Exercise 2.2.4.

a) $(-1, 1, -1, 1, -1, \dot{)}$

b) Impossibru. An infinite number of ones requires a "long-term" behavior where the sequence features 1. If the sequence doesn't converge to 1, it also has to feature other numbers–but then the sequence is oscillating between 1s and these other numbers and hence must be divergent or these other numbers must get so close to 1 that the sequence converges to 1.

c) $(1, 2, 2, 3, 3, 3, 4, 4, 4, 4, \dots)$

Exercise 2.2.5.

a) $\lim a_n = 0$. Let $\varepsilon > 0$. Choose $n \in \mathbb{N}$ such that $n \geq 5/\varepsilon + 1$. Then,

$$|a_n - 0| = a_n \leq \left[\left[\frac{5}{(5/\varepsilon)+1}\right]\right] \leq \frac{5}{(5/\varepsilon)+1} < \frac{5}{5/\varepsilon} = \varepsilon$$

as required.

b) $\lim a_n = 1$. Choose $N \in \mathbb{N}$ such that $N > 12/(3\varepsilon - 1)$ and let $n \geq \mathbb{N}$. Then,

$$\begin{aligned}
\left|a_n - \frac{4}{3}\right| &< \left|\left[\left[\frac{12 + 4\,(12/(3\varepsilon-1))}{3\,(12/(3\varepsilon-1))}\right]\right] - 1\right| \\
&\leq \frac{12 + 4\,(12/(3\varepsilon-1))}{3\,(12/(3\varepsilon-1))} - 1 \\
&= \frac{12 + (12/(3\varepsilon-1))}{3\,(12/(3\varepsilon-1))} \\
&= \frac{3\varepsilon}{3} \\
&= \varepsilon
\end{aligned}$$

as required.

Exercise 2.2.6. Suppose $\lim a_n = a$ and also that $\lim a_n = b$ with $a \neq b$. Since $a \neq b$, there exists $\delta > 0$ such that $|a - b| = \delta$. Now, by Definition 2.2.3, for every $\varepsilon > 0$, it follows that $|a_n - a| < \varepsilon$ and $|a_m - b| < \varepsilon$ when $n \geq N$ and $m \geq M$ for some M, N. Choose $\varepsilon = \delta/4$ and set M and N appropriately. Now, let $R = \max\{M, N\}$ and set $r \geq R$. Then, $|a_r - a| < \delta/4$ and $|a_r - b| < \delta/4$. By the triangle inequality,

$$|a - b| \leq |a_r - a| + |a_r - b| < \delta/2 = \frac{|a-b|}{2}$$

which is nonsense because $a - b \neq 0$. By contradiction, $a = b$.

Exercise 2.2.7.

a) Only frequently since $(-1)^n = -1$ for all odd n.

b) Eventually implies frequently.

c) A sequence (a_n) converges to a if, given any ε-neighborhood $V_\varepsilon(a)$ of a, (a_n) is eventually in $V_\varepsilon(a)$.

d) No, the sequence $(-2)^n$ contains an infinite number of 2s but is not eventually in the interval (1.9, 2.1). It is, however, frequently in (1.9, 2.1). Indeed, any sequence containing an infinite number of 2s must be frequently in (1.9, 2.1). If this were not the case, there would be some $N \in \mathbb{N}$ such that for all $n \geq N$, $a_n \neq 2$. But then there would be at most N 2s in the sequence.

Exercise 2.3.1.

a) By the Algebraic Limit Theorem,

$$0 = \lim(x_n) = \lim(\sqrt{x_n}\sqrt{x_n}) = \lim(\sqrt{x})\lim(\sqrt{x})$$

so, $\lim(\sqrt{x}) = 0$.

b) Follows by the same argument in part a).

Exercise 2.3.3. By the Order Limit Theorem, $\lim y_n \leq \lim z_n = l$. Also, $l = \lim x_n \leq \lim y_n$. So $l \leq \lim y_n \leq l$ and we conclude $\lim y_n = l$.

Exercise 2.3.5. Suppose (z_n) is convergent, i.e. $\lim z_n = z$. Then, for all $\varepsilon > 0$ there's an $N \in \mathbb{N}$ such that for all $n \geq N$, $|z_n - z| < \varepsilon$. If n is odd, this is the same as $|x_{((n+1)/2)} - z| < \varepsilon$. If n is even, this is the same as $|y_{n/2} - z| < \varepsilon$. Define $n_x = 2n + 1$ and $n_y = 2n$. Clearly, $n_x \geq 2N + 1$ and $n_y \geq 2N$. Additionally, $|x_{n_x} - z| < \varepsilon$ and $|y_{n_y} - z| < \varepsilon$. We conclude that $\lim x_n = \lim y_n = \lim z_n = z$.

Now suppose that $\lim x_n = \lim y_n = z$. Let $\varepsilon > 0$. Then, there exists $N_x, N_y \in \mathbb{N}$ such that for all $n_x \geq N_x$ and $n_y \geq N_y$, $|x_{n_x} - z| < \varepsilon$ and $|y_{n_y} - z| < \varepsilon$. Set $N = \max\{2N_x, 2N_y\}$. Choose $n \geq N$. Clearly, $n \geq 2n_x - 1$ and $n \geq 2n_y$. If n is odd, then $|z_n - z| = |x_{(n+1)/2} - z|$. But $(n+1)/2 \geq n_x$ so $|x_{(n+1)/2} - z| < \varepsilon$. Similarly, if n is even, $|z_n - z| = |y_{n/2} - z|\varepsilon$. Hence, $|z_n - z| < \varepsilon$ for all $n \geq N$.

Exercise 2.3.7.

a) Let $(x_n) = (n)$ and let $(y_n) = (-n)$. Then, $(x_n + y_n) = (n + (-n)) = (0)$, which obviously converges.

b) Impossible. By the Algebraic Limit Theorem, $\lim(y_n) = \lim(y_n + x_n - x_n) = \lim(y_n + x_n) - \lim(x_n)$.

c) Let $(b_n) = (1/n)$. By Exercise 2.3.6, $\lim(1/n) = 0$.

d) Suppose $(a_n - b_n)$ is bounded. Then, there exists $M > 0$ such that $|a_n - b_n| \leq M$ for all $n \in \mathbb{N}$. Similarly, by Theorem 2.3.2, there eixsts $B > 0$ such that $|b_n| \leq B$ for all $n \in \mathbb{N}$. Since (a_n) is unbounded, for any $K \in \mathbb{R}$, there exists $n_0 \in \mathbb{N}$ such that $|a_{n_0}| > K$. So, choose $n_1 \in \mathbb{N}$ such that $|a_{n_1}| > M + B$. Then, $|a_{n_1} - b_{n_1}| > M + B - b_{n_1} > M$, which is a contradiction.

e) Let $(a_n) = (0)$ and $(b_n) = (-1)^n$. Clearly $(a_n b_n) = (0)$ converges, but (b_n) does not.

Exercise 2.3.8.

a) $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m$. By the Algebraic Limit Theorem,

$$\begin{aligned}\lim(p(x_n)) &= \lim(a_0) + \lim(a_1 x_n) + \lim(a_2 x_n^2) + \cdots + \lim(a_n x_n^m) \\ &= \lim(a_0) + \lim(a_1)\lim(x_n) + \lim(a_2)\lim(x_n)^2 + \cdots + \lim(a_m)\lim(x_n)^m \\ &= a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m \\ &= p(x)\end{aligned}$$

b) Let $f(x) = [[x]]$ and $(x_n) = (1.5)$. Clearly, $\lim f(x_n) = 1$ and $\lim(x_n) = 1.5$.

Exercise 2.3.11.

a) Let $\varepsilon > 0$ and $\lim x_n = x$. We need to find an $N > 0$ such that for all $n \geq N$,

$$\begin{aligned}|y_n - x| &= \left|\frac{x_1 + x_2 + \cdots + x_n}{n} - x\right| \\ &= \left|\frac{x_1 + x_2 + \cdots + x_n - nx}{n}\right| \\ &= \frac{1}{n}\left|(x_1 - x) + (x_2 - x) + \cdots + (x_n - x)\right| \\ &\leq \frac{1}{n}\left(|x_1 - x| + |x_2 - x| + \cdots + |x_n - x|\right) < \varepsilon\end{aligned}$$

Since (x_n) converges, there is an $M > 0$ such that $|x_n - x| < \varepsilon/2$ for all $n > M$. Hence, the above becomes

$$\begin{aligned}&\frac{1}{n}\left(|x_1 - x| + |x_2 - x| + \cdots + |x_n - x|\right) \\ &= \frac{1}{n}\left(|x_1 - x| + |x_2 - x| + \cdots + |x_{M-1} - x|\right) + \frac{1}{n}\left(|x_M - x| + \cdots + |x_n - n|\right) \\ &< \frac{1}{n}\left(|x_1 - x| + |x_2 - x| + \cdots + |x_{M-1} - x|\right) + \frac{\varepsilon}{2}\end{aligned}$$

Now $(|x_1 - x| + \cdots + |x_{M-1} - x|)$ is finite, so we can choose some $R > 0$ large enough such that—with the $1/n$ factor—it's less than $\varepsilon/2$ for all $n \geq R$. Namely,

$$R = \left[\left[\frac{2(|x_1 - x| + \cdots + |x_{M-1} - x|)}{\varepsilon}\right]\right] + 1$$

We choose $N = \max\{R, M\}$ and then have

$$\begin{aligned}|y_n - x| &< \frac{1}{n}\left(|x_1 - x| + |x_2 - x| + \cdots + |x_{M-1} - x|\right) + \frac{\varepsilon}{2} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon\end{aligned}$$

for all $n \geq N$, as required.

b) Consider $(x_n) = (-1)^n$. Then

$$y_n = \frac{(-1) + 1 + (-1) + \cdots + (-1)^n}{n} = \begin{cases} 0 & n \text{ even} \\ -1/n & n \text{ odd}\end{cases}$$

Clearly, (y_n) converges to 0 even though (x_n) does not converge.

Exercise 2.3.12.

a) True, follows immediately by part (iii) for the Order Limit Theorem.

b) True. Suppose $a \in (0,1)$. Then, $|a_n - a| > 0$ for all n since $a_n \notin (0,1)$. But then we can choose $\varepsilon = \text{argmin}_n(|a_n - a|/2)$ and have $|a_n - a| > \varepsilon$ for all n, contradicting the existence of a.

c) False. The sequence where the nth term consists of the best decimal approximation of $\sqrt{2}$ to n places clearly converges to $\sqrt{2}$.

Exercise 2.4.1.

a) We'll use induction to show that $x_n \geq x_{n+1}$ for all $n \in \mathbb{N}$.

Base $(n = 1)$: Clearly, $3 \geq 1/(4-3) = 1$.

Inductive step: Assume $x_n \geq x_{n+1}$ for all $n \leq K$. Then,

$$\begin{aligned} x_K &\leq x_{K+1} \\ 4 - x_K &\geq 4 - x_{K+1} \\ \frac{1}{4 - X_K} &\leq \frac{1}{4 - x_{K+1}} \\ x_{K+1} &\leq x_{K+2} \end{aligned}$$

as required. Hence, (x_n) is decreasing. Since $3 \geq x_n$ for all $n \in \mathbb{N}$, we have that $x_n \geq 0$ and conclude that $|x_n| \leq 3$. By Theorem 2.4.2, (x_n) converges.

b) The sequence (x_{n+1}) is just (x_n) without the first term–clearly, the converge to the same limit.

c) We have,

$$\begin{aligned} \lim x_{n+1} &= \lim \left(\frac{1}{4 - x_n} \right) \\ &= \frac{1}{4 - \lim x_n} && \text{(by the Algebraic Limit Theorem)} \\ &= \frac{1}{4 - \lim x_{n+1}} \end{aligned}$$

Hence, $\lim x_n(4 - \lim x_n) = 1$ or $\lim x_n{}^2 - 4 \lim x_n + 1 = 0$. The roots are $\frac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}$. Since $x_{n+1} < x_n$ and $x_1 = 3$, we have $\lim x_n = 2 - \sqrt{3}$.

Exercise 2.4.2.

a) The argument assumes that the limit exists in the first place (and it does not).

b) Yes, because the limit exists since it is monotone and bounded above (by 3).

Exercise 2.4.3.

a) We have

$$\begin{aligned}
x_n &= \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\cdots\sqrt{2+\sqrt{2}}}}}} \\
&< \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\cdots\sqrt{2+2}}}}} \quad \text{(The final } \sqrt{2} \text{ was replaced with 2)} \\
&= \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\cdots 2}}}} \\
&= \sqrt{2+\sqrt{2+\sqrt{2+2}}} \\
&= 2
\end{aligned}$$

Hence, 2 is an upper bound for the sequence. Additionally, clearly $x_{n+1} > x_n$ for all n, so the sequence is monotone and by Theorem 2.4.2, the sequence must converge and $\lim x_n$ exists. Now, $x_{n+1} = \sqrt{2+x_n}$, So,

$$\begin{aligned}
\lim x_{n+1}^2 &= \lim(2+x_n) \\
\lim x_n^2 &= 2 + \lim x_n \\
&\text{(By the ALT and since } \lim x_{n+1} = \lim x_n\text{)} \\
\lim x_n^2 - \lim x_n - 2 &= 0 \\
(\lim x_n + 1)(\lim x_n - 2) &= 0
\end{aligned}$$

Hence, $\lim x_n = 2$.

b) The n-th term in the sequence contains n (nested) square roots and assume $n > 3$. Then,

$$x_n = \sqrt{2\sqrt{2\sqrt{2\cdots}}} = \prod_{k=1}^{n} 2^{1/2^k} = 2^{\sum_{k=1}^{n} 1/2^k}$$

But $\sum_{k=1}^{\infty} 1/2^k = 1$, so that means that the sequence $(1/2^k)$ must be bounded, and, correspondingly that x_n must be bounded. Hence, $\lim x_n$ must exist. Namely, $\lim x_n = 2$.

Exercise 2.4.4.

a) Suppose that $\mathbb{N}$ is bounded from above and consider the sequence (n). Clearly, (n) is monotone so by the Monotone Congergence Theorem, $\lim n$ exists and we set $\lim n = \alpha$. Now, by the Algebraic Limit Theorem, $\lim(n+1) = \lim n + 1 = \alpha + 1$. But $\lim n = \lim(n+1)$, hence $\alpha = \alpha + 1$, which is a contradiction. We conclude that $\mathbb{N}$ is unbounded from above.

b) In the proof of Theorem 1.4.1, we consider the sequence (a_n), which is clearly monotone and bounded above by any b_n. Hence the limit $\lim a_n = \alpha$ must exist.

To complete the proof, we need to show that for any convergent increasing sequence (x_n), $x_n \leq \lim x_n$ for all n. Suppose there exists an n_0 such that

$\lim x_n < x_{n_0}$. Since the sequence is increasing, this require that for all n, $\lim x_n < x_{n+n_0}$. But $\lim x_n = \lim x_{n+n_0}$ and, by the Order Limit Theorem, $\lim x_n < \lim x_{n+n_0} = \lim_x$, which is a contradiction.

So, we now have that $a_n \leq \alpha \leq b_n$ for all n, as required, and the rest of the proof follows just as it did in the AoC version.

Exercise 2.4.5.

a) Clearly, $x_1^2 = 4 \geq 2$. For the iterative case, we'll show that $x_{n+1}^2 - 2 \geq 0$.

$$\begin{aligned}
x_{n+1}^2 - 2 &= \left(\frac{1}{2}\left(x_n + \frac{2}{x_n}\right)\right)^2 - 2 \\
&= \frac{1}{4}\left(x_n^2 + 4 + \frac{4}{x_n^2}\right) - 2 \\
&= \frac{x_n^2}{4} - 1 + \frac{1}{x_n^2} \\
&= \left(\frac{x_n}{2} - \frac{1}{x_n}\right)^2 \\
&\geq 0
\end{aligned}$$

The fact that the sequence is decreasing (i.e., $x_n - x_{n+1} \geq 0$ for all $n \in \mathbb{N}$), now follows:

$$\begin{aligned}
x_n - x_{n+1} &= x_n - \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) \\
&= \frac{x_n^2 - 2}{2x_n} \\
&\geq 0 && \text{(since } x_n^2 \geq 2\text{)}
\end{aligned}$$

By the Monotone Convergence Theorem, the sequence converges and $x = \lim(x_n)$ for some x. By the definition of convergence and the Algebraic Limit Theorem,

$$\begin{aligned}
x &= \lim(x_n) \\
&= \lim(x_{n+1}) \\
&= \lim\left(\frac{1}{2}\left(x_n + \frac{2}{x_n}\right)\right) \\
&= \frac{1}{2}\lim(x_n) + \frac{1}{\lim(x_n)} \\
&= \frac{x}{2} + \frac{1}{x}
\end{aligned}$$

this means that $0 = \frac{1}{x} - \frac{x}{2}$ or $x^2 = 2$, which yields that $x = \sqrt{2}$.

b) The sequence

$$\begin{aligned}
x_1 &= c^2 \\
x_{n+1} &= \lim\left(\frac{1}{2}\left(x_n + \frac{c}{x_n}\right)\right)
\end{aligned}$$

converges to $\sqrt{c}$, which we can show by following the same steps from part a). Showing that $x_{n+1}^2 \geq c$ for all n:

$$\begin{aligned}
x_{k+1}^2 - c &= \left(\frac{1}{2}\left(x_k + \frac{c}{x_k}\right)\right)^2 - c \\
&= \frac{1}{4}\left(x_k^2 + 2c + \frac{c^2}{x_k^2}\right) - c \\
&= \frac{1}{4}\left(x_k^2 - 2c + \frac{c^2}{x_k^2}\right) \\
&= \frac{1}{4}\left(x_k - \frac{c}{x_k}\right)^2 \\
&\geq 0
\end{aligned}$$

Showing the sequence is decreasing:

$$\begin{aligned}
x_n - x_{n+1} &= x_n - \frac{1}{2}\left(x_n + \frac{c}{x_n}\right) \\
&= \frac{x_n}{2} - \frac{c}{2x_n} \\
&= \frac{x_n^2 - c}{2x_n} \\
&\geq 0 && \text{(since } x_n^2 \geq c\text{)}
\end{aligned}$$

And finally, we have that

$$\begin{aligned}
x &= \lim(x_n) \\
&= \lim(x_{n+1}) \\
&= \lim\left(\frac{1}{2}\left(x_n + \frac{c}{x_n}\right)\right) \\
&= \frac{1}{2}\lim(x_n) + \frac{c}{2\lim(x_n)} \\
&= \frac{x}{2} + \frac{c}{2x}
\end{aligned}$$

which means that $0 = \frac{c}{2x} - \frac{x}{2}$ or $x^2 = c$, which yields that $x = \sqrt{c}$.

Exercise 2.4.7.

a) Since (a_n) is bounded there exists an $M > 0$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}$. This means that each $\{a_k : k \geq n\}$ is bounded above by M. Now,

$$\{a_k : k \geq n\} \subseteq \{a_k : k \geq n+1\}$$

so by Exercise 1.3.11 a (which states that for all nonempty, bounded sets A, B with $A \subseteq B$, $\sup A \leq \sup B$), we have $y_n \leq y_{n+1}$, so (y_n) is decreasing. Additionally, (y_n) is bounded by M since each is $\{a_k : k \geq n\}$. By the Monotone Convergence Theorem, (y_n) converges.

b) The *limit inferior* of (a_n), or $\liminf a_n$, is defined by

$$\liminf a_n = \lim z_n,$$

where $z_n = \inf\{a_k \mid k \geq n\}$. The argument that z_n converges is virtually identical to the argument in a), except with sup replaced with inf and $\leq$ replaced with $\geq$.

c) For any non-empty, bounded set A, $\inf A \leq \inf A$. (Suppose this were not the case: then there would exist a least lower bound i and least upper bound s such that $i \leq a$ for all $a \in A$ and $i > s$. But, $a \leq s$ so $i \leq a \leq s$, which is a contradiction.) Hence, $z_n \leq y_n$ for all n—by the Order Limit Theorem, $\liminf a_n \leq \limsup a_n$.

The inequality is strict for "oscillating" sequences, like $((-1)^n)$.

d) ($\Longrightarrow$) Suppose $\liminf a_n = \limsup a_n = a$. Let $\varepsilon > 0$ and choose $n \in N$ such that $|z_n - a| < \varepsilon$ and $|y_n - a| < \varepsilon$ (such a choice is possible by the convergence of the respective sequences).

As a quick detour that I thought was relevant to this problem but actually isn't and should be ignored entirely, note that for any bounded and decreasing (and hence convergent) sequence (b_n) with limit b, $b_n \geq b$ for all n. To prove this, suppose that there exists an m such that $b_m < b$. Then for all $n \geq m$, we have $b_n < b$ since the sequence is decreasing. Since (b_n) is convergent, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ we have $b - b_n \leq b - b_m$. But for any $n \geq m$ we have $b_m \geq b_n$, and hence $b - b_m \leq b - b_n \leq b - b_m$ which requires that $b_n = b_m$. But then $b - b_n$ is a constant and thus must be 0, i.e. $b = b_n$, which is a contradiction. A similar claim for increasing sequences can be proved almost identically.

Since $z_n \leq x \leq y_n$ for all $x \in \{a_k \mid k \geq n\}$, $|x - a| < \varepsilon$ (since $z_n - a \leq x - a \leq y_n - a < \varepsilon$ and $\varepsilon > a - z_n \geq a - x \geq a - y_n$). Hence, (a_n) converges to a.

($\Longleftarrow$) Suppose $\lim a_n = a$. Then, for any $\varepsilon > 0$, there exists N such that $|a_n - a| < \varepsilon$ for all $n \geq N$. But then

$$-\varepsilon < a_n - a < \varepsilon$$
$$a - \varepsilon < a_n < a + \varepsilon$$

which means that $a - \varepsilon \leq y_n \leq a + \varepsilon$ and $a - \varepsilon \leq z_n \leq a + \varepsilon$. But, then, by the Order Limit Theorem, $a - \varepsilon \leq y \leq a + \varepsilon$ and $a - \varepsilon \leq z \leq a + \varepsilon$ and so, by Theorem 1.2.6, we conclude that $\liminf a_n = \limsup a_n = a$.

Exercise 2.4.9. Suppose $\sum_{n=0}^{\infty} 2^n b_{2^n}$ diverges and consider the partial sums of the sequence (s_m) (corresponding to the sum $\sum_{n=1}^{\infty} b_n$) indexed by 2^k:

$$
\begin{aligned}
s_{2^k} &= b_1 + b_2 + (b_3 + b_4) + \cdots + \left(b_{2^{k-1}+1} + \cdots + b_{2^k}\right) \\
&\geq b_1 + b_2 + (b_4 + b_4) + \cdots + \left(b_{2^{k-1}} + \cdots + b_{2^k}\right) \qquad \text{(since } (b_n) \text{ is decreasing)} \\
&= b_1 + b_2 + 2b_4 + \cdots + 2^{k-1}\left(b_{2^{k-1}} + \cdots + b_{2^k}\right) \\
&= \frac{1}{2}\left(2\left(b_1 + b_2 + 2b_4 + \cdots + 2^{k-1}\left(b_{2^{k-1}} + \cdots + b_{2^k}\right)\right)\right) \\
&= \frac{1}{2}\left(b_1 + \left(b_1 + 2b_2 + 4b_4 + \cdots + 2^k\left(b_{2^{k-1}} + \cdots + b_{2^k}\right)\right)\right) \\
&= \frac{1}{2}\left(b_1 + t_k\right)
\end{aligned}
$$

where t_k is the k-th partial sum of $\sum_{n=0}^{\infty} 2^n b_{2^n}$. Now, since t_k is unbounded, s_{2^k} must be unbounded and we conclude that $\sum_{n=1}^{\infty} b_n$ diverges.

Exercise 2.5.1.

a) Impossible. By the Bolzano-Weierstrass Theorem, the bounded subsequence contains a convergent subsequence, which is also a subsequence of the original sequence.

b) By Example 2.5.3, let $0 < b < 1$, then (b^n) converges to 0. By the Algebraic Limit Theorem, $(1 - b^n)$ converges to 1. Hence, the sequence

$$(b^0, 1 - b^0, b^1, 1 - b^1, \dots)$$

contains subsequences which converge to both 0 and 1.

c) The sequence

$$(1, 1, 1/2, 1, 1/2, 1/3, 1, 1/2, 1/3, 1/4, \dots)$$

works.

d) Impossible, there will always be a subsequence that converges to 0. Consider the intervals $I_k = [0, 1/k]$. Each I_k contains an infinite number of points. (Otherwise the sequence would not contain a subsequence which converges to $1/(k+1)$.) Choose $a_{n_k} \in I_{k+1}$. Then, $|a_{n_k} - 0| < 1/k$. Let $\varepsilon > 0$ and choose N such that $N > 1/\varepsilon$. Then, for all $l > N$, we have $|a_{n_l} - 0| < 1/l < \varepsilon$. Hence, $(a_{n_k}) \to 0$.

Exercise 2.5.2.

a) True. The subsequence $(x_2, x_3, \dots)$ is a proper subsequence and converes to the same limit as (x_n).

b) True. If (x_n) converged to some a, then for any $\varepsilon > 0$, all but a finite number of terms would be contained in $V_\varepsilon(a)$, including all the terms of the divergent subsequence. But this implies that the divergent subsequence converges, which is a contradiction.

c) True. Suppose that there exists only a single subsequence of (x_n) which converges. This implies that in the interval halving process described in the proof of Theorem 2.5.5, all but a finite number of points are contained in a single half for every I_k–otherwise there would be an interval in which both halves contain an infinite number of points, in which case the process described would yield two convergent subsequences by application to each half.

But this implies that the sequence converges to to the same value as the subsequence, which contradicts the fact that it's divergent.

d) True. Suppose (a_{n_k}) converges to a_s. Then, for any $\varepsilon > 0$, there exists an K such that for all $k > K$, $a_{n_k} \in V_\varepsilon(a_s)$. Let $m > n_k$. Then, (taking (a_n) to be decreasing without loss of generality) there exists a k' such that $a_{n_k} \leq a_m \leq a_{n_{k'}}$ since (a_n) is decreasing. But this implies that $a_m \in V_\varepsilon(a_s)$.

Exercise 2.5.3.

a) Let

$$\begin{aligned} b_1 &= a_1 + \cdots + a_{n_1} \\ b_2 &= a_{n_1+1} + \cdots + a_{n_2} \\ &\vdots \end{aligned}$$

Clearly, the partial sums of (b_n) is a subsequence of the partial sums of (a_n). Since the series (a_n) converges, by Theorem 2.5.2, so does the series (b_n).

b) The infinite series at the end of Section 2.1 doesn't converge.

Exercise 2.5.4. Let $S \subset \mathbb{R}$ be bounded from above by M and let $a_1 \in S$. Consider $I_1 = [a_1, M]$ and construct I_2 by bisecting I_1 and choosing the right half (i.e., $[(M - a_1)/2, M]$) if there exists an $a_2 \in S, [(M - a_1)/2, M]$. Otherwise, choose the left half. Continuing in the fashion, we obtain a series of nested intervals I_k, the intersection of which the Nested Interval Property guarantees is non-empty–let $x \in I_k$ for all k. Let $a \in S$. If $a \leq a_1$, clearly $x \geq a$. If $a \geq a_1$, then $a \in [a_1, M]$. Consider the largest k for which $a \in I_k$. That is, $a \notin I_l$ with $l > k$. The only way this can happen is if a is in the left bisection of I_k and the right bisection is chosen. So, in this case, $x \geq a$. If $a \in I_k$ for all k, then $a = x$. This follows since the length of the interval I_k is $(M - a_1)/2^n$ and by the Algebraic Limit Theorem, $((M - a_1)/2^n) \to 0$.

Hence, for any $a \in S$, $x \geq a$ and we've established x as an upper bound for S. Now consider an upper bound y for S and suppose $y < x$. This requires that there exists a k such that y is on the left bisection of I_k and x is on the right with y being strictly less than the midpoint. (If this were not the case, then both x and y would be in I_k for all k, and we'd have $x = y$.) But the right bisection is only chosen when there exists an $a_k \in S$ in the right bisection, meaning that $y < a_k$, which contradicts the fact that y is an upper bound for S. We conclude that $x \leq y$, establishing x as the supremum of S as required.

Exercise 2.5.5. By the Bolzano-Weierstrass Theorem, (a_n) contains a convergent subsequence. Now, consider the proof of the Bolzano-Weierstrass Theorem. At each interval I_k, we're tasked with bisecting it and choosing a half which contains an infinite number of terms. Suppose both halves of I_k contain an infinite number of terms.

If there are an infinite number of terms equal to the midpoint m_k of I_k, then there exists a bounded sequence converging to m_k and hence, by assumption, all subsequences converge to m_k. But this means that, excluding the midpoint, each half only contains a finite number of points–otherwise both the left and right bisections of I_{k+1} will contain an infinite number of points and hence yield bounded sequences converging to different values, which is a contradiction. By a nearly identical argument, both halves of I_k cannot contain an infinite number of terms. Hence, for any k, there exists an N such that for all $n > N$, $a_n \in I_k$. Let $\varepsilon > 0$ and choose N such that $a_n \in I_k$ for all $k > \log_2(M/\varepsilon)$ and $n > N$. Then,

$$|I_k| = M/2^k < M/2^{(\log_2(M/\varepsilon))} = \varepsilon$$

Hence, $|a_n - a| < \varepsilon$ and we have $(a_n) \to a$.

Exercise 2.6.1. Suppose $(x_n) \to x$. Let $\varepsilon > 0$. Since (x_n) converges, there exists an N such that for all $n, m > N$, $|x_n - x| < \varepsilon/2$ and $|x_m - x| < \varepsilon/$. Then,

$$\begin{aligned} |x_n - x_m| &= |x_n - x + x - x_m| \\ &\leq |x_n - x| + |x_m - x| \\ &< \epsilon/2 + \epsilon/2 \\ &= \epsilon. \end{aligned}$$

Hence, (a_n) is a Cauchy sequence.

Exercise 2.6.2.

a) $(1, -1, 1/2, -1/2, 1/4, -1/4, \dots)$ does the trick.

b) Impossible. By Theorem 2.5.2, subsequence of convergent sequences are convergent and by the Cauchy Criterion, a Cauchy sequence is a convergent sequence.

c) Impossible. A divergent monotone sequence is unbounded, but Lemma 2.6.3 says that Cauchy sequences are bounded. All subsequences (a_{n_k}) of divergent monotone sequences must also diverge, otherwise (assuming an increasing sequence without loss of generality) there would exist an M with $a_{n_k} \leq M$ for all k, which implies that $a_n \leq M$ for all n, since for any a_n, there exists a k such that $a_n \leq a_{n_k}$.

d) $(1, 2, 1, 3, 1, 4, 1, 5, \dots)$ does the trick.

Exercise 2.6.7.

a) Let (a_n) be monotone and bounded by M. With loss of generality, assume (a_n) is increasing. By the Bolzano-Weierstrass Theorem, there exists a convergent subsequence (a_{n_k}). By the argument in Exercise 2.6.2 c), (a_n) converges.

b) Let (a_n) be bounded by M. Consider the interval $I_0 = [-M, M]$. Bisecting this interval, we obtain two halves $[-M, M/2]$ and $[M/2, M]$, at least one of which will contain an infinite number of points in the sequence, which we call I_1. I_k is constructed by k such bisections.

   Consider the subsequence (a_{n_k}) where $a_{n_k} \in I_k$. Let $\varepsilon > 0$ and choose $K \in \mathbb{N}$ such that $K > 1 - \log_2(M/\varepsilon)$, which we can always do by the Archimedian Property. then, for all $k, j > K$, we have $M/2^{k-1} < \varepsilon$ and $M/2^{j-1} < \varepsilon$ and hence for any $a_{n_k}, a_{n_j} \in I_K$, we have $|a_{n_k} - a_{n_j}| < \varepsilon$ since $|I_K| = M/2^{K-1}$. Hence, (a_{n_k}) is Cauchy and by the Cauchy Criterion, converges.

c) The Archimedian Property is true on $\mathbb{Q}$ but the Axiom of Completeness does not hold on $\mathbb{R}$, so clearly the Archimedian Property is insufficient to prove the Axiom of Completeness.

Exercise 2.7.1.

a) Let $\varepsilon > 0$. Since (a_n) is decreasing and $(a_n) \to 0$, it's easy to show that $a_n \geq 0$ for all n. There exists an N such that for all $n \geq N$, $a_n < \epsilon$ since $(a_n) \to 0$. Choose $m \geq N$ such that m is odd. Then, since (a_n) is decreasing,

$$\begin{aligned}
&a_m - a_{m+1} + a_{m+2} - a_{m+3} + a_{m+4} - \cdots \pm a_{m+k} \\
&\leq \begin{cases} a_m - a_{m+2} + a_{m+2} - a_{m+4} + a_{m+4} \cdots + a_{m+k} & \text{if } k \text{ is even} \\ a_m - a_{m+2} + a_{m+2} - a_{m+4} + a_{m+4} \cdots - a_{m+k} & \text{if } k \text{ is odd} \end{cases} \\
&= \begin{cases} a_m & \text{if } k \text{ is even} \\ a_m - a_{m+k} & \text{if } k \text{ is odd} \end{cases} \\
&< \varepsilon
\end{aligned}$$

Hence, (s_n) is Cauchy.

b) Define

$$\begin{aligned}
I_0 &= [0, s_1] \\
I_k &= \begin{cases} [s_k, s_{k-1}] & k \text{ even} \\ [s_{k-1}, s_k] & k \text{ odd} \end{cases}
\end{aligned}$$

Then

$$|I_k| = |s_k - s_{k-1}| = a_k$$

and hence $(|I_k|) \to 0$. By the Nested Interval Property, there exists an $s \in I_k$ for all k. Clearly $s_n \in I_K$ for $n \geq K$. For any $\varepsilon > 0$, you can choose a k such that $|I_k| < \varepsilon$. Hence, $s_n \in V_\varepsilon(s)$ for all $n > k$.

c) Note,

$$\begin{aligned}
s_{2n} &= a_1 - a_2 + \cdots + a_{2n-1} - a_{2n} \\
s_{2n+1} &= s_{2n} + a_{2n+1}
\end{aligned}$$

Now, it follows from part a) that both s_{2n} and s_{2n+1} are increasing. Furthermore, $s_{2n} \leq a_1$ and $s_{2n+1} \leq a_1$. So, both sequences are bounded and increasing and by the Monotone Convergence Theorem, must converge. Additionally, both sequences must converge to the same limit since $s_{2n} - s_{2n+1} = a_{2n+1}$, so $(s_{2n} - s_{2n+1}) \to 0$. But since $s_n = s_{2(n/2)}$ for even n and $s_n = s_{2((n-1)/2)+1}$ for odd n, s_n must also converge to the same limit as the two subsequences.

Exercise 2.7.2.

a) $1/(2^n + n) < 1/2^n$ for all $n \in \mathbb{N}$. $\sum_{n=1}^{\infty} 1/2^n$ converges by Example 2.7.5, so by the Comparison Test, $\sum_{n=1}^{\infty} 1/(2^n + n)$ converges.

b) Since $-1 \leq \sin(n) \leq 1$, this series is bounded above by $\sum_{n=1}^{\infty} 1/n^2$, which we know converges. Hence, the series converges.

c) Presumably diverges since $(a_n) \to 1/2$, but it's unlcear to me how to actually show this since the Alternating Series Test says nothing about necessisity.

Exercise 2.7.9.

a) Suppose there exists no N such that $n \geq N$ implies $|a_{n+1}| \leq |a_n| r'$.

$$\lim \left| \frac{a_{n+1}}{a_n} \right| = r$$

means that for any $\varepsilon > 0$, there exists an N such that $n \geq N$ implies

$$\left| \left| \frac{a_{n+1}}{a_n} \right| - r \right| < \varepsilon$$

so

$$-(\varepsilon + r) < \left| \frac{a_{n+1}}{a_n} \right| < \varepsilon + r$$

which we relax to obtain

$$\left| \frac{a_{n+1}}{a_n} \right| \leq \varepsilon + r$$

Since $0 \leq r < r' < 1$, we can choose $\varepsilon = r' - r$ and obtain

$$\left| \frac{a_{n+1}}{a_n} \right| \leq r'$$

as required.

b) $0 < r' < 1$, so this converges by Example 2.7.5.

c) From part a), there exists an N such that for all $n \geq N$, we have $|a_{n+1}| \leq |a_n| r'$. But then

$$|a_{N+m}| \leq |a_{N+(m-1)}| r' \leq |a_{N+(m-2)}| (r')^2 \leq \cdots \leq |a_N| (r')^m$$

Hence, by the Comparison Test and part b) and the Absolute Convergence Test, the series converges.

Exercise 2.7.13.

a)

$$\begin{aligned} \sum_{k=1}^{n} x_k y_k &= s_n y_{n+1} - s_0 y_0 + \sum_{k=1}^{n} s_k (y_k - y_{k+1}) \\ &= s_n y_{n+1} + \sum_{k=1}^{n} s_k (y_k - y_{k+1}) \end{aligned}$$

b) Since $(s_n) \to s$, there exists a $B \geq 0$ such that $s_n \leq B$ for all n. (Since convergent sequences are bounded.) Additionally, (y_k) is decreasing. Hence,

$$s_k(y_k - y_{k+1}) \leq B(y_k - y_{k+1}) \leq B(y_1 - y_{k+1})$$

The series $\sum B(y_1 - y_{k+1})$ converges by the Algebraic Limit Theorem and hence $\sum s_k(y_k - y_{k+1})$ converges absolutely, since $y_k - y_{k+1} \geq 0$, by the Comparison Test.

This immediately leads to a proof of Abel's Test by the Algebraic Limit Theorem.