# System Identification

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## Contents

# 1 System Identification

# 2 Definitions

**Definition 1.** *A system is said to be* ***time invariant*** *if the response to a certain input is not depending on absolute time.*

**Definition 2.** *A system is said to be* ***linear*** *if its output response to a linear combination of inputs is the same as the linear combination of the output responses of the individual inputs.*

**Definition 3.** *A system is said to be* ***causal*** *if the output at a certain time depends on the input up to that time only.*

**Definition 4.** *A process is said to be* ***stationary*** *if it does not depend on time.*

# 3 Frequency Domain Methods

## 3.1 Sampling Operation

$\boxed{y(k) = y(t)|_{t=kT,k=0,1,2,\ldots}}$ Sampling with period $T$

## 3.2 Fourier Series of Periodic Signals

$$\boxed{X(e^{j\omega_m}) = \sum_{k=0}^{M-1} x(k)e^{-j\omega_m k}}$$

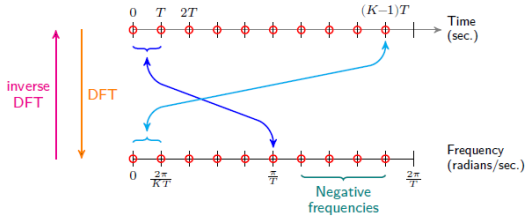
$$\omega_m = \frac{2\pi m}{M} = \omega_0$$

Non-negative frequencies are $m = 0$ to $m = M/s$.
They correspond to: $\omega_m = 0, \frac{2\pi}{M}, \frac{4\pi}{M}, \ldots, \frac{2\pi(M/2-1)}{M}, \pi$.

| | | |
|---|---|---|
| $M$ | number of samples | |
| $\omega_0 = \frac{2\pi}{M}$ | fundamental frequency ($y(k)$) | [rad] |
| $T$ | sampling time | [s] |
| $\tau_p = MT$ | period | [s] |
| $\omega_0 = \frac{2\pi}{\tau_p}$ | fundamental frequency ($y(t)$) | [rad s$^{-1}$] |

$$0, \underbrace{\frac{2\pi}{\tau_p}}_{\text{Fundamental frequency}}, \underbrace{2\left(\frac{2\pi}{\tau_p}\right), \ldots, \frac{M}{2}\left(\frac{2\pi}{\tau_p}\right)}_{\text{Harmonics}}$$

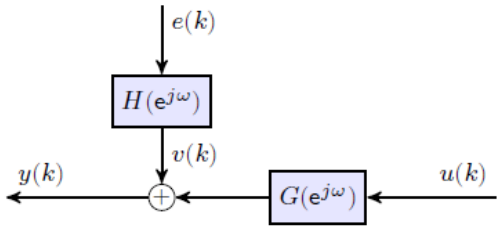
**Definition 5.** *The highest frequency $\omega_u = \omega_{M/2} = \frac{\pi}{T}$ is called the* ***Nyquist frequency****.*




```
%Non negative frequency vector
omega = omega_n(N,'p0');
%Discrete Fourier Transform
fft(u); %Matlab
DFT(u); %Lecture
```

Note that the definition of the fft in the Matlab documentation does not describe the actual implementation perfectly. The index actually runs from 0 to $N-1$ as in the definition made in class.

# 4 Spectral Estimation




$Y(j\omega) = G(j\omega)U(j\omega)$ Transfer function

$Y(e^{j\omega} = G(e^{j\omega}U(e^{j\omega})$ Discrete time TF

$$\frac{\omega_u}{2\pi} = \frac{r}{NT}$$

| | | |
|---|---|---|
| $\omega_u$ | input frequency | [rad s$^{-1}$] |
| $N$ | calculation length | [ ] |
| $T$ | experiment duration? | [s] |
| $r$ | some integer | [ ] |

$u(k) = \alpha\cos(\omega_u k),\ k = 0, 1, \ldots, K-1$ with $K \geq N$ Input

$y(k) = \alpha\left|G(e^{j\omega_u})\right|\cos(\omega_u k + \theta(\omega_u)) + v(k) + \text{transient}$ Output

where $\theta(\omega_u) = arg(G(e^{j\omega_u})$

### 4.1 Sinusoidal correlation methods

Correlation functions:

$$I_c(N) = \frac{1}{N}\sum_{k=0}^{N-1} y(k)\cos(\omega_u k)$$

$$I_s(N) = \frac{1}{N}\sum_{k=0}^{N-1} y(k)\sin(\omega_u k)$$

To calculate those from the data:

$$I_c(N) = \frac{\alpha}{2}\left|G(e^{j\omega_u})\right|\cos(\theta(\omega_u) + \frac{\alpha}{2}\left|G(e^{j\omega_u})\right|\frac{1}{N}\sum_{k=0}^{N-1}\cos(2\omega_u k + \theta(\omega_u)) + \frac{1}{N}\sum_{k=0}^{N-1} v(k)\cos(\omega_u k)$$

If the noise, $v(k)$ is sufficiently uncorrelated then the variance satisfies,

$$\lim_{N\to\infty} \text{var}\left\{\frac{1}{N}\sum_{k=0}^{N-1} v(k)\cos(\omega_u k)\right\} = 0$$

with a convergence rate of $1/N$.
Thus in the limit $N \to \infty$,

$$E\{I_c(N)\} \to \frac{\alpha}{2}\left|G(e^{j\omega_u})\right|\cos(\theta(\omega_u))$$
$$E\{I_s(N)\} \to -\frac{\alpha}{2}\left|G(e^{j\omega_u})\right|\sin(\theta(\omega_u))$$

and since $\lim_{N\to\infty} \text{var}\{I_c(N)\} = 0$, $\lim_{N\to\infty} \text{var}\{I_s(N)\} = 0$
The transfer function can be estimated via:

$$\boxed{\hat{G}_N(e^{j\omega_u}) = \frac{I_c(N) - jI_s(N)}{\alpha/2}}$$

- Advantages
  - Energy is concentrated at the frequencies of interest.
  - Amplitude of $u(k)$ can easily be tuned as a function of frequency.
  - Easy to avoid saturation and tune signal/noise (S/N) ratio.
- Disadvantages
  - A large amount of data is required.
  - Significant amount of time required for experiments.
  - Some processes won't allow sinusoidal inputs.

# 5 Frequency Domain Methods

$$x(k),\ k = -\infty, \ldots, \infty \quad \text{Discrete-time domain signal}$$

$$\boxed{X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x(k)e^{-j\omega k}} \text{ Fourier Transform}$$

- $X(e^{j\omega})$ is $2\pi$ periodic.
- If $\sum_{k=-\infty}^{\infty} |x(k)| < \infty$ then $X(e^{j\omega})$ converges.

$$\boxed{x(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega k}d\omega} \text{ Inverse Fourier Transform}$$

where $k = -\infty, \ldots, \infty$

### 5.1 Finite Energy Signal

#### 5.1.1 Energy Spectral Density (Finite Energy Signal)

If $x(k)$ is a finite energy signal,

$$||x(k)||_2^2 = \sum_{k=-\infty}^{\infty} |x(k)|^2 < \infty$$

$$\boxed{S_x(e^{j\omega}) = |X(e^{j\omega})|^2} \text{ Energy Spectral Density}$$

- For finite energy signals the energy spectral density is easily calculated, for signals of infinite length and thus infinite energy however, this is not possible. For that reason the **power** spectral density is calculated instead!

#### 5.1.2 Autocorrelation (Finite Energy Signal)

$$\boxed{R_x(\tau) = \sum_{k=-\infty}^{\infty} x(k)x(k-\tau), \quad \tau = -\infty, \ldots, 0, \ldots, \infty}$$

The spectral density is the Fourier Transform of the autocorrelation:

$$\sum_{\tau=-\infty}^{\infty} R_x(\tau)e^{-j\omega\tau} = S_x(e^{j\omega})$$

```
% autocorrelation for a non-periodic, finite energy signal
Correlation('finen',u); %Lecture
xcorr(u); %Matlab

% crosscorrelation for non-periodic, finite energy signals
Correlation('finen',u,y); %Lecture
xcorr(y,u); %Matlab
```

## 5.2 Discrete Periodic Signal

$$x(k) = x(k+M), \quad \forall\, k \in \{-\infty, \infty\} \quad \text{Periodic signal}$$

$$\omega_0 = \tfrac{2\pi}{M} \quad \text{Fundamental frequency}$$

- There are only $M$ unique harmonics of the sinusoid $e^{j\omega_0}$.
- The non-negative harmonic frequencies are,

$$e^{jn\omega_0},\ n = 0, 1, \ldots, M/2$$

```
u_period = sqrt(variance)*randn(N,1)+bias;
u = repmat(u_period,periods,1);
```

### 5.2.1 Discrete Fourier Series (Discrete Periodic Signal)

$$\boxed{X(e^{j\omega_n}) = \sum_{k=0}^{N-1} x(k)e^{-j\omega_n k},\ \text{where } \omega_n = \tfrac{2\pi n}{N} = n\omega_0}$$

$$\boxed{x(k) = \tfrac{1}{N}\sum_{k=0}^{N-1} X(e^{j\omega_n})e^{j\omega_n k}} \quad \text{Inverse Transform}$$

### 5.2.2 Autocorrelation (Discrete Periodic Signal)

$$\boxed{R_x(\tau) = \tfrac{1}{N}\sum_{k=0}^{N-1} x(k)x(k-\tau)}$$

The Fourier transform of $R_x(\tau)$ is now defined as the **power spectral density**, since it is normalized with the signal length.

$$\boxed{\phi_x(e^{j\omega_n}) = \sum_{\tau=0}^{N-1} R_x(\tau)e^{-j\omega_n\tau} = \tfrac{1}{N}|X(e^{j\omega_n})|^2}$$

The energy in a single period is:

$$\sum_{k=0}^{N-1}|x(k)|^2 = \sum_{n=0}^{N-1}\phi_x(e^{j\omega_n})$$

```
Correlation('periodic',u);
```

### 5.2.3 Cross-Correlation (Discrete Periodic Signal)

$$\boxed{R_{yu}(\tau) = \tfrac{1}{N}\sum_{k=0}^{N-1} y(k)u(k-\tau)}$$

The Fourier transform of $R_{yu}(\tau)$ is now defined as the **cross-spectral density**.

$$\boxed{\phi_{yu}(e^{j\omega_n}) = \sum_{\tau=0}^{N-1} R_{yu}(\tau)e^{-j\omega_n\tau}) = \tfrac{1}{N}Y(e^{j\omega_n})U^*(e^{j\omega_n})}$$

```
Correlation('periodic',u,y);
```

## 5.3 Random Signal

Normally distributed noise:

$$\boxed{e(k) \in \mathcal{N}(0,\lambda) \Rightarrow \begin{cases}\mathbb{E}\left[e(k)\right] = 0 \text{ (zero mean)}\\ \mathbb{E}\left[|e(k)|^2\right] = \lambda \text{ (variance)}\end{cases}}$$

The $e(k)$ are independent and identically distributed (i.i.d.).

```
standard_deviation = 2;
variance = standard_deviation^2;
bias = 0;
N = 1024;
u = randn(N,1)*standard_deviation + bias;
```

### 5.3.1 Autocovariance (Random Signal)

$$\begin{aligned} R_x(\tau) &= \mathbb{E}\left[x(k)x(k-\tau)\right] \\ &= \mathbb{E}\left[x(k)x^*(k-\tau)\right] \text{ (in the complex case)} \\ &= \mathbb{E}\left[x(k)x^*(x-\tau)\right] \text{ (in the multivariable case)} \end{aligned}$$

General (non-stationary, non-zero mean) case:

$$\begin{aligned} R_x(s,t) &= \mathbb{E}\left[(x(s) - \mathbb{E}\left[x\right])(x(t) - \mathbb{E}\left[E\right])\right] \\ &= \mathbb{E}\left[x(s)x(t)\right] \text{ (if zero mean)} \\ &= R_x(s-t) \text{ (if stationary)} \end{aligned}$$

Further properties are

- $R_x(-\tau) = R_x^*(\tau)$
- $R_x(0) \geq |R_x(\tau)| \; \forall \tau > 0$

```
xcorr(u)/N; %Lecture 3.37
```

### 5.3.2 Power Spectral Density (Random Signal)

$$\boxed{\phi_x(e^{j\omega}) := \sum_{\tau=-\infty}^{\infty} R_x(\tau)e^{-j\omega\tau} \text{ where } \omega \in [-\pi, \pi)}$$

For a zero-mean random signal:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} |x(k)|^2 = \text{Var}\,(x(k)) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi_x(e^{j\omega}) d\omega$$

Further properties are

- $\phi_x(e^{j\omega}) \in \mathbb{R}$
- $\phi_x(e^{j\omega}) \geq 0 \; \forall \; \omega$
- $\phi_x(e^{j\omega}) = \phi_x(e^{-j\omega})$ for all real-valued $x(k)$

```
fft(xcorr(u)/N)) %Lecture 3.37 (?)
```

### 5.3.3 Cross-Covariance (Random Signal)

$$\boxed{R_{yu}(\tau) = \mathbb{E}\left[(y(k) - \mathbb{E}\left[y(k)\right])(u(k-\tau) - \mathbb{E}\left[u(k)\right]\right]}$$

For zero mean signals:

$$R_{yu}(\tau) = \mathbb{E}\left[y(k)u(k-\tau)\right]$$

Joint stationarity is required to make the definition dependent on $\tau$ only.
If $R_{yu}(\tau) = 0$ for all $\tau$ then $y(k)$ and $u(k)$ are uncorrelated.

```
xcorr(u,y)/N; %Lecture 3.37 (?)
```

### 5.3.4 Cross Power Spectral Density (Random Signal)

$$\boxed{\phi_{yu}(e^{j\omega}) = \sum_{\tau=-\infty}^{\infty} R_{yu}(\tau)e^{-j\omega\tau}, \; \omega \in [-\pi, \pi)}$$

The inverse is,

$$R_{yu}(\tau) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi_{yu}(e^{j\omega})e^{j\omega\tau} d\omega$$

```
fft(xcorr(y,u)/N); %Lecture 3.37 (?)
```

## 5.4 Finite Length Signal

### 5.4.1 Discrete-Fourier Transform (Finite Length Signal)

$$\boxed{X_N(e^{j\omega_n}) = \sum_{k=0}^{N-1} x(k)e^{-j\omega_n k}, \text{ where } \omega_n = \frac{2\pi n}{N}}$$

The inverse DFT is

$$x(k) = \frac{1}{N} \sum_{n=0}^{N-1} X_N(e^{j\omega_n})e^{j\omega_n k}, \quad k = 0, \ldots, N-1$$

### 5.4.2 Periodogram (Finite Length Signal)

$$\boxed{\frac{1}{N} \left|V_N(e^{j\omega})\right|^2}$$

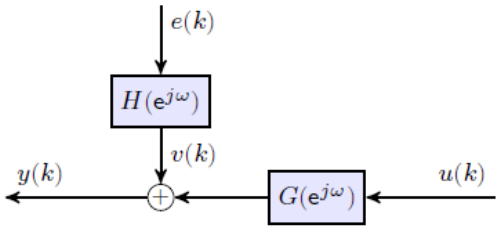
An asymptotically unbiased estimator of the spectrum is

$$\lim_{N\to\infty} \mathbb{E}\left[\frac{1}{N}|V_N(e^{j\omega}|^2\right] = \phi_v(\omega)$$

This assumes that the autocorrelation decays quickly enough:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{\tau=-N}^{N} |\tau R_v(\tau)| = 0$$

# 6 ETFE




Linear, time-invariant system, $g(l)$:

$$y(k) = \sum_{l=0}^{\infty} g(l)u(k-l) + v(k), \quad k = 0, 1, \ldots$$

Assumptions:

1. causal system: $g(l) = 0, \forall l < 0$
2. noise: $E\{v(k)\} = 0$, zero mean, stationary

Given $\{u(k), y(k)\}$ find an estimate $\hat{G}(e^{j\omega})$ such that it fits the $G(e^{j\omega})$.

$$\boxed{\text{Bias}(\hat{G}))G - E\{\hat{G}\}} \text{ Bias}$$

$$\boxed{\text{var}((\hat{G}) = E\left\{|\hat{G} - E\{\hat{G}\}|^2\right\}} \text{ Variance}$$

$$\boxed{\text{MSE}(\hat{G}) = E\left\{|G - \hat{G}|^2\right\}} \text{ Mean-square error}$$

Note that MSE($\hat{G}$) =var($\hat{G}$)+Bias$^2$($\hat{G}$).

## 6.1 Input-output relationship

For finite energy signals:

$$y(k) = \sum_{l=0}^{\infty} g(l)u(k-l) + v(k)$$

$$Y(e^{j\omega}) = G(e^{j\omega})U(e^{j\omega}) + V(e^{j\omega})$$

which in the idealized case leads to:

$$\boxed{\frac{Y(e^{j\omega})}{U(e^{j\omega})} = G(e^{j\omega}) + \frac{V(e^{j}\omega)}{U(e^{j\omega})} \approx G(e^{j\omega})}$$

In reality we only have $N$ samples:

$$\underbrace{Y_N(e^{j\omega_n})}_{\text{length-N DFT}} = \sum_{k=0}^{N-1} y(k)e^{-j\omega_n k} \approx \sum_{k=-\infty}^{\infty} y(k)e^{-j\omega_n k} = Y(e^{j\omega_n}$$

$$\underbrace{U_N(e^{j\omega_n})}_{\text{length-N DFT}} = \sum_{k=0}^{N-1} u(k)e^{-j\omega_n k} \approx \sum_{k=-\infty}^{\infty} u(k)e^{-j\omega_n k} = U(e^{j\omega_n})$$

$$\boxed{\hat{G}_N(e^{j\omega_n}) := \frac{Y_N(e^{j\omega_n})}{U_N(e^{j\omega_n})}} \text{ ETFE}$$

## 6.2 Periodic input case

Period $M$ inputs: $u(k) = u(k+M)$
If $sM = N$ for an integer $s$, the fourier series over N samples is equal to the real fourier series!

$$U_N(e^{j\omega_n}) = U(e^{j\omega_n}) \forall \omega_n = \frac{2\pi n}{N}, \; n = 0, \ldots, N-1$$

Then

$$Y_N(e^{j\omega_n}) = G(e^{j\omega_n})U_N(e^{j\omega_n}) + V_N(e^{j\omega_n})$$

$$\hat{G}_N(e^{j\omega_n}) = G(e^{j\omega_n}) + \frac{V_N(e^{j\omega_n})}{U_N(e^{j\omega_n})}$$

**Bias:**

$$E\{\hat{G}_N(e^{j\omega_n})\} = G(e^{j\omega_n}) + E\left\{\frac{V_N(e^{j\omega_n})}{U_N(e^{j\omega_n})}\right\} = G(e^{j\omega_n})$$

when assuming zero mean noise. Thus for periodic inputs with $N$ being an integer number of periods, **the ETFE is unbiased**.
**Variance:**
For the unbiased case:

$$E\left\{|\hat{G}_N(e^{j\omega_n}) - G(e^{j\omega_n})|^2\right\} = \frac{\phi_v(e^{j\omega_n}) + \frac{2}{N}c}{\frac{1}{N}|U_N(e^{j\omega_n})|^2}$$

where $|c| \leq C = \sum_{\tau=1}^{\infty} |\tau R_v(\tau)|$ is assumed to be finite.
For estimates at different frequencies ($\omega_n \neq \omega_i$):

$$E\left\{(\hat{G}_N(e^{j\omega_n}) - G(e^{j\omega_n}))(\hat{G}_N(e^{-j\omega_i}) - G(e^{-j\omega_i}))\right\} = 0$$

**Transient responses:**
Initial transient corrupts the measurement

$$y(k) = G(u_{periodic}(k)W_{[0,N-1]}(k)) + v(k)$$

with the window function:

$$W_{[0,N-1]}(k) = \begin{cases} 1 & \text{if } 0 \leq 0 < N \\ 0 & \text{otherwise} \end{cases}$$

For all outputs up to time $k = N - 1$

$$y(k) = Gu_{periodic}(k) - \underbrace{G(u_{periodic}W_{(-\infty,-1)})}_{r(k)} + v(k)$$

$$Y_N(e^{j\omega_n}) = G(e^{j\omega_n})U_N(e^{j\omega_n}) + R_N(e^{j\omega_n}) + V_N(e^{j\omega_n})$$

The input in negative time, which is present in a ideal periodic input, and missing in a real periodic input, has an influence on positive time, which is described by $r(k)$.

When using a periodic signal multiple times the resulting DFT does not contain more information, since in a periodic signal there are only a certain number of frequencies contained, but the energy in those frequencies increases!

**Transient bias error:**

$$\hat{G}(e^{j\omega_n}) = \frac{Y_N(e^{j\omega_n})}{U_N(e^{j\omega_n})} = G(e^{j\omega_n}) + \frac{R_N(e^{j\omega_n})}{U_N(e^{j\omega_n})} + \frac{V_N(e^{j\omega_n})}{U_N(e^{j\omega_n})}$$

**For periodic** $u(k)$
As $N = mM, m \to \infty$

$$|U_N(e^{j\omega_n})| = m|U_M(e^{j\omega_n})|$$

**For random** $u(k)$
As $N \to \infty$

$$E\{|U_N(e^{j\omega_n})|\} \to \sqrt{N}\sqrt{\phi_u(e^{j\omega_n})}$$

Thus

$$\boxed{\left|\frac{R_N(e^{j\omega_n})}{U_N(e^{j\omega_n})}\right| \to 0 \text{ with rate } \begin{cases} \frac{1}{N} & \text{for periodic input} \\ \frac{1}{\sqrt{N}} & \text{for random inputs} \end{cases}}$$

A fix for getting rid of the influence of the transient response: Get rid of the first period.

```
r = 4;
N_split = N/r;
u_re = reshape(u(N_split+1:end),N_split,[]);
y_re = reshape(y(N_split+1:end),N_split,[]);

U_re = fft(u_re);
Y_re = fft(y_re);

G_re = Y_re./U_re;

G_avg = mean(G_re,2);
```

## 6.3 Spectral Transformations

If $v(k) = 0$

$$\phi_y(e^{j\omega_n}) = G(e^{j\omega_n})\phi_u(e^{j\omega_n})G^T(e^{j\omega_n})$$

where $G^T(e^{j\omega_n})$ is the complex conjugate of $G(e^{j\omega_n})$.
If $v(k) \neq 0$ and uncorrelated

$$\phi_y(e^{j\omega_n}) = |G(e^{j\omega_n})|^2\phi_u(e^{j\omega_n}) + |H(e^{j\omega_n})|^2$$

But this approach has no more phase information. For that reason use the cross spectrum:

$$\phi_{yu}(e^{j\omega_n}) = G(e^{j\omega_n})\phi_u(e^{j\omega_n}) + \phi_{uv}(e^{j\omega_n}) = G(e^{j\omega_n})\phi_u(e^{j\omega_n})$$

if $u(k)$ and $v(k)$ are uncorrelated.

$$\boxed{\hat{G}(e^{j\omega}) = \frac{\hat{\phi}_{yu}(e^{j\omega_n})}{\hat{\phi}_u(e^{j\omega_n})}} \text{ Spectral estimation methods}$$

where

$$\phi_y(e^{j\omega_n}) = |G(e^{j\omega_n})|^2\phi_u(e^{j\omega_n}) + \phi_v(e^{j\omega_n})$$

$$\phi_{yu}(e^{j\omega_n}) = G(e^{j\omega_n})\phi_u(e^{j\omega_n})$$

## 6.4 Approaches to spectral estimation

### 6.4.1 Spectral estimation (Periodogram)

The periodogram is an asymptotically unbiased estimator of the spectrum given $\lim_{n\to\infty} \frac{1}{N} \sum_{\tau=-N}^{N} |\tau R_v(\tau)| = 0$

$$\frac{1}{N}|V_N(e^{j\omega_n})|^2 \quad \text{Periodogram}$$

$$\boxed{\lim_{N\to\infty} E\left\{\frac{1}{N}|V_N(e^{j\omega_n})|^2\right\} = \phi_v(e^{j\omega_n})}$$

which is under the assumption

$$\lim_{N\to\infty} \frac{1}{N} \sum_{\tau=-N}^{N} |\tau R_v(\tau)| = 0$$

### 6.4.2 Spectral estimation (via covariances)

The autocovariance of the noise for stochastic $v(k)$ is described as:

$$\hat{R}_v(\tau) = \begin{cases} \frac{1}{N-|\tau|} \sum_{k=\tau}^{N_1} v(k)v(k-\tau), & \text{for } \tau \geq 0 \\ \frac{1}{N-|\tau|} \sum_{k=0}^{N+\tau-1} v(k)v(k-\tau), & \text{for } \tau < 0 \end{cases}$$

This is an unbiased estimator of $R_v(\tau)$: $\mathrm{E}\left\{\hat{R}_v(\tau)\right\} = R_v(\tau)$

$$\hat{\phi}_v(e^{j\omega_n}) = \sum_{\tau=-N+1}^{N-1} \hat{R}_v(\tau)e^{-j\omega\tau}$$

```
xcorr(a-mean(a))-xcov(a) == 0 %true
```

The functions above both calculate the zero-mean shifted autocorrelation of a signal which is **not** equivalent to the autocovariance of the signal! They lack the normalization by $N - |\tau|$!

```
Covariance('zero-mean',a);
Covariance('zero-mean',a,b);
```

### 6.4.3 Spectral estimation (periodic signals)

Periodic signal $x(k)$ with period $M$, $N = mM$ for some integer $m$

$$R_x(\tau) = \frac{1}{M} \sum_{k=0}^{M-1} x(k)x(k-\tau)$$

The power spectral density can be calculated and is equal to the periodogram:

$$\phi_x(e^{j\omega_n}) = \sum_{\tau=0}^{M-1} R_x(\tau)e^{-j\omega_n\tau} = \frac{1}{M}\left|X_M(e^{j\omega_n})\right|^2$$

```
Ru = Correlation('periodic',u);
```

### 6.4.4 Spectral estimation (more general case)

Alternative autocorrelation estimate:

$$\hat{R}_x(\tau) = \begin{cases} \frac{1}{N} \sum_{k=\tau}^{N-1} x(k)x(k-\tau), & \text{for } \tau \geq 0 \\ \frac{1}{N} \sum_{k=0}^{N+\tau-1} x(k)x(k-\tau), & \text{for } \tau < 0 \end{cases}$$

Periodic $x(k)$: unbiased (exact) if $N = mM$

Random $x(k)$ biased $\mathrm{E}\left\{\hat{R}_x(\tau)\right\} = \frac{N-|\tau|}{N}R_x(\tau)$.
asymptotically biased as $N \to \infty, \tau/N \to 0$

```
Ru = xcorr(u)/N;
```

- The bias when estimating $R_x$ for a random signal has the form a triangular weighting across $\tau$. A scaling of the signal would remove the bias, which is effectively done by the spectral estimation for via covariances 6.4.2.

# 7 Averaging and Smoothing

Multiple experiments $u_r(k), y_r(k), r = 1\ldots, R, k = 0, \ldots, K-1$

$$\hat{G}(e^{j\omega_n}) = \sum_{r=1}^{R} \alpha_r \hat{G}_r(e^{j\omega_n})$$

where $\sum_{r=1}^{G} \alpha_r = 1$ and for calculating the average $\alpha_r = \frac{1}{R}$.

The averaging can be optimized by selecting $\alpha_r$ such that the variance $\sigma_r^2(e^{j\omega_n})$ is minimized.

$$\mathrm{Var}\left(\hat{G}(e^{j\omega_n})\right) = \mathrm{Var}\left(\sum_{r=1}^{R} \alpha_r(e^{j\omega_n})\hat{G}_r(e^{j\omega_n})\right) = \sum_{r=1}^{R} \alpha_r^2\sigma_r^2(e^{j\omega_n})$$

This is minimized by

$$\alpha_r(e^{j\omega_n}) = \frac{1/\sigma_r^2(e^{j\omega_n})}{\sum_{r=1}^{T} 1/\sigma_r^2(e^{j\omega_n})}$$

Thus the signal is weighted inversely proportional to the variance.

Thus if $\mathrm{Var}\left(\hat{G}_r(e^{j\omega_n})\right) = \frac{\phi_v(e^{j\omega_n})}{\frac{1}{N}|U_r(e^{j\omega_n})|^2}$ then $\alpha_r(e^{j\omega_n}) = \frac{|U_r(e^{j\omega_n})|^2}{\sum_{r=1}^{R}|U_r(e^{j\omega_n})|^2}$.

The best result is obtained if the input is the same for all $r$, which will lead to a reduction of the variance as follows:

$$\mathrm{Var}\left(\hat{G}(e^{j\omega_n})\right) = \frac{\mathrm{Var}\left(\hat{G}_r(e^{j\omega_n})\right)}{R}$$

Biased estimates will reduce the improvement in variance.

- Since we are adding complex numbers the magnitude of the average is not equal to the average of the magnitudes $r_i$.

## 7.1 Bias-variance trade-offs in data record splitting

Divide a data record into smaller parts for averaging:

$$\{u(k), y(k)\}, k = 0, \ldots, K-1$$

Choose $R$ records and calculation length $N$, such that $NR \leq K$:

$$u_r(n) = u(rN + n)$$

And average the resulting estimates:

$$\boxed{\hat{G}(e^{j\omega_n}) = \frac{1}{R}\sum_{r=0}^{R-1}\hat{G}_r(e^{j\omega_n}) = \frac{1}{R}\sum_{r=0}^{R-1}\frac{\hat{Y}_r(e^{j\omega_n})}{\hat{U}_r(e^{j\omega_n})}}$$

As $R$ increases:

- The number of points calculated, $N$ decreases.
- The variance decreases (by up to $1/R$).
- The bias increases (due to non-periodicity transients).

Mean-square error

- Transient bias grows linearly with the number of data splits.
- Variance decays with a rate of up to 1/(number of averages).

What if there is no option of running periodic input experiments? $\rightarrow$ exploit the assumed smoothness of the underlying system.

## 7.2 Smoothing the ETFE

Assume the true system to be close to constant for a range of frequencies: $G(e^{j\omega_{n+r}}) \approx G(e^{j\omega_n})$ for $r = 0, \pm 1, \ldots, \pm r$.
The minimum variance smoothed estimate is:

$$\tilde{G}_N(e^{j\omega_n}) = \frac{\sum_{r=-R}^{R}\alpha_r\hat{G}_N(e^{j\omega_{n+r}})}{\sum_{r=-R}^{R}\alpha_r}, \qquad \alpha_r = \frac{\frac{1}{N}|U_N(e^{j\omega_r+n})|^2}{\phi_v(e^{j\omega_{n+r}})}$$

The summation above can then be approximated by an integral:

$$\approx \frac{\int_{\omega_{n-r}}^{\omega_{n+r}}\alpha(e^{j\zeta})\hat{G}_N(e^{j\zeta})d\zeta}{\int_{\omega_{n-r}}^{\omega_{n+r}}\alpha(e^{j\zeta})d\zeta}, \qquad \text{with } \alpha(e^{j\zeta}) = \frac{\frac{1}{N}|U_N(e^{j\zeta})|^2}{\phi_v(e^{j\zeta})}$$

Which can be reformulated using a smoothing window:

$$\tilde{G}_N(e^{j\omega_n}) = \frac{\frac{1}{2\pi}\int_{-\pi}^{\pi}W_\gamma(e^{j(\zeta-\omega_n)})\alpha(e^{j\zeta})\hat{G}_N(e^{j\zeta})d\zeta}{\frac{1}{2\pi}\int_{-\pi}^{\pi}W_\gamma(e^{j(\zeta-\omega_n)})\alpha(e^{j\zeta})d\zeta} \qquad \text{with } \alpha(e^{j\zeta}) = \frac{\frac{1}{N}|U_N(e^{j\zeta})|^2}{\phi_v(e^{j\zeta})}$$

### 7.2.1 Assumptions on $\phi_v(e^{j\omega})$

Assume $\phi_v(e^{j\omega})$ is also a smooth function of frequency.

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}W_\gamma(e^{j(\zeta-\omega_n)})\left|\frac{1}{\phi_v(e^{j\zeta})} - \frac{1}{\phi(e^{j\omega_n})}\right|d\zeta \approx 0$$

Then use,

$$\alpha(e^{j\zeta}) = \frac{\frac{1}{N}|U_N(e^{j\zeta})|^2}{\phi_v(e^{j\omega_n})}$$

to get

$$\boxed{\tilde{G}_N(e^{j\omega_n}) = \frac{\frac{1}{2\pi}\int_{-pi}^{\pi}W_\gamma(e^{-j(\zeta-\omega_n)})\frac{1}{N}|U_N(e^{j\zeta})|^2\hat{G}_N(e^{j\zeta})d\zeta}{\frac{1}{2\pi}\int_{-\pi}^{\pi}W_\gamma(e^{j(\zeta-\omega_n)})\frac{1}{N}|U_N(e^{j\zeta})^2d\zeta}}$$

The wider the frequency window (decreasing $\gamma$)

- the more adjacent frequencies included in the smoothness estimate.
- the smoother the result.
- the lower the noise induced variance.
- the higher the bias.

### 7.2.2 Characteristic windows

$$\boxed{W_\gamma(e^{j\omega}) = \frac{1}{\gamma}\left(\frac{\sin\gamma\omega/2}{\sin\omega/2}\right)^2} \text{ Bartlett}$$

$$\boxed{W_\gamma(e^{j\omega}) = \tfrac{1}{2}D_\gamma(\omega) + \tfrac{1}{4}D_\gamma(\omega-\pi/\gamma) + \tfrac{1}{4}D_\gamma(\omega+\pi/\gamma)} \text{ Hann}$$

where

$$D_\gamma(\omega) = \frac{\sin\omega(\gamma+0.5)}{\sin\omega/2}$$

**Properties of window functions:**

- $\frac{1}{2\pi}\int_{-\pi}^{\pi} W_\gamma(e^{j\zeta})d\zeta = 1$
- $\int_{-\pi}^{\pi} \zeta W_\gamma(e^{j\zeta})d\zeta = 0$
- $M(\gamma) := \int_{-\pi}^{\pi} \zeta^2 W_\gamma(e^{j\zeta})d\zeta$
- $\bar{W}(\gamma) := 2\pi\int_{-\pi}^{\pi} W_\gamma^2(e^{j\zeta})d\zeta$

| | | | |
|---|---|---|---|
| Bartlett | $M(\gamma) = \frac{2.78}{\gamma}$, | $\bar{W}(\gamma) \approx 0.67\gamma$ | (for $\gamma > 5$) |
| Hamming | $M(\gamma) = \frac{\pi^2}{2\gamma^2}$, | $\bar{W}(\gamma) \approx 0.75\gamma$ | (for $\gamma > 5$) |

- $M(\gamma)$ gives an idea of the bias effect.
- $\bar{W}(\gamma)$ gives an idea of the variance effect.

### 7.2.3 Asymptotic bias properties

$$\mathrm{E}\left\{\tilde{G}(e^{j\omega_n}) - \mathrm{E}\left\{G(e^{j\omega_n})\right\}\right\} = \mathrm{E}\left\{\tilde{G}(e^{j\omega_n}) - G(e^{j\omega_n})\right\} =$$
$$M(\gamma)\left(\tfrac{1}{2}\underbrace{G''(e^{j\omega_n})}_{\text{curvature}} + \underbrace{G'(e^{j\omega_n})}_{slope}\frac{\phi_u'(e^{j\omega_n})}{\phi_u(e^{j\omega_n})}\right) + H.O.T.$$

Increasing $\gamma$

- makes the frequency window smaller.
- averages over fewer frequency values.
- makes $M(\gamma)$ smaller
- reduces the bias of the smoothed estimate $\tilde{G}(e^{j\omega_n})$

### 7.2.4 Asymptotic variance properties

$$\mathrm{E}\left\{(\tilde{G}(e^{j\omega_n}) - \mathrm{E}\left\{\tilde{G}(e^{j\omega_n})\right\})^2\right\} = \tfrac{1}{N}\bar{W}(\gamma)\frac{\phi_v(e^{j\omega_n})}{\phi_u(e^{j\omega_n})} + H.O.T.$$

Increasing $\gamma$

- makes the frequency window narrower.
- averages over fewer frequency values.
- makes $\bar{W}_\gamma$ larger.
- increases the variance of the smoothed estimate $\tilde{G}(e^{j\omega_n})$.

### 7.2.5 Asymptotic MSE properties

$$\mathrm{E}\left\{|\tilde{G}(e^{j\omega_n}) - G(e^{j\omega_n})|^2\right\} \approx M^2(\gamma)|F(e^{j\omega_n})|^2 + \tfrac{1}{N}\bar{W}(\gamma)\frac{\phi_v(e^{j\omega_n})}{\phi_u(e^{j\omega_n})}$$

where

$$F(e^{j\omega_n}) = \tfrac{1}{2}G''(e^{j\omega_n}) + G'(e^{j\omega_n})\frac{\phi_u'(e^{j\omega_n})}{\phi_u(e^{j\omega_n})}$$

If $M(\gamma) = M/\gamma^2$ and $\bar{W}(\gamma) = \bar{W}\gamma$ then MSE is minised by:

$$\boxed{\gamma_{optimal} = \left(\frac{4M^2|F(e^{j\omega_n})|^2\phi_u(e^{j\omega_n})}{\bar{W}\phi_v(e^{j\omega_n})}\right)^{1/5} N^{1/5}}$$

and

$$\text{MSE at } \gamma_{optimal} \approx CN^{-4/5}$$

## 8 Windowing and Input Signals

$$\phi_{yu}(e^{j\omega}) = G(e^{j\omega})\phi_u(e^{j\omega})$$

$$\boxed{\hat{G}(e^{j\omega_n}) = \frac{\hat{\phi}_{yu}(e^{j\omega_n})}{\hat{\phi}_u(e^{j\omega_n})}}$$

Recall that the smoothed ETFE is:

$$\tilde{G}_N(e^{j\omega_n}) = \frac{\frac{1}{2\pi}\int_{-\pi}^{\pi} W_\gamma(e^{j(\zeta-\omega_n)})\frac{1}{N}|U_N(e^{j\zeta})|^2\hat{G}_N(e^{j\zeta})d\zeta}{\frac{1}{2\pi}\int_{-\pi}^{\pi} W_\gamma(e^{j(\zeta-\omega_n)})\frac{1}{N}|U_n(e^{j\zeta})|^2 d\zeta}$$

The denominator term approaches $\frac{1}{2\pi}\int_{-\pi}^{\pi} W_\gamma(e^{j(\zeta-\omega_n)})\phi(e^{j\omega_n})d\zeta$ as $N \to \infty$.

If in addition $W_\gamma(e^{j\omega})$ is concentrated around $\zeta = 0$ (i.e. $\gamma/N \to 0$) then the denominator term approaches $\phi_u(e^{j\omega_n})$ as $N \to \infty$.

This motivates the smoothed spectral estimate:

$$\boxed{\tilde{\phi}_u(e^{j\omega_n}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} W_\gamma(e^{j(\zeta-\omega_n)})\frac{1}{N}|U_N(e^{j\omega})|^2 d\zeta}$$

Similarly the numerator approaches $\phi_{yu}$ as $N \to \infty$:

$$\boxed{\tilde{\phi}_{yu}(e^{j\omega_n}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} W_\gamma(e^{j(\zeta-\omega_n)})\frac{1}{N}|U_N(e^{j\omega})|^2 \hat{G}_N(e^{j\omega}) d\zeta}$$

For this reason the smoothed ETFE is equal to the smoothed spectral estimate for $N \to \infty$.

## 8.1 Frequency domain smoothing in matlab

```
gamma = 80;
U = fft(u); Y = fft(y);
G_est = Y./U;
G_est_smooth = G_est*0;

[om,Wg] = WfHann(g,N);
%shift to start at zero
zidx = find(om == 0);
omega = [om(zidx:N);om(1:zidx-1)];
Wg = [Wg(zidx:N) Wg(1:zidx-1)];
%variance weighting
a = U.*conj(U);
```

```
for wn = 1:N
  %reset normalization
  Wnorm = 0;
    for xi = 1:N
      %wrap window index
        widx = mod(xi-wn,N)+1;
        G_est_smooth(wn) = G_est_smooth(wn) +...
            Wg(widx)*G_est(xi)*a(xi);
        Wnorm = Wnorm + Wg(widx)*a(xi);
    end
    %weigh normalisation
    G_est_smooth(wn) = G_est_smooth(wn)/Wnorm;
end
```

## 8.2 Time domain windows

Define, via the inverse Fourier transform a time domain window:

$$\omega_\gamma(\tau) = \frac{1}{2\pi}\int_{-\pi}^{\pi} W_\gamma(e^{j\omega})e^{j\zeta\tau} d\zeta$$

Then the smoothed input spectral estimate $\tilde{\phi}_u(e^{j\omega_n})$ is:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} W_\gamma(e^{j(\zeta-\omega_n)})\frac{1}{N}|U_N(e^{j\omega})|^2 d\zeta \approx \sum_{\tau-\infty}^{\infty} \omega_\gamma(\tau)\hat{R}_u(\tau)e^{-j\tau\omega_n}$$

where

$$\omega_\gamma = \begin{cases} 0 & \text{for } \tau < -\gamma \\ > 0 & \text{for } -\gamma \le \tau \le \gamma \\ 0 & \text{for } \tau > \gamma \end{cases}$$

where often $\gamma << N$, which enables the faster calculated redefinition:

$$\boxed{\frac{1}{2\pi}\int_{-\pi}^{\pi} W_\gamma(e^{j(\zeta-\omega_n)})\frac{1}{N}|U_N(e^{j\omega})|^2 d\zeta \approx \sum_{\tau-\gamma}^{\gamma} \omega_\gamma(\tau)\hat{R}_u(\tau)e^{-j\tau\omega_n}}$$

The cross spectral estimate can also be formulated as a convolution in the frequency domain which leads to the analogous formulation to the spectral estimate of $u$:

$$\boxed{\tilde{\phi}_u(e^{j\omega_n}) = \sum_{\tau=-\gamma}^{\gamma} \omega_\gamma(\tau)\hat{R}_u(\tau)e^{-j\tau\omega_n}}$$

$$\boxed{\tilde{\phi}_{yu}(e^{j\omega_n}) = \sum_{\tau=-\gamma}^{\gamma} \omega_\gamma(\tau)\hat{R}_{yu}(\tau)e^{-j\tau\omega_n}}$$

### 8.2.1 Time domain smoothing in matlab

```
gamma = 80;
[~,Wg] = WtHann(gamma,N);
R_u = xcorr(u,N/2)/N; %Lecture 3.37
R_yu = xcorr(y,u,N/2)/N;

omega = Omega_n(N);
phi_u = zeros(size(omega));
phi_yu = zeros(size(omega));

tau = -gamma:gamma;
ind = tau+N/2;
for i = 1:N
%Lecture 5.9 and 5.11
    phi_u(i) = sum(Wg(tau+N/2).*R_u(ind).*exp(-1j.*tau.'.*
       omega(i)));
    phi_yu(i) = sum(Wg(tau+N/2).*R_yu(ind).*exp(-1j*tau.'.*
       omega(i)));
end
G_est_smooth = phi_yu./phi_u;
```

### 8.2.2 Window characteristics

Decreasing $\gamma$: narrower $\omega_\gamma(\tau)$, wider $W_\gamma(e^{j\omega})$

- the more frequencies, $\hat{G}(e^{j\omega_n})$ included in the smoothing.
- the fewer $\hat{R}(\tau)$ estimates included in the smoothing.
- the smoother the result.
- the lower the noise induced variance.
- the **higher the bias**.

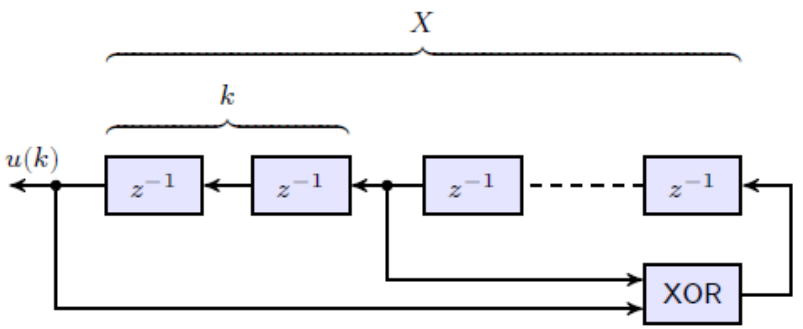
## 8.3 Input Signals

- Steps
- Doublet
- Sinusiods, Chirpts, Multi-Sines
- Filtered white noise
- Pseudo-Random Binary Signals (PRBS)

### 8.3.1 PRBS

$$u(k) = a \text{ or } -a$$

Shift-register generation




| $X$ | $k$ |
|---|---|
| 3 | 1 or 2 |
| 4 | 1 or 3 |
| 5 | 2 or 3 |
| 6 | 1 or 5 |
| 7 | 1 or 6 |
| 8 | — |
| 9 | 4 or 5 |
| 10 | 3 or 7 |

Periodicity

Periodic with period equal to at most $M = 2^X - 1$.

$$R_u(\tau) = \frac{1}{N}\sum_{k=0}^{N-1} u(k)u(k-\tau) = \begin{cases} a^2 & \text{if } \tau = 0 \\ \frac{-a^2}{2^X-1} & \text{if } \tau \neq 0 \end{cases}$$

**Definition 6.** ***Run length*** *defines how long the signal stays high.*

The run length distribution of $u(k)$ is then:

1/2 runs of length 1

1/4 runs of length 2

1/8 runs of length 3

⋮

Other properties:

- Equal energy at all frequencies.
- The maximum period of a PRBS signal can be found based on the discrete time dynamic system that generates the signal:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 2\gamma \end{bmatrix}$$

$$z = Ax(k)$$
$$x(k+1) = \mathrm{mod}_2(z(k))$$
$$y = Cx(k) - \gamma$$

where $\gamma$ is the amplitude of the signal.

Now the maximum length period result from the idea that $x(k)$ and therefore the output $y(k)$ can only change so many times until it reaches a state it has already held before. From that point on, since the system is first order, the signal repeats. Thus in the longest case, all possible values contained in $x$ are taken exactly once. Thus the maximum periodic length is equal to the number of possible binary numbers with 6 bits, where 6 is the order of the PRBS signal.

$\boxed{2^6 - 1}$ Maximum Length Period

### 8.3.2 PRBS in matlab

```
signal_order = 8;
singal_length = 2.^signal_order-1;
u = idinput(signal_length,'PRBS');
```

- Note that `idinput` only allows the choice of the total length of the signal and derives the fitting signal order such that the signal is at least of that chosen length. A warning is made if the chosen signal length is not equivalent to the run length.

### 8.3.3 Multi-sinusoidal signals

$$u(k) = \sum_{s=1}^{S} \sqrt{2\alpha_s} \cos(\omega_s k T + \phi_s)$$

where $T$ is the sampling period, $\omega_s = \frac{2\pi}{T_P}$, $\frac{T_P}{T} = N$, $S \leq \frac{N}{2}$.

**Choose $N$ to be a power of 2 for efficient FFT calculations.**

$$\sum_{s=1}^{S} \alpha_s = 1 \quad \text{Total signal power}$$

**Schroeder phasing**
Select the phases $\phi_s$ such that the minimize the peak amplitude:

$$\phi_s = 2\pi \sum_{j=1}^{s} j\alpha_s.$$

for equal power in each sinusoids:

$$\alpha_s = 1/S \text{ and } \phi_s = \tfrac{\pi(s^2+s)}{S}$$

# 9 Residual Spectra, Coherency, Aperiodicty, Offsets and Drifts

## 9.1 Residual Spectrum

### 9.1.1 Estimating $\phi_v(e^{j\omega_n})$

$$v(k) = y(k) - G(e^{j\omega})u(k)$$

$$\tilde{\phi}_v(e^{j\omega_n}) \approx \tfrac{1}{N}\tfrac{1}{2\pi}\int_{-\pi}^{\pi} W_\gamma(e^{j(\zeta-\omega_n)}) \left|Y_N(e^{j\omega}) - \tilde{G}(e^{j\omega})U_N(e^{j\omega})\right|^2 d\zeta \approx \tilde{\phi}_y(e^{j\omega_n}) - \tfrac{\left|\tilde{\phi}_{yu}(e^{j\omega_n})\right|^2}{\tilde{\phi}_u(e^{j\omega_n})}$$

How much energy is accounted for by the model? How much by noise?

$$\phi_v(e^{j\omega_n}) = \phi_y(e^{j\omega_n})\left(1 - \tfrac{|\phi_{yu}((e^{j\omega_n}))|^2}{\phi_y(e^{j\omega_n})\phi_u(e^{j\omega_n})}\right)$$

$$\hat{\kappa}_{yu}(e^{j\omega_n}) = \sqrt{\tfrac{|\hat{\phi}_{yu}(e^{j\omega_n})|^2}{\hat{\phi}_y(e^{j\omega_n})\hat{\phi}_u(e^{j\omega_n})}}$$ Coherency Spectrum

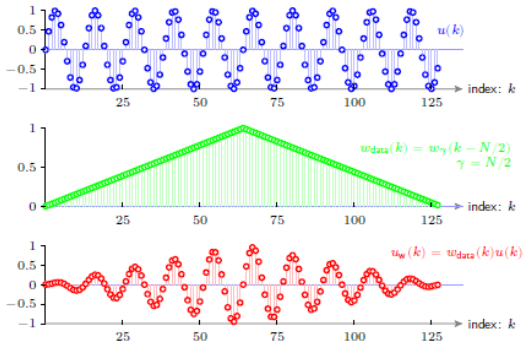
- If all of the energy in the output is due to the model for a frequency $\omega_n$ then $\hat{\kappa}_{yu}(e^{j\omega_n}) = 1$.
- This can be used as a measure of effectiveness of the modelling at a particular frequency.
- Theoretically, $0 \leq \hat{\kappa}_{yu}(e^{j\omega_n}) \leq 1$. One should aim to keep the coherency spectrum as high as possible. It can be adjusted by adjusting the smoothing.

## 9.2 Time-domain data windowing

Putting a time domain window directly on the data.

$$U_w(e^{j\omega_n}) = \sum_{k=0}^{N-1} w_{data}(k)u(k)e^{-jk\omega_n}$$

often with $w_{data}(k) = w_\gamma(k - N/2)$ (shifted to middle). Typically $\gamma = N/2$ such that all of the data is used.

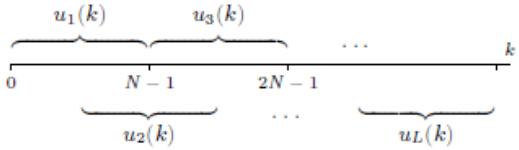



For an estimation of the periodogram **scaling is necessary!**

$$\frac{1}{E_{scl}}\frac{1}{N}|U_w(e^{j\omega_n})|^2 \quad \text{where } E_{scl} = \frac{\sum_{k=0}^{N-1}|w_{data}(k)u(k)|^2}{\sum_{k=0}^{N-1}|u(k)|^2} \approx \frac{1}{N}\sum_{k=0}^{N-1}|w_{data}(k)|^2 \quad \text{Periodogram}$$

- Transients influence the periodogram of a sequence, since by the application of the DFT, the signal is assumed to be periodic. Thus even if a sinusoid is sampled - if the periodic extension does not represent the same sinusoid - the DFT will show a range of frequencies, instead of a single distinct one.
- Time-domain data windowing can help reducing the influence of transients.

### 9.2.1 Welch's Method




1. Split the data record into $L$ overlapping segments of length $N$.
2. $U_l(e^{j\omega_n}) = \sum_{k=0}^{N-1} w_{data}(k)u_l(k)e^{j\omega_n k}$
3. $\tilde{\phi}_u(e^{j\omega_n}) = \frac{1}{NLE_{scl}}\sum_{l=1}^{L}\left|U_l(e^{j\omega_n})\right|^2$

\+ Windowing can reduce transient response effects.

\+ Noise reduction from averaging and windowing.

\+ Variance error can be reduced.

\- Windowing can cause energy leakage to adjacent frequencies.

\- Frequency resolution deteriorates.

\- Bias error can be increased.

\- Noise on $u_l(k)$ and $u_{l+1}$ is not uncorrelated.

- Do not use `welch()`, since it does not fit the definition here.

### 9.2.2 Drifts and Offsets

- Time domain windowing when an offset is present can lead to the introduction of additional frequencies close to zero, which is undesirable.
- Time domain windowing when a drift is present can lead to the introduction of additional frequencies close to the peak of the sampled sinusoid, which is undesirable as well.
- A possible solution is **preprocessing** the data via the assumption

  $$u_d(k) = u(k) - (alphak + \beta)$$

  where $\alpha k + \beta$ is the best linear fit to $u(k)$.

```
1 detrend(u);
```

\+ `detrend` can completely remove the effect of drifts and offsets.

\- It must be applied carefully, since if no drift/offset is present or if it is detected wrongly, large errors can be introduced.

# 10 Frequency Domain Subspace ID

$$x(k+1) = Ax(k) + Bu(k) \qquad A \in \mathbb{R}^{n_x \times n_x}$$
$$y(k) = Cx(k) + Du(k) \qquad D \in \mathbb{R}^{n_y \times n_u}$$

$$g(k) = \begin{cases} 0 & k = 0 \\ D & k = 0 \\ CA^{k-1} & k > 0 \end{cases} \quad \text{Pulse response coefficients}$$

$$G(e^{j\omega_n}) = \sum_{k=0}^{\infty} g(k)e^{-j\omega k} \quad 0 \le \omega \le \pi$$

Given

$$G(n) = G(e^{j\omega_n}) + V(e^{j\omega_n}), \quad n = 0, \ldots, N/2$$

Find

$$\hat{G}(e^{j\omega_n}) = \hat{C}\left(e^{j\omega}I - \hat{A}\right)^{-1}\hat{B} + \hat{D}$$

such that

$$\lim_{N\to\infty} ||\hat{G}(e^{j\omega_n}) - G(e^{j\omega_n})||_\infty = 0$$

$$\boxed{\mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{q-1} \end{bmatrix} \in \mathbb{R}^{n_y q \times n_x}} \text{ Extended observability}$$

$\text{rank}(\mathcal{O}) = n_x$ for all $n_y q \geq n_x$

$$\boxed{\mathcal{C} = \begin{bmatrix} B & AB & \cdots & A^{r-1}B \end{bmatrix} \in \mathbb{R}^{n_x \times n_u r}} \text{ Extended controllability}$$

$\text{rank}(\mathcal{C})=n_x$ for all $n_u r \geq n_x$
The ETFE matches the true transfer function only of there is no noise. In that case the inverse Fourier transform results in the time-aliased impulse response of the system $h_k$:

$$\boxed{h_k = \frac{1}{N}\sum_{n=0}^{N-1} G(e^{j\omega_n})e^{j2\pi kn/N} = CA^{k-1}\left(\sum_{l=0}^{\infty} A^{Nl}\right)B = CA^{k-1}\left(I - A^N\right)^{-1}B}$$

- We do not get the exact impulse response $g(k)$ since we do not have all the data until $N \to \infty$.

Now the **Hankel matrix** is formulated with the ultimate goal to extract estimates of the state space matrices.

$$H = \begin{bmatrix} h_1 & h_2 & h_3 & \cdots & h_r \\ h_2 & h_3 & \ddots & \ddots & h_{r+1} \\ h_3 & \ddots & \ddots & \ddots & \\ & \ddots & \ddots & & \vdots \\ h_q & h_{q+1} & & \cdots & h_{q+r-1} \end{bmatrix} \quad \text{choose } q > n_x, r > n_x \text{ and } q + r - 1 \leq N - 1$$

which leads to:

$$\boxed{H = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{q-1} \end{bmatrix}\left(I - A^N\right)^{-1}\begin{bmatrix} B & AB & \cdots & A^{r-1}B \end{bmatrix}}$$

Next singular value decomposition can be used to decompose the Hankel matrix, which allows to calculate the **rank** of the state space, as the dimension of $\Sigma_1$.

$$H = U\Sigma V^T = \begin{bmatrix} U_1 & U_2 \end{bmatrix}\begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}, \quad \Sigma_1 \in \mathbb{R}^{n_x \times n_x}$$

Since $U_2$ is multiplied with $\mathbf{0}$ the $\text{range}(H) = \text{range}(U_1) = \mathcal{O}$. Which directly leads to $\text{span}(\{U_1\}) = \text{span}(\mathcal{O})$, which means that the observability matrix is some linear transform of $U_1$. Or differently said, for some choice of $A$ they are equal. probably not correct!?

Next an estimate of $\hat{A}$ is made using the observability matrix, for which the selection matrix $J$ is introduced:

$$J_1\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{q-1} \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{q-2} \end{bmatrix} \text{ and } J_2\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{q-1} \end{bmatrix} = \begin{bmatrix} CA \\ \vdots \\ CA^{q-1} \end{bmatrix}$$

Then $j_1\mathcal{O}A = J_2\mathcal{O}$
So $\hat{A}$ is the least squares solution to $J_1\hat{U}_1\hat{A} = J_2\hat{U}_1$.

And $\hat{C}$ can be found as: $J_e\mathcal{O} = C \implies J_3\hat{U}_1 = \hat{C}$.
Next, using the initial formulation:

$$\hat{G}(e^{j\omega_n}) = \hat{C}\left(e^{j\omega}I - \hat{A}\right)^{-1}\hat{B} + \hat{D}$$

$\hat{B}$ and $\hat{D}$ can be found as the solution to the minimization:

$$\boxed{\hat{B}, \hat{D} = \underset{B,D}{\text{argmin}} \sum_{n=0}^{N} ||G(e^{j\omega_n}) - D - \hat{C}(e^{j\omega_n}I - \hat{A})^{-1}B||_F^2}$$

## 10.1 Summarizing the subspace identification algorithm

Uniform data spacing case:

$$G(n), \quad \omega_n = \tfrac{\pi n}{N}, \quad n = 0, \ldots, N/2$$

1. Extend data to negative frequencies
$$G(n) = \bar{G}(N - n), \quad n = N/2 + 1, \ldots, N - 1$$
2. Calculate inverse DFT to obtain the time-aliased pulse response
$$\hat{h}_k = \frac{1}{N}\sum_{n=0}^{N-1} G(n)e^{j2\pi kn/N}, \quad k = 0, \ldots, N - 1$$
3. Form a block-Hakel matrix:

$$\hat{H} = \begin{bmatrix} \hat{h}_1 & \hat{h}_2 & \hat{h}_3 & \cdots \hat{h}_r & \\ \hat{h}_2 & \hat{h}_3 & \ddots & \ddots & \hat{h}_{r+1} \\ \hat{h}_3 & \ddots & \ddots & \ddots & \\ & \ddots & \ddots & & \vdots \\ \hat{h}_q & \hat{h}_{q+1} & & \cdots & \hat{h}_{q+r+1} \end{bmatrix} \in \mathbb{R}^{n_y q \times \nu_r}$$

4. Calculate a singular value decomposition:

$$\hat{H} = \hat{U}\hat{\Sigma}\hat{V}^T$$

5. Select a model order, $\hat{n}_x$ and partition the SVD:

$$\hat{U}\hat{\Sigma}\hat{V}^T = \begin{bmatrix} \hat{U}_1 & \hat{U}_2 \end{bmatrix} \begin{bmatrix} \hat{\Sigma}_1 & 0 \\ 0 & \hat{\Sigma}_2 \end{bmatrix} \begin{bmatrix} \hat{V}_1^T \\ \hat{V}_2^T \end{bmatrix}, \quad \hat{\Sigma} \in \mathbb{R}^{\hat{n}_x \times \hat{n}_x}$$

6. Estimate $\hat{A}$ via:

$$J_1 = \begin{bmatrix} I_{n_y(q-1)} & O_{n_y(q-1)\times n_y} \end{bmatrix}$$
$$J_2 = \begin{bmatrix} 0_{n_q(q-1)} \times n_y & I_{n_y(q-1)} \end{bmatrix}$$

Solve for $\hat{A}$ via LS:

$$\boxed{J_1\hat{U}_1\hat{A} = J_2\hat{U}_1}$$

7. Estimate $\hat{C}$ via

$$J_3 = \begin{bmatrix} I_{n_y} & 0_{n_y \times n_y(q_1)} \end{bmatrix}$$

$$\boxed{\hat{C} = J_3\hat{U}_1}$$

8. Find $\hat{B}$ and $\hat{D}$ via least squares:

$$\boxed{\hat{B}, \hat{D} = \underset{B,D}{\operatorname{argmin}} \sum_{n=0}^{N} ||G(e^{j\omega_n}) - D - \hat{C}(e^{j\omega_n} I - \hat{A})^{-1} B||_F^2}$$

9. Form the estimate

$$\boxed{\hat{G}(z) = \hat{D} + \hat{C}(zI - \hat{A})^{-1}\hat{B}}$$

- To get real valued $B$ and $D$:

$$\begin{bmatrix} \text{real}\left((\hat{C}((e^{j\omega_n})I - \hat{A})^{-1}\right) & I \\ \text{imag}\left((\hat{C}((e^{j\omega_n})I - \hat{A})^{-1}\right) & 0 \end{bmatrix} \begin{bmatrix} B \\ D \end{bmatrix} = \begin{bmatrix} \text{real}(G(n)) \\ \text{imag}(G(n)) \end{bmatrix}$$

### 10.1.1 Properties

- Asymptotic convergence

$$\lim_{N\to\infty} ||\hat{G}(e^{j\omega_n}) - G(e^{j\omega_n})||_\infty = 0, \quad n = 0, \ldots, N-1 \text{ w.p. } 1$$

- The algorithm is „correct“. If $V(e^{j\omega_n}) = 0$ then there exists a data length $N_0 < \infty$ such that:

$$||\hat{G}_N(e^{j\omega}) - G(e^{j\omega})||_\infty = 0 \text{ for all } N > N_0$$

\+ Time- and frequency-domain versions available. $N4SID$, etc

\+ Many variants which depend on weighting for noise.

\+ gives a state-space model directly.

\+ Can be effective in determining system order.

\+ Works equally well for MIMO systems.

\- Unusual noise weighting in frequency-domain case.

\- Truncated SVD reconstrutions are not Hankel.

\- $\hat{U}_1$ does not have the „shift“structure.

\- Least-squares noise assumptions are not correct.

\- Can give unstable models for stable systems.

## 10.2 Nonuniformly spaced frequencies

Time domain

$$x(k+1) = Ax(k) + Bu(k)$$
$$y(k) = Cx(k) + Du(k)$$

Frequency domain

$$e^{j\omega}X(\omega) = AX(\omega) + BU(\omega)$$
$$Y(\omega) = CX(\omega) + BU(\omega)$$

For a specific frequency $\omega$ on each channel

$$U_i(\omega) = e_i \quad i = 1, \ldots n_u$$

Resulting system equations

$$e^{j\omega}X_i(\omega) = AX_i(\omega) + BU_i(\omega)$$
$$Y_i(\omega) = CX_i(\omega) + DU_i(\omega)$$

Defining

$$X_c(\omega) = \begin{bmatrix} X_1(\omega) & \cdots & X_{n_u}(\omega) \end{bmatrix}$$

and stacking the equations column-wise gives:

$$e^{j\omega} X_c(\omega) = A X_c(\omega) + B$$
$$G(\omega) = C X_c(\omega) + D$$

Multiplying by $e^{j\omega}$ and substituting, repeating and stacking row-wise:

$$\begin{bmatrix} G(e^{j\omega}) \\ e^{j\omega} G(e^{j\omega}) \\ \vdots \\ e^{j(q-1)\omega} G(e^{j\omega}) \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{q-1} \end{bmatrix} X_c(\omega) + \Gamma \begin{bmatrix} I_{n_u} \\ e^{j\omega} I_{n_u} \\ \vdots \\ e^{j(q-1)\omega} I_{n_u} \end{bmatrix}$$

where

$$\Gamma = \begin{bmatrix} D & & & \\ CB & D & 0 & 0 \\ CAB & \ddots & \ddots & \\ \vdots & & \ddots & \\ CA^{q-2}B & \cdots & CB & D \end{bmatrix}$$

Repeat for all frequencies $\omega_i$:

$$\mathcal{G} = \frac{1}{\sqrt{N}} \begin{bmatrix} G(e^{j\omega_1}) & \cdots & G(e^{j\omega_N}) \\ e^{j\omega_1} G(e^{j\omega_1}) & & e^{j\omega_N} G(e^{j\omega_N}) \\ \vdots & & \vdots \\ e^{j(q-1)\omega_1} G(e^{j\omega_1}) & \cdots & e^{j(q-1)\omega_N} G(e^{j\omega_N}) \end{bmatrix}$$

$$\mathcal{W} = \frac{1}{\sqrt{N}} \begin{bmatrix} I & \cdots & I \\ e^{j\omega_1} & & e^{j\omega_N} I \\ \vdots & & \vdots \\ e^{j(q-1)\omega_1} & \cdots & e^{j(q-1)\omega_N} I \end{bmatrix}$$

$$\mathcal{X}_c = \frac{1}{\sqrt{N}} \begin{bmatrix} X_c(\omega_1) \cdots & X_c(\omega_N) \end{bmatrix}$$

$$\boxed{g = \mathcal{O}\mathcal{X}_c + \Gamma\mathcal{W}}$$

Now as $\mathcal{O}$ and $\Gamma$ are real valued

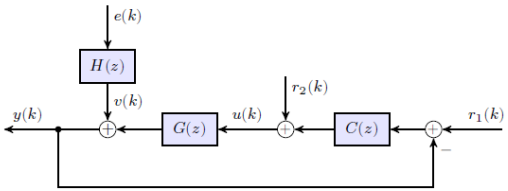
$$\underbrace{\begin{bmatrix} \text{real}(\mathcal{G}) & \text{imag}(\mathcal{G}) \end{bmatrix}}_{\mathcal{G}_r} = \mathcal{O} \underbrace{\begin{bmatrix} \text{real}(\mathcal{X}_c) & \text{imag}(\mathcal{X}_c) \end{bmatrix}}_{\mathcal{X}_{cr}} + \Gamma \underbrace{\begin{bmatrix} \text{real}(\mathcal{W}) & \text{imag}(\mathcal{W}) \end{bmatrix}}_{\mathcal{W}_r}$$

If $n_y q < n_u r$ then $\exists \mathcal{W}_r^\perp$ such that $\mathcal{W}_r \mathcal{W}_r^\perp$.

$$\mathcal{G}_r \mathcal{W}_r^\perp = (\mathcal{O}\mathcal{X}_{rc} + \Gamma\mathcal{W}_r)\mathcal{W}_r^\perp = \mathcal{O}\mathcal{X}_{cr}\mathcal{W}_r^\perp$$
$$\text{range}\left(\mathcal{G}_r \mathcal{W}_r^\perp\right) = \text{range}(\mathcal{O})$$

```
%To solve linear least squares problem: Ax = b
x = A\b;
```

# 11 Closed-Loop ID




$r_1(k), r_2(k)$ Excitation
$y(k), u(k), r_2(k), r_1(k)$ Measurements

$$r(k) = r_2(k) + C(z) r_1(k)$$

**Motivation for closed-loop ID**

- Unstable systems must be operated in closed-loop.
- Operational constraints may require closed-loop.
- Closed-loop controller maintains the system close to the operating point.
- Easier to focus identification on specific operation points.
- Will emphasize plant dynamics close to the cross-over frequency range.
- Closed-loop operation can remove a large-scale zero-frequency response.

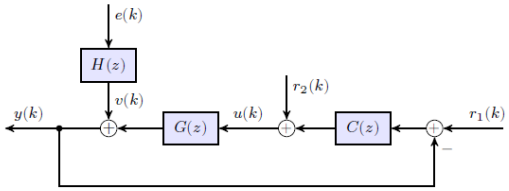
**Methods overview**

1. Direct (open-loop) methods:
$$\hat{G} = \hat{Y}_N / \hat{U}_N \text{ or } \hat{G} = \hat{\phi}_{yu} / \hat{\phi}_u$$
2. Indirect methods
$$y = \frac{G(z)}{(1+G(z)C(z))} r$$
3. Input-output methods:
$$y = \frac{G(z)}{(1+G(z)C(z))} r \text{ and } u = \frac{1}{(1+G(z)C(z))} r$$
4. Dual-Youla methods.

## 11.1 Direct Methods

$$\hat{\phi}_{yu}(e^{j\omega_n}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} W_\gamma(\omega_n - \zeta)\frac{1}{N}Y_N(e^{j\zeta})\bar{U}_N(e^{j\zeta})d\zeta$$
$$\hat{\phi}_{u}(e^{j\omega_n}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} W_\gamma(\omega_n - \zeta)\frac{1}{N}|U_N(e^{j\zeta})|^2 d\zeta$$
$$\hat{G}(e^{j\omega_n}) = \frac{\hat{\phi}_{yu}(e^{j\omega_n})}{\hat{\phi}_{u}(e^{j\omega_n})}$$

Closed loop transfer functions:




$$y = v + G(r_2 + C(r_1 - y))$$
$$y = v + Gr_2 + GCr_1 - GCy$$
$$y(1+GC) = v + Gr_2 + GCr_1 \quad \text{where } S = \frac{1}{1+GC}$$
$$y = Sv + \underbrace{SGr_2 + SGCr_1}_{SGr} \quad \text{where } r = r_2 + Cr_1$$

$$u = r_2 + C(r_1 - (Gu + v))$$
$$u = r_2 + Cr_1 - CGu - Cv$$
$$u + CG = r_2 + Cr_1 - Cv$$
$$u = \underbrace{Sr_2 + SCr_1}_{Sr} - SCv$$

$$T_{yr} = SG$$
$$T_{ur} = S$$

```
Gdz = c2d(Gs,T_s,'zoh');
Cdz = ...;
Sdz = 1/(1+Gdz*Cdz);
Tdz_y_r1 = 1-Sdz;
Tdz_y_r = feedback(Gdz,Cdz);
Tdz_y_r = Sdz*Gdz;
T_u_r = Sdz;
%Use minreal() to reduce the TF
```

Further these identities can be shown if $r_2, v = 0$:

```
y = lsim(Tdz_y_r1,r_1,t);
y = lsim(Tdz_y_r,lsim(Cdz,r_1,t),t);
u = lsim(Sdz*Cdz,r_1,t);
u = lsim(Sdz,lsim(Cdz,r_1,t),t);
```

Assume that $\phi_{rv} = 0$

$$\hat{G} = \frac{\hat{\phi}_{yu}}{\hat{\phi}_u} \approx \frac{|S|^2 G\hat{\phi}_r - |S|^2\bar{C}\hat{\phi}_v}{|S|^2\hat{\phi}_r + |S|^2|C|^2\hat{\phi}_v} \approx \frac{G\hat{\phi}_r - \bar{C}\hat{\phi}_v}{\hat{\phi}_r + |C|^2\hat{\phi}_v}$$

- For system identification the used controller $C$ should be not the optimal one for the use case, since a good controller suppresses information Find better explanation.
- Reducing the excitation to zero will only deliver information on $C$.

## 11.2 Input-Output Methods

**Identification problems:**
If $\phi_{vr} = 0$:

$$\hat{T}_{yr}(e^{j\omega_n}) = \frac{Y_N(e^{j\omega_n})}{R_N(e^{j\omega_n})} \text{ (asymptotically unbiased)}$$

$$\hat{T}_{ur}(e^{j\omega_n}) = \frac{U_n(e^{j\omega_n})}{R_N(e^{j\omega_n})} \text{ (asymptotically unbiased)}$$

**Closed-loop identification approach:**

$$\boxed{\frac{T_{yr}}{T_{ur}} = \frac{SG}{S} = G \longrightarrow \hat{G}(e^{j\omega_n}) = \frac{\hat{T}_{yr}(e^{j\omega_n})}{\hat{T}_{ur}(e^{j\omega_n})}}$$

- The estimates $\hat{T}_{yr}$ and $\hat{T}_{ur}$ may be unbiased. **Their ratio is not.**
- The estimated spectra are weighted by $S(e^{j\omega_n})$ or $S(e^{j\omega_n})C(e^{j\omega_n})$.
- The noise enters in a complicated manner.

### 11.2.1 Ratio distributions

Two normal substitutions $v \in \mathcal{N}(0,1)$ and $w \in \mathcal{N}(0,1)$.
The ratio $z = v/w$ is a stochastic variable with PDF:

$$\boxed{f_z(z) = \frac{1}{\pi}\frac{1}{1+z^2}} \text{ Cauchy Distribution}$$

- The variance of a Cauchy distribution is infinite.

### 11.2.2 Averaging closed-loop estimates

$$U_l(e^{j\omega_n}), Y_l(e^{j\omega_n})$$

$$\tilde{G}(e^{j\omega_n}) = \frac{1}{L}\sum_{l=1}^{L}\left(\frac{Y_l(e^{j\omega_n})}{U_l(e^{j\omega_n})}\right)$$

or

$$\tilde{G}(e^{j\omega_n}) = \frac{\tilde{Y}(e^{j\omega_n})}{\tilde{U}(e^{j\omega_n})}$$

- Either top or below is correct which is frequency dependent. Find better explanation.

## 11.3 Dual-Youla Methods

### 11.3.1 Youla Parametrizations

**Coprime factorizations**

$$G_0(s) = \frac{N_o(s)}{D_0(s)}$$

with $N_0(s), D_0(s)$ stable and coprime (no common zeros). **Coprime factorizations are not unique**.

**Bezout identity**
The transfer functions $N_0(s)$ and $D_0(s)$ are coprime iff there exists $U(s)$ and $V(s)$ such that

$$U(s)N_0(s) + V(s)D_0(s) = I$$

**Normalized Coprime Factorizations**
A coprime factorization is normalized if

$$D_0^*(s)D_0(s) + N_0^*(s)N_0(s) = I$$

```
sncfbal()
```

**All stablizing controllers**
If we have a controller $C_0$ which stablizes $G_0$ with

$$C_0 = \frac{X_0}{Y_0} \text{ } (X_0, Y_0 \text{ a coprime factorization})$$

then, all controllers, $C$, stablizing $G_0 = N_0/D_0$ have the form:

$$\boxed{C_Q = \frac{X_0 + QD_0}{Y_0 - QN_0}, \text{ with } Q \text{ stable}}$$

### 11.3.2 Dual-Youla methods

**Control design:**
Given $G(s)$ select $C(s)$ form the set of controllers stabilizing $G(s)$.
**Closed-loop identification:**
Given a particular controller $C(s)$ select $G(s)$ from the set of all plants stabilized by $C(s)$.
**Both problems can be formulated as a search over stable $Q(s)$.**

1. **Formulation**

   $$y = Gu + He \rightarrow Dy = Nu + Fe$$

   where $e \in \mathcal{N}(0,1)$ and $D, N, F$ stable. Now find $D, N$ and possibly $F$ from the data, knowing $C_0 = X_0/Y_0$.

2. **Parametrization with $R$ and $F$**

   $$G_R = \frac{N}{D} = \frac{N_0 + RY_0}{D_0 - RX_0} \text{ } R \text{ is stable}$$

   $$H_{R,F} = \frac{F}{D} = \frac{F}{D_0 - RX_0} \text{ } F \text{ is stable and stably invertible}$$

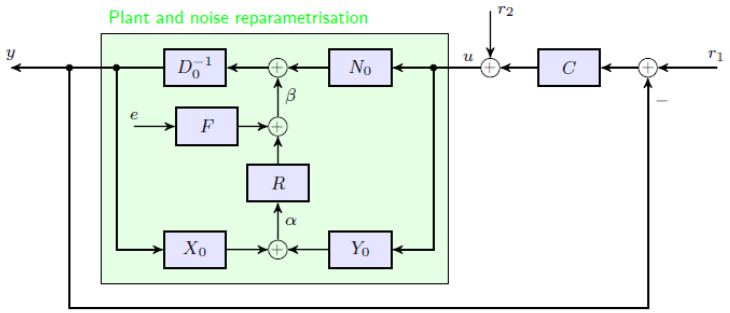
3. **Equivalent open-loop ID experiment:**

   $$(D_0 - RX_0)y = (N_0 + RY_0)u + Fe$$

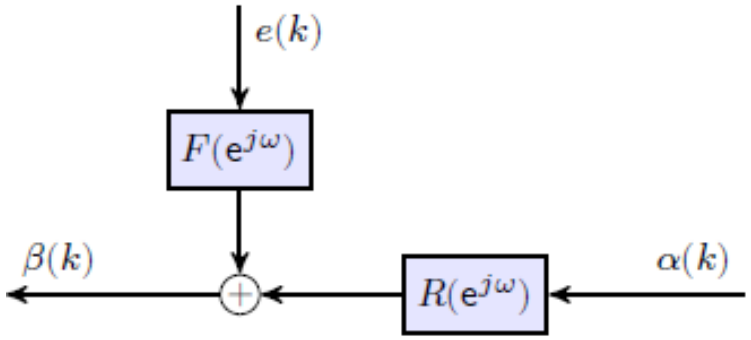
   and rearranging

   $$\underbrace{D_0 y - N_0 u}_{=:\beta} = R\underbrace{(X_0 y + Y_0 u)}_{=:\alpha} + Fe$$

   Open-loop system: $\beta = R\alpha + Fe$ with $R$ and $F$ stable.




- The transfer function from $\beta$ to $\alpha$ is zero. Thus the system is actually open loop! No feedback!




$\beta = D_0 y - N_0 u$ (filtered input and output signals)

$$\begin{aligned}\alpha &= X_0 y + Y_0 u \\ &= X_0 y + Y_0 \left( r_2 + \frac{X_0}{Y_0}(r_1 - y) \right) = X_0 y + Y_0 \left( r - \frac{X_0}{Y_0} y \right) \\ &= Y_0 r \text{ (filtered excitation signal)}\end{aligned}$$

### 11.3.3 Summary

1. Factorise: $C_0 = X_0 / Y_0$.
2. Choose excitation: $r$ (Note $\alpha = Y_0 r$ filtering).
3. Run closed-loop experiments with $C_0$, measuring $y$ and $u$.
4. Choose an initial model, $P_0 = N_0 / D_0$ (must be stabilised by $C_0$).
5. Filter measurements, $\beta = D_0 y - N_0 u$ (time or frequency domain).
6. Filter excitation, $\alpha = Y_0 r$.
7. Esimate $\hat{R}$ (and $\hat{F}$) from $\beta = R\alpha + Fe$.
8. Calculate plan estimate, $\hat{G} = (N_0 + \hat{R} Y_0)/(D_0 - \hat{R} X_0)$.

# 12 Time-Domain Correlation Methods

## 12.1 Parametrised Model Sets

We are looking for the plant, $G$, in a parametrised set

$$\{G(\theta)\}$$

where $\theta \in \mathbb{R}^d$ is the parameter vector.

| Model structure | Parameter vector $\theta \in \mathbb{R}^d$ |
|---|---|
| Pulse response: $g(k)$ | $[g(0) \quad g(1) \quad \ldots]$ |
| Transfer function: $\frac{B(z)}{A(z)}$ | $[a_1 \quad \ldots \quad b_1 \quad \ldots]$ |
| State-space: $\left[\begin{array}{c\|c} \text{A} & \text{B} \\ \hline \text{C} & \text{D} \end{array}\right]$ | $[A_{ij} \quad \ldots \quad B_{ij} \quad \ldots \quad C_{ij} \ldots \quad D_{ij} \quad \ldots]$ |

## 12.2 Identification Framework

$$Z_K = \{u(0), y(0), \ldots, u(K-1), y(K-1)\} \quad \text{Measurement data}$$

$$J(\theta, Z_K) \quad \text{Objective}$$

$$\hat{\theta} = \arg\min J(\theta, Z_K) \quad \text{General optimization formulation}$$

### 12.2.1 Possible Objectives

**Residual error objectives:**

$$e(k, \theta) = y(k) - G(\theta) u(k) \quad \text{error}$$

$$J(\theta) = ||e(\theta)||_2^2 \text{ or } ||e(\theta)||_\infty \text{ or } ||e(\theta)||_1$$

**Parametric error objective:**

$$J(\theta) = ||\theta - \theta_0||_2 \text{ or } E\{\theta - \theta_0\}$$

**Prediction error objective:**

$$J(\theta) = E\{y(k+1) - \hat{y}(k+1, \theta | k)\}$$

## 12.3 Correlation-Based Methods

**Input-Output relationship:**

$$y(k) = \sum_{i=0}^{\infty} g(i) u(k-i) + v(k)$$

$$\boxed{R_{yu}(\tau) = g(k) * R_u(\tau)}$$

This can be written in matrix form (Toeplitz):

$$\begin{bmatrix} R_{yu}(0) \\ R_{yu}(1) \\ R_{yu}(2) \\ \vdots \end{bmatrix} = \begin{bmatrix} R_u(0) & R_u(-1) & R_u(-2) & \cdots \\ R_u(1) & R_u(0) & R_u(-1) & \\ R_u(2) & R_u(1) & \ddots & \ddots \\ \vdots & & \ddots & \ddots \end{bmatrix} \begin{bmatrix} g(0) \\ g(1) \\ g(2) \\ \vdots \end{bmatrix}$$

Having only a finite data estimate:

$$\underbrace{\begin{bmatrix} \hat{R}_{yu}(0) \\ \vdots \\ \hat{R}_{yu}(N-1) \end{bmatrix}}_{\hat{R}_{yu}} = \underbrace{\begin{bmatrix} \hat{R}_u(0) & \cdots & \hat{R}_u(-(N-1)) \\ \vdots & & \vdots \\ \hat{R}_{yu}(N-1) & \cdots & \hat{R}_u(0) \end{bmatrix}}_{\bar{R}_N} \begin{bmatrix} \hat{g}(0) \\ \vdots \\ \hat{g}(N-1) \end{bmatrix}$$

- Note that $R_u(-\tau) = R_u(\tau)$.
- In the periodic (noise-free) and FIR case this is exact.
- $\hat{g}$ is uniquely determined, if $\bar{R}_N$ is invertible, which is given if $u(k)$ is persistently exciting.

## 12.4 Persistency of Excitation

A stationary input $u(k)$ is persistently exciting of order $n$ if

$$\bar{R}_n = \begin{bmatrix} R_u(0) & \cdots & R_u(-(n-1)) \\ \vdots & & \vdots \\ R_u(n-1) & \cdots & R_u(0) \end{bmatrix}$$

is positive definite.

- This is sufficient to uniquely determine the first $n$ coefficients of the pulse response, $\hat{g}k$) via the correlation approach.
- The definition also applies to deterministic signals.
- A signal is called **persistently exciting** if this holds for all $n$

**Spectra of persistently exciting signals**
$u(k)$ is persistently exciting of order $n$ if $\phi_u(e^{j\omega}) \neq 0$ for at least $n$ frequencies.
**Moving average (MA) filtering**

$$M_n(z) = m_1 z^{-1} + \cdots + m_n z^{-n} \quad n^{th} \text{ order MA filter}$$

If for all $n^{th}$ order MA filters,

$$|M_n(e^{j\omega})|^2 \phi(u)(e^{j\omega}) = 0 \Rightarrow M(e^{j\omega}) = 0$$

then $u(k)$ is persistently exciting of order at least $n$.

- A step function is persistently exciting of order 1.
- A $PRBS$ signal is persistently exciting of order $M$.
- A sum of sinusoids $u(k) = \sum_{s=1}^{S} \alpha_s \cos(\omega_s k + \phi_s)$ is persistently exciting of order.

$$\begin{cases} 2S & \text{if } 0 < \omega_s < \pi,\ s = 1, \ldots, S \\ 2S-1 & \text{if } \omega = 0 \text{ or } \omega = \pi \in \{\omega_s, s = 1, \ldots, S\} \\ 2S-2 & \text{if } \omega = 0 \text{ and } \omega = \pi \in \{\omega_s, s = 1, \ldots, S\} \end{cases}$$

## 12.5 Autoregressive moving average models

Model form:

$$G(z) = \frac{b_1 z^{-1} + \ldots + b_m z^{-m}}{1 + a_1 z^{-1} + \ldots + a_n z^{-n}}$$

Input-Output relationship:

$$\begin{aligned} y(k) &= G(z)u(k) \\ &= -a_1 y(k-1) - \cdots - a_n y(k-n) + b_1 u(k-1) + \cdots + b_m u(k-m) \\ &= \phi^T(k)\theta \end{aligned}$$

where

$$\begin{aligned} \phi(k) &= \begin{bmatrix} -y(k-1) & \cdots & -y(k-n) & u(k-1) & \cdots & u(k-m) \end{bmatrix}^T \\ \theta &= \begin{bmatrix} a_1 & \cdots & a_n & b_1 & \cdots & b_m \end{bmatrix}^T \end{aligned}$$

where $\phi(k)$ is called the **regressor vector** and $\theta$ is called the **parameter vector**.

### 12.5.1 ARX or ARMAX

Consider $y(k) = \phi^T(k)\theta$ for $k = 0, \ldots, N-1$

$$\underbrace{\begin{bmatrix} y(0) \\ \cdots \\ y(N-1) \end{bmatrix}}_{Y} = \underbrace{\begin{bmatrix} \phi^T(0) \\ \cdots \\ \phi^T(N-1) \end{bmatrix}}_{\Phi} \theta$$

or in matrix form: $Y = \Phi\theta$
**Least squares solution:**

$$\hat{\theta} = \left(\Phi^T\Phi\right)^{-1}\Phi^T Y$$

```
Theta = Phi\Y;
```

The least squares solution $\hat{\theta}$ solves the problem:

$$\begin{aligned} \underset{\theta}{minimise} &||\epsilon||_2 \\ \text{subject to} &Y = \Phi\theta + \epsilon \end{aligned}$$

Thus the cost function is defined as

$$J(\theta, Z_N) = ||Y - \Phi\theta||_2 = ||\epsilon||_2$$

### 12.5.2 Statistical Properties of the LS Estimate

$$Y = \Phi\theta + \epsilon, \quad \epsilon = \begin{bmatrix} \epsilon(0) \\ \vdots \\ \epsilon(N-1) \end{bmatrix} \quad \text{Model}$$

Error assumptions: $E\{e\} = 0$ and $E\{\epsilon\epsilon^T\} = \sigma^2 I$.

$$\boxed{E\{\hat{\theta}\} = \theta \text{ (unbiased estimator)}}$$

$$\boxed{\text{cov}\{\theta\} = E\{(\hat{\theta} - \theta)(\hat{\theta} - \theta)^T\} = \sigma^2(\Phi^T\Phi)^{-1}}$$

**For a model with correlated noise:**

$$E\{\epsilon\epsilon^T\} = R$$

$$\boxed{E\{\hat{\theta}\} = \theta \text{ (unbiased estimator)}}$$

$$\boxed{\text{cov}\{\theta\} = E\{(\hat{\theta} - \theta)(\hat{\theta} - \theta)^T\} = (\Phi^T\Phi)^{-1}\Phi^T R\Phi(\Phi^T\Phi)^{-1}}$$

Thus the variance of the error depends on the observations $\Phi$!

### 12.6 Best linear unbiased estimator (BLUE or Markov estimator)

**For a model with correlated noise:**

$$E\{\epsilon\epsilon^T\} = R$$

Best linear estimator:

$$\hat{\theta} = Z^T Y \text{ where } Z = R^{-1}\Phi(\Phi^T R^{-1}\Phi)^{-1}$$

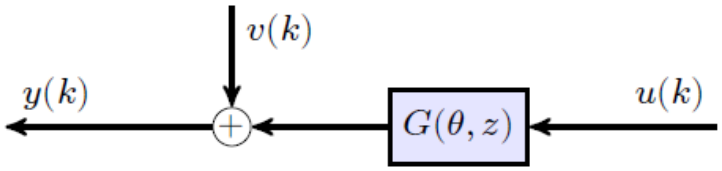
satisfies:

$$\boxed{E\{\hat{\theta}\} = \theta \text{ (unbiased estimator)}}$$

$$\boxed{\text{cov}\{\hat{\theta}_Z\} = (\Phi^T R{-}1\Phi)^{-1} \leq \text{cov}\{\hat{\theta}\} \text{ for any unbiased estimate}}$$

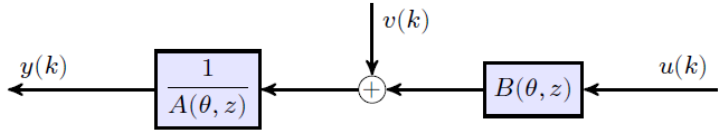
- BLUE requires knowledge of the error covariance, $R$, in order to reduce the variance of the error of the estimation.

### 12.7 Pitfall of Noisy Case




The setup above is only valid for the noise-free case. As soon as there is noise the equation derived from the setup above delivers biased results.

In actuality the problem can only be solved as a linear least squares problem if the setup can be described like this:




If this is not the case we get biased results. A possible remedy would be:

1. Assume the system to be of the form:

$$\begin{bmatrix} y(0) \\ \vdots \\ y(N-1) \end{bmatrix} = \begin{bmatrix} \phi^T(0) \\ \vdots \\ \phi^T(N-1) \end{bmatrix} \theta + \begin{bmatrix} v(0) \\ \vdots v(N-1) \end{bmatrix}$$

   where $\phi^T(k) = \begin{bmatrix} -y(k-1) & \cdots & -y(k-N) & u(k-1) & \cdots & u(k-m) \end{bmatrix} 1 \begin{bmatrix} -v(k-1) & \cdots \end{bmatrix}$

2. Assume zero noise influence in the regressor $\phi(k)$.
3. Solve with linear least squares.
4. Calculate the $\hat{V} = Y - \Phi\hat{\theta}$ as the residuals.
5. Recalculate taking into account the estimate of the noise $\hat{v}(k)$.
6. Repeat.

## 13 Prediction Error Methods

Given $Z_K = \{u(0), y(0), \ldots, u(K-1), y(K-1)\}$ what is the best estimate of $y(K)$?
Typical assumptions

- $G(z)$ and $H(z)$ stable.

- $H(z)$ is stably invertible (no zeros outside the unit disk).

  Given $v(k),\ k = 0, \ldots, K-1$ can we determine $e(k)$?

  $$H_{inv}(z): \quad e8k) = \sum_{i=0}^{\infty} h_{inv}(i)v(k-i) \quad \text{Inverse filter}$$

  Note that the inverse filter has to be causal: $h_{inv}(k) = 0, k < 0$ and stable: $\sum_{k=0}^{\infty} |h_{inv}(k)| < \infty$.

  If $H(z)$ has no zeros for $|z| \geq 1$ then $H_{inv}(z) = \frac{1}{H(z)}$.

- $e(k)$ has known statistics.

## 13.1 One step ahead prediction

Assume $H(z)$ is known, $H(z)$ monic ($h(0) = 1$).

$$\begin{aligned} v(k) &= \sum_{i=0}^{\infty} h(i)e(k-i) \\ &= \underbrace{e}_{h(0)v(k)}(k) + \underbrace{\sum_{i=1}^{\infty} h(i)e(k-i)}_{=m(k-1)\ \text{„observed“}} \end{aligned}$$

The above simplification of $h(0)v(k) = e(k)$ only applies if $H(z)$ is monic, i.e. $h(0) = 1$.

$$\hat{v}(k|k-1) \quad \text{Prediction based on measurement}$$

which leads to

$$\hat{v}(k|k-1) = m(k-1) = \sum_{i=1}^{\infty} h(i)e(k-i)$$

The error in this prediction is then $e(k)$ which cannot be reduced.

In general:

$$\text{Prob}\{x \leq e(k) \leq x + \delta x\} = \int_x^{x+\delta x} f_e(x)dx \approx f_e(x)\delta x$$

$$\text{Prob}\{x \leq v(k) \leq x + \delta x | v_{-\infty}^{k-1}\} \leq f_e(x - m(k-1))\delta x$$

which is equivalent to a shift of the distribution with $m(k-1)$. The expected value, the mean or the most likely value (which is not the same for an asymmetric distribution) of the shifted distribution would be a valid choice for $\hat{v}(k|k-1)$.

$$\boxed{\hat{v}(k|k-1) = m(k-1) = \sum_{i=1}^{\infty} h(i)e(k-i) = -\sum_{i=1}^{\infty} h_{inv}(i)v(k-i) \approx -\sum_{i=1}^{k} h_{inv}(i)v(k-i)}$$

### 13.1.1 Moving Average Model

$$v(k) = e(k) + ce(k-1) \Rightarrow H(z) = 1 + cz^{-1}$$

$$H_{inv}(z) = \frac{1}{1 + cz^{-1}} = \sum_{i=0}^{\infty} (-c)^i z^{-i}$$

where $|c| < 1$ for stable invertibility.

$$\boxed{\hat{v}(k|k-1) = cv(k-1) - c^2 v(k-2) + c^3 v(k-3) + \cdots + -(-c)^k v(0)}$$

Written recursively:

$$\begin{aligned} H(z)\hat{v}(k|k-1) &= (H(z) - 1)v(k) \\ \hat{v}(k|k-1) + c\hat{v}(k-1|k-2) &= cv(k-1) \\ \hat{v}(k|k-1) &= c\underbrace{(v(k-1) - \hat{v}(k-1|k-2))}_{\epsilon(k-1)} \\ &= c\epsilon(k-1) \end{aligned}$$

### 13.1.2 Autoregressive Noise Model

$$v(k) = \sum_{i=0}^{\infty} a^i e(k-i) \quad |a| < 1 \text{ for stablity}$$

$$H(z) = \sum_{i=0}^{\infty} a^i z^{-i} = \frac{1}{1 - az^{-1}}$$

$$H_{inv}(z) = 1 - az^{-1}$$

$$\boxed{\hat{v}(k|k-1) = (1 - H_{inv}(z))v(k) = av(k-1)}$$

## 13.2 Output Prediction

$$\begin{aligned} y(k) &= G(z)u(k) + v(k) \\ \hat{y}(k|k-1) = \text{E}\{y(k)|Z_K\} &= G(z)u(k) + \hat{v}(k|k-1) \\ &= G(z)u(k) + (1 - H_{inv}(z))v(k) \\ &= H_{inv}(z)G(z)u(k) + (1 - H_{inv}(z))y(k) \end{aligned}$$

Note that the last line is obtained by replacing $v(k) = y(k) - G(z)u(k)$.

The prediction error would then be:

$$\begin{aligned} y(k) - \hat{y}(k|k-1) &= -H_{inv}(z)G(z)u(k) + H_{inv}(z)y(k) \\ &= H_{inv}(z)(y(k) - G(z)u(k)) = H_{inv}(z)v(k) \\ &= e(k) \end{aligned}$$

The one-step ahead predictor and the prediction error are parametrised by $\theta$. This leads to an optimisation problem:

$$J(\theta, Z_K) = \frac{1}{K}\sum_{k=0}^{K-1} l(\epsilon_F(k,\theta)) \text{ where typically } l(\epsilon_F(k,\theta)) = ||\epsilon_F(k,\theta)||_2$$
$$\hat{\theta} = \underset{\theta}{\text{argmin}} J(\theta, Z_K)$$

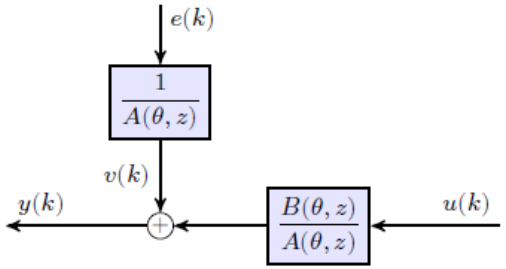
where $\epsilon_F$ is an optionally filtered error signal.

## 13.3 Different Models

| Polynomials used | Name of the Model Structure |
|---|---|
| B | FIR |
| AB | ARX |
| ABC | ARMAX |
| AC | ARMA |
| ABD | ARARX |
| ABCD | ARARMAX |
| BF | OE (output error) |
| BFCD | BJ (Box Jenkins) |
| ABCDF | General Model |

Table 1: Model naming terminology

### 13.3.1 Equation Error Model Structure: ARX




$$\boxed{y(k) + a_1 y(k-1) + \cdots + a_{n_a} y(k-n_a) = b_1 u(k-1) + \cdots + b_{n_b} u(k-n_b) + e(k)}$$

$$\theta = \begin{bmatrix} a_1 & a_2 & \cdots & a_{n_a} & b_1 & \cdots & b_{n_b} \end{bmatrix}^T$$

\+ The one step ahead predictor directly defines a linear regression.

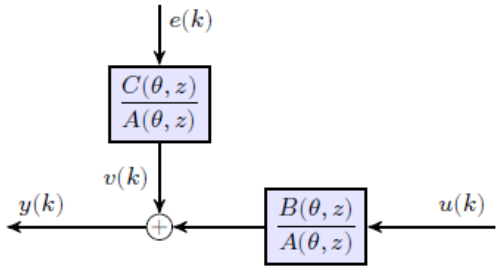
\- Lack of adequate freedom in describing the properties of the disturbance term.

$$\begin{aligned} \hat{y}(k|\theta) &= H_{inv}(\theta,z)G(\theta,z)u(k) + (1 - H_{inv}(\theta,z))y(k) \\ &= B(z)u(k) + (1 - A(z))y(k) \\ &= \theta^T\phi(k) = \phi^T(k)\theta \end{aligned}$$

$$\boxed{\text{vector of prediction errors}}\ Y - \phi\theta = \epsilon$$

- In this case the smallest error is also the smallest prediction error which is not generally the case.

### 13.3.2 Equation error structure: ARMAX




The ARMAX model structure expands the flexibility of the ARX by describing the equation error as a moving average of white noise.

$$\boxed{\begin{aligned} &y(k) + a_1 y(k-1) + \cdots + a_{n_a} y(k-n_a) = b_1 u(k-1) + \\ &\cdots + b_{n_b} u(k-n_b) + e(k) + c_1 e(k-1) + \cdots + c_{n_c} e(k-n_c) \end{aligned}}$$

with $C(z) = 1 + c_1 z^{-1} + \cdots + c_{n_c} z^{-n_c}$. The whole thing can be written as:

$$A(z)y(t) = B(z)u(t) + C(z)e(t)$$

$$
\begin{aligned}
\hat{y}(k|\theta) &= \frac{B(z)}{C(z)}u(k) + \left(1 - \frac{A(z)}{C(z)}\right)y(k) \\
C(z)\hat{y}(k|\theta) &= B(z)u(k) + (C(z) - A(z))y(k) \\
\hat{y}(k|\theta) &= B(z)u(k) + (1 - A(z))y(k) + (C(z) - 1)\underbrace{(y(k) - \hat{y}(k|\theta))}_{\epsilon(k)} \\
&= \begin{bmatrix} b_1 & \cdots & a_1 & \cdots & c_1 & \cdots \end{bmatrix} \\
&\quad \begin{bmatrix} u(k-1) & \cdots & -y(k-1) & \cdots & \epsilon(k-1) & \cdots \end{bmatrix} \\
&= \phi^T(\theta, k)\theta
\end{aligned}
$$

This is not linear in $\theta$, since $\epsilon$ depends on the chosen parametrisation, but is called pseudo-linear.

**Optimisation:**

$$
\begin{aligned}
&\underset{\theta,\epsilon}{\text{minimise}} ||\epsilon||_2 \quad \text{(or more generally, } l(\epsilon)\text{)} \\
&\text{subject to } Y = \Phi(\epsilon)^T\theta + \epsilon \quad \text{(nonlinear equality constraint)}
\end{aligned}
$$

**Constrained minimisation code for ARMAX**

$$
\begin{aligned}
A(z) &= 1 + a_1 z^{-1} + a_2 z^{-2} \\
B(z) &= b_1 z^{-1} + b_2 z^{-2} \\
C(z) &= 1 + c_1 z^{-1} + c_2 z^{-2} \\
\theta &= \begin{bmatrix} b_1 & b_2 & a_1 & a_2 & c_1 & c_2 \end{bmatrix}^T
\end{aligned}
$$

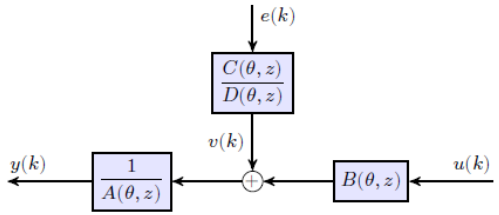
Fix code, probably not running!

```
%Create data part of regressor. Assume plant at rest
PhiTyu(1,:) = [0,0,0,0];
PhiTyu(2,:) = [u(1),0,-y(1),0];
for i = 3:K,
  PhiTyu(i,:) = [u(i-1),u(i-2),-y(i-1),-y(i-2)];
end

[x,fval] = fmincon(@(x)ARMAXobjective(x),x0,...
[],[],[],[],[],[],@(x)ARMAXconstraint(x,y,PhiTyu));
function [f] = ARMAXobjective(x) % x = [theta; e]
f = sqrt(x(7:end)'*x(7:end));
function [c,ceq] = ARMAXconstraint(x,y,PhiTyu)
e = x(7:end);
PhiTe = zeros(K,2);
PhiTe(2,1) = e(1);
for j = 3:K,
PhiTe(j,:) = [e(j-1), e(j-2)];
end
ceq = y - [PhiTyu, PhiTe] * theta - e; c = [];
```

### 13.3.3 Equation error structure: ARARMAX

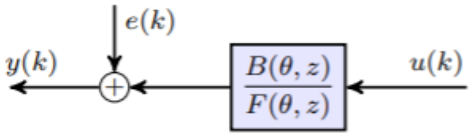
Instead of modelling the equation error as a moving average in can also be described as an autoregression, leading to a ARARX structure. A further generalization using an ARMA description of the equation error leads to the ARARMAX structure below:




$$
\begin{aligned}
G(\theta, z) &= \frac{B(\theta, z)}{A(\theta, z)} \\
H(\theta, z) &= \frac{C(\theta, z)}{A(\theta, z)D(\theta, z)}
\end{aligned}
$$

### 13.3.4 Output Error Model Structure

Contrary to the equation error model structure, this approach parametrizes the transfer functions independently.




$$\boxed{\begin{aligned} &w(t,\theta) + f_1 w(t-1,\theta) + \cdots + f_{n_f} w(t-n_f,\theta) \\ &= b_1 u(t-1) + \cdots + b_{n_b} u(t-n_b) \end{aligned}}$$

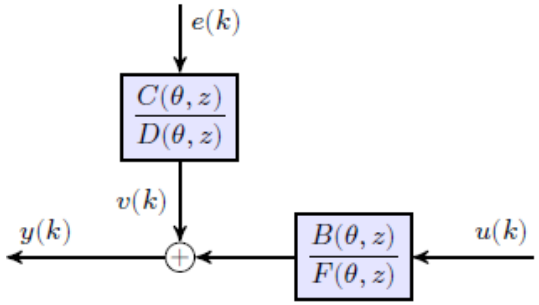
$$\begin{aligned} G(\theta,z) &= \frac{B(\theta,z)}{F(\theta,z)} \\ H(\theta,z) &= 1 \\ \hat{y}(k|\theta) &= \frac{B(\theta,z)}{F(\theta,z)} u(k) = \phi(k,\theta)^T \theta \end{aligned}$$

$$\theta = \begin{bmatrix} b_1 & b_2 & \cdots & b_{n_b} & f_1 & f_2 & \cdots & f_{n_f} \end{bmatrix}$$

### 13.3.5 Box-Jenkins Model Structure

The Box-Jenkins Model Structure naturally ensues the Output Error, by further modelling the properties of the output error with an ARMA model:

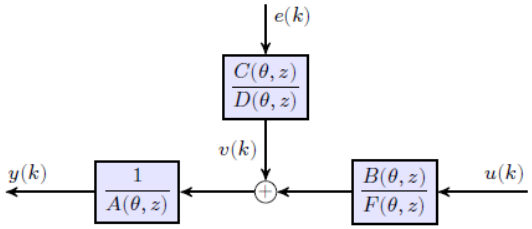
$$\boxed{y(t) = \tfrac{B(z)}{F(z)} u(t) + \tfrac{C(z)}{D(z)} e(t)}$$




$$\begin{aligned} G(\theta,z) &= \frac{B(\theta,z)}{F(\theta,z)} \\ H(\theta,z) &= \frac{C(\theta,z)}{D(\theta,z)} \\ \hat{y}(k|\theta) &= \frac{D(z)}{C(z)} \frac{B(z)}{F(z)} u(k) + \left(1 - \frac{D(z)}{C(z)}\right) y(k) \end{aligned}$$

### 13.3.6 General Model Structure

The total of the five transfer functions $A(z), B(z), C(z), D(z), E(z)$ enables a total




$$\begin{aligned} G(\theta,z) &= \frac{B(\theta,z)}{A(\theta,z)F(\theta,z)} \\ H(\theta,z) &= \frac{C(\theta,z)}{A(\theta,z)D(\theta,z)} \\ \hat{y}(k|\theta) &= \frac{D(z)}{C(z)} \frac{B(z)}{F(z)} u(k) + \left(1 - \frac{D(z)A(z)}{C(z)}\right) y(k) \end{aligned}$$

**Pseudolinear regression:**

$$\epsilon(k,\theta) = y(k) - \hat{y}(k|\theta) = \tfrac{D(z)}{C(z)} \left[ A(z) y(k) - \tfrac{B(z)}{F(z)} u(k) \right]$$

we newly define: $w(k,\theta) = \frac{B(z)}{F(z)} u(k)$ and $v(k,\theta) = A(z)y(t) - w(k,\theta)$, thus

$$\epsilon(k,\theta) = \tfrac{D(z)}{C(z)} v(k,\theta)$$

State vector:

$$\begin{aligned} \phi(k,\theta) = [&-y(k-1),\ \cdots,\ -y(k-n_a),\ u(k-1),\ \cdots,\ u(k-n_b) \\ &-w(k-1,\theta),\ \cdots,\ -w(k-n_f,\theta),\ \epsilon(k-1,\theta),\ \cdots,\ \epsilon(k-n_c,\theta)\ ^T \\ &-v(k-1,\theta),\ \cdots,\ -v(k-n_d,\theta)] \end{aligned}$$

with parameter vector:

$$\theta = \begin{bmatrix} a_1 & \cdots & a_{n_a} & b_1 & \cdots & b_{n_b} & f_1 & \cdots & f_{n_f} & c_1 & \cdots & c_{n_c} & d_1 & \cdots & d_{n_d} \end{bmatrix}^T$$

### 13.4 Known noise model (with ARMAX dynamics)

Assume known noise: $v(k) = L(z)e(k)$
Thus

$$A(z)y(k) = B(z)u(k) + L(z)e(k)$$

$$y_L(k) = L^{-1}(z)y(k)$$
$$u_L(k) = L^{-1}(z)u(k)$$
$$A(z)y_L(k) = B(z)u_L(k) + e(k)$$

for which least squares gives consistent estimates.

### 13.5 High-order model fitting

Assume an ARARX structure:

$$A(z)y(k) = B(z)u(k) + \tfrac{1}{D(z)}e(k) \quad e(k) \sim \mathcal{N}(0, \lambda)$$

Fitting a high order model (order of $D(z)$ is $n_d$)

$$A(z)D(z)y(k) = B(z)D(z)u(k) + e(k)$$

Least squares estimate with orders $n + n_d$ and $m + n_d$. This gives a consistent estimate of

$$\tfrac{B(z)D(z)}{A(z)D(z)} = \tfrac{B(z)}{A(z)}$$

This amounts to making the noise model sufficiently rich to capture additional autoregressive features in the noise. In practice though, the cancellation will not be exact. $\hat{A}(z)$ and $\hat{B}(z)$ will be high order.

## 14 Parameter estimation statistics

### 14.1 Model description

As found in chapter 4, Ljung:

$$\boxed{y(t) = G(z,\theta)u(t) + H(z,\theta)e(t), \quad f_e(x,\theta)}$$

where $f_e(x,\theta)$ is the PDF of $e(t)$ and $e(t)$ is assumed to be white noise.

**Definition 7.** *The **moments** of a PDF describe generalized concept of expected values:*

$$Ee^m = \in x^m f_e(x)dx = 0$$

*where $m$ is a positive integer. For $m = 1$, the first moment, is the expected value of a PDF, for $m = 2$, the second moment, is the variance of a PDF.*

### 14.2 Basic setup

$$G = G(\theta, z), \quad H = H(\theta, z) \quad \text{Parametrised model}$$

$$\hat{\theta} = \underset{\theta}{\text{argmin}} J(\theta, Z_K) \quad \text{Estimation}$$

#### 14.2.1 Maximum Likelihood approach

Consider $K$ observations. Each is a realisation of a random variable, with joint probability distribution

$$f(x_1, \ldots, x_K|\theta)$$

For independent variables

$$f(x_1, \ldots, x_K; \theta) = f_1(x_1; \theta)f_2(x_2; \theta)\cdots f_K(x_K; \theta) = \prod_{i=1}^{n} f_i(x_i; \theta)$$

Substituting the observation $Z_K$ gives a function of $\theta$

$$\mathcal{L}(\theta) = f(x_1, \ldots, x_k; \theta)|_{x_i = z_i, i=1,\ldots,K} \quad \text{Likelihood function}$$

$$\hat{\theta}_{ML} = \underset{\theta}{\text{argmax}} \mathcal{L}(\theta) \quad \text{Maximum likelihood estimator}$$

The log-likelihood is often mathematically easier to consider

$$\hat{\theta}_{ML} = \underset{\theta}{\text{argmax}} \ln \mathcal{L}(\theta)$$

It will give the same $\hat{\theta}$ since ln is monotonic.

### 14.3 Bayesian approach

Consider $\theta$ to be a random variable with pdf: $f_\theta(x)$, where $x_i = z_i$. This is an **a priori** distribution assumed before the experiment. Thus we have some knowledge about the variable that is to be estimated.

$$\mathbb{P}(\theta|z_1, \ldots, z_K) = \tfrac{\mathbb{P}(Z_K \cap \theta)}{\mathbb{P}(Z_K)} \quad \text{Conditional Probability}$$

$$\mathbb{P}(Z_K \cap \theta) = \mathbb{P}(Z_K|\theta)\,\mathbb{P}(\theta) \quad \text{Since } \theta \text{ is not independent from } Z_K$$

which combined results in

$$\boxed{\mathbb{P}(\theta|z_1, \ldots, z_K) = \tfrac{\mathbb{P}(Z_K|\theta)\mathbb{P}(\theta)}{\mathbb{P}(Z_K)}} \text{ Bayes' Theorem}$$

where $\mathbb{P}(\theta)$ contains the prior knowledge.
So

$$\boxed{\underset{\theta}{\text{argmax}}\ f(\theta|z_1, \ldots, z_K) = \underset{\theta}{\text{argmax}}\ f(Z_K|\theta)f_\theta(\theta)}$$

where $f_\theta(\theta)$ weights the maximum likelihood estimator towards the expected/most realistic value.

## 14.4 Maximum a posteriori (MAP) estimation

The MAP is closely related to ML estimation but in addition employs an augmented optimization objective which incorporates a prior distribution that quantifies the additionally available information on the estimated quantity.
Given the data $Z_K$

$$\hat{\theta}_{MAP} = \underset{\theta}{\operatorname{argmax}}\ f(Z_K|\theta) f_\theta(\theta)$$

The maximum likelihood estimator can be interpreted as:

$$\begin{aligned}\theta_{ML} &= \underset{\theta}{\operatorname{argmax}} f(x_1, \ldots, x_K|\theta)\Big|_{x_i=z_i, i=1,\ldots,K} \\ &= \underset{\theta}{\operatorname{argmax}} f(Z_K|\theta)\end{aligned}$$

This estimate coincides with the simple maximum likelihood estimate if $\theta$ is assumed to be uniformly distributed.

## 14.5 Cramér-Rao bound

The Cramér-Rao bound expresses a lower bound on the variance of unbiased estimators of a deterministic (fixed, though unknown) parameter.

$$P = \mathrm{E}\left\{(\hat{\theta}(Z_K) - \theta_0)(\hat{\theta}(Z_K) - \theta_0)^T\right\} \quad \text{Mean-square error matrix}$$

Assume $\mathrm{E}\left\{\hat{\theta}(Z_K)\right\} = \theta_0$ and $Z_K \subset \mathcal{R}^K$. Then $P \geq M^{-1}$ where $M$ is the **Fischer information Matrix**

$$\begin{aligned}M &= \mathrm{E}\left\{(\frac{d}{d\theta}\ln f(Z_K|\theta))(\frac{d}{d\theta}\ln f(Z_K|\theta)^T\right\}\Big|_{\theta=\theta_0} \\ &= -\mathrm{E}\left\{\frac{d^2}{d\theta^2}\ln f(Z_K|\theta)\right\}\Big|_{\theta=\theta_0}\end{aligned}$$

Consider a parametrised family of pdfs $f(x_1, \ldots, x_K|\theta) = \prod_{i=1}^{K} f_i(x_i|\theta)$
Then

$$\lim_{K\to\infty} \hat{\theta}_{ML} \overset{w.p.\ 1}{\longrightarrow} \theta_0$$

and

$$\lim_{K\to\infty} \sqrt{K}(\hat{\theta}_{ML}(Z_K) - \theta_0) \sim \mathcal{N}(0, M^{-1})$$

## 14.6 Prediction error statistics

The prediction error framework is set up as follows:

$$\epsilon(k, \theta) = y(k) - \hat{y}(k, \theta)$$

where the error $\epsilon(k, \theta)$ is assumed to be independent and identically distributed variables with pfd $f_e(x; \theta)$.
The joint pdf writes as

$$f(X_K; \theta) = \prod_{k=1}^{K} f_e(\epsilon(k, \theta); \theta)$$

$$\begin{aligned}\hat{\theta}_{ML} &= \underset{\theta}{\operatorname{argmax}}\ f(X_K; \theta)|_{X_K=Z_K} \\ &= \underset{\theta}{\operatorname{argmax}}\ \mathcal{L}(\theta) \\ &= \underset{\theta}{\operatorname{argmax}}\ \ln f(Z_K|\theta) \\ &= \underset{\theta}{\operatorname{argmax}}\ \frac{1}{K}\sum_{k=1}^{K} \ln f_e(\epsilon(k, \theta); \theta)\end{aligned}$$

The prediction error methods give the same estimate if the prediction error cost function is chosen as

$$l(\epsilon, \theta) = -\ln f_\epsilon(\epsilon|\theta)$$

Thus

$$\boxed{\hat{\theta}_{PE} = \underset{\theta}{\operatorname{argmin}}\ \tfrac{1}{K}\sum_{k=1}^{K} l(\epsilon(k, \theta); \theta) = \hat{\theta}_{ML}}$$

## 14.7 Linear regression statistics

One-step ahead predictor

$$\hat{y}(k|\theta) = \phi^T(k)\theta + \mu(k)$$

In the ARX case $\mu(k) = e(k)$, in other cases $\mu(k)$ can depend on $Z_K$.

$$\epsilon(k) = y(k) - \phi^t(k)\theta$$

A typical cost function is:

$$J(\theta, Z_K) = \tfrac{1}{K}\sum_{k=0}^{K-1} \tfrac{\epsilon(k)^2}{2}$$

Least-squares criterion:

$$\hat{\theta}_{LS} = \underbrace{\left(\frac{1}{K}\sum_{k=0}^{K-1}\phi(k)\phi^T(k)\right)^{-1}}_{R_K^{-1}\in\mathbb{R}^{d\times d}}\underbrace{\frac{1}{K}\sum_{k=0}^{K-1}\phi(k)y(k)}_{f_K\in\mathcal{R}^d}$$

Asymptotic bias:

$$\lim_{K\to\infty}\hat{\theta}_{LS}-\theta_0 = \lim_{K\to\infty} R_K^{-1}\tfrac{1}{K}\sum_{k=0}^{K-1}\phi^T(k)v(k) = (R^*)^{-1}f^*$$

$$R^* = \mathrm{E}\left\{\phi(k)\phi^T(k)\right\}, \quad f^* = \mathrm{E}\left\{\phi(k)v(k)\right\}$$

Under the assumption that $\lim_{K\to\infty}\hat{\theta}_{LS}=\theta_0$ we require $(R^*)^{-1}f^*=0$

So

1. $R^*$ must be non-singular. Persistency of excitation requirement.
2. $f^* = \mathrm{E}\left\{\phi(k)v(k)\right\} = 0$ This happens if either
   a) $v(k)$ is zero mean and independent of $\phi(k)$
   b) $u(k)$ is independent of $v(k)$ and $G$ is FIR ($n=0$)

This gives

$$\lim_{K\to\infty}\sqrt{K}(\hat{\theta}_{LS}-\theta_0)\sim\mathcal{N}(0,\sigma_0^2(R^*)^{-1})$$

## 14.8 Correlation methods

Ideally the sequence of prediction errors is white. Thus the prediction errors are uncorrelated with the data.

**Approach:**

1. Select a sequence $\zeta(k)$, derived from the past data $Z_K$
2. Require that the error $\epsilon(k,\theta)$ is uncorrelated with $\zeta(k)$
   $$\tfrac{1}{K}\sum_{k=0}^{K-1}\zeta(k)\epsilon(k,\theta)=0 \text{ could also use } \alpha(\epsilon)$$
3. The ID problem can be then viewed as finding $\theta$ such that this relationship is satisfied
   The values $\zeta(k)$ are known as **instruments**.
   Typically $\zeta(k)\in\mathbb{R}^{d\times n_y}$ where $\theta\in\mathbb{R}^d$ and $y(k)\in\mathcal{R}^{n_y}$.

**Procedure:**

1. Choose a linear fitter for the prediction errors
   $$\epsilon_F(k,\theta)=F(z)\epsilon(k,\theta)$$
2. Choose a sequence of correlation vectors $\zeta(k,Z_K,\theta)$ constructed from the data
3. Choose a function $\alpha(\epsilon)$ (default is $\alpha(\epsilon)=\epsilon$) Then
   $$\hat{\theta}=\theta \text{ solving } f_K(\theta,Z_K)=\tfrac{1}{K}\sum_{k=0}^{K-1}\zeta(k,\theta)\alpha(\epsilon(k,\theta))=0$$

### 14.8.1 Pseudo-Linear Regressions

For ARX, ARMAX, etc., model structures we can write the predictor

$$\hat{y}(k|\theta)=\phi^T(k,\theta)\theta$$

This can be solved via LS methods, but alternatively correlation based solutions can be found:

$$\boxed{\hat{\theta}_{PLR}=\theta \text{ solving } \tfrac{1}{K}\sum_{k=0}^{K-1}\phi(k,\theta)\underbrace{(y(k)-\phi^T(k,\theta)\theta)}_{\text{prediction error}}=0}$$

The prediction errors are orthogonal to the regressor, $\phi(k,\theta)$.

## 14.9 Instrumental Variable Methods

$$\hat{\theta}_{IV}=\theta \text{ solving } \tfrac{1}{K}\sum_{k=0}^{K-1}\zeta(k,\theta)(y(k)-\phi^T(k,\theta)\theta)=0$$

This is solved by

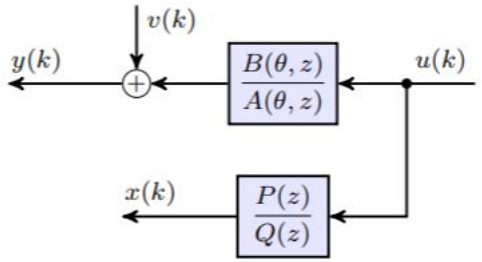
$$\hat{\theta}_{IV}=\left(\tfrac{1}{K}\sum_{k=0}^{K-1}\zeta(k)\phi^T(k)\right)^{-1}\tfrac{1}{K}\sum_{k=0}^{K-1}\zeta(k)y(k)$$

For consistency we require

$$\mathrm{E}\left\{\zeta(k)\phi^T(k)\right\} \quad \text{to be nonsingular}$$

and

$$\mathrm{E}\left\{\zeta(k)v(k)\right\}=0 \quad \text{(uncorrelated w.r.t. prediction error)}$$




1. Estimate $\hat{\theta}_{LS}$ via linear regression.

2. Select $Q(z) = \hat{A}_{LS}$ and $P(z) = \hat{B}_{LS}$, since the procedure works well when $P$ and $Q$ are close to $B$ and $A$.

3. Calculate $\hat{\theta}_{IV}$

- Variance and MSE depend on the choice of instruments.
- Consistency (asymptotically unbiased) is lost if:
  - Noise and instruments are correlated (for example, in closed-loop generating instruments from $u$)
  - Model order selection is incorrect.
  - Filter dynamics cancel plant dynamics.
  - True system is not in the model set.
- Closed-loop approaches: generate instruments from the excitation $r$.

# 15 Nomenclature

| | | |
|---|---|---|
| $y(k) = Gu(k)$ | output signal | [ ] |
| $u(k)$ | input signal | [ ] |
| $G$ | plant | [ ] |
| $\hat{G} = \frac{y}{u}$ | estimated plant | [ ] |
| $Y(e^{j\omega})$ | output spectrum | [ ] |
| $U(e^{j\omega})$ | input spectrum | [ ] |
| ZOH | zero order hold | |
| DAC | digital analog converter | |
| ADC | analog digital converter | |

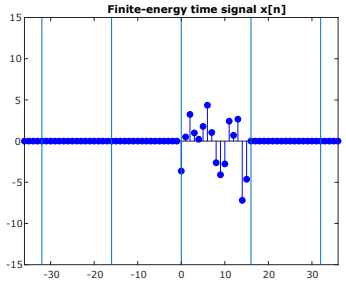
# 16 Overviews

## 16.1 Transfer Functions

### Nonperiodic Time Domain Signal

$$x[n] = \int_0^1 \hat{x}(\theta) e^{2\pi i n \theta} d\theta$$

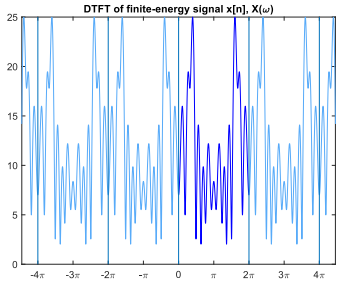
$$x(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega k} d\omega$$




### Discrete Time Fourier Transform (DTFT)

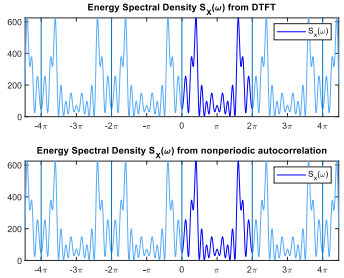
$$\hat{x}(\theta) = \sum_{n=-\infty}^{\infty} x[n] e^{-2\pi i n \theta}$$

$$X(\omega) = \sum_{k=-\infty}^{\infty} x(k) e^{-j\omega k}$$



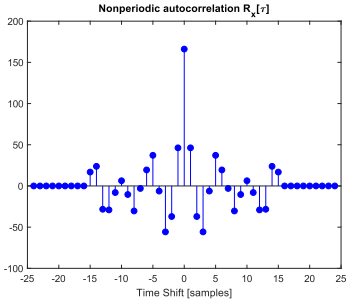

### Energy Spectral Density

$$S_x(\omega) = |X(\omega)|^2 = \sum_{\tau=-\infty}^{\infty} R_x(\tau) e^{-j\omega\tau}$$




### Nonperiodic Autocorrelation

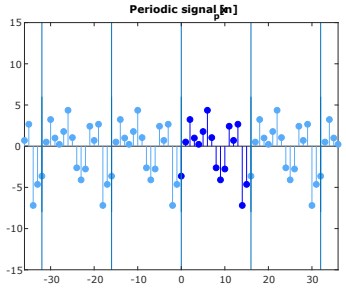
$$R_x(\tau) = \sum_{k=-\infty}^{\infty} x(k) x(k-\tau)$$




### Periodic Time Domain Signal

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} \hat{x}[k] e^{2\pi i n \frac{k}{N}}$$

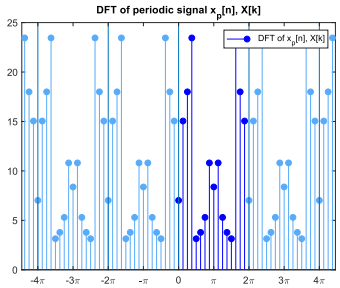
$$x(k) = \frac{1}{N} \sum_{n=0}^{N-1} X(\omega_n) e^{j\omega_n k}, \ \omega_n = 2\pi \frac{n}{N}$$




### Discrete Fourier Transform

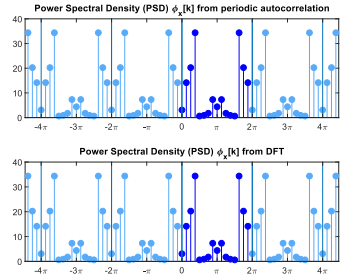
$$\hat{x}[k] = \sum_{n=0}^{N-1} x[n] e^{-2\pi i k \frac{n}{N}}$$

$$X(\omega_n) = \sum_{k=0}^{N-1} x(k) e^{-j\omega_n k}, \ \omega_n = 2\pi \frac{n}{N}$$



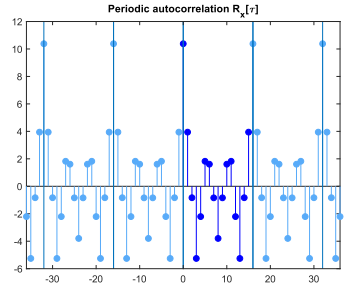

### Power Spectral Density (PSD)

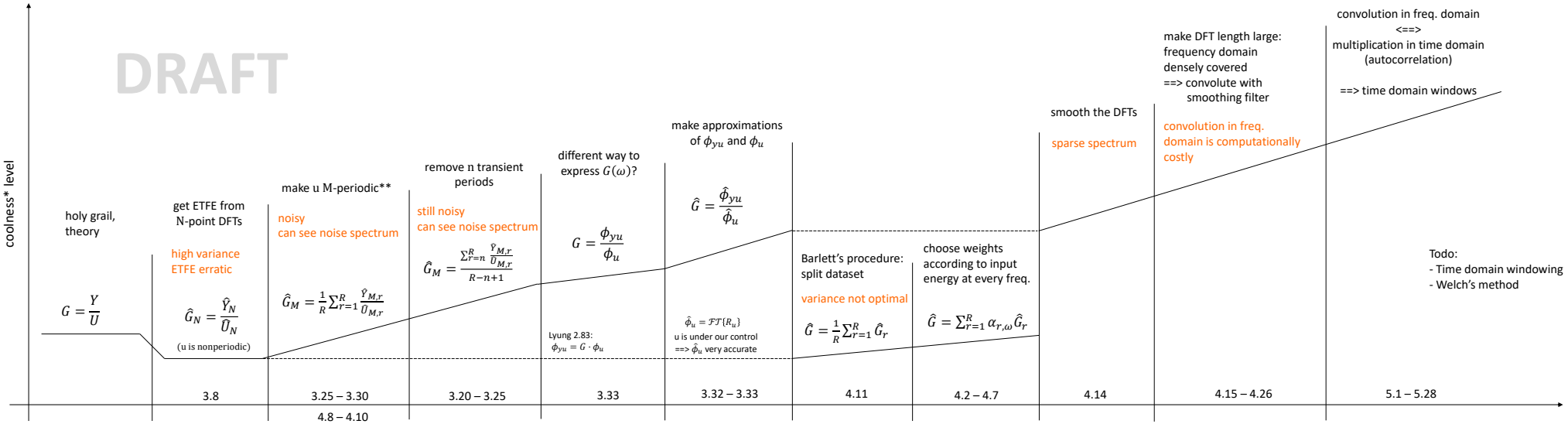
$$\phi(\omega_n) = \frac{1}{N} |X(\omega_n)|^2 = \sum_{\tau=0}^{N-1} R_x(\tau) e^{-j\omega_n \tau}$$




### Periodic Autocorrelation

$$R_x(\tau) = \frac{1}{N} \sum_{k=0}^{N-1} x(k) x(k-\tau)$$






*) ok, maybe better "usefulness"

**) Ch.3 uses DFTs of length $N \cdot M$, whereas Ch. 4 uses DFTs of length M, which is usually equivalent when averaged.