Zusammenfassung Analysis III

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Inhaltsverzeichnis

1 Laplacetransformation

1.1 Definition

$\mathscr{L}f(s) = \int_0^\infty f(t)e^{-st}dt = F(s) \qquad \mathscr{L}f = \mathscr{L}g \Rightarrow f(t) = g(t)$

1.2 Existenz

1. Stückweise kontinuierlich
2. $\exists \lim_{x\to\infty} f(x)\ \forall x_0$ die Endpunkte sind
3. $\exists\, k, M > 0\ |f(t)| \leqslant Me^{kt} \forall\ t \geqslant 0$

1.3 Eigenschaften

1. $\mathscr{L}(\alpha f + \beta g) = \alpha\mathscr{L}(f) + \beta\mathscr{L}(g)$
   $\mathscr{L}^{-1}(\alpha F + \beta G) = \alpha\mathscr{L}^{-1}(F) + \beta\mathscr{L}^{-1}(G)$
2. $\lim_{s\to\infty} \mathscr{L}f(s) = 0$
3. $sF(s)$ is bounded for $s \to \infty$
4. $\mathscr{L}f$ is continuos $\forall t \in [\alpha,\beta], \alpha > k$

1.4 s-shift

$\mathscr{L}(e^{at}f(t)) = F(s-a) \Leftrightarrow e^{at}f(t) = \mathscr{L}^{-1}(F(s-a))$

1.5 t-shift

$\mathscr{L}(u(t-a)f(t-a)) = e^{-as}\mathscr{L}f(s)$
$u(t-a)f(t-a) = \mathscr{L}^{-1}(e^{-as}\mathscr{L}f(s))$
$\mathscr{L}(u(t-a)\cdot 1) = \frac{e^{-as}}{s}$

1.6 Sifting

$\int_0^\infty g(t)\delta(t-a)dt = g(a) \qquad \mathscr{L}(\delta(t-a)) = e^{-as}$
$\mathscr{L}(\delta(t-a)g(t)) = g(a)e^{-as}$

1.7 Derivative

$\mathscr{L}(f^{(n)}) = s^n\mathscr{L}(f) - \sum_{j=0}^{n-1} s^{n-1-j}f^{(j)}(0)$
$\mathscr{L}(f') = s\mathscr{L}(f) - f(0) \qquad \mathscr{L}(f'') = s^2\mathscr{L}(f) - sf(0) - f'(0)$

1.8 Integral

$\mathscr{L}(\int_0^t f(x)dx) = \frac{1}{s}F(s)$
$\mathscr{L}(t\cdot g(t)) = -\mathscr{L}'(g(t)) = -\frac{d}{ds}\mathscr{L}(g(t))$

1.9 Convolution

$\mathscr{L}(f*g) = \mathscr{L}(f)\mathscr{L}(g)$
$f*g = \int_0^t f(t')g(t-t')dt'$
$f*g = g*f \qquad f*(g+h) = f*g + f*h$
$f*0 = 0 \qquad f*1 \neq 1 \qquad f*f \geq 0$

1.10 Identitäten

$\mathscr{L}(1) = \frac{1}{s} \qquad \mathscr{L}(t^n) = \frac{n!}{s^{n+1}} \qquad \mathscr{L}(e^{-at}) = \frac{1}{s+a}$
$\mathscr{L}(\sin(\omega t)) = \frac{\omega}{s^2+\omega^2} \qquad \mathscr{L}(\cos(\omega t)) = \frac{s}{s^2+\omega^2}$
$\mathscr{L}(\sinh(\omega t)) = \frac{w}{s^2-\omega^2} \qquad \mathscr{L}(\cosh(\omega t)) = \frac{s}{s^2-\omega^2}$
$\mathscr{L}(\sin^2(\omega t)) = \frac{2\omega^2}{s(s^2+4\omega^2)}$

2 Fourier-Reihen

2.1 Definition

$f(x) = a_0 + \sum_{k=1}^\infty [a_n\cos(\frac{n\pi}{L}x) + b_n\sin(\frac{n\pi}{L}x)]$
Periode p $= 2L$
$a_0 = \frac{1}{2L}\int_{-L}^L f(x)dx$
$a_n = \frac{1}{L}\int_{-L}^L f(x)\cos(\frac{n\pi}{L}x)dx$
$b_n = \frac{1}{L}\int_{-L}^L f(x)\sin(\frac{n\pi}{L}x)dx$
$R(f)(x_0) = \frac{1}{2}(f(x_0^-) + f(x_0^+)) \qquad f(x_0^\pm) = \lim_{\epsilon\to 0} f(x\pm\epsilon)$

2.2 Komplexe Fourierreihe

$e^{i\omega x} = \cos(\omega x) + i\cdot\sin(\omega x)$
$\cos(t) = \frac{e^{it}+e^{-it}}{2} \qquad \sin(t) = \frac{e^{it}-e^{-it}}{2i}$
$f(x) = \sum_{n=-\infty}^\infty C_n e^{\frac{in\pi x}{L}} \qquad C_n = \frac{1}{2L}\int_{-L}^L f(x)e^{\frac{-i\pi nx}{L}}dx$

Umformungen:

a) komplex $\to$ reell	b) reell $\to$ komplex
$a_0 = 2\cdot c_0$	$c_0 = \frac{a_0}{2}$
$a_n = c_n + c_{-n}$	$c_n = \frac{1}{2}(a_n - ib_n)$
$b_n = i(c_n - c_-)$	$c_{-n} = \frac{1}{2}(a_n + ib_n)$

2.3 Gerade

$f(x) = f(-x)$
$a_0 = \frac{1}{L}\int_0^L f(x)dx$
$a_n = \frac{2}{L}\int_0^L f(x)\cos(\frac{n\pi x}{L})dx$
$b_n = 0$

2.4 Ungerade

$f(-x) = -f(x)$
$a_{0,n} = 0$
$b_n = \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi x}{L})dx$

2.5 Orthogonalität

$$\int_{-L,0}^{L,2\pi}\cos(\tfrac{n\pi x}{L,\pi})\cos(\tfrac{m\pi x}{L,\pi})dx = \begin{cases} 0 & n\neq m \\ L,\pi & n=m\neq 0 \\ 2L,2\pi & n=m=0 \end{cases}$$

$$\int_{-L,0}^{L,2\pi}\sin(\tfrac{n\pi x}{L,\pi})\sin(\tfrac{m\pi x}{L,\pi})dx = \begin{cases} 0 & n\neq m \\ L,\pi & n=m\neq 0 \\ 0 & n=m=0 \end{cases}$$

$$\int_{-L,0}^{L,2\pi}\sin(\tfrac{n\pi x}{L,\pi})\cos(\tfrac{m\pi x}{L,\pi})dx = 0$$

2.6 Vereinfachungen

$\cos(n\pi) = (-1)^n$

$$\cos(n\tfrac{\pi}{2}) = \begin{cases} 0 & n = 2m-1 \\ (-1)^n & 2m \end{cases} \qquad \forall m = 1,2,3,...$$

$\sin(n\pi) = 0$

$$\sin(n\tfrac{\pi}{2}) = \begin{cases} 0 & n = 2m \\ (-1)^{n+1} & n = 2m-1 \end{cases} \qquad \forall m = 1,2,3,...$$

2.7 Fehler der Annäherung durch trigo-Polynom

$E = \int_{-\pi}^{\pi} (f - F)^2 dx$

Fehler wird kleiner mit steigendem N (Grad des Polynoms)

$E^* = \int_{-\pi}^{\pi} f^2 dx - \pi[2a_0 + \sum_{n=1}^{N} (a_n^2 + b_n^2)]$

2.8 Fourierintegral

$f(x) \xrightarrow{L\to\infty} \frac{1}{\pi} \int_0^\infty A(\omega)\cos(\omega x) + B(\omega)\sin(\omega x) d\omega$

$A(\omega) = \int_{-\infty}^{\infty} f(v)\cos(\omega v) dv$

$B(\omega) = \int_{-\infty}^{\infty} f(v)\sin(\omega v) dv$

2.8.1 falls Gerade

$A(\omega) = \frac{2}{\pi} \int_0^\infty f(v)\cos(\omega v) dv$

$B(\omega) = 0$

2.8.2 falls Ungerade

$A(\omega) = 0$

$B(\omega) = \frac{2}{\pi} \int_0^\infty f(v)\sin(\omega v) dv$

2.8.3 Existenz

1. $f(x)$ stückweise stetig auf endlichen Intervallen
2. $f(x)$ stetig differentierbar
3. $\int_{-\infty}^{\infty} |f(x)| dx < \infty$

dann gilt: $f(x) = FI(f)(x)$

an der Sprungstelle x_0:

$FI(f)(x_0) = \frac{1}{2}(\lim_{x\to x_0^-} f(x) + \lim_{x\to x_0^+} f(x))$

2.9 Fourier Transform

$\mathscr{F}(f(x))(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ix\omega} dx$

$f(x) = \mathscr{F}^{-1}(\mathscr{F}(f(x))(\omega)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathscr{F}(f(x))(\omega) e^{ix\omega} d\omega$

$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathscr{F}(f(x))(\omega) e^{i\omega(A(x,t,\ldots))} d\omega \Rightarrow f(A(x,t,\ldots))$

2.10 Eigenschaften

1. Linearität

   $\mathscr{F}(af(x) + bg(x))(\omega) = a\mathscr{F}(f(x))(\omega) + b\mathscr{F}(g(x))(\omega)$

2. Ableitung

   $\mathscr{F}(f'(x))(\omega) = i\omega\mathscr{F}(f(x))(\omega)$

   $\mathscr{F}(f''(x))(\omega) = -\omega^2\mathscr{F}(f(x))(\omega)$

3. $\mathscr{F}(f * g) = \sqrt{2\pi}\mathscr{F}(f)\mathscr{G}(g)$

   $(f * g) = \int_{-\infty}^{\infty} \mathscr{F}(f)\mathscr{G}(g) e^{i\omega x} d\omega$

   $wobei: (f * g)(x) = \int_{-\infty}^{\infty} f(p)g(x-p) dp$

4. t-shift

   $\mathscr{F}(f(x-a)) = e^{-ia\omega}\mathscr{F}(f(x))$

5. f-shift

   $\mathscr{F}(f(x))(\omega - a) = \mathscr{F}(e^{iax}f(x))$

2.11 Identitäten

$\mathscr{F}(e^{-ax^2}) = \frac{1}{\sqrt{2a}} e^{-\frac{\omega^2}{4a}}$

$\mathscr{F}(xe^{x^2}) = \frac{-i\omega}{2\sqrt{2}} e^{-\frac{\omega^2}{4}}$

3 PDE

3.1 Wichtige PDE

1-D Wellengleichung	:	$u_{tt} = c^2 u_{xx}$
1-D Wärmegleichung	:	$u_t = c^2 u_{xx}$
2-D Laplacegleichung	:	$\Delta u = u_{xx} + u_{yy} = 0$
2-D Poissongleichung	:	$u_{xx} + u_{yy} = f(x, y)$
2-D Wellengleichung	:	$u_{tt} = c^2(u_{xx} + u_{yy})$
2-D Wärmeleitgleichung	:	$u_t = c^2(u_{xx} + u_{yy})$
3-D Laplacegleichung	:	$\Delta u = u_{xx} + u_{yy} + u_{zz} = 0$

3.2 Klassifizierung

linear	:	$u_{xy} + u_z + u_{tt} = f(x, y, z, t)$
nichtlinear	:	$u_x u_{xy} = f(x, y, z, t)$
homogen	:	$f(u_{x,y,z}, u_t, \ldots) = 0$

Falls u_t oder u_{tt} vor kommt $y = ct$ bzw. $y = c^2 t$ einsetzen.

$Au_{xx} + 2Bu_{xy} + Cu_{yy} = F(x, a, u, u_x, u_y)$.

(1)	$AC - B^2 < 0 \Rightarrow$	Hyperbel	(Wellengleichung)
(2)	$AC - B^2 = 0 \Rightarrow$	Parabel	(Hitzegleichung)
(3)	$AC - B^2 > 0 \Rightarrow$	Ellipse	(Laplace / Poisson)

3.3 Methode 1: Trennung der Variablen

1. Ansatz: $u(t, x) = F(x)G(t)$
2. Ansatz einsetzen
3. 2 ODE eine mit $F(x)$ eine mit $G(x)$
4. Lösungen für $G(x)$ und $F(x)$ einsetzen.

3.4 1-D Wellengleichung mit Fourier Reihen

$$\begin{cases} u_{tt} = c^2 u_{xx} & \\ u(0,t) = u(L,t) = 0 & t < 0 \\ u(x,0) = f(x) & 0 \leq x \leq L \\ u_t(x,0)) = g(x) & 0 \leq x \leq L \end{cases}$$

$u(x,t) = F(x)G(t) \to u_{tt} = F\ddot{G}$ and $u_{xx} = F''G$

$F\ddot{G} = c^2 F''G \to \frac{\ddot{G}}{c^2 G} = \frac{F''}{F} = k \to \begin{cases} F'' & = kF \\ \ddot{G} & = c^2 kG \end{cases}$

$$u(0,t) = F(0)G(t) = 0 \forall t \geq 0 \qquad \Rightarrow F(0) = 0$$

$$u(L,t) = F(L)G(t) = 0 \forall t \geq 0 \qquad \Rightarrow F(L) = 0$$

wähle: $k < 0 \to F(x) = A\cos(\sqrt{-k}x) + B\sin(\sqrt{-k}x)$

aufgrund der Anfangsbedingungen folgt dass $\sqrt{-k}L = n\pi$

daraus: $F_n(x) = \sin(\frac{n\pi}{L}x)$

$$\ddot{G} = -c^2(\frac{n\pi}{L})^2 G$$
$$G_n(t) = B_n\cos(\frac{cn\pi}{L}t) + B_n^*\sin(\frac{cn\pi}{L}t)$$

allgemeine Lösung durch einsetzen:
$u(x,t) = F(x)G(t)$
$u_n(x,t) = (B_n\cos(\frac{cn\pi}{L}t) + B_n^*\sin(\frac{cn\pi}{L}t))\cdot\sin(\frac{n\pi}{L}x)$
Fourierreihe
$f(x) = u(x,0) = \sum_{n=1}^{\infty} B_n\sin(\frac{n\pi}{L}x)$

$\boxed{B_n = \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi}{L}x)dx}$

$g(x) = u_t(x,0) = \sum_{n=1}^{\infty}\frac{cn\pi}{L}B_n^*\sin(\frac{n\pi}{L}x)$

$\boxed{B_n^* = \frac{2}{cn\pi}\int_0^L g(x)\sin(\frac{n\pi}{L}x)dx}$

3.5 D'Alembert-Lösung der Wellengleichung

$$\begin{cases} u_{tt} = c^2u_{xx} \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}$$

$\boxed{u(x,t) = \frac{1}{2}(f(x+ct)+f(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(v)dv}$

3.6 Methode der Charakteristiken

1. $Au_{xx} + 2Bu_{xy} + Cu_{yy} = F(x,y,u_x,u_y,u)$
2. Klassifizieren
3. Charakteristische Gleichung

   $Ay'^2 - 2By' + C = 0$

   falls A,B,C konstant:

   $\rightarrow y'_{1,2} = \frac{B\pm\sqrt{B^2-AC}}{A} = \lambda_{1,2}$
4. Bestimmen der Charakteristiken

   aus den beiden Lösungen der charakteristischen Gleichung:

   $\int y'_{1,2}dx = y \Rightarrow y = \lambda_{1,2}x + c_{1,2}$

   $\Phi(x,y) = y - \lambda_1 x = c_1$

   $\Psi(x,y) = y - \lambda_2 x = c_2$
5. neue Koordinaten definieren

H:	$v := \Phi$	$w := \Psi$
P:	$v := x$	$w := \Phi = \Psi$
E:	$v := \frac{1}{2}(\Phi+\Psi)$	$w := \frac{1}{2i}(\Phi-\Psi)$

6. Koordinatentransformation $u(v(x,y),w(x,y))$

   $\rightarrow$Normalformen:

H:	$u_{vw} = F(v,w,u,u_v,u_w)$
P:	$u_{vv} = F(v,w,u,u_v,u_w)$
E:	$u_{vv} + u_{ww} = F(v,w,u,u_v,u_w)$

7. Lösung

H:	$u(v,w) = f(v) + g(w) - F(v,w,\dots)$
P:	$u(v,w) = f(w)v + g(w) - F(v,w,\dots)$
E:	$\rightarrow$ Laplacegleichung

8. Rücktransformation

3.7 Wärmegleichung mit Fourierreihe

$$\begin{cases} u_t = c^2u_{xx} \\ u(0,t) = u(L,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

1. Ansatz $u(x,t) = F(x)G(t)$
2. Einsetzen

   $\frac{\dot{G}}{c^2G} = \frac{F''}{F} = k$
3. Lösung von F

   $k < 0$ führt zu

   $F'' = -p^2F \rightarrow F(x) = A\cos(px) + B\sin(px)$

   $F(0) = F(L) = 0 \rightarrow F_n(x) = \sin(\frac{n\pi}{L}x)$ und $p = \frac{n\pi}{L}$
4. Lösung von G

   $G_n(t) = B_ne^{-(\frac{cn\pi}{L})^2t}$
5. allgemeine Lösung

   $u_n(t,x) = B_nsin(\frac{n\pi}{L}x)e^{-(\frac{cn\pi}{L})^2t}$

   $u(t,x) = \sum_{n=1}^{\infty}u_n(x,t)$

   $f(x) = u(x,0) = \sum_{n=1}^{\infty}B_n\sin(\frac{n\pi}{L}x)$

   $\rightarrow \boxed{B_n = \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi}{L}x)dx}$

3.8 2D-Wärmegleichung : Dirichlet Rechteck

$$\begin{cases} \nabla^2u = u_{xx} + u_{yy} = 0 \\ u(0,y) = u(a,y) = u(x,0) = 0 \\ u(x,b) = f(x) \end{cases}$$

$R = \{(x,y) :=\leq x \leq a, 0 \leq y \leq b\}$

1. Ansatz $u(x,t) = F(x)G(y)$
2. Einsetzen

   $F''G + FG'' = 0$ daraus $\frac{F''}{F} = -\frac{G''}{G} = -k$
3. Lösung von F

   $k < 0$ führt zu

   $F'' = -kF \rightarrow F(x) = A\cos(\sqrt{k}x) + B\sin(\sqrt{k}x)$

   $F(0) = F(a) = 0 \rightarrow F_n(x) = \sin(\frac{n\pi}{a}x)$
4. Lösung von G

   $G_n(y) = A_n^*e^{\frac{n\pi}{a}y} + B_n^*e^{-\frac{n\pi}{a}y}$

   $G_n(0) = 0 \rightarrow A_n^* = -B_n^* \rightarrow G_n(y) = 2A_n^*\sinh(\frac{n\pi}{a}y)$

5. ALLGEMEINE LÖSUNG

$u_n(x,y) = A_n \sin(\frac{n\pi}{a}x)\sinh(\frac{n\pi}{a}y)$

$u(x,y) = \sum_{n=1}^{\infty} u_n(x,y)$

$f(x) = u(x,b) \rightarrow \boxed{A_n = \frac{2}{a\sinh(\frac{n\pi}{a}b)} \int_0^a f(x)\sin(\frac{n\pi}{a}x)dx}$

3.9 1D-WÄRMEGLEICHUNG : UNENDLICHER BALKEN

$$\begin{cases} u_t = c^2 u_{xx} \\ u(x,0) = f(x) \end{cases}$$

1. ANSATZ $u(x,t) = F(x)G(t)$
2. EINSETEZN

   $\frac{F''}{F} = \frac{1}{c^2}\frac{\dot{G}}{G} = -k$
3. LÖSUNG VON F UND G

   $k > 0 \rightarrow k = p^2$

   $$\begin{cases} F_p(x) = A(p)\cos(px) + B(p)\sin(px) \\ G_p(t) = e^{-c^2p^2t} \end{cases}$$
4. ALLGEMEINE LÖSUNG

   $u(x,t) = \int_0^{\infty} [A(p)\cos(px) + B(p)\sin(px)]e^{-c^2p^2t}dp$

   ANFANGSBED. $\rightarrow$ FOURIERINTEGRAL:

   $\rightarrow f(x) = \int_0^{\infty} [A(p)\cos(px) + B(p)\sin(px)]dp$

   $A(p) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos(pv)dv = 0$

   $B(p) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin(pv)dv = 0$

   $f(v)$ GERADE: $A(p) = \frac{2}{\pi}\int_0^{\infty} f(v)\cos(pv)dp \;\; B(p) = 0$

   $f(v)$ UNGRD.: $B(p) = \frac{2}{\pi}\int_0^{\infty} f(v)\sin(pv)dp \;\; A(p) = 0$

   [...]

   $\boxed{u(x,t) = \frac{1}{2c\sqrt{\pi t}}\int_{-\infty}^{\infty} f(v)exp[-(\frac{x-v}{2c\sqrt{t}})^2]dv}$

3.10 UNENDLICHER BALKEN: FOURIERTRANSFORMATION

$$\begin{cases} u_t = c^2 u_{xx} \\ u(x,0) = f(x) \end{cases}$$

$$\boxed{\mathscr{F}(f(x))(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx}$$

1. FOURIERTRANSFORMATION NACH X ANWENDEN

   $$\left.\begin{array}{l} \mathscr{F}(u_{xx}(x,t)) = -w^2\hat{u}(\omega,t) \\ \mathscr{F}(u_t(x,t)) = \frac{d}{dt}\hat{u}(\omega,t) \end{array}\right\} \hat{u}_t(\omega,t) = -c^2\omega^2\hat{u}(\omega,t)$$
2. LÖSUNG DER ODE

   $\hat{u}(\omega,t) = C(\omega)e^{-c^2\omega^2t}$
3. ANFANGSBEDINGUNG: $\hat{u}(x,0) = \hat{f}(\omega)$

   $\hat{u}(\omega,t) = \hat{f}(\omega)e^{-c^2\omega^2t}$
4. INVERSE FOURIERTRANSFORMATION

   $u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(\omega)e^{-c^2\omega^2t}e^{i\omega x}d\omega$

   [...]

   $\boxed{u(t,x) = \frac{1}{2c\sqrt{\pi t}}\int_{-\infty}^{\infty} f(p)exp(-\frac{(x-p)^2}{4c^2t})dp}$

3.11 2D-WÄRMEGLEICHUNG : DIRICHLET KREISSCHEIBE

$$\begin{cases} \nabla^2 u = 0 & \{(x,y) : x^2+y^2 < R^2\} \\ u = f & \{(x,y) : x^2+y^2 = R^2\} \end{cases}$$

1. KOORDINATENTRANSFORMATION

   $$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \Leftrightarrow \begin{cases} r = (x^2+y^2)^{1/2} \\ \theta = \arctan\frac{y}{x} \end{cases}$$

   $$\begin{cases} u_{rr} + u_{\theta\theta}\frac{1}{r^2} + u_r\frac{1}{r} = 0 \\ u(R,\theta) = f(\theta) \end{cases}$$
2. ANSATZ $u(r,\theta) = F(r)G(\theta)$

   $$\begin{cases} r^2F'' + rF' - kF = 0 \\ G'' + kG = 0 \end{cases}$$

   $G(0) = G(2\pi)$ UND $G'(0) = G'(2\pi)$
3. LÖSUNG VON G $k = n^2$

   $G_n(\theta) = A_n\cos(n\theta) + B_n\sin(n\theta)$
4. LÖSUNG VON F (EULERGLEICHUNG)

   $F_n(r) = P_nr^n$
5. ALLGEMEINE LÖSUNG

   $u_n(r,\theta) = r^n(A_n\cos(n\theta) + B_n\sin(n\theta))$

   $u(r,\theta) = \sum_{n=0}^{\infty} u_n(r,\theta)$

   $$u(R,\theta) = f(\theta) \rightarrow \begin{cases} A_0 & \frac{1}{2\pi}\int_0^{2\pi} f(\phi)d\phi \\ A_n & \frac{1}{R^n\pi}\int_0^{2\pi} f(\phi)\cos(n\phi)d\phi \\ B_n & \frac{1}{R^n\pi}\int_0^{2\pi} f(\phi)\sin(n\phi)d\phi \end{cases}$$

   $u_r(r,\theta) = \sum_{n=0}^{\infty} n\cdot r^{n-1}(A_n\cos(n\theta) + B_n\sin(n\theta))$
6. POISSONGLEICHUNG DURCH EINSETZEN UND UMFORMEN

   $\boxed{u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} K(r,\theta,R,\phi)f(\phi)d\phi}$

   $\boxed{K(r,\theta,R,\phi) := \frac{R^2-r^2}{R^2-2rR\cos(\theta-\phi)+r^2}}$

3.12 MITTELWERTEIGENSCHAFT

Sei $u(x,y)$ harmonisch $\rightarrow \nabla^2 u = 0$ dann gilt:
$u(x_0,y_0) = u(0,\theta) = \int_0^{2\pi} K(0,...)u(a,\phi)d\phi = \frac{1}{2\pi}\int_0^{2\pi} u(a,\phi)d\phi = \frac{1}{2\pi}\int_0^{2\pi} u(x_0 + a\cos(\phi), y_0 + a\sin(\phi)d\phi$
In Worten: Der Wert einer harmonischen Funktion ist gleich dem Durchschnitt der Werte eines Kreises mit beliebigem Radius um (x_0, y_0).

3.13 Maximumswertprinzip

Wenn eine harmonische Funktion einen Maximalwert auf dem Bereich $\mathscr{D}$ hat, ist sie konstant.

4 A

4.1 le Logarithmus

$\ln(ab) = \ln(a) + \ln(b)$

4.2 Derivatives

$(ln(x))' = \frac{1}{x}$

4.3 Integrals

$\int ln(x)dx = x \cdot ln(x) - x + C$

$\int e^x dx = e^x$ $\quad \int a^x dx = \frac{a^x}{\ln(a)} + C$

$\int \frac{1}{\cos^2(x)} dx = \tan(x) + C$

$\int \frac{1}{\sin^2(x)} dx = -\cot(x) + C$

$\int \sin^2(x) dx = \frac{1}{2}(\sin(x)\cos(x) - x) + C$

$\int \cos^2(x) dx = \frac{1}{2}(\sin(x)\cos(x) + x) + C$

$$\int \frac{1}{1+x^2} dx = \begin{cases} \arctan(x) + C_1 \\ -arccot(x) + C_2 \end{cases}$$

$$\int \frac{1}{1-x^2} = \begin{cases} \operatorname{artanh}(x) + C_1 = \frac{1}{2}\ln(\frac{1+x}{1-x} + C_1) & |x| < 1 \\ \operatorname{arcoth}(x) + C_2 = \frac{1}{2}\ln(\frac{x+1}{x-1} + C_2) & |x| > 1 \end{cases}$$

$$\int \frac{1}{\sqrt{1-x^2}} dx = \begin{cases} \arcsin(x) + C_1 \\ -\arccos(x) + C_2 \end{cases}$$

$\int \frac{1}{\sqrt{x^2-1}} dx = \operatorname{arcosh}(x) + C = \ln|x + \sqrt{x^2-1}| + C \quad |x| > 1$

$\int \frac{1}{\sinh^2(x)} dx = -\coth(x) + C$

$\int \frac{1}{\cosh^2(x)} dx = \tanh(x) + C$

$\int \frac{1}{\sqrt{x^2+1}} dx = \operatorname{arsinh}(x) + C = \ln|x + \sqrt{x^2+1}| + C$

$\int \frac{1}{x^2+a^2} dx = \frac{1}{a}\arctan(\frac{x}{a}) + C$

4.4 Euleridentität

$\Im(e^{ix}) = \sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$ $\quad e^{i\omega x} = \cos(\omega x) + i\sin(\omega x)$

$\Re(e^{ix}) = \cos(x) = \frac{e^{ix} + e^{-ix}}{2}$

4.5 Trigonometric Identities

$\sin^2(x) + \cos^2(x) = 1$

$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta)$

$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$

$\sin(\alpha) + \sin(\beta) = 2\sin(\frac{\alpha+\beta}{2})\cos(\frac{\alpha-\beta}{2})$

$\sin(\alpha) - \sin(\beta) = 2\cos(\frac{\alpha+\beta}{2})\sin(\frac{\alpha-\beta}{2})$

$\cos(\alpha) + \cos(\beta) = 2\cos(\frac{\alpha+\beta}{2})\cos(\frac{\alpha-\beta}{2})$

$\cos(\alpha) - \cos(\beta) = 2\sin(\frac{\alpha+\beta}{2})\sin(\frac{\alpha-\beta}{2})$

$\sin(2x) = 2\sin(x)\cos(x)$

$\sin(3x) = 3\sin(x) - 4\sin^3(x)$

$\sin(4x) = \sin(x)(8\cos^3(x) + 4\cos(x))$

$\sin(5x) = 5\sin(x) - 20\sin^3(x) + 16\sin^5(x)$

$\cos(2x) = \cos^2(x) - \sin^2(x) = 1 - 2\sin^2(x)$

$\cos(3x) = 4\cos^3(x) - 3\cos(x)$

$\cos(4x) = 8\cos^4(x) - 8\cos^2(x) + 1$

$\cos(5x) = 16\cos^5(x) - 20\cos^3(x) + 5\cos(x)$

$\tan(2x) = \frac{2\tan(x)}{1-\tan^2(x)}$

$\tan(3x) = \frac{3\tan(x) - \tan^3(x)}{1 - 3\tan^2(x)}$

$\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$

$\sin^3(x) = \frac{1}{4}(3\sin(x) - \sin(3x))$

$\sin^4(x) = \frac{1}{8}(\cos(4x) - 4\cos(2x) + 3)$

$\sin^n(x) = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\cos((n-2k)(x - \frac{\pi}{2}))$

$\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$

$\cos^3(x) = \frac{1}{4}(3\cos(x) + \cos(3x))$

$\cos^4(x) = \frac{1}{8}(3 + 4\cos(2x) + cos(4x))$

$\cos^n = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\cos((n-2k)x)$

$\sin(ax)\sin(bx) = \frac{1}{2}(\cos(x(a-b)) - \cos(x(a+b)))$

$\cos(ax)\cos(bx) = \frac{1}{2}(\cos(x(a-b)) + \cos(x(a+b)))$

$\sin(ax)\cos(bx) = \frac{1}{2}(\sin(x(a-b)) + \sin(x(a+b)))$

$\sinh(x) = \frac{e^x - e^{-x}}{2}$ $\quad \cosh(x) = \frac{e^x + e^{-x}}{2}$

$\cosh^2(x) - \sinh^2(x) = 1$ $\quad \cosh(x) + \sinh(x) = e^x$

$\sinh(2x) = 2\sinh(x)\cosh(x)$

$\sinh(3x) = 4\sinh^3(x) + 3\sinh(x)$

$\cosh(2x) = \cosh^2(x) + \sinh^2(x) = 2\cosh^2(x) - 1$

$\cosh(3x) = 4\cosh^3(x) - 3\cosh(x)$

5 B

5.1 Partialbruchzerlegung

Gegeben: $\frac{P(x)}{Q(x)}$

1. $p = grad(P(x))$ und $q = grad(Q(x))$
2. falls $p > q > 2 \rightarrow$ Polynomdivision. $\rightarrow \frac{P(x)}{Q(x)} = A(x) + \frac{R(x)}{Q(x)}$ falls $grad(Q(x)) > 1 \rightarrow 3$.
3. $\frac{P(x)}{\prod_i (x-ai)\prod_i (x-b_i)^{n_i}} = \sum_{a_i}\frac{A_i}{(x-a_i)} + \sum_i\sum_{j=1}^{n_i}\frac{B_{i,j}}{(x-b_i)^j}$
4. Koeffizientenvergleich führt zu A_i und $B_{i,j}$

5.2 Faktorisierung

$1 + x^{n+1} = (1+x)\sum_{k=0}^{n}(-1)^k x^k$

5.3 Ableitungsregeln

Quotientenregel : $(\frac{u}{v})' = \frac{u'v - uv'}{v^2}$

Umkehrfunktion : $(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$

5.4 Even / Odd

Even	Odd	Even	Odd
$g_1 + g_2$	$u_1 + u_2$	u_1'	g_1'
$g_1 \cdot g_2$	$u_1 \cdot g_1$	$g_1 \circ g_2$	$u_1 \circ u_2$
$u_1 \cdot u_2$		$g_1 \circ u_1$	
		$u_1 \circ u_2$	