Autonomous $F(U(t))$	Non-Autonomous $F(U(t),t)$
F does not depend on t	$u'(t) = u(t) + t$

$\boxed{u'(t^n) \approx \frac{u(t^{n+1})-u(t^n)}{\Delta t} \approx \frac{U_{n+1}-U_n}{\Delta t}}$ FE Method (Consistent form)

$\boxed{U_{n+1} = U_n + \Delta t F(t^n, U_n)}$ FE Method (Update form)

$\boxed{u'(t^{n+1}) \approx \frac{u(t^{n+1})-u(t^n)}{\Delta t} \approx \frac{U_{n+1}-U_n}{\Delta t}}$ BE Method

$\boxed{\int_a^b f(x)dx \approx (b-a)\left(\frac{f(a)+f(b)}{2}\right)}$ Trapezoidal rule

$$\boxed{U_{n+1} = u_n + \frac{\Delta t}{2}F(t^n, U_n) + \frac{\Delta t}{2}F(t^{n+1}, U_{n+1})}$$

$\boxed{u'(t^n) \approx \frac{U_{n+1}-U_{n-1}}{2\Delta t} \quad u''(t^n) \approx \frac{u_{j+1}-2u_j+u_{j-1}}{\Delta x^2}}$ Midpoint rule

In the beginning we only have one initial value, but need two. Possible solution: $U_1 = u_0 + \Delta F(0, u_0)$ (Forward Euler method)

Truncation error

$$\boxed{T_n = u'(t) - F(t, u(t))}$$

Replace $u'(t)$ with approximation defined by applied method. (Consistent form)

$$\begin{aligned} u(t)|_{t_0=t^n} &= u(t^n) + (t-t^n)\cdot u'(t^n) + (t-t^n)^2\frac{u''(t^n)}{2} + \cdots \\ u'(t)|_{t_0=t^n} &= u'(t^n) + (t-t^n)\cdot u''(t^n) + (t-t^n)^2\frac{u'''t^n)}{2} + \cdots \end{aligned}$$

$T_n^{Midpoint} = T_n^{Trapezoidal} = \mathcal{O}(\Delta t^2)$
$T_n^{BE} = T_n^{FE} = \mathcal{O}(\Delta t)$

One Step Error / Global Error

Inserting the exact solution into the update form of a given method → local „One Step Error“: $L_j = \mathcal{O}((\Delta t)^{q+1})$
Adding all one step errors over all t yields the „Global Error“as an upper bound to the total error.

$$\boxed{E_N := u(t^N) - U_N \leq \sum_{j=0}^{N-1} |L_j|}$$

$$\boxed{E_N \approx \mathcal{O}((\Delta t)^{q+1})\cdot N = \mathcal{O}((\Delta t)^{q+1}\cdot \frac{T}{\Delta t} \approx \mathcal{O}((\Delta t)^q)}$$

Newtons Method

$$\begin{aligned} \vec{G}(x) &= 0 \\ D\vec{G}(x_k)\Delta x_k &= -\vec{G}(x_k) \\ x_{k+1} &= x_k + \Delta x_k \end{aligned}$$

Runge-Kutta-N (RK-N)

$$\boxed{\begin{aligned} Y_1 &= U_n + \Delta t \sum_{j=1}^{s} a_{1j}F(t^n + c_j\Delta t, Y_j) \\ Y_2 &= U_n + \Delta t \sum_{j=1}^{s} a_{2j}F(t^n + c_j\Delta t, Y_j) \\ &\;\;\vdots \\ Y_s &= U_n + \Delta t \sum_{j=1}^{s} a_{sj}F(t^n + c_j\Delta t, Y_j) \\ U_{n+1} &= U_n + \Delta t \sum_{j=1}^{s} b_j F(t^n + c_j\Delta t, Y_j) \\ U_0 &= u_0 \end{aligned}}$$

$$\begin{array}{c|cccc} c_1 & a_{11} & a_{12} & \cdots & a_{1s} \\ c_2 & a_{21} & a_{22} & \cdots & a_{2s} \\ \vdots & \vdots & \vdots & & \vdots \\ c_s & a_{s1} & a_{s2} & \cdots & a_{ss} \\ \hline & b_1 & b_2 & \cdots & b_s \end{array}$$

$\sum_{j=1}^{s} a_{ij} = c_i \qquad \sum_{j=1}^{s} b_j = 1$

consistency

$\sum_{j=1}^{s} b_j c_j = \frac{1}{2}$

2^nd order accurate

$$\begin{aligned} \sum_{j=1}^{s} b_j c_j^2 &= \frac{1}{3} \\ \sum_{j=1}^{s}\sum_{i=1}^{s} b_i a_{ij} c_j &= \frac{1}{6} \end{aligned}$$

3^rd order accurate

Explicit RK

$\boxed{a_{ij} = 0 \text{ if } j \geq i}$

A strictly lower triangular.

DIRK

$\boxed{a_{ij} = 0 \text{ if } j > i,\ a_{ii} \neq 0 \text{ for some i}}$

A lower triangular.

Multi-Step Methods

- **Advantage:** F only needs to be evaluated once each time step. RK methods need multiple evaluations.
- **Disadvantage:** Variable time-steps difficult to implement. Require several starting values.

$\boxed{\sum_{j=0}^{\gamma} \alpha_j U_{n+j} = \Delta t \sum_{j=0}^{\gamma} \beta_j F(t^{n+j}, U_{n+j})}$ linear multi-step method

Adam's Method

$$\begin{aligned} \alpha_\gamma &= 1 \\ \alpha_{\gamma-1} &= -1 \\ \alpha_j &= 0 \end{aligned}$$

Adams-Bashforth

$\beta_\gamma = 0 \rightarrow$ explicit

$$\boxed{\begin{array}{ll} (AB1) & U_{n+1} = U_n + \Delta t F(U_n) \\ (AB2) & U_{n+2} = U_{n+1} + \frac{\Delta t}{2}(-F(U_n) + 3F(U_{n+1}) \\ (AB3) & U_{n+3} = U_{n+2} + \frac{\Delta t}{12}(5F(U_n) - 16F(U_{n+1}) + 23F(U_{n+2}) \end{array}}$$

$\beta_\gamma \neq 0 \rightarrow$ implicit: **Adam's Moultons Method**

$$\boxed{\begin{array}{ll} (AM1): & U_{n+1} = U_n + \frac{\Delta t}{2}(F(U_n) + F(U_{n+1})) \\ (AM2): & U_{n+2} = U_{n+1} + \frac{\Delta t}{12}(-F(U_n) + 8F(U_{n+1}) + 5F(U_{n+2})) \\ (AM3): & U_{n+3} = U_{n+2} + \frac{\Delta t}{24}(F(U_n) - 5F(U_{n+1}) + 19F(U_{n+2}) \\ & \qquad +9F(U_{n+3})) \end{array}}$$

The resulting methods are $(\gamma+1)$-order accurate.

Starting Values

A γ-step method requires γ starting values $U_0, U_1, \ldots, U_{\gamma-1}$. → RK method used on initial condition.

explicit Adams-Bashfort	RK of order $\gamma - 1$
implicit Adams-Moulton	RK of order γ

Twice RK with one order of accuracy less than the multi-step method. For AB, RK -one-step error: $\mathcal{O}(\Delta t^\gamma) \rightarrow$ total error $(\gamma-1)\cdot\mathcal{O}(\Delta t^\gamma) \rightarrow$ the global error still $\mathcal{O}(\Delta t^\gamma)$.

Truncation Error

$$\boxed{\begin{aligned} T_{n+\gamma} &= \frac{1}{\Delta t}\left(\sum_{j=0}^{\gamma}\alpha_j u(t^{n+j}) - \Delta t\sum_{j=0}^{\gamma}\beta_j F(u(t^{n+j}))\right) \\ &= \frac{1}{\Delta t}\left(\sum_{j=0}^{\gamma}\alpha_j u(t^{n+j}) - \Delta t\sum_{j=0}^{\gamma}\beta_j u'(t^{n+j})\right) \end{aligned}}$$

Substitute with taylor expansions to end up with:

$$\boxed{\begin{aligned} T_{n+\gamma} = \quad & \frac{1}{\Delta t}(\sum_{j=0}^{\gamma}\alpha_j)u(t^n) + (\sum_{j=0}^{\gamma} j(\alpha_j - \beta_j))u'(t^n) \\ & +\Delta t(\sum_{j=0}^{\gamma}(\frac{j^2\alpha_j}{2} - j\beta_j))u''(t^n) + \ldots \\ & \Delta t^{k-1}(\sum_{j=0}^{\gamma}(\frac{j^k\alpha_j}{k!} - \frac{j^{k-1}\beta_j}{(k-1)!}))u^{(k)}(t^n) \end{aligned}}$$

$\sum_{j=0}^{\gamma}\alpha_j = 0 \qquad \sum_{j=0}^{\gamma} j\alpha_j = \sum_{j=0}^{\gamma}\beta_j$ Consistency

$\boxed{\sum_{j=0}^{\gamma}\frac{j^q}{q!}\alpha_j = \sum_{j=0}^{\gamma}\frac{j^{q-1}}{(q-1)!}\beta_j,\ \forall\ q \leq k+1}$ Tunctuation error of order Δt^k

Stability of Numerical Methods for ODEs

$\boxed{\lim_{\substack{\Delta t \to 0 \\ N\cdot\Delta t = T}} U_N = u(T)}$ Convergence condition

For a one-step method, the starting value coincides with the initial condition. For multi-step methods we need a condition for starting values:

$$\boxed{\lim_{\Delta t \to 0} U_j(\Delta t) = u_0 \quad \forall\ 0 \leq j \leq \gamma - 1}$$

Absolute Stability

$$\begin{aligned} u'(t) &= \lambda u(t) \\ u(0) &= u_0 \end{aligned}$$

$\boxed{|U_{n+1}| \leq |U_n|}$ Absolute stability

Name	Method	A-stable if
FE	$U_{n+1} = (1+\lambda\Delta t)U_n$	$\lvert 1+\lambda\Delta t\rvert \leq 1$
BE	$U_{n+1} = \frac{1}{1-\lambda\Delta t}U_n$	$\lambda\Delta t \in (-\infty, 0] \cup [2, \infty)$
TR	$U_{n+1} = \frac{1+\frac{\lambda\Delta t}{2}}{1-\frac{\lambda\Delta t}{2}}U_n$	$\left\lvert\frac{1+\frac{\lambda\Delta t}{2}}{1-\frac{\lambda\Delta t}{2}}\right\rvert \leq 1$

Derivation of the Poissonequation

1. Elastic body on Ω,min $\left(J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega u f dx\right) \to u(x)$
2. Dirichlet BC: $u|_{\partial\Omega} \equiv 0$
3. Euler-Lagrange equation:
   $J'(u,v) = \lim_{\tau\to 0} \frac{J(u+\tau v)-J(u)}{\tau} \overset{!}{=} \int_\Omega \langle \nabla u, \nabla v\rangle dx - \int_\Omega f v dx = 0$
   v is direction in which we calculate the derivative.
4. Integration by parts yields:
   $-\int_\Omega v\Delta u dx - \int_\Omega v f dx + \int_{\partial\Omega} v \frac{\partial u}{\partial v} ds(x) = 0$
   $\frac{\partial u}{\partial v} = \nabla u \cdot v$ v: unit outward vector, normal to boundary, $v \equiv 0$ on $\partial\Omega$
   $\Rightarrow \int_\Omega (-\Delta u - f) v dx = 0\ \forall v \Rightarrow \boxed{-\Delta u = f u|_{\partial\Omega \equiv 0}}$

Poisson Equation: Solution in 1D

$$-u''(x) = f(x), \quad \forall\, x \in (0,1)$$
$$u(0) = u(1) = 0$$

1. $u'(y) = C_2 + \int_0^y u''(z)dz = C_2 - \int_0^y f(z)dz$
2. $u(x) = C_1 + \int_0^x u'(y)dy = C_1 + C_2 x - \int_0^x \int_0^y f(z)dzdy$
3. $F(y) = \int_0^y f(z)dz \Rightarrow F'(y) = f(y)$
4. IBP: $\int_0^x F(y)dy = \int_0^x y' F(y)dy = xF(x) - \int_0^x yF'(y)dy$
   $= \int_0^x (x-y)f(y)dy$
5. $u(x) = C_1 + C_2 x - \int_0^x (x-y)f(y)dy$
6. $u(x) = \int_0^1 x(1-y)f(y)dy - \int_0^x (x-y)f(y)dy$
   $C_1 = u(0) = 0 \qquad C_2 = u(1) = \int_0^1 (1-y)f(y)dy$
7. $\boxed{G(x,y) = \begin{cases} y(1-x) & 0 \le y \le x \\ x(1-y) & x \le y \le 1 \end{cases}}$ Greens function
8. $u(x) = \int_0^1 G(x,y)f(y)dy$

- The integral to find $u(x)$ is not always possible to evaluate exactly $\to$ numerical quadrature rule.
- Slight perturbations in the poisson equation invalidate our solution. Thus a **general form of the PE** is used often:
  $-(a(x)u'(x))' + b(x)u'(x) + c(x)u(x) = f(x), \quad \forall x \in (0,1)$
  $u(0) = u(1) = 0$
- Greens function representation are only available in 1D.

1D Finite Difference Scheme

$$\boxed{u'(x_j) \approx \frac{u_{j+1}-u_j}{\Delta x} \qquad u''(x_j) \approx \frac{u_{j+1}-2u_j+u_{j-1}}{\Delta x^2}}$$

$$-u_{j+1} + 2u_j - u_{j-1} = \Delta x^2 f_j, \quad \forall\, j = 1,\dots,N \quad u|_{\partial\Omega} = 0$$

Only internal points! $x_0 = 0,\ x_{N+1} = 1,\ u_0 = u_{N+1} = 0$

$$AU = F \text{ where } A = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -1 & 2 & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{bmatrix} \in \mathbb{R}^{N\times N}$$

$$\boxed{\|u^{\Delta x}\|_\infty \le \frac{1}{8}\|f^{\Delta x}\|_\infty \qquad \|E^{\Delta x}\|_\infty \le \frac{\Delta x^2}{96}\max_{0\le x\le 1}|f''(x)|}$$

where $u^{\Delta x} = [u_1, u_2, \dots, u_N]$, $\quad f^{\Delta x} = [f_1, f_2, \dots, f_N]$ $\quad E^{\Delta x} = [E_1, \dots, E_N]$

FDS for 2-D Poisson Equation

$$\boxed{\begin{matrix} -(u_{xx}(x) + u_{yy}(x)) = f(x) & \text{for } x \in \Omega = (0,1)^2 \\ u(x) = 0 & \text{for } x \in \delta\Omega \end{matrix}}$$

$$\begin{array}{ll|ll} x_i = i\Delta x & \forall\ 1 \le i \le N & y_j = j\Delta y & \forall\ 1 \le j \le M \\ x_0 = 0 & x_{N+1} = 1 & y_0 = 0 & y_{M+1} = 1 \end{array}$$

$$-\left(\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta y^2}\right) = f_{ij}$$

for $N = M \Leftrightarrow \Delta x = \Delta y$

$$U = [u_{1,1}, u_{2,1}, \dots, u_{N,1}, u_{1,2}, \dots, u_{N,2}, u_{1,N}, \dots, u_{N,N}]^T$$
$$F = \Delta x^2 [f_{1,1}, f_{2,1}, \dots, u_{N,1}, u_{1,2}, \dots, u_{N,2}, u_{1,N}, \dots, u_{N,N}]^T$$

$$AU = F,\ A \in \mathbb{R}^{N^2\times N^2}, B \in \mathbb{R}^{N\times N}$$

$$A = \begin{bmatrix} B & -I & 0 & \cdot & 0 \\ -I & B & -I & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & -I & B & -I \\ 0 & \cdot & 0 & -I & B \end{bmatrix} \qquad B = \begin{bmatrix} 4 & -1 & 0 & \cdot & 0 \\ -1 & 4 & -1 & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & -1 & 4 & -1 \\ 0 & \cdot & 0 & -1 & 4 \end{bmatrix}$$

FEM for 1D-Poisson Variational Principle

$$\min_u J(u) \qquad J(u) = \frac{1}{2}\int_0^1 |u'(x)|^2 dx - \int_0^1 u(x)f(x)dx$$

$\int_0^1 |u'(x)|^2 dx < \infty$ $\qquad \left|\int_0^1 u(x)f(x)dx\right| < \infty$ Energy is well defined

$$H_0^1([0,1]) := \left\{u : [0,1] \to \mathbb{R} : u(0) = u(1) = 0 \text{ and } \int_0^1 |u'(x)|^2 dx < \infty\right\}$$

$H_0^1([0,1])$ is the set of all functions vanishing at the boundary and have the property that the integral of the square of their derivative is bounded. *Sobolev space*

$$\|u\|_{H_0^1([0,1])} := \left(\int_0^1 |u'(x)|^2 dx\right)^{\frac{1}{2}} \quad \text{Norm on } H_0^1([0,1])$$

Use the Cauchy-Schwarz inequality: $|v \cdot w| \le \|v\|\|w\|$

$$\left|\int_0^1 u(x)f(x)dx\right| \le \int_0^1 |u(x)||f(x)|dx \le \left(\int_0^1 |u(x)|^2 dx\right)^{\frac{1}{2}} \cdot \left(\int_0^1 |f(x)|^2 dx\right)^{\frac{1}{2}}$$

$$\boxed{\begin{matrix} L^2([0,1]) := \{g : [0,1] \to \mathbb{R} : \int_0^1 |g(x)|^2 dx < \infty\} \\ \text{Norm on L: } \|g\|_{L^2([0,1])} = \left(\int_0^1 |g(x)|^2 dx\right)^{\frac{1}{2}} \end{matrix}}$$

$$\boxed{u, f \in L^2([0,1]) \Rightarrow \left|\int_0^1 u(x)f(x)dx\right| < \infty}$$

Constraints for well defined dirichlet energy

$$\boxed{\begin{matrix} u \in H_0^1([0,1]), & u \in L^2([0,1]) \\ f \in L^2([0,1]) & \end{matrix}}$$

But by the *Poincaré inequality*: $\|u\|_{L^2([0,1])} \le \|u\|_{H_0^1([0,1])}$

Precise problem formulation:

Given $f \in L^2([0,1])$, find $u \in H_0^1([0,1])$, such that u minimises the energy functional $J(v)$ for all $v \in H_0^1([0,1])$.

Variational Formulation

Find $u \in H_0^1([0,1])$: $J'(u,v) = 0 = \lim_{\tau\to 0} \frac{J(u+\tau v)-J(u)}{\tau}$

Thus u must satisfy: $\int_0^1 u'(x)v'(x)dx = \int_0^1 v(x)f(x)dx \quad (A)$

- (A) is the variational formulation of the 1D Poisson equation. Also known as principle of virtual work.
- (A) is well defined since
  $\left|\int_0^1 u'(x)v'(x)dx\right| \le \|u\|_{H_0^1([0,1])}\|v\|_{H_0^1([0,1])} < \infty$
  and since $f \in L^2([0,1])$, using Cauchy-Schwarz and Poincaré:
  $\left|\int_0^1 v(x)f(x)dx\right| \le \|v\|_{L^2([0,1])}\|f\|_{L^2(0,1)}$
  $\le \|v\|_{H_0^1([0,1])}\|f\|_{L^2([0,1])} < \infty$
- If we set $u = v$ and use Cauchy-Schwarz and Poincaré:
  $\|u\|^2_{H_0^1([0,1])} = \int_0^1 |u'(x)|^2 dx = \int_0^1 u(x)f(x)dx$
  $\le \|u\|_{L^2([0,1])}\|f\|_{L^2([0,1])} \le \|u\|_{H_0^1([0,1])}\|f\|_{L^2([0,1])}$
  $\Rightarrow \|u\|_{H_0^1([0,1])} \le \|f\|_{L^2([0,1])}$
  Thus we are provided with a stability estimate on the solution. The deflection of the membrane scales smaller than the applied load.

Derivation of the variational formulation:

1. $-u''(x) = f(x)$
2. Multiply with test function v: $\quad -u''(x)v(x) = f(x)v(x)$
3. Integrate: $\quad -\int_0^1 u''(x)v(x)dx = \int_0^1 f(x)v(x)dx$
4. Look at LHS:
   $\int_0^1 -u''(x)v(x)dx \overset{IBP}{=} [-u'(x)v(x)] - 0^1 - \int -u'(x)v'(x) = \int_0^1 u'(x)v'(x)dx.$
5. Thus: $\int_0^1 u'(x)v'(x)dx = \int_0^1 f(x)v(x)dx \quad \forall\ v \in V$

FEM Formulation

Notation: $(g,h) = \int_0^1 g(x)h(x)dx$

$$\int_0^1 u'(x)v'(x)dx = \int_0^1 v(x)f(x)dx \longrightarrow (u', v') = (f, v) \quad (A)$$

Domain $\Omega = [0,1]$, $h > 0$, $N = \frac{1}{h} - 1$, $N+2$ points:
$x_0 = 0, \quad x_{N+1} = 1, \quad x_j = jh,\ \forall\ j \in (1,N)$

V^h is the set of all **continuous**, **piecewise linear** functions on $[0,1]$ with respect to the partition $[0,1] = \bigcup_{j=1}^{N+1}[(j-1)h, jh]$.

$$w(x) = \sum_{j=1}^N w_j \phi_j(x) \quad \phi_j(x) = \begin{cases} \frac{x - x_{j-1}}{h} & x \in [x_{j-1}, x_j) \\ \frac{x_{j+1} - x}{h} & x \in [x_j, x_{j+1}) \\ 0 & \text{otherwise} \end{cases}$$

Now find $u_h \in V^h$ such that $(u_h', v') = (f, v)\ \forall\ v \in V^h$

$$(u_h', v') = (f, v) \Leftrightarrow \left(u_h', \left(\sum_{j=1}^N v_j\phi_j(x)\right)'\right) = \left(f, \sum_{j=1}^N v_j\phi_j(x)\right)$$

By linearity: $\sum_{j=1}^N v_j(u_h', \phi_j') = \sum_{j=1}^N v_j(f, \phi_j)$

$$(u_h', \phi_j') = (f, \phi_j) \text{ must hold for all } j = 1,\dots,N$$

And $u_h \in V^h$: $u_h = \sum_{i=1}^N u_i\phi_i(x) \Rightarrow \sum_{i=1}^N u_i(\phi_i', \phi_j') = (f, \phi_j)$

Matrix formulation

$$A = \{A_{ij}\}_{i,j=1,\dots,N} \quad A_{ij} = (\phi_i', \phi_j')$$

$U = \{u_j\}_{j=1}^N \qquad F = \{F_j\}_{j=1}^N \qquad \sum_{i,j} A_{ij} u_j = F_j$

A is termed the stiffness matrix, F is termed the load vector and U is termed the solution vector.

1. A is symmetric: $A_{ij} = (\phi_i, \phi_j) = (\phi_j, \phi_i)$
2. A is positive definite:
3. **A is invertible and FEM well defined**.

FEM for 2D-Poisson

$$-\Delta u = f \text{ in } \Omega \qquad u|_{\partial\Omega} \equiv 0$$

$$J(w) = \frac{1}{2}\int_\Omega |\nabla w|^2 dx - \int_\Omega w(x)f(x)dx \quad \text{Dirichlet Energy}$$

Find $u \in V = H_0^1(\Omega)$ such that for all $v \in V$ it holds that

$$(u,v)_{H_0^1(\Omega)} = (f,v)_{L^2(\Omega)} \text{ or } \int_\Omega \langle \nabla u, \nabla v\rangle dx = \int_\Omega f(x)v(x)dx$$

Triangulation

$$\bar{\Omega} = \bigcup_{K\in T_h} \bar{K} = \bar{K}_1 \cup \bar{K}_2 \cup \ldots \cup \bar{K}_M \qquad h = \max_{K\in T_h} \operatorname{diam}(K)$$

K_i are triangles. $k_i \cap k_j = \phi$ = common vertices = common edges
$\operatorname{diam}(K)$: length of the longest edge of K $\quad \mathcal{N}_i$: set of nodes (vertices)
h is the mesh width

$$\boxed{V^h = \{v : \Omega \to \mathbb{R} : v \text{ cont.}, v|_K \text{ linear for each } K \in T_h,\ v = 0 \text{ on } \partial\Omega\}}$$

V^h: space of continuous, piecewise linear functions vanishing on the boundary $\partial\Omega$.

Concrete Realisation of FEM

$$\boxed{\phi_j(x) \in V^h \quad \phi_j(\mathcal{N}_i) = \begin{cases} 1 & \text{if } j = 1 \\ 0 & \text{otherwise}\end{cases}} \text{ Hat functions}$$

$$\boxed{v(x) = \sum_{j=1}^N v_j\phi_j(x)} \text{ Basis for } V^h$$

where $v_j = v(\mathcal{N}_j)\ \forall\ j = 1, \dots, N$
Thus we can formulate the problem as:

$$\begin{aligned}(u_h, \phi_j)_{H_0^1(\Omega)} &= (f, \phi_j)_{L^2(\Omega)} \\ \int_\Omega \langle \nabla u_h, \nabla\phi_j\rangle dx &= \int_\Omega f(x)\phi_j(x), \quad \forall 1 \le j \le N \\ u_h &= \sum_{i=1}^N u_i\phi_i \\ \sum_{i=1}^N u_i \int_\Omega \langle\nabla\phi_i, \nabla\phi_j\rangle dx &= \int_\Omega f(x)\phi_j(x)dx\end{aligned}$$

$$\boxed{\begin{array}{ll} U = \{u_i\}_{i=1}^N & F = \{F_j\}_{j=1}^N \\ A = \{A_{ij}\}_{i,j=1}^N & \\ F_j = \int_\Omega f(x)\phi_j(x)dx & A_{ij} = \int_\Omega \langle\nabla\phi_i, \nabla\phi_j\rangle dx \\ & AU = F \end{array}}$$

For A and B see FDS for 2-D Poisson Equation.

- A is symmetric.
- A is positive definite.

Implementation of the FEM
Step 1: Triangulation

$$Z \in \mathbb{R}^{2\times N} \qquad T \in \mathbb{R}^{3\times M}$$

- $Z(\cdot, j)$ refers to node $\mathcal{N}_j$
- $Z(1,j)$, $Z(2,j)$ refer to x and y of $\mathcal{N}_j$
- $T(\cdot, j)$ refers to the j^{th} triangle K_j
- $T(i,j), (i = 1,2,3)$ represent the indices of the nodes of K_j

Often an additional vector denoting boundary nodes is added, enabling boundary conditions.

2: Element Stiffness Matrices and Element Load Vectors

$$A_{ij} = \int_\Omega \langle\nabla\phi_i, \nabla\phi_j\rangle dx = \sum_{m=1}^M \int_{K_m} \langle\nabla\phi_i, \nabla\phi_j\rangle dx$$

$\int_{K_m} \langle\nabla\phi_i, \nabla\phi_j\rangle dx \neq 0$ only iff $\mathcal{N}_i$ and $\mathcal{N}_j$ are vertices of the same triangle!

$$T(\alpha, m),\ \alpha = 1,2,3 \text{ - Labels of the vertices of } K_m$$

$$Z(i, T(\alpha, m)),\ i = 1,2,\ \alpha = 1,2,3 \text{ - Coordinates of each vertex } T(\alpha, m)$$

$$\boxed{\phi_\alpha\left(\mathcal{N}_{T(\beta,m)}\right) = \begin{cases} 1 & \text{if } \alpha = \beta \\ 0 & \text{if } \alpha \neq \beta\end{cases}} \text{ Local shape functions}$$

$$\boxed{A_{\alpha,\beta}^m = \int_{K_m} \langle\nabla\phi_\alpha, \nabla\phi_\beta\rangle dx} \text{ Stiffness matrix } A^m \text{ for } K_m$$

- $A^m \in \mathbb{R}^{3\times 3}$ is symmetric.

$$\boxed{F_\alpha^m = \int_{K_m} f(x)\phi_\alpha(x)dx} \text{ Element load vector } F^m \text{ for } K_m$$

Reference Triangle

$$\Phi_K : \hat{K} \to K \quad \text{Mapping}$$

$$\boxed{\begin{aligned} x = \Phi_K(\hat{x}) &= \begin{pmatrix}\mathcal{N}_b - \mathcal{N}_a & \mathcal{N}_c - \mathcal{N}_a\end{pmatrix}\hat{x} + \mathcal{N}_a \\ &= J_K\hat{x} + \mathcal{N}_a\end{aligned}}$$

$$\boxed{F_\alpha^K = \int_K f(x)\phi_\alpha(x)dx = \int_{\hat{K}} f(\Phi_K(\hat{x}))\hat{\phi}_\alpha(\hat{x})|\det(J_K)|d\hat{x}}$$

$$\boxed{\begin{aligned} A_{\alpha,\beta}^K &= \int_K \langle\nabla\phi_\alpha, \nabla\phi_\beta\rangle dx \\ & \int_{\hat{K}} \langle J_K^{-T}\hat{\nabla}\hat{\phi}_\alpha, J_K^{-T}\hat{\nabla}\hat{\phi}_\beta\rangle |\det J_K|\, d\hat{x}\end{aligned}}$$

- $\hat{\phi}_\alpha$ local shape functions of the reference triangle.
- $\phi_\alpha(x) = \hat{\phi}_\alpha(\hat{x}) \qquad \nabla\phi_\alpha = J_K^{-T}\hat{\nabla}\hat{\phi}_\alpha(\hat{x})$

Step 3: Assembly

A = zeros(N,N); f = zeros(N);
looping over all triangles:
 fetch A^m and F^m
 A(T(α,m),T(β,m))=A(T(α,m),T(β,m))+$A_{\alpha,\beta}^m$
 F(T(α,m)=F(α,m)+F_α^m

Parabolic PDE

$Au_{xx} + 2Bu_{xt} + Cu_{tt} = F(x,t,u,u_x,u_t) \qquad AC - B^2 = 0 \to$ parabolic.

$$\boxed{\begin{array}{lll} u_t - \Delta u = 0 & & \text{on } \Omega \times (0,T) \\ u(x,0) & = u_0(x) & \text{on } \Omega \\ u(0,t) & = u(1,t) = 0 & \text{on } (0,T)\end{array}} \text{ Heat Equation}$$

$u = u(x,t) = \mathcal{T}(t)\mathcal{X}(x)$ Solving these two ODE using BC:

$$\frac{\mathcal{T}'(t)}{\mathcal{T}(t)} = \frac{\mathcal{X}''(x)}{\mathcal{X}(x)} = -\lambda_k \qquad \begin{aligned}\mathcal{X}_k(x) &= \sin(k\pi x),\ \lambda_k = (k\pi)^2\ \forall k \in \mathbb{Z} \\ \mathcal{T}(t) &= e^{-(k\pi)^2 t}\end{aligned}$$

$$\boxed{u(x,t) = \mathcal{X}(x)\mathcal{T}(t) = e^{-(k\pi)^2 t}\sin(k\pi x)}$$

$$\boxed{u(x,t) = \sum_{k=1}^\infty u_k^0 e^{-(k\pi)^2 t}\sin(k\pi x)}$$

This solution fulfils the initial PDE as well as the boundary conditions.

Evaluation of Seperation of Variables

1. Expand u_0 using the Fourier series:
 $u_0^N(x) = \sum_{k=1}^N u_k^0 \sin(k\pi x)$
 The error $|u_0 - u_0^N|$ is small for large N and if u_0 is smooth and satisfies $u_0(0) = u_0(1) = 0$
2. Calculate the coefficients $\{u_k^0\}_{k=1}^N$
 $u_k^0 = 2\int_0^1 u_0(x)\sin(k\pi x)dx$
3. $u_N(x,t) = \sum_{k=1}^N u_k^0 e^{-(k\pi)^2 t}\sin(k\pi x)$
 Two error sources: Error due to the finite-truncation of the Fourier series, i.e. the series is only exact if infinite!
 Error due to the use of quadrature rules to approximate integrals.

Application of Gauss-Green

$$\boxed{\frac{d}{dt}\int_\omega u(x,t)dx = -\int_{\partial\omega} F \cdot v d\sigma(x)}$$

Rate of change of u in ω = Flux over the boundary + Sources - Sinks

u(x,t) $\to$ some quantity, $F \to$ flux, $v \to$ unit outward normal
$\int_\omega u_t dx = -\int_{\partial\omega} F \cdot v d\sigma(x)$
Gauss-Green (Integration by parts): $\int_{\partial\omega} F \cdot v d\sigma(x) = \int_\omega \nabla(F)dx$
$\int_\omega (u_t + \nabla F)dx = 0\ \forall\ \omega \in \Omega \Rightarrow \boxed{u_t + \nabla(F) = 0}$ where $F = F(t,x,u,\nabla u, \dots)$
Special case: heat conduction: $F \propto -\nabla u \Rightarrow u_t = \Delta u$

Energy Estimate

$$\mathcal{E}(t) := \frac{1}{2}\int_0^1 |u(x,t)|^2 dx \quad \text{Energy estimate}$$

$\frac{d\mathcal{E}}{dt} = \frac{1}{2}\int_0^1 (u^2)_t dx = \int_0^1 uu_t dx = \int_0^1 uu_{xx}dx =$
$= -\int_0^1 u_x^2 dx + [uu_x]_0^1 \overset{\text{BC}}{=} -\int_0^1 u_x^2 dx$

$$\boxed{\frac{d\mathcal{E}}{dt} = -\int_0^1 u_x^2 dx \le 0 \Rightarrow \mathcal{E}(t) \le \mathcal{E}(0)}$$

Maximum Principles

$$\boxed{\max_{\substack{0\le x\le 1,\\ 0\le t\le T}} u(x,t) \le \max(0, \max_x(u_0(x)))}$$

Explicit Finite Difference Schemes for the Heat Equation

1. Discretising the Domain:
   $\Delta x > 0 \quad \delta t > 0 \quad N = \frac{1}{\Delta x} - 1$
   $x_0 = 0, \quad x_j = j\Delta x, \; j = 1, \dots, N, \quad x_{N+1} = 1$
   $\rightarrow N+2$ equally spaced points
   $t_0 = 0, \quad t^n = n\Delta t, \; n = 1, \dots, M, \quad t^{M+1} = T$
2. Discretising the Solution u $U_j^n \approx u(x_j, t^n)$
3. Discretising the Derivatives like in 1D FDS page 1
4. The FDS: $\frac{U_j^{n+1} - U_j^n}{\Delta t} - \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{\Delta x^2} = 0$
   $\lambda = \frac{\Delta t}{\Delta x^2} \rightarrow U_j^{n+1} = (1-2\lambda)U_j^n + \lambda U_{j+1}^n + \lambda U_{j-1}^n$

IC and BC: $\begin{aligned} U_j^0 &= u(x_j, 0) = u_0(x_j), \quad && \forall\, 1 \le j \le N \\ U_0^n &= U_{N+1}^n \equiv 0 && \forall\, 0 \le n \le M+1 \end{aligned}$

Called „explicit“since we use the explicit euler method for time stepping.

$\boxed{\Delta t \le \frac{1}{2}\Delta x^2}$ Stability condition

Truncation Error

$\tau_j^n = \frac{u_j^{n+1} - u_j^n}{\Delta t} - \frac{u_{j-1}^n - 2u_j^n + u_{j+1}^n}{\Delta x^2}$ or equivalently $\tau_j^n = D_t^+ u_j^n - D_x^- D_x^+ u_j^n$

$$\boxed{|\tau_j^n| \le C(\Delta t + \Delta_x) \text{ or } \sqrt{\frac{\Delta x}{2}\sum_{j=1}^{N} |u_j^n - U_j^n|^2} \le \bar{C}(\Delta t + \Delta x^2}$$

Thus the explicit finite difference scheme has a first-order rate of convergence in time and a second-order rate of convergence in space.

An Implicit Finite Difference Scheme

$$D_t^- U_j^{n+1} = D_x^- D_x^+ U_j^{N+1}$$

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = \frac{U_{j-1}^{n+1} - 2U_j^{n+1} + U_{j+1}^{n+1}}{\Delta x^2}$$

$\lambda = \frac{\Delta t}{\Delta x^2}$

$$-\lambda U_{j-1}^{n+1} + (1+2\lambda)U_j^{n+1} - \lambda U_{j+1}^{n+1} = U_j^n$$

$$AU^{n+1} = F^n$$

$$\boxed{U^{n+1} = \{U_j^{n+1}\}_{j=1}^N \qquad F^n = \{U_j^n\}_{j=1}^N}$$

$$\boxed{A = \begin{bmatrix} 1+2\lambda & -\lambda & 0 & \cdot & 0 \\ -\lambda & 1+2\lambda & -\lambda & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & -\lambda & 1+2\lambda & -\lambda \\ 0 & \cdot & 0 & -\lambda & 1+2\lambda \end{bmatrix}}$$

$\boxed{\text{Unconditional stability!}}$

Crank-Nicloson Scheme

$$D_t^+ U_j^n = \tfrac{1}{2} D_x^- D_x^+ U_j^n + \tfrac{1}{2} D_x^2 D_x^+ U_j^{n+1}$$

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = \frac{U_{j-1}^n - 2U_j^n + U_{j+1}^n}{2\Delta x^2} + \frac{U_{j-1}^{n+1} - 2U_j^{n+1} + U_{j+1}^{n+1}}{2\Delta x^2}$$

Crank-Nicholson is the formal average of the explicit finite difference scheme and the implicit finite difference scheme.

$$U_0^n = U_{N+1}^n = 0 \qquad U_j^0 = u_j^0 = u_0(x_j) \qquad \lambda = \frac{\Delta t}{\Delta x^2}$$

$$-\tfrac{\lambda}{2} U_{j-1}^{n+1} + (1+\lambda)U_j^{n+1} - \tfrac{\lambda}{2} U_{j+1}^{n+1} = \tfrac{\lambda}{2} U_{j-1}^n + (1-\lambda)U_j^n - \tfrac{\lambda}{2} U_{j+1}^n = F_j^n$$

$$AU^{n+1} = F^n$$

$$\boxed{U^{n+1} = \{U_j^{n+1}\}_{j=1}^N \qquad F^n = \{F_j^n\}_{j=1}^N}$$

$$\boxed{A = \begin{bmatrix} 1+\lambda & -\frac{\lambda}{2} & 0 & \cdot & 0 \\ -\frac{\lambda}{2} & 1+\lambda & -\frac{\lambda}{2} & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & -\frac{\lambda}{2} & 1+\lambda & -\frac{\lambda}{2} \\ 0 & \cdot & 0 & -\frac{\lambda}{2} & 1+\lambda \end{bmatrix}}$$

$\boxed{\text{Unconditionally stable!}}$

Despite the unconditional stability the CN-scheme **only satisfies the discrete maximum principle if** $\lambda = \frac{\Delta t}{\Delta x^2} \le 1$. For large values of λ the solution may contain spurious oscillations.

Truncation error

$$\tau_j^n = \frac{u_j^{n+1} - u_j^n}{\Delta t} - \frac{u_{j-1}^n - 2u_j^n + u_{j+1}^n}{2\Delta x^2} - \frac{u_{j-1}^{n+1} - 2u_j^{n+1} + u_{j+1}^{n+1}}{2\Delta x^2}$$

$$\tau_j^n = D_t^+ u_j^n - \left(\tfrac{1}{2} D_x^- D_x^+ u_j^n + \tfrac{1}{2} D_x^- D_x^+ u_j^{n+1}\right)$$

$$\boxed{|\tau_j^n| \le C(\Delta t^2 + \Delta x^2}$$

CN has a second-order rate of convergence in both time and space.

Linear Transport Equations (Hyperbolic PDEs)

$$u_t + a(x,t)u_x = 0 \; \forall (x,t) \in \mathbb{R} \times \mathbb{R}_+$$

Method of Characteristics

Assume that we have a curve along which the solution u is constant.

$$u(x,0) = u_0(x)$$

$$\begin{aligned} 0 &= \tfrac{d}{dt} u(x(t), t) \\ &= u_t(x(t), t) + u_x(x(t), t)x'(t) \end{aligned}$$

where x(t) is a characteristic curve.

Since this has to fulfil the linear transport equation we know that:

$$\begin{aligned} x'(t) &= a(x(t), t) \\ x(0) &= x_0 \end{aligned}$$

Gronwall's inequality

$$u'(t) \le \beta(t)u(t) \quad \forall\, t \in (a,b)$$

where $\beta(t)$ continuous, $u(t)$ differentiable on some $[a,b]$.

$$\boxed{u(t) \le u(a)\exp\left(\int_a^t \beta(t)dt\right) \quad \forall\, t \in [a,b]}$$

Let $u(x,t)$ be a smooth solution, $\lim_{|x|\to\infty} u(x,t) = 0$. Then u fulfils the following energy bound:

$$\boxed{\int_{\mathbb{R}} u^2(x,t)dx \le e^{||a||_{C^1} t} \int_{\mathbb{R}} u_0^2(x)dx}$$

Finite difference schemes Discretization

$[x_l, x_r]$ is discretized with a mesh size Δx into a sequence of $N+1$ points.
Divide $[0,T]$ into M points $t^n = n\Delta t$
Then a finite difference scheme is applied:

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} + \frac{a(U_{j+1}^n - U_{j-1}^n)}{2\Delta x} = 0$$

By an energy analysis we conclude that this central difference scheme does not produce valid but oscillatory solutions with increasing energies.

Upwind schemes

The central scheme does not respect the direction of propagation of information for the transport equation. This is corrected by the upwind scheme.
As found by the method of characteristics the characteristic curves are defined by:

$$x'(t) = a$$

Thus if $a > 0$ the direction of information propagation is from left to right, then we use a backward difference in space to obtain our scheme. We do the opposite if $a < 0$.

$$a^+ = \max\{a, 0\}, \quad a^- = \min\{a, 0\}, \quad |a| = a^+ - a^-$$

Thus the combined (backwards/forwards) upwind scheme can be written as:

$$\boxed{\frac{U_j^{n+1} - U_j^n}{\Delta t} + \frac{a^+(U_j^n - U_{j-1}^n)}{\Delta x} + \frac{a^-(U_{j+1}^n - U_j^n)}{\Delta x} = 0}$$

or

$$\boxed{\frac{U_j^{n+1} - U_j^n}{\Delta t} + \frac{a(U_{j+1}^n - U_{j-1}^n)}{2\Delta x} = \underbrace{\frac{|a|}{2\Delta x}(U_{j+1}^n - 2U_j^n + U_{j-1}^n)}_{\text{numerical viscosity}}}$$

However the upwind scheme still is only stable for some $\frac{\Delta t}{\Delta x}$.

Stability for the upwind scheme

If $|a|\frac{\Delta t}{\Delta x} \le 1$ the upwind scheme satisfies the energy estimate $E^{n+1} \le E^n$ and is thus **conditionally stable.**