# FINITE TIME BLOW UP FOR A 1D MODEL FOR 3D EULER EQUATIONS

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ABSTRACT. We analyzed a family of one-dimensional models for the three-dimensional Euler equation. These models interpolates the Constantin-Lax-Majda equation ($\alpha = 0$), where solutions admit a finite time blow up, and the de Gregorio model ($\alpha = 1$), where the solutions are conjectured to be globally regular. We provide numerical evidence that show solutions form singularities in finite time when $0 < \alpha < 1$.

## 1. INTRODUCTION

The Euler equation is one of the most fundamental system in fluid mechanics. A long standing open problem is the global regularity of the system in three dimensions: whether solutions stay smooth in all time, or there could be singularity formations in finite time.

Many simplified systems have been introduced and analyzed in order to provide hints for either global regularity or finite time blow up. We are interested in the following family of one-dimensional equations in vorticity form

$$\partial_t \omega + au\partial_x \omega = \omega H\omega. \tag{1}$$

When $a = 0$, the equation (1) is introduced by Constantin, Lax and Majda in [1]. The term $\omega H\omega$ is a 1D representation of vorticity stretching, where $H$ is the Hilbert transform defined as

$$H\omega(x) = \frac{1}{\pi} \text{ p.v.} \int_{-\infty}^{\infty} \frac{\omega(y)}{x-y} dy. \tag{2}$$

The Constantin-Lax-Majda model is well-studied. It is known that solutions can generate vorticity concentration in finite time.

When $a \neq 0$, the term $au\partial_x\omega$ stands for a drift. The velocity field $u$ satisfies the 1D version of the Biot-Savart law

$$u(x,t) = \frac{1}{\pi} \int \log|x-y|\omega(y,t)dy, \tag{3}$$

so that $\partial_x u = H\omega$.

In the case when $a < 0$, the drift intends to help the vorticity concentration. A similar finite time blowup machanism as the Constantin-Lax-Majda model is studied in [2] for $a = -1$.

The case when $a > 0$ is, however, less understood. In particular, when $a = 1$, the equation (1) is proposed by De Gregorio in [3]. It is easy to check that the periodic function $\sin x$ is a steady state of the equation. It indicates that the drift term compete with the stretching term, and prevents the finite time blowup. For general initial data, global regularity of De Gregorio model is still open.

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In this paper, we investigate equation (1) for $a \in [0,1]$ through numerical simulations. The goal is to provide numerical evidence of the blow up mechanism as the drift becomes stronger, from the Constantin-Lax-Majda model ($a = 0$) to the De Gregorio model ($a = 1$).

In section 2, we design a numerical scheme which successfully captures the blowup behavior of the Constantin-Lax-Majda model. In section 3, we apply the scheme to the general equation with $a \in [0,1]$. We observe that finite time blowup occurs for all $a < 1$. However, the blowup time becomes longer as $a$ approaches 1.

## 2. The Constantin-Lax-Majda model

We start our discussion with a special case of our main system (1), known as the Constantin-Lax-Majda model, where the drift coefficient $a = 0$:

$$\partial_t \omega = \omega H \omega, \qquad \omega(x,0) = \omega_0(x). \tag{4}$$

2.1. **The explicit solution.** The Constantin-Lax-Majda model (4) has an explicit solution (consult *e.g.* [4] for derivation)

$$\omega(x,t) = \frac{4\omega_0(x)}{(2 - tH\omega_0(x))^2 + t^2\omega_0^2(x)}, \tag{5}$$

and

$$H\omega(x,t) = \frac{2H\omega_0(x)(2 - tH\omega_0(x)) - 2t\omega_0^2(x)}{(2 - tH\omega_0(x))^2 + t^2\omega_0^2(x)}. \tag{6}$$

The solution blows up when $(2 - tH\omega_0(x))^2 + t^2\omega_0^2(x) = 0$. This happens at time $t$ and location $x$ where

$$t = \frac{2}{H\omega_0(x)}, \quad \text{and} \quad \omega_0^2(x) = 0. \tag{7}$$

In particular, we consider the 1-periodic initial condition

$$\omega_0(x) = -\sin(2\pi x). \tag{8}$$

The blowup condition (7) indicates that the blowup happens at time $t = 2$ and location $x = 0$.

2.2. **Numerical scheme.** To calculate the vorticity stretching term $\omega H \omega$, we apply pseudo-spectral method. The Hilbert transform (2) is a singular integral operator. So, it is hard to approximate using standard quadrature rules. On the other hand, it is a Fourier multiplier

$$\mathcal{F}(H\omega)(k) = -i \ \mathrm{sign}(k)\mathcal{F}(\omega)(k).$$

Hence, the pseudo-spectral method has advantages of being accurate and efficient.

For the time discretization, we apply the fourth order Runge-Kutta method. When the solution is smooth, we can choose a fixed time step. When the solution is close to blowup, we take adaptive time steps, which will be described later.

2.3. **Accuracy for smooth solutions.** We test our numerical scheme with initial condition (8). At time $T = 1$, the solution is smooth, and has an explicit form

$$\omega(x,t) = \frac{-4\sin(2\pi x)}{t^2 + 4 - 4t\cos(2\pi x)} \tag{9}$$

First, we verify the accuracy in time for our scheme. Fix the spatial discretization $n = 400$. Take a fixed time step $dt = T/m$, where $m$ is chosen to be 10, 20, 30, ... , 390, 400. We compare the numerical solution at time $T = 1$ with the analytical solution (9), and compute the error in $l^\infty$ norm. Figure 1 is the loglog plot between $m$ and the error. It clearly shows that the error is of order $m^{-4}$, verifying the fourth order accuracy in time.

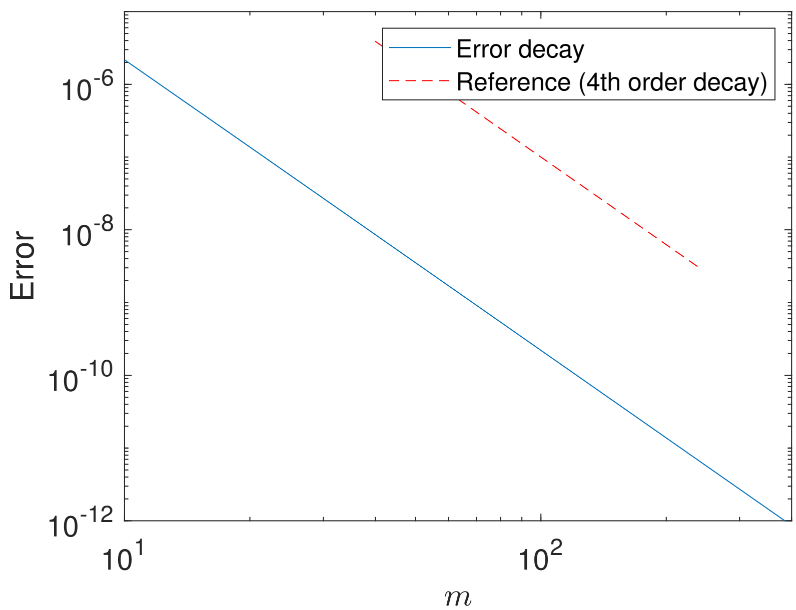


FIGURE 1. loglog plot of the error against $m$

Next, we verify the spatial accuracy for our scheme. Fix the time step with $m = 1000$. The value of $n$ is taken from 4, 8 to 128. Figure 2 is the loglog plot of the error (again in $l^\infty$ norm) against $n$. One can observe a fast decay, showing the spectral accuracy.

2.4. **Blow up time and location.** When the solution approaches the singularity, the discrete problem becomes more stiff. An adaptive time step is required to ensure stability. We take

$$dt = \min\left\{\frac{T}{m}, \frac{1}{10\|H\omega(t)\|_\infty}\right\}.$$

We first check whether the scheme captures the blowup time. Consider the initial condition (8). We know from (7) that blowup happens at $T = 2$. Fix $m = 2000$, and take $n = 64, 128, \cdots, 4096$. We record the time when $\|H\omega(t)\|_\infty$ reaches $10^8$. Note that the analytical solution will reach $10^8$ at $T = 2 - 2\times 10^{-8}$. We compare the difference between the numerical results and the analytical result in Figure 3. It shows that as we refine the spatial discretization, the scheme better captures the blowup time.

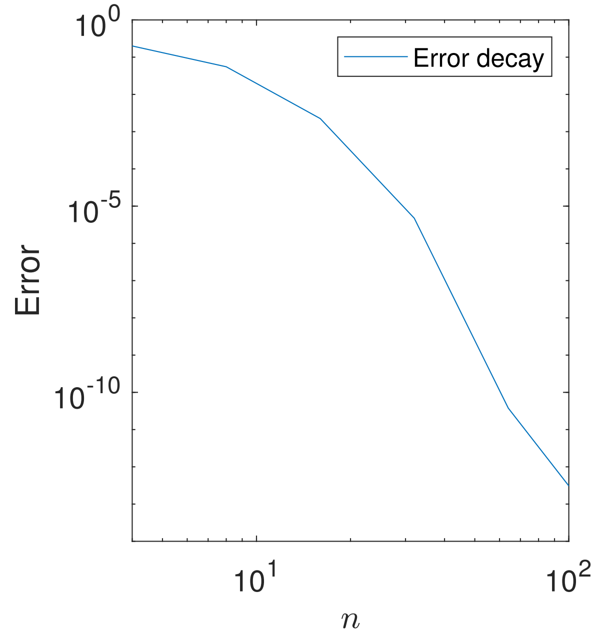


FIGURE 2. loglog plot of the error against $n$

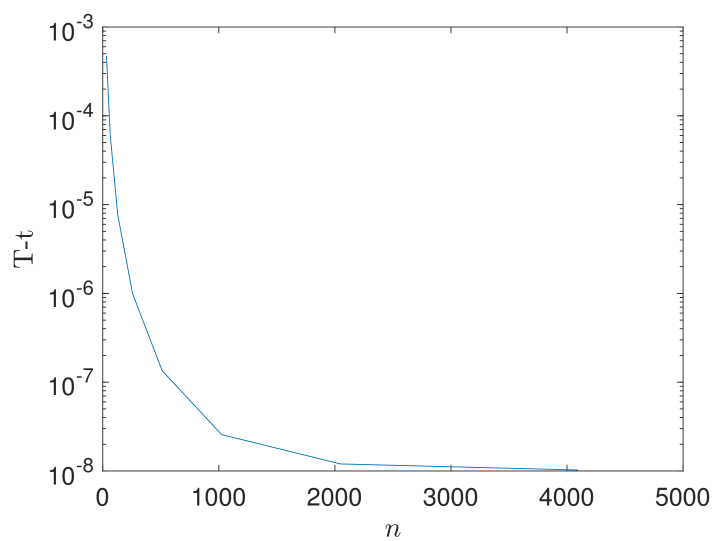


FIGURE 3. Error in blowup time when $n$ increases

Next, we testify whether the scheme captures the rate of blowup. It is easy to check that for initial condition (8), we have

$$\|H\omega(\cdot,t)\|_{L^\infty} = H\omega(0,t) = \frac{2}{2-t} = \mathcal{O}((T-t)^{-1}).$$

Therefore, the blow up behaves like $(T-t)^{-1}$ near the blowup time $T$.

We recorded time $t$ when $\|H\omega(t)\|_\infty$ reaches $10^4, 10^5, 10^6$ and $10^7$. Figure 4 is a loglog plot of $T-t$ against $\|H\omega(t)\|_\infty$. It indicates that the numerical blowup rate is indeed $(T-t)^{-1}$.

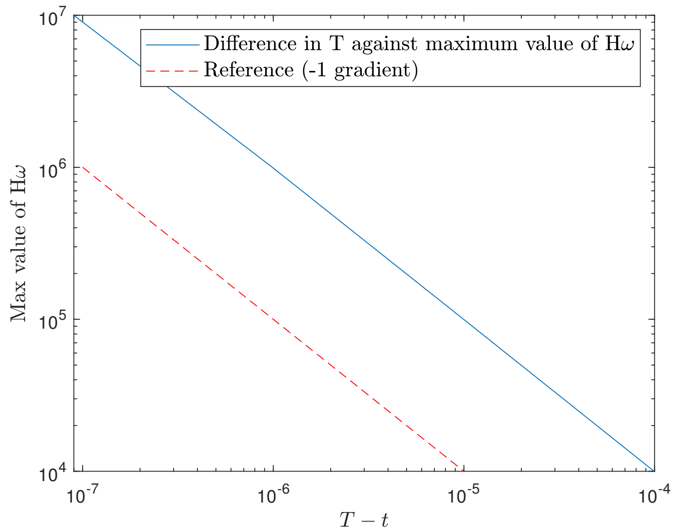


FIGURE 4. loglog plot of $\|H\omega(t)\|_\infty$ against $T-t$

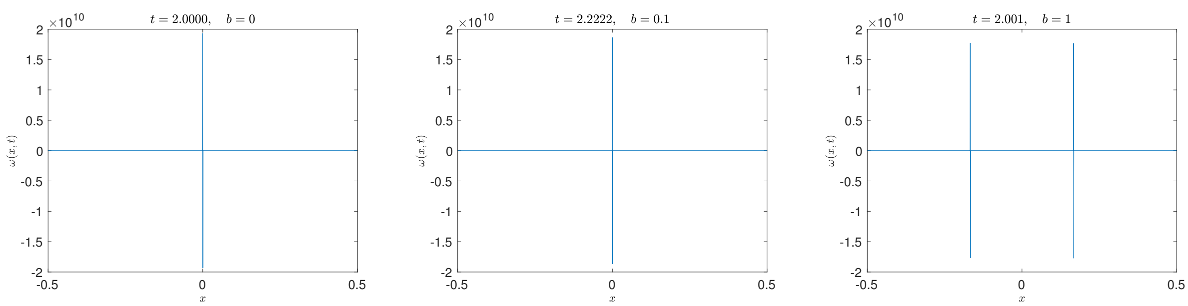


FIGURE 5. Blowup profiles with different initial conditions $b = 0, 0.1$ and 1.

Finally, we check the profile $\omega(x,t)$ near the blowup time. Let us consider a more general type of 1-periodic initial conditions

$$\omega_0(x) = -\sin(2\pi x) + b\sin(4\pi x). \tag{10}$$

Take $b = 0, 0.1$ and 1. By condition (7), we can obtain the analytical blowup time and locations:

- For $b = 0$, blowup happens at $T = 2$ and $x = 0$;
- For $b = 0.1$, blowup happens at $T = 20/9$ and $x = 0$;
- For $b = 1$, blowup happens at $T = 2$ and $x = \pm 1/6$.

Figure 5 shows the numerical solution with respect to the three initial conditions at the time when $\|H\omega\|_\infty$ reaches $10^{10}$. It shows that the numerical result matches the analytical result nicely. It indicates that our numerical scheme can capture the correct blowup profile.

## 3. GENERAL MODELS

In this section, we apply our numerical scheme to the general equation (1) with $a \in [0,1]$. While the Constantin-Lax-Majda model can be solved explicitly, the generalized model with $a \neq 0$ does not have an explicit solution. Moreover, whether the solution becomes singular in finite time is unknown.

The extra drift term $au\partial_x\omega$ can be treated similarly as the stretching term using pseudo-spectral method. The time step will be subjected to an additional CFL condition to ensure stability.

$$dt = \min\left\{\frac{T}{m}, \frac{1}{10\|H\omega(t)\|_\infty}, \frac{dx}{a\|u(t)\|_\infty}\right\}.$$

3.1. **Blow up time.** We perform numerical experiments for the general equation (1) with different drift coefficient $a = 0, 0.2, 0.4, 0.6, 0.8$ and 1, with the initial condition chosen as (10) with $b = 0, 0.1$ and 1.

The following table shows the blowup time under each setup. Here, we use $n = 4096$, and $T/m = 10^{-3}$.

| Blowup Time | | $a$ | | | | | |
|---|---|---|---|---|---|---|---|
| | | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1 |
| $b$ | 0 | 2.0000 | 2.3101 | 2.7858 | 3.6500 | 6.0719 | No Blowup |
| | 0.1 | 2.2222 | 2.5934 | 3.1722 | 4.2456 | 7.2966 | No Blowup |
| | 1 | 2.000 | 2.406 | 3.058 | 4.284 | 7.662 | No Blowup |

One can see that the drift intend to slow down the blow up. However, the numerical solutions still blow up in finite time as long as $a < 1$. Therefore, the numerical experiments strongly support the conjecture that finite time blowup happens when $a < 1$.

3.2. **Blow up profile.** Now, we investigate how the solution approaches the singularity. We shall take the initial condition (8).

Recall that for the Constantin-Lax-Majda model ($a = 0$), solution blows up at $T = 2$ at a single point $x = 0$; while for the De Gregorio model ($a = 1$), the solution stays the same in all time.

Figure 6 shows an interesting transition of blowup profiles when $a$ changes from 0 to 1.

When $a$ is close to 0, the blow up still happens near $x = 0$. But as $a$ becomes larger, the drift intends to push the profile away from 0, so a larger interval around 0 blows up. When $a = 0.8$, the graph shows that almost the whole profile grows dramatically at the time near blowup. When $a$ becomes even larger, it takes a much longer time for the whole profile to grow. When $a$ reaches 1, there is no growth any more, resulting a global existence of solution.

## References


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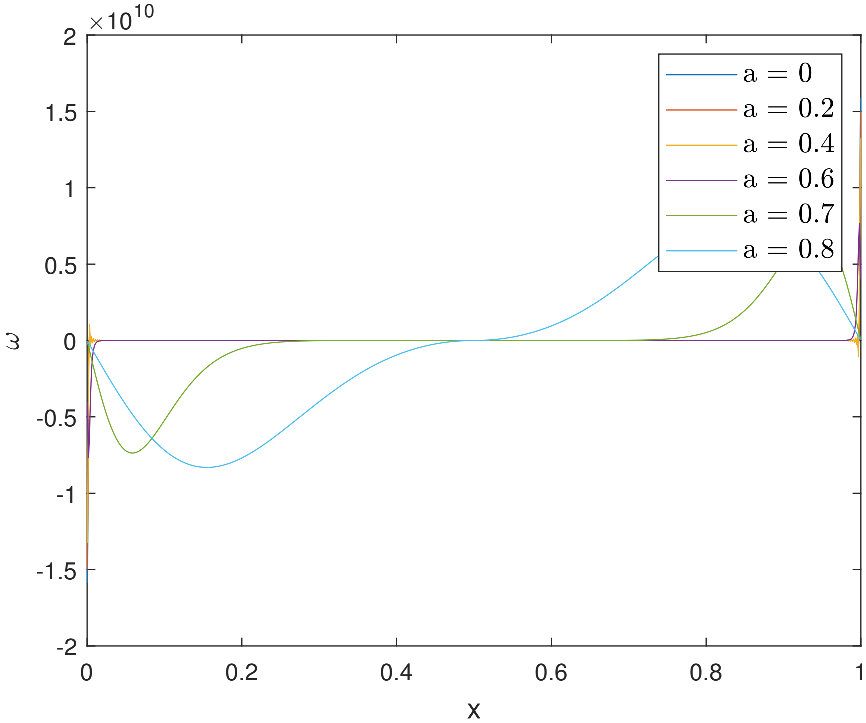


FIGURE 6. graph plotted with different drift term $a$