

Signals and Systems

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NICE TO KNOW

$r - k$ is a multiple of $N \Leftrightarrow r = k \pmod N$

DEFINITIONS

$$e^{Mt} = \mathbb{I} + Mt + \frac{(Mt)^2}{2!} + \dots = \sum_{k=0}^{\infty} \frac{((Mt)^k}{k!} \quad \text{Matrix exponential}$$

$$M \in \mathbb{C}^{n \times n} \quad \exists k : M^k \equiv 0, M^{k-1} \neq 0 \quad \text{Nilpotent matrix, to degree } k$$

→ Matrix exponential is easy to calculate! Only $k-1$ non zero terms in the sum.

NOTATION

x	signal, function of time
$x[n]$	value of x at discrete time n
$x(t)$	value of x at continuous time t
$\{x[n]\}$	entire sequence

SIGNAL REPRESENTATION

- Graph
- Rule: $x[n] := \begin{cases} (\frac{1}{2})^n & n \geq 0 \\ 0 & n < 0 \end{cases}$
- Sequence: $\{x[n]\} = \{\dots, 0, 1, \frac{1}{2}, \dots\}$ ↑ indicates index 0

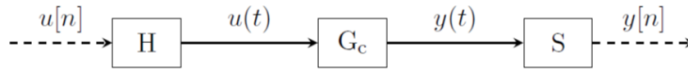
1 DISCRETIZATION OF CT SIGNALS & SYSTEMS

- **Uniform sampling:** $x[n] = x(nT_s)$
sampling period: T_s , sampling frequency: $f_s = \frac{1}{T_s}$
- **Zero-order hold:** $x(t) = x[n] \quad nT_s \leq t < (n+1)T_s$
Zero order hold does not require a future point for interpolation. Higher order holds are not causal.

1.1 HOLD AND SAMPLE OPERATORS

$u[n], u(t), y(t), y[n]$ refer to entire signals, G_c is a real world continuous system represent G_c as a state-space description:

$$G_c : \begin{cases} \dot{q}(t) &= A_c q(t) + B_c u(t) \\ y(t) &= C_c q(t) + D_c u(t) \end{cases}$$



Is there a system G_d such that $G_d = SG_cH$?

$$\begin{aligned} & \text{Block diagram: } u[n] \rightarrow G_d \rightarrow y[n] \\ & \begin{bmatrix} \dot{q}(t) \\ \dot{u}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} A_c & B_c \\ 0 & 0 \end{bmatrix}}_{\mathbf{M}} \cdot \begin{bmatrix} q(t) \\ u(t) \end{bmatrix} \\ & B \in \mathbb{R}^{n \times m}, \quad A \in \mathbb{R}^{n \times n} \end{aligned}$$

because $u(t) = u[0] \quad 0 \leq t < T_s$ therefore $\dot{u}(t) \equiv 0$.

The solution at T_s^- , just before sampling is:

$$\begin{bmatrix} q(T_s^-) \\ u(T_s^-) \end{bmatrix} = F \begin{bmatrix} q(0) \\ u(0) \end{bmatrix} \quad \text{with } F = e^{MT_s}.$$

$$A_d = F(1 : n, 1 : n), \quad B_d = F(1 : n, n+1 : n+m), \quad C_d = C_c, \quad D_d = D_c.$$

$$\begin{bmatrix} q[n+1] \\ y[n] \end{bmatrix} = \begin{bmatrix} A_d q[n] + B_d u[n] \\ C_d q[n] + D_d u[n] \end{bmatrix}$$

This is an exact discretization as opposed to the approximative **Euler discretization**: $\dot{q}(t) \approx \frac{q(t+T_s) - q(t)}{T_s} = \frac{q[n+1] - q[n]}{T_s}$

Euler is good as long as T_s is small.

2 CLASSIFICATION OF SYSTEMS

- **Memoryless**
Output at time n only depends on input at the same timestep.
- **Causal**
Output at time n only depends on past and present inputs.
- **Linear**
 $G\{\alpha_1 u_1[n] + \alpha_2 u_2[n]\} = \alpha_1 G\{u_1[n]\} + \alpha_2 G\{u_2[n]\}$
- **Time-invariant**
 $\{y_2[n]\} = \{y_1[n-k]\}, \quad y_1 = Gu_1, \quad y_2 = Gu_2,$
 $\{u_2[n]\} = \{u_1[n-k]\}, \quad \forall k, u_1[n]$

2.1 STABILITY OF LINEAR SYSTEMS, BIBO

Bounded sequence: $u[n] : |u[n]| \leq M \quad \forall n$

Stability: $u[n], y[n] = Gu[n], \exists M : |u[n]| \leq 1 \quad \forall n, |y[n]| \leq M$

BIBO: Bounded input bounded output.

3 LTI SYSTEM RESPONSE TO INPUTS

3.1 IMPULSE RESPONSE OF A SYSTEM

3.1.1 USEFUL SIGNALS

- Impulse sequence: $\{\delta[n]\} := \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$
- Step sequence: $\{s[n]\} := \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$

	Integration	Differentiation
Continuous	$s(t) = \int_{-\infty}^t \delta\tau d\tau$	$\frac{d}{dt}s(t) = \lim_{\epsilon \rightarrow 0} \frac{s(t) - s(t-\epsilon)}{\epsilon} = \delta(t)$
Discrete	$s[n] = \sum_{k=-\infty}^n \delta[k]$	$\{s[n]\} - \{s[n-1]\} = \{\delta[n]\}$

3.1.2 REPRESENTING A SEQUENCE WITH IMPULSES

$$\begin{aligned} x[n] &= \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \quad \forall n \\ \{x[n]\} &= \sum_{k=-\infty}^{\infty} xk\{\delta[n-k]\} \end{aligned}$$

3.1.3 RESPONSE TO ARBITRARY INPUTS

Impulse response $\{h[n]\} := G\{\delta[n]\}$

Any sequence can be written as a sum of time-shifted impulses.

$$\begin{aligned} \{y[n]\} &= G\left\{\sum_{k=-\infty}^{\infty} u[k]\delta[n-k]\right\} \stackrel{L}{=} \sum_{k=-\infty}^{\infty} u[k]G\{\delta[n-k]\} \stackrel{TI}{=} \\ &\sum_{k=-\infty}^{\infty} u[k]\{h[n-k]\} \end{aligned}$$

3.1.4 CONVOLUTION

$$x * h = \{x[n]\} * \{h[n]\} := \sum_{k=-\infty}^{\infty} x[k]\{h[n-k]\}$$

- Commutative: $x * h = h * x$
- Associative: $(x * h_1) * h_2 = x * (h_1 * h_2)$
- Distributive: $x * (h_1 + h_2) = x * h_1 + x * h_2$

3.2 STEP RESPONSE

$$\begin{aligned} \{r[n]\} &:= \{h[n]\} * \{s[n]\} = \sum_{k=-\infty}^{\infty} h[k]\{s[n]\} = \left\{ \sum_{k=-\infty}^{\infty} h[k] \right\} \\ r[n] - r[n-1] &= \sum_{k=-\infty}^{\infty} h[k] - \sum_{k=-\infty}^{\infty} h[k] = h[n], \quad \forall n \end{aligned}$$

3.3 CAUSALITY

$$y[n] = \sum_{k=-\infty}^{\infty} u[k]h[n-k], \quad \forall n$$

$$\text{System is causal} \Leftrightarrow h[n] = 0, \quad \forall n < 0$$

causal input: $u[n] : u[n] = 0 \quad \forall n \leq 0$

$$y[n] = \sum_{k=0}^n u[k]h[n-k] = \sum_{k=0}^n h[k]u[n-k], \quad \forall n$$

3.4 STABILITY OF AN LTI SYSTEM

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

4 LINEAR CONSTANT COEFFICIENT DIFFERENCE EQUATIONS

4.1 DEFINITION

$$\sum_{k=0}^N a_k y[n-k] = \sum_{k=0}^M b_k u[n-k], \quad a_k, b_k \in \mathbb{R}$$

a_0 and b_0 are not both zero!

If the system is causal ($a_0 \neq 0$)

$$y[n] = \frac{1}{a_0} \left(\sum_{k=0}^M b_k u[n-k] - \sum_{k=1}^N a_k y[n-k] \right)$$

4.2 CONVERTING FROM LCCDE TO STATE-SPACE

$$\begin{aligned} \text{SS:} \quad q[n+1] &= Aq[n] + Bu[n] \\ y[n] &= Cq[n] + Du[n] \end{aligned}$$

Special case of LCCDE: $y[n] + a_1 y[n-1] + \dots + a_N y[n-N] = b_0 u[n]$
To calculate $y[n]$ at time n we need N past outputs as well as the current input.

$$\left. \begin{aligned} q_1[n] &= y[n-N] \\ q_2[n] &= y[n-(N-1)] = y[n-N+1] \\ &\dots = \dots \\ q_N[n] &= y[n-1] \end{aligned} \right\} q[n] = \begin{bmatrix} q_1[n] \\ \vdots \\ q_N[n] \end{bmatrix}$$

Therefore:

$$q_1[n+1] = q_2[n], \quad q_2[n+1] = q_3[n], \dots, \quad q_{N-1}[n+1] = q_N[n]$$

$$q_N[n+1] = y[n] = b_0 u[n] - a_N q_1[n] - \dots - a_1 q_N[n]$$

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \cdot & 0 \\ & & & \ddots & \\ & & & & 1 \\ -a_N & -a_{N-1} & -a_{N-2} & \dots & -a_1 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ b_0 \end{bmatrix}$$

$$C = \begin{bmatrix} -a_N & -a_{N-1} & -a_{N-2} & \dots & -a_1 \end{bmatrix} \quad D = \begin{bmatrix} b_0 \end{bmatrix}$$

4.3 IMPULSE RESPONSE OF A DT LTI SYSTEM IN SS

$$\begin{aligned} q[1] &= Aq[0] + Bu[0] \\ q[2] &= Aq[1] + Bu[1] = A^2 q[0] + ABu[0] + Bu[1] \end{aligned}$$

\vdots

$$q[n] = A^n q[0] + \sum_{k=0}^{n-1} A^{n-k-1} Bu[k], \quad n \geq 0$$

$$y[n] = Cq[n] + Du[n] = CA^n q[0] + C \sum_{k=0}^{n-1} A^{n-k-1} Bu[k] + Du[n], \quad n \geq 0$$

$$h = \{D, CB, CAB, \dots, CA^{n-1}B, \dots\}$$

4.4 FINITE IMPULSE RESPONSE (FIR) VS. INFINITE IMPULSE RESPONSE (IIR)

$$\exists N : h[n] = 0 \quad \forall n \geq N \quad \text{FIR}$$

A system that can be written in the non-recursive form has a FIR.

$$y[n] = \frac{1}{a_0} \sum_{k=0}^M b_k u[n-k] \quad \text{Non-recursive Form}$$

5 PERIODIC SIGNALS

$$x[n+N] = x[n]$$

5.1 PERIODICITY CONSTRAINT

$CT : \cos(\omega t)$ is periodic with $T = \frac{2\pi}{|\omega|}$, sampled with T_s the resulting DT signal $\{x[n]\} = \{\cos(\Omega n)\}$ has the frequency $\Omega = \omega T_s$ is periodic iff

$$\frac{\Omega}{2\pi} = \frac{m}{N} \quad \text{for some integers } m, N$$

If $\frac{m}{N}$ is an irreducible fraction, then N is the fundamental period of the signal.

5.2 EIGENFUNCTIONS OF LTI SYSTEMS

$$\begin{aligned} \{y[n]\} = G\{z^n\} &= \sum_{k=-\infty}^{\infty} h[k]\{z^{n-k}\} \\ &= \sum_{k=-\infty}^{\infty} h[k]z^{-k}\{z^n\} \\ &= H(z)\{z^n\} \end{aligned}$$

$$H(z) := \sum_{k=-\infty}^{\infty} h[k]z^{-k}$$

6 THE Z-TRANSFORM

$$X(z) := \sum_{n=-\infty}^{\infty} x[n]z^{-n}, \quad z \in \mathbb{C}$$

Properties:

- Linearity $a_1\{x_1[n]\} + a_2\{x_2[n]\} \leftrightarrow a_1X(z) + a_2X(z)$
- Time-shifting $\{x[n-1]\} \leftrightarrow z^{-1}X(z)$
- Convolution $\{x_1[n]\} * \{x_2[n]\} \leftrightarrow X_1(z) \cdot X_2(z)$
- Accumulation $\left\{ \sum_{k=-\infty}^{\infty} x[k] \right\} \leftrightarrow \frac{z}{z-1}X(z)$
- Special case $\{u[n]\} = z^n \leftrightarrow G\{u[n]\} = H(z) \cdot \{z^n\}$

6.1 CONVERGENCE AND NON-UNIQUENESS

$$x[n] = \begin{cases} a^n & n \geq 0, a \in \mathbb{R}, a \neq 0 \\ 0 & \text{otherwise} \end{cases} \quad x[n] = \begin{cases} -a^n & n < 0, a \in \mathbb{R}, a \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n \quad X(z) = -\sum_{n=-\infty}^{-1} = -\sum_{n=1}^{\infty} \left(\frac{z}{a}\right)^n$$

$$X(z) = \frac{z}{z-a} \quad X(z) = \frac{z}{z-a}$$

$X(z)$ converges if $\left|\frac{a}{z}\right| < 1$ $X(z)$ converges if $\left|\frac{z}{a}\right| < 1$

The z-Transform must also include the R.O.C. in order to uniquely specify

R.O.C. - Region of convergence

6.2 TRANSFER FUNCTIONS OF LTI SYSTEMS

$$\{y[n]\} = \{u[n]\} * \{h[n]\} \longleftrightarrow Y(z) = U(z) \cdot H(z)$$

$$H(z) = \frac{Y(z)}{U(z)}$$

6.3 TRANSFER FUNCTION FROM LCCDE

$$\sum_{k=0}^N a_k y[n-k] = \sum_{k=0}^M b_k u[n-k] \leftrightarrow \sum_{k=0}^N a_k z^{-k} Y(z) = \sum_{k=0}^M b_k z^{-k} U(z)$$

$$\frac{Y(z)}{U(z)} = \frac{b_0 + b_1 z^{-1} + \dots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \dots + a_N z^{-N}} = H(z)$$

6.4 TRANSFER FUNCTION FROM SS

$$\begin{aligned} q[n+1] &= Aq[n] + Bu[n] \\ y[n] &= Cq[n] + Du[n] \\ &\updownarrow \\ zQ(z) &= AQ(z) + BU(z) \\ Q(z) &= (zI - A)^{-1}BU(z) \\ Y(z) &= CQ(z) + DU(z) \end{aligned}$$

$$H(z) = \frac{Y(z)}{U(z)} = C(zI - A)^{-1}B + D$$

6.5 STABILITY AND CAUSALITY

Given a transfer function $H(z)$, there exists a stable and causal interpretation for the underlying system iff all poles of $H(z)$ are inside the unit circle. That is, given pole p (a value p for which $|H(p)| = \infty$), then $|p| < 1$.

7 FOURIER TRANSFORM

7.1 DT FOURIER TRANSFORM

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty \quad \text{only for absolutely summable signals}$$

7.1.1 DEFINITION

$$X(\Omega) = \mathcal{F}x := \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}, \quad -\pi < \Omega \leq \pi$$

$$\{x[n]\} \longleftrightarrow \underbrace{X(\Omega)}_{\text{refers to whole function}}$$

$$X(\Omega) = |X(\Omega)| \cdot e^{j\Theta_X(\Omega)}$$

The DT Transform is equal to the z-Transform if

$$z = e^{j\Omega}, \quad X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

7.2 INVERSE DT FOURIER TRANSFORM

$$\{x[n]\} = \mathcal{F}^{-1}X := \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega)e^{j\Omega n} d\Omega \right\}$$

7.3 PROPERTIES OF THE DT FOURIER TRANSFORM

- Linearity: $a_1\{x_1[n]\} + a_2\{x_2[n]\} \longleftrightarrow a_1X_1(\Omega) + a_2X_2(\Omega)$
- Convolution: $\{x_1[n]\} * \{x_2[n]\} \longleftrightarrow X_1(\Omega) \cdot X_2(\Omega)$
- Parseval's theorem: $\sum_{-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\Omega)|^2 d\Omega$

8 FREQUENCY RESPONSE OF LTI SYSTEMS

$$y = u * h \longleftrightarrow Y(\Omega) = U(\Omega)H(\Omega)$$

$$H(\Omega) = \frac{Y(\Omega)}{U(\Omega)} \quad \text{Frequency Response}$$

$$\begin{aligned} H(\Omega) &= |H(\Omega)|e^{j\Theta_H(\Omega)} \\ |Y(\Omega)| &= |U(\Omega)||H(\Omega)| \\ \Theta_Y(\Omega) &= \Theta_U(\Omega) + \Theta_H(\Omega) \end{aligned}$$

8.1 FREQUENCY RESPONSE FROM LCCDE

$$H(\Omega) = H(z)|_{z=e^{j\Omega}}$$

$$H(\Omega) = \frac{\sum_{k=0}^M b_k e^{-j\Omega k}}{\sum_{k=0}^N a_k e^{-j\Omega k}}$$

8.2 RESPONSE TO COMPLEX EXPONENTIAL SEQUENCES

$$\{u[n]\} = \{z^n\} = \{e^{j\Omega_0 n}\}$$

$$\begin{aligned} \{y[n]\} &= G\{z^n\} = H(z)\{z^n\} \\ \rightarrow y[n] &= H(z = e^{j\Omega_0})e^{j\Omega_0 n} = H(\Omega = \Omega_0)e^{j\Omega_0 n} \\ &= |H(\Omega_0)|e^{j(\Omega_0 n + \Theta_H(\Omega_0))} \end{aligned}$$

8.3 RESPONSE TO REAL SINUSOIDS

$$\begin{aligned} y &= Gu \\ &= G(u_1 + ju_2) \\ &= Gu_1 + jGu_2 \\ &= y_1 + jy_2 \end{aligned}$$

$$\begin{aligned} u[n] &= e^{j\Omega_0 n} \Rightarrow u_1[n] = \cos(\Omega_0 n) \\ y[n] &= H(\Omega_0)e^{j\Omega_0 n} = |H(\Omega_0)|e^{j(\Omega_0 n + \Theta_H(\Omega_0))} \\ y_1[n] &= |H(\Omega_0)|\cos(\Omega_0 n + \Theta_H(\Omega_0)) \end{aligned}$$

9 DISCRETE FOURIER SERIES / TRANSFORM

9.1 DFS REPRESENTATION OF A PERIODIC SIGNAL

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]e^{jk\frac{2\pi}{N}n} \quad X[k] = \sum_{n=0}^{N-1} x[n]e^{-jk\frac{2\pi}{N}n}$$

$$\mathcal{F}_s, \quad X = \mathcal{F}_s x, \quad x = \mathcal{F}_s^{-1}X \quad \text{The DFS operator}$$

X is periodic with period N:

$$X[k+N] = \sum_{n=0}^{N-1} x[n]e^{-j(k+N)\frac{2\pi}{N}n} = \sum_{n=0}^{N-1} x[n]e^{-jk\frac{2\pi}{N}n} \underbrace{e^{-j2\pi n}}_{=1 \forall n} = X[k]$$

The DFS operator is invertible: $\mathcal{F}_s \mathcal{F}_s^{-1} = \mathbb{I} \quad \mathcal{F}_s^{-1} \mathcal{F}_s = \mathbb{I}$

Orthogonality of complex exponentials:

$$\frac{1}{N} \sum_{n=0}^{N-1} e^{j(r-k)\frac{2\pi}{N}n} = \begin{cases} 1 & \text{for } r-k = mN, m \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$$

9.2 PROPERTIES

- Linearity: $a_1\{x_1[n]\} + a_2\{x_2[n]\} \longleftrightarrow a_1\{X_1[k]\} + a_2\{X_2[k]\}$
- Parseval's theorem: $\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X[k]|^2$

9.3 DFS COEFFICIENTS OF A REAL SIGNAL

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-jk\frac{2\pi}{N}n}$$

Then for a real signal $X[N-k] = X^*[k]$

To prove start with: $X[N-\lambda] \dots$

$$\begin{aligned} X[N] &= X^*[0] \\ \text{Periodicity} \quad X[N] &= X[0] \\ \text{Thus } X[0] \text{ always real:} \quad X[0] &= X^*[0] \end{aligned}$$

If N is even $X[N/2]$ is always real. $X[N-N/2] = X[N/2] = X^*[N/2]$

9.4 RESPONSE TO COMPLEX EXPONENTIAL SEQUENCES

$$\left\{ \frac{1}{N} \sum_{k=0}^{N-1} Y[k]e^{jk\frac{2\pi}{N}n} \right\} = G\left\{ \frac{1}{N} \sum_{k=0}^{N-1} U[k]e^{jk\frac{2\pi}{N}n} \right\}$$

$$Y[k] = H(e^{jk\frac{2\pi}{N}})U[k]$$

9.5 RELATION BETWEEN DFS AND THE DT FOURIER TRANSFORM

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]e^{jk\frac{2\pi}{N}n}$$

$$X(\Omega) = \frac{2\pi}{N} \sum_{k=0}^{N-1} X[k]\delta(\Omega - k\frac{2\pi}{N})$$

9.6 DISCRETE FOURIER TRANSFORM (DFT)

$\{x[n]\}$ sequence of finite length N

$x_e[n] = x[n \bmod N] \forall n$ periodic extension of $\{x[n]\}$

$$x_e[n] = \frac{1}{N} \sum_{k=0}^{N-1} X_e[k]e^{jk\frac{2\pi}{N}n} \forall n$$

$$X_e[k] = \sum_{n=0}^{N-1} x_e[n]e^{-jk\frac{2\pi}{N}n} \forall k$$

$$x_n[n] = \frac{1}{N} \sum_{k=0}^{N-1} X_e[k]e^{jk\frac{2\pi}{N}n} \forall n \in (0, N-1)$$

To reconstruct $\{x[n]\}$ find the DFS coefficients of its periodic extension which are equal to the DFT coefficients.

9.7 DFT OF NON-PERIODIC SIGNALS

DFT coefficient $X[k_0]$ expresses the energy/power of $\{x[n]\}$ at the frequency $\Omega_0 = k_0 2\pi/N$

$$\{x[n]\} = \{e^{j\Omega_0 n}\} \Leftrightarrow X(\Omega) = 2\pi\delta(\Omega - \Omega_0)$$

If Ω_0 is an integer multiple of $\frac{2\pi}{N}$, $\exists k_0 \in [0, N-1] : k_0 \frac{2\pi}{N} = \Omega_0 \Rightarrow X[k_0]$ is located at the location of the delta function and captures all of the signals power.

9.7.1 EXAMPLE: $N = 10$, $\Omega_0 = \frac{\pi}{3}$

Ω_0 is not a multiple of $\frac{2\pi}{10}$

Even though the signal is periodic, choosing N wrongly leads to a periodic extension that involves many different frequencies instead of only $\pi/3$.

$$\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X[k]|^2 \quad \text{Parseval's Theorem}$$

Parseval indicates that the energy in the frequency $\pi/3$ has to be conserved when transformed.

UNFINISHED?

9.8 EFFECT OF CAUSAL INPUTS

$$u[n] = \begin{cases} e^{j\Omega n} & n \geq 0 \\ 0 & n < 0 \end{cases}$$

$$y[n] \rightarrow H(z = e^{j\Omega}) e^{j\Omega n} \text{ as } n \rightarrow \infty$$

10 ALIASING

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega) e^{j\Omega n} d\Omega \quad \text{Discrete time}$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \quad \text{Continuous time}$$

When uniformly sampling $x(t) = e^{j\omega t}$:

$$\{x[n]\} = \{e^{j\Omega n}\} \text{ where } \Omega = \omega T_s.$$

When uniformly sampling $x(t) = e^{j(\omega + \frac{2\pi k}{T_s})t}$:

$$\{x[n]\} = x(nT_s) = \{e^{j(\omega + \frac{2\pi k}{T_s})nT_s}\} = \{e^{j\omega n T_s} \underbrace{e^{jn2\pi}}_{=1}\}$$

$$-\frac{\pi}{T_s} < \omega < \frac{\pi}{T_s} \quad \text{Allowed frequency range}$$

$$\omega_N = \frac{\pi}{T_s} = \pi f_s \quad f_N = \frac{\omega_N}{2\pi} = \frac{f_s}{2} \quad \text{Nyquist frequency}$$

11 FILTERING

x : scalar, continuous, random with probability density function (PDF)

$$\int_{-\infty}^{\infty} p(x) dx = 1 \text{ and } p(x) \geq 0 \forall x \in \mathbb{R} \quad \text{must be satisfied}$$

$$\mathbb{E}[x] := \int_{-\infty}^{\infty} x p(x) dx \quad \text{expected value}$$

$$\text{Var}(x) := \mathbb{E}[(x - \text{expex})^2] \quad \text{variance}$$

11.1 WHITE NOISE

$$\mathbb{E}[x[n]] = 0 \quad \mathbb{E}[x[n]x[l]] = \begin{cases} 0 & \text{for } n \neq l \\ 1 & \text{for } n = l \end{cases}$$

$x[n]$ is a random variable with zero mean and $\{x[n]\}$ is uncorrelated across time.

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-jkn \frac{2\pi}{N}}$$

$$\mathbb{E}[X^*[k]X[q]] = \sum_{n=0}^{N-1} e^{j(k-q)n \frac{2\pi}{N}} = \begin{cases} N & \text{for } k = q \\ 0 & \text{otherwise} \end{cases}$$

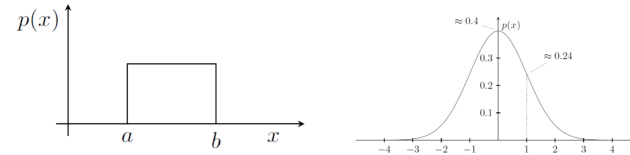
Thus all frequencies are equally represented and uncorrelated.

11.2 WHITE NOISE FROM PDF

11.2.1 UNIFORM DISTRIBUTION

$$p(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

- Matlab: **rand**
- Zero mean assumption: $a = -b$, $b > 0$
- Unit variance assumption: $a = -b$, $b = \sqrt{3}$

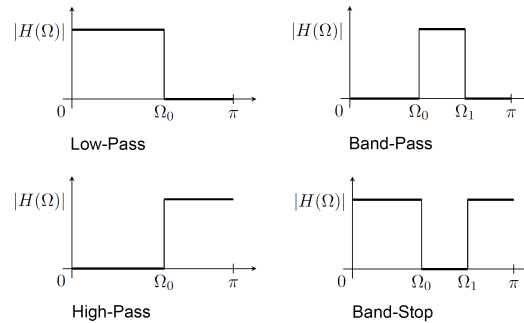


11.2.2 NORMAL DISTRIBUTION

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma^2} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

- Matlab: **randn**
- For white noise $\mu = 0$ and $\sigma^2 = 1$

11.3 MAIN TYPES OF FILTERS



11.4 NON-CAUSAL FILTERING

For real time applications only causal filters can be used.

Workflow for non causal filters:

1. $\{u[n]\} \rightarrow \{U[k]\}$ DFT
2. $\{U[k]\} \rightarrow \{Y[k]\}$ Manipulate in the frequency domain
3. $\{Y[k]\} \rightarrow \{y[n]\}$ Inverse DFT

11.4.1 NON-CAUSAL FILTERING WITH CAUSAL FILTERS

- G causal, LTI filter with TF $H(z)$
- \tilde{G} anti-causal LTI filter with TF $H(z^{-1})$

$$Y(e^{j\Omega}) = H(e^{j\Omega})H(e^{j\Omega})U(e^{j\Omega}) = |H(e^{j\Omega})|^2 U(e^{j\Omega})$$

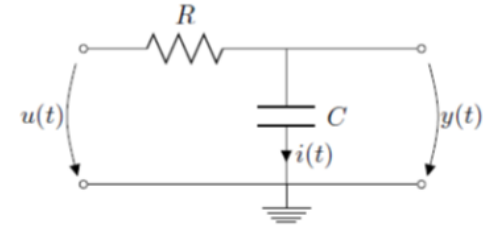
11.5 NON-LINEAR FILTER: MEDIAN

$$y[n] = \text{median}(u[n - M/2], \dots, u[n], \dots, u[n + M/2])$$

With M : any positive integer

To get a moving average filter replace **median** by **mean**.

11.6 ANTI-ALIASING



$$i(t) = C\dot{y}(t), \quad u(t) = Ri(t) + y(t) = RC\dot{y}(t) + y(t)$$

$$\frac{Y(s)}{U(s)} = \frac{1}{RCs+1} = \frac{\omega_C}{s+\omega_C}, \quad \omega_C = \frac{1}{RC}$$

11.6.1 CORNER FREQUENCY

The corner frequency is the frequency for which the voltage drops to $1/\sqrt{2}$ of its input value. This is equivalent to a reduction by $\frac{1}{2}$ in power.

12 FIR FILTERS

$$y[n] = \sum_{k=0}^{M-1} b_k u[n - k]$$

$$\text{FIR} = h = \{b_0, b_1, \dots, b_{M-1}\}$$

These filters are absolutely stable because h is absolutely summable.

12.1 MOVING AVERAGE FILTER

$$y[n] = \frac{1}{M} \sum_{k=0}^{M-1} u[n-k]$$

$$H(\Omega) = \frac{1}{M} \sum_{k=0}^{M-1} e^{-j\Omega k}$$

$H(0) = 1 \rightarrow$ a constant signal remains unchanged.

$$e^{-j\Omega} H(\Omega) = \frac{1}{M} \sum_{k=0}^{M-1} e^{-j\Omega(k+1)}$$

$$H(\Omega)(1 - e^{-j\Omega}) = \frac{1}{M} (1 - e^{-j\Omega M})$$

$$H(\Omega) = \frac{1}{M} \frac{(1 - e^{j\Omega M})}{(1 - e^{-j\Omega})}$$

$$H(\Omega) = 0 \text{ if } \Omega = 2\pi k/M$$

12.1.1 PHASE RESPONSE

For small values of Ω :

$$H(\Omega) \approx \frac{1 + (1-j\Omega) + \dots + (1-j\Omega(M-1))}{M}$$

$$\Re(H(\Omega)) = 1 \quad \Im\left(-\frac{\Omega(M-1)}{2}\right)$$

$$\angle(H(\Omega)) \approx \arctan\left(-\frac{\Omega(M-1)}{2}\right) \approx -\frac{\Omega(M-1)}{2}$$

using the small angle approximation. This approximation is exact until the first zero of $H(\Omega)$.

12.1.2 MAGNITUDE RESPONSE

$$|H(\Omega)| = \frac{\sin^2(\frac{\Omega M}{2})}{M^2 \sin^2(\frac{\Omega}{2})}$$

$$\frac{\sin(x)}{x} = \text{sinc}(x)$$

$$\left| \frac{\text{sinc}(\frac{\Omega}{2})}{\text{sinc}(\frac{\Omega}{2})} \right| \approx |\text{sinc}(\frac{\Omega}{2})| \text{ for small } \Omega.$$

12.2 WEIGHTED MOVING AVERAGE FILTER

$$y[n] = \frac{1}{S} \sum_{k=0}^{M-1} w_k u[n-k]$$

w_k is a decreasing function of k and denotes the weight given to the input $u[n-k]$.

$$\text{A common choice would be: } w_k = (M-k) \quad S = \frac{M(M+1)}{2}$$

12.3 NON-CAUSAL MOVING AVERAGE FILTER

$$h = \{\dots, 0, \frac{1}{M}, \dots, \frac{1}{M}, \dots, \frac{1}{M}, 0, \dots\}$$

$$h[n] = \frac{1}{S} \tilde{h}[n] \text{ for all times } n, \text{ where } S = \sum_{k=-\infty}^{\infty} \tilde{h}[n]$$

This results in a Low-Pass filter with a zero-phase.

12.4 PHASE IS IMPORTANT

$$-\angle(H(\Omega))/\Omega \quad \text{Phase delay}$$

12.5 DIFFERENTIATION USING FIR FILTERS

$$\begin{array}{ll} \text{causal:} & y(t) \approx \frac{u(t) - u(t-\tau)}{\tau} \\ \text{anti-causal:} & y(t) \approx \frac{u(t+\tau) - u(t)}{\tau} \\ \text{non-causal:} & y(t) \approx \frac{u(t+\tau) - u(t-\tau)}{2\tau} \end{array} \quad \left| \begin{array}{l} y_C[n] = \frac{1}{T_s} (u[n] - u[n-1]) \\ y_A[n] = \frac{1}{T_s} (u[n+1] - u[n]) \\ y_N[n] = \frac{1}{2T_s} (u[n+1] - u[n-1]) \end{array} \right.$$

T_0 is the desired drop time to e^{-1}

13 INFINITE IMPULSE RESPONSE FILTERS

$$y[n] = \sum_{k=0}^{M-1} b_k u[n-k] - \sum_{k=1}^{N-1} a_k y[n-k] \quad \text{Causal IIR filters}$$

- In contrast to FIR filters, IIR filters also depend on previous outputs. (some $a_k \neq 0$).
- Not necessarily stable.
- Meet a given set of filter specifications at a lower filter order compared to FIR.

$$H(z) = \frac{\sum_{k=0}^{M-1} b_k z^{-k}}{1 + \sum_{k=1}^{N-1} a_k z^{-k}} \quad H(\Omega) = \frac{\sum_{k=0}^{M-1} b_k e^{-j\Omega k}}{1 + \sum_{k=1}^{N-1} a_k e^{-j\Omega k}}$$

13.1 FIRST ORDER LOW PASS FILTER

$$y[n] = \alpha y[n-1] + (1-\alpha)u[n]$$

$$H(z) = \frac{1-\alpha}{1-\alpha z^{-1}} \quad H(\Omega) = \frac{1-\alpha}{1-\alpha e^{-j\Omega}}$$

- Very low frequency signals remain unaltered since: $H(\Omega=0) = \frac{1-\alpha}{1-\alpha e^{-j0}} = 1$
- $|H(\Omega)| = \frac{1-\alpha}{\sqrt{(1-\alpha \cos \Omega)^2 + \alpha^2 \sin^2 \Omega}}$
- Magnitude is monotonically non-increasing:
 $\frac{d|H(\Omega)|}{d\Omega} \leq 0, \forall \Omega \in (0, \pi)$
- $\angle H(\Omega) = \arctan\left(\frac{\overbrace{-\alpha \sin \Omega}^{\text{Always negative}}}{\underbrace{1-\alpha \cos \Omega}_{\text{Always positive}}}\right), \forall \Omega \in (0, \pi)$
- $-\frac{\pi}{2} \leq \angle H(\Omega) \leq 0$

13.1.1 DESIGN CONSIDERATIONS

How much time does it take $y[n]$ to decay to the value e^{-1} ? Supposing $y[0] = 1$ and $u[n] = 0$.

$$\begin{array}{ll} y[0] &= 1 \\ y[1] &= \alpha y[0] = \alpha \\ y[2] &= \alpha^2 \\ \vdots & \\ y[n] &= \alpha^n \end{array}$$

$$\alpha = e^{-\frac{1}{n}} \quad n = \frac{T_0}{T_s} \Rightarrow \alpha = e^{-\frac{T_0}{T_s}}$$

13.1.2 CONNECTION TO CT SYSTEMS

$$H(s) = \frac{1}{\tau s + 1} \quad \text{CT first-order low-pass filter}$$

$$\dot{y}(t) = -\frac{1}{\tau}(y(t) - u(t)) \xrightarrow{u(t) \equiv 0} y(t) = y(0)e^{-\frac{t}{\tau}}$$

Discretization of the above differential equation:

$$\begin{bmatrix} \dot{y} \\ \dot{u} \end{bmatrix} = \begin{bmatrix} -\frac{1}{\tau} & \frac{1}{\tau} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix} \quad 0 \leq t \leq T_s$$

solution found by matrix exponential:

$$\begin{bmatrix} y(T_s^-) \\ u(T_s^-) \end{bmatrix} = \exp\left(\begin{bmatrix} -\frac{T_s}{\tau} & \frac{T_s}{\tau} \\ 0 & 0 \end{bmatrix}\right) \begin{bmatrix} y(0) \\ u(0) \end{bmatrix} = \begin{bmatrix} e^{-\frac{T_s}{\tau}} & 1 - e^{-\frac{T_s}{\tau}} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y(0) & u(0) \end{bmatrix}$$

$$y[n] = e^{-\frac{T_s}{\tau}} y[n-1] + (1 - e^{-\frac{T_s}{\tau}}) u[n-1] = \alpha y[n-1] + (1-\alpha)u[n-1]$$

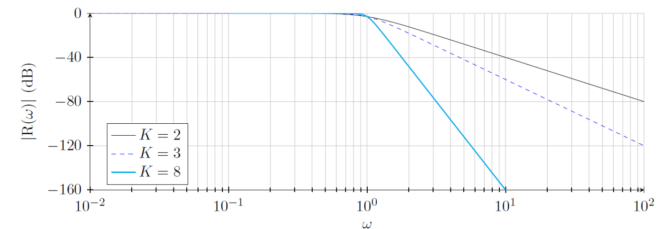
This is the same difference equation we obtained for the DT first order low pass filter.

14 IIR FILTER DESIGN

14.1 CT BUTTERWORTH FILTER DESIGN

$$R(\omega) = \frac{1}{\sqrt{1+\omega^{2K}}}$$

$R(\omega)$ represents the desired frequency response, where K is the order of the Filter and the corner frequency is 1 rad s^{-1}



- Butterworth filters have no ripples: $R = (1 + \omega^{2K})^{-1/2} \approx 1 - \frac{1}{2}\omega^{2K}$
- Thus the filter is maximally flat.

$$H(s) = \frac{1}{\prod_{k=1}^K (s - s_k)}$$

where $s_k = e^{\frac{j(2k+K-1)\pi}{2K}}$, $k = 1, \dots, K$ and s_i are the poles of $H(s)$ and the fact that they fall on the unit circle with $\omega_c = 1$ is a coincidence.

To get a different cutoff frequency $s \rightarrow \frac{s}{\omega_c}$

14.2 BILINEAR TRANSFORM

$$s = \frac{2}{T_s} \left(\frac{z-1}{z+1} \right) \quad z = \frac{1+s\frac{T_s}{2}}{1-s\frac{T_s}{2}}$$

14.2.1 MOTIVATION

1. $Y(z) = zU(z) \Leftrightarrow Y(s) = e^{sT_s}U(s)$
2. $y[n] = u[n-1] \Leftrightarrow y(t) = u(t+T_s)$
3. Equivalence of the two operators: $z = e^{sT_s}$
4. Approximation $e^{sT_s} = \frac{e^{s\frac{T_s}{2}}}{e^{-s\frac{T_s}{2}}} \approx \frac{1+s\frac{T_s}{2}}{1-s\frac{T_s}{2}} = z$
The approximation is valid for small T_s .

14.2.2 DT-CT FREQUENCY MAPPING

$$|z| = \left| \frac{1+j\omega\frac{T_s}{2}}{1-j\omega\frac{T_s}{2}} \right| = 1$$

The imaginary axis in the s-plane is therefore mapped to the unit circle in the z-plane.

$$\angle e^{j\Omega} = 2 \arctan(\omega \frac{T_s}{2}) \text{ for small } \Omega T_s \Rightarrow 2(\omega \frac{T_s}{2}) = \omega T_s$$

$$\begin{aligned} \omega = 0 & \Rightarrow \Omega = 0 \\ \omega = \infty & \Rightarrow \Omega = \pi \\ \omega = \frac{T_s}{2} & \Rightarrow \Omega = \frac{\pi}{2} \end{aligned}$$

14.3 OVERVIEW: DISCRETIZATION METHODS

Method	Transfer function	Filter parameter
Direct	$H(z) = \frac{1-\alpha}{1-\alpha z^{-1}}$	$\alpha = e^{-\frac{T_s}{\tau}}$ (decay time τ)
Sample and Hold	$H(z) = \frac{(1-\alpha)z^{-1}}{1-\alpha z^{-1}}$	$\alpha = e^{-\frac{T_s}{\tau}}$ (time constant τ)
Bilinear	$H(z) = \frac{(1-\alpha)(\frac{1+z^{-1}}{2})}{1-\alpha z^{-1}}$	$\alpha = \frac{1-\frac{T_s}{2\tau}}{1+\frac{T_s}{2\tau}}$ (time constant τ)

15 APPLIED CONCEPTS

15.1 HIGH-PASS FILTER DESIGN

$$H_{HPI}(\omega) = \begin{cases} 0 & 0 \leq \omega < \omega_c \\ 1 & \omega_c \leq \omega \end{cases}$$

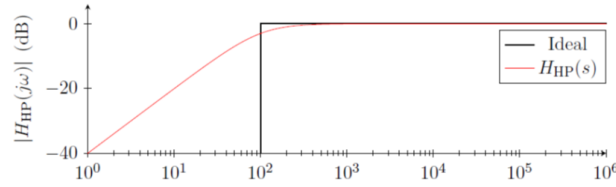
$$H_{LPI}(\omega) = \begin{cases} 1 & 0 \leq \omega < \omega_c \\ 0 & \omega_c \leq \omega \end{cases}$$

$$H_{HPI}(\omega) = 1 - H_{LPI}(\omega)$$

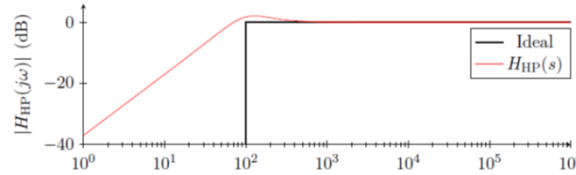
The I in HPI stands for ideal.

Thus one might be inclined to calculate high-pass filters from the TF of low pass filters.

$$H_{LP}(s) = \frac{\omega_c}{s+\omega_c} \rightarrow H_{HP}(s) = 1 - \frac{\omega_c}{s+\omega_c} = \frac{s}{s+\omega_c}$$



$$H_{LP}(s) = \frac{\omega_c^2}{s^2 + \sqrt{2}\omega_c s + \omega_c^2} \rightarrow H_{HP}(s) = 1 - H_{LP}(s) = \frac{s^2 + \sqrt{2}\omega_c s}{s^2 + \sqrt{2}\omega_c s + \omega_c^2}$$



We see that $H_{HP} = 1 - H_{LP}$ only holds for ideal filters. In the case shown the HP filter does not roll-off at 40 dB/decade and its response is not flat as would be expected of a second-order Butterworth filter.

15.2 DESIGNING A HP FILTER IN CT

Need a transformation that:

1. preserves stability - open left halfplane should be mapped to open left halfplane.
2. maps the $j\omega$ axis to the $j\omega$ axis
3. maps $\omega = 0$ to $\omega = \infty$ and $\omega = \infty$ to $\omega = 0$

$$s \rightarrow s^{-1}$$

1. $s = a + jb$ then $\frac{1}{s} = \frac{1}{a+jb} = \frac{a-jb}{a^2+b^2}$
therefore $\Re(s) = a$, $\Re(s^{-1}) = \frac{a}{a^2+b^2}$ and $\Re(s) < 0 \Leftrightarrow \Re(s^{-1})$

2. $s = j\omega$ then $\frac{1}{s} = -j\frac{1}{\omega}$

Positive frequencies are mapped to negative frequencies.

3. $\omega = 0 \rightarrow \omega = \infty$

$$H_{HPI}(\omega) = \begin{cases} 0 & 0 \leq \omega < \omega_c \\ 1 & \omega_c \leq \omega \end{cases}$$

$$H_{HPI}(\omega) = H_{LPI}(-1/\omega) = \begin{cases} 0 & 0 \leq |\omega| \leq \omega_c \\ 1 & \omega_c < |\omega| \end{cases}$$

15.3 DESIGNING A HP FILTER IN DT

Need a transformation that:

1. preserves stability - the inside of the unit circle mapped on the inside of the unit circle
2. maps the unit circle to the unit circle
3. for $z = e^{j\Omega}$, maps $\Omega = 0$ to $\Omega = \pi$ and $\Omega = \pi$ to $\Omega = 0$.

$$z = -z$$

1. $|z| < 1 \Leftrightarrow |-z| = |z| < 1$
2. $|z| = 1 \Leftrightarrow |-z| = |z| = 1$
3. $z = e^{j0} = 1 \Rightarrow -z = -1 = e^{j\pi}$
 $z = e^{j\pi} = -1 \Rightarrow -z = 1 = e^{j0}$

$$z \rightarrow -z \Rightarrow e^{j\Omega} \rightarrow -e^{j\Omega} = e^{j\pi} e^{j\Omega} = e^{j(\Omega+\pi)}$$

This transformation causes the frequency response to be shifted by π .

DT design process:

1. Given: desired HP corner Ω_c
2. design DT LP filter with corner $\pi - \Omega_c$
3. calculate $H_{HP}(z) = H_{LP}(-z)$

OR just convert filter specs to CT and design there.

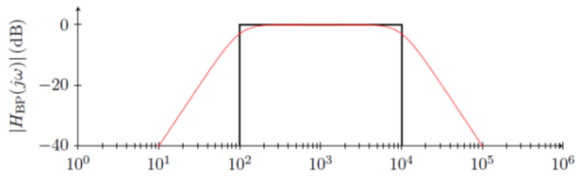
15.4 BAND-PASS FILTER DESIGN

$$H_{BPI}(\omega) = \begin{cases} 0 & 0 \leq \omega < \omega_0 \\ 1 & \omega_0 \leq \omega \leq \omega_1 \\ 0 & \omega_1 < \omega \end{cases}$$

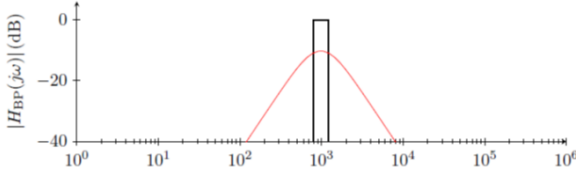
$$H_{BPI} = H_{LPI}(\omega)H_{HPI}(\omega)$$

One might therefore expect to obtain a band-pass filter as follows:
 $H_{BP}(s) = H_{LP}(s)H_{HP}(s)$

This works for $\omega_1/\omega_0 \gg 1$



But if ω_1 and ω_0 are relatively close:



15.5 LOW-PASS TO BAND-PASS FILTER TRANSFORMATION IN CT

1. given passband $\omega_0 \leq \omega \leq \omega_1$
2. design LP with corner $\omega_c = \omega_1 - \omega_0$
3. calculate $H_{BP} = H_{LP}(s \rightarrow \frac{s^2 + \omega_s^2}{s})$ where $\omega_s = \sqrt{\omega_0 \omega_1}$

1. Low frequencies of the band-pass are mapped to high frequencies of the low-pass:

$$\lim_{s \rightarrow 0} \left(\frac{s^2 + \omega_s^2}{s} \right) = \infty$$

Therefore: $H_{BPI}(0) = H_{LPI}(\infty) = 0$

2. High frequencies of the band-pass are mapped to high frequencies of the low pass:

$$\lim_{s \rightarrow \infty} \left(\frac{s^2 + \omega_s^2}{s} \right) = \infty$$

Therefore: $H_{BPI}(\infty) = H_{LPI}(\infty) = 0$

3. The frequency ω_s of the band-pass is mapped to a frequency of 0 on the low-pass:

$$\left. \frac{s^2 + \omega_s^2}{s} \right|_{s=j\omega_s} = 0$$

Therefore: $H_{BPI}(\omega_s) = H_{LPI}(0) = 1$

4. The corner frequencies of the band-pass are mapped to the corner frequency of the low-pass:

$$\left. \frac{s^2 + \omega_s^2}{s} \right|_{s=j\omega_1} = -j \frac{-\omega_1^2 + \omega_0 \omega_1}{\omega_1} = j(\omega_1 - \omega_0) = j\omega_c$$

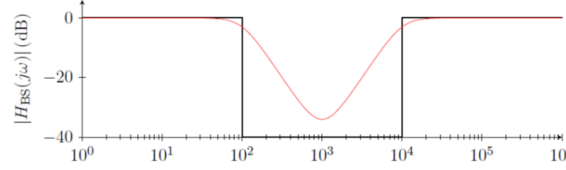
15.6 BAND-STOP FILTER DESIGN

$$H_{BSI}(\omega) = \begin{cases} 1 & 0 \leq \omega \leq \omega_0 \\ 0 & \omega_0 < \omega < \omega_1 \\ 1 & \omega_1 \leq \omega \end{cases}$$

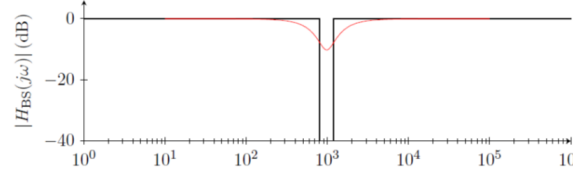
$$H_{BSI}(\omega) = H_{LPI}(\omega) + H_{HPI}(\omega)$$

One might expect to obtain a band stop filter as follows: $H_{BS}(s) = H_{LP}(s) + H_{HP}(s)$

This works as long as $\omega_1/\omega_0 \gg 1$.



But if ω_1 and ω_2 are close:



15.7 HIGH-PASS TO BAND-STOP FILTER TRANSFORMATION IN CT

1. given passband $\omega_0 \leq \omega \leq \omega_1$
2. design HP with corner $\omega_c = \omega_1 - \omega_0$
3. calculate $H_{BS} = H_{HP}(s \rightarrow \frac{s^2 + \omega_s^2}{s})$ where $\omega_s = \sqrt{\omega_0 \omega_1}$

1. Low frequencies of the band-stop are mapped to high frequencies of the high-pass:

$$\lim_{s \rightarrow 0} \left(\frac{s^2 + \omega_s^2}{s} \right) = \infty$$

Therefore: $H_{BSI}(0) = H_{HPI}(\infty) = 0$

2. High frequencies of the band-stop are mapped to high frequencies of the high-pass:

$$\lim_{s \rightarrow \infty} \left(\frac{s^2 + \omega_s^2}{s} \right) = \infty$$

Therefore: $H_{BSI}(\infty) = H_{HPI}(\infty) = 0$

3. The frequency ω_s of the band-stop is mapped to a frequency of 0 on the high-pass:

$$\left. \frac{s^2 + \omega_s^2}{s} \right|_{s=j\omega_s} = 0$$

Therefore: $H_{BSI}(\omega_s) = H_{HPI}(0) = 1$

4. The corner frequencies of the band-stop are mapped to the corner frequency of the high-pass:

$$\left. \frac{s^2 + \omega_s^2}{s} \right|_{s=j\omega_1} = -j \frac{-\omega_1^2 + \omega_0 \omega_1}{\omega_1} = j(\omega_1 - \omega_0) = j\omega_c$$

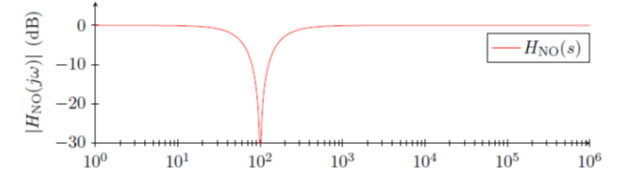
15.8 NOTCH FILTER DESIGN

$$H_{NOI}(\omega) = \begin{cases} 1 & 0 \leq \omega < \omega_c - \epsilon \\ 0 & \omega_c - \epsilon \leq \omega \leq \omega_c + \epsilon \\ 1 & \omega_c + \epsilon < \omega \end{cases}$$

$$H_{NO}(s) = \frac{s^2 + \omega_c^2}{s^2 + \sqrt{2}\omega_c s + \omega_c^2}$$

Motivation:

1. $|H_{NO}(\pm j\omega_c)| = 0$, we therefore require the terms $(s + j\omega_c)(s - j\omega_c) = s^2 + \omega_c^2$ in the numerator.
2. $|H_{NO}(0)| = 1$ and $|H_{NO}(\pm j\infty)| = 1$ thus motivating the denominator terms ω_c^2 and s^2
3. Stability: we therefore damp the filter's poles through the introduction of $\sqrt{2}\omega_c s$ in the denominator to give it the same poles as a Butterworth filter.



Let $H_{BS}(s)$ be the transfer function of a band-stop filter with $(\omega_0, \omega_1) = (\omega_c - \epsilon, \omega_c + \epsilon)$ if now $\epsilon \rightarrow 0$ we get back to the Notch filter.

15.9 FREQUENCY WARPING AND THE BILINEAR TRANSFORM

$$s = \frac{2}{T_s} \left(\frac{z-1}{z+1} \right)$$

Frequencies in CT are mapped to the following DT frequencies:

$$\Omega = 2\arctan\left(\omega \frac{T_s}{2}\right), \quad -\pi < \Omega \leq \pi$$

However, a sinusoid of DT frequency Ω corresponds to a sinusoid at CT frequency Ω/T_s . For small frequencies this is no issue since:

$$2\arctan\left(\omega \frac{T_s}{2}\right) \frac{1}{T_s} \rightarrow \omega \text{ as } \omega \rightarrow 0$$

For higher frequencies warping occurs - see lecture notes for example.

To avoid this we do pre-warping:

1. Let ω_c be the desired CT corner frequency of a DT filter.
2. Let T_s be the underlying sampling period.

$$\bar{\omega}_c = \frac{2}{T_s} \tan\left(\frac{\omega_c T_s}{2}\right) \quad \text{Warped } \omega_c$$

3. Design CT filter with frequency $\bar{\omega}_c$ and obtain $H(s)$.
4. Apply the bilinear transform

$$s = \frac{2}{T_s} \left(\frac{z-1}{z+1} \right)$$

16 SYSTEM IDENTIFICATION

Focusing on **black box** identification i.e. no physics based model is available.

16.1 FREQUENCY DOMAIN SYSTEM IDENTIFICATION



- u_e is a known input exciting the system.
- u_d is an unknown process noise, assumed to be white.
- y_d is an unknown measurement noise, assumed to be white.
- y_m given by $y_m = Gu_e + y_d + Gu_d$ is a measurement of the system's output, corrupted by process and measurement noise.

16.1.1 IDENTIFICATION WITHOUT NOISE

$$y_m = Gu_e$$

1. Let $\{u_e[n]\} = \{\delta[n]\}$
2. Then $\{y_m[n]\} = \{h[n]\}$ and $H(\Omega) = \sum_{n=0}^{\infty} y_m[n]e^{j\Omega n}$

By taking some measurement with a unit impulse input and taking the DFT of the collected data we get an approximate frequency response.

$$Y_m[k] = \sum_{n=0}^{N-1} y_m[n]e^{-j\frac{2\pi k}{N}n}$$

$$\hat{H}(\Omega_k) := Y_m[k] = H(\Omega_k) - \underbrace{\sum_{n=N}^{\infty} h[n]e^{j\Omega_k n}}_{H_N(\Omega_k)}$$

Where $H_N(\Omega_k)$ is the error made because we only measure N steps of the impulse response, and N chosen such that the impulse response has decayed significantly before N is reached.

16.1.2 IDENTIFICATION WITH NOISE

$$y_m[n] = h[n] + y_d[n]$$

$$\mathbb{E}[y_d[n]] = 0 \text{ and } \mathbb{E}[y_d[n]y_d[m]] = \sigma_y^2\delta[n-m]$$

The second condition meaning that the noise is uncorrelated across time.

$$\hat{H}(\Omega_k) = Y_m[k] = H(\Omega_k) - H_N(\Omega_k) + Y_d[k]$$

where $\mathbb{E}[Y_d[k]] = 0$ and $\mathbb{E}[|Y_d[k]|^2] = N\sigma_y^2$

$$\mathbb{E}[\hat{H}(\Omega_k) - H(\Omega_k)] = -H_N(\Omega_k)$$

which approaches 0 as $N \rightarrow \infty$

$$\mathbb{E}\left[\left|\hat{H}(\Omega_k) - H(\Omega_k)\right|^2\right] = H_N^2(\Omega_k) + N\sigma_y^2$$

And $N\sigma_y^2 \rightarrow \infty$ as $N \rightarrow \infty$!

16.2 IDENTIFICATION USING SINUSOIDAL INPUTS

$$y_m + Gu_e + y_d$$

Now $u_e[n] = e^{j\frac{2\pi}{N}ln}$ $n = 0, 1, \dots, N_T + N - 1$ where u_e is thus a sinusoid with frequency $\Omega_l = 2\pi l/N$ and N_T large enough such that the transient has died down sufficiently.

$$y_e[n] = H(\Omega_l)u_e[n] + e_e[n], \quad n \geq N_T$$

$$Y_e[l] = \sum_{n=N_T}^{N_T+N-1} y_e[n]e^{-j\frac{2\pi}{N}ln}$$

$$U_e[l] = \sum_{n=N_T}^{N_T+N-1} u_e[n]e^{-j\frac{2\pi}{N}ln}$$

$$E_e[l] = \sum_{n=N_T}^{N_T+N-1} e_e[n]e^{-j\frac{2\pi}{N}ln}$$

then

$$Y_e[l] = H(\Omega_l)U_e[l] + E_e[l]$$

where $E_e[l] \rightarrow 0$ as $N_T \rightarrow \infty$

$$Y_m[l] = \sum_{n=N_T}^{N_T+N-1} y_m[n]e^{-j\frac{2\pi}{N}ln}$$

$$Y_d[l] = \sum_{n=N_T}^{N_T+N-1} y_d[n]e^{-j\frac{2\pi}{N}ln}$$

$$\hat{H}(\Omega_l) := \frac{Y_m[l]}{U_e[l]} = H(\Omega_l) + \frac{E_e[l]}{N} + \frac{Y_d[l]}{N}$$

Therefore

$$\mathbb{E}[\hat{H}(\Omega_l) - H(\Omega_l)] = \frac{E_e[l]}{N}$$

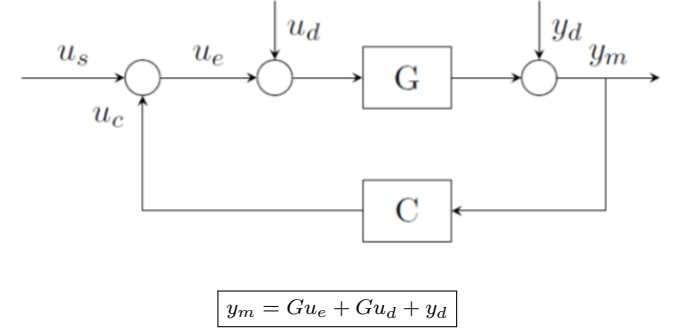
which approaches zero as $N_T \rightarrow \infty$

$$\mathbb{E}[|\hat{H}(\Omega_l) - H(\Omega_l)|^2] = \frac{E_e^2[l]}{N^2} + \frac{\sigma_y^2}{N}$$

16.2.1 EXPERIMENTAL PROCEDURE

- Choose N_T large enough to let transient die down. Large N \rightarrow long experiments but smaller error.
- Chose l , $\Omega_l = \frac{2\pi l}{N}$
- Calculate $Y_m[l] = \sum_{n=N_T}^{N_T+N-1} y_m[n]e^{-j\Omega_l n}$ and $U_e[l] = \sum_{n=N_T}^{N_T+N-1} u_e[n]e^{-j\Omega_l n} = \frac{NA}{2}$
- $\hat{H}(\Omega_l) := \frac{Y_m[l]}{U_e[l]}$
- Repeat for other frequencies.

16.2.2 CLOSED-LOOP SYSTEM IDENTIFICATION



where $u_e = u_c + u_s$

The bias and mean-squared error of the estimate $\hat{H}(\Omega_l) := \frac{Y_m[l]}{U_e[l]}$ approach 0 as $N \rightarrow \infty$.

16.3 IDENTIFYING THE TRANSFER FUNCTION

$$H(z) = \frac{\sum_{k=0}^{B-1} b_k z^{-k}}{1 + \sum_{k=1}^{A-1} a_k z^{-k}}$$

$$H(\Omega) = \frac{\sum_{k=0}^{B-1} b_k e^{-j\Omega k}}{1 + \sum_{k=1}^{A-1} a_k e^{-j\Omega k}}$$

Setting $\hat{H}(\Omega_l) = H(\Omega_l)$ at all measurement frequencies yields:

$$\left(1 + a_1 e^{-j\Omega_l} + \dots + a_{A-1} e^{-j(A-1)\Omega_l}\right) \hat{H}(\Omega_l) = b_0 + b_1 e^{-j\Omega_l} + \dots + b_{B-1} e^{-j(B-1)\Omega_l}$$

This gives two times l linear equations once for the real and once for the imaginary part.

$$R_l \cos(\phi_l) + a_1 R_l \cos(\phi_l - \Omega_l) + \dots + a_{A-1} R_l \cos(\phi_l - (A-1)\Omega_l) = b_0 + b_1 \cos(\Omega_l) + \dots + b_{B-1} \cos((B-1)\Omega_l)$$

$$R_l \sin(\phi_l) + a_1 R_l \sin(\phi_l - \Omega_l) + \dots + a_{A-1} R_l \sin(\phi_l - (A-1)\Omega_l) = -b_1 \sin(\Omega_l) - \dots - b_{B-1} \sin((B-1)\Omega_l)$$

This system of equations can be converted to the least squares problem of minimizing:

$$(F\Theta - G)^T(F\Theta - G)$$

where $\Theta = [a_1 \ a_2 \ \dots \ a_{A-1} \ b_0 \ b_1 \ \dots \ b_{B-1}]$

Θ size $(A+B-1)$ vector, unknown.

F size $(2L) \times (A+B-1)$ matrix, known.

G size $(2L)$ vector, known.

$$\Theta^* = (F^T F)^{-1} F^T G$$

$$F\Theta = G \Rightarrow WF\Theta = WG$$

$$W = \text{diag}(w_0, w_0, w_1, w_1, \dots, w_L)$$