System Modeling

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Inhaltsverzeichnis

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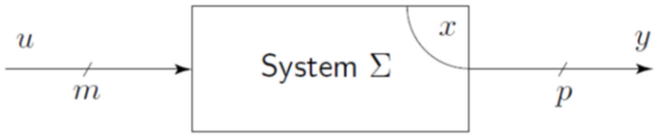
1 Basics



$$\dot{x}(t) = f(x(t), u(t), t), \qquad x(t) \in \mathbb{R}^n,\ u(t) \in \mathbb{R}^m$$
$$y(t) = g(x(t), u(t), t), \qquad y(t) \in \mathbb{R}^p$$

Transfer function:
$Y(s) = \left[D + C(s \cdot I - A)^{-1}B\right] U(s), \quad y(t) \in \mathbb{C}^p,\ u(t) \in \mathbb{C}^m$

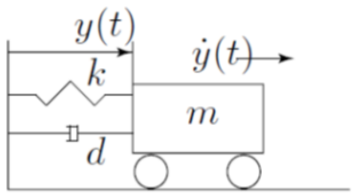
1.1 Model types

- **black box model:** derived from experiments only
- **grey-box model:** model-based, experiments need for parameter identification, model variation
- **white-box model:** no experiments at all

Describing a system by a model based on physical principles allows to **extrapolate the system behaviour** and is useful if the real system is not available.

1.2 Parametric model

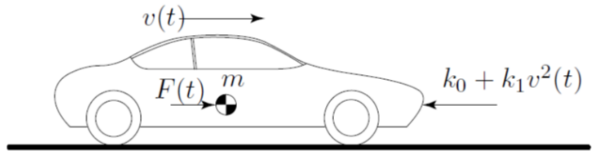
Using a physical model describing the system.



$\boxed{m\ddot{y}(t) + d\dot{y}(t) + ky(t) = F(t)}$ Differential equation

With the parameters: mass: m, viscous damping: d, spring constant: k

1.2.1 Forward causality



$$m\frac{d}{dt}v(t) = -\{k_0 + k_1 v(t)^2\} + F(t)$$

Input: Traction force F Output: actual fuel mass flow $\overset{*}{m}(t)$

$$\overset{*}{m}(t) = \{\mu + \epsilon F(t)\}v(t)$$

1.2.2 Backward causality

Do an experiment, record speed history and invert the causality chain to reconstruct the applied forces.
$v(t_i) = v_i, \quad i = 1, \ldots, N, \quad t_i - t_{i-1} = \delta$

$$F(t_i) \approx m\frac{v(t_i) - v(t_{i-1})}{\delta} + k_0 + k_1\left(\frac{v(t_i) + v(t_{i-1)}}{2}\right)^2$$

inserting $F(t_i)$ and $v(t_i)$ into $\overset{*}{m}(t) = \{\mu + \epsilon F(t)\}v(t)$:

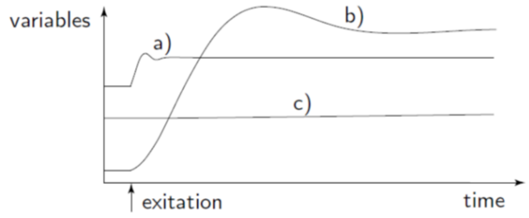
$$M_{tot} = \sum_{i=1}^{N} \overset{*}{m}(t_i)\delta$$

1.3 Non parametric model

Using a mathematical model fitting data extracted from the real system.
Drawbacks

- require the system to be accessible for experiments.
- cannot predict the behaviour of the system if modifieed.
- not useful for systematic design optimization.

1.4 Relevant dynamics



- b) signals with **relevant** dynamics
- a) signals with **fast** dynamics
- c) signals with **slow** dynamics

2 Modelling methodology

1. Define the **system-boundaries, inputs / outputs**.
2. Identify the **relevant reservoirs** and corresponding **level variables**.

   Don't forget sensor dynamics!
3. Formulate the **differential equations** for all relevant reservoirs:

   $\frac{d}{dt}$(reservoir content) $= \sum$ inflows $- \sum$ outflows
4. Formulate the **algebraic relations that express the flows** between the reservoirs as functions of the level variables.
5. Resolve implicit algebraic loops, if possible. Simplify the resulting mathematical relations.
6. **Identify the unknown system parameters** using some experiments.
7. **Validate the model** with experiments that have not been used to identify the system parameters.

2.1 Normalization

Replace the physical variables z(t),v(t) and w(t) by **normalized variables x(t), u(t) and y(t)**, which have a magnitude of ≈1.

$z_i(t) = z_{i,0} \cdot x_i(t), \qquad v(t) = v_0 \cdot u(t), \qquad w(t) = w_0 \cdot y(t)$

$$\begin{aligned} \frac{d}{dt}x(t) &= f_0(x(t), u(t)) \\ y(t) &= g_0(x(t), u(t)) \end{aligned}$$

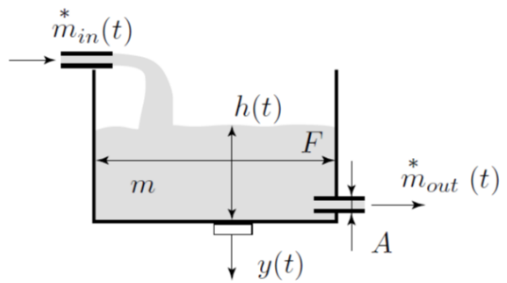
2.2 Linearization

Linearize the system around an equilibrium point (x_e, u_e), where $\frac{d}{dt}\vec{x}(t) = 0$ and $\frac{d}{dt}y(t) = 0$.

$$\begin{aligned} x_i(t) &= x_{i,e} + \delta x_i(t) \text{ with } |\delta x_i(t)| \ll 1, \\ u(t) &= u_e + \delta u(t) \quad \text{with } |\delta u(t)| \ll 1, \\ y(t) &= y_e + \delta y(t) \quad \text{with } |\delta y(t)| \ll 1 \end{aligned}$$

$$\begin{aligned} \frac{d}{dt}\delta x(t) &= \frac{\partial f_0}{\partial x}|_{x=x_e,u=u_e} \cdot \delta x(t) + \frac{\partial f_0}{\partial u}|_{x=x_e,u=u_e} \cdot \delta u(t) \\ \delta y(t) &= \frac{\partial g_0}{\partial x}|_{x=x_e,u=u_e} \cdot \delta x(t) + \frac{\partial g_0}{\partial u}|_{x=x_e,u=u_e} \cdot \delta u(t) \end{aligned}$$

2.3 Example: Water Tank



1. Input: $u(t) = \overset{*}{m}_{in}(t)$
   Output: $y(t) = h(t)$
2. Reservoir: mass of water: $m(t)$

   level variable: $h(t)$
3. mass balance: $\frac{d}{dt}m(t) = u(t) - \overset{*}{m}_{out}(t)$
4. Water massflow leaving tank with Bernoulli's law:

   $dm_{out} = \rho A dx \quad \frac{dm_{out}}{dt} = \rho A \frac{dx}{dt}$

   $p_S + \frac{1}{2}\rho v_S^2(t) + \rho g h_S(t) = p_O + \frac{1}{2}\rho v_O^2(t) + \rho g h_O$

   $\overset{*}{m}_{out}(t) = A_O \rho v_O(t), \quad v_O(t) = \sqrt{2gh(t)}$

   Where A_O - area of the outlet, ρ - density of the water, $v_O(t)$ velocity of the water in the outlet.

3 Mechanical Systems: Energy and Power

$\boxed{T_t(t) = \frac{1}{2}m\left(v_{x,cg}^2 + v_{y,cg}^2\right)}$ Kinetic energy: translation

$\boxed{R_r(t) = \frac{1}{2}J\omega^2(t) = \frac{1}{2}\Theta\omega^2(t)}$ Kinetic energy: rotation

Where J or Θ is the moment of inertia [m^2 kg].

$$\frac{d}{dt}R_r = \omega\dot{\omega}\Theta + \frac{1}{2}\dot{\Theta}\omega^2(t) = P_{torque} = M\cdot\omega \Rightarrow \dot{\omega}\Theta + \frac{1}{2}\dot{\Theta}\omega = M_{ext}$$

$\boxed{J_C = \int_V r_\perp^2 \rho dV}$ Moment of Inertia

Where $r_\perp$ is the distance to the axis of rotation.

$\boxed{J_A = J_C + M r_C A^2}$ Parallel axis theorem (Steiner's theorem)

Where C is the center of gravity and A is the point under consideration.

$\boxed{U(t) = U(x(t), y(t))}$ Potential energy

Only dependent on the body's coordinates, not on its velocity oder acceleration.

$\boxed{\text{E(t)=T(t)+U(t)}}$ total energy

$\boxed{\frac{dE(t)}{dt} = \sum_{i=1}^k P_i(t)}$ Mechanical power balance

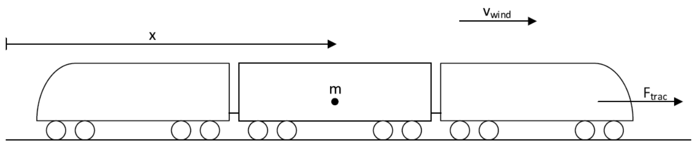
Where P_i are the mechanical powers acting on the body.

$\boxed{P = \mathbf{F}\cdot\mathbf{v} = Fv\cos(\theta) = \int_p \mathbf{F}\cdot\frac{\mathbf{dl}}{dt}}$ Power of a force

Where θ is the angle between $\mathbf{v}$ and $\mathbf{F}$.

$\boxed{P = \mathbf{T}\cdot\omega}$ Power of a torque

3.1 Example: Train



1. Input: Traction force F_{trac}
   Output: Velocity $\dot{x}(t)$
2. Reservoir: $E_{kin}(t) = \frac{1}{2}m\dot{x}^2(t)$
3.

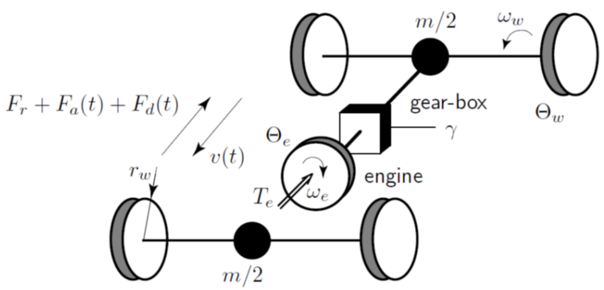
$$\frac{d}{dt}E_{kin} = P_+ - P_-$$

$$m\dot{x}\ddot{x} = (F_{trac} - F_{drag} - F_{roll} + F_{grade})\cdot\dot{x}$$

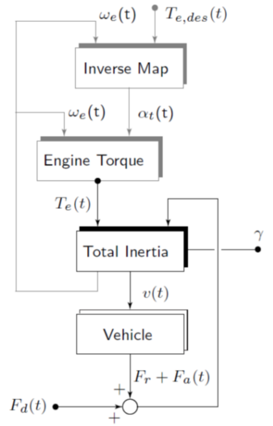
$$\boxed{\frac{d}{dt}\begin{pmatrix}\dot{x}(t)\\ x(t)\end{pmatrix} = \begin{pmatrix}\frac{1}{m}(F_{trac} - F_{drag} - F_{roll} + F_{grade})\\ \dot{x}(t)\end{pmatrix}}$$

$$\begin{aligned} F_{drag} &= \tfrac{1}{2}\rho c_w A(\dot{x}(t) - v_{wind}(t))^2\\ F_{roll} &= c_r m g\\ F_{grade} &= mg\sin(\alpha)\end{aligned}$$

3.2 Example: Vehicle

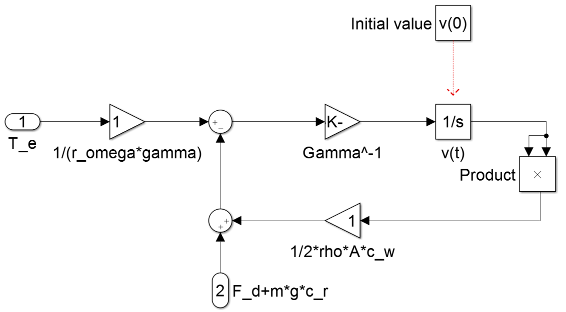


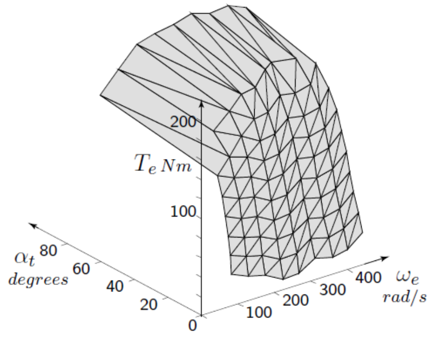
- The **clutch is engaged** such that the gear ration γ is piecewise constant.
- **No drivetrain elastics and no wheel slip** effects need to be considered. $\omega_w(t) = \gamma\omega_e(t)$ and $v(t) = r_w\omega_w(t)$.
- the vehicle has to overcome:
  - **rolling friction** $F_r = c_r m g$
  - **aerodynamic drag** $F_a(t) = \frac{1}{2}\rho c_w A v^2(t)$
- All other forces are paced into an unknown disturbance $F_d(t)$
- The kinetic energy divided in pure rotation and pure translation.
- No potential energy effects need to be considered.
- The vehicle mass m includes the mass of the engine flywheel and the wheels.



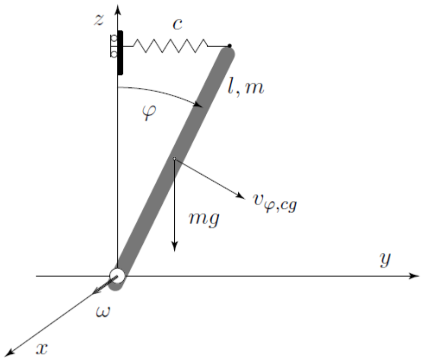
Work out dynamic subsystem: Car model

1. Input: Engine torque T_e [N m]
   Output: velocity of car $v(t)$ [m s^{-1}]
2. Reservoir: $E_{tot} = \frac{1}{2}mv^2(t) + \frac{1}{2}J_e\omega_e(t)^2 + 4\frac{1}{2}J_w\omega_w(t)^2$
   No slip assumption: $v(t) = r_w\omega_w(t)$
   gear box ratio γ : $\omega_w(t) = \gamma\omega_e(t)$
   $\omega_e^2(t) = \left(\frac{v(t)}{r_w\gamma}\right)^2 \qquad \omega_w(t)^2 = \left(\frac{v(t)}{r_w}\right)^2$
   $$\boxed{\frac{1}{2}\Gamma v(t)^2 = \frac{1}{2}\left(m + \frac{J_e}{r_w^2\gamma^2} + \frac{4J_w}{r_w^2}\right)v(t)^2}$$
3. $\frac{dE_{Tot}}{dt} = \sum_i P_{in,i} - \sum_j P_{out,j} =$
   $$= T_e\omega_e(t) - \left(\vec{F}_d + \vec{F}_a + \vec{F}_r\right)\cdot\vec{v}(t)$$
   $$\begin{aligned} F_r &= c_r m g\\ F_a &= \tfrac{1}{2}\rho c_w A v^2(t)\end{aligned} \qquad \omega_e = \frac{\omega_w}{\gamma} = \frac{v(t)}{r_w\gamma}$$
   $$\boxed{\frac{dv(t)}{dt} = \Gamma^{-1}\left(\frac{T_e}{r_w\gamma} - \left(F_d + \frac{1}{2}\rho A c_w v^2(t) + c_r m g\right)\right)}$$





3.3 Example: Nonlinear Pendulum



1. Input $\vec{F} = \vec{0}$ only initial conditions relevant
   Output $\phi(t)$
2. Reservoirs: Kinetic energy: $\frac{1}{2}J_0\dot{\varphi}^2(t)$

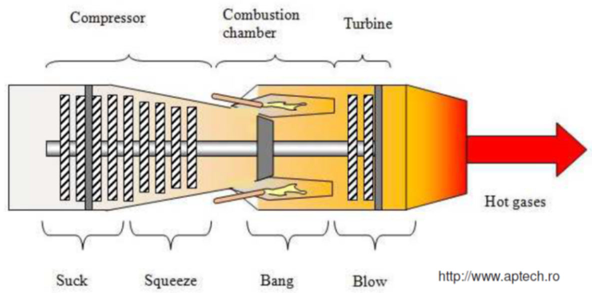
   Potential energy: $mg\frac{l}{2}\cos(\varphi(t)) + \frac{1}{2}cx(t)^2$

   $E_{Tot} = \frac{1}{2}J_0\dot{\varphi}^2(t) + mg\frac{l}{2}\cos(\varphi(t)) + \frac{1}{2}cl^2\sin^2(\varphi(t))$
3. $0 \equiv \frac{dE_{Tot}}{dt} = J_0\dot{\varphi}(t)\ddot{\varphi}(t) +$

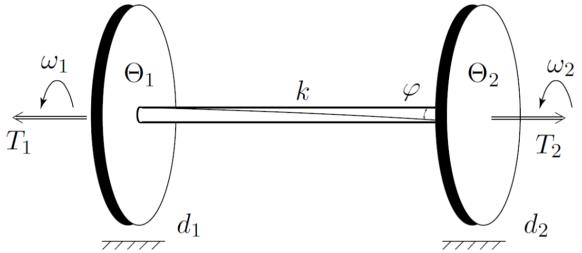
   $+mg\frac{l}{2}\left(-\sin(\varphi(t))\dot{\varphi}(t)\right) + 2\sin(\varphi(t))\cos(\varphi(t))\dot{\varphi}(t)$

$$\boxed{J_0\ddot{\varphi}(t) = mg\frac{l}{2}\sin(\varphi(t)) - cl^2\sin(\varphi(t))\cos(\varphi(t))}$$

3.4 Example: Gas Turbine



- **Rotor 2:** Turbine stage, driving torque T_2, M.o. inertia: Θ_2
- **Rotor 1:** Compressor stage, breaking torque T_1, M.o. inertia Θ_1
- **Shaft:** Elasticity constant: k
- **Friction losses:** d_1 and d_2



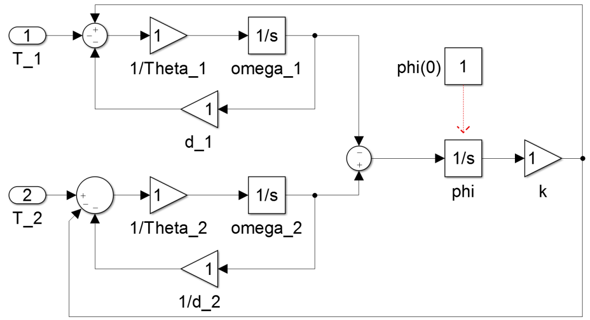
1. Input: T_1 and T_2
   Output: Rotor speed: ω_1
2. Reservoirs:
   a) kinetic energy of the turbine: $E_2(t)$, level: ω_2
   b) kinetic energy of the compressor: $E_1(t)$, level: ω_1
   c) potential energy of the shaft: $U_{shaft}(t)$, level: φ
3. Dynamic equation

$$\begin{aligned}
P_{mech,1} &= \text{compressor power} &&= T_1\cdot\omega_1\\
P_{mech,2} &= \text{friction loss in bearing 1} &&= d_1\omega_1\cdot\omega_1\\
P_{mech,3} &= \text{power of the shaft elasticity, rotor 1} &&= k\varphi\cdot\omega_1\\
P_{mech,4} &= \text{power of the shaft elasticity, rotor 2} &&= k\varphi\cdot\omega_2\\
P_{mech,5} &= \text{friction loss in bearing 2} &&= d_2\omega_2\cdot\omega_2\\
P_{mech,6} &= \text{turbine power} &&= T_2\cdot\omega_2
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dt}\left(\frac{1}{2}\Theta_1\omega_1^2(t)\right) &= -P_{m,1}(t) - P_{m,2}(t) + P_{m,3}(t)\\
\frac{d}{dt}\left(\frac{1}{2}\Theta_2\omega_2^2(t)\right) &= -P_{m,4}(t) - P_{m,5}(t) + P_{m,6}(t)\\
\frac{d}{dt}\left(\frac{1}{2}k\varphi^2(t)\right) &= -P_{m,3}(t) + P_{m,4}(t)
\end{aligned}$$

4. Algebraic relations

$$\begin{aligned}
\Theta_1\frac{d}{dt}\omega_1(t) &= -T_1(t) - d_1\cdot\omega_1(t) + k\cdot\varphi(t)\\
\Theta_2\frac{d}{dt}\omega_2(t) &= T_2(t) - d_2\cdot\omega_2(t) - k\varphi(t)\\
\frac{d}{dt}\varphi(t) &= \omega_2(t) - \omega_1(t)
\end{aligned}$$



4 Lagrange Formalism

4.1 Recipe

1. Define inputs and outputs
2. Define the generalized coordinates
   $q(t) = [q_1(t), q_2(t), \dots, q_n(t)]$ and
   $\dot{q}(t) = [\dot{q}_1(t), \dot{q}_2(t), \dots, \dot{q}_n(t)]$
3. Build the Lagrange function

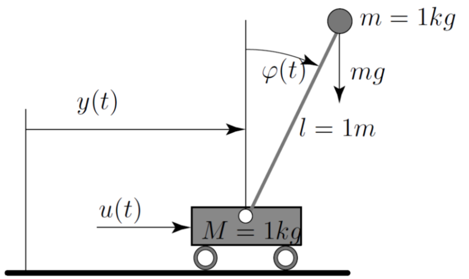
$$\boxed{L(q,\dot{q}) = \sum_i T_i(q,\dot{q}) - U_i(q)}$$

4. System dynamic equations

$$\boxed{\frac{d}{dt}\left\{\frac{\partial L}{\partial \dot{q}_k}\right\} - \frac{\partial L}{\partial q_k} = Q_k, \quad k = 1,\dots,n}$$

- Q_k represents the k_{th} „generalized force or torque“acting on the k_{th} generalized coordinate variable q_k
- n: number of degrees of freedom in the system
- always n generalized variables

4.2 Example: Pendulum on a Cart



1. Input: force acting on the cart: $u(t)$
   Output: angle of the pendulum: $\varphi(t)$
2. System's coordinate variables $q_1 = y, \quad \dot{q}_1 = \dot{y}$; $q_2 = \varphi, \quad \dot{q}_2 = \dot{\varphi}$
3. Lagrange functions

$$\begin{aligned} L_1(t) &= T_1(t) - U_1(t) \\ L_2(t) &= T_2(t) - U_2(t) \qquad \to n = 2 \\ L(t) &= L_1(t) + L_2(t) \end{aligned}$$

$T_1 = \frac{1}{2}M\dot{y}^2 = \frac{1}{2}M\dot{q}_1^2$
$U_1 = 0$ (potential energy of the cart)
$T_2 = \frac{1}{2}mv_B^2 = \frac{1}{2}m(\dot{q}_1^2 - 2\dot{q}_1\dot{q}_2 l\cos(q_2) + \dot{q}_2^2 l^2)$
$U_2 = mgl\cos(\varphi)$

$$\begin{aligned} \vec{v}_B &= \vec{v}_a + \vec{\Omega} \times \vec{AB} \\ &\dot{y}\vec{e}_y + \dot{\varphi}\vec{e}_x \times [l\cos(\varphi)\vec{e}_z - l\sin(\varphi)\vec{e}_y] \\ &\dot{q}_1\vec{e}_y + \dot{q}_2 l(-\cos(q_2)\vec{e}_y - \sin(q_2)\vec{e}_z \\ \vec{v}_B &= (\dot{q}_1 - \dot{q}_2 l\cos(q_2))\vec{e}_y - \sin(q_2)\dot{q}_2 l\vec{e}_z \\ v^2 &= \dot{q}_1^2 - 2\dot{q}_1\dot{q}_2 l\cos(q_2) + \dot{q}_2^2 l^2 \end{aligned}$$

$$\boxed{L = \frac{1}{2}M\dot{q}_1^2 + \frac{1}{2}m\left(\dot{q}_1^2 + 2\dot{q}_1\dot{q}_2 l\cos(q_2) + \dot{q}_2^2 l^2\right) - mgl\cos(q_2)}$$

4. System's dynamic equations

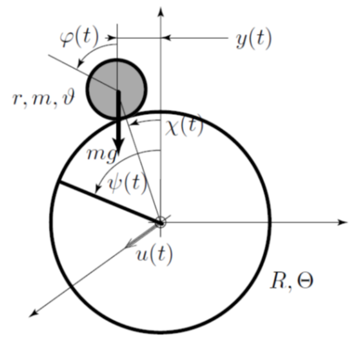
$$\begin{aligned} \frac{d}{dt}\left\{\frac{\partial L}{\partial \dot{q}_1}\right\} - \frac{\partial L}{\partial q_1} &= Q_1 \\ \frac{d}{dt}\left(\frac{1}{2}M(2\dot{q}_1 + \frac{1}{2}m(2\dot{q}_1 + 2\dot{q}_2 l\cos(q_2))\right) - 0 &= u \\ (M+m)\ddot{q}_1 + ml\left(\ddot{q}_2\cos(q_2) - \dot{q}_2^2\sin(q_2)\right) &= u \\ \frac{d}{dt}\left\{\frac{\partial L}{\partial \dot{q}_2}\right\} - \frac{\partial L}{\partial q_2} &= Q_2 \\ \frac{1}{2}m(2\dot{q}_1 l\cos(q_2) + l^2) - (mgl\sin(q_2) + \\ + \frac{1}{2}m(2\dot{q}_1\dot{q}_2 l(-\sin(q_2)) &= 0 \end{aligned}$$

4.3 Lagrange Equations for Constrained Systems

$$\boxed{\frac{d}{dt}\left\{\frac{\partial L}{\partial \dot{q}_k}\right\} - \frac{\partial L}{\partial q_k} - \sum_{j=1}^{\nu} \mu_j\alpha_{j,k} = Q_k, \quad k = 1,\dots,n}$$

- Constraints use Lagrange multipliers μ_j
- Number of constraints $\nu < n$
- $n + \nu$ coupled equations to solve

4.3.1 Example: Ball on Wheel



1. Input: $u(t)$
   Output: $y(t) = (R + r)\sin(\chi)$
2. Rotational degrees of freedom
   - $\psi(t), \ \chi(t), \ \varphi(t)$
3. Lagrange function

$$L(t) = T(t) - U(t)$$

4. Differential equations including constraints
   $n = 3, \ \nu = 1, \ q_1 = \psi, \ q_2 = \chi, \ q_3 = \phi$

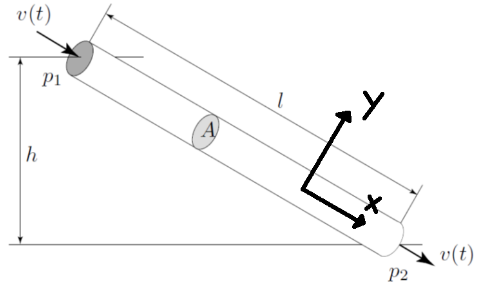
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{dq_k} - \mu\alpha_k = Q_k\right.$$

   $Q_1 = u(t), \ Q_2 = Q_3 = 0, \ \alpha_1\dot{q}_1 + \alpha_2\dot{q}_2 + \alpha_3\dot{q}_3 = 0, \ \alpha_1 = R, \alpha_2 = -(R + r), \ \alpha_3 = r$ (Kinematic constraint - no slip condition)
5. Results

$$\begin{bmatrix} \Theta + \vartheta\frac{R^2}{r^2} & -\vartheta\frac{R(R+r)}{r^2} \\ -\vartheta\frac{R(R+r)}{r^2} & m(R+r)^2 + \vartheta\frac{(R+r)^2}{r^2} \end{bmatrix} \begin{bmatrix} \ddot{\psi} \\ \ddot{\chi} \end{bmatrix} = \begin{bmatrix} u \\ mg(R+r)\sin(\chi) \end{bmatrix}$$

5 Hydraulic Systems

5.1 Water duct



$$\sin(\alpha) = \frac{dh}{dl}$$

$$\begin{aligned} \frac{d\vec{p}}{dt} = m\frac{d\vec{v}}{dt} \quad &= \vec{F}_{pressure} &&+\vec{F}_{gravity} &&-\vec{F}_{friction} \\ &= [P_1 A - P_2 A]\vec{x} &&+\int_{tube}\vec{g}dm &&-\vec{F}_r \end{aligned}$$

$$\begin{aligned} \int_{tube}\vec{g}dm &= g\int_{tube}(-\cos(\alpha)\vec{y} + \sin(\alpha)\vec{x})\rho \cdot A \cdot dl \\ &= \rho \cdot g \cdot A\left[\int_0^h -\frac{\cos(\alpha)}{\sin(\alpha)}\vec{y} + \int_0^h \frac{\sin(\alpha)}{\sin(\alpha)}dh\vec{x}\right] \\ &= -\rho g A(\tan\alpha)^{-1}h\vec{y} + \rho g A h\vec{x} \\ F_{r,x}(t) &= \frac{1}{2}\rho v^2(t)\mathrm{sign}[v(t)] \cdot \lambda(v(t))\frac{Al}{d} \end{aligned}$$

$$\boxed{\rho A l\frac{dv(t)}{dt} = A(P_1 - P_2) + \rho \cdot g \cdot A \cdot h - F_{R,x}}$$

Dynamics along $\vec{x}$-axis ($\vec{v} = v\vec{x}$)

If $\Delta p = \rho g h$:

$$\boxed{\frac{d}{dt}v = \frac{g \cdot \Delta h(t)}{l_T} - \frac{F_{friction}}{\rho \cdot l_T \cdot A_T}}$$

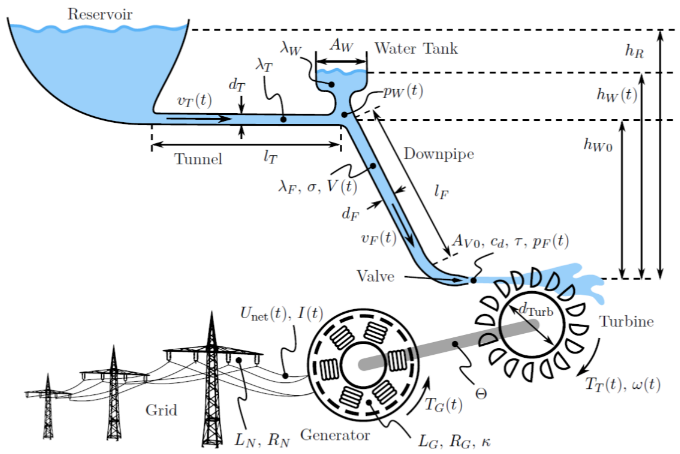
5.1.1 Example: Compressibility in Downpipe (HEPP)

$$\begin{aligned} \Delta V(t) &= V(t) - V_0 \\ \frac{dV(t)}{dt} &= \overset{*}{V}_{in} - \overset{*}{V}_{out} = v_F(t) \cdot A_F - v_v(t) \cdot A_V(t) \end{aligned}$$

The pressure in the downpipe can be calculated with the modulus of elasticity ($\rightarrow$ the pressure due to compression) σ_0 and the static pressure:

$$p_F(t) = \frac{\Delta V}{\sigma_0 \cdot V_0} + p_{stat} = \frac{V(t) - V_0}{\sigma_0 \cdot V_0} + \rho \cdot g \cdot h_R$$

5.1.2 Example: Downpipe of a HEPP



$$A_F \cdot \rho \cdot l_F\frac{dv_F(t)}{dt} = A_F \cdot (p_W(t) - p_F(t)) + A_F \cdot \rho \cdot g \cdot h_{w0} - F_f(t)$$

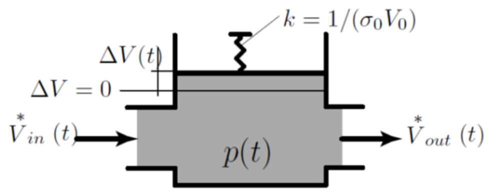
with the friction force:

$$F_f(t) = A_F \cdot \lambda_F \cdot \frac{l_F \cdot \rho}{2 \cdot d_F} \cdot \mathrm{sign}(v_F(t)) \cdot v_F^2(t)$$

leading to:

$$\frac{dv_F(t)}{dt} = \left(\frac{p_W(t) - p_F(t)}{\rho \cdot l_F} + \frac{g \cdot h_{w0}}{l_F}\right) - \frac{\lambda_F}{2 \cdot d_F} \cdot \mathrm{sign}(v_F(t)) \cdot v_F^2(t)$$

5.2 Compressibility

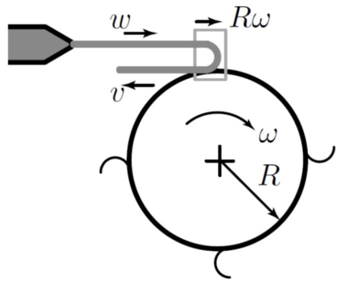


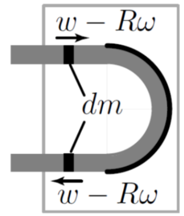
$$\sigma_0 = \frac{1}{V_0}\frac{dV}{dP}$$

Where V_0: nominal volume, P: pressure, σ_0 compressibility.

$$\boxed{\begin{aligned} \frac{d}{dt}V(t) &= \overset{*}{V}_{in}(t) - \overset{*}{V}_{out}(t) = A_{in}v_{in}(t) - A_{out}v_{out}(t) \\ \Delta P(t) &= k\Delta V(t) = \frac{1}{\sigma_0 V_0}\Delta V(t) \\ \Delta V(t) &= V(t) - V_0 \end{aligned}}$$

5.3 Pelton Turbine





$$\vec{P}_1 = dm\vec{w} \quad \vec{P}_2 = dm(\vec{w} - 2R\omega)$$
$$dp = \vec{P}_1 - \vec{P}_2 = dm\vec{w} + dm(\vec{w} - 2R\omega = 2\vec{w} - 2R\omega$$
$$\vec{F} = \frac{d\vec{P}}{dt} = 2\frac{dm}{dt}(w - R\omega)\vec{x} \qquad \frac{dm}{dt} = \rho\dot{V}$$
$$\vec{F} = 2\rho\dot{V}(w - R\omega)\vec{x} \qquad \vec{T} = 2\rho\dot{V}R(w - R\omega)(-\vec{z})$$

$$P = |\vec{T}| \cdot \omega = \underbrace{2\rho\dot{V}Rw}_{\alpha_1}\omega - \underbrace{w\rho\dot{V}R^2}_{\alpha_2}\omega^2$$

$$\frac{dP}{d\omega} = \alpha_1 - 2\alpha_2\omega = 0 \qquad \omega_0 = \frac{\alpha_1}{2\alpha_2} = \frac{w}{2R} \longrightarrow v \approx 0$$

$$P_{max} = \rho\dot{V}w^2\frac{1}{2}$$

6 Electric Systems

6.1 RLC-Networks

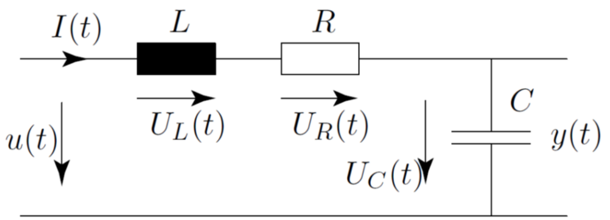
Two classes of reservoirs:

- **magnetic energy:** stored in magnetic fields B
- **electric energy:** stored in electric fields E

Element	Capacitance	Inductance
Energy	$W_E = \frac{1}{2}C \cdot U^2(t)$	$W_M = \frac{1}{2}L \cdot I^2(t)$
Level variable	$U(t)$	$I(t)$
Conservation law	$C \cdot \frac{d}{dt}U(t) = I(t)$	$L \cdot \frac{d}{dt}I(t) = U(t)$

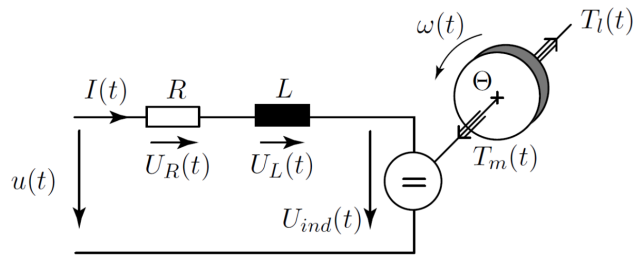
Kirchhoff's laws:

- The algebraic sum of all currents in each network node is zero.
- The algebraic sum of all voltages following a closed network loop is zero.



1. Input: $u(t)$ Output: $y(t)$
2. Reservoirs: Magnetic energy in L, electric energy in C
3. Kirchhoff rule: $U_L(t) + U_R(t) + U_C(t) = u(t)$
4. C and L law:
   $U_L(t) = L \cdot \frac{d}{dt}I(t), \qquad I(t) = C \cdot \frac{d}{dt}U_C(t)$
   and Ohm's law: $U_R(t) = R \cdot I(t)$
5. Definition: $y(t) = U_C(t),\ I(t) = \frac{d}{dt}Q(t)$
   Reformulation: $I(t) = C \cdot \frac{d}{dt}y(t),\ \frac{d}{dt}I(t) = C \cdot \frac{d^2}{dt^2}y(t)$
   Result: $L \cdot C \cdot \frac{d^2}{dt^2}y(t) + R \cdot C \cdot \frac{d}{dt}y(t) + y(t) = u(t)$

- **Classical DC drives** have a mechanical commutation of the current in the rotor coils and constant (permanent magnets) or time-varying stator fields (external excitation).
- **Brushless drives** have an electronic commutation of the stator current and permanent magnet on the rotor.
- **AC drives** have an electronic commutation of the stator current and use self-inductance to build up the rotor fields.



1. Input: $u(t), T_l(t)$ Output: $\omega(t)$
2. - the magnetic energy stored in the rotor coil, $I(t)$
   - the kinetic energy stored in the rotor, $\omega(t)$
3. $$\boxed{\begin{aligned} L_A \cdot \tfrac{d}{dt}I(t) &= -R_A \cdot I(t) - U_{ind}(t) + u(t) \\ \Theta \cdot \tfrac{d}{dt}\omega(t) &= T_m(t) - T_l(t) - d \cdot \omega(t) \end{aligned}}$$
4. $$\boxed{\begin{aligned} U_{ind}(t) \cdot I(t) &= \kappa \cdot \omega(t) \cdot I(t) \Rightarrow U_{ind} = k \cdot \omega \\ T_m(t) \cdot \omega(t) &= \kappa \cdot I(t) \cdot \omega(t) \Rightarrow T_m = k \cdot I(t) \end{aligned}}$$

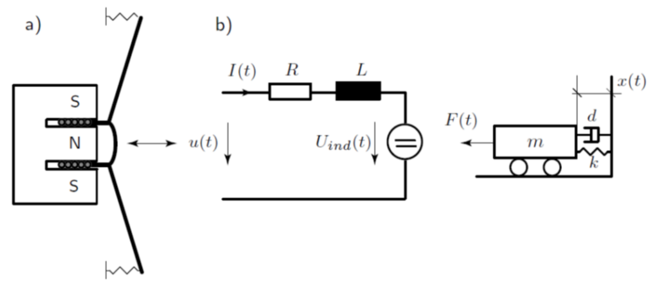
6.2 Lorentz & Faraday

$$\boxed{\vec{F} = I(\vec{l} \times \vec{B})}$$

$$\boxed{\vec{F} = q(\vec{v} \times \vec{B})}$$

$$\boxed{\vec{U} = -v(\vec{l} \times \vec{B})}$$

6.3 Example: Loudspeaker



$$\begin{aligned} d\vec{F} &= Id\vec{l} \times \vec{B} \\ \vec{F}_1 &= I(t)BD\pi\vec{n} \end{aligned}$$

$$\boxed{\vec{F}(t) = \underbrace{n\pi DB}_{\kappa} I(t)\vec{n}} \quad \text{Total force}$$

$$m \cdot \ddot{x}(t) = F(t) - k \cdot x(t) - d\dot{x}(t) \quad \text{Mechanical part}$$

$$L \cdot \tfrac{d}{dt}I(t) = -R \cdot I(t) - U_{ind}(t) + u(t) \quad \text{Electric part}$$

$$F(t) = B \cdot n \cdot d \cdot \pi \cdot I(t) = \kappa \cdot I(t) \quad \text{Motor law}$$

$$U_{ind}(t) = \kappa v(t) = \kappa \cdot \dot{x}(t) \quad \text{Generator law}$$

7 Thermodynamic Systems

7.1 Internal Energy

$$\boxed{dU = \partial W + \partial Q}$$

For a closed system: During an arbitrary process, the variation of U is the sum of the work of external forces + thermal energy transferred by/to the system.

- **adiabatic process:** $\partial Q = 0,\ dU = \partial W$
- **isochoric process:** $\partial W = 0,\ dU = \partial Q$
- **isolated system:** $dU = 0$

$$\boxed{\partial W = -P_{ext} dV} \text{ Work of external forces}$$

$$\boxed{\begin{array}{ll} dQ = mC_v dT & \text{if process without change of volume} \\ dQ = mC_p dT & \text{if process without change of pressure} \end{array}}$$

$$\boxed{\dot{Q} = \dot{m} \cdot L_f \quad [L_f] = \mathrm{J\,kg^{-1}}} \text{ Melting}$$

With $[C_v] = [C_p] = \mathrm{J\,K^{-1}\,kg^{-1}}$.

$$U(T_1) = m \cdot C \cdot T_1 \quad \text{Internal energy}$$

7.2 Enthalpy

$$\boxed{H = U + PV}$$

$$\begin{aligned} dH &= dU + P \cdot dV + V \cdot P \\ &= \partial W + \partial Q + P \cdot dV + V \cdot dP \\ &= -PdV + mC_p dT + P \cdot dV + V \cdot dP \\ &= mC_p dT + V \cdot dP \end{aligned}$$

$$\boxed{dH = mC_p dT + V \cdot dP}$$

$$\boxed{dH = mC_p dT = dU + PdV} \text{ isobaric Process}$$

$$\overset{*}{H}(t) = \overset{*}{m}(t) C_p(T(t)) T(t)$$

$$\boxed{dU = \partial Q_v = mC_v dT} \text{ isochoric Process}$$

7.3 Ideal gases

2 laws of Joule:

- Internal energy U only depends on T.
- Enthalpy H only depends on T.

$$\begin{aligned} \partial Q &= mC_v dT + PdV \\ \partial Q &= mC_p dT - PdV \\ dU &= mC_v dT \\ dH &= mC_p dT \end{aligned}$$

$$\boxed{PV = n\bar{R}T = mRT}$$

Where Pressure P in [Pa], Volume V in [m^3], Quantity n in [mol],
Universal gas constant $\bar{R} = 8.314\,\mathrm{J\,K^{-1}\,mol^{-1}}$, Temperature T in [K],
Modified gas constant $R = \bar{R}/M_{gas}$, Molar mass M_{gas}.

$$\boxed{R = C_p - C_v}$$

7.3.1 Adiabatic process

$PV^\gamma = const \qquad \gamma = \frac{C_p}{C_v}$
Where $\gamma = \frac{5}{3}$ for mono-atomic gas and $\gamma = \frac{7}{5}$ for di-atomic gas.
$PV^\gamma = const, \quad TV^{\gamma-1} = const, \quad P^{1-\gamma}T^\gamma = const$
$\gamma > 1$

$$\boxed{PV = const} \text{ Isothermal process}$$

7.4 Heat Transfer

$$\boxed{\overset{*}{Q} = \frac{\kappa A}{l} \cdot (T_1 - T_2)} \text{ Fouriers law}$$

Where κ : thermal conductivity in $[\mathrm{W\,K^{-1}\,m^{-1}}]$.

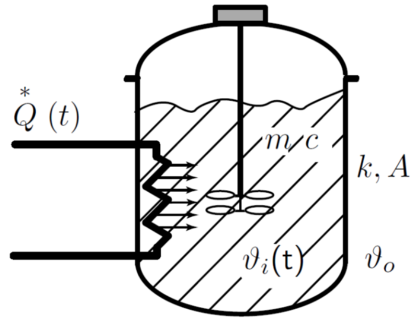
$$\boxed{\overset{*}{Q} = k \cdot A \cdot (T_1 - T_2)} \text{ Newtons law}$$

Where k: heat transfer coefficient in $[\mathrm{W\,K^{-1}\,m^{-2}}]$.

$$\boxed{\overset{*}{Q} = \epsilon \cdot \sigma \cdot A \cdot (T_1^4 - T_2^4)} \text{ Stefan-Boltzmanns law}$$

Where
ϵ: Emissivity
σ: Stefan-Boltzmann constant $\sigma = 5.6703 \times 10^{-8}\,\mathrm{W\,m^{-2}\,K^{-4}}$

7.5 Example: Stirred Reactor System



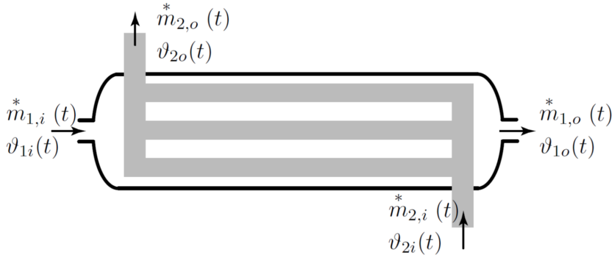
ϑ_i, ϑ_o =	temperature inside and outside	K
m =	mass in the reactor	kg
c =	specific heat of the reactor liquid	$\mathrm{J\,kg^{-1}\,K^{-1}}$
A =	active heat exchange surface	$\mathrm{m^2}$
k =	heat transfer coefficient	$\mathrm{W\,m^{-2}\,K^{-1}}$

Assumptions:

- Reactor fluid has uniform temperature distribution and the temperature of the environment is constant.
- Heat exchanger can impose an arbitrary heat flux to the fluid.
- Heat flows through the reactors poorly insulated wall.
- Only relevant reservoir is the thermal heat stored in the liquid.
- Reaction is taking place inside the reactor is assumed to be neutral - no heat is generated or absorbed inside the reactor.

1. Input: controller input heat flow $u(t) = \overset{*}{Q}_{in}(t)$
   Output: internal reactor temperature $\vartheta(t) = \vartheta_i(t) - \vartheta_o$
2. Reservoir: Internal energy stored $U(t) = m \cdot c \cdot \vartheta(t)$
3. $\frac{d}{dt}U(t) = mc\frac{d}{dt}\vartheta(t) = \overset{*}{Q}_{in}(t) - \overset{*}{Q}_{out}(t)$
4. $\overset{*}{Q}_{in}(t) = u(t)$
   $\overset{*}{Q}_{out} = k \cdot A \cdot \vartheta(t)$
5. $mc\frac{d\vartheta(t)}{dt} = u(t) - kA\vartheta(t)$

7.6 Example: Heat exchanger

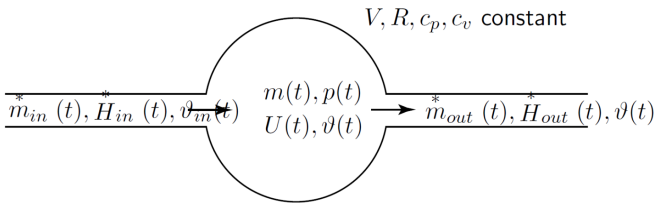


$$\frac{d}{dt}U_{1,j} = m_1 c_1 \frac{d\vartheta_{1o,j}(t)}{dt} = \overset{*}{m}_1 c_1(\vartheta_{1i,j}(t) - \vartheta_{1o,j}(t)) - \overset{*}{Q}_j(t)$$
$$\frac{d}{dt}U_{2,j} = m_2 c_2 \frac{d\vartheta_{2o,j}(t)}{dt} = \overset{*}{m}_2 c_2(\vartheta_{2i,j}(t) - \vartheta_{2o,j}(t)) + \overset{*}{Q}_j(t)$$

$$\overset{*}{Q}_j(t) = kA(\vartheta_{1o,j}(t) - \vartheta_{2o,j}(t))$$

7.7 Example: Gas receiver

Needed for every time variant pressure in a system!



Adiabatic conditions are assumed.

1. Input/Output: mass flows $\overset{*}{m}_{in/out}(t)$ and enthalpy flows $\overset{*}{H}_{in/out}(t)$

2. Reservoirs: mass: $m(t)$ internal energy $U(t)$ Level: $\vartheta(t)$

3. $\frac{d}{dt}U(t) = \overset{*}{H}_{in}(t) - \overset{*}{H}_{out}(t)$

   $\frac{d}{dt}m(t) = \overset{*}{m}_{in}(t) - \overset{*}{m}_{out}(t)$

4. $\frac{d}{dt}\vartheta = \frac{\vartheta R}{pVc_v}\left\{c_p\overset{*}{m}_{in}\vartheta_{in} - c_p\overset{*}{m}\vartheta - (\overset{*}{m}_{in} - \overset{*}{m}_{out})c_v\vartheta\right\}$

   $\frac{d}{dt}p(t) = \frac{\kappa R}{V}\left\{\overset{*}{m}_{in}(t)\vartheta_{in}(t) - \overset{*}{m}_{out}(t)\vartheta(t)\right\}$

   If isothermal:
   $\frac{d}{dt}\vartheta(t) = 0 \Rightarrow \frac{d}{dt}p(t) = \frac{R\vartheta}{V}\left\{\overset{*}{m}_{in}(t) - \overset{*}{m}_{out}(t)\right\}$

7.8 1. Principle (Conservation of Energy)

$\boxed{dU + dK = \delta W + \delta Q}$ Closed System

dU: internal energy variation
dK: kinetic energy variation
δW: mechanical energy: work of pressure forces, work of gravity forces
δQ: thermal energy exchanged with the surrounding.

$\boxed{dH + dK = \delta\tau + \partial Q}$ Open System

We use $H = U + PV$ where PV takes into account the work of fluid transport.
$\delta\tau$ is the „useful“work.

7.9 2. Principle (Entropy)

$$\boxed{\left(\frac{\partial S}{\partial U}\right)_V = \frac{1}{T}}$$

$$\boxed{\left(\frac{\partial S}{\partial V}\right)_U = \frac{P}{T}}$$

$$\boxed{\begin{aligned} dS &= \left(\frac{\partial S}{\partial U}\right)_U dU + \left(\frac{\partial S}{\partial V}\right)_U dV \\ &= \left(\frac{1}{T}\right) dU + \left(\frac{P}{T}\right) dV \\ &= mC_v\frac{dT}{T} + mR\frac{dV}{V} \text{ (ideal gas)} \\ &= mC_v\frac{dT}{T} - mR\frac{dP}{P} \text{ (ideal gas)} \end{aligned}}$$

7.10 Isentropic process for a perfect gas

$$\boxed{pV = nRT \qquad pV = mR_S T \qquad pv = RT}$$

Isentropic when $dS = 0$

$$\boxed{PV^\gamma = const \qquad \gamma = \frac{C_{p,m}}{C_{v,m}} = \frac{C_p}{C_v}}$$

Where $\gamma = 5/3$ for mono-atomic gas and $\gamma = 7/5$ for di-atomic gas.

$$\begin{aligned} (P,V): \; PV^\gamma &= const \\ (T,V): \; TV^{\gamma-1} &= const \\ (P,T): \; P^{\gamma-1}T^\gamma &= const \\ (P,T): \; P^{\frac{1-\gamma}{\gamma}}T &= const \end{aligned}$$

8 Valves

8.1 Incompressible

$\boxed{M = \frac{u}{c}}$ Mach number

With u: local flow velocity and c: speed of sound in the medium.

$\boxed{\overset{*}{m}(t) = c_d A\sqrt{2\rho}\sqrt{p_{in}(t) - p_{out}(t)}}$ Bernoulli's law

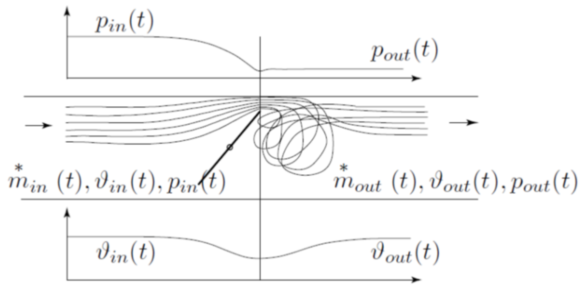
For derivation: $P_{in} + \frac{1}{2}\rho v_{in}^2 = P_{out} + \frac{1}{2}\rho v_{out}^2$
c_d accounts for flow restrictions, friction & other losses.

8.2 Compressible

Isenthalpic process:
A fluid circulates in a tube with:

- No moving wall
- No heat exchange

$\rightarrow dH = 0$ no enthalpy variation.



1. Assumption: Isentropic process
   $mC_pT_{in} + \frac{1}{2}mv_{in}^2 = mC_pT_{out} + \frac{1}{2}mv_{out}^2, \qquad v_{out} >> v_{in}$
   $v_{out} = \sqrt{2C_p(T_{in} - T_{out})}$

2. $C_p = f(\gamma, R)$
   $\gamma = \frac{C_p}{C_v} \qquad C_p - C_v = R$
   $C_p = \frac{R\cdot\gamma}{\gamma-1}$
   $\rightarrow \frac{T_{out}}{T_{in}} = \left(\frac{P_{out}}{P_{in}}\right)^{\frac{\gamma-1}{\gamma}}$

$$\boxed{v_2 = \sqrt{2R\frac{\gamma}{\gamma-1}\left(1-\Pi^{\frac{\gamma-1}{\gamma}}\right)}} \quad \Pi = \frac{P_{out}}{P_{in}}$$

$$\boxed{\overset{*}{m} = A(t)c_d\frac{P_{in}}{\sqrt{RT_{in}}}\Psi(P_{in},P_{out})}$$

Exact definition of Ψ

$$\Psi = \begin{cases} \sqrt{\kappa\left(\frac{2}{\kappa+1}\right)^{\frac{\kappa+1}{\kappa-1}}} & \text{for } p_{out} < p_{cri} \\ \left(\frac{p_{out}}{p_{in}}\right)^{1/\kappa}\sqrt{\frac{2\kappa}{\kappa-1}\left[1-\left(\frac{p_{out}}{p_{in}}\right)^{\frac{\kappa-1}{\kappa}}\right]} & \text{for } p_{out} \geq p_{crit} \text{ subsonic} \end{cases}$$

$$p_{crit} = \left[\frac{2}{\kappa+1}\right]^{\frac{\kappa}{\kappa-1}} \cdot p_{in}$$

Approximation (air and many other gases OK)

$$\Psi = \begin{cases} \frac{1}{\sqrt{2}} & \text{for } p_{out} < 0.5 \cdot p_{in} \\ \sqrt{\frac{2\cdot p_{out}}{p_{in}}\left[1-\frac{p_{out}}{p_{in}}\right]} & \text{for } p_{out} \geq 0.5 \cdot p_{in} \end{cases}$$

Laminar flow condition: $\Pi_{tr} := \frac{p_{out}}{p_{in}} < 1$

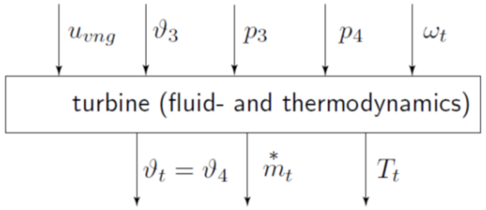
If larger pressure ratios occur, then use a smooth approximation

$$\tilde{\Pi} = a \cdot (\Pi - 1)^3 + b \cdot (\Pi - 1)$$

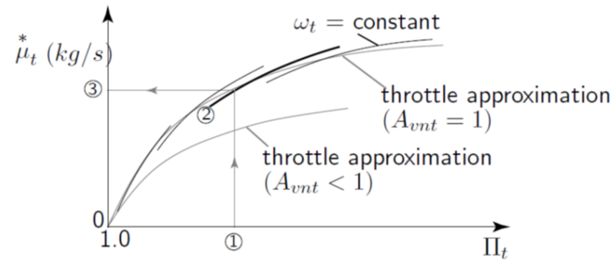
$$\begin{aligned} a &= \frac{\Psi'_{tr}\cdot(\Pi_{tr}-1)-\Psi_{tr}}{2\cdot(\Psi_{tr}-1)^3} \\ b &= \Psi'_{tr} - 3 \cdot a \cdot (\Pi_{tr}-1)^2 \end{aligned}$$

Where Ψ_{tr} is the value of Ψ and Ψ'_{tr} the value of the gradient of Ψ at the threshold Π_{tr}.

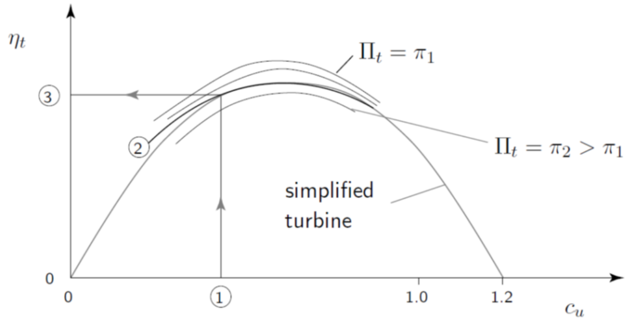
8.3 Turbine



ϑ_3: Temperature before turbine
P_3: Pressure before turbine
P_4: Pressure after turbine
ω_t: Turbine speed
u_{vnt}: Variable nozzle geometry: control input
$\vartheta_t = \vartheta_4$: Temperature after turbine
$\overset{*}{m}_t$: gas mass flow
T_t: shaft torque



$\overset{*}{\mu}_t = \overset{*}{m}_t\sqrt{\vartheta_3/\vartheta_{3,ref}}/(p_3/p_{3,ref})$: Normalized mass flow
$\Pi_t = \frac{p_3}{p_4}$: Turbine pressure ratio
A_{vnt}: Turbine inlet area



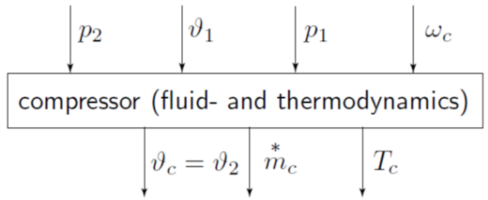
$c_u = \frac{r_t\omega_t}{c_{us}} \qquad c_{us} = \sqrt{2c_p\vartheta_3[1-\Pi_t^{(1-\kappa)/\kappa}]}$
η_t : Efficiency of the turbine

$$\boxed{T_t = \frac{\eta_t\overset{*}{m}_t c_p\vartheta_3}{\omega_t}\left[1-\Pi_t^{(1-\kappa)/\kappa}\right]}$$

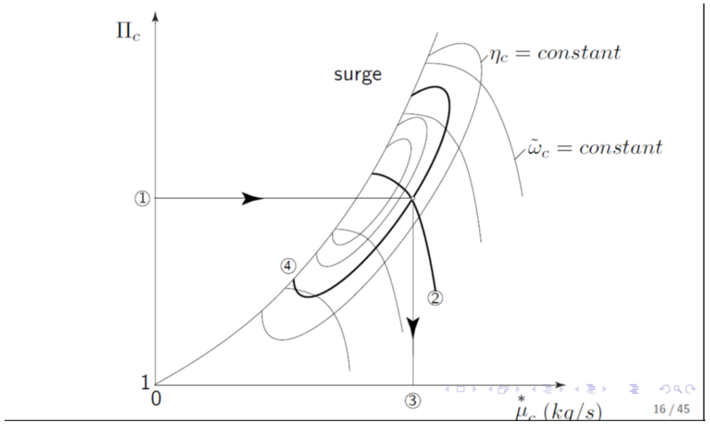
$$\boxed{\vartheta_4 = \vartheta_3 \cdot \left[1-\eta_t\cdot(1-\Pi_t^{(1-\kappa)/\kappa})\right]}$$

The temperature of the gas leaving the turbine is slightly higher than it would be under ideal isentropic conditions. This is due to the imperfect efficiency n_t.

8.4 Compressor



P_2: Pressure at compressor output
P_1: Pressure at compressor input
ϑ_1: Temperature at compressor input
ω_c: Compressor turn speed
ϑ_2: Temperature at compressor output
$\overset{*}{m}_c$: Gas mas flow
T_c: Torque absorbed by compressor



Maximum power is not at maximum efficiency.

$$\boxed{T_c = \frac{\overset{*}{m}_c c_p\vartheta_1}{\eta_c\omega_c}\left[\Pi_c^{(\kappa-1)/\kappa}-1\right]}$$

$$\boxed{\vartheta_2 = \vartheta_1 \cdot \left[1+\frac{1}{\eta_c}\left(\Pi_c^{(\kappa-1)/\kappa}-1\right)\right]}$$

$\Pi_C = \frac{p_2}{p_1}$: Compressor pressure ratio

9 Chemical systems

$$\alpha A + \beta B \leftrightarrow \gamma C + \delta D$$

$$\boxed{n_A = 1\,\text{mol} = 6.022 \times 10^{23}}$$

$\boxed{\frac{d^-}{dt}[A] = -\alpha \cdot r^- \cdot [A]^\alpha \cdot [B]^\beta}$ Rate of formation left to right

$\boxed{\frac{d^+}{dt}[A] = -\alpha \cdot r^- \cdot [C]^\gamma \cdot [D]^\delta}$ Rate of formation right to left

$$\boxed{\frac{d}{dt}[A] = \alpha\left(r^+ \cdot [C]^\gamma \cdot [D]^\delta - r^- \cdot [A]^\alpha \cdot [B]^\beta\right)}$$

Total rate of formation

More A → stronger decomposition of A → pay attention to sign when negative feedback is happening!

$$\boxed{\text{Do not forget additional terms for open systems!}}$$

$$\frac{d}{dt}[C](t) = \frac{\overset{*}{m}_C}{M_C \cdot V_C}$$

r^+ and r^- depend on pressure and temperature:

$$r^+ = k^+(\vartheta, p, \ldots) \cdot e^{-E^+/(R\vartheta)}$$

$$r^- = k^-(\vartheta, p, \ldots) e^{-E^-/(R\vartheta)}$$

Where k^+ pre-exponential factor, E^+ activation energy, Boltzmann term: $exp\{-E^+/(R\vartheta)\}$ fraction of collisions that have sufficient energy to react.

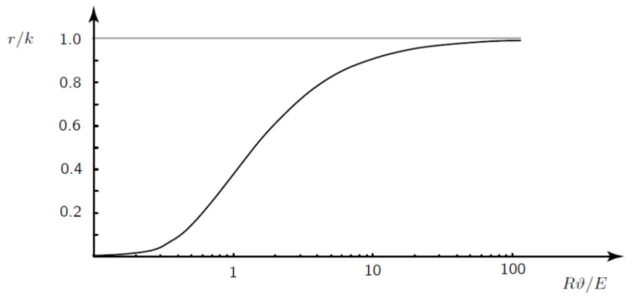


Figure: Arrhenius function.

$\boxed{\overset{*}{H}_{flow} = \overset{*}{m} \cdot c_p \cdot \vartheta}$ Enthalpy flow of a fluid

9.1 Example: Continuously stirred tank reactor

- The concentration [B] remains constant.
- Dissociation $A + B \leftarrow C$ is negligible.
- Mass m and density ρ are constant.
- Perfect insulation.

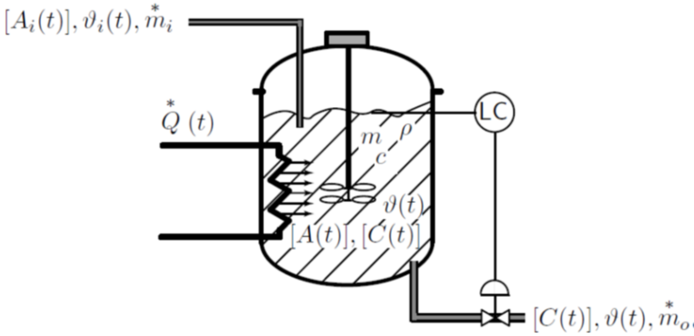


Figure: Continuous chemical reactor.

1. Input: $\overset{*}{Q}(t)$ rate of heat transferred by the exchanger.
   Outputs: concentration C and temperature $\vartheta(t)$
2. 3 Reservoirs:
   - n_A amount of species A, level variable [A]
   - n_C amount of species C, level variable [C]
   - internal energy U, level variable ϑ
3. Conservation laws:
   - $\frac{d}{dt}n_A(t) = \overset{*}{V}[A_i(t)] - \overset{*}{V}[A(t)] - Vk^-[B]e^{-E/(R\vartheta(t))}[A(t)]$
   - $\frac{d}{dt}n_C(t) = -\overset{*}{V}[C(t)] + Vk^-[B]e^{-E/(R\vartheta)}[A(t)]$
   - $\frac{d}{dt}U(\vartheta(t), n_A(t), n_B(t), n_C(t)) = \overset{*}{H}_i(\vartheta_i(t)) - \overset{*}{H}_o(\vartheta(t)) + \overset{*}{Q}(t)$
4. $$\begin{aligned} dU(\vartheta, n_A, n_B, n_C) &= \frac{\partial U}{\partial \vartheta}d\vartheta + \frac{\partial U}{\partial n_A}dn_A + \frac{\partial U}{\partial n_B}dn_B + \frac{\partial U}{\partial n_c}dn_C \\ &= \rho V c_v d\vartheta + H_A dn_A + H_B dn_B + H_C dn_C \end{aligned}$$
5. $$\begin{aligned} \tau\frac{d}{dt}[A(t)] &= [A_i(t)] - (1 + \tau k e^{-E/(R\vartheta(t))})[A(t)] \\ \tau\frac{d}{dt}[C(t)] &= -[C(t)] + \tau k e^{-E/(R\vartheta(t))}[A(t)] \\ \tau\frac{d}{dt}\vartheta(t) &= \vartheta_i(t) - \vartheta(t) + \frac{1}{\rho c_v}\frac{\overset{*}{Q}(t)}{\overset{*}{V}} + \tau H_0 \frac{\kappa}{c_v \rho} e^{-E/(R\vartheta(t))}[A(t)] \end{aligned}$$

Static behaviour of the CSTR:

- $\overset{*}{Q} = 0$
- $\vartheta_i = const.$
- $[A_i] = const.$

$$\boxed{\overset{*}{H}_{flow}(\vartheta) + \overset{*}{Q}_{chem}(\vartheta) = 0}$$

$$\boxed{\overset{*}{Q}_{chem}(\vartheta) = H_0 \frac{Vke^{-E/(R\vartheta)}}{1 + \tau ke^{-E/(R\vartheta)}}[A_i]}$$

10 Model Parametrization

10.1 Planning experiments

Planning experiments is about knowing:

- Choice of correct input signals
- Choice of sensor(s) (location)
- Measurement for (non?)linear model identification
- Frequency content of excitation signals
- Noise level at input and output
- Safety issues

Experimentally obtained data may be used to:

- Identify unknown system structures and system parameters.
- Validate the results of the system modeling and parameter identification.

$\boxed{\text{Never use the same set of data for both purposes!}}$

10.2 Least squares estimation

Used to fit the parameters of a **linear** and **static** model.

$$y(k) = \mathbf{h}[\mathbf{u}(k)]^T \cdot \pi + e(k)$$

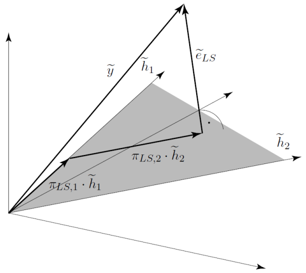
- Index of discrete time: k
- Input vector: $\mathbf{u}(k) \in \mathbb{R}^m$
- Output signal: $y(k) \in \mathbb{R}$
- Vector of unknown parameter: $\pi \in \mathbb{R}^q$
- Regressor: $\mathbf{h}[.] \in \mathbb{R}^q$
- r = number of measurements taken
- Typically $r \gg q$

y should be chosen such that it contains the measurement with the most significant error which we want to minimise. We want to minimise the following equation:

$$\epsilon = \tilde{e}^T \cdot W \cdot \tilde{e}$$

Where $W = \text{diag}(w_1, ..., w_n)$ is a symmetric, positive definite weighting matrix. If all measurements are equally reliable, one can choose $W = \mathbb{I}$.

$$\boxed{\pi_{LS} = [H^T \cdot W \cdot H]^{-1} H^T \cdot W \cdot \tilde{y}}$$



Where H must have full column rank $\rightarrow$ all q parameters $(\pi_1, \pi_2, \ldots, \pi_q)$ are required to explain the data.

$\boxed{M^\dagger = (M^T \cdot M)^{-1} \cdot M^T}$ Moore-Penrose inverse

Where $M \in \mathbb{R}^{r \times q},\ r > q,\ \text{rank}\{M\} = q.$

If the error e is an **uncorrelated white noise signal** with **mean value 0** and **variance** σ.
Then:

- expected value: $\mathbb{E}[\pi_L S] = \pi_{true}$
- covariance matrix: $\sum = \sigma^2 \cdot (H^T \cdot W \cdot H)^{-1}$

$$\boxed{\text{A}^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}$$

10.3 Iterative Least Squares

$$\pi_{LS} = [H^T \cdot W \cdot H]^{-1} H^T \cdot W \cdot \tilde{y}$$

Motivation: If an additional measurement is taken, we don't want to recompute the inverse $[H^T \cdot W \cdot H]^{-1}$

$$\pi_{LS(r+1)} = f(\pi_{LS(r)}, y_{(r+1)}) \quad \text{Iterative approach}$$

1. **Start:** $\pi_{LS} = [H^T \cdot W \cdot H]^{-1} H^T \cdot W \cdot \tilde{y}$
2. **Simplification:** $W = I \quad \rightarrow \quad \pi_{LS} = [H^T \cdot H]^{-1} \cdot \tilde{y}$
3. **Formulate matrix products as sums:**
   $$\pi_{LS(r)} = \left[\sum_{k=1}^{r} h_{(k)} \cdot h_{(k)}^T\right]^{-1} \cdot \sum_{k=1}^{r} h_{(k)} \cdot y_{(k)}$$
4. **Use matrix inversion Lemma**
   Notation: $\Omega_{(r)} = \left[\sum_{k=1}^{r} h_{(k)} \cdot h_{(k)}^T\right]^{-1}$
   $$\Omega_{(r+1)} = \left[\sum_{k=1}^{r} h_{(k)} \cdot h_{(k)}^T + h_{(r+1)} \cdot h_{(r+1)}^T\right]^{-1}$$
   $\rightarrow$ Matrix inversion Lemma leads to:
5. **How to recursively compute the estimate:**
   $$\pi_{LS(r+1)} = \Omega_{(r)} \cdot \sum_{k=1}^{r} h_{(k)} y_{(k)}$$
   $$\pi_{LS(r+1)} = \Omega_{(r+1)} \cdot \sum_{k=1}^{r+1} h_{(k)} y_{(k)}$$
   $$\boxed{\begin{aligned} &\pi_{LS(r+1)} = \pi_{LS(r)} + \\ &+ \underbrace{\frac{1}{1+c_{(r+1)}} \Omega_{(r)} h_{(r+1)}}_{A} \underbrace{\left(y_{(r+1)} - h_{(r+1)}^T \pi_{LS(r)}\right)}_{B} \\ &\Omega_{(r+1)} = \Omega_{(r)} - \\ &-\tfrac{1}{1+c_{(r+1)}} \cdot \Omega_{(r)} \cdot h_{(r+1)} \cdot h_{(r+1)}^T \cdot \Omega_{(r)} \\ &\text{where } c_{(r+1)} = h_{(r+1)}^T \cdot \Omega_{(r)} \cdot h_{(r+1)} \end{aligned}}$$
   A: Indicates the correction direction.
   B: Innovation term or prediction error.

10.3.1 Matrix inversion Lemma

Suppose $M \in \mathbb{R}^{n \times n}$ regular $(\det(M) \neq 0)$
$v \in \mathbb{R}^n : \quad 1 + v^T \cdot M^{-1} \cdot v \neq 0$
$[M + v \cdot v^T]^{-1} = M^{-1} - \frac{1}{1+v^T \cdot M^{-1} \cdot v} \cdot M^{-1} \cdot v \cdot v^T \cdot M^{-1}$

10.4 Exponential Forgetting

$$\epsilon_{(r)} = \sum_{k=1}^{r} \lambda^{r-k} [y_{(k)} - h_{(k)}^T \cdot \pi_{LS(k)}]^2, \quad \lambda < 1$$

$$\boxed{\begin{aligned} &\pi_{LS(r+1)} = \pi_{LS(r)} + \tfrac{1}{\lambda + c_{(r+1)}} \Omega_{(r)} h_{(r+1)} \left[y_{(r+1)} - h_{(r+1)}^T \pi_{LS(r)}\right] \\ &\Omega_{(r+1)} = \tfrac{1}{\lambda} \Omega_{(r)} \left[\mathbb{I} - \tfrac{1}{\lambda + c_{(r+1)}} h_{(r+1)} h_{(r+1)}^T \Omega_{(r)}\right] \end{aligned}}$$

10.5 Simplified Recursive LS Algorithm

Each new prediction error $\epsilon_{(r+1)} = y_{(r+1)} - h_{(r+1)}^T \cdot \pi_{(r)}$ contains new information on π only in the direction of $h_{(r+1)}$. Therefore $\pi_{(r+1)}$ is sought, which requires the smallest possible change $\pi_{(r+1)} - \pi_{(r)}$ to explain the new observation.
Cost function to minimize:

$$J(\pi) = $$
$$\tfrac{1}{2} \cdot [\pi_{(r+1)} - \pi_{(r)}]^T \cdot (\pi_{(r+1)} - \pi_{(r)}) + \mu \cdot [y_{(r+1)} - h_{(r+1)}^T \cdot \pi_{(r+1)}]$$

Necessary conditions for the minimum:

$$\frac{\partial J}{\partial \pi_{(r+1)}} = 0 \qquad \frac{\partial J}{\partial \mu} = 0$$

Solution:

$$\boxed{\pi_{(r+1)} = \pi_{(r)} + \frac{h_{(r+1)}}{h^t_{(r+1)} \cdot h_{(r+1)}} \cdot [y_{(r+1)} - h^T_{(r+1)} \cdot \pi_{(r)}]}$$

Modification with $0 < \gamma < 2$ for convergence, $0 < \lambda < 1$ for forgetting

$$\boxed{\pi_{(r+1)} = \pi_{(r)} + \frac{\gamma \cdot h_{(r+1)}}{\lambda + h^t_{(r+1)} \cdot h_{(r+1)}} \cdot [y_{(r+1)} - h^T_{(r+1)} \cdot \pi_{(r)}]}$$

- Kaczmarz' projection algorithm requires less computational effort.
- It converges much slower than regular LS algorithms.

11 Analysis of Linear Systems

11.1 Normalization

$$\bar{x}_i(t) = \frac{x_i(t)}{x_{i,0}}, \quad \bar{u}_j(t) = \frac{u_j(t)}{u_{j,0}}, \quad \bar{y}_k(t) = \frac{y_k(t)}{y_{k,0}}$$

With the scaling factors: $x_{i,0}, u_{j,0}, y_{k,0}$.
The normalized variables $\bar{x}_i(t)$, $\bar{u}_j(t)$ and $\bar{y}_k(t)$ have **no physical units** and are **close to 1**.

$$x = T \cdot \bar{x}, \quad T = \mathrm{diag}\{x_{1,0}, \dots, x_{n,0}\}$$

Where T is the **similarity transform matrix**.
After transforming the system has the form:

$$\begin{aligned} \tfrac{d}{dt}\bar{x}(t) &= \dot{\bar{x}}(t) = f_0(\bar{x}(t), \bar{u(t)}, t) \\ \bar{y}(t) &= g_0(\bar{x}(t), u(\bar{t}), t) \end{aligned}$$

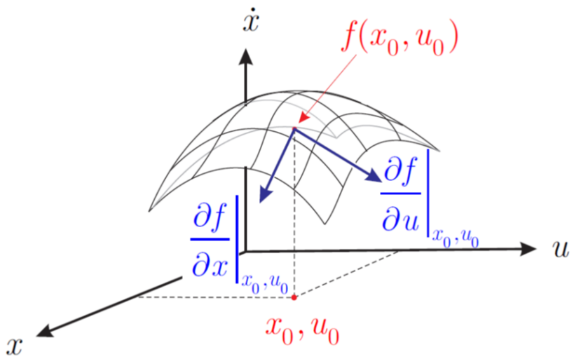
Where $\bar{x}(t) \in \mathbb{R}^n$, $\bar{u}(t) \in >$, $\bar{y}(t) \in \mathbb{R}^p$.

11.2 Linearisation

$B_r := \{x \in \mathbb{R}^n \mid ||x - x_0||^2 + ||u - u_0||^2 \leq r\}$ operating point $\{x_0, u_0\}$

$B_r := \{x \in \mathbb{R}^n \mid ||x - x_e||^2 + ||u - u_e||^2 \leq r\}$ equilibrium point $\{x_e, u_x\}$

Around a chosen equilibrium point $\{x_e, u_e\}$, $f_o(x_e, u_e, t) = 0$.



$$\dot{x} = f(x,u) = f(x_0,u_0) + \frac{\partial f}{\partial x}\Big|_{x_0,u_0} \underbrace{(x - x_0)}_{\delta x} + \frac{\partial f}{\partial u}\Big|_{x_0,u_0} \underbrace{(u - u_0)}_{\delta u} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\Big|_{x_0,u_0} (x - x_0)^2 + \dots$$

$$\begin{aligned} \tilde{x}(t) &= x(t) - x_e \\ \tilde{u}(t) &= u(t) - u_e \\ \tilde{y}(t) &= y(t) - g_0(x_e, u_e, t) \\ \tfrac{d}{dt}\tilde{x}(t) &= \tilde{f}_0(\tilde{x}(t), \tilde{u}(t), t) \\ \tilde{y}(t) &= \tilde{g}_0(\tilde{x}(t), \tilde{u}(t), t) \end{aligned}$$

Where $\tilde{f}_0(0, 0, t) = 0$.
Introduction of new variables:

$$\begin{aligned} x_i(t) &= x_e + \delta x_i(t) \text{ with } |\delta x_i| \ll 1 \\ u_i(t) &= u_e + \delta u_i(t) \text{ with } |\delta u_i| \ll 1 \\ y_i(t) &= y_e + \delta y_i(t) \text{ with } |\delta y_i| \ll 1 \end{aligned}$$

Taylor series neglecting all terms of second and higher order yields:

$$\begin{aligned} \tfrac{d}{dt}\delta x(t) &= \frac{\partial f_0}{\partial x}\Big|_{x_e,u_e} \delta x(t) + \frac{\partial f_0}{\partial u}\Big|_{x_e,u_e} \delta u(t) \\ \delta y(t) &= \frac{\partial g_0}{\partial x}\Big|_{x_e,u_e} \delta x(t) + \frac{\partial g_0}{\partial u}\Big|_{x_e,u_e} \delta u(t) \end{aligned}$$

$$\boxed{\begin{aligned} \tfrac{d}{dt}x(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}}$$

$$\boxed{e^{At} = \mathbb{I} + \tfrac{1}{1!}At + \tfrac{1}{2!}(At)^2 + \dots + \tfrac{1}{n!}(At)^n + \cdots}$$

- $\frac{de^{At}}{dt} = Ae^{At} = e^{At}A$
- $e^A \cdot e^B \neq e^{A+B}$
- If A and B commute: $e^A + e^B = e^{A+B}$

Solving the differential equation for x yields:

$$x(t) = e^{At}x(0) + \int_0^t e^{At-\sigma}Bu(\sigma)d\sigma$$

11.3 Stability

11.3.1 Lyapunov Stability

- asymptotically stable if $\lim_{t\to\infty} ||x(t)|| = 0$
- stable if $||x(t)|| < \infty \;\forall\, t \in [0, \infty]$
- unstable if $\lim_{t\to\infty} ||x(t)|| = \infty$

11.3.2 Eigenvalues

$$\boxed{Av_i = \lambda_i v_i}$$

Where eigenvectors v_i and eigenvalues λ_i.

$$T = [v_1, \cdots, v_n] \rightarrow AT = T\Lambda \Rightarrow T^{-1}AT = \Lambda$$

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}$$

- multiplicity of $\lambda_i = r_i$
- rank loss of $[\lambda_i - A]$: ρ_i

Stability

- All $\Re(\lambda_i) < 0 \;\forall\, i \Rightarrow$ system is asymptotically stable
- $\Re(\lambda_i) \leq 0 \;\forall\, i \Rightarrow$ system is Lyapunov stable
- $\Re(\lambda_i > 0 \Rightarrow$ system is unstable

11.4 Continuous time transition matrix

$$\Phi(t) = e^{At} = I + At + \frac{(At)^2}{2!} + \dots + \frac{(At)^n}{n!} + \cdots$$

$$\boxed{x(t) = \Phi(t)x(0) + \int_0^t \Phi(t - \sigma)Bu(\sigma)d\sigma}$$

For stability analysis $u = 0$ and $x(t) = \Phi(t)x(0)$!

11.5 Reachability

$$\boxed{x(\tau) = e^{A\tau} \int_0^\tau e^{-A\sigma}Bu(\sigma)d\sigma}$$

All the states that can be reached within τ.

$$\boxed{\mathcal{R}_n = \begin{bmatrix} B & AB & A^2B & A^3B & \cdots & A^{n-1}B \end{bmatrix}}$$

If $\mathrm{rank}(R_n) = n$ the system is reachable for all $x(0)$.

11.6 Controllability

Is it possible to reconstruct $x(0)$ using the output signal $y(t)$ only?

$$\boxed{\mathcal{O}_n = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}}$$

If rank($\mathcal{O}_n = n$) the system is observable.

12 Balanced Realization and Order Reduction

The Gramian matrices only exist if the system is asymptotically stable!

$$\boxed{W_R = \int_0^\infty e^{A\sigma} BB^T e^{A^T\sigma} d\sigma} \text{ Controllability Gramian}$$

The closer W_R is to a singular matrix (det close to 0), the less controllable the system will be.

$$\boxed{W_O = \int_0^\infty e^{A^T\sigma} C^T C e^{A\sigma} d\sigma} \text{ Observability Gramian}$$

The closer W_O is to a singular matrix (det close to 0), the less observable the system will be.
→ Check which element in the Gramian matrix is the smallest and reduce that state.

12.1 Hurwitz systems

$$\boxed{\text{All EV have strictly negative real parts.}} \text{ Hurwitz stable}$$

Then the Gramian matrices are the solutions of the two Lyapunov equations:

$$\begin{aligned} AW_R + W_R A^T &= -BB^T \\ A^T W_O + W_O A &= -C^T C \end{aligned}$$

Assume that the last ν elements σ_j with $j = n - \nu + 1$ are substantially smaller than the other first $n - \nu$ elements σ_i with $i = 1, \dots, n - \nu$. Then the contribution of the last ν balanced modes may be negleted.

Bad idea: Simply delete those system parts with minor contribution.

Good idea: Find a coordinate transformation $T \cdot x_b = x$ that yields a system with diagonal Gramians.

- Calculate W_R, W_O by solving the Lyapunov equations.
- Find the coordinate transformation such that W_R, W_O =diag(w_i) and $W_R = W_O$.
- $$\begin{aligned} \tilde{A} &= T^{-1}AT \\ \tilde{B} &= T^{-1}B \\ \tilde{C} &= CT \\ \tilde{D} &= D \end{aligned}$$

Normalize the system in advance!

1. System partitioning Last ν elements σ_j are substantially smaller than the other first.

   $$\begin{aligned} \frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} &= \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} &+ \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \\ y(t) &= \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} &+ Du(t) \end{aligned}$$

   Where $x_1 \in \mathbb{R}^{n-\nu}$ and $x_2 \in \mathbb{R}^\nu$.
2. Reduction

   $$\begin{aligned} \tfrac{d}{dt}x_1(t) &= A_{11}x_1(t) &+ B_1u(t) \\ y(t) &= C_1x_1(t) &+ Du(t) \end{aligned}$$

   This yields good agreement in the frequency domain, but in general the DC gains and the reduced order system will be different!

12.1.1 Calculation of the DC-Gain

$$P(s) = C(sI - A)^{-1}B + D$$

$$\boxed{P(s = 0) = CA^{-1}B + D} \text{ DC-Gain}$$

12.2 Singular perturbation

Neglect the dynamics of the last ν states but not their DC contributions. → DC gain does not change.

$$\tfrac{d}{dt}x_2(t) = 0 \rightarrow x_2(t) = -A_{2,2}^{-1}[A_{2,1}x_1(t) + B_2u(t)]$$

$$\begin{aligned} \tfrac{d}{dt}x_1(t) &= [A_{1,1} - A_{1,2}A_{2,2}^{-1}A_{2,1}]x_1(t) &+ [B_1 - A_{1,2}A_{2,2}^{-1}B_2]u(t) \\ y(t) &= [C_1 - C_2A_{2,2}^{-1}A_{2,1}]x_1(t) &+ [D - C_2A_{2,2}^{-1}B_2]u(t) \end{aligned}$$

This is always possible if the original system was asymptotically stable.

12.3 Example: Equation of eliminated state

$\dot{\tilde{x}}_1 = -1.9\tilde{x}_1 - 0.06\tilde{x}_2 - 0.08\tilde{x}_3 + 0.41\tilde{u}_4$
$\frac{d}{dt}\tilde{x}_1 = 0 \Rightarrow \tilde{x}_1 = -0.03\tilde{x}_2 - 0.04\tilde{x}_3 + 0.22\tilde{u}$

13 Zero Dynamics

$$\begin{aligned} \tfrac{d}{dt}x(t) = \quad \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

$\boxed{P(s) = C[sI - A]^{-1} \cdot B + D}$ Transfer function

$$P(s) = \frac{Y(s)}{U(s)} = k\frac{s^{n-r} + b_{n-r-1}s^{n-r-1} + \dots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \dots + a_2 s^2 + a_1 s + a_0}$$

- The order of the highest power is n.
- Input gain: k
- The relative degree r

State-Space Representation

$$\frac{d}{dt}x(t) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & . & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ k \end{bmatrix} u(t)$$

$$y(t) \quad = \begin{bmatrix} b_0 & \cdots & b_{n-r-1} & 1 & 1 & 0 \cdots & 0 \end{bmatrix} x(t) = Cx(t)$$

This is the controller canonical form.

13.1 Zero Dynamics - Definition

The zero dynamics of a system correspond to its behaviour for special non-zero inputs $u^*(t)$ and initial conditions x^* for which $y(t)$ is identical to zero for i finite interval.

- Study the influence of zeros on the dynamics of the system.
- Study the internal dynamics / Analyse the stability of system states which are not directly controlled by the input.

The relative degree r is the **number of differentiations needed** to have the input $u(t)$ explicitly appear in the output $y^{(r)}(t)$

$$\begin{aligned} y(t) &= Cx(t) \\ y(t) &= C\dot{x}(t) = CAx(t) + CBu(t) = CAx(t) \\ &\vdots \\ y^{(r-1)}(t) &= CA^{r-1}x(t) + CA^{r-2}Bu(t) = CA^{r-1}x(t) \\ y^{(r)}(t) &= CA^r x(t) + ku(t) \end{aligned}$$

Then we transform coordinates (with Φ) as follows:

$$\begin{aligned} z_1 &= y &&= Cx \\ z_2 &= \dot{y} &&= CAx \\ \vdots &= &&= \\ z_r &= y^{(r-1)} &&= CA^{r-1}x \\ &= y^{(r)} &&= CA^r x + ku \end{aligned}$$

The remaining n-r coordinates are chosen such that the transformation Φ is regular and such that their derivatives dont depend on u.

$$\begin{aligned} z_{r+1} &= x_1 \\ z_{r+2} &= x_2 \\ &\vdots \\ z_n &= x_n - r \end{aligned}$$

Then the vector z is partitioned into subvectors:

$$z = \begin{bmatrix} \zeta \\ \eta \end{bmatrix}, \quad \zeta = \begin{bmatrix} z_1 \\ \vdots \\ z_r \end{bmatrix}, \quad \eta = \begin{bmatrix} z_{r+1} \\ \vdots \\ z_n \end{bmatrix}$$

New form of the system:

$$\begin{bmatrix} \dot{\zeta} \\ \dot{\eta} \end{bmatrix} = \left[\begin{array}{ccccc|ccccc} 0 & 1 & 0 & . & 0 & 0 & . & . & . & 0 \\ 0 & 0 & 1 & . & 0 & 0 & . & . & . & 0 \\ 0 & . & . & 0 & 1 & 0 & . & . & . & 0 \\ - & - & r^T & - & - & - & - & s^T & - & - \\ \hline 0 & . & . & . & 0 & 0 & 1 & 0 & . & 0 \\ 0 & . & . & . & 0 & 0 & 0 & 1 & . & 0 \\ 0 & . & . & . & 0 & 0 & . & . & 0 & 1 \\ - & - & p^T & - & - & - & - & q^T & -/- & \end{array}\right] \begin{bmatrix} \zeta \\ \eta \end{bmatrix} + \begin{bmatrix} 0 \\ . \\ 0 \\ \mathrm{k} \\ \hline 0 \\ . \\ . \\ 0 \end{bmatrix} u$$

and $y = z_1$

To achieve a vanishing output we choose:

$$\zeta^* = 0, \quad u^*(t) = -\frac{1}{k}s^T\eta^*(t)$$

Where $\eta_0^* \neq 0$ can be chosen arbitrarily.
Then the internal states (zero dynamic states) evolve as follows:

$$\frac{d}{dt}\eta(t) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ - & - & q^T & - & - \end{bmatrix} \eta^*(t) = Q\eta^*(t), \quad \eta^*(0) = \eta_0^*$$

13.2 Minimum Phase

If the matrix Q is asymptotically stable the system is **minimum phase**.
As soon as there is a zero with a positive real part:

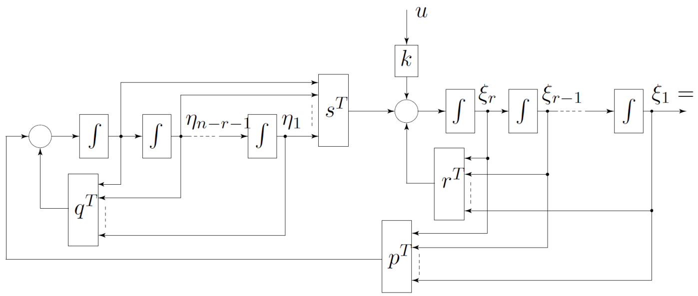
- The system is non-minimum phase.
- The system has unstable zero dynamics.
- its internal states η can diverge without $y(t)$ being affected.

Consequences:

- The input $u(t)$ may not be chosen such that $y(t)$ is (almost) zero before the states η are (almost) zero.
- Feedback control is more difficult to design.
- This imposes a constraint on the bandwidth of the closed-loop system: The controller must be significantly slower than the slowest non-minimumphase zero.

$$\begin{aligned} \dot{z}_n &= \dot{x}_{n-r} \\ &= x_{n-r+1} \\ &= z_1 - b_0 x_1 \ldots - b_{n-r-1}x_{n-r} \\ &= z_1 - b_0 z_{r+1} \ldots - b_{n-r-1}z_n \\ &= z_1 + q^T\eta \end{aligned}$$

Therefore the EV of Q coincide with the transmission zeros of the original system and with the roots of the numerator of its transfer function.



13.3 Summary of the procedure

1. Convert the plant's transfer function into a state-space controller canonical form.
2. Find r and do the coordinate transform sucht that $z = \Phi^{-1} \cdot x$
3. Find the transformation matrices Φ^{-1} and then compute Φ
4. Build a new state-space representation in $z = \begin{bmatrix} \zeta \\ \eta \end{bmatrix}$.
5. Study the submatrix Q of $\tilde{A} = \Phi^{-1} A \Phi$ corresponding to the zero-dynamics vector η.

13.4 Example: Small SISO System

1. Transfer function → state-space controller canonical form

$$P(s) = \frac{Y(s)}{U(s)} = k \frac{b_1 s + b_0}{a_3 s^3 + a_2 s^2 + a_1 s + a_0}$$

$$\begin{aligned} \frac{d}{dt} x(t) &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 \end{bmatrix} \cdot x(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ k \end{bmatrix} \cdot u(t) \\ y(t) &= \begin{bmatrix} b_0 & b_1 & 1 & 0 \end{bmatrix} \cdot x(t) + [0] \cdot u(t) \end{aligned}$$

2. Coordinate transformation
$r = 2 \Longrightarrow$

$$\begin{aligned} y(t) &= b_0 x_1(t) + b_1 x_2(t) + x_3(t) \\ \dot{y}(t) &= b_0 x_2(t) + b_1 x_3(t) + x_4(t) \\ \ddot{y}(t) & -a_0 x_1(t) - a_1 x_2(t) + (b_0 - a_2) x_3(t) + \\ & +(b_1 - a_3) x_4(t) + ku(t) \end{aligned}$$

The coordinate transform $z = \Phi^{-1} \cdot x$ has the form:

$$\begin{aligned} z_1 = y \quad &= b_0 x_1 + b_1 x_2 + x_3 \\ z_2 = \dot{y} \quad &= b_0 x_2 + b_1 x_3 + x_4 \\ z_3 = x_1 & \\ z_4 = x_2 & \end{aligned}$$

3. Find Φ^{-1} : $z = \Phi^{-1} \cdot x$ and then compute Φ
Alternatively solve $z = \Phi^{-1} \cdot x$ for $x_i(z_i)$ to get Φ.

$$\Phi^{-1} = \begin{bmatrix} b_0 & b_1 & 1 & 0 \\ 0 & b_0 & b_1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

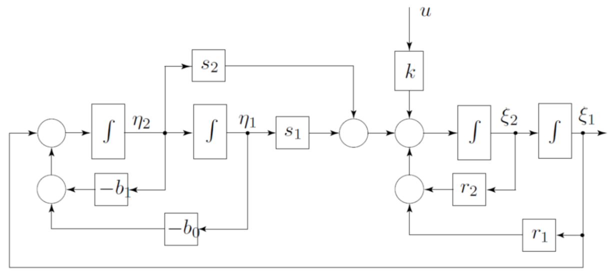
$$\Phi = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -b_0 & -b_1 \\ -b_1 & 1 & b_0 b_1 & b_1^2 - b_0 \end{bmatrix}$$

4. New state-space representation in $z = \begin{bmatrix} \zeta \\ \eta \end{bmatrix}$

$$\zeta = \begin{bmatrix} z_1 \\ z_1 \end{bmatrix}, \ \eta = \begin{bmatrix} z_3 \\ z_4 \end{bmatrix}$$

$$\begin{aligned} \frac{d}{dt} z(t) &= \Phi^{-1} A \Phi z(t) + \Phi^{-1} B u(t), \ y(t) = C \Phi z(t) \\ \frac{d}{dt} \begin{bmatrix} \zeta_1(t) \\ \zeta_2(t) \\ \zeta_3(t) \\ \zeta_4(t) \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ r_1 & r_2 & s_1 & s_2 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -b_0 & -b_1 \end{bmatrix} \cdot \begin{bmatrix} \zeta_1(t) \\ \zeta_2(t) \\ \zeta_3(t) \\ \zeta_4(t) \end{bmatrix} + \begin{bmatrix} 0 \\ k \\ 0 \\ 0 \end{bmatrix} \cdot u(t) \end{aligned}$$

$$\begin{aligned} r_1 &= b_0 - a_2 - b_1(b_1 - a_3) \\ r_2 &= b - 1 - a_3 \\ s_1 &= b_0 b_1 (b_1 - a_3) - a_0 - b_0(b_0 - a_2) \\ s_2 &= (b_1 - a_3)(b_1^2 - b_0) - a_1 - (b_0 - a_2) b - 1 \end{aligned}$$



5. Study Q of $\tilde{A} = \Phi^{-1} A \Phi$ corresponding to the zero dynamics vector η
Choosing $\zeta_1^*(0) = \zeta_2^*(0) = 0$ and $u^*(t) = -\frac{1}{k}[s_1 \eta_1^*(t) + s_2 \eta_2^*(t)]$ yields $y(t) = 0$.
$\eta_1^*(0) \neq 0$ and $\eta_2^*(0) \neq 0$ may be chosen arbitrarily.

$$\boxed{\frac{d}{dt} \eta^*(t) = \begin{bmatrix} 0 & 1 \\ -b_0 & -b_1 \end{bmatrix} \cdot \eta^*(t) = Q \cdot \eta^*(t)}$$

6. Conclude on the conditions to have Q asymptotically stable.

14 Nonlinear Systems

$$\frac{d}{dt}x(t) = f(x(t), u(t), t), \quad x(t_0) = x_0 \neq 0 \quad \text{Nonlinear differential equation}$$

$$x_e : f(x_e, t) = 0 \; \forall \; t \quad \text{Equilibrium}$$

The point x_e is **Uniformly Lyapunov Stable** if for each $R > 0$ there is $r(R) > 0$: $||x_0|| < r$ for which the corresponding solution satisfies: $||x(t)|| < R \; \forall \; t > t_0$

The same point is **asymptotically stable** if it is **ULS and attractive**: $\lim\limits_{t\to\infty} x(t) = x_e$.

A system is **exponentially asymptotically stable** if there exist constant scalars $a > 0$ and $b > 0$: $||x(t)|| \leq a \cdot e^{-b(t-t_0)} \cdot ||x_0||$
In general only an exponentially asymptotically stable equilibrium is acceptable for technical applications since it is robust with respect to modeling errors.

For linear systems an equilibrium set can be either one isolated point, entire subspaces or periodic orbits (same frequency but arbitrary amplitude).
Nonlinear systems can have (infinitely) many isolated equilibrium points. An equilibrium point can

- have a finite region of attraction
- be non-exponentially asymptotically stable
- be unstable $\Rightarrow$ the state of the system can „escape to infinity"in finite time

14.1 Stability of Nonlinear First-Order Systems

$$\frac{d}{dt}x(t) = f(x(t), u(t)), \; x, u \in \mathbb{R}$$

$$\boxed{\int \frac{dx}{f(x)} = \int dt = t + c}$$

Thus we can find $x(t)$ and assess stability as follows:

$$\frac{d}{dt}x(t) = -x^3(t) + u(t), \quad x(0) = x_0$$

$$x(t) = x_0 \cdot (2tx_0^2 + 1)^{-1/2}$$

But the solution approaches the equilibrium slower than exponentially:

$$||x(t)|| = ||x_0|| \cdot (2tx_0^2 + 1)^{-1/2} \leq a \cdot e^{-bt} \cdot ||x_0||$$

For $\lim t \to \infty$ this inequality is proved wrong, thus we have no exponential asymptotic stability.

14.2 Stability of Nonlinear Second-Order Systems

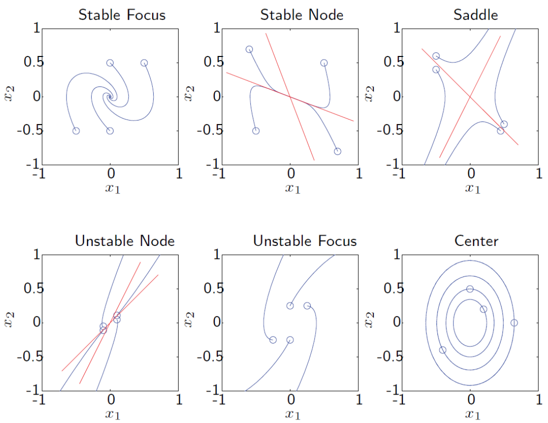
$$\begin{aligned} \tfrac{d}{dt}x_1(t) &= f_1(x_1, x_2), \; x_1(0) = x_{1,0} \\ \tfrac{d}{dt}x_2(t) &= f_2(x_1, x_2), \; x_2(0) = x_{2,0} \end{aligned}$$

Linearization yields the linear system:

$$\frac{d}{dt}\delta x(t) = A \cdot \delta x(t), \quad \delta x(t) = [\delta x_1(t), \delta x_2(t)]^T, \quad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Excluding the case where the matrix A has two eigenvalues with zero real part, the local behaviour of the nonlinear system and the linearized system are topologically equivalent.

eigenvalues	linearized system	nonlinear system
$\lambda_1 \in \mathbb{C}_-, \lambda_2 \in \mathbb{C}_-$	Stable Focus	Stable Focus
$\lambda_1 \in \mathbb{R}_-, \lambda_2 \in \mathbb{R}_-$	Stable Node	Stable Node
$\lambda_1 \in \mathbb{R}_+, \lambda_2 \in \mathbb{R}_-$	Saddle	Saddle
$\lambda_1 \in \mathbb{R}_+, \lambda_2 \in \mathbb{R}_+$	Unstable Node	Unstable Node
$\lambda_1 \in \mathbb{C}_+, \lambda_2 \in \mathbb{C}_+$	Unstable Focus	Unstable Focus
$\Re(\lambda_{1,2}) = 0$	Center	???



14.3 Example: Critical Nonlinear System

$$\begin{aligned} \tfrac{d}{dt}x_1 &= -x_1 + x_2 \\ \tfrac{d}{dt}x_2 &= x_2^3 \end{aligned}$$

- System has only one isolated equilibrium at $x_{e,1} = x_{e,2} = 0$
- Linearization has one EV at -1 and one at 0
- The solution of the linear system is stable (not asymptotically)
- The nonlinear system is unstable (has even finite escape times)

$$x_2(t) = \frac{x_{2,0}}{\sqrt{1 - 2tx_{2,0}^2}}$$

Where $\lim t \to 1/2x_{2,0}^2 \Rightarrow x_2(t) \to$ escapes to infinity.

14.4 Lyapunov Principle - General Systems

- The Lyapunov Principle is valid for all finite-order systems: as long as the linearized system has no eigenvalues on the imaginary axis.
- The local stability properties of an arbitrary-order nonlinear system are fully understood once the eigenvalues of the linearization are known.
- Particularly, if the linearization of a nonlinear system around an isolated equilibrium point x_e is asymptotically stable (unstable resp.) then this equilibrium is an asymptotically stable (unstable reps.) equilibrium of the nonlinear system.

14.5 Lyapunov Theory

Local stability properties of equilibrium $x = 0$ of the system

$$\dot{x}(t) = f(x(t)), \quad x(0) \neq 0$$

are fully described by

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0}$$

provided A has no eigenvalues with zero real part. Else use Lyapunovs direct method:

14.5.1 Definitions

A scalar function $\alpha(p)$ with $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is a *nondecreasing* function if $\alpha(0) = 0$ and $\alpha(p) \geq \alpha(q) \; \forall \; p > q$.
A function $V : \mathbb{R}^{n+1} \to \mathbb{R}$ is a candidate global Lyapunov function if:

- the function is strictly positive, i.e. $V(x,t) > 0 \; \forall \; x \neq 0, \; \forall \; t$ and $V(0) = 0$ and
- there are two nondecreasing functions α and β that satisfy the inequalities $\beta(||x||) \leq V(x,t) \leq \alpha(||x||)$.

If these conditions are met only in a neighborhood of the equilibrium point $x = 0$ only local assertions can be made.

14.5.2 Theorem 1

The system

$$\dot{x}(t) = f(x(t), t), \quad x(t_0) = x_0 \neq 0$$

is globally/locally stable in the sense of Lyapunov if there is a global/local Lyapunov function candidate $V(x,t)$ for which the following inequality hold true $\forall\ x(t) \neq 0$ and $\forall\ t$

$$\boxed{\dot{V}(x(t), t) = \frac{\partial V(x,t)}{\partial t} + \frac{\partial V(x,t)}{\partial x} f(x(t), t) \leq 0}$$

14.5.3 Theorem 2

The system

$$\dot{x}(t) = f(x(t), t), \quad x(t_0) = x_0 \neq 0$$

is globally/locally asymptotically stable if there is a global/local Lyapunov function candidate $V(x,t)$ such that $-\dot{V}(x(t), t)$ satisfies all conditions of a global/local Lyapunov function candidate.

14.5.4 Finding suitable candidate functions

Using physical insight, Lyapunov functions can be seen as generalized energy functions.
For a linear system:

$$V(x) = x^T P x$$

where $P = P^T > 0$ is the solution of the Lyapunov equation

$$PA + A^T P = -Q$$

For arbitrary $Q = Q^T > 0$ a solution to this equation exists iff A is a Hurwitz matrix.
Lyapunov theorems provide sufficient but not necessary conditions. Many extensions proposed (LaSalle, Hahn, ect.).

14.5.5 Example

$$\begin{aligned} \tfrac{d}{dt}x_1 &= x_1(x_1^2 + x_2^2 - 1) - x_2 \\ \tfrac{d}{dt}x_2 &= x_1 + x_2(x_1^2 + x_2^2 - 1) \end{aligned}$$

One isolated equilibrium at $x_1 = x_2 = 0$.
Linearizing the system around this equilibrium yields:

$$A = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$$

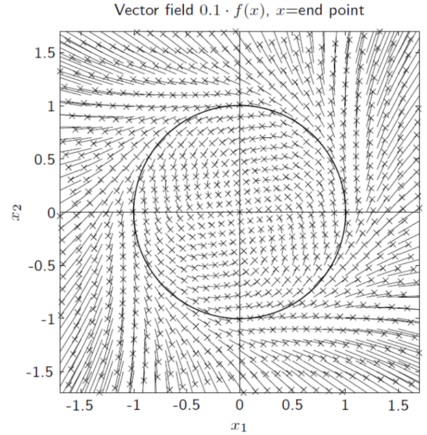
Eigenvalues are $\lambda_{1,2} = -1 \pm j \rightarrow$ the system is locally asymptotically stable (Lyapunov principle).

The system has a finite region of attraction. Use Lyapunov analysis to obtain a conservative estimation of the region of attraction.

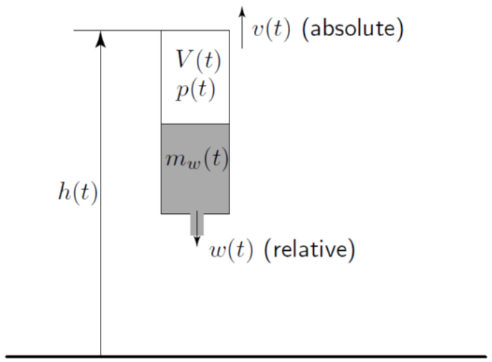
Candidate Lyapunov function: $V(x) = x_1^2 + x_2^2$

Time derivative along a trajectory:
$\dot{V}(t) = 2(x_1^2 + x_2^2)(x_1^2 + x_2^2 - 1)$

Therefore at least the region $||x||^2 = x_1^2 + x_2^2 < 1$ must be part of the region of attraction.



15 Example: Water-propelled rocket



$$V_W(0) = \frac{m_w(0)}{\rho_w} \qquad V_a(0) = V_l - V_w$$

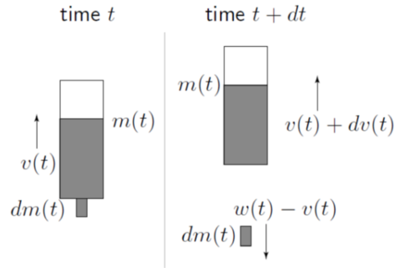
$0 < t < t_1$:	Lift force due to water jet.
$t_1 < t < t_2$:	Thrust due to pressurized air.
$t > t_2$:	Ballistic mode

A Hybrid System changes its dynamic behavior depending on discrete events.

Assumptions

- Only vertical motion is modelled.
- Only gravity and thrust forces are considered. No aerodynamic forces.
- Isentropic expansion of the air.
- $m_a \ll m_w$.
- The fluid flow through the nozzle is modelled with Bernoulli's law. (Incompressible fluid without friction)

15.1 Phase 1: Water-thrust $0 < t < t_1$

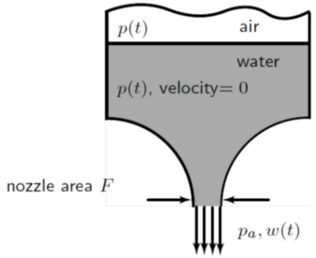


$$dB(t) = F_{ext}(t) \cdot dt = -g \cdot m(t) \cdot dt =$$
$$m(t) \cdot dv(t) - dm(t) \cdot w(t) \quad \text{Momentum conservation law}$$

$$\frac{dm(t)}{dt} = \overset{*}{m} = \rho \cdot F \cdot w(t) \quad \text{Water mass flow}$$

Dynamic equations:

$$\boxed{\begin{aligned} m(t) \cdot \frac{d}{dt} v(t) &= -g \cdot m(t) + \underbrace{\rho \cdot F \cdot w^2(t)}_{T_w} \\ \frac{d}{dt} m_R(t) &= -\rho \cdot F \cdot w(t) \end{aligned}}$$



$$\boxed{\frac{1}{2} \cdot \rho \cdot w^2(t) + p_a = p(t) \Rightarrow w(t) = \sqrt{\frac{2}{\rho}} \cdot \sqrt{p(t) - p_a}}$$

$$\boxed{V(t) = V_l - \frac{m_w(t)}{\rho} \qquad p(t) = \left(\frac{V(0)}{V(t)}\right)^{\kappa} \cdot p(0)}$$

15.2 Phase 2: Air-thrust, $t_1 < t < t_2$

$$m_{air}(t_1) = m_{air}(0)$$

$$dB(t) = m(t) \cdot dv(t) - \underbrace{dm(t) \cdot w(t)}_{=0}$$

$$m_R \cdot \frac{dv(t)}{dt} = -m_R \cdot g + \underbrace{\rho_{air} \cdot F \cdot w^2(t)}_{T_{air}}$$

$$\boxed{T_{air}(t) = \underbrace{\rho_{air} \cdot F \cdot w(t)}_{\overset{*}{m}_{air}(t)} \cdot w(t) = \overset{*}{m}_{air}(t) \cdot \frac{\overset{*}{m}_{air}(t)}{\rho_{air} \cdot F}}$$

$$\overset{*}{m}_{air}(t) = c_d \cdot F \frac{p_{in}(t)}{\sqrt{R \cdot \vartheta_{in}(t)}} \Psi(p_{in}(t), p_{out}(t))$$

$$\Psi(p_{in}(t), p_{out}(t)) = \begin{cases} \frac{1}{\sqrt{2}} & \text{for } 2p_{out} < p_{in} \\ \sqrt{\frac{2p_{out}}{p_{in}} \left[1 - \frac{p_{out}}{p_{in}}\right]} & \text{for } 2p_{out} \geq p_{in} \end{cases}$$

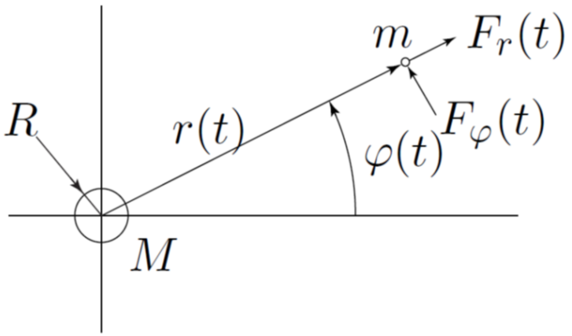
Where p_{in} pressure inside the rocket, p_{out} pressure of atmosphere, ϑ_{in} temperature of air in the rocket

$$\frac{d}{dt} p(t) = \frac{\kappa R}{V} \{-\overset{*}{m}_{out}(t) \vartheta(t)$$

Where V_l : total volume of the rocket $\vartheta(t)$ temperature of air in the rocket.

$$\frac{d}{dt}\vartheta = \frac{\vartheta(t)(t)R}{p(t) \cdot V_l c_v} \{-c_p \overset{*}{m}_{out}(t)\vartheta(t) + \overset{*}{m}_{out}(t)\vartheta(t)\} =$$
$$-\frac{\vartheta^2(t)R^2}{p(t)V_l c_v} \cdot \overset{*}{m}_{out}(t)$$

16 Example: Geostationary Satellite



R :	radius of the earth
M :	mass of the earth
m :	mass of the satellite
$r(t)$:	dist. Earth center to satellite
$\varphi(t)$:	orbit angle of the satellite
$F_r(t)$:	radial force
$F_\varphi(t)$:	tangential force

16.1 Assumptions

- No other celestial bodies considered.
- $M \gg m$ C.O.G. located at center of the Earth
- satellite always remains in the equatorial plane$\rightarrow$ 2 variables are sufficient to describe its position $r(t)$ and $\varphi(t)$
- the attitude (orientation) of the satellite is kept constant (by an inner control system) $\rightarrow F_r(t)$ and $F_\varphi(t)$ are independent

16.2 Modelling

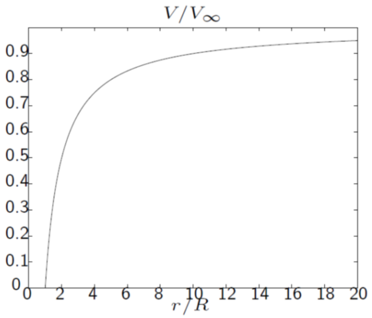
1.

Inputs:	radial force	$F_r(t)$
	tangential force	$F_\varphi(t)$
Outputs	Earth to satellite distance	$r(t)$
	orbit angle	$\varphi(t)$

2. Energies involved:
   Kinetic energy $T(r,\dot{r},\dot{\varphi}) = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}m(r\dot{\varphi})^2$
   Potential energy V: $\mathbf{F}_{grav}/r) = m\mathbf{G}_{Earth}(r) = m\frac{GM}{r^2}\mathbf{u}$
   Where $G = 6.673 \times 10^{-11}\,\mathrm{m^3\,s^{-2}\,kg^{-1}}$
   $M = 5.974\,10 \times 10^{24}\,\mathrm{kg}$
   $R = 6.367\,19 \times 10^6\,\mathrm{m}$

3. Energy needed to bring the satellite from R to r:

$$V(r) = \int_R^r F(\rho)d\rho = GMm\left(\frac{1}{R} - \frac{1}{r}\right) \qquad r > R$$



Where $V_\infty = GMm(\frac{1}{R}$.

4. Mimimum Speed to reach orbit:

$$\frac{1}{2}mv_0^2 = GMm\left(\frac{1}{R} - \frac{1}{r}\right) = (\Delta V_{grav})_{R\rightarrow r} \quad \text{energy balance}$$

$$\boxed{v_0(r) = \sqrt{2GM\left(\frac{1}{R} - \frac{1}{r}\right)}}$$

Where the escape velocity is $v_0(r \rightarrow \infty) = \sqrt{\frac{2GM}{R}} = 11.2\,\mathrm{km\,s^{-1}}$
It is the velocity that is required to completely leave the influence of the gravitational field of the earth.

5. Lagrange formalism

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{r}}\right] - \frac{\partial L}{\partial r} = F_r$$
$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{\varphi}}\right] - \frac{\partial L}{\partial \varphi} = F_\varphi r$$

$$L = T - V = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}m(r\dot{\varphi})^2 - GMm\left(\frac{1}{R} - \frac{1}{r}\right)$$

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r} \qquad \frac{\partial L}{\partial \dot{\varphi}} = mr^2\dot{\varphi}$$
$$ddt\left(\frac{\partial L}{\partial \dot{r}}\right) = m\ddot{r} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) = mr^2\ddot{\varphi} + 2mr\dot{\varphi}\dot{r}$$
$$\frac{\partial L}{\partial r} = mr\dot{\varphi}^2 - GMm\frac{1}{r^2} \qquad \frac{\partial L}{\partial \varphi} = 0$$

$$\boxed{\begin{aligned} m\ddot{r} &= mr\dot{\varphi}^2 - GMm\tfrac{1}{r^2} + F_r \\ mr^2\ddot{\varphi} &= -2mr\dot{\varphi}\dot{r} + F_\varphi r \end{aligned}}$$

Control accelerations: $u_r = \frac{F_r}{m}$ and $u_\varphi = \frac{F_\varphi}{m}$

$$\boxed{\begin{aligned} \ddot{r} &= r\dot{\varphi}^2 - GM\tfrac{1}{r^2} + u_r \\ \ddot{\varphi} &= -2\dot{\varphi}\dot{r}\tfrac{1}{r} + \tfrac{1}{r}u_\varphi \end{aligned}}$$

Geostationary Conditions:

$$u_r = 0, \quad \ddot{r} = 0, \quad \dot{r} = 0, \quad r = r_0$$
$$u_\varphi = 0, \quad \ddot{\varphi} = 0, \quad \dot{\varphi} = \omega_0, \quad \varphi = \omega_0 t$$

$$\boxed{\omega_0 = \frac{2\pi}{day} = \omega_0 = 7.2910 \times 10^{-5}\,\mathrm{rad\,s^{-1}}}$$

Where 1 sidereal day = 23h 56 min 4.1s

$$\boxed{r_0 = \left(\frac{GM}{\omega_0^2}\right)^{1/3} \approx 4.22 \times 10^7\,\mathrm{m}}$$

- r_0 is approximately 6.2 times the radius of the earth.
- The energy required is more than 80% of the escape energy.
- The resulting tangential speed is $v_\varphi = r_0\omega_0 \approx 10\,800\,\mathrm{km\,h^{-1}}$

6. State space formulation

$$x_1(t) = r,\ x_2(t) = \dot{r},\ u_1(t) = u_r$$
$$x_3(t) = \varphi,\ x_4(t) = \dot{\varphi},\ u_2(t) = u_\varphi$$

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} \qquad u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

$$\frac{d}{dt}x(t) = f(x(t), u(t))$$

$$f(t) = \begin{bmatrix} x_2(t) \\ x_1x_4^2(t) - GM/x_1^2(t) + u_1(t) \\ x_4(t) \\ -2x_2(t)x_4(t)/x_1(t) + u_2(t)/x_1(t) \end{bmatrix}$$

$$y(t) = h(x(t)) = \begin{bmatrix} x_1(t)/r_0 \\ x_3(t) \end{bmatrix}$$

Linearization around $x_0(t) \begin{bmatrix} r_0 \\ 0 \\ \omega_0 t \\ \omega_0 \end{bmatrix} \qquad u_0(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Thus we end up with:

$$\left[\begin{array}{c|c} \mathrm{A} & \mathrm{b} \\ \hline \mathrm{c} & \mathrm{d} \end{array}\right] = \left[\begin{array}{cccc|cc} 0 & 1 & 0 & 0 & 0 & 0 \\ 3\omega_0^2 & 0 & 0 & 2r_0\omega_0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -2\omega_0/r_0 & 0 & 0 & 0 & 1/r_0 \\ \hline 1/r_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right]$$

7. System analysis

   - The system is completly controllable and observable if all sensors and actuators function.
   - If the radial thruster fails the satellite remains completely controllable.
   - If the tangential thruster fails the satellite is no longer completely controllable.
   - If the radial sensor fails the satellite remains completely observable.
   - If the tangential sensor fails the satellite is no longer completely observable.