Empirical Economics and Econometrics

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I have a question!

Contents

Recall that our problem is to solve the following equation as an inverse problem

$$\mathcal{A}(F,\phi) = 0 \Rightarrow K_F\phi = r_F$$

where K_F is a linear operator and r_F is a known function. We will solve this equation in a functional space. We will use the following notation

1 Linear equation in functional space

We will introduce the following terminology in the context of functional spaces:

- Spaces
- Linear operator
- Solution of linear equations

1.1 Spaces

Let's define a space (of functions) $\mathcal{E}$ on $\mathbb{R}$ as a linear space if it satisifies the following properties:

- $\forall f, g \in \mathcal{E},\ f + g \in \mathcal{E}$
- $\forall f \in \mathcal{E}$, $\forall \alpha \in \mathbb{R}$, $\alpha f \in \mathcal{E}$

Now let's define a norm on $\mathcal{E}$ as a function $\|\cdot\| : \mathcal{E} \to \mathbb{R}$ such that

- $\|f\| \geq 0$ and $\|f\| = 0$ if and only if $f = 0$
- $\|\alpha f\| = |\alpha|\,\|f\|$
- $\|f + g\| \leq \|f\| + \|g\|$

Definition 1.1 (complete space)**.** A space $\mathcal{E}$ is called a complete space if every Cauchy sequence in $\mathcal{E}$ converges to a limit in $\mathcal{E}$.

Definition 1.2 (Banach space)**.** A space $\mathcal{E}$ is called a Banach space if it is a complete space with respect to the norm $\|\cdot\|$.

Definition 1.3 (scalar product)**.** A scalar product on $\mathcal{E}$ is a function $\langle \cdot, \cdot \rangle : \mathcal{E} \times \mathcal{E} \to \mathbb{R}$ such that

- $\langle f, g \rangle = \langle g, f \rangle$
- $\langle \alpha f, g \rangle = \alpha \langle f, g \rangle$
- $\langle f + g, h \rangle = \langle f, h \rangle + \langle g, h \rangle$
- $\langle f, f \rangle \geq 0$ and $\langle f, f \rangle = 0$ if and only if $f = 0$

If $\mathcal{E}$ is equipped with a scalar product, then it is a Hilbert space.

Definition 1.4 (Hilbert space)**.** A space $\mathcal{E}$ is called a Hilbert space if it is a complete space with respect to the norm $\|\cdot\|$ induced by the scalar product $\langle\cdot,\cdot\rangle$.

The relationship between the norm and the scalar product is given by the following equation:

$$\|f\| = \sqrt{\langle f, f\rangle}$$

Remark. A Banach space B is a complete normed vector space. In terms of generality, it lies somewhere in between a metric space M (that has a metric, but no norm) and a Hilbert space H (that has an inner-product, and hence a norm, that in turn induces a metric).

Example 1.1. $L^p(\Omega, \mathcal{F}, \mu)$ is a space of functions such that $\int |f|^p < \infty$. It is a Banach space with the norm $\|f\|_p = \left(\int |f|^p\right)^{1/p}$. Also if μ is a probability measure, then we have the inclusion $L^p(\Omega, \mathcal{F}, \mu) \subset L^q(\Omega, \mathcal{F}, \mu)$ for $p \geq q$.

Definition 1.5 (Sobolev space)**.** Let $\Omega \subset \mathbb{R}^d$ be an open set. The Sobolev space $W^{k,p}(\Omega)$ is the space of functions $f : \Omega \to \mathbb{R}$ such that

$$\|f\|_{W^{k,p}} = \left(\sum_{|\alpha|\leq k} \int_\Omega |\partial^\alpha f|^p\right)^{1/p} < \infty$$

where α is a multi-index and $\partial^\alpha f$ is a partial derivative of order $|\alpha|$.

Definition 1.6 (subspace)**.** Let $\mathcal{E}$ be a space and $\mathcal{H}$ be a subspace of $\mathcal{E}$. Then $\mathcal{H}$ is a subspace of $\mathcal{E}$ if it satisfies the following properties:

- $\forall f, g \in \mathcal{H}$, $f + g \in \mathcal{H}$
- $\forall f \in \mathcal{H}$, $\forall \alpha \in \mathbb{R}$, $\alpha f \in \mathcal{H}$

Proposition 1.1. $calh$ is closed if for every sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{H}$ such that $f_n \to f$ in $\mathcal{E}$, we have $f \in \mathcal{H}$.

Remark. In a finite dimensional space, every subspace is closed. However, in an infinite dimensional space, a subspace can be closed or not.

Definition 1.7 (Orthogonal subspace)**.** Let $\mathcal{E}$ be a space and $\mathcal{H}$ be a subspace of $\mathcal{E}$. Then $\mathcal{H}^\perp$ is the orthogonal subspace of $\mathcal{H}$ if

$$\mathcal{H}^\perp = \{f \in \mathcal{E} : \langle f, g\rangle = 0, \forall g \in \mathcal{H}\}$$

Remark. The orthogonal subspace of a subspace is always closed.

1.2 Linear operator

Definition 1.8 (linear operator)**.** Let $\mathcal{E}$ and $\mathcal{H}$ be two Hilbert spaces (equipped with scalar product). A linear operator $K : \mathcal{E} \to \mathcal{H}$ is a function such that

$$K(\alpha_1\phi_1 + \alpha_2\phi_2) = \alpha_1 K\phi_1 + \alpha_2 K\phi_2$$

for all $\phi_1, \phi_2 \in \mathcal{E}$ and $\alpha_1, \alpha_2 \in \mathbb{R}$.

Example 1.2. Let X, Y, Z be three random variables defined on the $(\Omega, \mathcal{F}, \mathbb{P})$. Let $X = Y \times Z$. We construct three L^2 spaces L^2_X, L^2_Y, L^2_Z. Define a linear operator K such that $K\phi = \mathbb{E}(\phi(y) \mid Z = z)$. Then K is a linear operator from L^2_Y to L^2_Z.

For a linear operator, we define the corresponding subspaces defined as domain, range and kernel as follows:

Definition 1.9 (domain, range and kernel)**.** Let $\mathcal{E}$ and $\mathcal{H}$ be two Hilbert spaces and $K : \mathcal{E} \to \mathcal{H}$ be a linear operator. Then

- The domain of K is defined as
$$\mathcal{D}(K) = \{\phi \in \mathcal{E} : K\phi \in \mathcal{H}\}$$
- The range of K is defined as
$$\mathcal{R}(K) = \{K\phi : \phi \in \mathcal{M}(K)\}$$
- The kernel of K is defined as
$$\mathcal{N}(K) = \{\phi \in \mathcal{M}(K) : K\phi = 0\}$$

For completeness, we define injection and surjection as follows:

Definition 1.10 (injection and surjection)**.** Let $\mathcal{E}$ and $\mathcal{H}$ be two Hilbert spaces and $K : \mathcal{E} \to \mathcal{H}$ be a linear operator. Then

- K is called an injection if for all $\phi_1, \phi_2 \in \mathcal{E}$, $K\phi_1 = K\phi_2$ implies $\phi_1 = \phi_2$.
- K is called a surjection if for all $h \in \mathcal{H}$, there exists $\phi \in \mathcal{E}$ such that $K\phi = h$.

Boundness and continuity

Definition 1.11 (boundness)**.** Let $\mathcal{E}$ and $\mathcal{H}$ be two Hilbert spaces and $K : \mathcal{E} \to \mathcal{H}$ be a linear operator. Then K is called bounded if there exists a constant $C > 0$ such that

$$\|K\phi\|_{\mathcal{H}} \leq C \|\phi\|_{\mathcal{E}}$$

for all $\phi \in \mathcal{E}$.

Definition 1.12 (continuity)**.** Let $\mathcal{E}$ and $\mathcal{H}$ be two Hilbert spaces and $K : \mathcal{E} \to \mathcal{H}$ be a linear operator. Then K is called continuous if for all $\phi_n \to \phi$ in $\mathcal{E}$, we have $K\phi_n \to K\phi$ in $\mathcal{H}$.

We will look at an example where k is not continuous.

Example 1.3. Let $\mathcal{E} = \mathcal{C}^0_{[0,1]}$ be the space of continuous functions on $[0, 1]$. Define a linear operator $K : \mathcal{E} \to \mathbb{R}$ such that $K\phi = \phi(x_0)$.

Adjoint operator Similar to the transpose of a matrix, we define the adjoint operator of a linear operator.

Definition 1.13 (adjoint operator)**.** Let $\mathcal{E}$ and $\mathcal{H}$ be two Hilbert spaces and $K : \mathcal{E} \to \mathcal{H}$ be a linear operator. Then the adjoint operator $K^* : \mathcal{H} \to \mathcal{E}$ is defined as

$$\langle K\phi, h\rangle_{\mathcal{H}} = \langle \phi, K^* h\rangle_{\mathcal{E}}$$

for all $\phi \in \mathcal{E}$ and $h \in \mathcal{H}$.

Remark. We can show that matrix transpose is a special case of adjoint operator. For example, let A be a matrix of dimension $m \times n$ and x, y be vectors of dimension n, m respectively. Then we have $\langle Ax, y\rangle = (Ax)^\top y = x^\top A^\top y = \langle x, A^* y\rangle$.

Example 1.4 (integral operator)**.** Let $\mathcal{E} = \mathcal{C}^0\left([0,1]\right) = \mathcal{H}$. Define a linear operator $K : \mathcal{E} \to \mathcal{H}$ such that $K\phi\left(x\right) = \int_0^1 \phi\left(y\right) k\left(x,y\right) dy = h\left(y\right)$ where $k\left(x,y\right)$ is a given function. Then $\langle K\phi, h\rangle_{\mathcal{H}} = \int_0^1 \int_0^1 \phi\left(y\right) k\left(x,y\right) dydx = \int_0^1 \int_0^1 k\left(x,y\right) \phi\left(y\right) dxdy = \langle \phi, K^* h\rangle_{\mathcal{E}}$. Then we have $K^* h\left(x\right) = \int_0^1 k\left(x,y\right) h\left(y\right) dy$. We say K is **self-adjoint** if $k\left(x,y\right) = k\left(y,x\right)$, that is $K^* = K$.