Approximation

Assume that we are selecting the top α percent.

$$\theta_\alpha = G(1-\alpha)$$

The local FDR is defined as

$$\begin{aligned}
\text{LFdr}_i &= \frac{P(\theta_i < \theta_\alpha, \delta_i = 1)}{P(\delta_i = 1)} \\
&= \frac{P(\theta_i < \theta_\alpha, \text{stat}_i > \lambda)}{P(\text{stat}_i > \lambda)*}
\end{aligned}$$

which can be approximated by

$$\begin{aligned}
P(\theta_i < \theta_\alpha, \text{stat}_i > \lambda) &= P(\theta_i < \theta_\alpha | \text{stat}_i > \lambda) P(\text{stat}_i > \lambda) \\
&\approx P(\theta_i < \theta_\alpha | Y = y_i, S = s_i) P(\text{stat}_i > \lambda) \\
&\approx \frac{1}{n} \sum_{i=1}^{n} 1\{\text{stat}_i > \lambda\} P(\theta_i < \theta_\alpha | Y = y_i, S = s_i) \\
&= \frac{1}{n} \sum_{i=1}^{n} 1\{\text{stat}_i > \lambda\} (1 - v_\alpha(y_i, s_i))
\end{aligned}$$

or we can show the steps in this way

$$\begin{aligned}
P(\theta_i < \theta_\alpha, \text{stat}_i > \lambda) &\approx \frac{1}{n} \sum_{i=1}^{n} 1\{\text{stat}_i > \lambda, \theta_i < \theta_\alpha\} \\
&\approx \frac{1}{n} \sum_{i=1}^{n} 1\{\text{stat}_i > \lambda\} P(\theta_i < \theta_\alpha | Y = y_i, S = s_i) \\
&= \frac{1}{n} \sum_{i=1}^{n} 1\{\text{stat}_i > \lambda\} (1 - v_\alpha(y_i, s_i))
\end{aligned}$$

The `ThreshFDR` function

```
function (lambda, stat, v)
{
    mean((1 - v) * (stat > lambda))/mean(stat > lambda)
}
```

To find the threshold λ_1

```
lambda_1<-try(Finv(gamma, ThreshFDR, interval = c(0.1, 0.9), stat = RANKING_STAT, v =
↪  TAIL_PROB), silent = TRUE)
```

Both ways look sort of dubious to me…

Right and left tail selection

Rule: Posterior tail probability

	Right	Left
θ_α	$G(1-\alpha)$	$G(\alpha)$
$v_\alpha(y)$	$P(\theta_i > \theta_\alpha \| y)$	$P(\theta_i < \theta_\alpha \| y)$
cap constr approx	$\alpha = \frac{1}{n}\sum 1\{v_\alpha(y) > \lambda_2\}$	$\alpha = \frac{1}{n}\sum 1\{v_\alpha(y) > \lambda_2\}$
fdr constr approx	$\gamma = \frac{\frac{1}{n}\sum(1-v_\alpha(y_i))1\{v_\alpha(y_i)>\lambda_1\}}{\frac{1}{n}\sum 1\{v_\alpha(y_i)>\lambda_1\}}$	$\gamma = \frac{\frac{1}{n}\sum(1-v_\alpha(y_i))1\{v_\alpha(y_i)>\lambda_1\}}{\frac{1}{n}\sum 1\{v_\alpha(y_i)>\lambda_1\}}$

Thus, for both left and right, we pick the max of λ_1 and λ_2 as the threshold for the two constraints.

Rule : Posterior mean

	Right	Left
θ_α	$G(1-\alpha)$	$G(\alpha)$
$v_\alpha(y)$	$P(\theta_i > \theta_\alpha \| y)$	$P(\theta_i < \theta_\alpha \| y)$
$T(y)$	$E(\theta_i \| y) = y + \frac{f(y)}{f'(y)}$	~
cap constr approx	$\alpha = \frac{1}{n}\sum 1\{T(y_i) > \lambda_2\}$	$\alpha = \frac{1}{n}\sum 1\{T(y_i) < \lambda_2\}$
fdr constr approx	$\gamma = \frac{\frac{1}{n}\sum(1-v_\alpha(y_i))1\{T(y_i)>\lambda_1\}}{\frac{1}{n}\sum 1\{T(y_i)>\lambda_1\}}$	$\gamma = \frac{\frac{1}{n}\sum(1-v_\alpha(y_i))1\{T(y_i)<\lambda_1\}}{\frac{1}{n}\sum 1\{T(y_i)<\lambda_1\}}$

MLE and James-Stein rule

Same to the posterior mean rule, but with different $T(y)$

Thus, for PM and MLE, the FDR is

$$\mathrm{LFdr}_i = \frac{P(\theta_i > \theta_\alpha, \delta_i = 1)}{P(\delta_i = 1)} = \frac{P(\theta_i > \theta_\alpha, \mathrm{stat}_i < \lambda)}{P(\mathrm{stat}_i < \lambda)}$$

which is approximated by

$$\frac{1}{n}\sum_{i=1}^{n} 1\{\mathrm{stat}_i < \lambda\}(1 - v_\alpha(y_i, s_i))$$

To find the threshold λ_1

```
lambda_1 <- - try(Finv(gamma, ThreshFDR, interval = c(0.1, 0.9), stat = -RANKING_STAT, v
↪  = TAIL_PROB), silent = TRUE)
#  minus sign
```

because $stat < \lambda$ is equivalent to $-stat > -\lambda$