Definitions

$$\boxed{e^{Mt} = \mathbb{I} + Mt + \frac{(Mt)^2}{2!} + \cdots = \sum_{k=0}^{\infty} \frac{((Mt)^k}{k!}}$$

Hold and Sample Operators

$$G_d = SG_cH \quad \begin{bmatrix}\dot q(t)\\ \dot u(t)\end{bmatrix} = \begin{bmatrix}A_c & B_c\\ 0 & 0\end{bmatrix}\cdot\begin{bmatrix}q(t)\\ u(t)\end{bmatrix} = M\cdot\begin{bmatrix}q(t)\\ u(t)\end{bmatrix}$$

$u(t) = u[0] \quad 0 \le t < T_s \Rightarrow \dot u(t) \equiv 0$
T_s^- just before sampling

$$\begin{bmatrix}q(T_s^-)\\ u(T_s^-)\end{bmatrix} = F\begin{bmatrix}q(0)\\ u(0)\end{bmatrix}. \text{ with } F = e^{MT_s}.$$

$$A_d = F(1:n, 1:n), \quad B_d = F(1:n, n+1:n+m)$$
$$C_d = C_c, \quad D_d = D_c.$$

$$\boxed{\begin{aligned} q[n+1] &= A_dq[n] + B_du[n]\\ y[n] &= C_dq[n] + D_du[n]\end{aligned}}$$

Euler: $\dot q(t) \approx \frac{q(t+T_s)-q(t)}{T_s} = \frac{q[n+1]-q[n]}{T_s}$
Euler is good as long as T_s is small.

Classification of Systems

- **Memoryless**: $y[n]$ only depends on $u[n]$
- **Causal**: $y[n]$ only depends on past an present inputs.
- **Lin.**: $G\{\alpha_1u_1[n]+\alpha_2u_2[n]\} = \alpha_1G\{u_1[n]\}+\alpha_2G\{u_2[n]\}$
- **Time-invariant**
  $\{y_2[n]\} = \{y_1[n-k]\}, \quad y_1 = Gu_1,\ y_2 = Gu_2,$
  $\{u_2[n]\} = \{u_1[n-k]\}, \quad \forall\ k, u_1[n]$

Stability of linear systems, BIBO

Bounded sequence: $u[n]:\ |u[n]| \le M\ \forall\ n$
Stability: $\exists M:\ |u[n]| \le 1\ \forall\ n,\ |y[n]| \le M$
BIBO: Bounded input bounded output.

Useful signals

$$\{\delta[n]\} := \begin{cases}1 & n = 0\\ 0 & n \neq 0\end{cases} \quad \{s[n]\} := \begin{cases}1 & n \ge 0\\ 0 & n < 0\end{cases}$$

	Integration	Differentiation
C	$s(t) = \int_{-\infty}^{t}\delta\tau d\tau$	$\frac{d}{dt}s(t) = \lim_{\epsilon\to0}\frac{s(t)-s(t-\epsilon)}{\epsilon} = \delta(t)$
D	$s[n] = \sum_{k=-\infty}^{n}\delta[k]$	$\{s[n]\} - \{s[n-1]\} = \{\delta[n]\}$

Representing a sequence with impulses

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]\ \forall n \quad \{x[n]\} = \sum_{k=-\infty}^{\infty} xk\{\delta[n-k]\}$$

Convolution

$$\boxed{x*h = \{x[n]\}*\{h[n]\} := \sum_{k=-\infty}^{\infty} x[k]\{h[n-k]\}}$$

$x*h = h*x \qquad (x*h_1)*h_2 = x*(h_1*h_2)$
$x*(h_1+h_2) = x*h_1 + x*h_2$

$x*\delta = x \qquad \{x[n]\}*\{\delta[n-n_0]\} = \{x[n-n_0]\}$

$x*s = \{\sum_{k=-\infty}^{n} x[k]\} \quad \{x[n]\}*\{s[n-n_0]\} = \{\sum_{k=-\infty}^{n-n_0} x[k]\}$

Step Response

$$\{r[n]\} := \{h[n]\}*\{s[n]\} = \sum_{k=-\infty}^{\infty} h[k]\{s[n]\} = \{\sum_{k=-\infty}^{\infty} h[k]\}$$

$$r[n] - r[n-1] = \sum_{k=-\infty}^{\infty} h[k] - \sum_{k=-\infty}^{\infty} h[k] = h[n],\ \forall\ n$$

Causality

$$\boxed{\text{System is causal} \Leftrightarrow\ h[n] = 0,\ \forall\ n < 0}$$

causal input: $u[n] : u[n] = 0\ \forall\ n \le 0$
$y[n] = \sum_{k=0}^{n} u[k]h[n-k] = \sum_{k=0}^{n} h[k]u[n-k],\ \forall n$

Stability of an LTI system

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

LCCDE Definition

$$\sum_{k=0}^{N} a_ky[n-k] = \sum_{k=0}^{M} b_ku[n-k], \qquad a_k, b_k \in \mathbb{R}$$
If the system is causal ($a_0 \neq 0$)

Converting from LCCDE to state-space

SS: $\begin{aligned} q[n+1] &= Aq[n] + Bu[n]\\ y[n] &= Cq[n] + Du[n]\end{aligned}$

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdot & 0\\ 0 & 0 & 1 & \cdot & 0\\ & & & \cdot & \\ & & & & 1\\ -a_N & -a_{N-1} & -a_{N-2} & \cdot & -a_1 \end{bmatrix} \qquad B = \begin{bmatrix}0\\ \cdot\\ \cdot\\ 0\\ b_0\end{bmatrix}$$
$$C = \begin{bmatrix}-a_N & -a_{N-1} & -a_{N-2} & \cdot & -a_1\end{bmatrix} \qquad D = [b_0]$$

Impulse response of a DT LTI system in SS

$$\boxed{h = \{D, CB, CAB, \ldots, CA^{n-1}B, \ldots\}}$$

FIR vs. IIR

$$\boxed{\exists N : h[n] = 0\ \forall\ n \ge N} \text{ FIR}$$

If a system can be written in non-recursive form it has a FIR.

$$\boxed{y[n] = \frac{1}{a_0}\sum_{k=0}^{M} b_ku[n-k]} \text{ Non-recursive Form}$$

Periodicity constraint

CT : $\cos(\omega t)$ is periodic with $T = \frac{2\pi}{|\omega|}$, sampled with T_s the resulting DT signal $\{x[n]\} = \{\cos(\Omega n)\}$ has the frequency $\Omega = \omega T_s$ is periodic iff

$$\frac{\Omega}{2\pi} = \frac{m}{N} \text{ for some integers } m, N$$

If $\frac{m}{N}$ is an irreducible fraction, then N is the fundamental period of the signal.

Eigenfunctions of LTI Systems

$$\{y[n]\} = G\{z^n\} \qquad = \sum_{k=-\infty}^{\infty} h[k]\{z^{n-k}\}$$

$$\boxed{H(z) := \sum_{k=-\infty}^{\infty} h[k]z^{-k}} \qquad \Leftarrow H(z)\{z^n\}$$

The z-Transform

$$\boxed{X(z) := \sum_{n=-\infty}^{\infty} x[n]z^{-n}, \quad z \in \mathbb{C}}$$

- Lin. $a_1\{x_1[n]\} + a_2\{x_2[n]\} \leftrightarrow a_1X(z) + a_2X_2(z)$
- T-shift. $\{x[n-1]\} \leftrightarrow z^{-1}X(z)$
- Conv. $\{x_1[n]\}*\{x_2[n]\} \leftrightarrow X_1(z)\cdot X_2(z)$
- Acc. $\{\sum_{k=-\infty}^{\infty} x[k]\} \leftrightarrow \frac{z}{z-1}X(z)$
- Step $s[n] \leftrightarrow \frac{z}{z-1}$

The z-Transform must also include the R.O.C. in order to uniquely specify the sequence in the time domain.

Transfer functions

$$\{y[n]\} = \{u[n]\}*\{h[n]\} \longleftrightarrow Y(z) = U(z)\cdot H(z)$$

$$\sum_{k=0}^{N} a_ky[n-k] = \sum_{k=0}^{M} b_ku[n-k]$$

$$\boxed{\frac{Y(z)}{U(z)} = \frac{b_0+b_1z^{-1}+\cdots+b_Mz^{-M}}{a_0+a_1z^{-1}+\cdots+a_Nz^{-}} = H(z)}$$

$$\boxed{H(z) = \frac{Y(z)}{U(z)} = C(zI-A)^{-1}B + D}$$

Stability and causality

Given a transfer function $H(z)$, there exists a stable and causal interpretation for the underlying system iff all poles of $H(z)$ are inside the unit circle.

DT Fourier Transform

$$\boxed{\sum_{n=-\infty}^{\infty} |x[n]| < \infty} \text{ only absolutely summable signals}$$

$$\boxed{X(\Omega) = \mathcal{F}x := \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}, \quad -\pi < \Omega \le \pi}$$

$$\boxed{\{x[n]\} = \mathcal{F}^{-1}X := \{\frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega)e^{j\Omega n}d\Omega\}}$$

$z = e^{j\Omega}, \quad X(z) = \sum_{-\infty}^{\infty} x[n]z^{-n}$

- Lin. : $a_1\{x_1[n]\} + a_2\{x_2[n]\} \longleftrightarrow a_1X_1(\Omega) + a_2X_2(\Omega)$
- Conv. : $\{x_1[n]\}*\{x_2[n]\} \longleftrightarrow X_1(\Omega)\cdot X_2(\Omega)$
- Parseval : $\sum_{-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\Omega)|^2d\Omega$
- Frequ-shift : $e^{i\Omega_0n}x[n] \longleftrightarrow X[\Omega-\Omega_0]$

Frequency Response of LTI Systems

$$H(\Omega) = \frac{Y(\Omega)}{U(\Omega)} \quad \text{Frequency Response}$$

$$H(\Omega) = |H(\Omega)|e^{j\Theta_H(\Omega)}$$
$$|Y(\Omega)| = |U(\Omega)||H(\Omega)|$$
$$\Theta_Y(\Omega) = \Theta_U(\Omega) + \Theta_H(\Omega)$$

$$H(\Omega) = H(z)|_{z=e^{j\Omega}} \quad \text{from LCCDE}$$

Response to complex exponential sequences

$\{u[n]\} = \{z^n\} = \{e^{j\Omega_0n}\}$

$$\begin{aligned}\{y[n]\} &= G\{z^n\} = H(z)\{z^n\}\\ \to y[n] &= H(z = e^{j\Omega_0})e^{j\Omega_0n = H(\Omega=\Omega_0)e^{j\Omega_0n}}\\ &= |H(\Omega_0)|e^{j(\Omega_0n+\Omega_H(\Omega_0))}\end{aligned}$$

Response to real sinusoids

$y = Gu = G(u_1 + ju_2) = Gu_1 + jGu_2 = y_1 + jy_2$

$$\begin{aligned}u[n] &= e^{j\Omega_0n} \Rightarrow u_1[n] = \cos(\Omega_0n)\\ y[n] &= H(\Omega_0)e^{j\Omega_0n} = |H(\Omega_0)|e^{j(\Omega_0n+\Theta_H(\Omega_0))}\\ y_1[n] &= |H(\Omega_0)|\cos(\Omega_0n + \Theta_H(\Omega_0))\end{aligned}$$

DFS representation of a periodic signal

$$\boxed{x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{jk\frac{2\pi}{N}n} \qquad X[k] = \sum_{n=0}^{N-1} x[n]e^{-jk\frac{2\pi}{N}n}}$$

$$\frac{1}{N}\sum_{n=0}^{N-1} e^{j(r-k)\frac{2\pi}{N}n} = \begin{cases}1 & \text{for } r-k = mN, m \in \mathbb{Z}\\ 0 & \text{otherwise}\end{cases}$$

- $a_1\{x_1[n]\} + a_2\{x_2[n]\} \leftrightarrow a_1\{X_1[k]\} + a_2\{X_2[k]\}$
- Parseval's theorem: $\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N}\sum_{k=0}^{N-1} |X[k]|^2$

DFS coefficients of a real signal

$\boxed{X[N-k] = X^*[k]}$ To proove start with: $X[N-\lambda]$...

$$X[N] = X^*[0] \quad X[N] = X[0] \Rightarrow X[0] = X^*[0]$$

If N is even $X[N/2]$ is always real.
$X[N-N/2] = X[N/2] = X^*[N/2]$

Response to Complex Exponential Sequences

$$\{\frac{1}{N}\sum_{k=0}^{N-1} Y[k]e^{jk\frac{2\pi}{N}n}\} = G\{\frac{1}{N}\sum_{k=0}^{N-1} U[k]e^{jk\frac{2\pi}{N}n}\}$$

$$\boxed{Y[k] = H(e^{jk\frac{2\pi}{N}})U[K]}$$

Relation between DFS and DT FT

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{jk\frac{2\pi}{N}n} \quad \boxed{X(\Omega) = \frac{2\pi}{N}\sum_{k=0}^{N-1} X[k]\delta(\Omega - k\frac{2\pi}{N})}$$

Discrete Fourier Transform (DFT)

$\{x[n]\}$ sequence of finite length N

$x_e[n] = x[n \mod N]\ \forall\ n$ periodic extension of $\{x[n]\}$

See DFS!

DFT of Non-Periodic Signals

$$\boxed{\{x[n]\} = \{e^{j\Omega_0n}\} \Leftrightarrow X(\Omega) = 2\pi\delta(\Omega-\Omega_0)}$$

If Ω_0 is an integer multiple of $\frac{2\pi}{N}$, $\exists k_0 \in [0, N-1] : k_0\frac{2\pi}{N} = \Omega_0 \Rightarrow X[k_0]$ is located at the location of the delta function and captures all of the signals power.
If that is not true the coefficient „overflows“:

Example: $N=10,\ \Omega_0=\frac{\pi}{3}$

Ω_0 is not a multiple of $\frac{2\pi}{10}$
Even though the signal is periodic, choosing N wrongly leads to a periodic extension that involves many different frequencies instead of only $\pi/3$.
Parceval indicates that the energy in the frequency $\pi/3$ has to be conserved when transformed.

Aliasing

Sampling uniformly: $x(t)=e^{j\omega t}\Rightarrow x[n]=\{e^{j\omega T_s n}\}$

When uniformly sampling $x(t)=e^{j(\omega+\frac{2\pi k}{T_s})t}$:

$\{x[n]\}=x(nT_s)=\{e^{j(\omega+\frac{2\pi k}{T_s})nT_s}\}=\{e^{j\omega n T_s}\underbrace{e^{jn2\pi}}_{=1}\}$

$\rightarrow$ Different frequencies map to one and the same!

$\boxed{-\frac{\pi}{T_s}<\omega<\frac{\pi}{T_s}}$ Allowed frequency range

$\boxed{\omega_N=\frac{\pi}{T_s}=\pi f_s \quad f_N=\frac{\omega_N}{2\pi}=\frac{f_s}{2}}$ Nyquist frequency

Non-causal filtering with causal filters

- G causal, LTI filter with TF $H(z)$
- $\tilde{G}$ anti-causal LTI filter with TF $H(z^{-1})$

$$Y(e^{j\Omega}=H(e^{j\Omega})H(e^{j\Omega})U(e^{j\Omega}=|H(e^{j\Omega})|^2U(e^{j\Omega})|$$

Non-Linear Filter: Median

$y[n]=\texttt{median}(u[n-M/2],\dots,u[n],\dots,u[n+M/2])$

FIR filters

$\boxed{y[n]=\sum_{k=0}^{M-1}b_k u[n-k]}$ These filters are absolutely stable because h is absolutely summable.

FIR $=h=\{b_0,b_1,\dots,b_{M-1}\}$

Moving Average Filter

$\boxed{y[n]=\frac{1}{M}\sum_{k=0}^{M-1}u[n-k]}$ $\boxed{H(\Omega)=\frac{1}{M}\frac{(1-e^{j\Omega M})}{(1-e^{-j\Omega})}}$

$H(\Omega)=0$ iff $\Omega=2\pi k/M$

$\boxed{\angle(H(\Omega))\approx-\frac{\Omega(M-1)}{2}}$ $\boxed{|H(\Omega)|=\frac{\sin^2(\frac{\Omega M}{2})}{M^2\sin^2(\frac{\Omega}{2})}}$

exact until 1. zero of $H(\Omega)$.

$\boxed{\left|\frac{\mathrm{sinc}(\frac{\Omega M}{2})}{\mathrm{sinc}(\frac{\Omega}{2})}\right|\approx|\mathrm{sinc}(\frac{\Omega}{2})| \text{ for small } \Omega.}$ $\frac{\sin(x)}{x}=\mathrm{sinc}(x)$

Non-Causal Moving Average Filter

$$\boxed{h=\{\dots,0,\frac{1}{M},\dots,\underset{\uparrow}{\frac{1}{M}},\dots,\frac{1}{M},0,\dots\}}$$

$$H(\Omega)=\frac{1}{M}\sum_{k=0}^{M-1}e^{-j\Omega(k-\frac{M-1}{2})}=e^{j\Omega(\frac{M-1}{2})}H_{MA}(\Omega)$$

Non-Causal Weighted Moving Average Filter

$$\boxed{h[n]=\frac{1}{S}\tilde{h}[n] \text{ for all times n, where } S=\sum_{k=-\infty}^{\infty}\tilde{h}[n]}$$

Results in a LP with zero-phase. $\boxed{-\angle(H(\Omega))/\Omega}$ Phase delay

Differentiation using FIR Filters

1: $y(t)\approx\frac{u(t)-u(t-\tau)}{\tau}$ | $y_C[n]=\frac{1}{T_s}(u[n]-u[n-1])$
2: $y(t)\approx\frac{u(t+\tau)-u(t)}{\tau}$ | $y_A[n]=\frac{1}{T_s}(u[n+1]-u[n])$
3: $y(t)\approx\frac{u(t+\tau)-u(t-\tau)}{2\tau}$ | $y_N[n]=\frac{1}{2T_s}(u[n+1]-u[n-1])$

1. causal 2. anti-causal 3. non-causal

IIR-Filters: First Order Low Pass Filter

$$y[n]=\alpha y[n-1]+(1-\alpha)u[n]$$
$$H(z)=\frac{1-\alpha}{1-\alpha z^{-1}} \qquad H(\Omega)=\frac{1-\alpha}{1-\alpha e^{-j\Omega}}$$

How much time does it take $y[n]$ to decay to the value e^{-1}? Supposing $y[0]=1$ and $u[n]=0$.

$y[n]=\alpha^n\Rightarrow\alpha=e^{-\frac{1}{n}} \qquad n=\frac{T_0}{T_s}\Rightarrow\alpha=e^{-\frac{T_s}{T_0}}$

T_0 is the desired drop time to e^{-1}

IIR Filters: CT Butterworth filter design

$$\boxed{H(s)=\frac{1}{\prod_{k=1}^{K}(s-s_k)}}$$

To get a differenct cutoff frequency $s\rightarrow\frac{s}{\omega_c}$

Bilinear Transform

$$\boxed{s=\frac{2}{T_s}\left(\frac{z-1}{z+1}\right) \qquad z=\frac{1+s\frac{T_s}{2}}{1-s\frac{T_s}{2}}}$$

Designing a HP filter in CT

$\boxed{H_{HP}=1-H_{LP}}$ Only works for ideal filters

$$\boxed{H_{HPI}(\omega)=H_{LPI}(-1/\omega)=\begin{cases}0 & 0\le|\omega|\le\omega_c\\ 1 & \omega_c<|\omega|\end{cases}}$$

Designing a HP filter in DT

$$\boxed{z=-z}$$

Maps unit circle to itself, the inside to the inside for stability, $\Omega=0\rightarrow\Omega_n=\pi$ and vice versa.
DT design process:

1. Given: desired HP corner Ω_c
2. design DT LP filter with corner $\pi-\Omega_c$
3. calculate $H_{HP}(z)=H_{LP}(-z)$

Band-Pass Filter Design

$$H_{BPI}=H_{LPI}(\omega)H_{HPI}(\omega) \quad \text{If ideal!}$$

$\boxed{H_{BP}(s)=H_{LP}(s)H_{HP}(s)}$ for $\omega_1/\omega_0\gg1$

Low-pass to band-pass filter transformation in CT

1. given passband $\omega_0\le\omega\le\omega_1$
2. design LP with corner $\omega_c=\omega_1-\omega_0$
3. $H_{BP}=H_{LP}(s\rightarrow\frac{s^2+\omega_s^2}{s})$ where $\omega_s=\sqrt{\omega_0\omega_1}$

Band-Stop Filter Design

$$H_{BSI}(\omega)=H_{LPI}(\omega)+H_{HPI}(\omega) \quad \text{if ideal!}$$

$\boxed{H_{BS}(s)=H_{LP}(s)+H_{HP}(s)}$ for $\omega_1/\omega_0\gg1$

High-Pass to Band-Stop Filter Transformation in CT

1. given passband $\omega_0\le\omega\le\omega_1$
2. design HP with corner $\omega_c=\omega_1-\omega_0$
3. $H_{BS}=H_{HP}(s\rightarrow\frac{s^2+\omega_s^2}{s})$ where $\omega_s=\sqrt{\omega_0\omega_1}$

Notch Filter Design

$$\boxed{H_{NO}(s)=\frac{s^2+\omega_c^2}{s^2+\sqrt{2}\omega_c s+\omega_c^2}}$$

Pre-Warping

$\boxed{\bar{\omega}_c=\frac{2}{T_s}\tan\left(\frac{\omega_c T_s}{2}\right)}$ Warped ω_c, sampling period T_s

Identification without noise

1. Let $\{u_e[n]\}=\{\delta[n]\}$
2. Then $\{y_m[n]\}=\{h[n]\}$ and $H(\Omega)=\sum_{n=0}^{\infty}y_m[n]e^{j\Omega n}$

$$Y_m[k]=\sum_{n=0}^{N-1}y_m[n]e^{-j\frac{2\pi k}{N}n}$$

$$\boxed{\hat{H}(\Omega_k):=Y_m[k]=H(\Omega_k)-\underbrace{\sum_{n=N}^{\infty}h[n]e^{j\Omega_k n}}_{H_N(\Omega_k)}}$$

Identification using Sinusoidal Inputs

$$y_m=Gu_e+y_d$$

$u_e[n]=e^{j\frac{2\pi}{N}ln} \quad n=0,1,.,N_T+N-1 \quad \Omega_l=2\pi l/N.$

$$y_e[n]=H(\Omega_l)u_e[n]+e_e[n], \quad n\ge N_T$$

$$Y_e[l]=\sum_{n=N_T}^{N_T+N-1}y_e[n]e^{-j\frac{2\pi}{N}ln}$$
$$U_e[l]=\sum_{n=N_T}^{N_T+N-1}u_e[n]e^{-j\frac{2\pi}{N}ln}$$
$$E_e[l]=\sum_{n=N_T}^{N_T+N-1}e_e[n]e^{-j\frac{2\pi}{N}ln}$$

$$Y_e[l]=H(\Omega_l)U_e[l]+E_e[l]$$

where $E_e[l]\rightarrow0$ as $N_T\rightarrow\infty$

$$Y_m[l]=\sum_{n=N_T}^{N_T+N-1}y_m[n]e^{-j\frac{2\pi}{N}ln}$$
$$Y_d[l]=\sum_{n=N_T}^{N_T+N-1}y_d[n]e^{-j\frac{2\pi}{N}ln}$$

$$\hat{H}(\Omega_l):=\frac{Y_m[l]}{U_e[l]}=H(\Omega_l)+\frac{E_e[l]}{N}+\frac{Y_d[l]}{N}$$

$$\mathbb{E}[\hat{H}(\Omega_l)-H(\Omega_l)]=\frac{E_e[l]}{N}$$

which approaches zero as $N_T\rightarrow\infty$

$$\mathbb{E}[|\hat{H}(\Omega_l)-H(\Omega_l)|^2]=\frac{E_e^2[l]}{N^2}+\frac{\sigma_y^2}{N}$$

Experimental Procedure

- Choose N_T large enough to let transient die down. Large N $\rightarrow$ long experiments but smaller error.
- Chose l, $\quad\Omega_l=\frac{2\pi l}{N}$
- Calculate $Y_m[l]=\sum_{n=N_T}^{N_T+N-1}y_m[n]e^{-j\Omega_l}$ and $U_e[l]=\sum_{n=N_t}^{N_T+N_1}u_e[n]e^{-j\Omega_l n}=\frac{NA}{2}$
- $\hat{H}(\Omega_l):=\frac{Y_m[l]}{U_e[l]}$

Identifying the Transfer Function

$$H(z)=\frac{\sum_{k=0}^{B-1}b_k z^{-k}}{1+\sum_{k=1}^{A-1}a_k z^{-k}} \qquad H(\Omega)=\frac{\sum_{k=0}^{B-1}b_k e^{-j\Omega k}}{1+\sum_{k=1}^{A_1}a_k e^{-j\Omega k}}$$

Setting $\hat{H}(\Omega_l)=H(\Omega_l)$ at all measurement frequencies yields:

$$\left(1+a_1e^{-j\Omega_l}+\dots+a_{A-1}e^{-j(A-1)\Omega_k}\right)\hat{H}(\Omega_l)=$$
$$b_0+b_1e^{-j\Omega_l}+\dots+b_{B-1}e^{-j(B-1)\Omega_l}$$

This gives two times l linear equations once for the real and once for the imaginary part.
$R_l\cos(\phi_l)+a_1R_l\cos(\phi_l-\Omega_l)+\dots+a_{A-1}R_l\cos(\phi_l-(A-1)\Omega_l)=b_0+b_1\cos(\Omega_l)+\dots+b_{B-1}\cos((B-1)\Omega_l)$
$R_l\sin(\phi_l)+a_1R_l\sin(\phi_l-\Omega_l)+\dots+a_{A-1}R_l\sin(\phi_l-(A-1)\Omega_l)$
$=-b_1\sin(\Omega_l)-\dots-b_{B-1}\sin((B-1)\Omega_l)$
This system of equations can be converted to the least squares problem of minimizing:

$$(F\Theta-G)^T(F\Theta-G)$$

where $\Theta=[a_1\quad a_2\quad\cdots\quad a_{A-1}\quad b_0\quad b_1\quad\cdots\quad b_{B-1}]$
Θ size (A+B-1) vector, unkown.
F size $(2L)\times(A+B-1)$ matrix, known.
G size $(2L)$ vector, known.

$$\boxed{\Theta^*=(F^TF)^{-1}F^TG}$$

$$F\Theta=G\Rightarrow WF\Theta=WG \quad \text{Weighted least squares}$$

$$W=\mathrm{diag}(w_0,w_0,w_1,w_,\cdots,w_L)$$