Signals and Systems

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Contents

Nice to know

$r-k$ is a multiple of $N \Leftrightarrow r=k \mod N$

Definitions

$\boxed{e^{Mt}=\mathbb{I}+Mt+\frac{(Mt)^2}{2!}+\cdots=\sum_{k=0}^{\infty}\frac{((Mt)^k}{k!}}$ Matrix exponential

$\boxed{M\in\mathbb{C}^{n\times n} \quad \exists\, k: M^k\equiv 0, M^{k-1}\neq 0}$ Nilpotent matrix, to degree k

→ Matrix exponential is easy to calculate! Only $k-1$ non zero terms in the sum.

Notation

x	signal, function of time
$x[n]$	value of x at discrete time n
$x(t)$	value of x at continuous time t
$\{x[n]\}$	entire sequence

Signal Representation

- Graph
- Rule: $x[n] := \begin{cases}(\frac{1}{2})^n & n\geq 0\\ 0 & n<0\end{cases}$
- Sequence: $\{x[n]\}=\{\ldots,0,\underset{\uparrow}{1},\frac{1}{2},\ldots\}$ ↑ indicates index 0

1 Discretization of CT Signals & Systems

- **Uniform sampling:** $x[n]=x(nT_s)$

  sampling period: T_s, sampling frequency: $f_s=\frac{1}{T_s}$

- **Zero-order hold:** $x(t)=x[n] \quad nT_s\leq t<(n+1)T_s$

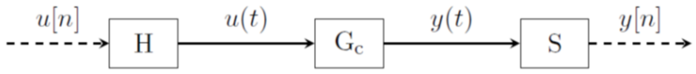
  Zero order hold does not require a future point for interpolation. Higher order holds are not causal.

1.1 Hold and Sample Operators

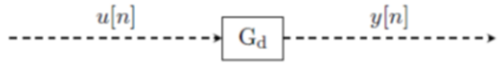
$u[n], u(t), y(t), y[n]$ refer to entire signals, G_c is a real world continuous system

represent G_c as a state-space description:

$$G_c:\begin{aligned}\dot{q}(t)&=A_cq(t)+B_cu(t)\\ y(t)&=C_cq(t)+D_cu(t)\end{aligned}$$



Is there a system G_d such that $G_d=SG_cH$?



$$\begin{bmatrix}\dot{q}(t)\\ \dot{u}(t)\end{bmatrix}=\underbrace{\begin{bmatrix}A_c & B_c\\ 0 & 0\end{bmatrix}}_{M}\cdot\begin{bmatrix}q(t)\\ u(t)\end{bmatrix}$$

$$B\in\mathbb{R}^{n\times m},\quad A\in\mathbb{R}^{n\times n}$$

because $u(t)=u[0] \quad 0\leq t<T_s$ therefore $\dot{u}(t)\equiv 0$.

The solution at T_s^-, just before sampling is:

$$\begin{bmatrix}q(T_s^-)\\ u(T_s^-)\end{bmatrix}=F\begin{bmatrix}q(0)\\ u(0)\end{bmatrix}.\ \text{with } F=e^{MT_s}.$$

$$A_d=F(1:n,1:n),\quad B_d=F(1:n,n+1:n+m),\quad C_d=C_c,\quad D_d=D_c.$$

$$\boxed{\begin{aligned}q[n+1]&=A_dq[n]+B_du[n]\\ y[n]&=C_dq[n]+D_du[n]\end{aligned}}$$

This is an exact discretization as opposed to the approximative **Euler discretization:** $\dot{q}(t)\approx\frac{q(t+T_s)-q(t)}{T_s}=\frac{q[n+1]-q[n]}{T_s}$

Euler is good as long as T_s is small.

2 Classification of Systems

- **Memoryless**

  Output at time n only depends on input at the same timestep.

- **Causal**

  Output at time n only depends on past an present inputs.

- **Linear**

  $G\{\alpha_1u_1[n]+\alpha_2u_2[n]\}=\alpha_1G\{u_1[n]\}+\alpha_2G\{u_2[n]\}$

- **Time-invariant**

  $\{y_2[n]\}=\{y_1[n-k]\},\quad y_1=Gu_1,\ y_2=Gu_2,$

  $\{u_2[n]\}=\{u_1[n-k]\},\quad \forall\ k,u_1[n]$

2.1 Stability of linear systems, BIBO

Bounded sequence: $u[n]:\ |u[n]|\leq M\ \forall\ n$

Stability: $u[n],\ y[n]=Gu[n],\ \exists M:\ |u[n]|\leq 1\ \forall\ n,\ |y[n]|\leq M$

BIBO: Bounded input bounded output.

3 LTI System response to Inputs

3.1 Impulse response of a system

3.1.1 Useful signals

- Impulse sequence: $\{\delta[n]\}:=\begin{cases}1 & n=0\\ 0 & n\neq 0\end{cases}$
- Step sequence: $\{s[n]\}:=\begin{cases}1 & n\geq 0\\ 0 & n<0\end{cases}$

	Integration	Differentiation
Continuous	$s(t)=\int_{-\infty}^{t}\delta\tau d\tau$	$\frac{d}{dt}s(t)=\lim_{\epsilon\to 0}\frac{s(t)-s(t-\epsilon)}{\epsilon}=\delta(t)$
Discrete	$s[n]=\sum_{k=-\infty}^{n}\delta[k]$	$\{s[n]\}-\{s[n-1]\}=\{\delta[n]\}$

3.1.2 Representing a sequence with inpulses

$x[n]=\sum_{k=-\infty}^{\infty}x[k]\delta[n-k] \quad \forall\ n$

$\{x[n]\}=\sum_{k=-\infty}^{\infty}xk\{\delta[n-k]\}$

3.1.3 Response to arbitrary inputs

Impulse response $\{h[n]\}:=G\{\delta[n]\}$

Any sequence can be written as a sum of time-shifted impulses.

$\{y[n]\}=G\{\sum_{k=-\infty}^{\infty}u[k]\delta[n-k]\overset{L}{=}\sum_{k=-\infty}^{\infty}u[k]G\{\delta[n-k]\}\overset{TI}{=}\sum_{k=-\infty}^{\infty}u[k]\{h[n-k]\}$

3.1.4 Convolution

$x*h=\{x[n]\}*\{h[n]\}:=\sum_{k=-\infty}^{\infty}x[k]\{h[n-k]\}$

- Commutative: $x*h=h*x$
- Associative: $(x*h_1)*h_2=x*(h_1*h_2)$
- Distributive: $x*(h_1+h_2)=x*h_1+x*h_2$

3.2 Step Response

$\{r[n]\}:=\{h[n]\}*\{s[n]\}=\sum_{k=-\infty}^{\infty}h[k]\{s[n]\}=\{\sum_{k=-\infty}^{\infty}h[k]\}$

$r[n]-r[n-1]=\sum_{k=-\infty}^{\infty}h[k]-\sum_{k=-\infty}^{\infty}h[k]=h[n],\ \forall\ n$

3.3 Causality

$$y[n] = \sum_{k=-\infty}^{\infty} u[k]h[n-k],\ \forall\ n$$

$$\boxed{\text{System is causal} \Leftrightarrow\ h[n]=0,\ \forall\ n<0}$$

causal input: $u[n] : u[n] = 0\ \forall\ n \leq 0$

$$y[n] = \sum_{k=0}^{n} u[k]h[n-k] = \sum_{k=0}^{n} h[k]u[n-k],\ \forall n$$

3.4 Stability of an LTI system

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

4 Linear constant coefficient difference equations

4.1 Definition

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k u[n-k], \qquad a_k, b_k \in \mathbb{R}$$

a_0 and b_0 are not both zero!

If the system is causal ($a0 \neq 0$)

$$y[n] = \frac{1}{a_0}\left(\sum_{k=0}^{M} b_k u[n-k] - \sum_{k=1}^{N} a_k y[n-k]\right)$$

4.2 Converting from LCCDE to state-space

$$\text{SS:}\quad \begin{aligned} q[n+1] &= Aq[n] + Bu[n] \\ y[n] &= Cq[n] + Du[n] \end{aligned}$$

Special case of LCCDE: $y[n] + a_1 y[n-1] + \cdots + a_N y[n-N] = b_0 u[n]$

To calculate $y[n]$ at time n we need N past outputs as well as the current input.

$$\left.\begin{aligned} q_1[n] &= y[n-N] \\ q_2[n] &= y[n-(N-1)] = y[n-N+1] \\ \cdots &= \cdots \\ q_N[n] &= y[n-1] \end{aligned}\right\} q[n] = \begin{bmatrix} q_1[n] \\ \vdots \\ q_N[n] \end{bmatrix}$$

Therefore:

$q_1[n+1] = q_2[N],\ q_2[n+1] = q_3[n], \ldots,\ q_{N-1}[n+1] = q_N[n]$

$q_N[n+1] = y[n] = b_0 u[n] - a_N q_1[n] - \cdots - a_1 q_N[n]$

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdot & 0 \\ & & & \ddots & \\ & & & & 1 \\ -a_N & -a_{N-1} & -a_{N-2} & \cdots & -a_1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ b_0 \end{bmatrix}$$

$$C = \begin{bmatrix} [-a_N & -a_{N-1} & -a_{N-2} & \cdots & -a_1] \end{bmatrix} \qquad D = \begin{bmatrix} b_0 \end{bmatrix}$$

4.3 Impulse response of a DT LTI system in SS

$$\begin{aligned} q[1] &= Aq[0] + Bu[0] \\ q[2] &= Aq[1] + Bu[1] = A^2 q[0] + ABu[0] + Bu[1] \\ &\ \vdots \\ q[n] &= A^n q[0] + \sum_{k=0}^{n-1} A^{n-k-1} Bu[k], \quad n \geq 0 \\ y[n] &= Cq[n] + Du[n] = CA^n q[0] + C\sum_{k=0}^{n-1} A^{n-k-1} Bu[k] + Du[n],\ n \geq 0 \end{aligned}$$

$$\boxed{h = \{D, CB, CAB, \ldots, CA^{n-1}B, \ldots\}}$$

4.4 Finite Impulse Response (FIR) vs. Infinite Impulse Response (IIR)

$$\boxed{\exists N : h[n] = 0\ \forall\ n \geq N}\ \text{FIR}$$

A system that can be written in the non-recursive form has a FIR.

$$\boxed{y[n] = \frac{1}{a_0}\sum_{k=0}^{M} b_k u[n-k]}\ \text{Non-recursive Form}$$

5 Periodic signals

$$x[n+N] = x[n]$$

5.1 Periodicity constraint

CT : $\cos(\omega t)$ is periodic with $T = \frac{2\pi}{|\omega|}$, sampled with T_s the resulting DT signal $\{x[n]\} = \{\cos(\Omega n)\}$ has the frequency $\Omega = \omega T_s$ is periodic iff

$$\frac{\Omega}{2\pi} = \frac{m}{N}\ \text{for some integers}\ m, N$$

If $\frac{m}{N}$ is an irreducible fraction, then N is the fundamental period of the signal.

5.2 Eigenfunctions of LTI Systems

$$\begin{aligned} \{y[n]\} = G\{z^n\} &= \sum_{k=-\infty}^{\infty} h[k]\{z^{n-k}\} \\ &= \sum_{k=-\infty}^{\infty} h[k]z^{-k}\{z^n\} \\ &= H(z)\{z^n\} \end{aligned}$$

$$\boxed{H(z) := \sum_{k=-\infty}^{\infty} h[k]z^{-k}}$$

6 The z-Transform

$$\boxed{X(z) := \sum_{n=-\infty}^{\infty} x[n]z^{-n}, \quad z \in \mathbb{C}}$$

Properties:

- Linearity $\quad a_1\{x_1[n]\} + a_2\{x_2[n]\} \leftrightarrow a_1 X(z) + a_2 X_2(z)$
- Time-shifting $\quad \{x[n-1]\} \leftrightarrow z^{-1}X(z)$
- Convolution $\quad \{x_1[n]\} * \{x_2[n]\} \leftrightarrow X_1(z) \cdot X_2(z)$
- Accumulation $\quad \{\sum_{k=-\infty}^{\infty} x[k]\} \leftrightarrow \frac{z}{z-1}X(z)$
- Special case $\quad \{u[n]\} = z^n \leftrightarrow G\{u[n]\} = H(z) \cdot \{z^n\}$

6.1 Convergence and non-uniqueness

$$x[n] = \begin{cases} a^n & n \geq 0,\ a \in \mathbb{R}, a \neq 0 \\ 0 & \text{otherwise} \end{cases} \qquad x[n] = \begin{cases} -a^n & n < 0,\ a \in \mathbb{R}, a \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n \qquad X(z) = -\sum_{n=-\infty}^{-1} = -\sum_{n=1}^{\infty} \left(\frac{z}{a}\right)^n$$

$$X(z) = \frac{z}{z-a} \qquad X(z) = \frac{z}{z-a}$$

$X(z)$ converges if $\left|\frac{a}{z}\right| < 1$ $\qquad$ $X(z)$ converges if $\left|\frac{z}{a}\right| < 1$

$$\boxed{\text{The z-Transform must also include the R.O.C. in order to uniquely specify}}$$

R.O.C. - Region of convergence

6.2 Transfer functions of LTI systems

$$\{y[n]\} = \{u[n]\} * \{h[n]\} \longleftrightarrow Y(z) = U(z) \cdot H(z)$$

$$\boxed{H(z) = \frac{Y(z)}{U(z)}}$$

6.3 Transfer function from LCCDE

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k u[n-k] \leftrightarrow \sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} U(z)$$

$$\boxed{\frac{Y(z)}{U(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \cdots + a_N z^{-}} = H(z)}$$

6.4 Transfer function from SS

$$\begin{aligned} q[n+1] &= Aq[n] + Bu[n] \\ y[n] &= Cq[n] + Du[n] \\ &\ \updownarrow \\ zQ(z) &= AQ(z) + BU(z) \\ Q(z) &= (zI - A)^{-1} BU(z) \\ Y(z) &\quad CQ(z) + DU(z) \end{aligned}$$

$$\boxed{H(z) = \frac{Y(z)}{U(z)} = C(zI - A)^{-1}B + D}$$

6.5 Stability and causality

Given a transfer function $H(z)$, there exists a stable and causal interpretation for the underlying system iff all poles of $H(z)$ are inside the unit circle. That is, given pole p (a value p for which $|H(p)| = \infty$), then $|p| < 1$.

7 Fourier Transform

7.1 DT Fourier Transform

$$\boxed{\sum_{n=-\infty}^{\infty} |x[n]| < \infty} \text{ only for absolutely summable signals}$$

7.1.1 Definition

$$\boxed{X(\Omega) = \mathcal{F}x := \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}, \quad -\pi < \Omega \leq \pi}$$

$$\{x[n]\} \longleftrightarrow \underbrace{X(\Omega)}_{\text{refers to whole function}}$$

$$X(\Omega) = |X(\Omega)| \cdot e^{j\Theta_X(\Omega)}$$

The DT Transform is equal to the z-Transform if

$z = e^{j\Omega}, \quad X(z) = \sum_{-\infty}^{\infty} x[n]z^{-n}$

7.2 Inverse DT Fourier Transform

$$\boxed{\{x[n]\} = \mathcal{F}^{-1}X := \{\tfrac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega)e^{j\Omega n}d\Omega\}}$$

7.3 Properties of the DT Fourier Transform

- Linearity: $a_1\{x_1[n]\} + a_2\{x_2[n]\} \longleftrightarrow a_1X_1(\Omega) + a_2X_2(\Omega)$
- Convolution: $\{x_1[n]\} * \{x_2[n]\} \longleftrightarrow X_1(\Omega) \cdot X_2(\Omega)$
- Parseval's theorem: $\sum_{-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\Omega)|^2 d\Omega$

8 Frequency Response of LTI Systems

$$y = u * h \longleftrightarrow Y(\Omega) = U(\Omega)H(\Omega)$$

$$H(\Omega) = \frac{Y(\Omega)}{U(\Omega)} \quad \text{Frequency Response}$$

$$H(\Omega) = |H(\Omega)|e^{j\Theta_H(\Omega)}$$
$$|Y(\Omega)| = |U(\Omega)||H(\Omega)|$$
$$\Theta_Y(\Omega) = \Theta_U(\Omega) + \Theta_H(\Omega)$$

8.1 Frequency response from LCCDE

$$H(\Omega) = H(z)|_{z=e^{j\Omega}}$$

$$\boxed{H(\Omega) = \frac{\sum_{k=0}^{M} b_k e^{-j\Omega k}}{\sum_{k=0}^{N} a_k e^{-j\Omega k}}}$$

8.2 Response to complex exponential sequences

$$\{u[n]\} = \{z^n\} = \{e^{j\Omega_0 n}\}$$

$$\begin{aligned} \{y[n]\} &= G\{z^n\} = H(z)\{z^n\} \\ \rightarrow y[n] &= H(z = e^{j\Omega_0})e^{j\Omega_0 n} = H(\Omega = \Omega_0)e^{j\Omega_0 n} \\ &= |H(\Omega_0)|e^{j(\Omega_0 n + \Omega_H(\Omega_0))} \end{aligned}$$

8.3 Response to real sinusoids

$$\begin{aligned} y &= Gu \\ &= G(u_1 + ju_2) \\ &= Gu_1 + jGu_2 \\ &= y_1 + jy_2 \end{aligned}$$

$$\begin{aligned} u[n] &= e^{j\Omega_0 n} \Rightarrow u_1[n] = \cos(\Omega_0 n) \\ y[n] &= H(\Omega_0)e^{j\Omega_0 n} = |H(\Omega_0)|e^{j(\Omega_0 n + \Theta_H(\Omega_0))} \\ y_1[n] &= |H(\Omega_0)|\cos(\Omega_0 n + \Theta_H(\Omega_0)) \end{aligned}$$

9 Discrete Fourier Series / Transform

9.1 DFS representation of a periodic signal

$$\boxed{x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{jk\frac{2\pi}{N}n} \qquad X[k] = \sum_{n=0}^{N-1} x[n]e^{-jk\frac{2\pi}{N}n}}$$

$$\mathcal{F}_s, \qquad X = \mathcal{F}_s x, \quad x = \mathcal{F}_s^{-1}X \quad \text{The DFS operator}$$

X is periodic with period N:

$$X[k+N] = \sum_{n=0}^{N-1} x[n]e^{-j(k+N)\frac{2\pi}{N}n} = \sum_{n=0}^{N-1} x[n]e^{-jk\frac{2\pi}{N}n}\underbrace{e^{-j2\pi n}}_{=1\forall\ n} = X[k]$$

The DFS operator is invertible: $\mathcal{F}_s\mathcal{F}_s^{-1} = \mathbb{I} \qquad \mathcal{F}_s^{-1}\mathcal{F} = \mathbb{I}$

Orthogonality of complex exponentials:

$$\frac{1}{N}\sum_{n=0}^{N-1} e^{j(r-k)\frac{2\pi}{N}n} = \begin{cases} 1 & \text{for } r-k = mN, m \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$$

9.2 Properties

- Linearity: $a_1\{x_1[n]\} + a_2\{x_2[n]\} \longleftrightarrow a_1\{X_1[k]\} + a_2\{X_2[k]\}$
- Parseval's theorem: $\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N}\sum_{k=0}^{N-1} |X[k]|^2$

9.3 DFS coefficients of a real signal

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-jk\frac{2\pi}{N}n}$$

Then for a real signal $X[N-k] = X^*[k]$

To proove start with: $X[N-\lambda]$...

$$\begin{aligned} & X[N] = X^*[0] \\ \text{Periodicity} \quad & X[N] = X[0] \\ \text{Thus } X[0] \text{ always real:} \quad & X[0] = X^*[0] \end{aligned}$$

If N is even $X[N/2]$ is always real. $X[N-N/2] = X[N/2] = X^*[N/2]$

9.4 Response to Complex Exponential Sequences

$$\{\frac{1}{N}\sum_{k=0}^{N-1} Y[k]e^{jk\frac{2\pi}{N}n}\} = G\{\frac{1}{N}\sum_{k=0}^{N-1} U[k]e^{jk\frac{2\pi}{N}n}\}$$

$$\boxed{Y[k] = H(e^{jk\frac{2\pi}{N}})U[K]}$$

9.5 Relation between DFS and the DT Fourier Transform

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{jk\frac{2\pi}{N}n}$$

$$\boxed{X(\Omega) = \frac{2\pi}{N}\sum_{k=0}^{N-1} X[k]\delta(\Omega - k\tfrac{2\pi}{N})}$$

9.6 Discrete Fourier Transform (DFT)

$$\{x[n]\} \quad \text{sequence of finite length N}$$

$$x_e[n] = x[n \mod N]\ \forall\ n \quad \text{periodic extension of } \{x[n]\}$$

$$x_e[n] = \frac{1}{N}\sum_{k=0}^{N-1} X_e[k]e^{jk\frac{2}{\pi}n}\ \forall n$$

$$X_e[k] = \sum_{n=0}^{N-1} x_e[n]e^{-jk\frac{2\pi}{N}n}\ \forall k$$

$$\boxed{x_n[n] = \frac{1}{N}\sum_{k=0}^{N-1} X_e[k]e^{jk\frac{2\pi}{N}n}\ \forall\ n \in (0, N-1)}$$

To reconstruct $\{x[n]\}$ find the DFS coefficients of its periodic extension which are equal to the DFT coefficients.

9.7 DFT of Non-Periodic Signals

DFT coefficient $X[k_0]$ expresses the energy/power of $\{x[n]\}$ at the frequency $\Omega_0 = k_0 2\pi/N$

$$\boxed{\{x[n]\} = \{e^{j\Omega_0 n}\} \Leftrightarrow X(\Omega) = 2\pi\delta(\Omega - \Omega_0)}$$

If Ω_0 is an integer multiple of $\frac{2\pi}{N}$, $\exists k_0 \in [0, N-1] : k_0\frac{2\pi}{N} = \Omega_0 \Rightarrow X[k_0]$ is located at the location of the delta function and captures all of the signals power.

9.7.1 Example: $N = 10,\ \Omega_0 = \frac{\pi}{3}$

Ω_0 is not a multiple of $\frac{2\pi}{10}$

Even though the signal is periodic, choosing N wrongly leads to a periodic extension that involves many different frequencies instead of only $\pi/3$.

$$\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N}\sum_{k=0}^{N-1} |X[k]|^2 \quad \text{Parseval's Theorem}$$

Parceval indicates that the energy in the frequency $\pi/3$ has to be conserved when transformed.
UNFINISHED?

9.8 Effect of Causal Inputs

$$u[n] = \begin{cases} e^{j\Omega n} & n \geq 0 \\ 0 & n < 0 \end{cases}$$

$$\boxed{y[n] \rightarrow H(z = e^{j\Omega})e^{j\Omega n} \text{ as } n \rightarrow \infty}$$

10 Aliasing

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega)e^{j\Omega n}d\Omega \quad \text{Discrete time}$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega \quad \text{Continuous time}$$

When uniformly sampling$x(t) = e^{j\omega t}$:
$\{x[n]\} = \{e^{j\Omega n}\}$ where $\Omega = \omega T_s$.
When uniformly sampling $x(t) = e^{j(\omega + \frac{2\pi k}{T_s})t}$:

$$\{x[n]\} = x(nT_s) = \{e^{j(\omega+\frac{2\pi k}{T_s})nT_s}\} = \{e^{j\omega n T_s}\underbrace{e^{jn2\pi}}_{=1}\}$$

$$\boxed{-\frac{\pi}{T_s} < \omega < \frac{\pi}{T_s}} \text{ Allowed frequency range}$$

$$\boxed{\omega_N = \frac{\pi}{T_s} = \pi f_s \quad f_N = \frac{\omega_N}{2\pi} = \frac{f_s}{2}} \text{ Nyquist frequency}$$

11 Filtering

x: scalar, continuous, random with probability density function (PDF)

$$\int_{-\infty}^{\infty} p(x)dx = 1 \text{ and } p(x) \geq 0 \forall x \in \mathbb{R} \quad \text{must be satisfied}$$

$$\boxed{\mathbb{E}[x] := \int_{-\infty}^{\infty} xp(x)dx} \text{ expected value}$$

$$\boxed{\text{Var}(x) := \mathbb{E}[(x - expex)^2]} \text{ variance}$$

11.1 White Noise

$$\mathbb{E}[x[n]] = 0 \qquad \mathbb{E}[x[n]x[l]] = \begin{cases} 0 & \text{for } n \neq l \\ 1 & \text{for } n = l \end{cases}$$

$x[n]$ is a random variable with zero mean and $\{x[n]\}$ is uncorrelated across time.

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-jkn\frac{2\pi}{N}}$$

$$\boxed{\mathbb{E}[X^*[k]X[q]] = \sum_{n=0}^{N-1} e^{j(k-q)n\frac{2\pi}{N}} = \begin{cases} N & \text{for } k = q \\ 0 & \text{otherwise} \end{cases}}$$

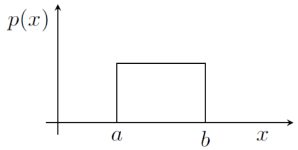
Thus all frequencies are equally represented and uncorrelated.

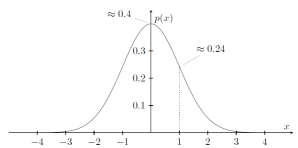
11.2 White noise from PDF

11.2.1 Uniform Distribution

$$p(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

- Matlab: **rand**
- Zero mean assumption: $a = -b, \quad b > 0$
- Unit variance assumption: $a = -b, \quad b = \sqrt{3}$



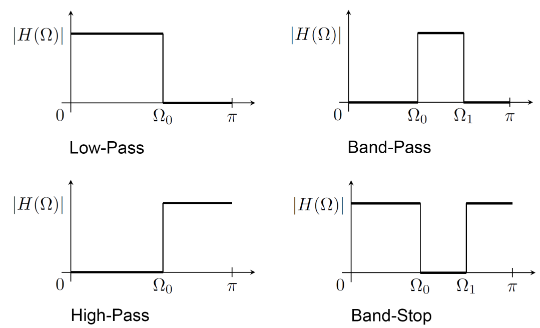


11.2.2 Normal Distribution

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

- Matlab: **randn**
- For white noise $\mu = 0$ and $\sigma^2 = 1$

11.3 Main Types of filters



11.4 Non-Causal Filtering

For real time applications only causal filters can be used.

Workflow for non causal filters:

1. $\{u[n]\} \rightarrow \{U[k]\}$ DFT
2. $\{U[k]\} \rightarrow \{Y[k]\}$ Manipulate in the frequency domain
3. $\{Y[k]\} \rightarrow \{y[n]\}$ Inverse DFT

11.4.1 Non-causal filtering with causal filters

- G causal, LTI filter with TF $H(z)$
- $\tilde{G}$ anti-causal LTI filter with TF $H(z^{-1})$

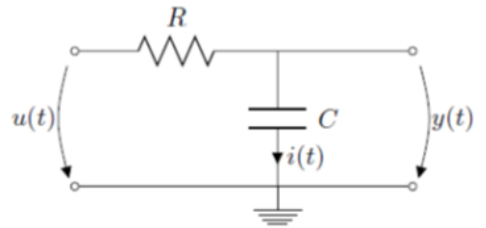
$$Y(e^{j\Omega} = H(e^{j\Omega})H(e^{j\Omega})U(e^{j\Omega} = |H(e^{j\Omega})|^2U(e^{j\Omega})|$$

11.5 Non-Linear Filter: Median

$y[n] =$`median`$(u[n - M/2], \ldots, u[n], \ldots, u[n + M/2])$
With M: any positive integer

To get a moving average filter replace **median** by **mean**.

11.6 Anti-Aliasing



$$i(t) = C\dot{y}(t), \quad u(t) = Ri(t) + y(t) = RC\dot{y}(t) + y(t)$$

$$\frac{Y(s)}{U(s)} = \frac{1}{RCs+1} = \frac{\omega_C}{s+\omega_C}, \quad \omega_C = \frac{1}{RC}$$

11.6.1 Corner frequency

The corner frequency is the frequency for which the voltage drops to $1/\sqrt{2}$ of its input value. This is equivalent to a reduction by $\frac{1}{2}$ in power.

12 FIR filters

$$\boxed{y[n] = \sum_{k=0}^{M-1} b_k u[n-k]}$$

FIR $= h = \{b_0, b_1, \ldots, b_{M-1}\}$

These filters are absolutely stable because h is absolutely summable.

12.1 Moving Average Filter

$$\boxed{y[n] = \frac{1}{M}\sum_{k=0}^{M-1} u[n-k]}$$

$H(\Omega) = \frac{1}{M}\sum_{k=0}^{M-1} e^{-j\Omega k}$

$H(0) = 1 \rightarrow$ a constant signal remains unchanged.

$e^{-j\Omega}H(\Omega) = \frac{1}{M}\sum_{k=0}^{M-1} e^{-j\Omega(k+1)}$

$H(\Omega)(1-e^{-j\Omega}) = \frac{1}{M}(1-e^{-j\Omega M}$

$$\boxed{H(\Omega) = \frac{1}{M}\frac{(1-e^{j\Omega M})}{(1-e^{-j\Omega})}}$$

$H(\Omega) = 0$ ii $\Omega = 2\pi k/M$

12.1.1 Phase response

For small values of Ω :

$H(\Omega) \approx \frac{1+(1-j\Omega)+\cdots+(1-j\Omega(M-1))}{M}$

$\Re(H(\Omega)) = 1 \qquad \Im(-\frac{\Omega(M-1)}{2})$

$\angle(H(\Omega)) \approx \arctan\left(-\frac{\Omega(M-1)}{2}\right) \approx -\frac{\Omega(M-1)}{2}$

using the small angle approximation. This approximation is exact until the first zero of $H(\Omega)$.

12.1.2 Magnitude response

$$|H(\Omega)| = \frac{\sin^2(\frac{\Omega M}{2})}{M^2\sin^2(\frac{\Omega}{2})}$$

$\frac{\sin(x)}{x} = \text{sinc}(x)$

$$\boxed{\left|\frac{\text{sinc}(\frac{\Omega}{2}}{)}\text{sinc}(\frac{\Omega}{2})\right| \approx |\text{sinc}(\frac{\Omega}{2})| \text{ for small } \Omega.}$$

12.2 Weighted Moving average Filter

$$y[n] = \frac{1}{S}\sum_{k=0}^{M-1} w_k u[n-k]$$

w_k is a decreasing function of k and denotes the weight given to the input $u[n-k]$.

A common choice would be: $w_k = (M-k) \qquad S = \frac{M(M+1)}{2}$

12.3 Non-Causal Moving Average Filter

$$\boxed{h = \{\ldots, 0, \frac{1}{M}, \ldots, \underset{\uparrow}{\frac{1}{M}}, \ldots, \frac{1}{M}, 0, \ldots\}}$$

$$\boxed{h[n] = \frac{1}{S}\tilde{h}[n] \text{ for all times n, where } S = \sum_{k=-\infty}^{\infty}\tilde{h}[n]}$$

This results in a Low-Pass filter with a zero-phase.

12.4 Phase is Important

$\boxed{-\angle(H(\Omega))/\Omega}$ Phase delay

12.5 Differentiation using FIR Filters

causal:	$y(t) \approx \frac{u(t)-u(t-\tau)}{\tau}$	$y_C[n] = \frac{1}{T_s}(u[n]-u[n-1])$
anti-causal:	$y(t) \approx \frac{u(t+\tau)-u(t)}{\tau}$	$y_A[n] = \frac{1}{T_s}(u[n+1]-u[n])$
non-causal:	$y(t) \approx \frac{u(t+\tau)-u(t-\tau)}{2\tau}$	$y_N[n] = \frac{1}{2T_s}(u[n+1]-u[n-1])$

13 Infinite Impulse Response Filters

$\boxed{y[n] = \sum_{k=0}^{M-1} b_k u[n-k] - \sum_{k=1}^{N-1} a_k y[n-k]}$ Causal IIR filters

- In contrast to FIR filters, IIR filters also depend on previous outputs. (some $a_k \neq 0$).
- Not necessarily stable.
- Meet a given set of filter specifications at a lower filter order compared to FIR.

$$\boxed{H(z) = \frac{\sum_{k=0}^{M-1} b_k z^{-k}}{1+\sum_{k=1}^{N-1} a_k z^{-k}} \qquad H(\Omega) = \frac{\sum_{k=0}^{M-1} b_k e^{-j\Omega k}}{1+\sum_{k=1}^{N-1} a_k e^{-j\Omega k}}}$$

13.1 First Order Low Pass Filter

$$y[n] = \alpha y[n-1] + (1-\alpha)u[n]$$

$$H(z) = \frac{1-\alpha}{1-\alpha z^{-1}} \qquad H(\Omega) = \frac{1-\alpha}{1-\alpha e^{-j\Omega}}$$

- Very low frequency signals remain unaltered since: $H(\Omega = 0) = \frac{1-\alpha}{1-\alpha e^{-j0}} = 1$
- $|H(\Omega)| = \frac{1-\alpha}{\sqrt{(1-\alpha\cos\Omega)^2+\alpha^2\sin^2\Omega}}$
- Magnitude is monotonically non-increasing: $\frac{d|H(\Omega)|}{d\Omega} \leq 0, \ \forall\ \Omega \in (0,\pi)$
- $\angle H(\Omega) = \arctan\left(\frac{\overbrace{-\alpha\sin\Omega}^{\text{Always negative}}}{\underbrace{1-\alpha\cos\Omega}_{\text{Always positive}}}\right), \ \forall\ \Omega \in (0,\pi)$
- $-\frac{\pi}{2} \leq \angle H(\Omega) \leq 0$

13.1.1 Design Considerations

How much time does it take $y[n]$ to decay to the value e^{-1}? Supposing $y[0] = 1$ and $u[n] = 0$.

$$\boxed{\begin{aligned} y[0] &= 1 \\ y[1] &= \alpha y[0] = \alpha \\ y[2] &= \alpha^2 \\ &\vdots \\ y[n] &= \alpha^n \end{aligned}}$$

$$\boxed{\alpha = e^{-\frac{1}{n}} \qquad n = \frac{T_0}{T_s} \Rightarrow \alpha = e^{-\frac{T_s}{T_0}}}$$

T_0 is the desired drop time to e^{-1}

13.1.2 Connection to CT systems

$$H(s) = \frac{1}{\tau s+1} \quad \text{CT first-order low-pass filter}$$

$$\dot{y}(t) = -\frac{1}{\tau}(y(t)-u(t)) \overset{u(t)\equiv 0}{\Longrightarrow} y(t) = y(0)e^{-\frac{t}{\tau}}$$

Discretization of the above differential equation:

$$\begin{bmatrix}\dot{y}\\ \dot{u}\end{bmatrix} = \begin{bmatrix}-\frac{1}{\tau} & \frac{1}{\tau}\\ 0 & 0\end{bmatrix}\begin{bmatrix}y\\ u\end{bmatrix} \qquad 0 \leq t \leq T_s$$

solution found by matrix exponential:

$$\begin{bmatrix}y(T_s^-)\\ u(T_s^-)\end{bmatrix} = \exp\left(\begin{bmatrix}-\frac{T_s}{\tau} & \frac{T_s}{\tau}\\ 0 & 0\end{bmatrix}\right)\begin{bmatrix}y(0)\\ u(0)\end{bmatrix} = \begin{bmatrix}e^{-\frac{T_s}{\tau}} & 1-e^{-\frac{T_s}{\tau}}\\ 0 & 1\end{bmatrix}\begin{bmatrix}y(0) & u(0)\end{bmatrix}$$

$$y[n] = e^{-\frac{T_s}{\tau}}y[n-1] + (1-e^{-\frac{T_s}{\tau}}u[n-1] = \alpha y[n-1] + (1-\alpha)u[n-1]$$

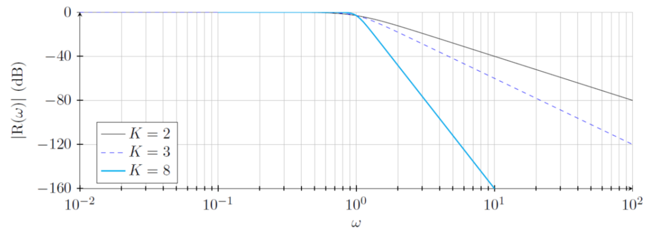
This is the same difference equation we obtained for the DT first order low pass filter.

14 IIR Filter Design

14.1 CT Butterworth filter design

$$R(\omega) = \frac{1}{\sqrt{1+\omega^{2K}}}$$

$R(\omega)$ represents the desired frequency response, where K is the order of the Filter and the corner frequency is $1\,\text{rad}\,\text{s}^{-1}$



- Butterworth filters have no ripples: $R = (1+\omega^{2K})^{-1/2} \approx 1 - \frac{1}{2}\omega^{2K}$
- Thus the filter is maximally flat.

$$\boxed{H(s) = \frac{1}{\prod_{k=1}^{K}(s-s_k)}}$$

where $s_k = e^{\frac{j(2k+K-1)\pi}{2K}}$, $k = 1,\dots,K$ and s_i are the poles of $H(s)$ and the fact that they fall on the unit circle with $\omega_c = 1$ is a coincidence.

To get a differenct cutoff frequency $s \to \frac{s}{\omega_c}$

14.2 Bilinear Transform

$$\boxed{s = \frac{2}{T_s}\left(\frac{z-1}{z+1}\right) \qquad z = \frac{1+s\frac{T_s}{2}}{1-s\frac{T_s}{2}}}$$

14.2.1 Motivation

1. $Y(z) = zU(z) \Leftrightarrow Y(s) = e^{sT_s}U(s)$
2. $y[n] = u[n-1] \Leftrightarrow y(t) = u(t+T_s)$
3. Equivalence of the two operators: $z = e^{sT_s}$
4. Approximation $e^{sT_s} = \frac{e^{s\frac{T_s}{2}}}{e^{-s\frac{T_s}{2}}} \approx \frac{1+s\frac{T_s}{2}}{1-s\frac{T_s}{2}} = z$
   The approximation is valid for small T_s.

14.2.2 DT-CT frequency mapping

$$|z| = \left|\frac{1+j\omega\frac{T_s}{2}}{1-j\omega\frac{T_s}{2}}\right| = 1$$

The imaginary axis in the s-plane is therefore mapped to the unit circle in the z-plane.

$$\angle e^{j\Omega} = 2\arctan(\omega\tfrac{T_s}{2}) \text{ for small } \Omega T_s \Omega = 2(\omega\tfrac{T_s}{2}) = \omega T_s$$

$$\begin{aligned} \omega &= 0 && \Rightarrow \Omega = 0 \\ \omega &= \infty && \Rightarrow \Omega = \pi \\ \omega &= \frac{T_s}{2} && \Rightarrow \Omega = \frac{\pi}{s} \end{aligned}$$

14.3 Overview: Discretization methods

Method	Transfer function	Filter parameter
Direct	$H(z) = \frac{1-\alpha}{1-\alpha z^{-1}}$	$\alpha = e^{-\frac{T_s}{\tau}}$ (decay time τ)
Sample and Hold	$H(z) = \frac{(1-\alpha)z^{-1}}{1-\alpha z^{-1}}$	$\alpha = e^{-\frac{T_s}{\tau}}$ (time constant τ)
Bilinear	$H(z) = \frac{(1-\alpha)(\frac{1+z^{-1}}{2})}{1-\alpha z^{-1}}$	$\alpha = \frac{1-\frac{T_s}{2\tau}}{1+\frac{t_s}{2\tau}}$ (time constant τ)

15 Applied Concepts

15.1 High-Pass Filter Design

$$H_{HPI}(\omega) = \begin{cases} 0 & 0 \le \omega < \omega_c \\ 1 & \omega_c \le \omega \end{cases}$$

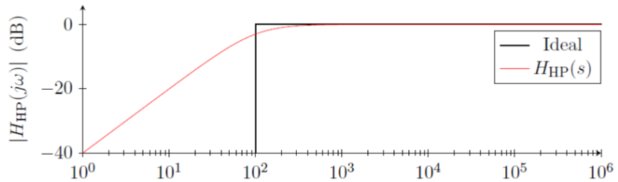
$$H_{LPI}(\omega) = \begin{cases} 1 & 0 \le \omega < \omega_c \\ 0 & \omega_c \le \omega \end{cases}$$

$$H_{HPI}(\omega) = 1 - H_{LPI}(\omega)$$

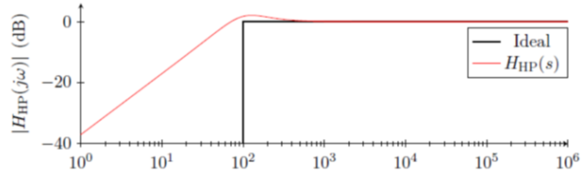
The I in HPI stands for ideal.

Thus one might be inclined to calcute high-pass filters from the TF of low pass filters.

$H_{LP}(s) = \frac{\omega_c}{s+\omega_c} \to H_{HP}(s) = 1 - \frac{\omega_c}{s+\omega_c} = \frac{s}{s+\omega_c}$



$H_{LP}(s) = \frac{\omega_c^2}{s^2+\sqrt{2}\omega_c s+\omega_c^2} \to H_{HP}(s) = 1 - H_{LP}(s) = \frac{s^2+\sqrt{2}\omega_c s}{s^2+\sqrt{2}\omega_c s+\omega_c^2}$



We see that $H_{HP} = 1 - H_{LP}$ only holds for ideal filters. In the case shown the HP filter does not roll-off at 40 dB/decade and its response is not flat as would be expected of a second-order Butterworth filter.

15.2 Designing a HP filter in CT

Need a transformation that:

1. preserves stability - open left halfplane should be mapped to open left halfplane.
2. maps the $j\omega$ axis to the $j\omega$ axis
3. maps $\omega = 0$ to $\omega = \infty$ and $\omega = \infty$ to $\omega = 0$

$$\boxed{s \to s^{-1}}$$

1. $s = a + jb$ then $\frac{1}{s} = \frac{1}{a+jb} = \frac{a-jb}{a^2+b^2}$
   therefore $\Re(s) = a$, $\Re(s^{-1}) = \frac{a}{a^2+b^2}$ and $\Re(s) < 0 \Leftrightarrow \Re(s^{-1})$
2. $s = j\omega$ then $\frac{1}{s} = -j\frac{1}{\omega}$
   Positive frequencies are mapped to negative frequencies.
3. $\omega = 0 \to \omega = \infty$

$$H_{HPI}(\omega) = \begin{cases} 0 & 0 \le \omega < \omega_c \\ 1 & \omega_c \le \omega \end{cases}$$

$$\boxed{H_{HPI}(\omega) = H_{LPI}(-1/\omega) = \begin{cases} 0 & 0 \le |\omega| \le \omega_c \\ 1 & \omega_c < |\omega| \end{cases}}$$

15.3 Designing a HP filter in DT

Need a transformation that:

1. preserves stability - the inside of the unit circle mapped on the inside of the unit circle
2. maps the unit circle to the unit circle
3. for $z = e^{j\Omega}$, maps $\Omega = 0$ to $\Omega = \pi$ and $\Omega = \pi$ to $\Omega = 0$.

$$\boxed{z = -z}$$

1. $|z| < 1 \Leftrightarrow |-z| = |z| < 1$
2. $|z| = 1 \Leftrightarrow |-z| = |z| = 1$
3. $z = e^{j0} = 1 \Rightarrow -z = -1 = e^{j\pi}$
   $z = e^{j\pi} = -1 \Rightarrow -z = 1 = e^{j0}$

$z \to -z \Rightarrow e^{j\Omega} \to -e^{j\Omega} = e^{j\pi}e^{j\Omega} = e^{j(\Omega+\pi)}$

This transformation causes the frequency response to be shifted by π.

DT design process:

1. Given: desired HP corner Ω_c
2. design DT LP filter with corner $\pi - \Omega_c$
3. calculate $H_{HP}(z) = H_{LP}(-z)$

OR just convert filter specs to CT and design there.

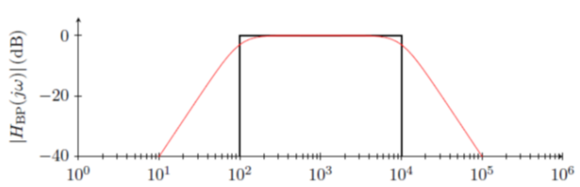
15.4 Band-Pass Filter Design

$$H_{BPI}(\omega) = \begin{cases} 0 & 0 \le \omega < \omega_0 \\ 1 & \omega_0 \le \omega \le \omega_1 \\ 0 & \omega_1 < \omega \end{cases}$$

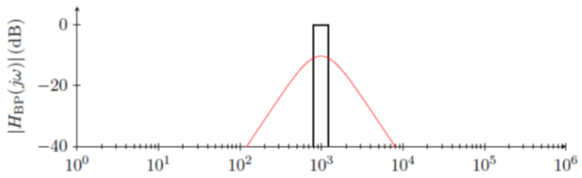
$$H_{BPI} = H_{LPI}(\omega)H_{HPI}(\omega)$$

One might therefore expect to obtain a band-pass filter as follows: $H_{BP}(s) = H_{LP}(s)H_{HP}(s)$

This works for $\omega_1/\omega_0 \gg 1$



But if ω_1 and ω_0 are relatively close:



15.5 Low-pass to band-pass filter transformation in CT

1. given passband $\omega_0 \leq \omega \leq \omega_1$
2. design LP with corner $\omega_c = \omega_1 - \omega_0$
3. calculate $H_{BP} = H_{LP}(s \to \frac{s^2+\omega_s^2}{s})$ where $\omega_s = \sqrt{\omega_0\omega_1}$

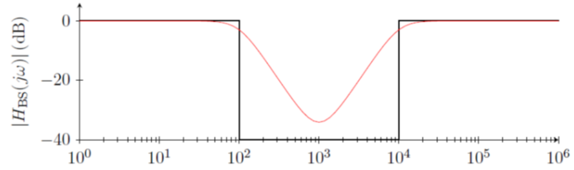
1. Low frequencies of the band-pass are mapped to high frequencies of the low-pass:
$$\lim_{s\to 0}\left(\frac{s^2+\omega_s^2}{s}\right) = \infty$$
Therefore: $H_{BPI}(0) = H_{LPI}(\infty) = 0$
2. High frequencies of the band-pass are mapped to high frequencies of the low pass:
$$\lim_{s\to\infty}\left(\frac{s^2+\omega_s^2}{s}\right) = \infty$$
Therefore: $H_{BPI}(\infty) = H_{LPI}(\infty) = 0$
3. The frequency ω_s of the band-pass is mapped to a frequency of 0 on the low-pass:
$$\left.\frac{s^2+\omega_s^2}{s}\right|_{s=j\omega_s} = 0$$
Therefore: $H_{BPI}(\omega_s) = H_{LPI}(0) = 1$
4. The corner frequencies of the band-pass are mapped to the corner frequency of the low-pass:
$$\left.\frac{s^2+\omega_s^2}{s}\right|_{s=j\omega_1} = -j\frac{-\omega_1^2+\omega_0\omega_1}{\omega_1} = j(\omega_1-\omega_0) = j\omega_c$$

15.6 Band-Stop Filter Design

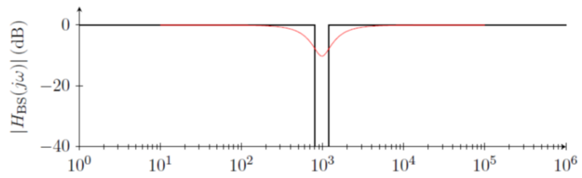
$$H_{BSI}(\omega) = \begin{cases} 1 & 0 \leq \omega \leq \omega_0 \\ 0\omega_0 < \omega < \omega_1 & \\ 1 & \omega_1 \leq \omega \end{cases}$$

$$H_{BSI}(\omega) = H_{LPI}(\omega) + H_{HPI}(\omega)$$

One might expect to obtain a band stop filter as follows: $H_{BS}(s) = H_{LP}(s) + H_{HP}(s)$
This works as long as $\omega_1/\omega_0 \gg 1$.



But if ω_1 and ω_2 are close:



15.7 High-Pass to Band-Stop Filter Transformation in CT

1. given passband $\omega_0 \leq \omega \leq \omega_1$
2. design HP with corner $\omega_c = \omega_1 - \omega_0$
3. calculate $H_{BS} = H_{HP}(s \to \frac{s^2+\omega_s^2}{s})$ where $\omega_s = \sqrt{\omega_0\omega_1}$

1. Low frequencies of the band-stop are mapped to high frequencies of the high-pass:
$$\lim_{s\to 0}\left(\frac{s^2+\omega_s^2}{s}\right) = \infty$$
Therefore: $H_{BSI}(0) = H_{HPI}(\infty) = 0$
2. High frequencies of the band-stop are mapped to high frequencies of the high-pass:
$$\lim_{s\to\infty}\left(\frac{s^2+\omega_s^2}{s}\right) = \infty$$
Therefore: $H_{BSI}(\infty) = H_{HPI}(\infty) = 0$
3. The frequency ω_s of the band-stop is mapped to a frequency of 0 on the high-pass:
$$\left.\frac{s^2+\omega_s^2}{s}\right|_{s=j\omega_s} = 0$$
Therefore: $H_{BSI}(\omega_s) = H_{HPI}(0) = 1$
4. The corner frequencies of the band-stop are mapped to the corner frequency of the high-pass:
$$\left.\frac{s^2+\omega_s^2}{s}\right|_{s=j\omega_1} = -j\frac{-\omega_1^2+\omega_0\omega_1}{\omega_1} = j(\omega_1-\omega_0) = j\omega_c$$

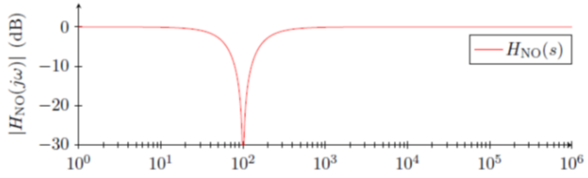
15.8 Notch Filter Design

$$H_{NOI}(\omega) = \begin{cases} 1 & 0 \leq \omega < \omega_c - \epsilon \\ 0 & \omega_c - \epsilon \leq \omega \leq \omega_c + \epsilon \\ 1 & \omega_c + \epsilon < \omega \end{cases}$$

$$\boxed{H_{NO}(s) = \frac{s^2+\omega_c^2}{s^2+\sqrt{2}\omega_c s+\omega_c^2}}$$

Motivation:

1. $|H_{NO}(\pm j\omega_c)| = 0$, we therefore require the terms $(s+j\omega_c)(s-j\omega_c) = s^2+\omega_c^2$ in the numerator.
2. $|H_{NO)}(0)| = 1$ and $|H_{NO}(\pm j\infty)| = 1$ thus motivating the denominator terms ω_c^2 and s^2
3. Stability: we therefore damp the filter's poles through the introduction of $\sqrt{2}\omega_c s$ in the denominator to give it the same poles as a Butterworth filter.



Let $H_{BS}(s)$ be the transfer function of a band-stop filter with $(\omega_0, \omega_1) = (\omega_c - \epsilon, \omega_c + \epsilon)$ if now $\epsilon \to 0$ we get back to the Notch filter.

15.9 Frequency Warping and the Bilinear Transform

$$s = \frac{2}{T_s}\left(\frac{z-1}{z+1}\right)$$

Frequencies in CT are mapped to the following DT frequencies:

$$\Omega = 2\text{arctan}\left(\omega\frac{T_s}{2}\right), \quad -\pi < \Omega \leq \pi$$

However, a sinusoid of DT frequency Ω corresponds to a sinusoid at CT frequency Ω/T_s. For small frequencies this is no issue since:

$$2\text{arctan}\left(\omega\frac{T_s}{2}\right)\frac{1}{T_s} \to \omega \text{ as } \omega \to 0$$

For higher frequencies warping occurs - see lecture notes for example. To avoid this we do pre-warping:

1. Let ω_c be the desired CT corner frequency of a DT filter.
2. Let T_s be the underlying sampling period.
$$\bar{\omega}_c = \frac{2}{T_s}\tan\left(\frac{\omega_c T_s}{2}\right) \quad \text{Warped } \omega_c$$
3. Design CT filter with frequency $\bar{\omega}_c$ and obtain $H(s)$.
4. Apply the bilinear transform
$$s = \frac{2}{T_s}\left(\frac{z-1}{z+1}\right)$$

16 System Identification

Focusing on **black box** identification i.e. no physics based model is available.

16.1 Frequency domain system identification

- u_e is a known input exciting the system.
- u_d is an unknown process noise, assumed to be white.
- y_d is an unknown measurement noise, assumed to be white.
- y_m given by $y_m = Gu_e + y_d + Gu_d$ is a measurement of the system's output, corrupted by process and measurement noise.

16.1.1 Identification without noise

$$y_m = Gu_e$$

1. Let $\{u_e[n]\} = \{\delta[n]\}$
2. Then $\{y_m[n]\} = \{h[n]\}$ and $H(\Omega) = \sum_{n=0}^{\infty} y_m[n]e^{j\Omega n}$

By taking some measurement with a unit impulse input and taking the DFT of the collected data we get an approximate frequency response.

$$Y_m[k] = \sum_{n=0}^{N-1} y_m[n]e^{-j\frac{2\pi k}{N}n}$$

$$\boxed{\hat{H}(\Omega_k) := Y_m[k] = H(\Omega_k) - \underbrace{\sum_{n=N}^{\infty} h[n]e^{j\Omega_k n}}_{H_N(\Omega_k)}}$$

Where $H_N(\Omega_k)$ is the error made because we only measure N steps of the impulse response, and N chosen such that the impulse response has decayed significantly before N is reached.

16.1.2 Identification with noise

$$y_m[n] = h[n] + y_d[n]$$

$$\mathbb{E}[y_d[n]] = 0 \text{ and } \mathbb{E}[y_d[n]y_d[m]] = \sigma_y^2\delta[n-m]$$

The second condition meaning that the noise is uncorrelated across time.

$$\hat{H}(\Omega_k) = Y_m[k] = H(\Omega_k) - H_N(\Omega_k) + Y_d[k]$$

where $\mathbb{E}[Y_d[k]] = 0$ and $\mathbb{E}[|Y_d[k]|^2] = N\sigma_y^2$

$$\mathbb{E}[\hat{H}(\Omega_k) - H(\Omega_k)] = -H_n(\Omega_k)$$

which approaches 0 as $N \to \infty$

$$\mathbb{E}[\left|\hat{H}(\Omega_k) - H(\Omega_k)\right|^2] = H_N^2(\Omega_k) + N\sigma_y^2$$

And $N\sigma_y^2 \to \infty$ as $N \to \infty$!

16.2 Identification using Sinusoidal Inputs

$$y_m + Gu_e + y_d$$

Now $u_e[n] = e^{j\frac{2\pi}{N}ln}$ $\quad n = 0, 1, \dots, N_T + N - 1$ where u_e is thus a sinusoid with frequency $\Omega_l = 2\pi l/N$ and N_T large enough such that the transient has died down sufficiently.

$$y_e[n] = H(\Omega_l)u_e[n] + e_e[n], \quad n \geq N_T$$

$$Y_e[l] = \sum_{n=N_T}^{N_T+N-1} y_e[n]e^{-j\frac{2\pi}{N}ln}$$
$$U_e[l] = \sum_{n=N_T}^{N_T+N-1} u_e[n]e^{-j\frac{2\pi}{N}ln}$$
$$E_e[l] = \sum_{n=N_T}^{N_T+N-1} e_e[n]e^{-j\frac{2\pi}{N}ln}$$

then

$$Y_e[l] = H(\Omega_l)U_e[l] + E_e[l]$$

where $E_e[l] \to 0$ as $N_T \to \infty$

$$Y_m[l] = \sum_{n=N_T}^{N_T+N-1} y_m[n]e^{-j\frac{2\pi}{N}ln}$$
$$Y_d[l] = \sum_{n=N_T}^{N_T+N-1} y_d[n]e^{-j\frac{2\pi}{N}ln}$$

$$\hat{H}(\Omega_l) := \frac{Y_m[l]}{U_e[l]} = H(\Omega_l) + \frac{E_e[l]}{N} + \frac{Y_d[l]}{N}$$

Therefore

$$\mathbb{E}[\hat{H}(\Omega_l) - H(\Omega_l)] = \frac{E_e[l]}{N}$$

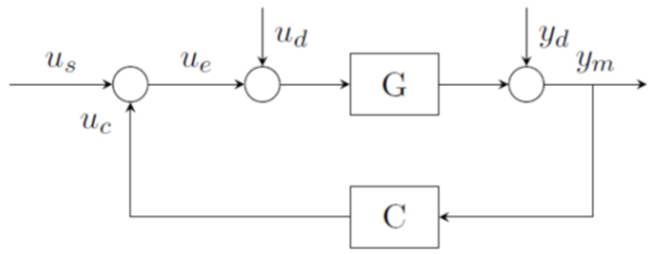
which approaches zero as $N_T \to \infty$

$$\mathbb{E}[|\hat{H}(\Omega_l) - H(\Omega_l)|^2] = \frac{E_e^2[l]}{N^2} + \frac{\sigma_y^2}{N}$$

16.2.1 Experimental Procedure

- Choose N_T large enough to let transient die down. Large N $\to$ long experiments but smaller error.
- Chose l, $\quad \Omega_l = \frac{2\pi l}{N}$
- Calculate $Y_m[l] = \sum_{n=N_T}^{N_T+N-1} y_m[n]e^{-j\Omega_l}$ and $U_e[l] = \sum_{n=N_t}^{N_T+N_1} u_e[n]e^{-j\Omega_l n} = \frac{NA}{2}$
- $\hat{H}(\Omega_l) := \frac{Y_m[l]}{U_e[l]}$
- Repeat for other frequencies.

16.2.2 Closed-loop System Identification



$$\boxed{y_m = Gu_e + Gu_d + y_d}$$

where $u_e = u_c + u_s$

The bias and mean-squared error of the estimate $\hat{H}(\Omega_l) := \frac{Y_m[l]}{U_e[l]}$ approach 0 as $N \to \infty$.

16.3 Identifying the Transfer Function

$$H(z) = \frac{\sum_{k=0}^{B-1} b_k z^{-k}}{1 + \sum_{k=1}^{A-1} a_k z^{-k}}$$

$$H(\Omega) = \frac{\sum_{k=0}^{B-1} b_k e^{-j\Omega k}}{1 + \sum_{k=1}^{A_1} a_k e^{-j\Omega k}}$$

Setting $\hat{H}(\Omega_l) = H(\Omega_l)$ at all measurement frequencies yields:

$$\left(1 + a_1 e^{-j\Omega_l} + \dots + a_{A-1}e^{-j(A-1)\Omega_k}\right)\hat{H}(\Omega_l) =$$
$$b_0 + b_1 e^{-j\Omega_l} + \dots + b_{B-1}e^{-j(B-1)\Omega_l}$$

This gives two times l linear equations once for the real and once for the imaginary part.

$R_l\cos(\phi_l) + a_1 R_l\cos(\phi_l - \Omega_l) + \dots + a_{A-1}R_l\cos(\phi_l - (A-1)\Omega_l) = b_0 + b_1\cos(\Omega_l) + \dots + b_{B-1}\cos((B-1)\Omega_l)$

$R_l\sin(\phi_l) + a_1 R_l\sin(\phi_l - \Omega_l) + \dots + a_{A-1}R_l\sin(\phi_l - (A-1)\Omega_l)$

$= -b_1\sin(\Omega_l) - \dots - b_{B-1}\sin((B-1)\Omega_l)$

This system of equations can be converted to the least squares problem of minimizing:

$$(F\Theta - G)^T(F\Theta - G)$$

where $\Theta = \begin{bmatrix} a_1 & a_2 & \cdots & a_{A-1} & b_0 & b_1 & \cdots & b_{B-1}\end{bmatrix}$
Θ size (A+B-1) vector, unkown.
F size $(2L) \times (A+B-1)$ matrix, known.

G size $(2L)$ vector, known.

$$\boxed{\Theta^* = (F^T F)^{-1}F^T G}$$

16.3.1 Weighted least squares

$$F\Theta = G \Rightarrow WF\Theta = WG$$

$$W = \mathrm{diag}(w_0, w_0, w_1, w_, \cdots, w_L)$$