For example,

$$\bigcup_{k=1}^{\infty} \{x \mid x \leq k\} = \mathbb{R}$$

Exercise 1.3.7.

Part i) of the definition is satisfied by assumption. For part ii), let b be an upper bound of A. Then, for all $a' \in A$, $a' \leq b$ and, crucially, $a \leq b$. Hence, $a = \sup A$.

Exercise 1.3.10.

a) By the Axiom of Completeness, $c = \sup A$ exists. Furthermore, since $a < b$ for all $a \in A, b \in B$, we have that $c \leq b$. But $a \leq c$ by definition, so we're done.

b) First, we need to construct disjoint nonempty sets A, B such that $A \cup B = \mathbb{R}$. Let $B = \{x \mid x > e \text{ for all } e \in E\}$. Since E is nonempty is bounded above by some b, B is nonempty (namely, it must contain $b+1$). Let $A = B^c = \{x \mid x \leq e \text{ for all } e \in E\}$.

   Now, for $a \in A, b \in B, e \in E$, we have $a \leq e$ and $b > e$. Hence, $a < b$ as required. By the Cut Property, there exists $c \in R$ such that $a \leq c$ and $c \leq b$. From the definition of A, this means that for all $e \in E$, $e \leq c$ such that c is an upper bound on E, satisfying part i) of the definition of the supremum. Additionally, suppose d is an upper bound on E. If $d \in A$, then $e \leq d$ and $d \notin B$, meaning that $d < b$. This requires that $d = c$: if $c < d$, then $a \not\leq c$ for all a. Otherwise, $d \in B$, from which it follows that $c \leq d$. We conclude that $c = \sup E$.

c) Let $A = \{x \in \mathbb{Q} \mid x < \sqrt{2}\}$ and $B = \{x \in \mathbb{Q} \mid x > \sqrt{2}\}$. Clearly, $A \cup B = \mathbb{Q}$ (since $\sqrt{2} \notin \mathbb{Q}$) and $a < b$ for all $a \in A$, $b \in B$. By the Cut Property, there exists $c \in \mathbb{Q}$ such that for all $a \in A$, $b \in B$, $a \leq c$ and $b \geq c$. Since $A \cup B = \mathbb{Q}$ and A and B are disjoint, either $c \in A$ or $c \in B$. If $c \in A$, then $1 < c < \sqrt{2}$ and

$$c < c + \frac{2 - c^2}{2} < \sqrt{2}$$

   But this means that there's a rational number (namely $c + (2 - c^2)/2$) in A larger than c, which is a contradiction. A similar argument can be made if $c \in B$.

Exercise 1.3.11.

a) True. Since A and B are nonempty and bounded, $\sup A$ and $\sup B$ exist. Since $A \subseteq B$, $a \leq \sup B$ for all $a \in A$, i.e., $\sup B$ is an upper bound for A. Additionally, $\sup A \leq b$ for all upper bounds b of A. Hence, $\sup A \leq \sup B$.

b) True. Since $\sup A$ and $\inf B$ exist, A and B must be nonempty and bounded. Suppose there does not exist $c \in \mathbb{R}$ such that $a < c < b$ for all $a \in A, b \in B$. Then, there exists $a' \in A$ and $b' \in B$ with $b' \leq a'$. (If $a' < b'$ then $a' < a' + (b' - a')/2 = (a' + b')/2 < b'$ so it must be the case that $b' \leq a'$.) But by assumption $a' \leq \sup A < \inf B \leq b'$, which is a contradiction.

c) False. Let $A = B = \varnothing$, then the assumption trivially holds but neither $\sup A$ nor $\inf B$ exists.

Exercise 1.4.1.

a) Let $a = p/q$ and $b = m/n$. Then $ab = (pm)/(qn)$. Since $\mathbb{Z}$ is closed under multiplication, $ab \in \mathbb{Q}$. Similar for $a + b$, given that $\mathbb{Z}$ is closed under addition.

b) Suppose $a + t \in \mathbb{Q}$. Then, there exists $p, q \in \mathbb{Z}$ such that $a + t = p/q$. Now, since $a = m/n$ for some $m, n\mathbb{Z}$, we have $t = p/q - m/n = (np - mq)/(qn)$. But this contradicts the irrationality of t, so $a + t \in \mathbb{I}$. Similar argument for $at \in \mathbb{I}$.

c) Nothing. By part (b), $1 - \sqrt{2}, \sqrt{2}/2 - 1, \sqrt{2}/2 + 1 \in \mathbb{I}$. But $(1 - \sqrt{(}2) + \sqrt{(}2) = 1 \in \mathbb{Q}$ and $(\sqrt{2}/2 - 1) + (\sqrt{2}/2 + 1) = \sqrt{2} \in \mathbb{I}$. Similarly, $\sqrt{2}\sqrt{2} \in \mathbb{Q}$ but $(\sqrt{2} + 1)\sqrt{2} = 2 + \sqrt{2} \in \mathbb{I}$. Hence, $\mathbb{I}$ is neither closed under addition nor multiplication.

Exercise 1.4.2. First, we show that s is an upper bound of A. Let $a' \in A$ and suppose $a' > s$, which implies $a' - s > 0$. By assumption, $s + 1/n \geq a'$ for all $n \in \mathbb{N}$. By the Archimedean Property, there exists $n_0 \in \mathbb{N}$ such that $1/n_0 < a' - s$. But this implies $s + 1/n_0 < a'$, which is a contradiction. Hence, $s \geq a$ for all $a \in A$.

Now, we want to show that s is a least upper bound. Let b be an upper bound for A. By assumption, $s - 1/n < b$ for all $n \in \mathbb{N}$ so $1/n > s - b$. Now, suppose $s > b$. Then $s - b > 0$ and so, by the Archimedean Property, there exists $n_0 \in \mathbb{N}$ with $1/n_0 < s - b$. But this is a contradiction, so $s \leq b$.

Exercise 1.4.3. Suppose there exists $x \in \cap_{n=1}^{\infty}(0, 1/n)$. Then, $0 < x < 1/n$ for all $n \in \mathbb{N}$. But, by the Archimedean Property, there exists $n_0 \in \mathbb{N}$ such that $1/n_0 < x$. This is a contradiction, so $\cap_{n=1}^{\infty}(0, 1/n) = \varnothing$.

Exercise 1.4.5. By Theorem 1.4.3, there exists $r \in \mathbb{Q}$ with $a - \sqrt{2} < r < b - \sqrt{2}$. Hence, $a < r + \sqrt{2} < b$. But, by Exercise 1.4.1 part (a), $r + \sqrt{2} \in \mathbb{I}$, so we're done.

Exercise 1.4.7. Choose $n_0 \in \mathbb{N}$ large enough so that

$$\frac{1}{n_0} < \frac{\alpha^2 - 2}{2\alpha}$$

Then

$$\alpha^2 - \frac{2\alpha}{n_0} > \alpha^2 - (\alpha^2 - 2) = 2.$$

But this contradicts the fact that α is a least upper bound of T. Hence, we conclude that $\alpha^2 = 2$.

Exercise 1.5.1. Let $n_m = \min\left\{n \in \left(\mathbb{N} \setminus \cup_{i=1}^{m-1} n_i\right) \mid f(n) \in A\right\}$ and let $g(m) = f(n_m)$. Now suppose $g(s) = f(n_s) = f(n_t) = g(t)$ for some $s, t \in \mathbb{N}$. Because f is 1-1, $n_s = n_t$. But if $n_s = n_t$ then $s = t$ because if $s < t$ then $n_s \notin \left(\mathbb{N} \setminus \cup_{i=1}^{t-1} n_i\right)$ and a similar argument can be made for $t < s$. Hence, g is 1-1. Additionally, g is onto: if $a \in A$, there exists some $k \in \mathbb{N}$ with $f(k) \in A$ because f is onto. But there there must be some m' such that $k = n_{m'}$ and hence $g(m') = f(k) = a$.

Exercise 1.5.2. NIP is only true for closed intervals on $\mathbb{R}$ and not $\mathbb{Q}$ (since the proof relies on AoC).

Exercise 1.5.3.

a) Suppose A_1 and A_2 are countable and let $B_2 = A_2 \setminus A_1$. Since $B_2 \subseteq A_2$, B_2 is countable or finite by Theorem 1.5.7. Additionally, $A_1 \cup A_2 = A_1 \cup B_2$. Let

$f_1 : \mathbb{N} \to A_1$ be 1-1 and onto. If B_2 is finite, then $B_2 = \{b_1, b_2, \ldots, b_s\}$ and we define

$$g(n) = \begin{cases} b_n & n \leq s \\ f_1(n-s) & \text{n > s} \end{cases}$$

g is 1-1: if $g(n) = g(m)$ then either $g(n), g(m) \in B_2$ or $g(n), g(m) \in A_1$ because A_1 and B_2 are disjoint. In the former case, we have $b_n = b_m$ and conclude $n = m$. In the later case, we have $f_1(n-s) = f_1(m-s)$ and have $n = m$ since f is 1-1.

g is onto: If $x \in B_2$ there is some n with $b_n = x$. Hence, $g(n) = x$. If $x \in A_1$ then the surjectivity of f gives an n with $f(n) = x$. But then $g(n+s) = f_1(n) = x$.

If B_2 is countable with bijection $f_2 : \mathbb{N} \to B_2$, define

$$g(n) = \begin{cases} f_1(n/2) & n \text{ even} \\ f_2((n-1)/2) & n \text{ odd} \end{cases}$$

g is 1-1: if $g(n) = g(m)$ and $g(n), g(m) \in A_1$ we have $f_1(n/2) = f_2(m/2)$ and conclude $n = m$. Otherwise, we have $f_2((n-1)/2) = f_2((m-1)/2)$ and have $n = m$ by the injectivity of f_2.

g is onto: Suppose $x \in A_1$. By the surjectivity of f_1, we have $n \in \mathbb{N}$ with $f_1(n) = x$. Hence, $g(2 * n) = f_1(n) = x$. For $x \in B_1$, we have some n with $f_2(n) = x$ and have $g(2 * n + 1) = f_2(n) = x$ (since $2 * n + 1$ is always odd).

The more general statement follows by induction on m. The inductive step is essentially the proof above.

b) Infinity isn't a number: induction can only be used to prove $\cup_{n=1}^{k} A_n$ is countable for any $k \in \mathbb{N}$.

c) Let R_1 be the set of integers appearing in the first row of the array, R_2 in the second, and so on. Clearly, these sets are all disjoint and there are an infinite number of them. Additionally, we'll annotate each integer in each R_i with its sequence number via a pairing. For example, $R_1 = \{(1,1), (2,3), (3,6), (4,10), \ldots\}$.

Let $f_i : \mathbb{N} \to A_i$ be a bijection for each A_i. We define our bijective function $g : \mathbb{N} \to \cup_{n=1}^{\inf} A_n$ as

$$g(n) = \begin{cases} f_1(s) & \text{if } (s,n) \in R_1 \text{ for some } s \in \mathbb{N} \\ f_2(s) & \text{if } (s,n) \in R_2 \text{ for some } s \in \mathbb{N} \\ & \vdots \end{cases}$$

g is 1-1: If $g(n) = g(m)$ then both $g(n), g(m) \in A_i$ for some i since the A_i's are disjoint. Hence, $g(n) = f_i(n) = f_i(m) = g(m)$. But each f_i is 1-1, so $n = m$.

g is onto: Let $z \in \cup_{n=1}^{\inf} A_n$. Then there is some $i \in \mathbb{N}$ such that $z \in A_i$. But since f_i is onto, there is a $n \in \mathbb{N}$ with $f_i(n) = z$. But since $(n,m) \in R_i$ this means that $g(m) = f_i(n) = z$.

Exercise 1.5.6.

a) The collection $\{(n-1, n) \mid n \in \mathbb{N}\}$ works with bijective function f given by $f(n) = (n-1, n)$.

b) No such collection exists. By Theorem 1.4.3, for any $a, b \in \mathbb{R}$ with $a < b$ there is a $r \in \mathbb{Q}$ with $a < r < b$. Hence, in any non-empty interval (a, b) we have $r \in (a, b)$. Since the collection consists of disjoint intervals, it is easy to construct a bijective function from $\mathbb{Q}$ to the collection. But $\mathbb{Q}$ is countable, so by Exercise 1.5.5, the collection must also be countable.

Exercise 1.5.11.

a) Define $h_g : A' \to B'$ as g^{-1} restricted to A'. That is, $h_g(a') = b'$ if $g(b') = a'$. Such a $b' \in B'$ exists for all $a' \in A$ because g maps B' onto A'. h_g is 1-1: if $h_g(a') = h_g(a'')$ then there are some $b', b'' \in B'$ such that $g(b') = a'$ and $g(b'') = a''$. Hence, $b' = h_g(a') = h_g(a'') = b''$ and we conclude that $a' = a''$. h_g is also onto: if $b' \in B'$ then $h_g(g(b')) = b'$. (We're guaranteed $g(b') \in A'$ because g maps B' onto A'.)

Define $h_f : A \to B$ as f restricted to A. h_f is onto and 1-1 by assumption.

Finally, define

$$h(x) = \begin{cases} h_g & \text{if } x \in A', \\ h_f & \text{if } x \in A \end{cases}$$

h is clearly onto and 1-1.

b) If $A_1 = \varnothing$, then g is onto and we're done (since $Y \sim X$ implies $X \sim Y$). So, assume $A_1 \neq \varnothing$. We show $A_n \cap A_{n+1} = \varnothing$ for all $n \in \mathbb{N}$ by induction on n.

Base case ($n = 1$): We need to show that $A_1 \cap A_2 = \varnothing$. $A_1 = X \setminus g(Y)$ and $A_2 = g(f(A_1)) = g(f(X \setminus g(Y)))$. A_1 consists precisely of all the elements that are *not* in g's range–since $A_2 \subseteq g(Y)$ we conclude that $A_1 \cap A_2 = \varnothing$.

Inductive step: Suppose $A_k \cap A_{k+1} = \varnothing$ for all $k < n$. Note that $f(A_k) \cap f(A_{k+1}) = \varnothing$: if $a_k \in A_k$ and $a_{k+1} \in A_{k+1}$ then $f(a_k) \neq f(a_{k+1})$ because f is 1-1 and A_k, A_{k+1} are disjoint. Similarly, $g(f(A_k)) \cap g(f(A_{k+1})) = \varnothing$ because g is 1-1. But $A_{k+1} = g(f(A_k))$ and $A_{k+2} = g(f(A_{k+1}))$, so we're done.

The fact that the collection $\{f(A_n) \mid n \in \mathbb{N}\}$ follows immediately by the fact that f is 1-1.

c) Observe that

$$A' = X \setminus A = X \setminus \cup_{n=1}^{\infty} A_n = X \setminus (A_1 \cup (\cup_{n=2}^{\infty} A_n)) = X \setminus ((X \setminus g(Y)) \cup (\cup_{n=2}^{\infty} A_2)) \subseteq g(Y)$$

because $X \setminus (X \setminus g(Y)) = g(Y)$. If $z \in A'$ then there is no $n \in \mathbb{Z}^{\geq}$ such that $(gf)^n(a_1) = z$ for all $a_1 \in A_1$. Since $A' \subseteq g(Y)$, there eixsts $b \in B$ with $g(b) = z$. What remains to be shown is that $b \in B'$. Suppose $b \notin B'$. Then $b \in \cup_{n=1}^{\infty} f(A_n)$. But $b = f((gf)^n a_1)$ for some $a_1 \in A_1$ and $n_0 \in \mathbb{N}^{\geq}$, so $g(b) = (gf)^{n_0+1}(a_1) = z$, which is a contradiction. We conclude that $b \in B'$ and thus that g maps B' onto A'.