1 Basics

1.0.1 LTI, CT, SS

$$\boxed{\begin{aligned}\dot{x} &= A^c x + B^c u\\ y &= Cx + Du\end{aligned}}$$

$$\boxed{x(t) = e^{A^c(t-t_0)}x_0 + \int_{t_0}^{t} e^{A^c(t-\tau)}Bu(\tau)d\tau}$$

where $e^{A^c t} := \sum_{n=0}^{\infty} \frac{(A^c t)^n}{n!}$ (always converges)

1.0.2 TI, DT, SS Systems

$$\boxed{x(k+N) = A^N x(k) + \sum_{i=0}^{N-1} A^i Bu(k+N-1-i)}$$

1.0.3 Euler Discretization of Nonlinear, TI Systems

$$\begin{aligned}\dot{x}^c(t) &= g^c(x^c(t), u^c(t))\\ y^c(t) &= h^c(x^c(t), u^c(t))\end{aligned}$$

$$\boxed{\dot{x}^c(t) \approx \frac{x^c(t+T_s)-x^c(t)}{T_s}}$$

$$\boxed{\begin{aligned}x(k+1) &= x(k) + T_s g^c(x(k), u(k)) = g(x(k), u(k))\\ y(k) &= h^c(x(k), u(k)) = h(x(k), u(k))\end{aligned}}$$

1.0.4 Euler Discretization of LTI Systems

$$\begin{aligned}x(k+1) &= Ax(k) + Bu(k)\\ y(k) &= Cx(k) + Du(t)\end{aligned}$$

$$\boxed{A = I + T_s A^c,\ B = T_s B^c,\ C = C^c,\ D = D^c}$$

1.0.5 Exact Discretization of LTI Systems

$$\begin{aligned}x(t_{k+1}) &= e^{A^c T_s}x(t_k) + \int_{t_k}^{t_{k+1}} e^{A^c(t_{k+1}-\tau)}B^c d\tau u(t_k)\\ &= \underbrace{e^{A^c T_s}}_{\triangleq A}x(t_k) + \underbrace{\int_0^{T_s} e^{A^c(T_s-\tau')}B^c d\tau'}_{\triangleq B}u(t_k)\end{aligned}$$

if A invertible: $B = (A^c)^{-1}(A-I)B^c$
Note that the first order approximation to the exact discretization is equivalent to the euler discretization.

$$e^{A^c T_s} = I + A^c T_s$$

1.1 Analysis of LTI DT Systems

1.1.1 Controllability

$$\boxed{\mathcal{C} = \begin{bmatrix}B & AB & \cdots & A^{n-1}B\end{bmatrix}}\ \text{Controllability Matrix}$$

$$\boxed{\text{The system } (A, B) \text{ is controllable if } \mathcal{C} \text{ has full rank.}}$$

A system is called **stabilizable** if there exits a input sequence that returns the state to the origin asymptotically, starting from any point. True if all uncontrollable modes are stable.

$$\boxed{\text{rank}([\lambda_j I - A|B]) = n\ \forall \lambda_j \in \Lambda_A^+ \Rightarrow (A, B) \text{ is stabilizable}}$$

where Λ_A^+ is the set of all eigenvalues of A lying on or outside the unit circle.

1.1.2 Observability

$$\boxed{\mathcal{O} = \begin{bmatrix}C^T & (CA)^T & \cdots & (CA^{n-1})^T\end{bmatrix}^T}$$

Observability matrix

$$\boxed{\text{The system } (C, A) \text{ is observable if } \mathcal{O} \text{ has full rank.}}$$

A system is called **detectable** if it is possible to construct from the measurement sequence a sequence of state estimates that converges to the true state asymptotically, starting from an arbitrary initial estimate. True if all of its unobservable modes are stable.

$$\boxed{\text{rank}([A^T - \lambda_j I|C^T]) = n\ \forall \lambda_j \in \Gamma_A^+ \Rightarrow (A, C)}$$

where Γ_A^+ is the set of all eigenvalues of A lying on or outside the unit circle.

1.2 Stability of nonlinear DT Systems

$$x(k+1) = g(x(k))$$

with an equilibrium point at 0 i.e. $g(0) = 0$.

$$\boxed{\forall\ \epsilon > 0\ \exists\ \delta(\epsilon)\ :\ ||x(0)|| < \delta(\epsilon) \rightarrow ||x(k)|| < \epsilon, \forall k \geq 0}$$

Lyapunov stability

An equilibrium point is **asymptotically stable** in $\Omega \subseteq \mathbb{R}^n$ if it is Lyapunov stable and **attractive**.

$$\boxed{\lim_{k\to\infty} x(k) = 0,\ \forall x(0) \in \Omega}\ \text{Attractivity}$$

and **globally asymptotically stable** if it is asymptotically stable and $\Omega = \mathbb{R}^n$.

1.2.1 Lyapunov function

Definition 1. *Equilibrium point $x = 0$, $\Omega \subset \mathbb{R}^n$ a closed an bounded set containing the origin. A function $V : \mathbb{R}^n \to \mathbb{R}$, continuous at the origin, finite for every $x \in \Omega$, and such that:*

$$\boxed{\begin{aligned}&V(0) = 0 \text{ and } V(x) > 0,\ \forall x \in \Omega \setminus \{0\}\\ &V(g(x)) - V(x) \leq -\alpha(x) \forall x \in \Omega \setminus \{0\}\end{aligned}}$$

where $\alpha : \mathbb{R}^n \to \mathbb{R}$ is continuous positive definite, is called a ***Lyapunov function.***

Theorem 1. *If a system admits a Lyapunov function $V(x)$, then $x = 0$ is* ***asymptotically stable*** *in Ω.*

Theorem 2. *If a system admits a Lyapunov function $V(x)$ for $\Omega = \mathbb{R}^n$, which additionally satisfies $||x|| \to \infty \Rightarrow V(x) \to \infty$ then $x = 0$ is* ***globally asymptotically stable***

Theorem 3. *If the linearization of a nonlinear system around an isolated equilibrium point x_e is stable (unstable),then this equilibrium is an asymptotically stable (unstable) equilibrium of the nonlinear system as well.* ***(Lyapunov indirect method.)***

Sum of L. funs is an L. fun as well.

1.2.2 Lyapunov Stability of LTI DT Systems

Candidate Lyapunov function: $V(x) = x^T Px$ with P positive definite.

$$\begin{aligned}V(Ax(k)) - V(x(k)) &= x^T(k)A^T PAx(k) - x^T Px(k)\\ &= x^T(k)(A^T PA - P)x(k) \leq -\alpha(x(k))\end{aligned}$$

Can choose $\alpha(x(k)) = x^T Qx(k), Q > 0$, need to find $P > 0$ solving

$$\boxed{A^T PA - P = -Q,\ Q > 0}\ \text{DT Lyapunov equation}$$

Theorem 4. *The discrete-time Lyapunov equation has a unique solution $P > 0$ if and only if A has all eigenvalues inside the unit circle, i.e. if and only if the system $x(k+1) = Ax(k)$ is stable.*

$$\boxed{\begin{aligned}\phi(x(0)) &= \sum_{k=0}^{\infty} x(k)^T Qx(k) = \sum_{k=0}^{\infty} x(0)^T (A^k)^T QA^k x(0)\\ &= x(0)^T Px(0)\end{aligned}}$$

To prove existence of Lyapunov function

$$\begin{aligned}x(k)^T \left(\sum_{i=1}^{\infty} (A^i)^T QA^i\right) x(k) &=\\ x(k+1)^T \left(\sum_{i=0}^{\infty} (A^i)^T QA^i\right) x(k+1)&\end{aligned}$$

Approach: 1. Linearize 2. Design LQR 3. If LQR closed-loop is stable (with observability and controllability) the system is stable.

1.2.3 Region of Attraction

1. Determine Lyapunov function.
2. Find for what regions it is valid.

2 Optimal Control

$$\begin{aligned}J^*(x(0)) &:= \min_U x_N^T Px_N + \sum_{i=0}^{N-1}(x_i^T Qx_i + u_i^T Ru_i\\ \text{subj. to } x_{i+1} &= Ax_i + Bu_i,\ i = 0, \ldots, N-1\\ x_0 &= x(0)\end{aligned}$$

- $P \succeq 0$ with $P = P^T$, is the **terminal** weight
- $Q \succeq 0$ with $Q = Q^T$, is the **state** weight
- $R \succ 0$ with $R = R^T$, is the **input** weight

2.0.1 Batch approach

$$\begin{bmatrix}x_0\\ x_1\\ \cdot\\ \cdot\\ \cdot\\ x_N\end{bmatrix} = \underbrace{\begin{bmatrix}I\\ A\\ \cdot\\ \cdot\\ \cdot\\ A^N\end{bmatrix}}_{\mathcal{S}^x} x(0) + \underbrace{\begin{bmatrix}0 & \cdots & \cdots & 0\\ B & 0 & \cdots & 0\\ AB & B & \cdots & 0\\ \vdots & \ddots & \ddots & 0\\ A^{N-1}B & \cdots & AB & B\end{bmatrix}}_{\mathcal{S}^u} \begin{bmatrix}u_0\\ u_1\\ \cdot\\ \cdot\\ \cdot\\ u_N\end{bmatrix}$$

$$X := \mathcal{S}^x x(0) + \mathcal{S}^u U$$

Setting the gradient equal to zero:

$$\begin{aligned}\nabla_U J(x(0), U) &= 2HU + 2F^T x(0) = 0\\ U^*(x(0)) &= -((\mathcal{S}^u)^T \bar{Q}\mathcal{S}^u + \bar{R})^{-1}(\mathcal{S}^u)^T \bar{Q}\mathcal{S}^x x(0)\end{aligned}$$

Optimal cost:

$$\begin{aligned}&J^*(x(0)) =\\ x(0)^T \Big((\mathcal{S}^x)^T \bar{Q}\mathcal{S}^x - (\mathcal{S}^x)^T \bar{Q}\mathcal{S}^u((\mathcal{S}^u)^T \bar{Q}\mathcal{S}^u + \bar{R})^{-1}\bullet&\\ \bullet(\mathcal{S}^u)^T \bar{Q}\mathcal{S}^x\Big)x(0)&\end{aligned}$$

2.0.2 Recursive Approach

$$J_j^*(x(j)) := \min_{U_{j\to N}} x_N^T Px_N + \sum_{i=j}^{N-1}(x_i^T Qx_i + u_i^T Ru_i)$$

$$\begin{aligned}\text{subj. to}\quad x_{i+1} &= Ax_i + Bu_i,\ i = j, \ldots, N-1\\ x_j &= x(j)\end{aligned}$$

Procedure:

1. Start at step N $J_N^*(x_N) := l_f(x_N)$
2. Iterate **backwards** for $i = N-1, \ldots, 0$ (DP iteration)
   $$J_I^*(x_i) := \min_{u_i} l(x_i, u_i) + J_{i+1}^*(Ax_i + Bu_i)$$
3. $J^*(x_0) := J_0^*(x_0)$, optimal controller is the optimizer $u_0^*(x_0)$

2.0.3 LQR

$$\boxed{u_i^* = -(B^T P_{i+1}B + R)^{-1}B^T P_{i+1}Ax(i) := F_i x_i \text{ for } i = 1 : N}$$

$$\boxed{P_i = A^T P_{i+1}A + Q - A^T P_{i+1}B(B^T P_{i+1}B + R)^{-1}B^T P_{i+1}A}$$

Discrete Time Riccati equation (RDE)

2.1 Comparison Batch/Recursive Approach

- Batch optimization returns a sequence of numeric values depending only on the inital state whereas the recursive approach yields feedback policies $u_i^* = F_i x_i$ depending on each x_i.
- They are identical if there are no disturbances.
- The recursive approach is more robust to disturbances and model errors, because if future states deviate the optimal input can still be computed.
- The recursive approach is computationally more attractive because it divides the problem into small calculations.
- Neither one method can deal with inequality constraints.

2.1.1 Infinite Horizon

$$\boxed{\begin{aligned}u^*(k) &= -(B^T P_\infty B + R)^{-1}B^T P_\infty Ax(k)\\ &:= F_\infty(x(k) - x_s) + u_s\end{aligned}}$$

$$\boxed{J_\infty(x(k)) = x^T(k)P_\infty x(k)}$$

$$\boxed{P_\infty = A^T P_\infty A + Q - A^T P_\infty B(B^T P_\infty B + R)^{-1}B^T P_\infty A}$$

- The constant feedback matrix F_∞ is referred to as the asymptotic form of the **Linear Quadratic Regulator (LQR)**.
- The closed-loop system with $u(k) = F_\infty x(k)$ is **guaranteed** to be stable if (A, B) is stabilizable and $(Q^{\frac{1}{2}}, A)$ is detectable.
- The infinite-horizon cost to go is actually a Lyapunov function for the system. Thus $\lim_{k\to\infty} x(k) = 0$
- Choices for the terminal cost:
  1. Equal to P_∞. To find it solve the are with $P_i = P_{i+1}$.
  2. Assuming no control action after the end of the horizon → solve the Lypunov equation for P:
     $$APA^T + Q = P$$
     This approach only makes sense if the system is asymptotically stable.
  3. If we want the state and the input both to be zero after the end of the finite horizon, no P but an additional constraint is needed:
     $$x_{i+N} = 0$$

3 Convex Optimization

3.1 Convex sets

- The **intersection** of two or more convex sets is convex.
- The **union** of two complex sets is not necessarily convex.

3.2 Convex function

A function $f : \text{dom}(f) \to \mathbb{R}$ is **convex** iff $\text{dom}(f)$ is convex and

$$\boxed{\begin{aligned}f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y),&\\ \forall \lambda \in (0, 1),\ \forall x, y \in \text{dom}(f)&\end{aligned}}$$

First order condition for convexity

$$\boxed{f(y) \geq f(x) + \nabla f(x)^T (y - x),\ \forall\ x, y \in \text{dom}(f)}$$

Second order condition for convexity

$$\boxed{\nabla^2 f(x) \succeq 0,\ \forall\ x \in \text{dom}(f)}$$

If $\text{dom}(f)$ is convex and $\nabla^2 f(x) \succ 0\ \forall x \in \text{dom}(f)$, then f is strictly convex.

3.3 Convex Optimization Problem

$$\begin{aligned}\min_{x\in\text{dom}(f)}\ & f(x)\\ \text{subj. to } g_i(x) &\leq 0 \quad i = 1, \ldots, m\\ h_i(x) &= 0 \quad i = 1, \ldots, p\end{aligned}$$

where it is necessary that, f convex, $\text{dom}(f)$ convex, g_i convex, $h_i(x) = a_i^T x - b$ affine!

Thus the problem can be rewritten as:

$$\begin{aligned}\min_{x\in\text{dom}(f(x))}\quad & f(x)\\ \text{subj. to}\quad & g_i(x) \leq 0\ i = 1, \ldots, m\\ & Ax = b \quad A \in \mathbb{R}^{p\times m}\end{aligned}$$

Feasible set of a convex optimization problem is convex.

4 Duality

4.1 The Lagrange Dual Problem

Primal Problem

$$\begin{aligned}\min_{x\in\text{dom}(f)}\ & f(x)\\ \text{subj. to } g_i(x) &\leq 0\ i = 1, \ldots m\\ h_i(x) &= 0\ i = 1, \ldots p\end{aligned}$$

$$L(x, \lambda, \nu) = f(x) + \sum_{i=1}^{m} \lambda_i g_i(x) + \sum_{i=1}^{p} \nu_i h_i(x)$$

Lagrange Dual function:

$$d(\lambda, \nu) = \inf_{x\in\text{dom}(f)} L(x, \lambda, \nu)$$

- To find d solve $\nabla_x L(x, \lambda, \nu) = 0$ and insert back into L.
- The dual function is always a **convex** function.

Dual Problem
Every $\nu \in \mathbb{R}^p$, $\lambda \geq 0$ produces a lower bound for p^*. Which is the best?

$$\begin{aligned}\max_{\lambda,\nu}\ & d(\lambda, \nu)\\ \text{subj. to } & \lambda \geq 0\end{aligned}$$

- Problem (D) is **convex** even if (P) is not.
- Problem (D) has optimal value $d^* \leq p^*$.

Assume one inequality constraint only:

$$\mathcal{G} := \{(u, t)\,|\, t = f(x),\ u = g(x),\ x \in \mathcal{X}\}$$

Primal problem:

$$p^\star = \min\{t\,|\,(u, t) \in \mathcal{G},\ u \leq 0\}$$

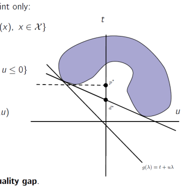
Dual function:

$$d(\lambda) = \min_{(u,t)\in\mathcal{G}}(t + \lambda u)$$

Dual problem:

$$d^\star = \max_{\lambda\geq 0} d(\lambda)$$

The quantity $p^\star - d^\star$ is the **duality gap**.



t-axis for $f(x)$, u-axis for $g(x)$ therefore only points with $u < 0$ are feasible.

4.1.1 Example: Dual of a Linear Program (LP)

$$\begin{aligned} \min_{x\in\mathbb{R}^n} \; & c^T x \\ \text{subj. to } & Ax = b \\ & Cx \le e \end{aligned}$$

$$\begin{aligned} d(\lambda,\nu) &= \min_{x\in\mathbb{R}^n} \left[c^T x + \nu^T(Ax-b) + \lambda^T(Cx-e)\right] \\ &= \min_{x\in\mathbb{R}^n} \left[(A^T\nu + C^T\lambda + c)^T x - b^T\nu - e^T\lambda\right] \\ &= \begin{cases} -b^T\nu - e^T\lambda & \text{if } A^T\nu + C^T\lambda + c = 0 \\ -\infty & \text{otherwise} \end{cases} \end{aligned}$$

Thus the dual problem is:

$$\begin{aligned} \max_{\lambda,\nu} \; & -b^T\nu - e^T\lambda \\ \text{subj. to } & A^T\nu + C^T\lambda + c = 0 \\ & \lambda \ge 0 \text{ dual feasibility} \end{aligned}$$

$$\boxed{\text{The dual of an LP is also an LP}}$$

4.1.2 Example: Dual of a Quadratic Program

$$\begin{aligned} \min_{x\in\mathbb{R}^n} \; & \tfrac{1}{2}x^T Qx + c^T x \\ \text{subj. to } & Cx \le e \end{aligned}$$

with $Q \succ 0$

The dual function is:

$$\begin{aligned} d(\lambda) &= \min_{x\in\mathbb{R}^n} \left[\tfrac{1}{2}x^T Qx + c^T x + \lambda^T(Cx-e)\right] \\ &= \min_{x\in\mathbb{R}^n} \left[\tfrac{1}{2}x^T Qx + (c + C^T\lambda)^T x - e^T\lambda\right] \end{aligned}$$

The optimal x satisfies: $Qx + c + C^T\lambda = 0$

Substituting $x = -Q^{-1}(c + C^T\lambda)$ into $d(\lambda)$

$$d(\lambda) = -\tfrac{1}{2}(c + C^T\lambda)^T Q^{-1}(c + C^T\lambda) - e^T\lambda$$

The **dual** problem is then:

$$\begin{aligned} \min_{\lambda} \; & \tfrac{1}{2}\lambda^T C^T Q^{-1} C\lambda + (CQ^{-1}c + e)^T\lambda + \tfrac{1}{2}c^T Q^{-1} c \\ & \text{subj. to } \lambda \ge 0 \end{aligned}$$

$$\boxed{\text{The dual of a QP is a QP as well!}}$$

4.2 Weak Duality

- It is **always** true that $d^* \le p^*$.
- Sometimes the dual is much easier to solve than the primal (or vice-versa).

If $p^* \ne d^*$ then $p^* - d^*$ is the **duality gap**.

4.3 Strong Duality

- It is sometimes true that $d^* = p^*$.
- Strong duality does not hold for non-convex problems.

4.3.1 Slater Condition (valid for conv.opt. prob.)

If there is at least one **strictly feasible point**, i.e.

$$\{\, x \mid Ax = b,\, g_i(x) < 0,\ \forall i \in \{1,\dots,m\}\} \ne \emptyset$$

$$\boxed{\text{Then } p^* = d^*}$$

4.4 Primal and Dual Solution Properties

Assume that strong duality holds, with optimal solution x^* and (λ^*,ν^*).

1. From strong duality: $d^* = p^* \Rightarrow d(\lambda^*,\nu^*) = f(x^*)$
2. From the definition of the dual function:

$$f(x^*) = d(\lambda^*,\nu^*) = f(x^*) + \underbrace{\sum_{i=1}^{m}\lambda_i^* g_i(x^*)}_{=0} + \underbrace{\sum_{i=1}^{p}\nu_i^* h_i(x^*)}_{=0}$$

3. **Complementary Slackness**

$$\begin{aligned} \lambda_i^* = 0 \text{ for every } g_i(x^*) < 0 &\rightarrow \textbf{inactive constraint} \\ g_i(x^*) = 0 \text{ for every } \lambda_i^* > 0 &\rightarrow \textbf{active constraint} \end{aligned}$$

4.5 Karush-Kuhn-Tucker Conditions

Assume that all g_i and h_i are differentiable. **Necessary** conditions for optimality:

1. Primal Feasibility:

$$\begin{aligned} g_i(x^*) &\le 0 & i &= 1,\dots,m \\ h_i(x^*) &= 0 & i &= 1,\dots,p \end{aligned}$$

2. Dual Feasibility:

$$\lambda^* \ge 0$$

- Introduce constraints in D s.t. $(\lambda,\nu) \in \mathrm{dom}(d)$!

3. Complementary Slackness:

$$\lambda_i^* g_i(x^*) = 0 \;\; i = 1,\dots,m$$

4. Stationarity:

$$\begin{aligned} &\nabla_x L(x^*,\lambda^*,\nu^*) = \\ &\nabla f(x^*) + \sum_{i=1}^{m}\lambda_i^*\nabla g_i(x^*) + \sum_{i=1}^{p}\nu_i^*\nabla h_i(x^*) = 0 \end{aligned}$$

For a convex optimization problem:

1. If (x^*,λ^*,ν^*) satisfy the KKT conditions, then $p^* = d^*$.
   - $p^* = f(x^*) = L(x^*,\lambda^*,\nu^*)$ (due to complementary slackness.
   - $d^* = g(\lambda^*,\nu^*) = L(x^*,\lambda^*,v^*)$ (due to convexity of the functions and stationarity)
2. If the Slater conditions holds ($\rightarrow$ strong duality), then
   - x^* is optimal **if and only if** there exist (λ^*,ν^*) satisfying the KKT conditions.

4.5.1 Example: KKT Conditions for a QP

Provide a possibility to check whether a point is an optimum.

$$\begin{aligned} \min_{x\in\mathbb{R}^2} \; & \tfrac{1}{2}x^T Qx + c^T x \\ \text{subj. to } & Ax = b \\ & x \ge 0 \end{aligned}$$

The **Lagr.** is $L(x,\lambda,\nu) = \frac{1}{2}x^T Qx + c^T x + \nu^T(Ax-b) - \lambda^T x$

The KKT conditions are:

$$\begin{aligned} \nabla_x L(x,\lambda,\nu) = Qx + A^T\nu - \lambda + c &= 0 && \text{[stationarity]} \\ Ax &= b && \text{[primal feasibility]} \\ x &\ge 0 && \text{[primal feasibility]} \\ \lambda &\ge 0 && \text{[dual feasibility]} \\ x_i\lambda_i = 0 \quad i &= 1..n && \text{[complementarity]} \end{aligned}$$

The final three conditions can be written as $0 \le x \perp \lambda \ge 0$

4.5.2 Global Sensitivity Analysis

λ_i^* large and $u_i < 0 \;\Rightarrow p^*(u,v)$ incr. greatly.

λ_i^* small and $u_i > 0 \;\Rightarrow p^*(u,v)$ does not decr. much.

$\begin{Bmatrix} v^* \text{ large, positive, } v_i < 0 \\ v^* \text{ large, negative, } v_i > 0 \end{Bmatrix} \Rightarrow p^*(u,v)$ incr. greatly.

$\begin{Bmatrix} v^* \text{ small, positive, } v_i > 0 \\ v^* \text{ small, negative, } v_i < 0 \end{Bmatrix} \Rightarrow p^*(u,v)$ does not decr. much.

Note that the results are not symmetrical and that we only found a lower bound on $p^*(u,v)$.

4.5.3 Local Sensitivity Analysis

Assume **strong duality** for the unperturbed problem with (ν^*,λ^*) dual optimal. Weak duality for the perturbed problem then implies

$$\begin{aligned} p^*(u,v) &\ge d^*(\nu^*,\lambda^*) - u^T\lambda^* - v^T\nu^* \\ &= p^*(0,0) - u^t\lambda^* - v^T\nu^* \end{aligned}$$

If in addition $p^*(u,v)$ is differentiable at $(0,0)$ then

$$\lambda_i^* = -\frac{\partial p^*(0,0)}{\partial u_i}, \qquad \nu_i^* = -\frac{\partial p^*(0,0)}{\partial v_i}$$

- λ_i^* is sensitivity of p^* relative to i^{th} inequality.
- ν_i^* is sensitivity of p^* relative to i^{th} equality.

5 Constrained Finite Time Optimal Control

5.1 Constrained Optimal Control: Quadratic Cost

$$J(x_0,U) = x_N^T P x_N + \sum_{i=0}^{N-1} x_i^T Q x_i + u_i^T R u_i$$

with $P \succeq 0,\ Q \succeq 0,\ R \succ 0$

CFTOC:

$$\begin{aligned} J^*(x(k)) &= \min_U J(x_0,U) \\ \text{sb.t.} \quad & x_{i+1} = Ax_i + Bu_i,\ i = 0\dots N-1 \\ & x_i \in \mathcal{X},\ u_i \in \mathcal{U},\ i = 0\dots N-1 \\ & x_N \in \mathcal{X}_f \\ & x_0 = x(k) \end{aligned}$$

5.1.1 Construction of the QP with substitution

1. Rewrite the cost as

$$\begin{aligned} J(x(k),U) &= U^T HU + 2x(k)^T FU + x(k)^T Yx(k) \\ &= \begin{bmatrix} U^T & x(k)^T \end{bmatrix} \begin{pmatrix} H & F^T \\ F & Y \end{pmatrix} \begin{bmatrix} U^T & x(k)^T \end{bmatrix}^T \end{aligned}$$

2. Rewrite the constraints compactly as

$$\mathcal{X} = \{x \mid A_x x \le b_x\} \quad \mathcal{U} = \{u \mid A_u u \le b_u\} \quad \mathcal{X}_f = \{x \mid A_f x \le b_f\}$$

$$GU \le w + Ex(k)$$

$$G = \begin{bmatrix} A_u & 0 & \dots & 0 \\ 0 & A_u & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_u \\ 0 & 0 & \dots & 0 \\ A_xB & 0 & \dots & 0 \\ A_xAB & A_xB & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_fA^{N-1}B & A_fA^{N-2}B & \dots & A_fB \end{bmatrix}, E = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -A_x \\ -A_xA \\ -A_xA^2 \\ \vdots \\ -A_fA^N \end{bmatrix}, w = \begin{bmatrix} b_u \\ b_u \\ \vdots \\ b_u \\ b_x \\ b_x \\ b_x \\ \vdots \\ b_f \end{bmatrix}$$

3. Rewrite the constrained optimal control problem as

$$\begin{aligned} J^*(x(k)) &= \min_U \begin{bmatrix} U^T & x(k)^T \end{bmatrix} \begin{pmatrix} H & F^T \\ F & Y \end{pmatrix} \begin{bmatrix} U^T & x(k)^T \end{bmatrix}^T \\ \text{sb.t.} \quad & GU \le w + Ex(k) \end{aligned}$$

   Then we can find a solution for every k which results in a piecewise affine solution.

5.1.2 Construction of the QP without substitution

Resulting QP problem:

$$\begin{aligned} J^*(x(k)) &= \min_z \begin{bmatrix} z^T & x(k)^T \end{bmatrix} \begin{pmatrix} \bar{H} & 0 \\ 0 & Q \end{pmatrix} \begin{bmatrix} z^T & x(k)^T \end{bmatrix}^T \\ \text{sb.t.} \quad & G_{in}z \le w_{in} + E_{in}x(k) \\ & G_{eq}z = E_{eq}x(k) \end{aligned}$$

where $z = \begin{bmatrix} x_1^T & \cdots & x_N^T & u_0^T & \cdots & u_{N-1}^T \end{bmatrix}^T$

Equalities from System dynamics: $x_{i+1} = Ax_i + Bu_i$

$$G_{\text{eq}} = \left[\begin{array}{cccc:cccc} I & & & & -B & & & \\ -A & I & & & & -B & & \\ & -A & I & & & & -B & \\ & & \ddots & \ddots & & & & \ddots \\ & & -A & I & & & & -B \end{array}\right], E_{\text{eq}} = \begin{bmatrix} A \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Inequalities: $G_{in}z \le w_{in} + E_{in}x(k)$

$$\mathcal{X} = \{x \mid A_x x \le b_x\} \quad \mathcal{U} = \{u \mid A_u u \le b_u\} \quad \mathcal{X}_f = \{x \mid A_f x \le b_f\}$$

$$G_{\text{in}} = \left[\begin{array}{ccccc:ccccc} 0 & & & & & 0 & & & & \\ & A_x & & & & & 0 & & & \\ & & \ddots & & & & & \ddots & & \\ & & & A_x & & & & & 0 & \\ & & & & A_f & & & & & 0 \\ \hdashline 0 & & & & & A_u & & & & \\ & 0 & & & & & A_u & & & \\ & & \ddots & & & & & \ddots & & \\ & & & 0 & & & & & A_u & \\ & & & & 0 & & & & & A_u \end{array}\right] \qquad w_{\text{in}} = \left[\begin{array}{c} b_x \\ b_x \\ \vdots \\ b_x \\ b_f \\ \hdashline b_u \\ b_u \\ \vdots \\ b_u \\ b_u \end{array}\right]$$

$$E_{\text{in}} = \begin{bmatrix} -A_x^\top & 0 & \cdots & 0 \end{bmatrix}^\top$$

Cost function from MPC $x_N^T P x_N + \sum_{i=0}^{N-1} x_i^T Q x_i + u_i^T R u_i$

$$\bar{H} = \left[\begin{array}{cccc:ccc} Q & & & & & & \\ & \ddots & & & & & \\ & & Q & & & & \\ & & & P & & & \\ \hdashline & & & & R & & \\ & & & & & \ddots & \\ & & & & & & R \end{array}\right]$$

Matlab hint:

```
barH = blkdiag(kron(eye(N-1),Q), P, kron(eye(N),R))
```

5.2 Constrained Optimal Control: 1-Norm and ∞-Norm

$$J(x_0,U) := ||PX_N||_p + \sum_{i=0}^{N-1} ||Qx_i||_p + ||Ru_i||_p$$

with $p = 1$ or $p = \infty$, P, Q, R full rank column matrices

5.2.1 l_∞ (Chebyshev) minimization

$$\begin{aligned} \min_{x\in\mathbb{R}^n} \; & ||x||_\infty \\ \text{sb.t. } & Fx \le g \end{aligned}$$

$$\begin{aligned} \min_{x\in\mathbb{R}^n} \; & [\max\{x_1,\dots,x_n,-x_1,\dots,-x_n\}] \\ \text{sb.t. } & Fx \le g \end{aligned}$$

$$\begin{aligned} \min_{x,t} \; & t \\ \text{sb.t.} \quad & x_i \le t \;\; i = 1\dots n \\ & -x_i \le t \;\; i = 1\dots n \\ & Fx \le g \end{aligned}$$

$$\begin{aligned} \min_{x,t} \; & \\ \text{sb.t.} \quad & -\mathbf{1}t \le x \le \mathbf{1}t \\ & Fx \le g \end{aligned}$$

5.2.2 l_1 minimization

Constrained l_1 minimization

$$\begin{aligned} \min_{x\in\mathbb{R}^n} \; & ||x||_1 \\ \text{sb.t. } & Fx \le g \end{aligned}$$

$$\begin{aligned} \min_{x\in\mathbb{R}^n} \; & \left[\sum_{i=1}^{n}\max\{x_i,-x_i\}\right] \\ \text{sb.t. } & Fx \le g \end{aligned}$$

$$\begin{aligned} \min_{x\in\mathbb{R}^n, t\in\mathbb{R}^n} \; & t_1 + \cdots + t_n \\ \text{sb.t. } x_i &\le t_i \;\; i = 1\dots m \\ -x_i &\le t_i \;\; i = 1\dots m \\ Fx &\le g \end{aligned}$$

$$\begin{aligned} \min_{x\in\mathbb{R}^n, t\in\mathbb{R}^n} \; & \mathbf{1}^T t \\ \text{sb.t. } & -t \le x \le t \\ & Fx \le g \end{aligned}$$

5.2.3 Construction of the LP for l_∞

Following the procedure above the original problem can be written as:

$$
\begin{aligned}
\min_z \quad & \epsilon_0^x + \dots + \epsilon_N^x + \epsilon_0^u + \dots + \epsilon_{N-1}^u \\
\text{subj. to} \quad & -\underline{1}_n \epsilon_i^x \leq \pm Q \left[A^i x_0 + \sum_{j=0}^{i-1} A^j B u_{i-1-j} \right] \\
& -\underline{1}_r \leq \pm P \left[A^N x_0 + \sum_{j=0}^{N-1} A^j B u_{N-1-j} \right] \\
& \underline{1}_m \epsilon_i^u \leq \pm R u_i \\
& A^i x_0 + \sum_{j=0}^{i-1} A^j B u_{i-1-j} \in \mathcal{X}, \; u_i \in \mathcal{U} \\
& A^N x_0 + \sum_{j=0}^{N-1} A^j B u_{N-1-j} \in \mathcal{X}_f \\
& x_0 = x(k), \; i = 0, \dots, N-1
\end{aligned}
$$

which in standard LP form can be written as

$$
\begin{aligned}
\min_z \quad & c^T z \\
\text{subj. to} \quad & \bar{G} z \leq \bar{w} + \bar{S} x(k)
\end{aligned}
$$

where $z := \{\epsilon_0^x, \dots, \epsilon_N^x, \epsilon_0^u, \dots, \epsilon_{N-1}^u, u_0^T, \dots, u_{N-1}^T\} \in \mathbb{R}^s$ and $s = (m+1)N + N + 1$

$$
\bar{G} = \begin{bmatrix} G_\epsilon & 0 \\ 0 & G \end{bmatrix}, \bar{S} = \begin{bmatrix} S_\epsilon \\ S \end{bmatrix}, \bar{w} = \begin{bmatrix} w_\epsilon \\ w \end{bmatrix}
$$

6 Invariance

Definition 2. *Let A and B be subsets of $\mathbb{R}^n$. The **Minkowski Sum** is:*

$$A \oplus B := \{x + y | x \in A, y \in B\}$$

$$[a, b] \oplus [c, d] = [a + c, b + d] \quad \textit{Scalar case}$$

Definition 3. *Let A and B be subsets of $\mathbb{R}^n$. The **Pontryagin Difference** is:*

$$A \ominus B := \{x | x + e \in A \forall e \in B\}$$

$$[a, b] \ominus [c, d] = [a - c, b - d] \quad \textit{Scalar case}$$

6.1 Invariance

$$\boxed{x(k) \in \mathcal{O} \Rightarrow x(k+1) \in \mathcal{O}, \; \forall \; k \in \{0, 1, \dots\}}$$

Positive Invariant Set

If the invariant set is within the constraints, it provides a set of initial states from which the trajectory will never violate the system constraints.

$$\boxed{\mathcal{O}_\infty \subset \mathcal{X}} \text{ Maximal Positive Invariant Set}$$

if $0 \in \mathcal{O}_\infty$, $\mathcal{O}_\infty$ is invariant and contains all invariant sets that contain the origin.

Pre Set: Given a set S and the dynamic system $x(k+1) = g(x(k))$, the pre set S is the set of states that evolve into the target set S in one time step.

$$\text{pre(s)} := \{x | g(x) \in S\}$$

Input: g, $\mathcal{X}$
Output: $\mathcal{O}_\infty$
$\Omega_0 \leftarrow \mathcal{X}$
loop
$\Omega_{i+1} \leftarrow \text{pre}(\Omega_i) \cap \Omega_i$
if $\Omega_{i+1} = \Omega_i$ **then**
return $\mathcal{O}_\infty = \Omega_i$
end if
end loop

Invariant Set Conditions: A set $\mathcal{O}$ is a positive invariant set if and only if:

$$\mathcal{O} \subseteq \text{pre}(\mathcal{O})$$

6.2 Controlled Invariance

Control Invariant Set: A set $\mathcal{C} \subseteq \mathcal{X}$ is said to be control invariant if:

$$x(k) \in \mathcal{C} \Rightarrow \exists u(k) \in \mathcal{U} \text{ s.t. } g(x(k), u(k)) \in \mathcal{C} \; \forall \; k \in \mathbb{N}^+$$

Maximum Control Invariant Set $\mathcal{C}_\infty$:
The set $\mathcal{C}_\infty$ is control invariant and contains all control invariant sets contained in $\mathcal{X}$.

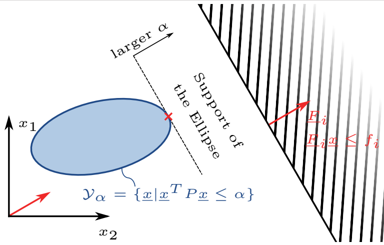
$$\boxed{\text{pre(S)} := \{x | \exists u \in \mathcal{U} \text{ s.t. } g(x, u) \in S\}} \text{ Pre-Set}$$

- For box constraints and linear system check **all** corner points to find the invariant controller.

6.2.1 Pre-Set Computation: Controlled System

$$
\begin{aligned}
x(k+1) &= Ax(k) + Bu(k) \\
u(k) \in \mathcal{U} &:= \{u | Gu \leq g\} \\
S &:= \{x | Fx \leq f\}
\end{aligned}
\qquad
\begin{aligned}
&\text{pre(S)} \\
&\{x | \exists u \in \mathcal{U}, \; Ax + Bu \in S\} \\
&\{x | \exists u \in \mathcal{U}, \; FAx + FBu \leq f\} \\
&\{x | \; \exists u, \begin{bmatrix} FA & FB \\ 0 & G \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \leq \begin{bmatrix} f \\ g \end{bmatrix}\}
\end{aligned}
$$

6.2.2 Example



Change of variables: $y := P^{\frac{1}{2}} x$

$$
\begin{aligned}
h_{Y_\alpha}(\gamma) &= \max_y \gamma^T P^{-\frac{1}{2}} y \\
\text{sb.t. } & y^T y \leq \sqrt{\alpha^2}
\end{aligned}
$$

which can be solved by inspection:

$$h_{Y_\alpha} = \gamma^T P^{-\frac{1}{2}} \frac{P^{-\frac{1}{2}} \gamma}{||P^{-\frac{1}{2}} \gamma||} \sqrt{\alpha} = ||P^{-\frac{1}{2}} \gamma|| \sqrt{\alpha}$$

The solution follows as:

$$
\begin{aligned}
\alpha^* &= \max_\alpha \alpha \text{ s.t. } ||P^{-1/2} F_i T||^2 \alpha \leq f_i^2 \; \forall i \in \{1, \dots, n\} \\
&= \min_{i \in \{1, \dots, n\}} \frac{f_i^2}{F_i P^{-1} F_i^T}
\end{aligned}
$$

7 Terminal Cost, Constraint and Controller

- **Infinite-Horizon**
  Solution of the RHC problem with $N = \infty \rightarrow$ open loop trajectories are the same as closed loop trajectories.
  - Problem feasible $\rightarrow$ closed loop trajectories will always be feasible.
  - Cost finite $\rightarrow$ states and inputs will converge to origin.
- **Finite-Horizon**
  RHC is „short-sighted“strategy approximating $N = \infty$-controller but:
  - **Feasibility**: After some steps the problem might become infeasible even without disturbance and modelling uncertainty.
  - **Stability**: The generated control inputs may not lead to convergent trajectories.

7.1 Proof of Feasibility and Stability

1. Prove recursive feasibility by showing the existence of a feasible control sequence at all time instants when starting from a feasible initial point.
2. Prove stability by showing that the optimal cost function is a Lyapunov function.

There are two possible cases:
1. Terminal constraint at zero: $x_N = 0$
2. Terminal constraint in some (convex) set: $x_N \in \mathcal{X}_f$

7.2 Stability of MPC - Main Result

Assumptions:
1. Stage cost positive definite.
2. Terminal set is invariant under the local control law $\kappa_f(x_i)$:
   $$x_{i+1} A x_i + B\kappa_f(x_i) \in \mathcal{X}_f \; \forall \; x_i \in \mathcal{X}_f$$
   All state and input constraints are satisfied in $\mathcal{X}_f$
   $$\mathcal{X}_f \subseteq \mathcal{X}, \; \kappa_f(x_i) \in \mathcal{U}, \; \forall \; x_i \in \mathcal{X}_f$$
3. Terminal cost is a continuous **Lyapunov function** in the terminal set $\mathcal{X}_f$
   $$l_f(x_{i+1} - l_f(x_i) \leq -l(x_i, \kappa_f(x_i)), \; \forall \; x_i \in \mathcal{X}_f$$

Under those 3 assumptions:
Theorem: The closed-loop system under the MPC control law $u_0^*(x)$ is asymptotically stable and the set $\mathcal{X}_f$ is positive invariant for the system.

7.3 Choice of terminal Sets and Cost - QP

- $\mathcal{X}_f = 0$ simplest choice but small region of attraction for small N
- Alternatively design LQR and get F_∞, P_∞.
- Terminal weight $P = P_\infty$, thus a possible invariant terminal set is
  $$\mathcal{X}_f^\beta = \{x \in \mathbb{R}^n | \underline{x}^T P_\infty \underline{x} \leq \beta\}$$
  thus a sublevel set of a Lyapunov function (invariant by definition). Use the ellipse-support approach to find β.
- $\mathcal{X}_f \rightarrow$ maximum invariant set for closed-loop:
  $$x_{k+1} = Ax_k + BF_\infty(x_k) \in \mathcal{X}_f \forall x_k \in \mathcal{X}_f$$

Then all the **Assumptions** of the Feasibility and Stability Theorem are verified.

8 Practical Issues

8.1 Reference Tracking

$$
\begin{aligned}
& \min u_s^T R u_s \\
\text{sb.t. } & \begin{bmatrix} I - A & -B \\ C & 0 \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} 0 \\ r \end{bmatrix} \\
& x_s \in \mathcal{X}, \; u_s \in \mathcal{U}
\end{aligned}
$$

If no solution exists compute reachable set point that is closest to r:

$$
\begin{aligned}
& \min (Cx_s - r)^T Q_s (Cx_s - r) \\
& \text{sb.t. } x_s = Ax_s + Bu_s \\
& x_s \in \mathcal{X}, \; u_s \in \mathcal{U}
\end{aligned}
$$

$$
\begin{aligned}
\min \quad & \sum_{i=0}^{N-1} \Delta x_i^T Q \Delta x_i + \Delta u_i^T R \Delta u_i + V_f(\Delta x_n) \\
\text{sb.t.} \Delta x_0 \quad & = \Delta x(k) \\
\Delta x_{i+1} \quad & = A \Delta x_i + B \Delta u_i \\
H_x \Delta x_i \quad & \leq k_x - H_x x_s \\
H u \Delta u_i \quad & \leq k_u - H_u u_s \\
\Delta x_N \quad & \in \mathcal{X}_f
\end{aligned}
$$

Convergence

Assume feasibility in $x_s \in \mathcal{X}, \; u_s \in \mathcal{U}$ and choose terminal weight $V_f(x)$ and constraint $\mathcal{X}_f$ satisfying:
- $\mathcal{X}_f \subset \mathcal{X}, \; Kx \in \mathcal{U} \; \forall \; x \in \mathcal{X}_f$
- $V_f(x^+) - V_f(x) \leq -l(x, Kx) \; \forall \; x \in \mathcal{X}_f$

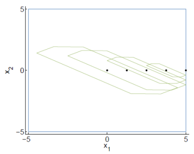
If in addition the target reference is such that

$$x_s \bigoplus \mathcal{X}_f \subset \mathcal{X}, \; K\Delta x + u_s \in \mathcal{U} \; \forall \; \Delta x \in \mathcal{X}_f$$

then the closed-loop system converges to the target reference.

8.2 Scaling the terminal Set

For tracking, if chosing $x_s \neq 0$ the terminal set has to be shifted with x_s. A large terminal set may only allow for a small set of feasible target since if it is moved to much its extreme states become infeasible. For that reason the moving terminal set is scaled down when getting close to state constraints.



8.3 Offset-free Tracking

Suppose the observer is stable and the number of outputs n_y equals the dimension of the constant disturbance n_d. The observer state satisfies:

$$\begin{bmatrix} A - I & B \\ C & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_\infty \\ u_\infty \end{bmatrix} = \begin{bmatrix} -B_d \hat{d}_\infty \\ y_{m,\infty} - C_d \hat{d}_\infty \end{bmatrix}$$

where $y_{m,\infty}$ and u_∞ are the steady state measured outputs and inputs.
$\Rightarrow$ Observer output $C\hat{x}_\infty + C_d \hat{d}_\infty$ tracks the measurement $y_{m,\infty}$ without offset.

This leads to a new condition at steady-state:

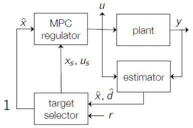
$$
\begin{aligned}
x_s &= Ax_s + Bu_s + B_d \hat{d}_\infty \\
y_s &= Cx_s + C_d \hat{d}_\infty = r
\end{aligned}
$$

Thus we adapt the target condition according to the disturbance:

$$\begin{bmatrix} A - I & B \\ C & 0 \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} -B_d \hat{d} \\ r - C_d \hat{d} \end{bmatrix}$$

In practice:
1. Estimate state and disturbance, $\hat{x}$, $\hat{d}$
2. Obtain (x_s, u_s) from steady state target problem using disturbance estimate.
3. Solve MPC problem for tracking using disturbance estimate $\hat{d}$:

$$
\begin{aligned}
\text{subj. to } & x_0 = \hat{x}(k) \\
& d_0 = \hat{d}(k) \\
& x_{i+1} = Ax_i + Bu_i + B_d d_i, \quad i = 0, \dots, N \\
& d_{i+1} = d_i, \quad i = 0, \dots, N \\
& x_i \in \mathcal{X}, \; u_i \in \mathcal{U}, \quad i = 0, \dots, N-1 \\
& x_N - x_s \in \mathcal{X}_f
\end{aligned}
$$



Main result:

- $\kappa(\hat{x}(k), \hat{d}(k), r(k)) = u_0^*$

- $n_d = n_y$
- RHC recursively feasible and unconstrained for $k \geq j$ with $j \in \mathbb{N}^+$.
- Closed-loop system:

$$
\begin{aligned}
x(k+1) &= f(x(k), \kappa(\hat{x}, \hat{d}, r) \\
\hat{x}(k+1) &= (A + L_x C)\hat{x} + (B_d + L_x C_d)\hat{d} \\
&\quad + B\kappa(\hat{x}, \hat{d}, r) - L_x y_m(k) \\
\hat{d}(k+1) &= L_d C \hat{x}(k) + (i + L_d C_d)\hat{d}(k) - L_d y_m(k)
\end{aligned}
$$

converges, i.e. $\hat{x} \rightarrow \hat{x}_\infty, \hat{d} \rightarrow \hat{d}_\infty, y_m \rightarrow y_{m,\infty}$

Then $y_m(k) \rightarrow r$ as $k \rightarrow \infty$

8.4 Enlarging the Feasible Set

The introduction of a terminal set reduces the feasible set. $\rightarrow$ MPC **without terminal constraint**, with guaranteed stability.

Possible if:
- initial state lies in sufficiently small subset of feasible set.
- N is sufficiently large.

such that the terminal state satisfies the terminal constraint without enforcing it in the optimization. Thus the solution of the finite horizon MPC problem corresponds to the infinite horizon solution.

Advantage: Controller defined in larger feasible set. **Disadvantage**: Characterization of region of attraction or specification of required horizon length extremly difficult.

With larger horizon length N, region of attraction approaches maximum control invariant set.

8.5 Soft Constraints

$$
\begin{aligned}
& \min_z f(z) \\
\text{sb.t. } & g(z) \leq 0 \quad \text{Original}
\end{aligned}
\qquad
\begin{aligned}
& \min_{z,\epsilon} f(z) + l_\epsilon(\epsilon) \\
& \text{sb.t. } g(z) \leq \epsilon \\
& \epsilon \geq 0 \quad \text{Softened}
\end{aligned}
$$

- Solution: Combine quadratic and linear cost:
  $$l_\epsilon(\epsilon) = v \cdot \epsilon + s \cdot \epsilon^2$$
  where $v \geq \lambda^*, \; s > 0$
  $$\boxed{v_i > \lambda_i^*} \text{ Exactness}$$

8.5.1 Simplification: Separation of Objectives

1. Minimize violation over horizon:
   $$
   \begin{aligned}
   \epsilon^{\min} \quad & = \text{argmin}_{U,\epsilon} \epsilon_i^T S \epsilon_i + v^T \epsilon_i \\
   & \text{s.t. } x_{i+1} = Ax_i + Bu_i \\
   & H_x x_i \leq K_x + \epsilon_i \\
   & H_u u_i \leq K_u \\
   & \epsilon_i \geq 0
   \end{aligned}
   $$
   Now fix the slack variables!
2. Optimize controller performance:
   $$
   \begin{aligned}
   \min_u \quad & \sum_{i=0}^{N-1} x_i^T Q x_i + u_i^T R u_i + x_N^T P x_N \\
   \text{s.t.} \quad & x_{i+1} = Ax_i + Bu_i \\
   & H_x x_i \leq k_x + \epsilon_i^{\min} \\
   & H_u u_i \leq k_u
   \end{aligned}
   $$

- **Advantage:** Simplifies tuning, constraints will be satisfied if possible.
- **Disadvantage:** Requires the solution of two optimization problems.

9 Robust MPC

9.0.1 Constraint Satisfaction

In order to robustly enforce constraints of a linear system the concept of robust invariance is developed:

First the MPC prediction is broken into two parts:

$$\begin{aligned} \phi_{i+1} & \quad A\phi_i + Bu_i + w_i \\ u_i & \in \mathcal{U} \\ \phi_i & \in \mathcal{X} \forall W \in \mathcal{W}^N \end{aligned}$$

- $i = 0 \dots N-1$
- Optimize over control actions.
- Enforce constraints explicitly by imposing $\phi_i \in \mathcal{X}$ and $u_i \in \mathcal{U}$ for all sequences W.

$$\begin{aligned} \phi_N & \in \mathcal{X}_f \\ \phi_{i+1} & = (A+BK)\phi_i + w_i \end{aligned}$$

- $i = N, \dots$
- Assume control law to be linear $u_i = K\phi_i$.
- Enforce constraints implicitly by constraining ϕ_N to be in a **robust invariant set** $\mathcal{X}_f \subseteq \mathcal{X}$ and $K\mathcal{X}_f \subseteq \mathcal{U}$ for the system $\phi_{i+1} = (A+BK)\phi_i + w_i$.

9.0.2 Robust Invariance

A set $\mathcal{O}^{\mathcal{W}}$ is said to be a robust positive invariant set for the autonomous system $x(k+1) = g(x(k), w(k))$ if
$x \in \mathcal{O}^{\mathcal{W}} \Rightarrow g(x, w) \in \mathcal{O}^{\mathcal{W}}$, for all $w \in \mathcal{W}$.

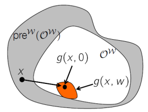
Robust Pre-Set: Given a set Ω and the dynamic system $x(k+1) = g(x(k), w(k))$, the pre-set of Ω is the set of states that evolve into the target set Ω in one time step for all values of the disturbance $w \in \mathcal{W}$:

$$\text{pre}^{\mathcal{W}}(\Omega) := \{x | g(x, w) \in \Omega \text{ for all } w \in \mathcal{W}\}$$

9.0.3 Robust Invariant Set Conditions

A set $\mathcal{O}^{\mathcal{W}}$ is a robust positive invariant set if and only if

$$\mathcal{O}^{\mathcal{W}} \subseteq \text{pre}^{\mathcal{W}}(\mathcal{O}^{\mathcal{W}})$$



For computing the maximum robust invariant set use the algorithm from the nominal case, replacing $\text{pre}(\Omega)$ by $\text{pre}^{\mathcal{W}}(\Omega)$.

9.0.4 Ensuring satisfaction of robust contstraints

Goal: Ensure that constraints are satisfied for the MPC sequence:

$$\phi_i(x_0, U, W) = \left\{ x_i + \sum_{j=0}^{i-1} A^j w_j \middle| W \in \mathcal{W}^i \right\} \subseteq \mathcal{X}$$

Assume that $\mathcal{X} = \{x | Fx \leq f\}$ then this is equivalent to:

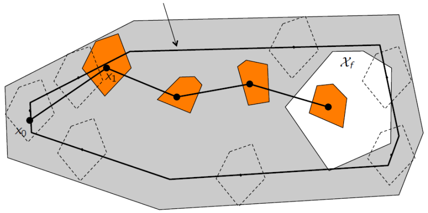
$$Fx_i + F\sum_{j=0}^{i-1} A^j w_k \leq f \forall W \in \mathcal{W}^i$$

This leads to:

$$Fx_i \leq f - \max_{W \in \mathcal{W}^i} F\sum_{j=0}^{i-1} A^j w_j = f - h_{\mathcal{W}^i}\left(F\sum_{j=0}^{i-1} A^j\right)$$

What this results in is a tightening of the constraints on the nominal system!

Tightened constraints for ϕ_1



For the terminal state constraint we can do exactly the same.

9.1 Open-Loop MPC

$$\min_U \sum_{i=0}^{N-1} l(x_i, u_i) + l_f(x_N)$$

$$\begin{aligned} \text{subj. to } x_{i+1} & = Ax_i + Bu_i \\ x_i & \in \mathcal{X} \ominus (\mathcal{W} \oplus A\mathcal{W} \oplus \dots \oplus A^{i-1}\mathcal{W}) \\ u_i & \in \mathcal{U} \\ x_N & \in \mathcal{X}_f \ominus (\mathcal{W} \oplus A\mathcal{W} \oplus \dots \oplus A^{i-1}\mathcal{W}) \end{aligned}$$

9.2 Tube-MPC

Seperate the availabe control authority into two parts:

1. A portion that determines the optimal trajectory to the origin for the nominal system.
$$z(k+1) = Az(k) + Bv(k)$$
2. A portion that compensates for deviations from this system, i.e. a 'tracking' controller, to keep the real trajectory close to the nominal.
$$u_i = K(x_i - z_i) + v_i$$
for some linear controller K, which stabilizes the nominal system.

9.2.1 Compute $\mathcal{E}$

What is the set F_i that contains all possible states x_i?

$$F_i = \mathcal{W} \bigoplus A\mathcal{W} \dots \bigoplus A^{i-1}\mathcal{W} = \bigoplus_{j=0}^{i-1} A^j\mathcal{W}, \ F_0 := \{0\}$$

Minimal Robust Invariant Set

Input: A
Output: F_∞
$\Omega_0 \leftarrow \{0\}$
loop
  $\Omega_{i+1} \leftarrow \Omega_i \oplus A^i\mathcal{W}$
  if $\Omega_{i+1} = \Omega_i$ **then**
    return $F_\infty = \Omega_i$
  end if
end loop

- A finite n does not always exist, but a 'large' n is a good approximation
- If n is not finite, there are other methods of computing small invariant sets, which will be slightly larger than F_∞

$$[a, b] \oplus [c, d] = [a+c, b+d] \qquad \sum_{i=0}^{\infty} x^i = \frac{1}{1-x} \text{ if } |x| < 1$$

9.2.2 Constraint Tightening

We want to work with the nominal system but ensure that the noisy system satisfies constraints!

$$\boxed{z_i \bigoplus \mathcal{E} \subseteq \mathcal{X} \Leftarrow z_i \in \mathcal{X} \ominus \mathcal{E}} \text{ Sufficient condition}$$

The set $\mathcal{E}$ is known offline - thus the tightened constraints can be computed offline.

For the input:

$$\boxed{u_i \in K\mathcal{E} \bigoplus v_i \subset \mathcal{U} \Leftarrow v_i \in \mathcal{U} \ominus K\mathcal{E}}$$

$$[a, b] \ominus [c, d] = [a-c, b-d]$$

9.2.3 Tube-MPC Problem Formulation

$$\mathcal{Z}(x_0) := \left\{ Z, V \middle| \begin{array}{ll} z_{i+1} = Az_i + Bv_i & i \in [0, N-1] \\ z_i \in \mathcal{X} \ominus \mathcal{E} & i \in [0, N-1] \\ v_i \in \mathcal{U} \ominus K\mathcal{E} & i \in [0, N-1] \\ z_N \in \mathcal{X}_f & \\ x_0 \in z_0 \bigoplus \mathcal{E} & \end{array} \right\}$$

$$J(Z, V := \sum_{i=0}^{N-1} l(z_i, v_i) + l_f(z_N)$$

$$(V^*(x_0), Z^*(x_0)) = \text{argmin}_{V,Z}\{J(Z, V | (Z, V) \in \mathcal{Z}(x_0)\}$$

$$\mu_{tube}(x) := K(x - z_0^*(x)) + v_0^*(x)$$

- Optimizing the nominal system, with tightened state and input constraints.
- First tube center is optimization variable → has to be within $\mathcal{E}$ of x_0.
- The cost is with respect to the tube centers (nominal system).
- The terminal set is with respect to the tightened constraints.

9.2.4 Tube-MPC Assumptions

1. The stage cost is a positive function, i.e. it is strictly positive and only zero at the origin.
2. The terminal set is invariant **for the nominal system** under the local control law $\kappa_f(z)$:
$$Az + B\kappa_f(z) \in \mathcal{X}_f \text{ for all } z \in \mathcal{X}_f$$
All **tightened state and input constraints** are satisfied in $\mathcal{X}_f$:
$$\mathcal{X}_f \in \mathcal{X} \ominus \mathcal{E}, \ \kappa_f(z) \in \mathcal{U} \ominus K\mathcal{E} \text{ for all } z \in \mathcal{X}_f$$
3. Terminal cost is a continuous Lyapunov function in the terminal set $\mathcal{X}_f$:
$$l_f(Az + B\kappa_f(z)) \leq -l(z, \kappa_f(z)) \text{ for all } z \in \mathcal{X}_f$$
And thus $\mathcal{X}_f$ is a level set of l_f.

9.3 Summary: Tube MPC

– Offline –

1. Choose a stabilizing controller K such that $||A + BK|| \leq 1$.
2. Compute the minimal robust invariant set $\mathcal{E} = F_\infty$ for the system $x(k+1) = (A+BK)x(k) + w(k), \ w \in \mathcal{W}^1$.
3. Compute the tightened constraints
$\tilde{\mathcal{X}} := \mathcal{X} \ominus \mathcal{E}, \ \tilde{\mathcal{U}} := \mathcal{U} \ominus K\mathcal{E}$
4. Choose terminal weight function l_f and constraint $\mathcal{X}_f$ satisfying the assumptions made.

– Online –

1. Measure / estimate state x.
2. Solve the problem
$(V^*(x), Z^*(x)) = \text{argmin}_{V,Z}\{J(Z, V) | (Z, V) \in \mathcal{Z}(x)\}$
3. Set the input to $u = K(x - z_0^*(x)) + v_0^*(x)$

Benefits:

- Less conservative than open-loop robust MPC, since we are actively compensation for the disturbance.
- Works for unstable systems.
- Optimization problems to solve are simple.

Cons:

- Sub-optimal MPC (optimal is extremely difficult).
- Reduced feasible set when compared to nominal MPC.
- We need to know what $\mathcal{W}$ is (this is usually not realistic).

10 Robustness of Nominal MPC

We want to control the noisy system

$$x(k+1) = Ax(k) + Bu(k) + w(k)$$

Now running standard MPC on that gives us the following closed-loop system:

$$x(k+1) = Ax(k) + Bu_0^*(x(k)) + w(k)$$

for which we can prove convergence to a neighbourhood of the origin (for linear systems), but depending on the noise realization it may not be feasible.

10.1 Do we still have Lyapunov decrease?

Nominally

$$J^*(Ax + Bu^*(x)) - J^*(x) \leq -l(x, u^*(x))$$

But now our state develops as follows:

$$x(k+1) = Ax(k) + Bu^*(x(k)) + w(k)$$

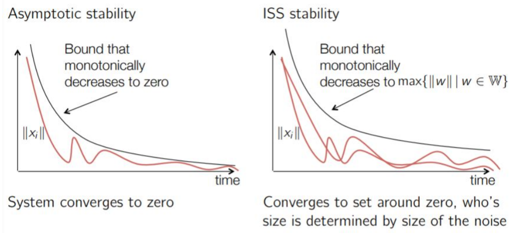
The optimal cost J^* is continuous for linear systems, convex constraints and continuous stage costs:

$$|J^*(Ax + Bu^*(x) + w) - J^*(Ax + Bu^*(x))| \leq \gamma||Ax + Bu^*(x) + w - (Ax + Bu^*(x))|| = \gamma||w||$$

Thus the Lyapunov decrease can be bounded as:

$$\begin{aligned} & J^*(Ax + Bu^*(x) + w) - J^*(x) \\ & = J^*(Ax + Bu^*(x) + w) - J^*(x) \\ & \quad - J^*(Ax + Bu^*(x)) + J^*(Ax + Bu^*(x)) \\ & \leq J^*(Ax + Bu^*(x)) - J^*(x) + \gamma||w|| \\ & \leq -l(x, u^*(x)) + \gamma||w|| \end{aligned}$$

- Amount of decrease grows with $||x||$
- Amount of increase is upper bounded by $\max\{||w|| \ |w \in \mathcal{W}\}$
- Thus we move towards the origin until there is a balance between the size of x and the size of w. Thus the system is **Input-to-State-Stable (ISS)**



11 Implementation

11.1 Explicit MPC

- Linear MPC + Quadratic or linear-norm cost ⇒ Controller is piecewise affine function
- We can pre-compute the controller offline
- Online evaluation of PWA is very fast
- This is only possible for very small systems (3-6 states)

When there is an explicit solution to the MPC problem posed, the optimization can be solved offline, resulting in a control law that is piecewise affine. Thus for finding the current control action, the system state has to be located within the partitioned feasible polyhedron. This search can be done sequentially or through a search tree:

11.1.1 Sequential Search vs. Search Tree

- **Sequential Search**
  - Very simple
  - Linear in number of regions
- **Search Tree**
  - Offline construction of a search tree by finding hyperplanes that separate regions into two equally sized parts and repeating that for the resulting subsets.
  - Potentially logarithmic
  - Significant offline processing (reasonable for <1000 regions)

11.1.2 Gradient Methods

Idea: Gradient ∇f gives direction of steepest local ascent. ⇒ Make steps of size h into anti-gradient direction.

$$\boxed{x^{(i+1)} = x^{(i)} - h^{(i)}\nabla f(x^{(i)})}$$

12 Nonlinear MPC

- Presented assumptions on the terminal set and cost did not rely on linearity
- Lyapunov stability is a general framework to analyze stability of nonlinear dynamic systems
- **Results can be directly extended to nonlinear systems**
- Computing the sets $\mathcal{X}_f$ and function l_f can be very difficult.

Practical approaches include:

- Choose zero terminal constraint (no terminal cost needed)
- Linearization (for quadratic cost)
  - Linearize system around origin, assuming the linearization is stabilizable.
  - Design auxiliary controller $\kappa_f(x) = Kx$, terminal cost $l_f(x) = x^T Px$ and constraint set $\mathcal{X}_f = \{x | x^T Px \leq \alpha\}$ for linearized system s.t.
    * $l_f(A' + B'K)x) - l_f(x) = -2x^T(Q + K^T RK)x \ \forall x \in \mathcal{X}_f$
    * All state and input constraints are satisfied in $\mathcal{X}_f$
    * α is small enough such that

$$\begin{aligned} & l_f(g(x, Kx)) - l_f((A' + B'K)x) \\ & \leq x^T(Q + K^T RK)x && \forall x \in \mathcal{X}_f \\ \Rightarrow \ & l_f(g(x, Kx)) - l_f(x) \\ & \leq -x^T(Q + K^T RK)x && \forall x \in \mathcal{X}_f \end{aligned}$$

Terminal cost is a Lyapunov function in the terminal set and terminal set is invariant also for the nonlinear system.

- At each time step: Linearize the system around a trajectory (usually solution from previous time step). Solve convex problem.