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Bicategorical Orthogonality Constructions for Linear Logic

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Résumé

Cette thèse porte sur la sémantique bicatégorique de la logique linéaire. Nous nous intéressons spécifiquement à la catégorification du modèle relationnel de la logique linéaire par le modèle des espèces généralisées introduit par Fiore, Gambino, Hyland et Winskel ainsi qu'à ses raffinements par des constructions d'orthogonalité.

Nous présentons dans un premier temps une généralisation bicatégorique du modèle des espaces de finitude introduit par Ehrhard où nous introduisons une orthogonalité sur la bicatégorie des espèces nous permettant d'obtenir une nouvelle bicatégorie où les interactions entre programmes et environnements sont finies.

Nous considérons ensuite la catégorification de la notion de stabilité des fonctions stables vers les foncteurs stables. Nous combinons les espèces de structure avec la stabilité grâce à une orthogonalité sur les sous-groupes d'endomorphismes pour chaque objet d'un groupoïde. Cette orthogonalité nous permet ensuite de restreindre les foncteurs analytiques associés aux espèces à des foncteurs stables et nous montrons qu'ils forment une bicatégorie cartésienne fermée.

Nous étudions dernièrement la catégorification du modèle de Scott de la logique linéaire et son lien avec le modèle des espèces. Nous commençons par montrer que la bicatégorie des profoncteurs équipée de la pseudo-comonade des coproduits finis est un modèle de la logique linéaire catégorifiant le modèle de Scott. Nous introduisons ensuite une orthogonalité entre la bicatégorie de Scott obtenue et la bicatégorie des espèces et obtenons une nouvelle bicatégorie constituant une première étape afin de relier la substitution linéaire et non-linéaire dans ce contexte.

Abstract

This thesis is concerned with the bicategorical semantics of linear logic. We are specifically interested in the categorification of the relational model of linear logic with the generalized species model introduced by Fiore, Gambino, Hyland and Winskel; and its refinements using orthogonality constructions.

We first present a bicategorical generalization of the model of finiteness spaces introduced by Ehrhard. We introduce an orthogonality construction on the bicategory of profunctors based on finite presentability to obtain a new bicategory where all interactions are enforced to be finite.

We then consider the categorication of the computational notion of stability from stable functions to stable functors. We bring together generalized species of structures and stability by refining the species model with an orthogonality on subgroups of endomorphisms for each object in a groupoid. We show that this orthogonality allows us to restrict the analytic functors to stable functors and prove that they form a cartesian closed bicategory.

We lastly study the categorification of the qualitative Scott model of linear logic and its connection with the quantitative species model of Fiore et al. We start by showing that the bicategory of profunctors with the finite coproduct pseudo-comonad is a model of linear logic that categorifies the Scott model. We then define an orthogonality between the Scott bicategory and the species bicategory that allows us to construct a new bicategory giving us a first step towards connecting linear and non-linear substitution in this setting.

Contents

Introduction

Denotational semantics

A cornerstone of theoretical computer science is the correspondence between models of computation and proof systems established by Curry and Howard [63]. Formally, there is a correspondence between simply typed λ-calculus and propositional intuitionistic logic which associates types to formulae, λ-terms to proofs and β-reduction to cut elimination. Lambek later gave a categorical counterpart to this correspondence by showing that cartesian closed categories model simply typed λ-calculus by interpreting types as objects and λ-terms as morphisms [82, 83]. The composition of morphisms now corresponds to cut elimination or program execution.

Denotational semantics is concerned with the study of invariants of this dynamic where we want to express the behavior of programs as morphisms between mathematical objects in a way that does not depend on the implementation. Among the desired properties of the interpretation of a program are monotonicity and continuity, i.e. the more a program has information on its input, the more it will provide information on its output and any finite part of the output can be attained through a finite computation. The semantics of programs is also required to be compositional i.e. the interpretation of a program can be recovered from the interpretation of its sub-programs. These features form the basis of domain theory developed by Scott and Strachey with the fundamental model of λ-calculus given by Scott-continuous functions (monotonous maps preserving directed suprema) between domains [103, 101, 102].

Domain theory does not take into account the quantitative aspects of resource management of programs. A major advance in this direction was Girard's linear logic which allows the study of how programs or proofs manage their resources by using exponential modalities, distinguishing linear arguments that can be used exactly once and non-linear ones that can be used an arbitrary number of times [53]. In quantitative models of linear logic, non-linear programs are thought of as analytic maps that are infinitely differentiable and represented by power series which can be approximated by polynomials. Viewing programs as series, a natural question was to understand the logical counterpart of differentiation, which led Ehrhard and Regnier to introduce differential linear logic and the syntactic notion of Taylor expansion which associates a formal sum of resource λ-terms to a given λ-term [35, 37].

Species of structures

Power series are also used in enumerative combinatorics to count the number of ways a certain combinatorial pattern (trees, partitions, cycles, etc.) can be formed. Generating functions are one of the tools used to describe these combinatorial structures. They are typically encoded as formal power series

$$x \mapsto \sum_n f_n \frac{x^n}{n!}$$

where f_n corresponds to the number of objects of size n for a given structure. Combinatorial species were introduced by Joyal in 1981 as a unified categorical framework for the theory of generating functions [72]. A combinatorial species is a class of structures on arbitrary finite sets of labels which is invariant under relabellings along bijections. Formally, a species is a functor $F : \mathbb{B} \to \mathbf{Set}$ from the category of finite sets and bijections $\mathbb{B}$ to the category $\mathbf{Set}$. The functoriality of F means that labels do not matter and one can equivalently define species as functors from the category $\mathbb{P}$ of integers and permutations to $\mathbf{Set}$. To a species F, we can associate the following exponential generating series:

$$x \mapsto \sum_{n=0}^{+\infty} f_n \frac{x^n}{n!} \quad \text{where } f_n := |F(\underline{n})| \,.$$

Species can be added together, multiplied, differentiated, and substituted one into another to form new species from given ones and these operations translate to the operations on their corresponding generating series. The information encoded by combinatorial species is however not fully captured by generating series since two non-isomorphic species can have the same series. Joyal therefore introduced the notion of analytic functor as the series counterpart of species of structures that fully encodes the data described by species [73].

The starting point of this thesis is the bicategory of generalized species of structures introduced by Fiore, Gambino, Hyland and Winskel [41]. It encompasses Joyal's combinatorial species and is also a categorification of the relational model for differential linear logic.

Categorifying models of computation

In recent years, there was a growing interest in the categorification of models of computation by replacing semantics where types are sets or preorders with richer categorical structures providing finer mathematical invariants. When using bidimensional categorical structures to model computations, rewriting steps between terms become 2-cells carrying information on transformations between programs. It has seen applications in game semantics [70], type theory [1, 92] and higher dimensional rewriting [62]. The bicategory of profunctors in particular has become increasingly important in theoretical computer science to model a wide range of two dimensional computational structures [23, 41, 6, 112, 113, 58].

In 1-categorical semantics of linear logic or other related systems, types are interpreted as objects in a category $\mathbb{C}$ and proofs by morphisms. The category $\mathbb{C}$ being a model of linear logic means that for a closed formula A, if two proofs π and π' of A are related by the cut elimination procedure, then they are interpreted by the same morphism $[\![\pi]\!] = [\![\pi']\!]$ in the category $\mathbb{C}$. In a bicategorical setting, the equality is replaced by an isomorphism carrying information on the reduction sequence $\pi \to \pi'$. While 1-categorical semantics of linear logic has been extensively studied (see Mellies [89] for a complete review of linear logic models and Ehrhard [35] for differential linear logic), no complete account of what is a bicategorical model of differential linear logic has been given yet. The standard categorical notions for the monoidal, additive and exponential structure in 1-categorical semantics have canonical bicategorical weakenings. A first attempt towards the formalization of bicategorical model of linear logic was done by Jacq in his thesis [69]. If we only focus on the λ-calculus fragment, Fiore and Saville lifted the Curry-Howard-Lambek correspondence between simply typed λ-calculus and cartesian closed categories to a bicategorical framework [44, 97].

In this thesis, we are interested in the categorification of the relational model of linear logic with the generalized species model by Fiore et al. and its refinements using orthogonality constructions.

Orthogonality

An orthogonality is a relation between programs and environments that enforces more control on their possible interactions [53]. It is inspired from orthogonality for bilinear forms in linear algebra and also carries intuitions similar to game semantics. Orthogonality is also used in realizability where the types are constructed using a duality between terms and contexts with respect to a pole containing "correct computations" [76].

It is a powerful tool to define new models of linear logic by refining an existing one: for a category $\mathbb{C}$ that is a model of linear logic with monoidal units $\mathbf{1}$ and $\perp$, the duality between programs and environments is translated to a duality between morphisms $\mathbf{1} \to c$ and morphisms $c \to \perp$ where c ranges over the objects of $\mathbb{C}$. This duality induces a new category refining the model $\mathbb{C}$ where the interactions between morphisms are controlled by the orthogonality. This method was given a general categorical treatment with the notion of orthogonality categories [68]. Orthogonality categories are a particular case of double-glueing constructions which provide a more general method to refine models of linear logic. Many models are instances of double glueing constructions (phase spaces [53], coherence spaces [52], totality spaces [87], finiteness spaces [32], probabilistic coherence spaces [28], non-uniform coherence spaces [19, 20, 18], ludics [57], geometry of interaction [55]).

Double glueing techniques are also used to study full completeness for linear logic models by refining an existing model to contain only morphisms corresponding to the interpretation of some proofs [99]. Orthogonality can further be used to compare two models of linear logic by doing the double glueing construction on the product of the two categories. Ehrhard showed for example that the Scott model of linear logic is the extensional collapse of the relational model using this method [34]. This construction has been used in the context of intersection types which characterize normalization properties of λ-calculus. The quantitative relational model corresponds to a non-idempotent intersection type system whereas the qualitative Scott model corresponds to an idempotent type system. The double glueing construction provides a connection between the two type systems that allows to translate non-idempotent normalization to the idempotent one [33].

Contributions

This thesis follows the line of research of categorifying models of linear logic by replacing the refinements on the relational model with refinements on the species model. In a bidimensional setting, an orthogonality controls interactions between programs and environments and also restricts the allowed reductions between terms. The constructions in this thesis give a first step towards obtaining a theory of orthogonality for bidimensional structures.

- Chapter 2 presents a bicategorical generalization of the model of finiteness spaces introduced by Ehrhard [32]. This model is based on a refinement of the relational model with an orthogonality that enforces finite interactions between programs and environments. It provided a semantics for finite non-determinism and it gave a semantical motivation for differential linear logic and the syntactic notion of Taylor expansion.

  We replace the relational model with the model of generalized species of structures introduced by Fiore et al. [41] and introduce an orthogonality construction on the bicategory of profunctors based on finite presentability to obtain a new bicategory where all interactions are enforced to be finite. We show that all the linear logic structure in the bicategory of profunctors can be refined to this new bicategory. We obtain a cartesian closed bicategory containing species whose analytic functors can have possibly infinite support but whenever an explicit computation is made, the result is always finite.

- Chapter 3 is based on joint work with Marcelo Fiore and Hugo Paquet. We tackle the categorication of the computational notion of stability from stable functions to stable functors. Stability was a key notion in the development of linear logic through dI-domains and coherence spaces and more recently in probabilistic programming [13, 52, 36].

  In this chapter, we bring together generalized species of structures and stability by refining the species model with an orthogonality on subgroups of endomorphisms for each object in a groupoid. We show that this orthogonality can also be translated to an orthogonality on the category of presheaves associated with a groupoid that allows us to restrict the analytic functors to stable functors and prove that they form a cartesian closed bicategory.

- We study in Chapter 4 the generalization of the collapse construction by Ehrhard between the quantitative differential relational model and the qualitative Scott model of linear logic. We start by showing that the bicategory of profunctors with the finite coproduct pseudo-comonad is a model of linear logic that categorifies the Scott model. Substitution in the obtained cartesian closed bicategory is non-linear as it allows for duplication and erasure whereas the monoidal structure of the exponential modality in the species model encodes linear substitution.

We then define an orthogonality between the Scott bicategory and the species bicategory that allows us to construct a new bicategory giving a connection between linear and non-linear substitution and constituting a first step towards connecting the idempotent and non-idempotent profunctor typing systems introduced by Olimpieri [92].

Chapter 1

Preliminaries

1.1 Linear Logic

We give in this section a short overview of linear logic and its categorical semantics. Linear logic was introduced by Girard as a refinement of both intuitionistic and classical logic based on the decomposition of the implication $A \Rightarrow B$ into two more primitive connectives $!A \multimap B$ allowing for a more sensitive treatment of resource usage [53]. It was later extended by Ehrhard and Regnier [38] with three additional rules to account for differentiation.

Linear logic has an additive and multiplicative version of each connective which have different interpretations in terms of resource usage. The additive conjunction $A \,\&\, B$ (with unit $\top$) means that we have to chose whether we want to use the resource A or the resource B whereas the multiplicative conjunction $A \otimes B$ (with unit $\mathbf{1}$) means that we can use both resources A and B at the same time. Dually, the connective $\oplus$ (called *plus*) corresponds to the additive disjunction with unit $\mathbf{0}$, whereas the multiplicative version is given by $\parr$ (called *par*) with unit $\bot$.

1.1.1 Formulas

We start with a countable set $\mathcal{A}$ of atoms $a, b, \ldots$ and a bijection $(-)^\perp : \mathcal{A} \to \mathcal{A}$ such that $a^{\perp\perp} = a$ for all $a \in \mathcal{A}$. The set of atoms contains four distinguished elements $\mathbf{0}, \mathbf{1}, \top$ and $\bot$ that verify $\mathbf{0}^\perp = \top$, $\top^\perp = \mathbf{0}$, $\mathbf{1}^\perp = \bot$ and $\bot^\perp = \mathbf{1}$. Formulae in **DiLL** are given by the following syntax:

$$A, B ::= a \mid a^\perp \mid \mathbf{0} \mid \mathbf{1} \mid \top \mid \bot \mid A \oplus B \mid A \,\&\, B \mid A \otimes B \mid A \parr B \mid\, !A \mid ?A$$

The negation of an arbitrary formula is defined inductively using De Morgan's equations:

$$\begin{aligned} (A \oplus B)^\perp &:= A^\perp \,\&\, B^\perp & (A \,\&\, B)^\perp &= A^\perp \oplus B^\perp \\ (A \otimes B)^\perp &:= A^\perp \parr B^\perp & (A \parr B)^\perp &= A^\perp \otimes B^\perp \\ (!A)^\perp &= ?(A^\perp) & (?A)^\perp &= !(A^\perp) \end{aligned}$$

Linear implication is defined as $A \multimap B := A^\perp \parr B$.

1.1.2 Sequent calculus

We present the monolateral sequent calculus for **DiLL** where sequents are of the form $\vdash \Gamma$ for a finite sequence of formulas Γ. They are built using the rules described below:

- Structural rules:

$$\frac{}{\vdash A, A^\perp}\,\textsc{ax} \qquad \frac{\vdash \Gamma, A, B, \Delta}{\vdash \Gamma, B, A, \Delta}\,\textsc{ex} \qquad \frac{\vdash \Gamma, A \qquad\qquad \vdash A^\perp, \Delta}{\vdash \Gamma, \Delta}\,\textsc{cut}$$

- Additive rules:

$$\frac{}{\vdash \Gamma, \top}\,\top \qquad \frac{\vdash \Gamma, A \qquad\qquad \vdash \Gamma, B}{\vdash \Gamma, A \,\&\, B}\,\& \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, A \oplus B}\,\oplus_1 \qquad \frac{\vdash \Gamma, B}{\vdash \Gamma, A \oplus B}\,\oplus_2$$

- Multiplicative rules:

$$\frac{}{\vdash 1}\,\mathbf{1} \qquad \frac{\vdash \Gamma}{\vdash \Gamma, \bot}\,\bot \qquad \frac{\vdash \Gamma, A \qquad\qquad \vdash \Delta, B}{\vdash \Gamma, A \otimes B, \Delta}\,\otimes \qquad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\,\parr$$

- Exponential rules:

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?A}\,\mathrm{w} \qquad \frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\,p \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\,\mathrm{der} \qquad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\,\mathrm{c}$$

- Differential rules:

$$\frac{}{\vdash !A}\,\overline{\mathrm{w}} \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, !A}\,\overline{\mathrm{der}} \qquad \frac{\vdash \Gamma, !A \qquad\qquad \vdash \Delta, !A}{\vdash \Gamma, \Delta, !A}\,\overline{\mathrm{c}}$$

1.1.3 Categorical semantics

There are several axiomatizations of the notion of categorical model of linear logic (Lafont category, Seely category, Linear category and Lafont-Seely category [79, 104, 14, 10, 11, 89]) that were unified with the notion of linear-nonlinear adjunction:

Definition 1.1.1 ([11])**.** A *linear-nonlinear adjunction* between a cartesian category $(\mathbb{M}, \&, \top)$ and a $*$-autonomous category $(\mathbb{L}, \otimes, \mathbf{1})$ is a symmetric monoidal adjunction between lax monoidal functors:

$$\begin{tikzcd}
(\mathbb{L}, \otimes, \mathbf{1}) \arrow[r, bend right, "R"'] \arrow[r, phantom, "\perp"] & (\mathbb{M}, \&, \top) \arrow[l, bend right, "L"']
\end{tikzcd}$$

The original definition is for models of intuitionistic linear logic and therefore only requires $\mathbb{L}$ to be symmetric monoidal closed [89]. We recall below only the Seely axiomatization which is what is use in the remainder of this thesis:

Definition 1.1.2. A *Seely category* is the data of

1. a a $*$-autonomous category $(\mathbb{L}, \otimes, \mathbf{1})$ with finite products $(\&, \top)$;
2. a comonad $(!, \mathrm{dig}, \mathrm{der}) : \mathbb{L} \to \mathbb{L}$;
3. natural isomorphisms called the *Seely isomorphisms*

$$s^2_{A,B} : !A \otimes !B \to !(A \& B) \qquad s^0 : \mathbf{1} \to !\top$$

such that $(!, s) : (\mathbb{L}, \&, \top) \to (\mathbb{L}, \otimes, \mathbf{1})$ is a symmetric monoidal functor and the following coherence diagram commutes:

$$\begin{tikzcd}
!A \otimes !B \arrow[r, "s^2_{A,B}"] \arrow[dd, "\mathrm{dig}_A \otimes \mathrm{dig}_A"'] & !(A \& B) \arrow[d, "\mathrm{dig}_{A\&B}"] \\
 & !!(A \& B) \arrow[d, "!\langle !\pi_1, !\pi_2 \rangle"] \\
!!A \otimes !!B \arrow[r, "s^2_{!A,!B}"'] & !(!A \& !B)
\end{tikzcd}$$

When these conditions hold, the co-Kleisli category $\mathbb{L}_!$ is cartesian closed and the comonadic adjunction

$$\begin{tikzcd} \mathbb{L} \arrow[rr, bend right, "R"'] \arrow[rr, phantom, "\perp"] & & \mathbb{L}_! \arrow[ll, bend right, "L"'] \end{tikzcd}$$

lifts to a symmetric monoidal adjunction [89].

Remark. In a Seely category $\mathbb{C}$, every object c can be equipped with a commutative comonoid structure

$$e_c : !\,\mathbf{1} \xrightarrow{!t} !\top \xrightarrow{(s^0)^{-1}} \mathbf{1} \qquad\qquad d_c : !c \xrightarrow{!\Delta} !(c \,\&\, c) \xrightarrow{(s^2_{c,c})^{-1}} !c \otimes !c$$

where $\Delta : c \to c \,\&\, c$ is the diagonal morphism and $t : \mathbf{1} \to \top$ is the unique morphism to the terminal object.

We now review the categorical semantics of differential linear logic (**DiLL**) which adds three co-structural rules (co-weakening, co-dereliction and co-contraction) to the sequent calculus of linear logic. We follow the presentation by Fiore [45] and refer the reader to [35] for Ehrhard's axiomatization in terms of exponential structures.

Definition 1.1.3. A *biproduct structure* on a category $\mathbb{C}$ consists of a monoidal structure $(\&, \top)$ together with natural transformations:

$$u_c : \top \to c \qquad n_c : c \to \top \qquad \nabla_c : c \,\&\, c \to c \qquad \Delta_c : c \to c \,\&\, c$$

such that (c, u_c, ∇_c) is a commutative monoid and (c, n_c, Δ_c) is a commutative comonoid for all $c \in \mathbb{C}$.

The biproduct structure allows us to interpret the co-weakening and co-contraction rules. The co-weakening $\overline{\mathrm{w}}$ has components $\overline{\mathrm{w}}_c : \mathbf{1} \to !c$ given by $\mathbf{1} \xrightarrow{s^0} !\top \xrightarrow{!u_c} !c$ and the cocontraction $\overline{\mathrm{c}}$ has components $\overline{\mathrm{c}}_c : !c \otimes !c \to !c$ given by:

$$!c \otimes !c \xrightarrow{s^2_{c,c}} !(c \,\&\, c) \xrightarrow{!\nabla_c} !c.$$

Definition 1.1.4. A biproduct structure $(\&, \top, u, n, \nabla, \Delta)$ and a symmetric monoidal structure $(\otimes, \mathbf{1})$ on a category $\mathbb{C}$ are *compatible* if the following diagrams commute:

$$\begin{tikzcd}
c \otimes d \arrow[r, "n_c \otimes \mathrm{id}"] \arrow[d, "n_{c\otimes d}"'] & \top \otimes d \arrow[d, "u_c \otimes \mathrm{id}"] \\
\top \arrow[r, "u_{c\otimes d}"'] & c \otimes d
\end{tikzcd}
\qquad
\begin{tikzcd}
c \otimes d \arrow[r, "\Delta_c \otimes \mathrm{id}"] \arrow[d, "\Delta_{c\otimes d}"'] & (c \,\&\, c) \otimes d \arrow[d, "\nabla_c \otimes \mathrm{id}"] \\
(c \otimes d) \,\&\, (c \otimes d) \arrow[ru, "\langle \pi_1 \otimes d, \pi_2 \otimes \mathrm{id} \rangle" description] \arrow[r, "\nabla_{c\otimes d}"'] & c \otimes d
\end{tikzcd}$$

Definition 1.1.5. A model of **DiLL** is a Seely category $\mathbb{C}$ such that the cartesian structure $(\&, \top)$ is a biproduct structure compatible with the symmetric monoidal structure together with a codereliction natural transformation $\overline{\mathrm{der}} : \mathrm{id}_{\mathbb{C}} \Rightarrow\, !$ satisfying the following axioms:

$$\begin{tikzcd}
 & c \arrow[ld, bend right, "\overline{\mathrm{der}}_c"'] \arrow[dd, "\mathrm{id}"] \\
!c \arrow[rd, bend right, "\mathrm{der}_c"'] & \\
 & c
\end{tikzcd}
\qquad
\begin{tikzcd}
c \arrow[r, "\overline{\mathrm{der}}_c"] \arrow[d, "\cong"] & !c \arrow[r, "\mathrm{dig}_c"] & !!c \\
c \otimes \mathbf{1} \arrow[r, "\overline{\mathrm{der}}_c \otimes \overline{\mathrm{w}}_c"'] & !c \otimes !c \arrow[r, "\mathrm{dig}_c \otimes \overline{\mathrm{der}}_{!c}"'] & !!c \otimes !!c \arrow[u, "\overline{\mathrm{c}}_{!c}"']
\end{tikzcd}$$

$$\begin{tikzcd}
c \otimes !d \arrow[r, "\overline{\mathrm{der}}_c \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes \mathrm{der}_d"'] & !c \otimes !d \arrow[d, "\mu_{c,d}"] \\
c \otimes d \arrow[r, "\overline{\mathrm{der}}_{c\otimes d}"'] & !(c \otimes d)
\end{tikzcd}$$

where the transformation $\mu_{c,d} : !c \otimes !d \to !(c \otimes d)$ is given by

$$!c \otimes !d \xrightarrow{s^2_{c,d}} !(c \,\&\, d) \xrightarrow{\mathrm{dig}_{c\&d}} !!(c \,\&\, d) \xrightarrow{!(s^2_{c,d})^{-1}} !(!c \otimes !d) \xrightarrow{!(\mathrm{der}_c \otimes \mathrm{der}_d)} !(c \otimes d).$$

In a model of **DiLL**, we can define a *deriving transformation* whose components $\delta_c : !c \otimes c \to c$ are calculated as

$$!c \otimes c \xrightarrow{\mathrm{id} \otimes \overline{\mathrm{der}}_c} !c \otimes !c \xrightarrow{\overline{\mathrm{c}}_c} !c$$

A morphism $f : !c \to d$ in the co-Kleisli category can then be differentiated by precomposing with δ_c:

$$\mathbf{D}\, f := !c \otimes c \xrightarrow{\delta_c} !c \xrightarrow{f} d.$$

Using the adjunction between $\otimes$ and $\multimap$, the derivative $\mathbf{D}\, f$ is a map $!c \to (c \multimap d)$ from c to the linear function space $c \multimap d$ corresponding to the usual mathematical intuition.

We present in the next two sections two examples of models of linear logic.

1.1.4 Relational model

The category of sets and relations **Rel** with the finite multiset comonad is one of the most basic models of linear logic. It provides a quantitative semantics of linear logic as the multiplicities in multisets carry information on the number of times a program or a proof uses its argument to compute a given output. Many models of linear logic are obtained as refinements of **Rel**.

Definition 1.1.6. The *category of relations*, denoted by **Rel**, has objects sets and morphisms binary relations. For a set A, the identity is given by the diagonal relation $\mathrm{id}_A := \{(a,a) \mid a \in A\} \in \mathbf{Rel}(A,A)$. For relations $R \in \mathbf{Rel}(A,B)$ and $S \in \mathbf{Rel}(B,C)$, their composite $S \circ R \subseteq A \times C$ is defined by

$$S \circ R := \{(a,c) \mid \exists b \in B, (a,b) \in R \text{ and } (b,c) \in S\}.$$

The category **Rel** can also be obtained from the category **Set** as the Kleisli category of the power set monad $\mathcal{P} : \mathbf{Set} \to \mathbf{Set}$. A relation $R \subseteq A \times B$ corresponds indeed to a function $A \to \mathcal{P}B$.

For the additive structure, the category **Rel** has biproducts given by the disjoint union of sets

$$A \,\&\, B := A \uplus B = (\{1\} \times A) \cup (\{2\} \times B)$$

with projections $\pi_A = \{((1,a),a) \mid a \in A\}$ and $\pi_B = \{((2,b),b) \mid b \in B\}$ and injections $\mathrm{inj}_A = \{(a,(1,a)) \mid a \in A\}$ and $\mathrm{inj}_B = \{(b,(2,b) \mid b \in B\}$ and zero object given by the empty set $\varnothing$.

The cartesian product in **Set** becomes a symmetric monoidal product in **Rel**: $A \otimes B := A \times B$ with unit the singleton set $\mathbf{1} = \{*\}$. The category **Rel** is isomorphic to its opposite category $\mathbf{Rel}^{\mathrm{op}}$ so the linear negation is trivial and **Rel** is in fact compact closed with dualizing object $\perp = \mathbf{1} = \{*\}$.

The relational model is considered to be degenerate as the linear negation is the identity and the biproduct structure means that the additive connectives (& and $\oplus$) have the same interpretation. Compact closure further implies that the interpretation of the multiplicative connectives (tensor $\otimes$ and par $\parr$) is identified.

The exponential comonad in the relational model is based on the finite multiset construction.

Definition 1.1.7. A *multiset* (or *bag*) over a set A is a function $m : A \to \mathbb{N}$. For an element $a \in A$, the number $m(a)$ is called the *multiplicity* of a. We use the notation $[a, a, b]$ for the multiset $a \mapsto 2, b \mapsto 1$. Two multisets m and m' over A can be added pointwise and we denote this operation by $m + m'$.

A multiset $m : A \to \mathbb{N}$ is said to be *finite* if it has finite support and we denote by $\mathcal{M}_{\text{fin}}A$ the set of all finite multisets over A. It induces a comonad on the category **Rel**:

Definition 1.1.8. The finite multiset comonad $\mathcal{M}_{\text{fin}} : \mathbf{Rel} \to \mathbf{Rel}$ maps a set A to $\mathcal{M}_{\text{fin}}A$, a relation $R \subseteq A \times B$ to the relation containing all pairs of multisets $(m, m') \in \mathcal{M}_{\text{fin}}A \times \mathcal{M}_{\text{fin}}B$ such that there exists enumerations $[a_1, \ldots, a_n]$ of m and $[b_1, \ldots, b_n]$ of m' with $(a_i, b_i) \in R$ for all $i \in \underline{n}$. The comultiplication or digging has components

$$\mathrm{dig}_A = \{(m, [m_1, \ldots, m_n]) \mid m = m_1 + \cdots + m_n\} \in \mathbf{Rel}(\mathcal{M}_{\text{fin}}A, \mathcal{M}_{\text{fin}}\mathcal{M}_{\text{fin}}A)$$

and the counit has components $\mathrm{der}_A = \{([a], a) \mid a \in A\} \in \mathbf{Rel}(\mathcal{M}_{\text{fin}}A, A)$.

The finite multiset comonad on **Rel** described above can also be obtained from the more primitive construction of the finite multiset monad on **Set**. This monad corresponds to the free commutative monoid completion for sets

$$\begin{tikzcd}
\mathbf{ComMon} \arrow[r, bend right, "U"'] \arrow[r, phantom, "\perp"] & \mathbf{Set} \arrow[l, bend right, "L"']
\end{tikzcd}$$

obtained from the adjunction above where U forgets the monoid structure and L maps a set A to the commutative monoid $(\mathcal{M}_{\text{fin}}A, +, [])$. The finite multiset monad on **Set** distributes over the power set monad and hence lifts to a monad in the Kleisli category $\mathbf{Rel} \cong \mathbf{Set}_{\mathcal{P}}$. Since **Rel** is self-dual, we obtain the desired comonad by dualization. Substitution in the obtained co-Kleisli category $\mathbf{Rel}_{\mathcal{M}_{\text{fin}}}$ is linear making the relational model quantitative.

Rel is also a model of differential linear logic, the differential of a relation $R \in \mathbf{Rel}(!A, B)$ is the relation $\mathbf{D}\,R \in \mathbf{Rel}(!A \times A, B)$ containing all the pairs $((m, a), b)$ such that $(m + [a], b)$ is in R.

1.1.5 The Scott model of linear logic

We present here the Scott model of linear logic which unlike **Rel** is a *qualitative* model with a substitution allowing for duplication and erasure.

Huth first showed that the Scott model of λ-calculus can be extended to a model of linear logic where the objects are prime algebraic complete lattices, the linear maps are functions preserving all suprema and the co-Kleisli maps are Scott-continuous functions [64, 65]. Independently, Winskel gave a simpler presentation based on preorders and ideal relations [118, 119]. In both cases, the co-Kleisli category is equivalent to the category of prime algebraic complete lattices and Scott-continuous functions between them. The obtained linear logic model is qualitative as it only provides information about which arguments were used to compute a given output but not how many times. The qualitative Scott model is connected to the quantitative differential relational model through an extensional collapse construction exhibited by Ehrhard [34]. The category of prime algebraic lattices and maps preserving all suprema gives rise to a model of linear logic whose associated co-Kleisli category is equivalent to the Scott model of prime algebraic lattices and Scott-continuous functions between them [64, 65]. It is however more convenient to manipulate linear logic constructions on preorders rather than on lattices and since any prime algebraic lattice can be obtained as the set of downward closed subsets of a preorder, we adopt the viewpoint of taking our objects to be preorders. The Kleisli category of this model is then equivalent to the Scott model of prime algebraic lattices [118, 119].

Definition 1.1.9. Let **ScottL** be the category whose objects are preordered sets $A = (|A|, \leq_A)$ and a morphism from A to B is an *ideal relation* $R \subseteq |A| \times |B|$, i.e. it is up-closed in $(|A|, \leq_A)$ and down-closed in $(|B|, \leq_B)$. Explicitly, it verifies that for all $a, a' \in |A|$ and $b, b' \in |B|$:

$$\big(a \leq_A a' \ \wedge\ (a, b) \in R \ \wedge\ b' \leq_B b\big) \ \Rightarrow\ (a', b') \in R.$$

The identity is given by $\mathrm{id}_A := \{(a, a') \mid a' \leq_A a\}$ and composition is the usual composition of relations.

ScottL can also be obtained from the following construction: let **Pord** denote the category of preorders and monotonous functions and let $\downarrow\colon \mathbf{Pord} \to$

Pord be the functor mapping a preorder A to its downclosure. It induces a monad on **Pord** and its Kleisli category $\mathbf{Pord}_{\downarrow}$ is isomorphic to the linear Scott category **ScottL**.

The forgetful functor sending a preorder $(|A|, \leq_A)$ to its underlying set $|A|$ has a left adjoint mapping a set S to the discrete preorder $(S, =)$. In this viewpoint, the category of sets and relations can also be obtained as the full subcategory of $\mathbf{Pord}_{\downarrow}$ containing discrete preorders.

The dual of a preordered set A is defined to be $A^{\perp} := (|A|, \geq_A)$. Every preordered set A induces a domain $\downarrow(A)$ of ideals (downward closed subsets of A) ordered by inclusion. Dually, the set of upward closed subsets of A can be defined by $\uparrow(A) := \downarrow(A^{\perp})$. Morphisms in the linear category $\mathbf{ScottL}(A, B)$ can now be seen as elements of $\downarrow(A^{\perp} \times B)$; they are also equivalent to functions from $\downarrow(A)$ to $\downarrow(B)$ that commute with all unions or functions $\uparrow(A) \to \uparrow(B)$ that commute with all intersections. For a relation R in $\mathbf{ScottL}(A, B)$, we denote the induced functions by $R_{\star} : \downarrow(A) \to \downarrow(B)$ and $R^{\star} : \uparrow(A) \to \uparrow(B)$.

ScottL is less degenerate than **Rel** in that the dual is not the identity but it is still a compact closed category where the interpretation of the connectives $\otimes$ and $⅋$ coincide. For preorders A and B, $A \otimes B$ and $A ⅋ B$ are given by $(|A| \times |B|, \leq_A \times \leq_B)$ and has the singleton preordered set $\mathbf{1} = (\{*\}, =)$ as a unit. The additive structure is given by the disjoint union of preorders $A \,\&\, B = A \oplus B := (|A| + |B|, \leq_A + \leq_B)$ with the empty preordered set $\mathbf{0} = (\varnothing, \varnothing)$ as zero object.

There are (at least) two comonad structures on **ScottL**, one that is *quantitative* corresponding to the one defined for the relational model in the previous section and one that is *qualitative* allowing for duplication and erasure. We denote the qualitative comonad by $! : \mathbf{ScottL} \to \mathbf{ScottL}$, it takes a preordered set A to the preordered set whose web $|!A|$ is the set of finite sequences of elements in $|A|$:

$$|!A| := \{\langle a_1, \ldots, a_n \rangle \mid a_i \in |A|, n \in \mathbb{N}\}$$

and the preorder relation is defined as follows:

$$\langle a_1, \ldots, a_n \rangle \leq_{!A} \langle b_1, \ldots, b_m \rangle \quad :\Leftrightarrow \quad \forall i \in \underline{n}, \exists j \in \underline{m}, a_i \leq_A b_j$$

On morphisms, a relation $R \in \mathbf{ScottL}(A, B)$ is mapped to

$$!R := \{(\langle a_1, \ldots a_n \rangle, \langle b_1, \ldots, b_m \rangle) \mid \forall j \in \underline{m}, \exists i \in \underline{n}, (a_i, b_j) \in R\}.$$

The obtained co-Kleisli category $\mathbf{ScottL}_!$ is then equivalent to the category of prime algebraic lattices and Scott-continuous functions between them since every relation in $\mathbf{ScottL}_!(A, B)$ corresponds to a Scott-continuous function $\downarrow(A) \to \downarrow(B)$.

Remark 1. We chose this presentation of the comonad instead of finite subsets [64, 118] or finite multisets [33, 34] since it is more convenient for the profunctorial generalization in Chapter 4. Note that for the three presentations, the associated lattices of downward closed subsets are all isomorphic and the associated co-Kleisli categories are all equivalent to the Scott model.

Differential linear logic is inherently quantitative and since substitution in this model is non-linear by allowing for duplication and erasure, **ScottL** is not a model of differential linear logic, a formal proof can be found in Proposition 2 of [33].

1.2 Glueing and orthogonality for models of linear logic

We give in this section a brief overview of single glueing and double glueing constructions for linear logic models and focus on the particular case of orthogonality categories. The content of this section is adapted from the general presentation by Hyland and Schalk [68], we also refer the reader to Tan and Hasegawa [109, 60] for earlier work on this subject.

1.2.1 Glueing for intuitionistic linear logic

Single glueing for intuitionistic linear logic is based on the comma construction that we recall below:

Definition 1.2.1. For functors $F : \mathbb{A} \to \mathbb{C}$ and $G : \mathbb{B} \to \mathbb{C}$, their *comma category*, denoted by F/G is given by:

- objects are triples (a, b, h) where $a \in \mathrm{Ob}(\mathbb{A})$, $b \in \mathrm{Ob}(\mathbb{B})$ and h is a morphism in $\mathbb{C}(F(a), G(b))$;
- a morphism from (a_1, b_1, h_1) to (a_2, b_2, h_2) is a pair (f, g) where $f \in \mathbb{A}(a_1, a_2)$ and $g \in \mathbb{B}(b_1, b_2)$ such that $h_2 \circ F(f) = G(g) \circ h_1$

$$\begin{tikzcd}
F(a_1) \arrow[r, "F(f)"] \arrow[d, "h_1"'] & F(a_2) \arrow[d, "h_2"] \\
G(b_1) \arrow[r, "G(g)"'] & G(b_2)
\end{tikzcd}$$

Composition is given componentwise by the composites in $\mathbb{A}$ and $\mathbb{B}$ and the identity of (a, b, h) is $(\mathrm{id}_a, \mathrm{id}_b)$.

There are canonical forgetful functors $\mathcal{U}_{\mathbb{A}} : F/G \to \mathbb{A}$ and $\mathcal{U}_{\mathbb{B}} : F/G \to \mathbb{B}$ and a natural transformation $\alpha : F \circ U_{\mathbb{A}} \Rightarrow G \circ U_{\mathbb{B}}$ exhibiting F/G as a strict 2-limit in the 2-category **Cat**:

Definition 1.2.2. Let $f : a \to c$ and $g : b \to c$ be 1-cells in a 2-category $\mathscr{A}$. A *comma object* from f to g is a tuple $(f/g, p_1, p_2, \alpha)$ where f/g is an object in $\mathscr{A}$, $p_1 : f/g \to a$, $p_2 : f/g \to b$ are 1-cells in $\mathscr{A}$ and $\alpha : f \circ p_1 \Rightarrow g \circ p_2$ is a 2-cell in $\mathscr{A}$

$$\begin{tikzcd}
f/g \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\swarrow_{\alpha}"] & a \arrow[d, "f"] \\
b \arrow[r, "g"'] & c
\end{tikzcd}$$

satisfying the following universality property: for every $(d, q_1 : d \to a, q_2 : d \to b, \beta : f \circ q_1 \Rightarrow g \circ q_2)$, there exists a unique 1-cell $u : d \to f/g$ such that $p_1 \circ u = q_1$, $p_2 \circ u = q_2$ and $u * \alpha = \beta$.

Theorem 1.2.3 (Section 4.1 in [68])**.** *Let $\mathbb{C}$ and $\mathbb{D}$ be models of intuitionistic linear logic and $L : \mathbb{C} \to \mathbb{D}$ be a functor. If $\mathbb{D}$ has pullbacks and L is monoidal and linearly distributive (i.e. there is a distributive law $!_{\mathbb{D}} L \Rightarrow L !_{\mathbb{C}}$), then the glueing category $\mathbb{G} := \mathrm{id}_{\mathbb{D}}/L$ is a model of intuitionistic linear logic and the forgetful functor $\mathbb{G} \to \mathbb{C}$ preserves the structure strictly.*

1.2.2 General double glueing

For functors $L : \mathbb{C} \to \mathbb{D}$ and $K : \mathbb{C} \to \mathbb{D}^{\mathrm{op}}$, the *double glueing category* $\mathbb{G}$ lies in the following lax limit diagram:

$$\begin{tikzcd}
& \mathbb{G} \arrow[dl] \arrow[d] \arrow[dr] \arrow[dl, phantom, shift left=6, "\Rightarrow"] \arrow[dr, phantom, shift right=6, "\Rightarrow"] & \\
\mathbb{D} & \mathbb{C} \arrow[l, "L"] \arrow[r, "K"'] & \mathbb{D}^{\mathrm{op}}
\end{tikzcd}$$

The category $\mathbb{G}$ is explicitly described by:

- objects are triples $(c, X \xrightarrow{f} L(c), Y \xrightarrow{g} K(c))$ where c is an object of $\mathbb{C}$, and $X \xrightarrow{h} L(c), Y \xrightarrow{k} K(c)$ are morphisms in $\mathbb{D}$;

- morphisms from the object $(c_1, X_1 \xrightarrow{h_1} L(c_1), Y_1 \xrightarrow{k_1} K(c_1))$ to the object $(c_2, X_2 \xrightarrow{h_2} L(c_2), Y_2 \xrightarrow{k_2} K(c_2))$ are triples (f, α, β) where $f : c_1 \to c_2$ is a morphism in $\mathbb{C}$, $\alpha : X_1 \to X_2$ and $\beta : Y_2 \to Y_1$ are in $\mathbb{D}$ and the following two diagrams commute:

$$\begin{tikzcd}
L(c_1) \arrow[r, "L(f)"] & L(c_2) & & K(c_2) \arrow[r, "K(f)"] & K(c_1) \\
X_1 \arrow[u, "h_1"] \arrow[r, "\alpha"'] & X_2 \arrow[u, "h_2"'] & & Y_2 \arrow[u, "k_2"] \arrow[r, "\beta"'] & Y_1 \arrow[u, "k_1"']
\end{tikzcd}$$

  The forgetful functor $\mathbb{G} \to \mathbb{C}$ maps an object $(c, X \xrightarrow{f} L(c), Y \xrightarrow{g} K(c))$ to c and a morphism (f, α, β) to f.

In order to obtain a model of classical linear logic, we require $\mathbb{C}$ to be equipped with a self-duality $(-)^* : \mathbb{C} \to \mathbb{C}^{\mathrm{op}}$ and that

$$K = \mathbb{C} \xrightarrow{(-)^*} \mathbb{C}^{\mathrm{op}} \xrightarrow{L^{\mathrm{op}}} \mathbb{D}^{\mathrm{op}}$$

but this condition is not necessary for the intuitionistic case.

Theorem 1.2.4 (Theorem 4.16 in [68]). *If $\mathbb{C}$ and $\mathbb{D}$ are models of classical linear logic, $\mathbb{D}$ has pullbacks, L is monoidal and linearly distributive (i.e. there is a distributive law $!_{\mathbb{D}} L \Rightarrow L !_{\mathbb{C}}$), then $\mathbb{G}$ is a model of classical linear logic and the forgetful functor $\mathbb{G} \to \mathbb{C}$ preserves the structure strictly.*

1.2.3 Orthogonality categories

We are interested in a more restrictive class of double glueing models, namely double glueing along hom-functors with an orthogonality. Let $\mathbb{C}$ be a model of classical linear logic with monoidal units $\mathbf{1}$ and $\perp$ and assume that $L = \mathbb{C}(\mathbf{1}, -)$ and $K = \mathbb{C}(-, \perp)$. We start with the glued category along L and K described in the previous section

$$\begin{tikzcd}
& & \mathbb{G} \arrow[ddll] \arrow[dd] \arrow[ddrr] & & \\
& \Rightarrow & & \Rightarrow & \\
\mathbf{Set} & & \mathbb{C} \arrow[ll, "\mathbb{C}(\mathbf{1}, -)"] \arrow[rr, "\mathbb{C}(-, \perp)"'] & & \mathbf{Set}^{\mathrm{op}}
\end{tikzcd}$$

and restrict to the full subcategory containing objects $(c, X \xrightarrow{f} \mathbb{C}(\mathbf{1}, c), Y \xrightarrow{g} \mathbb{C}(c, \perp))$ where the morphisms f and g are monomorphisms.

Objects of $\mathbb{G}$ are now triples (c, X, Y) where c is an object of $\mathbb{C}$, X is a subset of $\mathbb{C}(\mathbf{1}, c)$ and Y is a subset of $\mathbb{C}(c, \perp)$. A morphism from (c, X, Y) to (c', X', Y') is a morphism f in $\mathbb{C}(c, c')$ such that the following two diagrams commute

$$\begin{tikzcd}
\mathbb{C}(\mathbf{1}, c) \arrow[r, "f_\star = \mathbb{C}(\mathbf{1}, f)"] & \mathbb{C}(\mathbf{1}, c') \\
X \arrow[u, hook] \arrow[r] & X' \arrow[u, hook]
\end{tikzcd}
\qquad
\begin{tikzcd}
K(c') \arrow[r, "f^\star = \mathbb{C}(f, \perp)"] & K(c') \\
Y' \arrow[u, hook] \arrow[r] & Y \arrow[u, hook]
\end{tikzcd}$$

Definition 1.2.5. Let $(\mathbb{C}, \otimes, \mathbf{1})$ be a monoidal category with a self duality $(-)^* : \mathbb{C} \to \mathbb{C}^{\mathrm{op}}$. An *orthogonality* is a family $\{\perp_c\}_{c \in \mathbb{C}}$ of relations

$$\perp_c \subseteq \mathbb{C}(\mathbf{1}, c) \times \mathbb{C}(c, \perp)$$

indexed by the objects of $\mathbb{C}$ satisfying the following conditions:

1. for an isomorphism $f : c \to d$ and $(u, v) \in \mathbb{C}(\mathbf{1}, c) \times \mathbb{C}(c, \perp)$,

$$u \perp_c v \quad \Leftrightarrow \quad f \circ u \perp_d v \circ f^{-1}$$

2. for $(u, v) \in \mathbb{C}(\mathbf{1}, c) \times \mathbb{C}(c, \perp)$,

$$u \perp_c v \quad \Rightarrow \quad \mathrm{id}_{\mathbf{1}} \perp_{\mathbf{1}} v \circ u$$

3. for $u \in \mathbb{C}(\mathbf{1}, c)$, $u' \in \mathbb{C}(\mathbf{1}, c')$ and $v \in \mathbb{C}(c \otimes c', \perp)$, we have

$$(\mathbf{1} \xrightarrow{u} c) \perp_c (c \cong c \otimes \mathbf{1} \xrightarrow{\mathrm{id} \otimes u'} c \otimes c' \xrightarrow{v} \perp) \quad \text{and}$$
$$(\mathbf{1} \xrightarrow{u'} c') \perp_{c'} (c' \cong \mathbf{1} \otimes c' \xrightarrow{u \otimes \mathrm{id}} c \otimes c' \xrightarrow{v} \perp)$$

if and only if $(\mathbf{1} \cong \mathbf{1} \otimes \mathbf{1} \xrightarrow{u \otimes u'} c \otimes c') \perp_{c \otimes c'} (c \otimes c' \xrightarrow{v} \perp)$.

4. for $u \in \mathbb{C}(\mathbf{1}, c)$ and $v \in \mathbb{C}(c, \perp)$,

$$u \perp_c v \quad \Leftrightarrow \quad v^* \perp_{c^*} u^*.$$

Definition 1.2.6. Let $\{\perp_c\}_{c \in \mathbb{C}}$ be an orthogonality relation on a category $\mathbb{C}$. For a subset $X \subseteq \mathbb{C}(\mathbf{1}, c)$, define its *orthogonal* $X^\perp \subseteq \mathbb{C}(c, \perp)$ as

$$X^\perp := \{y : c \to \perp \mid \forall x \in X, x \perp_c y\}$$

and dually, for a subset $Y \subseteq \mathbb{C}(c, \perp)$, its orthogonal $Y^\perp \subseteq \mathbb{C}(\mathbf{1}, c)$ is given by

$$Y^\perp := \{x : \mathbf{1} \to c \mid \forall y \in Y, x \perp_c y\}.$$

It induces a Galois connection

$$\begin{tikzcd}
(\mathcal{P}(\mathbb{C}(\mathbf{1}, c)), \subseteq) \arrow[r, bend left, "(-)^\perp"] \arrow[r, phantom, "\perp"] & (\mathcal{P}(\mathbb{C}(c, \perp)), \subseteq)^{\mathrm{op}} \arrow[l, bend left, "(-)^\perp"]
\end{tikzcd}$$

between the posets $(\mathcal{P}(\mathbb{C}(\mathbf{1}, c)), \subseteq)$ and $(\mathcal{P}(\mathbb{C}(c, \perp)), \subseteq)^{\mathrm{op}}$ ordered by inclusion.

Definition 1.2.7. For a model of classical linear logic $\mathbb{C}$ and an orthogonality $\{\perp_c\}_{c \in \mathbb{C}}$ on $\mathbb{C}$, the *tight orthogonality category* $\mathbb{T}$ consists of

- objects are pairs (c, X) where c is an object of $\mathbb{C}$ and X is a subset of $\mathbb{C}(\mathbf{1}, c)$ verifying $X = X^{\perp\perp}$,
- a morphism from (c, X) to (c', X') is a morphism f in $\mathbb{C}(c, c')$ that preserves the orthogonality forward and backward, i.e. the following two diagrams commute

$$\begin{tikzcd}
\mathbb{C}(\mathbf{1}, c) \arrow[r, "f_\star"] & \mathbb{C}(\mathbf{1}, c') & \mathbb{C}(c', \perp) \arrow[r, "f^\star"] & \mathbb{C}(c, \perp) \\
X \arrow[u, hook] \arrow[r, dotted] & X' \arrow[u, hook] & X'^\perp \arrow[u, hook] \arrow[r, dotted] & X^\perp \arrow[u, hook]
\end{tikzcd}$$

It is a subcategory of the glued category $\mathbb{G}$ and the inclusion functor maps (c, X) in $\mathbb{T}$ to $(c, X, X^\perp)$ in $\mathbb{G}$.

One of the better behaved cases of orthogonality relations are the *focused* ones:

Definition 1.2.8. An orthogonality $\{\perp_c\}_{c\in\mathbb{C}}$ on a monoidal category $(\mathbb{C}, \otimes, \mathbf{1})$ is *focused* if there exists a distinguished *pole* $\perp\!\!\!\perp \subseteq \mathbb{C}(\mathbf{1}, \perp)$ such that $\perp_c \hookrightarrow \mathbb{C}(\mathbf{1}, c) \times \mathbb{C}(c, \perp)$ is given by:

$$\{(x, y) \in \mathbb{C}(\mathbf{1}, c) \times \mathbb{C}(c, \perp) \mid y \circ x \in \perp\!\!\!\perp\}$$

for all $c \in \mathbb{C}$.

When the orthogonality is focused, all conditions in Definition 1.2.5 are automatically verified. It also simplifies the conditions on the morphisms in the glueing category $\mathbb{G}$ as morphisms do not need to preserve the orthogonality forward and backward.

Definition 1.2.9. An orthogonality $\{\perp_c\}_{c\in\mathbb{C}}$ on a monoidal category $(\mathbb{C}, \otimes, \mathbf{1})$ is *reciprocal* if for all $x : \mathbf{1} \to c, f : c \to d$ and $y : d \to \perp$:

$$x \perp_c y \circ f \qquad \Leftrightarrow \qquad f \circ x \perp_d y$$

Lemma 1.2.10. *An orthogonality relation* $\{\perp_c\}_{c\in\mathbb{C}}$ *is focused if and only if it is reciprocal.*

Proof. Assume that the orthogonality is reciprocal and define $\perp\!\!\!\perp := \{f \in \mathbb{C}(\mathbf{1}, \perp) \mid \mathrm{id}_{\mathbf{1}} \perp_{\mathbf{1}} f\}$. For $c \in \mathbb{C}$ and $(x, y) \in \mathbb{C}(\mathbf{1}, c) \times \mathbb{C}(c, \perp)$, if $(x, y) \in \perp_c$, then we have by definition of an orthogonality relation, we have $\mathrm{id}_{\mathbf{1}} \perp_{\mathbf{1}} y \circ x$ so $y \circ x$ is in $\perp\!\!\!\perp$. If $y \circ x \in \perp\!\!\!\perp$, then using the reciprocal property, $\mathrm{id}_{\mathbf{1}} \perp_{\mathbf{1}} y \circ x$ implies $x \perp_c y$.

For the other direction, assume that the orthogonality is focused with pole $\perp\!\!\!\perp$. Let $x : \mathbf{1} \to c, f : c \to d$ and $y : d \to \perp$ be morphisms in $\mathbb{C}$, we have:

$$x \perp_c y \circ f \quad \Leftrightarrow \quad y \circ f \circ x \in \perp\!\!\!\perp \quad \Leftrightarrow \quad f \circ x \perp_d y$$

so that $\{\perp_c\}_{c\in\mathbb{C}}$ is reciprocal as desired. ☐

For example, in the relational model, there are only two possible focused orthogonalities as $\mathbf{Rel}(\mathbf{1}, \perp)$ is a singleton set. We will see how the bicategory of profunctors offers a much richer setting in future chapters. Since we cannot assume the orthogonality relation to be focused in general, we need the morphisms interpreting the linear logic structure to still verify this property.

Definition 1.2.11. Let $\{\perp_c\}_{c\in\mathbb{C}}$ be an orthogonality relation on a category $\mathbb{C}$, and $X \subseteq \mathbb{C}(\mathbf{1}, c)$ and $Y \subseteq \mathbb{C}(d, \perp)$ be two subsets. A morphism $f : c \to d$ in $\mathbb{C}$ is called:

1. *positive with respect to* X *and* Y if for all $x \in X$ and $y \in Y$,

$$f \circ x \perp_d y \qquad \Rightarrow \qquad x \perp_c y \circ f$$

2. *negative with respect to* X *and* Y if for all $x \in X$ and $y \in Y$,

$$x \perp_c y \circ f \qquad \Rightarrow \qquad f \circ x \perp_d y$$

3. *focused with respect to* X *and* Y if it is both positive and negative.

Additive structure

Finite products: Assume that $\mathbb{C}$ has finite products $(\&, \top)$ and that the projections are focused, then the tight orthogonality category $\mathbb{T}$ has finite products given by:

$$(c, X) \,\&\, (c', X') := (c \,\&\, c', X \,\&\, X')$$

where $X \,\&\, X' := \{\langle x, x'\rangle : \mathbf{1} \to c \,\&\, c' \mid x \in X, x' \in X'\}$. It corresponds to the following construction:

$$X \times X' \hookrightarrow \mathbb{C}(\mathbf{1}, c) \times \mathbb{C}(\mathbf{1}, c') \xrightarrow[\cong]{\langle -,- \rangle} \mathbb{C}(\mathbf{1}, c \,\&\, c')$$

The terminal object is given $(\top, \mathbb{C}(\mathbf{1}, \top))$.

Finite coproducts: Assume that $\mathbb{C}$ has finite coproducts $(\oplus, \mathbf{0})$ and that the injections are focused, then the tight orthogonality category $\mathbb{T}$ has finite coproducts given by:

$$(c, X) \oplus (c', X') := (c \oplus c', X \oplus X')$$

where $X \oplus X' := \{[x, x'] : c \oplus c' \to \perp \mid x \in X^\perp, x' \in X'^\perp\}$. It corresponds to the following construction:

$$\left(X^\perp \times X'^\perp \hookrightarrow \mathbb{C}(c, \perp) \times \mathbb{C}(c', \perp) \xrightarrow{[-,-]} \mathbb{C}(c \oplus c', \perp)\right)^\perp$$

The initial object is given $(\mathbf{0}, \mathbb{C}(\mathbf{0}, \perp)^\perp)$.

$*$-autonomous structure

If $\mathbb{C}$ is $*$-autonomous, we need to assume additional conditions on the orthogonality to obtain a $*$-autonomous structure for $\mathbb{T}$. The functor L is monoidal with transformation $m_{c,c'} : \mathbb{C}(\mathbf{1}, c) \times \mathbb{C}(\mathbf{1}, c') \to \mathbb{C}(\mathbf{1}, c \otimes c')$ given by:

$$(x : \mathbf{1} \to c, x' : \mathbf{1} \to c') \quad \mapsto \quad \mathbf{1} \cong \mathbf{1} \otimes \mathbf{1} \xrightarrow{x \otimes x'} c \otimes c'$$

The morphism $m_{c,c'}$ is not necessarily an inclusion so for $X \subseteq \mathbb{C}(\mathbf{1}, c)$ and $X' \subseteq \mathbb{C}(\mathbf{1}, c')$, we denote by $X \cdot X'$ the following set:

$$\{x \otimes x' : \mathbf{1} \cong \mathbf{1} \otimes \mathbf{1} \to c \otimes c' \mid x \in X, x' \in X'\}$$

it corresponds to the image of the following morphism:

$$\begin{tikzcd}
X \times X' \arrow[r, hook] \arrow[dr, two heads] & \mathbb{C}(\mathbf{1}, c) \times \mathbb{C}(\mathbf{1}, c') \arrow[r, "m_{c,c'}"] & \mathbb{C}(\mathbf{1}, c \otimes c') \\
& X \cdot X' \arrow[ur, hook] &
\end{tikzcd}$$

Likewise, for $X \subseteq \mathbb{C}(\mathbf{1}, c)$ and $Y \subseteq \mathbb{C}(c', \perp)$, we denote by $X \multimap Y$ the set

$$\{x \multimap y : c \multimap c' \to \mathbf{1} \multimap \perp \cong \perp \mid x \in X, y \in Y\}.$$

Definition 1.2.12. An orthogonality $\{\perp_c\}_{c \in \mathbb{C}}$ on a monoidal category $(\mathbb{C}, \otimes, \mathbf{1})$ is *stable* if for all $X \subseteq \mathbb{C}(\mathbf{1}, c)$, $X' \subseteq \mathbb{C}(\mathbf{1}, c')$ and $Y \subseteq \mathbb{C}(c', \perp)$, we have:

1. $(X^{\perp\perp} \cdot X'^{\perp\perp})^{\perp} = (X^{\perp\perp} \cdot X'^{\perp})^{\perp} = (X \cdot X'^{\perp\perp})^{\perp}$ and
2. $(X^{\perp\perp} \multimap Y^{\perp\perp})^{\perp} = (X \multimap Y^{\perp\perp})^{\perp} = (X^{\perp\perp} \multimap Y)^{\perp}$.

Proposition 1.2.13 (Proposition 5.11 in [68]). *If $\mathbb{C}$ is $*$-autonomous with a stable orthogonality then $\mathbb{T}$ is $*$-autonomous and the forgetful functor preserves the structure. The tensor product of two objects (c, X) and (c', X') is $(c \otimes c', X \otimes X')$ where*

$$X \otimes X' := (X \cdot X')^{\perp\perp} = \{x \otimes x' : \mathbf{1} \cong \mathbf{1} \otimes \mathbf{1} \to c \otimes c' \mid x \in X, x' \in X'\}^{\perp\perp}$$

and monoidal unit $(\mathbf{1}, \{\mathrm{id}_{\mathbf{1}}\}^{\perp\perp})$.

Exponential structure

Assume that $\mathbb{C}$ has a linear exponential comonad ! and that there is a distributive law $\kappa : \mathbb{C}(\mathbf{1}, -) \to \mathbb{C}(\mathbf{1}, !-)$ between the ! comonad in $\mathbb{C}$ and the identity comonad in **Set**. The morphisms κ_c are not necessarily inclusions so for $X \subseteq \mathbb{C}(\mathbf{1}, c)$, we denote by $X^!$ the following set:

$$\{\kappa_c(x) : \mathbf{1} \to !c \mid x \in X\}$$

it corresponds to the image of the following morphism:

$$\begin{tikzcd}
X \arrow[r, hook] \arrow[dr, two heads] & \mathbb{C}(\mathbf{1}, c) \arrow[r, "\kappa_c"] & \mathbb{C}(\mathbf{1}, !c) \\
& X^! \arrow[ur, hook] &
\end{tikzcd}$$

Proposition 1.2.14 (Proposition 5.13 in [68])**.** *If the orthogonality on* $\mathbb{C}$ *is stable and all the comonad morphisms* dig_c *and* der_c*, the comonoid morphisms* e_c *and* d_c*, and all morphisms* $!f$ *for* $f \in \mathbb{C}$ *are positive with respect to* κ*, then* $\mathbb{T}$ *has a linear exponential comonad and the forgetful functor preserves the structure. The exponential of an object* (c, X) *is given by* $(!c, !X)$ *where*

$$!X := (X^!)^{\perp\perp} = \{\kappa_c(x) : \mathbf{1} \to !c \mid x \in X\}^{\perp\perp}.$$

1.3 Bicategories for linear logic

We present in this section the bicategorical structure needed to interpret linear logic connectives. Most of the content is drawn from [84, 71, 100].

1.3.1 Bicategories, functors, transformations and modifications

Definition 1.3.1. A *bicategory* $\mathscr{B}$ consists of the following data:

1. A class $\mathscr{B}_0$ of *objects* or 0-*cells* $A, B, \ldots$.
2. For each pair of object A and B in $\mathscr{B}_0$, a (small) category $\mathscr{B}(A, B)$ whose objects $f, g : A \to B$ are called 1-*cells* and morphisms $\alpha : f \Rightarrow g$ are called 2-cells. The classes of all 1-cells and 2-cells are denoted by $\mathscr{B}_1$ and $\mathscr{B}_2$ respectively. The composition in this category is called *vertical composition*) and is denoted by $\beta\alpha$ for $\alpha : f \Rightarrow g$ and $\beta : g \Rightarrow h$.

$$\begin{tikzcd}
A \arrow[r, bend left=50, "f"] \arrow[r, "g" description] \arrow[r, bend right=50, "h"'] \arrow[r, phantom, shift left=3ex, "\Downarrow \alpha"] \arrow[r, phantom, shift right=3ex, "\Downarrow \beta"] & B & \mapsto & A \arrow[r, bend left=50, "f"] \arrow[r, bend right=50, "h"'] \arrow[r, phantom, "\Downarrow \beta\alpha"] & B
\end{tikzcd}$$

3. For each object $A \in \mathscr{B}_0$, an *identity functor* $I_A : \mathbf{1} \to \mathscr{B}(A,A)$ where $\mathbf{1}$ is the terminal category. I_A can be seen as specifying an identity 1-cell id_A in $\mathscr{B}(A,A)$.

4. For each objects $A, B, C \in \mathscr{B}_0$, a bifunctor, called *horizontal composition*

$$\begin{aligned} *_{ABC} : \mathscr{B}(A,B) \times \mathscr{B}(B,C) &\to \mathscr{B}(A,C) \\ (f,g) &\mapsto g \circ f \\ (\alpha,\beta) &\mapsto \beta * \alpha \end{aligned}$$

$$\begin{tikzcd}
A \arrow[r, bend left=50, "f"] \arrow[r, bend right=50, "f'"'] \arrow[r, phantom, "\Downarrow \alpha"] & B \arrow[r, bend left=50, "g"] \arrow[r, bend right=50, "g'"'] \arrow[r, phantom, "\Downarrow \beta"] & C & \mapsto & A \arrow[r, bend left=50, "g \circ f"] \arrow[r, bend right=50, "g' \circ f'"'] \arrow[r, phantom, "\Downarrow \beta * \alpha"] & C
\end{tikzcd}$$

5. For all objects A, B, C and D, a natural isomorphism referred to as the *associator*:

$$a : *_{ABD} \circ (*_{BCD} \times \mathrm{id}_{\mathscr{B}(A,B)}) \Rightarrow *_{ACD} \circ (\mathrm{id}_{\mathscr{B}(C,D)} \times *_{ABC})$$

whose components are invertible 2-cells:

$$a_{f,g,h} : (h \circ g) \circ f \Rightarrow h \circ (g \circ f)$$

6. For all objects A, B, natural isomorphisms called *left* and *right unitors*:

$$\begin{aligned} l :\ *_{ABB} \circ (I_B \times \mathrm{id}_{\mathscr{B}(A,B)}) &\Rightarrow \mathrm{id}_{\mathscr{B}(A,B)} \\ r :\ *_{AAB} \circ (\mathrm{id}_{\mathscr{B}(A,B)} \times I_A) &\Rightarrow \mathrm{id}_{\mathscr{B}(A,B)} \end{aligned}$$

whose components are invertible 2-cells:

$$l_f : \mathrm{id}_B \circ f \Rightarrow f \quad \text{and} \quad r_f : f \circ \mathrm{id}_A \Rightarrow f$$

satisfying the following coherence axioms:

- for all objects A, B, C, D, E and 1-cells $k, h, g, f \in \mathscr{B}(A,B) \times \mathscr{B}(B,C) \times \mathscr{B}(C,D) \times \mathscr{B}(D,E)$, the following diagram (called the *pentagon identity*) commutes:

$$\begin{tikzcd}
 & & (k \circ h)(g \circ f) \arrow[drr, "a_{k,h,(g\circ f)}"] & & \\
((k \circ h) \circ g) \circ f \arrow[urr, "a_{(k\circ h),g,f}"] \arrow[dr, "a_{k,h,g} * \mathrm{Id}_f"'] & & & & k \circ (h \circ (g \circ f)) \\
 & (k \circ (h \circ g)) \circ f \arrow[rr, "a_{k,h\circ g,f}"'] & & k \circ ((h \circ g) \circ f) \arrow[ur, "\mathrm{Id}_k * a_{h,g,f}"'] &
\end{tikzcd}$$

- for all objects A, B, C and 1-cells $g, f \in \mathscr{B}(A,B) \times \mathscr{B}(B,C)$, the following diagram (called the *triangle identity*) commutes:

$$\begin{tikzcd}
(g \circ \mathrm{id}_B) \circ f \arrow[rr, "a_{g,\mathrm{id}_B,f}"] \arrow[dr, "r_g * \mathrm{Id}_f"'] & & g \circ (\mathrm{id}_B \circ f) \arrow[dl, "\mathrm{Id}_g * l_f"] \\
 & g \circ f &
\end{tikzcd}$$

A bicategory in which the associators, left and right unitors are identities, so that composition is strictly associative, is called a *2-category*.

Definition 1.3.2. Let $\mathscr{A}$ and $\mathscr{B}$ be bicategories. A *pseudo-functor* $\mathscr{A} \to \mathscr{B}$ consists of:

1. a function $F : \mathscr{A}_0 \to \mathscr{B}_0$
2. for every objects $A, B \in \mathscr{A}_0$, a functor $F_{AB} : \mathscr{A}(A,B) \to \mathscr{B}(F(A), F(B))$
3. for every object $A \in \mathscr{A}_0$, a natural isomorphism

$$\begin{tikzcd}
& \mathscr{B}(FA,FA) \\
\mathbf{1} \arrow[ur, bend left, "I^{\mathscr{B}}_{FA}"] \arrow[dr, bend right, "I^{\mathscr{A}}_{A}"'] & \Downarrow \varphi_0 \\
& \mathscr{A}(A,A) \arrow[uu, "F_{A,A}"']
\end{tikzcd}$$

whose component is an invertible 2-cell $\varphi_0 : \mathrm{id}_{FA} \Rightarrow F(\mathrm{id}_A)$.

4. for every objects $A, B, C \in \mathscr{A}_0$, a natural isomorphism

$$\begin{tikzcd}
\mathscr{A}(A,B) \times \mathscr{A}(B,C) \arrow[r, "F_{A,B} \times F_{B,C}"] \arrow[d, "*^{\mathscr{A}}_{A,B,C}"'] \arrow[dr, phantom, "\Downarrow \varphi_2"] & \mathscr{B}(F(A),F(B)) \times \mathscr{B}(F(B),F(C)) \arrow[d, "*^{\mathscr{B}}_{FA,FB,FC}"] \\
\mathscr{A}(A,C) \arrow[r, "F_{A,C}"'] & \mathscr{B}(F(A),F(C))
\end{tikzcd}$$

whose components are invertible 2-cells $(\varphi_2)_{gf} : F(g) \circ F(f) \Rightarrow F(g \circ f)$.

They are required to make the following diagrams for all 1-cells $f \in \mathscr{A}(A,B)$, $g \in \mathscr{A}(B,C)$ and $h \in \mathscr{A}(C,D)$:

- Associativity:

$$\begin{tikzcd}
& F(h \circ g) \circ Ff \arrow[r, bend left, "\varphi_2"] & F((h \circ g) \circ f) \arrow[dr, "Fa^{\mathscr{A}}"] & \\
(Fh \circ Fg) \circ Ff \arrow[ur, bend left, "\varphi_2 * \mathrm{Id}_{Ff}"] \arrow[dr, bend right, "a^{\mathscr{B}}"'] & & & F(h \circ (g \circ f)) \\
& Fh \circ (Fg \circ Fh) \arrow[r, bend right, "\mathrm{Id}_{Fh} * \varphi_2"'] & Fh \circ F(g \circ f) \arrow[ur, bend right, "\varphi_2"'] &
\end{tikzcd}$$

- Left and right unity:

$$\begin{tikzcd}
& (Ff) \circ (\mathrm{id}^{\mathscr{B}}_{Fb}) \arrow[dl, bend right, "\mathrm{Id}_{Ff} * \varphi_0"'] \arrow[dr, bend left, "r^{\mathscr{B}}"] & & (\mathrm{id}^{\mathscr{B}}_{Fa}) \circ (Ff) \arrow[dl, bend right, "l^{\mathscr{B}}"'] \arrow[dr, bend left, "\varphi_0 * \mathrm{Id}_{Ff}"] & \\
(Ff) \circ (F\mathrm{id}^{\mathscr{A}}_{b}) \arrow[dr, bend right, "\varphi_2"'] & & Ff & & (F\mathrm{id}^{\mathscr{A}}_{a}) \circ (Ff) \arrow[dl, bend left, "\varphi_2"] \\
& F(f \circ \mathrm{id}^{\mathscr{A}}_{b}) \arrow[ur, bend right, "Fr^{\mathscr{A}}"'] & & F(\mathrm{id}^{\mathscr{A}}_{a} \circ f) \arrow[ul, bend left, "Fl^{\mathscr{A}}"] &
\end{tikzcd}$$

Definition 1.3.3. A *lax functor* is a pseudo-functor where the natural transformations φ_0 and φ_2 are not assumed to be isomorphisms. For a *colax functor*, the 2-cells φ_0 and φ_2 are in the opposite direction $F(\mathrm{id}_A) \Rightarrow \mathrm{id}_{FA}$ and $F(g \circ f) \Rightarrow F(g) \circ F(f)$. A *strict functor* is a pseudo-functor where the 2-cells φ_0 and φ_2 are identities.

Definition 1.3.4. Let $(F, \phi), (G, \psi) : \mathscr{A} \to \mathscr{B}$ be two homomorphisms between bicategories. A *pseudo-natural transformation* $\sigma : F \Rightarrow G$ is given by the data:

1. 1-cells $\sigma_A : F(A) \to G(A)$ for each object $A \in \mathscr{A}$,
2. for every pair of objects A, B in $\mathscr{A}$, a natural isomorphism $\sigma : (\sigma_A)^* \circ G_{AB} \to (\sigma_B)_* \circ F_{AB}$ whose components are invertible 2-cells $\sigma_f : Gf \circ \sigma_A \to \sigma_B \circ Ff$

$$\begin{tikzcd}
F(A) \arrow[r, "F(f)"] \arrow[d, "\sigma_A"'] \arrow[dr, phantom, "\Uparrow \sigma_f"] & F(B) \arrow[d, "\sigma_B"] \\
G(A) \arrow[r, "G(f)"'] & G(B)
\end{tikzcd}$$

such that the following diagrams are equal:

- naturality:

$$\begin{tikzcd}
FA \arrow[rr, bend left, "F(gf)"] \arrow[dd, "\sigma_A"'] & \Uparrow \sigma_{gf} & FC \arrow[dd, "\sigma_C"] \\
& & \\
GA \arrow[rr, bend left, "G(gf)"'] \arrow[dr, bend right, "G(f)"'] & \Uparrow \varphi_2^G & GC \\
& GB \arrow[ur, bend right, "G(g)"'] &
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[rr, bend left, "F(gf)"] \arrow[dr, bend right, "F(f)"'] \arrow[ddd, "\sigma_A"'] & \Uparrow \varphi_2^F & FC \arrow[ddd, "\sigma_C"] \\
& FB \arrow[ur, bend right, "F(g)"'] \arrow[ddd, "\sigma_B"] & \\
\Uparrow \sigma_f & & \Uparrow \sigma_g \\
GA \arrow[dr, bend right, "G(f)"'] & & GC \\
& GB \arrow[ur, bend right, "G(g)"'] &
\end{tikzcd}$$

- preservation of units:

$$\begin{tikzcd}
FA \arrow[rr, bend left, "F(\mathrm{id}_A)"] \arrow[dd, "\sigma_A"'] & \Uparrow \sigma_{\mathrm{id}_A} & FA \arrow[dd, "\sigma_A"] \\
& & \\
GA \arrow[rr, bend left, "G(\mathrm{id}_A)"] \arrow[rr, bend right, "\mathrm{id}_{GA}"'] & \Uparrow \varphi_0^G & GA
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[rr, bend left, "F(\mathrm{id}_A)"] \arrow[rr, bend right, "\mathrm{id}_{FA}"'] \arrow[dd, "\sigma_A"'] \arrow[ddrr, bend right, "\sigma_A"'] & \Uparrow \varphi_0^F & FA \arrow[dd, "\sigma_A"] \\
& & \Uparrow r^{-1} \\
GA \arrow[rr, bend right, "\mathrm{id}_{GA}"'] & \Uparrow l & GA
\end{tikzcd}$$

A *lax transformation* is when the 2-cells σ_f are not required to be isomorphisms, and a *colax transformation* is when they go in the other direction $\sigma_B \circ Ff \to Gf \circ \sigma_A$. A *strict transformation* is when they are identities.

Definition 1.3.5. Let $F, G : \mathscr{A} \to \mathscr{B}$ be two pseudo-functors between bicategories and let $\sigma, \theta : F \Rightarrow G$ be two pseudo-transformations between homomorphisms. A *modification* $\Gamma : \sigma \Rrightarrow \theta$ consists of 2-cells $\Gamma_A : \sigma_A \Rightarrow \theta_A$ for every object $A \in \mathscr{A}$, such that the following diagrams are equal for every 1-cell $f : A \to B$ in $\mathscr{A}$:

$$\begin{tikzcd}
FA \arrow[r, "F(f)"] \arrow[d, bend right=60, "\sigma_A"'] \arrow[dr, phantom, "\Uparrow \sigma_f"] & FB \arrow[d, bend right=40, "\sigma_A"'] \arrow[d, bend left=40, "\theta_B"] \arrow[d, phantom, "\overset{\Gamma_B}{\Longrightarrow}"] \\
GA \arrow[r, "G(f)"'] & GB
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[r, "F(f)"] \arrow[d, bend right=40, "\sigma_A"'] \arrow[d, bend left=40, "\theta_A"] \arrow[d, phantom, "\overset{\Gamma_A}{\Longrightarrow}"] \arrow[dr, phantom, "\Uparrow \theta_f"] & FB \arrow[d, bend left=60, "\theta_B"] \\
GA \arrow[r, "G(f)"'] & GB
\end{tikzcd}$$

1.3.2 Internal adjunctions, equivalences and monads

Definition 1.3.6. An *internal adjunction* between two objects A and B in a bicategory $\mathscr{B}$ consists of quadruple $(L, R, \eta, \varepsilon)$ where

1. $L : A \to B, R : B \to A$ are 1-cells and
2. $\eta : \mathrm{id}_A \Rightarrow R \circ L$ and $\varepsilon : L \circ R \Rightarrow \mathrm{id}_B$ are 2-cells in $\mathscr{B}(A, B)$ satisfying the triangle identities:

$$\begin{tikzcd}
\mathrm{id}_A R \arrow[r, "\eta * \mathrm{Id}_R"] \arrow[ddrr, "l_L"'] & (RL)R \arrow[r, "a"] & R(LR) \arrow[d, "\mathrm{Id}_R * \varepsilon"] \\
& & R\mathrm{id}_B \arrow[d, "r_L"] \\
& & R
\end{tikzcd}
\qquad
\begin{tikzcd}
L\mathrm{id}_A \arrow[r, "\mathrm{Id}_L * \eta"] \arrow[ddrr, "r_L"'] & L(RL) \arrow[r, "a^{-1}"] & L(RL) \arrow[d, "\varepsilon * \mathrm{Id}_L"] \\
& & \mathrm{id}_B L \arrow[d, "l_L"] \\
& & L
\end{tikzcd}$$

Definition 1.3.7. An *internal equivalence* between two objects A and B in a bicategory $\mathscr{B}$ is a quadruple $(f, g, \eta, \varepsilon)$ where

1. $f : A \to B, g : B \to A$ are 1-cells and
2. $\eta : \mathrm{id}_A \Rightarrow g \circ f, \varepsilon : f \circ g \Rightarrow \mathrm{id}_B$ are invertible 2-cells.

Definition 1.3.8. An *internal adjoint equivalence* between two objects A and B in a bicategory $\mathscr{B}$ is an adjunction $(f, g, \eta, \varepsilon)$ that is also an equivalence.

Notation 1. Given a 1-cell $f : A \to B$, since any two choices (g, η, ε), $(g', \eta', \varepsilon')$ making an adjoint equivalence are canonically isomorphic, we use the same notation as [100] and write g as $f^\bullet$ for an adjoint equivalence $(f, g, \eta, \varepsilon)$.

Definition 1.3.9. An *internal monad* in a bicategory $\mathscr{B}$ is a tuple (A, f, η, μ) where:

1. A is an object of $\mathscr{B}$,
2. f is a 1-cell in $\mathscr{B}(A, A)$,
3. $\eta : \mathrm{id}_A \Rightarrow f$ and $\mu : f^2 \Rightarrow f$ are 2-cells making the following diagrams commute:

$$\begin{tikzcd}
(f^2)f \arrow[r, "a"] \arrow[d, "\mu * \mathrm{Id}_f"'] & f(f^2) \arrow[r, "\mathrm{Id}_f * \mu"] & f \arrow[d, "\mu"] \\
f^2 \arrow[rr, "\mu"'] & & f
\end{tikzcd}
\qquad
\begin{tikzcd}
f & & f^2 \arrow[ll, "\mu"'] \arrow[rr, "\mu"] & & f \\
f * \mathrm{id}_A \arrow[u, "r"] \arrow[urr, "\mathrm{Id}_f * \eta"] & & & & \mathrm{id}_A * f \arrow[u, "l"'] \arrow[ull, "\eta * \mathrm{Id}_f"']
\end{tikzcd}$$

1.3.3 Biadjunctions and biequivalences

Definition 1.3.10. Two pseudo-functors $F : \mathscr{A} \rightleftarrows \mathscr{B} : G$ form a *biadjunction* $F \dashv G$ if for every objects $A \in \mathscr{A}$ and $B \in \mathscr{B}$, there is an equivalence

$$\mathscr{A}(A, G(B)) \simeq \mathscr{B}(F(A), B)$$

pseudo-natural in A and B.

Definition 1.3.11. Let $\mathscr{B}$ and $\mathscr{C}$ be bicategories, a pseudo-functor $F : \mathscr{B} \to \mathscr{C}$ is *fully faithful* if for all objects X, Y in $\mathscr{B}$, the functor

$$F_{X,Y} : \mathscr{B}(X, Y) \to \mathscr{C}(F(X), F(Y))$$

is an equivalence of categories.

Definition 1.3.12. A pseudo-functor $F : \mathscr{B} \to \mathscr{C}$ between bicategories $\mathscr{B}$ and $\mathscr{C}$ is said to be *essentially surjective* if for every object $C \in \mathscr{C}$, there is an object $B \in \mathscr{B}$ such that there is an equivalence $F(B) \simeq C$ in $\mathscr{C}$.

Definition 1.3.13. A pseudo-functor $F : \mathscr{B} \to \mathscr{C}$ between bicategories $\mathscr{B}$ and $\mathscr{C}$ is a *biequivalence* if it is both essentially surjective and fully faithful.

1.3.4 Cartesian closed bicategories

Definition 1.3.14. An object $\top$ in a bicategory $\mathscr{B}$ is *terminal* if for every object $A \in \mathscr{B}$, the category $\mathscr{B}(A, \top)$ is equivalent to the terminal category.

Definition 1.3.15. A bicategory $\mathscr{B}$ is said to be *cartesian* if it has a terminal object and the diagonal functor $\Delta : \mathscr{B} \to \mathscr{B} \times \mathscr{B}$ has a right biadjoint. Explicitly, if we denote by $\& : \mathscr{B} \times \mathscr{B} \to \mathscr{B}$ the said right biadjoint, for all objects A, B and X in $\mathscr{B}$, there are 1-cells $\pi_A : A\&B \to A$ and $\pi_B : A\&B \to B$ such that the functor

$$\mathscr{B}(X, A \,\&\, B) \xrightarrow{(\pi_A \circ(-), \pi_B \circ(-))} \mathscr{B}(X, A) \times \mathscr{B}(X, B)$$

is an adjoint equivalence of categories.

Definition 1.3.16. A cartesian bicategory $\mathscr{B}$ is *closed* if the pseudo-functor $(-) \,\&\, A : \mathscr{B} \to \mathscr{B}$ has a right biadjoint $A \Rightarrow (-) : \mathscr{B} \to \mathscr{B}$ for all objects A in $\mathscr{B}$. Explicitely, for every pair of objects $A, B \in \mathscr{B}$, we have:

1. an exponential object $A \Rightarrow B$ together with an evaluation map $\mathrm{Ev}_{A,B} \in \mathscr{B}((A \Rightarrow B) \,\&\, A, B)$
2. for every $X \in \mathscr{B}$, an adjoint equivalence

$$\begin{tikzcd}
\mathscr{B}(X, A \Rightarrow B) \arrow[r, bend left, "\mathrm{Ev}_{A,B} \circ ((-) \,\&\, A)"] \arrow[r, phantom, "\perp"] & \mathscr{B}(X \,\&\, A, B) \arrow[l, bend left, "\Lambda_A"]
\end{tikzcd}$$

pseudo-natural in A, B and X.

1.3.5 Symmetric monoidal bicategories

Going from monoidal categories to monoidal bicategories, we obtain an intermediate notion of syllepsis between the braided and symmetric cases.

Definition 1.3.17. A *monoidal bicategory* consists of a bicategory $\mathscr{M}$ together with the following data:

- a *monoidal unit object* $\mathbf{1} \in \mathscr{M}$,

- a *tensor product* pseudo-functor

$$\otimes = (\otimes, \varphi_2, \varphi_0) : \mathscr{M} \times \mathscr{M} \to \mathscr{M}$$

- a pseudo-natural transformation called the *associator*

$$\alpha = (\alpha_{ABC}, \alpha_{fgh}) : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$$

- two pseudo-natural transformations refered to as the *left* and *right unitors* that are equivalences:

$$\begin{aligned} l = (l_A, l_f) &: \mathbf{1} \otimes A \to A \\ r = (r_A, r_f) &: A \to A \otimes \mathbf{1} \end{aligned}$$

- an invertible modification π called the *pentagonator*

$$\begin{tikzcd}
& & (A \otimes B) \otimes (C \otimes D) \arrow[drr, bend left, "\alpha_{A,B,C} \otimes D"] & & \\
((A \otimes B) \otimes C) \otimes D \arrow[urr, bend left, "\alpha_{A\otimes B,C,D}"] \arrow[dr, bend left, "\alpha_{A,B,C} \otimes \mathrm{id}_D"'] & & \Downarrow \pi_{ABCD} & & A \otimes (B \otimes (C \otimes D)) \\
& (A \otimes (B \otimes C)) \otimes D \arrow[rr, bend right, "\mathrm{id}_A \otimes \alpha_{B,C,D}"'] & & A \otimes ((B \otimes C) \otimes D) \arrow[ur, bend right, "\alpha_{A,B\otimes C,D}"'] &
\end{tikzcd}$$

- three invertible modifications λ, μ and ρ called 2-unitors

$$\begin{tikzcd}
(A \otimes \mathbf{1}) \otimes B \arrow[r, "\alpha_{A\,\mathbf{1}\,B}"] \arrow[dr, phantom, "\Downarrow \mu_{AB}"] & A \otimes (\mathbf{1} \otimes B) \arrow[d, "I_A \otimes l_B"] \\
A \otimes B \arrow[u, "r_A \otimes I_B"] \arrow[r, "I_{a\otimes b}"'] & A \otimes B
\end{tikzcd}$$

$$\begin{tikzcd}
(\mathbf{1} \otimes A) \otimes B \arrow[rr, "l_A \otimes I_B"] \arrow[dr, "\alpha_{\mathbf{1}\,AB}"'] & \Downarrow \lambda_{AB} & A \otimes B \\
& \mathbf{1} \otimes (A \otimes B) \arrow[ur, "l_{A\otimes B}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes B \arrow[rr, "I_A \otimes r_B"] \arrow[dr, "r_{A\otimes B}"'] & \Downarrow \rho_{AB} & A \otimes (B \otimes \mathbf{1}) \\
& (A \otimes B) \otimes \mathbf{1} \arrow[ur, "\alpha_{AB\,\mathbf{1}}"'] &
\end{tikzcd}$$

subject to coherence laws that can be found in [100].

Definition 1.3.18. A *braided monoidal bicategory* is a monoidal bicategory $\mathscr{M}$ together with

1. a pseudo-natural equivalence called the *braiding*

$$\beta = (\beta_{AB}, \beta_{fg}) : A \otimes B \to B \otimes A$$

2. two invertible modifications R and S:

$$\begin{tikzcd}
& A \otimes (B \otimes C) \arrow[r, "\beta_{A,B\otimes C}"] & (B \otimes C) \otimes A \arrow[dr, "\alpha_{B,C,A}"] & \\
(A \otimes B) \otimes C \arrow[ur, "\alpha_{A,B,C}"] \arrow[dr, "\beta_{A,C} \otimes \mathrm{id}_C"'] \arrow[rrr, phantom, "\Downarrow R_{A,B,C}"] & & & B \otimes (C \otimes A) \\
& (B \otimes A) \otimes C \arrow[r, "\alpha_{B,A,C}"'] & B \otimes (A \otimes C) \arrow[ur, "\mathrm{id}_B \otimes \beta_{A,C}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & (A\otimes B)\otimes C \arrow[r, "\beta_{A\otimes B,C}"] & C\otimes(A\otimes B) \arrow[dr, "\alpha^{\bullet}_{C,A,B}"] & \\
A\otimes(B\otimes C) \arrow[ur, "\alpha^{\bullet}_{A,B,C}"] \arrow[dr, "\mathrm{id}_A\otimes\beta_{B,C}"'] & \arrow[r, phantom, "\Downarrow S_{A,B,C}"] & & (C\otimes A)\otimes B \\
 & A\otimes(C\otimes B) \arrow[r, "\alpha^{\bullet}_{A,C,B}"'] & (A\otimes C)\otimes B \arrow[ur, "\beta_{A,C}\otimes\mathrm{id}_B"'] &
\end{tikzcd}$$

subject to coherence laws that can be found in [100].

Definition 1.3.19. A *sylleptic monoidal bicategory* is a braided monoidal bicategory $\mathscr{M}$ together with an invertible modification ν called the *syllepsis* with components:

$$\begin{tikzcd}
 & B\otimes A \arrow[dr, "\beta_{BA}"] & \\
A\otimes B \arrow[ur, "\beta_{AB}"] \arrow[rr, bend right, "\mathrm{id}_{A\otimes B}"'] \arrow[rr, phantom, "\Downarrow \nu_{A,B}"] & & A\otimes B
\end{tikzcd}$$

satisfying coherence laws that can be found in [100].

Definition 1.3.20. A *symmetric monoidal bicategory* is a sylleptic monoidal bicategory $\mathscr{M}$ satisfying the following additional coherence law for all objects A and B in $\mathscr{M}$:

$$\begin{tikzcd}
A\otimes B \arrow[rr, "\beta_{AB}"] \arrow[dd, "\beta_{BA}"'] \arrow[dr, no head] & & B\otimes A & & A\otimes B \arrow[rr, "\beta_{AB}"] \arrow[dd, "\beta_{AB}"'] & & B\otimes A \arrow[dl, no head] \\
 & \mathrm{id}_{A\otimes B} \arrow[dr] \arrow[ur, phantom, "\Nearrow r"] & & = & & \mathrm{id}_{B\otimes A} \arrow[dl] \arrow[ul, phantom, "\Nwarrow l"] & \\
B\otimes A \arrow[rr, "\beta_{BA}"'] \arrow[ur, phantom, "\Nearrow \nu_{A,B}"] & & A\otimes B \arrow[uu, "\beta_{AB}"'] & & B\otimes A \arrow[rr, "\beta_{BA}"'] & & A\otimes B \arrow[uu, "\beta_{AB}"'] \arrow[ul, phantom, "\Nwarrow \nu_{B,A}"]
\end{tikzcd}$$

Definition 1.3.21. A symmetric monoidal bicategory $\mathscr{M}$ is *closed* if the pseudo-functor $- \otimes B : \mathscr{M} \to \mathscr{M}$ has a right biadjoint $B \multimap - : \mathscr{M} \to \mathscr{M}$ for all objects B in $\mathscr{M}$. Explicitly, for every pair of objects $A, B \in \mathscr{B}$, there is

1. an object $A \multimap B$ together with an evaluation map $\mathrm{ev}_{A,B} \in \mathscr{M}((A \multimap B) \otimes A, B)$
2. for every $X \in \mathscr{M}$, an adjoint equivalence

$$\begin{tikzcd} \mathscr{M}(X, A \multimap B) \arrow[r, bend left, "\mathrm{ev}_{A,B} \circ (- \otimes A)"] \arrow[r, phantom, "\perp"] & \mathscr{M}(X \otimes A, B) \arrow[l, bend left, "\lambda"] \end{tikzcd}$$

   pseudo-natural in A, B and X.

Definition 1.3.22. A symmetric monoidal bicategory $\mathscr{M}$ is *compact closed* if for every object A in $\mathscr{M}$, there exists an object A^* is equipped with:

1. 1-cells: $i_A : \mathbf{1} \to A \otimes A^*$ and $e_A : A^* \otimes A \to \mathbf{1}$
2. 2-cells with components: $\zeta_A : \mathrm{id}_A \to (\mathrm{id}_A \otimes e_A) \circ (\mathrm{id}_A \otimes i_A)$ and $\theta_A : \mathrm{id}_{A^*} \to (e_A \otimes \mathrm{id}_{A^*}) \circ (\mathrm{id}_{A^*} \otimes i_A)$

subject to coherence axioms that can be found in [107].

1.3.6 Kleisli bicategories

The content of this section is adapted from [78] and [24].

Definition 1.3.23. A *pseudo-comonad* on a bicategory $\mathscr{B}$ consists of the following data

- a pseudo-functor $T : \mathscr{B} \to \mathscr{B}$;
- two natural pseudo-transformations $\varepsilon : T \Rightarrow \mathrm{id}_{\mathscr{B}}$ and $\delta : T \Rightarrow T \circ T$ refered to as the *co-unit* and *co-multiplication* respectively;
- three invertible modifications $\alpha : (\delta_T * \delta) \Rrightarrow (T\delta * \delta)$, $\rho : \mathrm{id}_T \Rrightarrow (\varepsilon_T * \delta)$ and $\lambda : (T\varepsilon * \delta) \Rrightarrow \mathrm{id}_T$ refered to as the *associativity, right unit* and *left unit* respectively

$$\begin{tikzcd}
TA \arrow[r, "\delta_A"] \arrow[d, "\delta_A"'] \arrow[dr, phantom, "\Nearrow \alpha_A"] & T^2A \arrow[d, "T\delta_A"] \\
T^2A \arrow[r, "\delta_{TA}"'] & T^3A
\end{tikzcd}
\qquad
\begin{tikzcd}
TA & T^2A \arrow[l, "\varepsilon_{TA}"] \arrow[r, "T\varepsilon_A"'] & TA \\
& TA \arrow[ul, "\mathrm{id}_{TA}"] \arrow[u, "\delta_A"] \arrow[ur, "\mathrm{id}_{TA}"'] \arrow[ul, phantom, shift left=4, "\rho_A \Nearrow"] \arrow[ur, phantom, shift right=4, "\Nwarrow \lambda_A"] &
\end{tikzcd}$$

subject to the following coherence laws:

$$\begin{tikzcd}
T^4A & & T^2A \arrow[ll, "T^2\delta_A"'] \arrow[dl, phantom, "\Nearrow T\alpha_A"] & \\
& T^3A \arrow[ul, "T\delta_{TA}"] \arrow[dl, phantom, "\Nearrow \alpha_{TA}"] & & T^2A \arrow[ul, "T\delta_A"'] \arrow[ll, "T\delta_A"] \\
T^3A \arrow[uu, "\delta_{T^2A}"] & & \Nearrow \alpha_A & \\
& T^2A \arrow[ul, "\delta_{TA}"'] \arrow[uu, "\delta_{TA}"'] & & TA \arrow[ll, "\delta_A"] \arrow[uu, "\delta_A"']
\end{tikzcd}
=
\begin{tikzcd}
T^4A & \cong & T^3A \arrow[ll, "T^2\delta_A"'] & \\
& & & T^2A \arrow[ul, "T\delta_A"'] \\
T^3A \arrow[uu, "\delta_{T^2A}"] & & T^2A \arrow[ll, "T\delta_A"] \arrow[uu, "\delta_{TA}"] \arrow[ur, phantom, "\Nearrow \alpha_A"] & \\
& T^2A \arrow[ul, "\delta_{TA}"'] \arrow[ur, phantom, "\Nearrow \alpha_A"] & & TA \arrow[ll, "\delta_A"] \arrow[ul, "\delta_A"] \arrow[uu, "\delta_A"']
\end{tikzcd}$$

$$\begin{tikzcd}
& & T^2A \arrow[dl, "\mathrm{T}\delta_A"'] & \\
T^2A & T^3A \arrow[l, "T\varepsilon_{TA}"] & \Uparrow \alpha_A & TA \arrow[ul, "\delta_A"'] \arrow[dl, "\delta_A"] \\
& & T^2A \arrow[ul, "\delta_{TA}"] &
\end{tikzcd}
=
\begin{tikzcd}
& T^3A \arrow[dl, "\mathrm{T}\varepsilon_{TA}"'] \arrow[d, phantom, "\Uparrow T\rho_A"] & & \\
T^2A & & T^2A \arrow[ll, "\mathrm{id}_{T^2}"] \arrow[ul, "T\delta_A"'] \arrow[dl, "\delta_{TA}"] & TA \arrow[l, "\delta_A"] \\
& T^3A \arrow[ul, "T\varepsilon_{TA}"] \arrow[u, phantom, "\Uparrow \lambda_{TA}"] & &
\end{tikzcd}$$

Definition 1.3.24. Given a pseudo-comonad $(T, \varepsilon, \delta, \alpha, \lambda, \rho)$ on a bicategory $\mathscr{B}$, the *co-Kleisli bicategory of* T, denoted $\mathscr{B}_T$ is defined as follows:

$$(\mathscr{B}_T)_0 := \mathscr{B}_0 \qquad \mathscr{B}_T(A, B) := \mathscr{B}(TA, B).$$

Composition of 1-cells $f : TA \to B$ and $g : TB \to C$ in $\mathscr{B}_T(A, B)$ is given by

$$g \circ f := g \circ T(f) \circ \delta_A.$$

The identities in $\mathscr{B}_T$ are given by $\varepsilon_A : TA \to A$ and the coherence isomorphisms for the bicategorical structure of $\mathscr{B}_T$ are given by α, λ and ρ.

1.4 The bicategory of profunctors

We present in this section the generalized species model of **DiLL** introduced by Fiore, Gambino, Hyland and Winskel [41]. It is based on the bicategory of profunctors which is of central importance for this thesis. The bicategory of profunctors was first used to model linear logic by Cattani and Winskel, they used the finite colimit completion pseudo-comonad $\mathcal{C}$ to obtain a model that generalizes intuitions from the Scott model [22, 23]. Fiore et al. used the free symmetric strict monoidal completion pseudo-comonad $\mathcal{S}$ to obtain a model of **DiLL** that categorifies the relational model. The finite multiset monad on **Set** corresponding to the free commutative monoid construction for sets is now the free symmetric strict monoidal completion for categories.

1.4.1 Kan extensions

We give a brief reminder on Kan extensions before presenting profunctors.

Definition 1.4.1. For functors $F : \mathbb{A} \to \mathbb{C}$ and $H : \mathbb{A} \to \mathbb{B}$, the *left Kan extension* of F along H is a functor $\mathbf{Lan}_H F : \mathbb{B} \to \mathbb{C}$ equipped with a natural transformation $\eta : F \Rightarrow \mathbf{Lan}_H F \circ H$ such that for every other functor $G : \mathbb{B} \to \mathbb{C}$ and natural transformation $\gamma : F \to G \circ H$, there exists a unique natural transformation $\delta : \mathbf{Lan}_H F \Rightarrow G$ such that

$$\begin{tikzcd} \mathbb{A} \arrow[rr, "F"] \arrow[dr, "H"'] & \arrow[d, phantom, "\gamma \Downarrow"] & \mathbb{C} \\ & \mathbb{B} \arrow[ur, "G"'] & \end{tikzcd} = \begin{tikzcd} \mathbb{A} \arrow[rr, "F"] \arrow[dr, "H"'] & \arrow[d, phantom, "\Downarrow \eta"] & \mathbb{C} \\ & \mathbb{B} \arrow[ur, "\mathbf{Lan}_H F" sloped] \arrow[ur, bend right=60, "G"'] \arrow[ur, phantom, shift right=3, "\delta \Downarrow"] & \end{tikzcd}$$

Construction 1. It induces a functor $\mathbf{Lan}_H : [\mathbb{A}, \mathbb{C}] \to [\mathbb{B}, \mathbb{C}]$ which maps a functor $F : \mathbb{A} \to \mathbb{C}$ to $\mathbf{Lan}_H F$ and a natural transformation $\alpha : F \Rightarrow F'$ to the natural transformation $\delta : \mathbf{Lan}_H F \Rightarrow \mathbf{Lan}_H F'$ obtained from the universal property of the left Kan extension.

$$\begin{tikzcd}
\mathbb{A} \arrow[rr, bend left=40, "F"] \arrow[rr, bend right=20, "F'" description] \arrow[rr, phantom, shift left=4, "\Downarrow \alpha"] \arrow[rd, "H"'] & {} & \mathbb{C} & & \mathbb{A} \arrow[rr, "F"] \arrow[rd, "H"'] & & \mathbb{C} \\
& \mathbb{B} \arrow[ru, "\mathbf{Lan}_H F'"'] \arrow[u, phantom, "\Downarrow \gamma'"] & & = & & \mathbb{B} \arrow[ru, "\mathbf{Lan}_H F" description] \arrow[ru, bend right=60, "\mathbf{Lan}_H F'"'] \arrow[u, phantom, "\eta \Downarrow"] \arrow[ru, phantom, shift right=5, "\delta \Downarrow"] &
\end{tikzcd}$$

Proposition 1.4.2 ([74])**.** *Let $H : \mathbb{A} \to \mathbb{B}$ be a fully faithful functor from a small category $\mathbf{A}$. Then, for every functor $F : \mathbb{A} \to \mathbb{D}$ into a cocomplete category $\mathbb{D}$, the natural transformation $F \Rightarrow \mathbf{Lan}_H(F) \circ H$ is an isomorphism and the functor $\mathbf{Lan}_H : [\mathbb{A}, \mathbb{D}] \to [\mathbb{B}, \mathbb{D}]$ is fully faithful.*

1.4.2 Profunctors

For small categories $\mathbb{A}$ and $\mathbb{B}$, a profunctor $F : \mathbb{A} \nrightarrow \mathbb{B}$ is a functor $F : \mathbb{A} \times \mathbb{B}^{op} \to \mathbf{Set}$ or equivalently a functor $F : \mathbb{A} \to \widehat{\mathbb{B}}$ [9]. Profunctors can be seen as a generalization of **Rel** as a relation $R \subseteq A \times B$ corresponds to a profunctor between discrete categories such that each component is either the empty set or a singleton.

The composite of two profunctors $P : \mathbb{A} \nrightarrow \mathbb{B}$ and $Q : \mathbb{B} \nrightarrow \mathbb{C}$ is the profunctor $Q \circ P : \mathbb{A} \nrightarrow \mathbb{C}$ given by the composite $(\mathbf{Lan}_{\mathrm{y}_{\mathbb{B}}} Q)P$:

$$\begin{tikzcd}
 & \mathbb{B} \arrow[r, "Q"] \arrow[d, hook, "\mathrm{y}_{\mathbb{B}}"'] & \widehat{\mathbb{C}} \\
\mathbb{A} \arrow[r, "P"] & \widehat{\mathbb{B}} \arrow[ru, dashed, "\mathbf{Lan}_{\mathrm{y}_{\mathbb{B}}} Q"'] &
\end{tikzcd}$$

Pointwise, it can be computed using coend formula:

$$(a, c) \quad \mapsto \quad \int^{b \in \mathbb{B}} P(a, b) \times Q(b, c) \cong \left(\sum_{b \in \mathbb{B}} P(a, b) \times Q(b, c) \right) /\!\sim$$

where $\sim$ is the least equivalence relation such that

$$(b, P(a, f)(p), q) \sim (b', p, Q(f, c)(q))$$

for $p \in P(a, b')$, $q \in Q(b, c)$ and $f : b \to b' \in \mathbb{B}$.

Definition 1.4.3. We denote by **Prof** the bicategory given by:

- **objects:** small categories $\mathbb{A}$, $\mathbb{B}$;
- **1-cells:** profunctors $P : \mathbb{A} \nrightarrow \mathbb{B}$;
- **2-cells:** natural transformations.

Recall that **Rel** is isomorphic to the Kleisli category of the powerset monad on **Set**, in the categorified setting, we have to work with relative pseudo-monads due to size issues. Fiore et al. showed that the bicategory **Prof** can be obtained as the Kleisli bicategory of the relative pseudo-monad of presheaves $\widehat{(-)}$ over the inclusion functor $\mathbf{Cat} \to \mathbf{CAT}$ [42].

Definition 1.4.4. The *dual* of a profunctor $P : \mathbb{A} \nrightarrow \mathbb{B}$ is the profunctor $P^{\perp} : \mathbb{B}^{\mathrm{op}} \nrightarrow \mathbb{A}^{\mathrm{op}}$ given by:

$$(b, a) \mapsto P(a, b).$$

It induces a self-duality $(-)^{\perp} : \mathbf{Prof}^{\mathrm{op}} \to \mathbf{Prof}$.

Definition 1.4.5. For a profunctor $P : \mathbb{A} \nrightarrow \mathbb{B}$, we let $P^{\#} : \widehat{\mathbb{A}} \to \widehat{\mathbb{B}}$ be the left Kan extension of P along the Yoneda embedding $\mathrm{y}_{\mathbb{A}} : \mathbb{A} \hookrightarrow \widehat{\mathbb{A}}$:

$$\begin{tikzcd}
\mathbb{A} \arrow[rr, "\mathrm{y}_{\mathbb{A}}"] \arrow[dr, hook, "P"'] & {} \arrow[d, phantom, "\Downarrow"] & \widehat{\mathbb{B}} \\
& \widehat{\mathbb{A}} \arrow[ur, dotted, "P^{\#} = \mathbf{Lan}_{\mathrm{y}} P"'] &
\end{tikzcd}$$

Dually, we denote by $P_{\#}$ the functor $(P^{\perp})^{\#} : \widehat{\mathbb{B}^{\mathrm{op}}} \to \widehat{\mathbb{A}^{\mathrm{op}}}$.

Notation 2. Explicitly, for $X \in \widehat{\mathbb{A}}$ and $b \in \mathbb{B}$, the set $P^{\#}X(b)$ is given by

$$\int^{a \in \mathbb{A}} P(a, b) \times X(a)$$

and it contains equivalence class $p \underset{a}{\bowtie} x$ of triples $a \in \mathbb{A}$, $p \in P(a, b)$ and $x \in X(a)$. Two equivalence classes are equal $p_1 \underset{a_1}{\bowtie} x_1 = p_2 \underset{a_2}{\bowtie} x_2$ if and only if there exists $f : a_1 \to a_2$ in $\mathbb{A}$ such that $p_2 = f \cdot p_1$ and $x_1 = x_2 \cdot f$.

In the Scott model of linear logic, a Scott-continuous function is linear if it preserves all suprema, in this setting the morphisms in the linear bicategory **Prof** correspond to cocontinuous functors. The mapping $P \to P^{\#}$ induces a biequivalence between **Prof** and the 2-category of cocontinous functors:

Definition 1.4.6. We denote by **Cocont** the bicategory given by:

- **objects:** small categories $\mathbb{A}$, $\mathbb{B}$;
- **1-cells:** cocontinuous functors $F : \widehat{\mathbb{A}} \to \widehat{\mathbb{B}}$;
- **2-cells:** natural transformations.

Theorem 1.4.7. *The bicategory* **Prof** *and the* 2*-category* **Cocont** *are biequivalent.*

1.4.3 Additive structure

The bicategory **Prof** has biproducts given by the coproduct of categories that we denote $\mathbb{A} + \mathbb{B}$ with zero object the empty category $\mathbf{0}$. The coproduct inclusions $\iota_i : \mathbb{A}_i \nrightarrow \sum_i \mathbb{A}_i$ are given by

$$(a_i, (j, a)) \mapsto \begin{cases} \mathbb{A}_i(a, a_i) & \text{if } i = j \\ \emptyset & \text{otherwise} \end{cases}$$

and the projections $\pi_i : \sum_i \mathbb{A}_i \nrightarrow \mathbb{A}_i$ are dually given by

$$((j, a), a_i) \mapsto \begin{cases} \mathbb{A}_i(a_i, a) & \text{if } i = j \\ \emptyset & \text{otherwise} \end{cases}$$

1.4.4 Compact closed structure

The cartesian product in **Cat** becomes a symmetric monoidal product in **Prof** with unit $\mathbf{1}$, the category with a unique object and a unique arrow. Explicitely $\otimes : \mathbf{Prof} \times \mathbf{Prof} \to \mathbf{Prof}$ maps a pair $(\mathbb{A}, \mathbb{B})$ to $\mathbb{A} \times \mathbb{B}$. On 1-cells $P_1 : \mathbb{A}_{\mathbb{1}} \nrightarrow \mathbb{B}_{\mathbb{1}}$ and $P_2 : \mathbb{A}_{\mathbb{2}} \nrightarrow \mathbb{B}_{\mathbb{2}}$, the profunctor $P_1 \otimes P_1 : \mathbb{A}_{\mathbb{1}} \otimes \mathbb{A}_{\mathbb{2}} \nrightarrow \mathbb{B}_{\mathbb{1}} \otimes \mathbb{B}_{\mathbb{2}}$ is given by

$$((a_1, a_2), (b_1, b_2)) \mapsto P_1(a_1, b_1) \times P_2(a_2, b_2).$$

The bicategory **Prof** is symmetric monoidal closed with internal hom $\mathbb{A} \multimap \mathbb{B} := \mathbb{A}^{\mathrm{op}} \times \mathbb{B}$ and linear evaluation $\mathrm{ev}_{\mathbb{A},\mathbb{B}} : (\mathbb{A} \multimap \mathbb{B}) \times \mathbb{A} \nrightarrow \mathbb{B}$ is given by:

$$((a, b), a', b') \mapsto \mathbb{A}(a, a') \times \mathbb{B}(b', b).$$

For every $\mathbb{A}, \mathbb{B}$ and $\mathbb{C}$, it induces an adjoint equivalence

$$\begin{tikzcd}
\mathbf{Prof}(\mathbb{A} \otimes \mathbb{B}, \mathbb{C}) \arrow[r, bend right, "\lambda"'] \arrow[r, phantom, "\perp"] & \mathbf{Prof}(\mathbb{A}, \mathbb{B} \multimap \mathbb{C}) \arrow[l, bend right, "\mathrm{ev}_{\mathbb{B},\mathbb{C}} \circ (- \otimes \mathbb{B})"']
\end{tikzcd}$$

where λ maps a profunctor $P : \mathbb{A} \times \mathbb{B} \nrightarrow \mathbb{C}$ to the profunctor $\lambda(P) : (a, (b, c)) \mapsto P((a, b), c)$. The dualizer $(-)^{\perp} : \mathbb{A} \mapsto \mathbb{A}^{\mathrm{op}}$ provides **Prof** with a compact closed structure [107].

1.4.5 Exponential structure

The exponential structure in the species model relies on the free symmetric strict monoidal completion for a category:

Definition 1.4.8. For a category $\mathbb{A}$, define $\mathcal{S}\mathbb{A}$ as the category whose objects are finite sequences $\langle a_1, \ldots, a_n \rangle$ of objects of $\mathbb{A}$ and a morphism f between two sequences of the same length $u = \langle a_1, \ldots, a_n \rangle$ and $v = \langle b_1, \ldots b_n \rangle$ consists of a pair $(\sigma, (f_i)_{i \in \underline{n}})$ where $\sigma \in \mathfrak{S}_n$ and $(f_i : a_i \to b_{\sigma(i)})_{i \in \underline{n}}$ is a sequence of morphisms in $\mathbb{A}$. Equivalently, the homsets are given by

$$\mathcal{S}\mathbb{A}(u, v) = \begin{cases} \sum\limits_{\sigma \in \mathfrak{S}_n} \prod\limits_{i=1}^{n} \mathbb{A}(a_i, b_{\sigma(i)}) & \text{if } u \text{ and } v \text{ have the same length } n \\ \varnothing & \text{otherwise.} \end{cases}$$

The category $\mathcal{S}\mathbb{A}$ described above is symmetric monoidal with tensor product

$$\otimes \ : \ (u, v) \mapsto u \otimes v$$

given by the concatenation of sequences and unit the empty sequence $\langle\rangle$. The category $\mathcal{S}\mathbb{A}$ is the free symmetric strict monoidal completion of $\mathbb{A}$ in the following sense: for any functor $F : \mathbb{A} \to \mathbb{B}$ where $\mathbb{B}$ is a symmetric strict monoidal category, there exists a unique (up to natural isomorphism) symmetric monoidal strict functor $\widehat{F} : \mathcal{S}\mathbb{A} \to \mathbb{B}$ that makes the following diagram commute:

$$\begin{tikzcd} \mathbb{A} \arrow[rr, "\eta_{\mathbb{A}}"] \arrow[dr, "F"'] & & \mathcal{S}\mathbb{A} \arrow[dl, dotted, "\widehat{F}"] \\ & \mathbb{B} & \end{tikzcd}$$

where $\eta_{\mathbb{A}}$ is the functor $a \mapsto \langle a \rangle$. This construction induces a 2-monad on **CAT** which lifts to a pseudo-monad on **Prof** [42]. By dualization, one obtains a pseudo-comonad on **Prof** where the counit der and the comultiplication dig have the following components:

$$\begin{aligned} \mathrm{der}_{\mathbb{A}} &: \mathcal{S}\mathbb{A} \nrightarrow \mathbb{A} & \mathrm{dig}_{\mathbb{A}} &: \mathcal{S}\mathbb{A} \nrightarrow \mathcal{S}\mathcal{S}\mathbb{A} \\ (u, a) &\mapsto \mathcal{S}\mathbb{A}(\langle a \rangle, u) & (u, \langle u_1, \ldots, u_n \rangle) &\mapsto \mathcal{S}\mathbb{A}(u_1 \otimes \cdots \otimes u_n, u) \end{aligned}$$

For a profunctor $P : \mathbb{A} \nrightarrow \mathbb{B}$, the action of the pseudo-comonad for sequences $u \in \mathcal{S}\mathbb{A}$ and $v \in \mathcal{S}\mathbb{B}$ is given by

$$\mathcal{S}P : (u, v) \mapsto \begin{cases} \sum\limits_{\sigma \in \mathfrak{S}_n} \prod\limits_{i=1}^{n} P(u_i, v_{\sigma(i)}) & \text{if } u \text{ and } v \text{ have the same length } n \\ \varnothing & \text{otherwise} \end{cases}$$

Generalized species correspond to the 1-cells in the co-Kleisli bicategory $\mathbf{Prof}_{\mathcal{S}}$. For a species $F : \mathcal{S}\mathbb{A} \nrightarrow \mathbb{B}$, its comonadic lifting $F^{\mathcal{S}} : \mathcal{S}\mathbb{A} \nrightarrow \mathcal{S}\mathbb{B}$ is given by $(\mathcal{S}F) \circ \mathrm{dig}_{\mathbb{A}}$. The composite of two species $G : \mathcal{S}\mathbb{B} \nrightarrow \mathbb{C}$ and $F : \mathcal{S}\mathbb{A} \nrightarrow \mathbb{B}$ in $\mathbf{Prof}_{\mathcal{S}}$ is then given by $G \circ F^{\mathcal{S}} : \mathcal{S}\mathbb{A} \nrightarrow \mathbb{C}$.

1.4.6 Cartesian closed structure

The cartesian structure in the coKleisli bicategory $\mathbf{Prof}_{\mathcal{S}}$ is inherited from the linear bicategory **Prof** where they are given by the coproduct of categories and the terminal object in $\mathbf{Prof}_{\mathcal{S}}$ is the empty category $\mathbf{0}$. In a bicategorical setting, the Seely isomorphisms become merely equivalences:

Lemma 1.4.9. *[41] For categories $\mathbb{A}$ and $\mathbb{B}$, let $F : \mathbb{A} \& \mathbb{B} \to \mathcal{S}\mathbb{A} \otimes \mathcal{S}\mathbb{B}$ be the functor mapping $(1, a)$ to $(\langle a \rangle, \langle\rangle)$ and $(2, b)$ to $(\langle\rangle, \langle b \rangle)$. Using the universal property of $\mathcal{S}$*

$$\begin{tikzcd} \mathbb{A} \& \mathbb{B} \arrow[rr, "\eta_{\mathbb{A}\&\mathbb{B}}"] \arrow[dr, "F"'] & & \mathcal{S}(\mathbb{A} \& \mathbb{B}) \arrow[dl, dotted, "S_{\mathbb{A},\mathbb{B}}"] \\ & \mathcal{S}\mathbb{A} \otimes \mathcal{S}\mathbb{B} & \end{tikzcd}$$

it induces a functor $\mathcal{S}(\mathbb{A} \& \mathbb{B}) \to \mathcal{S}\mathbb{A} \otimes \mathcal{S}\mathbb{B}$ that we denote by $S_{\mathbb{A},\mathbb{B}}$. We use the same notation as in [41] and write $(w.1, w.2)$ for the image of $w \in \mathcal{S}(\mathbb{A} \& \mathbb{B})$ by $S_{\mathbb{A},\mathbb{B}}$. Let $I_{\mathbb{A},\mathbb{B}} : \mathcal{S}\mathbb{A} \otimes \mathcal{S}\mathbb{B} \to \mathcal{S}(\mathbb{A} \& \mathbb{B})$ be the functor defined by

$$(u, v) \mapsto (\mathcal{S}\mathrm{inj}_1 u) \otimes (\mathcal{S}\mathrm{inj}_2 v).$$

The functors $I_{\mathbb{A},\mathbb{B}}$ and $S_{\mathbb{A},\mathbb{B}}$ form an equivalence of categories $\mathcal{S}(\mathbb{A} \& \mathbb{B}) \simeq \mathcal{S}\mathbb{A} \otimes \mathcal{S}\mathbb{B}$. We also have $\mathcal{S}\,\mathbf{0} \simeq \mathbf{1}$.

The function space $\mathbb{A} \Rightarrow \mathbb{B}$ in $\mathbf{Prof}_{\mathcal{S}}$ is defined as $\mathcal{S}\mathbb{A} \multimap \mathbb{B}$ and the non-linear evaluation $\mathrm{Ev}_{\mathbb{A},\mathbb{B}} : \mathcal{S}((\mathbb{A} \Rightarrow \mathbb{B}) \& \mathbb{A}) \nrightarrow \mathbb{B}$ is given by the following composition:

$$\mathcal{S}((\mathbb{A} \Rightarrow \mathbb{B}) \& \mathbb{A}) \xrightarrow{S_{\mathbb{A}\Rightarrow\mathbb{B},\mathbb{A}}} \mathcal{S}(\mathbb{A} \Rightarrow \mathbb{B}) \otimes \mathcal{S}\mathbb{A} \xrightarrow{\mathrm{der}_{\mathbb{A}\Rightarrow\mathbb{B}} \otimes \mathrm{id}} (\mathbb{A} \Rightarrow \mathbb{B}) \otimes \mathcal{S}\mathbb{A} \xrightarrow{\mathrm{ev}_{\mathcal{S}\mathbb{A},\mathbb{B}}} \mathbb{B}$$

Explicitly, $\mathrm{Ev}_{\mathbb{A},\mathbb{B}}$ maps $(w, b) \in \mathcal{S}((\mathbb{A} \Rightarrow \mathbb{B}) \mathbin{\&} \mathbb{A}) \times \mathbb{B}^{\mathrm{op}}$ to the set

$$\mathcal{S}(\mathbb{A} \Rightarrow \mathbb{B})(\langle(w.2, b)\rangle, w.1).$$

For a species P in $\mathbf{Prof}_{\mathcal{S}}(\mathbb{A} \mathbin{\&} \mathbb{B}, \mathbb{C})$, its *currying* $\Lambda(P) \in \mathbf{Prof}_{\mathcal{S}}(\mathbb{A}, \mathbb{B} \Rightarrow \mathbb{C})$ is given by $\lambda(P \circ I_{\mathbb{A},\mathbb{B}})$ where

$$\lambda : \mathbf{SProf}(\mathcal{S}\mathbb{A} \otimes \mathcal{S}\mathbb{B}, \mathbb{C}) \to \mathbf{SProf}(\mathcal{S}\mathbb{A}, \mathcal{S}\mathbb{B} \multimap \mathbb{C})$$

is provided by the monoidal closed structure on **Prof**.

1.4.7 Differential structure

As mentioned in the introduction, a combinatorial species $F : \mathbb{P} \to \mathbf{Set}$ has an associated generating series $x \mapsto \sum_{n \geq 0} f_n \frac{x^n}{n!}$ where the coefficient f_n is equal to the cardinality of the set $F(\underline{n})$. This formal power series can be differentiated:

$$x \mapsto \sum_{n \geq 1} f_n n \frac{x^{n-1}}{n!} = \sum_{m \geq 0} f_{m+1} \frac{x^m}{m!}$$

and the corresponding species is the functor $F' : \underline{n} \mapsto F(\underline{n+1})$.

This operation extends to the universe of generalized species of structures where a species $F : \mathcal{S}\mathbb{A} \nrightarrow \mathbb{B}$ has a differential $\mathbf{D}F : \mathcal{S}\mathbb{A} \otimes \mathbb{A} \nrightarrow \mathbb{B}$ given by:

$$((u, a), b) \mapsto F(u \otimes \langle a \rangle, b).$$

We can retrieve the combinatorial construction by taking $\mathbb{A} = \mathbb{B} = \mathbf{1}$.

To recast this differentiation operation to the context of models of differential linear logic, the codereliction $\overline{\mathrm{der}}$ in **Prof** has components $\overline{\mathrm{der}}_{\mathbb{A}} : \mathbb{A} \nrightarrow \mathcal{S}\mathbb{A}$ mapping $(a, u) \in \mathbb{A} \times (\mathcal{S}\mathbb{A})^{\mathrm{op}}$ to $\mathcal{S}\mathbb{A}(u, \langle a \rangle)$. The components of the deriving pseudo-natural transformation $\delta_{\mathbb{A}} : \mathbb{A} \nrightarrow \mathcal{S}\mathbb{A} \otimes \mathbb{A}$ calculated as

$$\begin{tikzcd} \mathcal{S}\mathbb{A} \otimes \mathbb{A} \arrow[r, "\mathrm{id} \otimes \overline{\mathrm{der}}_{\mathbb{A}}"] & \mathcal{S}\mathbb{A} \otimes \mathcal{S}\mathbb{A} \arrow[r, "\bar{\mathrm{c}}_{\mathbb{A}}"] & \mathcal{S}\mathbb{A} \end{tikzcd}$$

are therefore given by $((u, a), u') \mapsto \mathcal{S}\mathbb{A}(u', u \otimes \langle a \rangle)$. The species $\mathbf{D}\, F$ defined above is now obtained by precomposing F with $\delta_{\mathbb{A}}$.

1.4.8 Analytic functors

As mentioned in the introduction, Joyal presented the notion of analytic functor as the Taylor series counterpart of combinatorial species [73].

Definition 1.4.10. An endofunctor $F : \mathbf{Set} \to \mathbf{Set}$ is said to be *analytic* if there exists a species $P : \mathbb{P} \to \mathbf{Set}$ such that F is isomorphic to the functor

$$X \in \mathbf{Set} \mapsto \sum_{n \in \mathbb{N}} P(\underline{n}) \times_{\mathfrak{S}_n} X^n$$

where an element σ of the symmetric group $\mathfrak{S}_n$ acts on $(p, f : \underline{n} \to X) \in P(\underline{n}) \times X^n$ as follows:

$$\sigma \cdot (p, f) := (P(\sigma)(p), f \circ \sigma^{-1}).$$

One can think of $P(\underline{n})$ as the coefficient of the monomial X^n and the quotient by $\mathfrak{S}_n$ as a factor $\frac{1}{n!}$ which provides the connection with generating series in enumerative combinatorics. Joyal also showed that analytic functors are closed under composition, sums, products and quotients and that these operations encode the ones on their corresponding combinatorial species.

Joyal gave an alternate characterization of analytic functors as endofunctors on **Set** that preserve filtered colimits, cofiltered limits and weak pullbacks (or equivalently filtered colimits and weak wide pullbacks as shown by Hasegawa [61]).

Theorem 1.4.11. *A functor $F : \mathbf{Set} \to \mathbf{Set}$ is analytic if and only if it is finitary and preserves wide weak pullbacks.*

Definition 1.4.12. Let $\mathbb{C}$ be a category, a *weak pullback* of a cospan $a \xrightarrow{f} c \xleftarrow{g} b$ in $\mathbb{C}$ is an object x together with maps $p : x \to a$ and $q : x \to b$ such that the following square commutes

$$\begin{tikzcd}
x \arrow[r, "q"] \arrow[d, "p"'] & b \arrow[d, "g"] \\
a \arrow[r, "f"'] & c
\end{tikzcd}$$

and for every other commutative square

$$\begin{tikzcd}
x' \arrow[r, "q'"] \arrow[d, "p'"'] & b \arrow[d, "g"] \\
a \arrow[r, "f"'] & c
\end{tikzcd}$$

there exists a morphism $h : x' \to x$ (not necessarily unique) such that $p \circ h = p'$ and $q \circ h = q'$.

Remark 2. Note that if the pullback $a \times_c b$ of the cospan $a \xrightarrow{f} c \xleftarrow{g} b$ exists in $\mathbb{C}$, then the triple (x, p, q) is a weak pullback if and only if the universal morphism $x \to a \times_c b$ is a split epimorphism.

Joyal further exhibited that analytic functors are the objects of a category where the morphisms are weak cartesian transformations between two analytic functors. A natural transformation is *weakly cartesian* if all its naturality squares are weak pullbacks. In this setting, the Taylor expansion uniqueness theorem translates to a more general statement:

Theorem 1.4.13 ([73]). *The category of combinatorial species and natural transformations between them is equivalent to the category of analytic functors.*

The theorem above implies in particular that if two species have the same analytic functors, then they are naturally isomorphic. Considering now generalized species of structure, Fiore introduced the notion of generalized analytic functor as their series counterpart [39].

Definition 1.4.14. For small categories $\mathbb{A}$ and $\mathbb{B}$, a functor $F : \widehat{\mathbb{A}} \to \widehat{\mathbb{B}}$ is said to be *analytic* if there exists a generalized species $P : \mathcal{S}\mathbb{A} \nrightarrow \mathbb{B}$ such that F is isomorphic to $\mathbf{Lan}_{s_{\mathbb{A}}} P$

$$\begin{tikzcd}
\mathcal{S}\mathbb{A} \arrow[rr, "P"] \arrow[dr, "s_{\mathbb{A}}"'] & \arrow[d, phantom, "\Downarrow"] & \widehat{\mathbb{B}} \\
& \widehat{\mathbb{A}} \arrow[ur, dotted, "\mathbf{Lan}_{s_{\mathbb{A}}} P"'] &
\end{tikzcd}$$

where $s_{\mathbb{A}} : \mathcal{S}\mathbb{A} \to \widehat{\mathbb{A}}$ is the functor mapping a list $\langle a_1, \dots a_n \rangle$ in $\mathcal{S}\mathbb{A}$ to the presheaf $\sum_{i=1}^{n} \mathbb{y}_{\mathbb{A}}(a_i)$ in $\widehat{\mathbb{A}}$.

The connection with analytic functors for combinatorial species is that Definition 1.4.10 can be reformulated by saying that a functor is analytic if it corresponds to the left Kan extension of a combinatorial species P along the inclusion functor $\mathbb{P} \hookrightarrow \mathbf{Set}$:

$$\begin{tikzcd}
\mathbb{P} \arrow[rr, "P"] \arrow[dr, "\iota"'] & \arrow[d, phantom, "\Downarrow"] & \mathbf{Set} \\
& \mathbf{Set} \arrow[ur, dotted, "\mathbf{Lan}_{\iota} P"'] &
\end{tikzcd}$$

Indeed, $\mathbf{Lan}_\iota P$ is given by the following coend formula which is isomorphic to the quotiented sum given in Definition 1.4.10:

$$\mathbf{Lan}_\iota P : X \in \mathbf{Set} \mapsto \int^{\underline{n} \in \mathbb{P}} P(\underline{n}) \times X^n \cong \sum_{n \in \mathbb{N}} P(\underline{n}) \times_{\mathfrak{S}_n} X^n$$

Note that the functor $s_{\mathbb{A}} : \mathcal{S}\mathbb{A} \to \widehat{\mathbb{A}}$ is not fully faithful, it maps morphisms in $\mathcal{S}\mathbb{A}$ to isomorphisms in $\widehat{\mathbb{A}}$:

Lemma 1.4.15. *Let $(a_i)_{i \in I}$ and $(b_j)_{j \in J}$ be two families of elements $\mathbb{A}$ for some index sets I and J. If the associated presheaves $\sum_{i \in I} \mathbb{y}_{\mathbb{A}}(a_i)$ and $\sum_{j \in J} \mathbb{y}_{\mathbb{A}}(b_j)$ are isomorphic in $\widehat{\mathbb{A}}$, then there exists a bijection $\sigma : I \to J$ and a family of isomorphisms $(f_i : a_i \to b_{\sigma(i)})_{i \in I}$ in $\mathbb{A}$.*

Proof. Note first that if two representables $\mathbb{y}_{\mathbb{A}}(a)$ and $\mathbb{y}_{\mathbb{A}}(b)$ are isomorphic in $\widehat{\mathbb{A}}$, then so are a to b in $\mathbb{A}$ (it is an easy corollary of the Yoneda lemma). Let us now consider the case of coproducts of representables:

$$\begin{aligned}
\mathbf{Hom}_{\widehat{\mathbb{A}}}(\sum_{i \in I} \mathbb{y}_{\mathbb{A}}(a_i), \sum_{j \in J} \mathbb{y}_{\mathbb{A}}(b_j) &\cong \prod_{i \in I} \mathbf{Hom}_{\widehat{\mathbb{A}}}(\mathbb{y}_{\mathbb{A}} a_i, \sum_{j \in J} \mathbb{y}_{\mathbb{A}}(b_j)) \\
&\cong \prod_{i \in I} \sum_{j \in J} \mathbb{y}_{\mathbb{A}}(b_j)(a_i) \quad \text{(Yoneda lemma)} \\
&\cong \prod_{i \in I} \sum_{j \in J} \mathbf{Hom}_{\mathbb{A}}(a_i, b_j) \\
&\cong \sum_{\sigma : I \to J} \prod_{i \in I} \mathbf{Hom}_{\mathbb{A}}(a_i, b_{\sigma(i)})
\end{aligned}$$

A morphism $\gamma : \sum_{i \in I} \mathbb{y}_{\mathbb{A}}(a_i) \to \sum_{j \in J} \mathbb{y}_{\mathbb{A}}(b_j)$ in $\widehat{\mathbb{A}}$ can be seen as the data of a function $\sigma : I \to J$ and a family of morphisms $f_i : a_i \to b_{\sigma(i)}$ in $\mathbb{A}$. If γ is an isomorphism, σ must be a bijection and each f_i an isomorphism. □

Corollary 1.4.16. *For two elements u and v in $\mathcal{S}\mathbb{A}$, if $\alpha : s_{\mathbb{A}}(u) \to s_{\mathbb{A}}(v)$ is an isomorphism in $\widehat{\mathbb{A}}$, then there exists an isomorphism $f : u \to v$ in $\mathcal{S}\mathbb{A}$ such that $\alpha \cong s_{\mathbb{A}}(f)$.*

We therefore cannot use Proposition 1.4.2 but Fiore extended Joyal's results in the setting of generalized species and showed that there is a biequivalence between the bicategory of generalized species $\mathbf{Prof}_\Sigma$ (restricted to groupoids) and the 2-category of analytic functors $\mathbf{An}$ [39].

Definition 1.4.17. The 2-category $\mathbf{An}$ is consists of:

- **0-cells:** small groupoids $\mathbb{A}, \mathbb{B}$,
- **1-cells:** analytic functors $F : \widehat{\mathbb{A}} \to \widehat{\mathbb{B}}$,
- **2-cells:** quasi cartesian natural transformations.

Definition 1.4.18. Let $\mathbb{C}$ be a category, a *quasi-pullback* of a cospan $a \xrightarrow{f} c \xleftarrow{g} b$ with pullback $(a \times_c b, \pi_a, \pi_b)$ in $\mathbb{C}$ is an object x together with maps $p : x \to a$ and $q : x \to b$ such that the following square commutes

$$\begin{tikzcd}
x \arrow[r, "q"] \arrow[d, "p"'] & b \arrow[d, "g"] \\
a \arrow[r, "f"'] & c
\end{tikzcd}$$

and the universal arrow $x \to a \times_c b$ is an epimorphism.

Remark 3. A morphism $f : a \to b$ in a category $\mathbb{C}$ is an epimorphism if and only if the square $(\mathrm{id}_b \circ f, \mathrm{id}_b \circ f)$ is a quasi-pullback.

A weak pullback is also a quasi-pullback but the converse is not true in general. In **Set**, assuming the axiom of choice, all epimorphisms are split which implies that the notions of quasi and weak pullbacks coincide. This property also holds for a presheaf category $\mathbf{Set}^A$ over a discrete category A so that the analytic functors $\mathbf{Set} \to \mathbf{Set}$ introduced by Joyal and the analytic functors $\mathbf{Set}^A \to \mathbf{Set}^A$ studied by Hasegawa can also be characterized as finitary functors that preserve quasi pullbacks.

In the case of analytic functors $\widehat{\mathbb{A}} \to \widehat{\mathbb{B}}$ for general categories, Fiore showed that they are always finitary but they preserve wide quasi-pullbacks only when we restrict to groupoids.

Lemma 1.4.19. *For categories $\mathbb{A}$ and $\mathbb{B}$ and a species $F : \mathcal{S}\mathbb{A} \nrightarrow \mathbb{B}$, the functor $\widetilde{F} : \widehat{\mathbb{A}} \to \widehat{\mathbb{B}}$ is finitary.*

Proof. let $D : \Delta \to \widehat{\mathbf{A}}$ be a filtered diagram, we have:

$$\begin{aligned}
\widehat{F}(\varinjlim_{\delta\in\Delta} D(\delta)) &= \int^{u=\langle a_1,\ldots,a_n\rangle} F(u,b) \times \widehat{\mathbf{A}}(s_{\mathbf{A}}(u), \varinjlim_{\delta\in\Delta} D(\delta)) && (1.1)\\
&= \int^{u} F(u,b) \times \prod_{i=1}^{n} \widehat{\mathbf{A}}(\mathrm{y}(a_i), \varinjlim_{\delta\in\Delta} D(\delta)) && (1.2)\\
&= \int^{u} F(u,b) \times \prod_{i=1}^{n} \varinjlim_{\delta\in\Delta} D(\delta)(a_i) && (1.3)\\
&= \int^{u} F(u,b) \times \varinjlim_{\delta\in\Delta} \prod_{i=1}^{n} D(\delta)(a_i) && (1.4)\\
&= \int^{u} F(u,b) \times \varinjlim_{\delta\in\Delta} \Big(\widehat{\mathbf{A}}(s_{\mathbf{A}}(u), D(\delta))\Big) && (1.5)\\
&= \int^{u} \varinjlim_{\delta\in\Delta} \Big(F(u,b) \times \widehat{\mathbf{A}}(s_{\mathbf{A}}(u), D(\delta))\Big) && (1.6)\\
&= \varinjlim_{\delta\in\Delta} \Big(\int^{u} F(u,b) \times \widehat{\mathbf{A}}(s_{\mathbf{A}}(u), D(\delta))\Big) && (1.7)
\end{aligned}$$

We obtain (5) by using the fact that filtered colimits commute with finite products, (7) holds since $(F(u,b)\times -)$ is a left adjoint so it preserves colimits and (8) is a consequence of the coend being a colimit and colimits being commutative. □

Lemma 1.4.20 (Theorem 6.8 in [39]). *For groupoids $\mathbb{A}$ and $\mathbb{B}$, a finitary functor $\widehat{\mathbb{A}} \to \widehat{\mathbb{B}}$ is analytic if and only if it preserves wide quasi-pullbacks.*

Chapter 2

Finiteness Species of Structures

The intuitions of linear logic originate from linear algebra, Girard wanted to interpret types as vector spaces such that linear maps between them correspond to programs using their arguments exactly once and analytic functions correspond to general programs. The exponential modality in linear logic however leads to infinite dimensional vector spaces which are no longer isomorphic to their double dual, a property required to model classical negation. Topological vector spaces were therefore considered to circumvent this issue [56, 17, 31]. In this setting, the series interpreting a program usually has infinite support describing all its possible behaviors for all possible inputs which allows for the study of non-deterministic languages.

Finiteness spaces are a model of linear logic introduced by Ehrhard as a way to enforce finite interactions between programs and reject infinite computations [32]. The construction of the finiteness spaces model is done in two steps: the first step is a double glueing construction (in the sense of Hyland and Schalk [68]) on the relational model **Rel**. The second step is parameterized by a fixed field (or commutative semi-ring) $\mathcal{R}$: for every finiteness space, one can define a vector space (or semi-module) whose vectors are linear combinations with finitary support, and this space is endowed with a topology induced by the duality. The finiteness space construction yields a model of controlled non-determinism: the objects can be infinite dimensional vector spaces and the morphisms are series with possibly infinite support but whenever an explicit computation is made, the result is always finite. It corresponds to the operational property that a program always has a finite number of reduction paths for a given input and output. Finiteness spaces were also used to characterize strongly normalizing terms in non-deterministic λ-calculus [95]. More recently, finiteness spaces were used in the theory of generalized power series rings and topological groupoids [15, 7].

In this chapter, we present a bicategorical extension of this construction where the relational model is replaced with the model of generalized species of structures and the finiteness property now relies on finite presentability. The categorification of the orthogonality relation allows us to work in a better behaved setting of *focused orthogonalities* where forward preservation is equivalent to backward preservation for morphisms preserving the finiteness structure [68]. We follow the same pattern of the double-glueing construction for 1-categories to obtain a bicategory of finiteness spaces and profunctors between them where computations are enforced to be finite and show that all the differential linear logic constructions in **Prof** can be refined to our bicategory.

2.1 Relational finiteness spaces

The model of relational finiteness spaces is obtained from **Rel** via a glueing construction along hom-functors using the following orthogonality relation:

Definition 2.1.1. For a countable set S, subsets $x \in \mathbf{Rel}(1, S) \cong \mathcal{P}(S)$ and $x' \in \mathbf{Rel}(S, 1) \cong \mathcal{P}(S)$, we say that x and x' are *orthogonal* if $x \cap x'$ is a finite set and we denote it by $x \perp_S x'$.

The idea is that morphisms in $\mathbf{Rel}(1, S)$ are thought of as closed programs of type S and morphisms in $\mathbf{Rel}(S, 1)$ correspond to counter-programs or environments. The orthogonality relation allows for more control on interactions between programs and environments as we require their interaction to always be finite even if the type S is infinite. For a subset $\mathscr{F} \subseteq \mathcal{P}(S)$, we define its *orthogonal* as

$$\mathscr{F}^{\perp} := \{x \in \mathcal{P}(S) \mid \forall x' \in \mathscr{F}, x \perp x'\} \subseteq \mathcal{P}(S).$$

This orthogonality relation induces a Galois connection on $\mathcal{P}\mathcal{P}(S)$

$$\begin{tikzcd} \mathcal{P}\mathcal{P}(S) \arrow[r, bend left, "(-)^{\perp}"] \arrow[r, phantom, "\perp"] & \mathcal{P}\mathcal{P}(S) \arrow[l, bend left, "(-)^{\perp}"] \end{tikzcd}$$

where finiteness spaces, introduced below, are its fixpoints $\mathscr{F} = \mathscr{F}^{\perp\perp}$.

Definition 2.1.2. A *relational finiteness space* is a pair $A = (|A|, \mathscr{F}(A))$ where $|A|$ is a countable set and $\mathscr{F}(A)$ is a subset of $\mathcal{P}(|A|)$ satisfying $\mathscr{F}(A) = \mathscr{F}(A)^{\perp\perp}$.

For any countable set S, the smallest finiteness structure is given by the set of finite subsets of S, $\mathcal{P}_{\text{fin}}(S)$ whose orthogonal is given by the whole powerset $\mathcal{P}(S)$. For a relational finiteness space A, while elements of $\mathscr{F}(A)$ may be infinite subsets of $|A|$, they are called *finitary subsets* as they "behave" like finite sets in that $\mathscr{F}(A)$ is closed under inclusion (for $x \in \mathscr{F}(A)$, if $x' \subseteq x$, then $x' \in \mathscr{F}(A)$) and finite unions.

Definition 2.1.3. The category **FinRel** has objects finiteness spaces and morphisms are relations that preserve the finitary structure forward and backward. Explicitly, for finiteness spaces $A = (|A|, \mathscr{F}(A))$ and $B = (|B|, \mathscr{F}(B))$, a relation $R \subseteq |A| \times |B|$ is said to be a *morphism of finiteness spaces* from A to B if for all $x \in \mathscr{F}(A)$, $R_\star \cdot x \in \mathscr{F}(B)$ and for all $y \in \mathscr{F}(B)^\perp$, $R^\star \cdot y \in \mathscr{F}(A)^\perp$.

$$\begin{tikzcd}
\mathcal{P}(|A|) \arrow[r, "R_\star"] & \mathcal{P}(|B|) & \mathcal{P}(|B|) \arrow[r, "R^\star"] & \mathcal{P}(|A|) \\
\mathscr{F}(A) \arrow[u, hook] \arrow[r, dotted] & \mathscr{F}(B) \arrow[u, hook] & \mathscr{F}(B)^\perp \arrow[u, hook] \arrow[r, dotted] & \mathscr{F}(A)^\perp \arrow[u, hook]
\end{tikzcd}$$

Formally, the category **FinRel** is the tight orthogonality category in the sense of Hyland and Schalk obtained from the orthogonality relation defined above [68]. Ehrhard showed that the linear logic structure from **Rel** can be lifted to **FinRel** which constitutes a model of differential linear logic [35]. Finiteness spaces do not provide a model of PCF since the fixpoint operator is not a morphism in the model. Vaux showed however that it allows for primitive recursion and is hence a model of Gödel's system T [114].

The morphisms in the co-Kleisli category of **FinRel** play the role of supports for power series for the second part of the construction: for a fixed field (or semi-ring) $\mathcal{R}$, we can define for every relational finiteness space $A = (|A|, \mathscr{F}(A))$, the following vector space (or semi-module):

$$\mathcal{R}\langle A\rangle := \{X \in \mathcal{R}^{|A|} \mid \text{support}(X) \in \mathscr{F}(A)\}.$$

Ehrhard showed that $\mathcal{R}\langle A\rangle$ can be endowed with a topology $\mathcal{T}_A$ such that a matrix $M \in \mathcal{R}\langle A \multimap B\rangle$ corresponds to a linear continuous map $\mathcal{R}\langle A\rangle \to \mathcal{R}\langle B\rangle$ and a matrix $M \in \mathcal{R}\langle !A \multimap B\rangle$ corresponds to an analytic map $\mathcal{R}\langle A\rangle \to \mathcal{R}\langle B\rangle$ [32] for which there is a natural notion of differentiation. This construction provided the semantical motivation for differential linear logic and the syntactic notion of Taylor expansion which associates a formal sum of resource terms to a given term [37, 35]. Finiteness spaces were also used to characterize strongly normalizing terms in non-deterministic λ-

calculus [95]. More recently, finiteness spaces were used in the theory of generalized power series rings and topological groupoids [15, 7].

2.2 Profunctorial finiteness spaces

We work in this chapter with a fragment of **Prof** where the objects are locally finite categories, it has the important consequence that finitely presentable presheaves are always finite presheaves as we will see below.

Definition 2.2.1. A small category $\mathbb{A}$ is said to be *locally finite* if it is enriched over finite sets i.e. for any objects $a, a' \in \mathbb{A}$, the homset $\mathbb{A}(a, a')$ is finite.

Definition 2.2.2. For a category $\mathbb{A}$, a presheaf $X : \mathbb{A}^{op} \to \mathbf{Set}$ is said to be *finite* if for all $a \in \mathbb{A}$, $X(a)$ is a finite set. We denote by $\widehat{\mathbb{A}}_{\mathrm{fin}} \hookrightarrow \widehat{\mathbb{A}}$ the full subcategory of finite presheaves. Note that the Yoneda embedding $\mathsf{y}_{\mathbb{A}}$ for a locally finite category $\mathbb{A}$ factors through the inclusion $\widehat{\mathbb{A}}_{\mathrm{fin}} \hookrightarrow \widehat{\mathbb{A}}$ by an embedding $\mathbb{A} \hookrightarrow \widehat{\mathbb{A}}_{\mathrm{fin}}$.

For presheaf categories, finitely presentable objects can be characterized as presheaves that are isomorphic to a finite colimit of representables. For a locally finite category $\mathbb{A}$, since a finite colimit of finite presheaves is also a finite presheaf, there is an embedding from the subcategory of finitely presentable objects $\widehat{\mathbb{A}}_{\mathrm{fp}}$ to $\widehat{\mathbb{A}}_{\mathrm{fin}}$.

Definition 2.2.3. A profunctor $F : \mathbb{A} \nrightarrow \mathbb{B}$ between two small categories $\mathbb{A}$ and $\mathbb{B}$ is said to be a *finite profunctor* if it can be factored as a functor $F : \mathbb{A} \to \widehat{\mathbb{B}}_{\mathrm{fin}}$ through the embedding $\widehat{\mathbb{B}}_{\mathrm{fin}} \hookrightarrow \widehat{\mathbb{B}}$. In other words, for all $a \in \mathbb{A}$ and $b \in \mathbb{B}$, $F(a, b)$ is a finite set. A finite profunctor will be denoted by $F : \mathbb{A} \nrightarrow_{\mathsf{f}} \mathbb{B}$.

Recall that the composite of two profunctors $F : \mathbb{A} \nrightarrow \mathbb{B}$ and $G : \mathbb{B} \nrightarrow \mathbb{C}$ is the profunctor $G \circ F : \mathbb{A} \nrightarrow \mathbb{C}$ given by the coend formula:

$$(a, c) \quad \mapsto \quad \int^{b \in \mathbb{B}} F(a, b) \times G(b, c) \cong \left(\sum_{b \in \mathbb{B}} F(a, b) \times G(b, c) \right) /\!\sim$$

where $\sim$ is the least equivalence relation such that

$$(b, F(a, f)(s), t) \sim (b', s, G(f, c)(t))$$

for $s \in F(a, b')$, $t \in G(b, c)$ and $f : b \to b' \in \mathbb{B}$. Note that the composite of two finite profunctors between locally finite categories need not to be finite (since the sum above can be infinite if $\mathbb{B}$ has an infinite object set for example) but we will see how finiteness structures will enable us to make this notion compositional.

Definition 2.2.4. Let $\mathbb{A}$ be a locally finite category, $X : \mathbb{A}^{op} \to \mathbf{Set}$ a presheaf and $X' : \mathbb{A} \to \mathbf{Set}$ a copresheaf, we say that X and X' are *orthogonal* and write $X \perp_{\mathbb{A}} X'$ if the following set is finite:

$$\langle X \mid X' \rangle := \int^{a \in \mathbb{A}} X(a) \times X'(a).$$

In the bicategorical case, presheaves in $\widehat{\mathbb{A}}$ or equivalently profunctors $\mathbf{1} \nrightarrow \mathbb{A}$ are thought of as closed programs of type $\mathbb{A}$ and co-presheaves in $\widehat{\mathbb{A}^{\mathrm{op}}}$ or profunctors $\mathbb{A} \nrightarrow \mathbf{1}$ correspond to environments. In our setting, the interaction between a program $X : \mathbb{A}^{\mathrm{op}} \to \mathbf{Set}$ and an environment $X' : \mathbb{A} \to \mathbf{Set}$ corresponds to their composition in **Prof**: $X' \circ X = \int^{a \in \mathbb{A}} X(a) \times X'(a)$. Adding the orthogonality structure on categories allows us to work in a setting where we enforce this composite to always be finite. Note that the condition in Definition 2.2.4 becomes

$$X' \circ X \in \mathbf{FinSet} \quad \hookrightarrow \quad \mathbf{Set} \cong \mathbf{Prof}(\mathbf{1}, \mathbf{1}).$$

Unlike the relational case, the orthogonality in the categorified setting becomes focused (Definition 1.2.8) so that the two preservation conditions for relations of Definition 2.1.3 reduce to a single preservation condition for profunctors as we will see in Definition 2.2.11. It simplifies the proofs significantly since we do not have to prove both directions every time:

Lemma 2.2.5. *For all $X : \mathbf{1} \nrightarrow_{\mathsf{f}} \mathbb{A}$, $Y : \mathbb{B} \nrightarrow_{\mathsf{f}} \mathbf{1}$ and $F : \mathbb{A} \nrightarrow_{\mathsf{f}} \mathbb{B}$, we have:*

$$F \circ X \perp_{\mathbb{B}} Y \qquad \Leftrightarrow \qquad X \perp_{\mathbb{A}} Y \circ F.$$

Proof. It follows from the fact that the sets $\langle F \circ X \mid Y \rangle$ and $\langle X \mid Y \circ F \rangle$ are both isomorphic to $\int^{a \in \mathbb{A}} \int^{b \in \mathbb{B}} F(a, b) \times X(a) \times Y(b)$. $\square$

For a set A considered as a discrete category, a subset $x \subseteq A$ can be viewed as a presheaf $x : A^{\mathrm{op}} \to \mathbf{Set}$ (or a copresheaf $x : A \to \mathbf{Set}$) that maps $a \in A$ to the singleton $\{\star\}$ if $a \in x$ and to the empty set otherwise. Hence, for $x \subseteq A$ viewed as a presheaf and $x' \subseteq A$ viewed as a copresheaf, $x \cap x'$ is finite is equivalent to the set $\int^{a \in A} x(a) \times x'(a)$ being finite. This analogy provides the connection between the bicategorical case and the relational case.

Definition 2.2.6. For a subcategory $\mathbb{C} \hookrightarrow \widehat{\mathbb{A}}_{\text{fin}}$, we denote by $\mathbb{C}^{\perp}$, the full subcategory of $\widehat{\mathbb{A}^{op}}_{\text{fin}}$ of finite copresheaves X' such that for all $X \in \mathbb{C}$, $X' \perp_{\mathbb{A}} X$.

Let $\mathbf{Sub}(\widehat{\mathbb{A}})$ be the poset of subcategories of $\widehat{\mathbb{A}}$, the orthogonality relation induces a Galois connection:

$$\begin{tikzcd}
\mathbf{Sub}(\widehat{\mathbb{A}}) \arrow[r, bend left, "(-)^{\perp}"] \arrow[r, phantom, "\perp"] & \mathbf{Sub}(\widehat{\mathbb{A}^{\text{op}}})^{\text{op}} \arrow[l, bend left, "(-)^{\perp}"]
\end{tikzcd}$$

whose fixed points are full subcategories $\mathbb{C}$ verifiying $\mathbb{C}^{\perp\perp} \cong \mathbb{C}$.

Definition 2.2.7. A *finiteness structure* is a pair $\mathbf{A} = (|\mathbf{A}|, \mathscr{F}\mathbf{A})$ of a locally finite category $|\mathbf{A}|$ and a full subcategory $\mathscr{F}\mathbf{A} \hookrightarrow \widehat{|\mathbf{A}|}_{\text{fin}}$ verifying $\mathscr{F}\mathbf{A} \cong \mathscr{F}\mathbf{A}^{\perp\perp}$.

Lemma 2.2.8. *For a finiteness structure $\mathbf{A} = (|\mathbf{A}|, \mathscr{F}\mathbf{A})$, the subcategory of finitely presentable objects $\widehat{|\mathbf{A}|}_{\text{fp}} \hookrightarrow \widehat{|\mathbf{A}|}_{\text{fin}}$ is always a full subcategory of $\mathscr{F}\mathbf{A}$.*

Proof. If X is finitely presentable, then X is isomorphic to a finite colimit of representables $X \cong \varinjlim_{i \in I} |\mathbf{A}|(-, a_i) : |\mathbf{A}|^{\text{op}} \to \mathbf{Set}$. For any $X' \in (\mathscr{F}\mathbf{A})^{\perp}$,

$$\langle X \mid X' \rangle = \int^{a \in |\mathbf{A}|} X(a) \times X'(a) \cong \varinjlim_{i \in I} \int^{a \in |\mathbf{A}|} |\mathbf{A}|(a, a_i) \times X'(a) \cong \varinjlim_{i \in I} X'(a_i).$$

Since a finite colimit of finite sets is finite, we obtain that $X \perp_{\mathbf{A}} X'$ as desired. ☐

The minimal finiteness structure is $(|\mathbf{A}|, \widehat{|\mathbf{A}|}_{\text{fp}})$ and its orthogonal is the maximal finiteness structure $(|\mathbf{A}|, \widehat{|\mathbf{A}|}_{\text{fin}})$ so for any finiteness structure $\mathbf{A} = (|\mathbf{A}|, \mathscr{F}\mathbf{A})$, we have

$$(|\mathbf{A}|, \widehat{|\mathbf{A}|}_{\text{fp}}) \hookrightarrow \mathbf{A} \hookrightarrow (|\mathbf{A}|, \widehat{|\mathbf{A}|}_{\text{fin}}).$$

Lemma 2.2.9. *If $\mathbb{A}$ is a finite category (both the object and morphism sets are finite), then there is a unique finiteness structure given by $\widehat{\mathbb{A}}_{\text{fin}}$.*

Proof. By Lemma 2.2.8, it suffices to show that if $\mathbb{A}$ is finite, then any finite presheaf $X : \mathbb{A}^{op} \to \mathbf{FinSet}$ is finitely presentable. If $\mathbb{A}$ is finite, then the category of elements $\int X$ of X is finite as well and since $X \cong \varinjlim(\int X \to \mathbb{A} \to \widehat{\mathbb{A}})$, X is a finite colimit of representables and hence is finitely presentable. ☐

In the relational case, for a finiteness structure $A = (|A|, \mathscr{F}A)$, $\mathscr{F}A$ can be larger than $\mathcal{P}_{\text{fin}}(|A|)$ but its elements "behave" like finite sets in the sense that $x \subseteq y \in \mathscr{F}(A)$ implies $x \in \mathscr{F}(A)$ and a finite union of finitary elements is finitary. In the categorical case, $\mathscr{F}(\mathbf{A})$ can be thought of as a category larger than $\widehat{|\mathbf{A}|}_{\text{fp}}$ but its elements "behave" like finitely presentable elements as $\mathscr{F}(\mathbf{A})$ is closed under retractions and finite colimits.

Lemma 2.2.10. *Let $\mathbf{A} = (|\mathbf{A}|, \mathscr{F}(\mathbf{A}))$ be a finiteness structure, then the following two properties hold:*

1. *if X' is a retract of an element $X \in \mathscr{F}(\mathbf{A})$, then $X' \in \mathscr{F}(\mathbf{A})$;*
2. *$\mathscr{F}(\mathbf{A})$ is closed under finite colimits.*

Proof. Let $\alpha : X \Rightarrow X'$ be a retraction in $\widehat{|\mathbf{A}|}$. Since a retraction is an epimorphism and colimits in $\widehat{|\mathbf{A}|}$ are computed pointwise, for every $a \in |\mathbf{A}|$, $\alpha_a : X(a) \to X'(a)$ is a surjection. Hence, for every $Y \in \mathscr{F}(\mathbf{A})^{\perp}$,

$$\langle X \mid Y \rangle = \int^{a \in |\mathbf{A}|} X(a) \times Y(a) \quad \twoheadrightarrow \quad \int^{a \in |\mathbf{A}|} X'(a) \times Y(a) = \langle X' \mid Y \rangle$$

which implies that $\langle X' \mid Y \rangle$ is a finite set as well so that $X' \in \mathscr{F}(\mathbf{A})$. The second property follows from the fact that a finite colimit of finite sets is finite. ☐

Definition 2.2.11. Given two finiteness structures $\mathbf{A} = (|\mathbf{A}|, \mathscr{F}\mathbf{A})$ and $\mathbf{B} = (|\mathbf{B}|, \mathscr{F}\mathbf{B})$, a finite profunctor $F : |\mathbf{A}| \nrightarrow_{\mathsf{f}} |\mathbf{B}|$ is called a *finiteness profunctor* if $F^{\#} := \mathbf{Lan}_{\mathsf{y}_{|\mathbf{A}|}} F : \widehat{|\mathbf{A}|} \to \widehat{|\mathbf{B}|}$ verifies $F^{\#}(\mathscr{F}\mathbf{A}) \hookrightarrow \mathscr{F}\mathbf{B}$ i.e if there exists a functor $\mathscr{F}\mathbf{A} \to \mathscr{F}\mathbf{B}$ making the diagram below commute:

$$\begin{tikzcd}
\widehat{|\mathbf{A}|} \arrow[r, "F^{\#}"] & \widehat{|\mathbf{B}|} \\
\mathscr{F}\mathbf{A} \arrow[u, hook] \arrow[r, dotted] & \mathscr{F}\mathbf{B} \arrow[u, hook]
\end{tikzcd}$$

Lemma 2.2.12. *Given two finiteness structures $\mathbf{A} = (|\mathbf{A}|, \mathscr{F}\mathbf{A})$ and $\mathbf{B} = (|\mathbf{B}|, \mathscr{F}\mathbf{B})$, a profunctor $F : |\mathbf{A}| \nrightarrow_{\mathsf{f}} |\mathbf{B}|$ is a finiteness profunctor $\mathbf{A} \nrightarrow_{\mathsf{f}} \mathbf{B}$ if and only if $F^{\perp} : (|\mathbf{B}|^{op}, \mathscr{F}\mathbf{B}^{\perp}) \nrightarrow_{\mathsf{f}} (|\mathbf{A}|^{op}, \mathscr{F}\mathbf{A}^{\perp})$ is also a finiteness profunctor.*

Proof. Direct consequence of Lemma 2.2.5. ☐

Since the categories $\mathbf{Prof}(\mathbf{1}, |\mathbf{A}|)$ and $\widehat{|\mathbf{A}|}$ are isomorphic, we will abuse notation and identify presheaves $X \in \widehat{|\mathbf{A}|}$ with profunctors $\mathbf{1} \nrightarrow |\mathbf{A}|$ and write $F \circ X$ instead of $F^{\#}X$. Under this isomorphism, we can reformulate the condition of Definition 2.2.11 as follows: F is a finiteness profunctor if for all presheaves X in $\mathscr{F}\mathbf{A}$, $F \circ X$ is in $\mathscr{F}\mathbf{B}$. Likewise, using the isomorphism $\mathbf{Prof}(|\mathbf{B}|, \mathbf{1}) \cong \widehat{|\mathbf{B}|^{\mathrm{op}}}$, $F^{\perp}$ is a finiteness profunctor if for all copresheaves Y in $\mathscr{F}\mathbf{B}^{\perp}$, $Y \circ F$ is in $\mathscr{F}\mathbf{A}^{\perp}$.

Definition 2.2.13. Define **FinProf** to be the bicategory whose 0-cells are finiteness structures, 1-cells are finiteness profunctors as in Definition 2.2.11 and 2-cells are natural transformations between such profunctors.

Proof. We show below that **FinProf** is indeed a bicategory.

Identity For a finiteness structure $\mathbf{A} = (|\mathbf{A}|, \mathscr{F}\mathbf{A})$, $\mathrm{id}_{|\mathbf{A}|} : |\mathbf{A}| \nrightarrow |\mathbf{A}|$ is a finite profunctor as $|\mathbf{A}|$ is a locally finite category. Since $\mathrm{id}_{|\mathbf{A}|}$ verifies $\widehat{id_{|\mathbf{A}|}} \cong id_{\widehat{|\mathbf{A}|}}$, it is a finiteness profunctor $\mathbf{A} \nrightarrow_{\mathsf{f}} \mathbf{A}$.

Composition Let $\mathbf{A} = (|\mathbf{A}|, \mathscr{F}\mathbf{A}), \mathbf{B} = (|\mathbf{B}|, \mathscr{F}\mathbf{B})$ and $\mathbf{C}(|\mathbf{C}|, \mathscr{F}\mathbf{C})$ be finiteness structures and $F : \mathbf{A} \nrightarrow_{\mathsf{f}} \mathbf{B}$ and $G : \mathbf{B} \nrightarrow_{\mathsf{f}} \mathbf{C}$ be finiteness profunctors. It is clear that if $F^{\#}(\mathscr{F}\mathbf{A}) \hookrightarrow \mathscr{F}\mathbf{B}$ and $G^{\#}(\mathscr{F}\mathbf{B}) \hookrightarrow \mathscr{F}\mathbf{C}$, then $\widehat{G \circ F}(\mathscr{F}\mathbf{A}) \cong G^{\#} \circ F^{\#}(\mathscr{F}\mathbf{A}) \hookrightarrow \mathscr{F}\mathbf{C}$. It remains to show that $G \circ F$ is a finite profunctor. For all $a \in |\mathbf{A}|$ and $c \in |\mathbf{C}|$, we have

$$(G \circ F)(a, c) = \int^{b \in |\mathbf{B}|} F(a, b) \times G(b, c) \cong G^{\#}(F^{\#}(\mathbb{y}(a)))(c).$$

Since $\mathbb{y}(a) \in \mathscr{F}\mathbf{A}$, $G^{\#}(F^{\#}(\mathbb{y}(a))$ is an element of $\mathscr{F}\mathbf{C}$ so it is a finite presheaf, which implies that $G^{\#}(F^{\#}(\mathbb{y}(a)))(c)$ is finite as desired. ☐

We obtain as a corollary of Lemma 2.2.12 that the mapping $\mathbf{A} \mapsto \mathbf{A}^{\perp} := (|\mathbf{A}|^{\mathrm{op}}, \mathscr{F}\mathbf{A}^{\perp})$ can be extended to a full and faithful functor $\mathbf{FinProf}^{\mathrm{op}} \to \mathbf{FinProf}$.

Lemma 2.2.14. *The forgetful functor $\mathcal{U} : \mathbf{FinProf} \to \mathbf{Prof}$ is locally fully faithful and injective on 1-cells. Explicitely, for finiteness structures $\mathbf{A}$ and $\mathbf{B}$, the induced functor $\mathbf{FinProf}(\mathbf{A}, \mathbf{B}) \to \mathbf{Prof}(|\mathbf{A}|, |\mathbf{B}|)$ is injective on objects and fully faithful.*

2.3 Linear logic structure

In this section, we prove that the differential linear logic structure in **Prof** can be lifted to **FinProf**. The proofs will make use of the lemma below that shows how certain families of adjoint equivalences needed for the linear logic structure can be lifted from **Prof** to **FinProf** using the fact that the forgetful functor is locally fully faithful.

Lemma 2.3.1. *Let* $\mathbb{A}, \mathbb{B}, \mathbb{C}, \mathbb{D}$ *be categories and* $(L : \mathbb{A} \to \mathbb{B}, R : \mathbb{B} \to \mathbb{A}, \eta, \varepsilon)$ *be an adjoint equivalence. Let* $L' : \mathbb{C} \to \mathbb{D}$*,* $R' : \mathbb{D} \to \mathbb{C}$*,* $F : \mathbb{C} \to \mathbb{A}$ *and* $G : \mathbb{D} \to \mathbb{B}$ *be functors such that* F *and* G *are fully faithful,* $GL' = LF$ *and* $FR' = RG$*. Then* L' *and* R' *are adjoint equivalent* $L' \dashv R'$*.*

$$\begin{tikzcd}
\mathbb{A} \arrow[r, bend left, "L"] \arrow[r, phantom, "\simeq \perp"] & \mathbb{B} \arrow[l, bend left, "R"] \\
\mathbb{C} \arrow[u, "F"] \arrow[r, bend left, "L'"'] & \mathbb{D} \arrow[u, "G"'] \arrow[l, bend left, "R'"]
\end{tikzcd}$$

Proof. For objects $c \in \mathbb{C}$ and $d \in \mathbb{D}$, using the hypotheses above, we have:

$$\begin{aligned}
\mathbb{C}(c, R'd) &\cong \mathbb{A}(Fc, FR'(d)) = \mathbb{A}(Fc, RGd) \\
&\cong \mathbb{B}(LFc, Gd) = \mathbb{B}(GL'c, Gd) \cong \mathbb{D}(L'c, d)
\end{aligned}$$

which implies that $L' \dashv R'$.

For $c \in \mathbb{C}$, the component of the unit η' of the adjunction $L' \dashv R'$ is the morphism η'_c determined by $F(\eta'_c) = \eta_{F(c)}$. It is an isomorphism since F is fully faithful and hence conservative. We can show that the counit of the adjunction $L' \dashv R'$ is an isomorphism in a similar fashion. □

2.3.1 Additive structure

Similarly to the 1-categorical case, **FinProf** is endowed with a finite biproduct structure. Under the isomorphism $\widehat{\&_i \mathbb{A}_i} \cong \prod_i \widehat{\mathbb{A}_i}$, we will often identify a presheaf $Z \in \widehat{\&_i \mathbb{A}_i}$ with a tuple of presheaves $(Z_i)_{i \in I} \in \prod_i \widehat{\mathbb{A}_i}$.

Lemma 2.3.2. *For a finite family of finiteness structures* $(\mathbf{A}_i)_{i \in I}$*,* $\&_i \mathbf{A}_i := (\&_i |\mathbf{A}_i|, \prod_i \mathscr{F}\mathbf{A}_i)$ *is a finiteness structure.*

Proof. It suffices to show that $(\prod_i \mathscr{F}\mathbf{A}_i)^\perp \cong \prod_i (\mathscr{F}\mathbf{A}_i)^\perp$. □

Definition 2.3.3. For a finite family of finiteness structures $(\mathbf{A}_i)_{i \in I}$, we define the finiteness structure $\oplus_i \mathbf{A}_i$ by $(\&_i |\mathbf{A}_i|, (\mathscr{F}(\&_i \mathbf{A}_i^\perp))^\perp)$.

Lemma 2.3.4. *The empty category* $\mathbf{0}$ *with its presheaf category* $(\mathbf{0}, \widehat{\mathbf{0}})$ *forms a finiteness structure that is the neutral for* $\&$ *and* $\oplus$*.*

Lemma 2.3.5. *For a finite family of finiteness structures* $(\mathbf{A}_i)_{i \in I}$*, the profunctors* $\pi_i : \&_i \left|\mathbf{A}_i\right| \nrightarrow \left|\mathbf{A}_i\right|$ *and* $\mathrm{inj}_i : \left|\mathbf{A}_i\right| \nrightarrow \&_i \left|\mathbf{A}_i\right|$ *are finiteness profunctors* $\&_i \mathbf{A}_i \nrightarrow_{\mathsf{f}} \mathbf{A}_i$ *and* $\mathbf{A}_i \nrightarrow_{\mathsf{f}} \oplus_i \mathbf{A}_i$ *respectively. They induce adjoint equivalences:*

$$\mathbf{FinProf}(\mathbf{X}, \&_i \mathbf{A}_i) \simeq \prod_i \mathbf{FinProf}(\mathbf{X}, \mathbf{A}_i) \quad \textit{and}$$
$$\mathbf{FinProf}(\oplus_i \mathbf{A}_i, \mathbf{X}) \simeq \prod_i \mathbf{FinProf}(\mathbf{A}_i, \mathbf{X}).$$

Proof. The profunctors π_i and inj_i are given by $\pi_i : ((i, a_i), a) \mapsto \left|\mathbf{A}_i\right|(a, a_i)$ and $\mathrm{inj}_i : (a, (i, a_i)) \mapsto \left|\mathbf{A}_i\right|(a_i, a)$ so they are finite profunctors since the category $\left|\mathbf{A}_i\right|$ is locally finite. For $Z \in \mathscr{F}(\&_i \mathbf{A}_i)$ and $X \in \mathscr{F}\mathbf{A}_i^{\perp}$, $\langle \pi_i Z \mid X\rangle \cong \langle Z_i \mid X\rangle \in \mathbf{FinSet}$ which implies that $\pi_i \in \mathbf{FinProf}(\&_i \mathbf{A}_i, \mathbf{A}_i)$. Likewise, for X in $\mathscr{F}\mathbf{A}_i$ and $Z \in \mathscr{F}(\oplus_i \mathbf{A}_i)^{\perp}$, $\langle \mathrm{inj}_i X \mid Z\rangle \cong \langle X \mid Z_i\rangle \in \mathbf{FinSet}$ so that $\mathrm{inj}_i \in \mathbf{FinProf}(\mathbf{A}_i, \oplus_i \mathbf{A}_i)$.

Using Lemma 2.3.1, the adjoint equivalences above follow from the biproduct structure in **Prof** where we have adjoint equivalences

$$\mathbf{Prof}(\left|\mathbf{X}\right|, \&_i \left|\mathbf{A}_i\right|) \simeq \prod_i \mathbf{Prof}(\left|\mathbf{X}\right|, \left|\mathbf{A}_i\right|) \quad \text{and}$$
$$\mathbf{Prof}(\&_i \left|\mathbf{A}_i\right|, \left|\mathbf{X}\right|) \simeq \prod_i \mathbf{Prof}(\left|\mathbf{A}_i\right|, \left|\mathbf{X}\right|). \qquad \square$$

2.3.2 $*$-Autonomous structure

We show in this section that the symmetric monoidal structure in **Prof** lifts to **FinProf**. Adding the orthogonality structure allows for a less degenerate model as the bicategory **FinProf** is now $*$-autonomous instead of compact closed. For the symmetric monoidal structure, it suffices to prove that the tensor product lifts to a pseudo-functor **FinProf** × **FinProf** → **FinProf** and that the symmetry, associator and left and right unitors pseudo-natural transformations have components in **FinProf**.

For relational finiteness spaces, the tensor product of $A = (\left|A\right|, \mathscr{F}(A))$ and $B = (\left|B\right|, \mathscr{F}(B))$ is the smallest structure that contains all products $x \times y$ of subsets $x \in \mathscr{F}(A)$ and $y \in \mathscr{F}(B)$. Since the set $\{x \times y \mid x \in \mathscr{F}(A), y \in \mathscr{F}(B)\}$ is not necessarily closed under double orthogonality $A \otimes B$ is defined as

$$(\left|A\right| \times \left|B\right|, \{x \times y \mid x \in \mathscr{F}(A), y \in \mathscr{F}(B)\}^{\perp\perp}).$$

In the categorified case, the construction is similar, for finiteness structures $\mathbf{A}$ and $\mathbf{B}$, $\mathscr{F}(\mathbf{A} \otimes \mathbf{B})$ is the smallest finiteness structure containing all products $X \times Y$ for $X \in \mathscr{F}(\mathbf{A})$ and $Y \in \mathscr{F}(\mathbf{B})$.

Definition 2.3.6. For finiteness structures $\mathbf{A}$ and $\mathbf{B}$, their tensor product is defined as $\mathbf{A} \otimes \mathbf{B} := (|\mathbf{A}| \times |\mathbf{B}|, \mathscr{F}(\mathbf{A} \otimes \mathbf{B}))$ where $\mathscr{F}(\mathbf{A} \otimes \mathbf{B})$ is the full subcategory of $\widehat{|\mathbf{A}| \times |\mathbf{B}|}_{\text{fin}}$ whose object set is given by

$$\{X \times Y \mid X \in \mathscr{F}\mathbf{A} \text{ and } Y \in \mathscr{F}\mathbf{B}\}^{\perp\perp}.$$

The closure under double orthogonality ensures that we obtain a finiteness structure.

Lemma 2.3.7. *For finiteness profunctors $F_1 : \mathbf{A}_1 \nrightarrow_{\mathsf{f}} \mathbf{B}_1$ and $F_2 : \mathbf{A}_2 \nrightarrow_{\mathsf{f}} \mathbf{B}_2$, the profunctor $F_1 \otimes F_2 : |\mathbf{A}_1| \times |\mathbf{A}_2| \nrightarrow |\mathbf{B}_1| \times |\mathbf{B}_2|$ given by $(F_1 \otimes F_2)((a_1, a_2), (b_1, b_2)) := F_1(a_1, b_1) \times F_2(a_2, b_2)$ is in $\mathbf{FinProf}(\mathbf{A}_1 \otimes \mathbf{A}_2, \mathbf{B}_1 \otimes \mathbf{B}_2)$.*

Proof. Using Lemma 2.2.12, we show that $(F_1 \otimes F_2)^\perp \mathscr{F}(\mathbf{B}_1 \otimes \mathbf{B}_2)^\perp \hookrightarrow \mathscr{F}(\mathbf{A}_1 \otimes \mathbf{A}_2)^\perp$. Let Z be in $\mathscr{F}(\mathbf{B}_1 \otimes \mathbf{B}_2)^\perp$ i.e. for all $Y_1 \in \mathscr{F}\mathbf{B}_1$ and $Y_2 \in \mathscr{F}\mathbf{B}_2$, $\langle Z \mid Y_1 \times Y_2 \rangle \in \mathbf{FinSet}$. $(F_1 \otimes F_2)^\perp(Z) \in \mathscr{F}(\mathbf{A}_1 \otimes \mathbf{A}_2)^\perp$ is equivalent to:

$$\begin{aligned} &\forall X_1 \in \mathscr{F}\mathbf{A}_1, \forall X_2 \in \mathscr{F}\mathbf{A}_2, \langle (F_1 \otimes F_2)^\perp(Z) \mid X_1 \times X_2 \rangle \in \mathbf{FinSet} \\ \Leftrightarrow &\forall X_1 \in \mathscr{F}\mathbf{A}_1, \forall X_2 \in \mathscr{F}\mathbf{A}_2, \langle Z \mid (F_1 X_1) \times (F_2 X_2) \rangle \in \mathbf{FinSet} \end{aligned}$$

Since $F_1 X_1$ is in $\mathscr{F}\mathbf{B}_1$ and $F_2 X_2$ is in $\mathscr{F}\mathbf{B}_2$, we obtain the desired result. □

Lemma 2.3.8. $(\mathbf{1}, \mathbf{FinSet})$ *is the tensor unit.*

Proof. Let $\mathbf{A}$ be a finiteness structure, we show that $\mathscr{F}(\mathbf{A})^\perp \cong \mathscr{F}(\mathbf{A} \otimes \mathbf{1})^\perp \cong \mathscr{F}(\mathbf{1} \otimes \mathbf{A})^\perp$ so that the components of the left unitor $l_{|\mathbf{A}|} : |\mathbf{A}| \times |\mathbf{1}| \nrightarrow |\mathbf{A}|$ and right unitor $r_{|\mathbf{A}|} : |\mathbf{1}| \times |\mathbf{A}| \nrightarrow |\mathbf{A}|$ are in $\mathbf{FinProf}$. Let $Y \in \mathscr{F}(\mathbf{A})^\perp$, $X \in \mathscr{F}(\mathbf{A})$ and $S \in \mathbf{FinSet}$. We have $\langle Y, X \times S \rangle \in \mathbf{FinSet} \Leftrightarrow \langle Y \times S, X \rangle \in \mathbf{FinSet}$. Since $\mathscr{F}(\mathbf{A})^\perp$ is closed under finite colimits, $Y \times S$ is in $\mathscr{F}(\mathbf{A})^\perp$ which implies the desired result. Now, for $Y \in \mathscr{F}(\mathbf{A} \otimes \mathbf{1})^\perp$ and $X \in \mathscr{F}(\mathbf{A})$, $\langle Y, X \rangle \cong \langle Y, X \times \{*\} \rangle \in \mathbf{FinSet}$ so that $Y \in \mathscr{F}(\mathbf{A})^\perp$ as desired. The proof for $\mathscr{F}(\mathbf{A})^\perp \cong \mathscr{F}(\mathbf{1} \otimes \mathbf{A})^\perp$ is similar. □

Lemma 2.3.9. *For finiteness structures $\mathbf{A} = (|\mathbf{A}|, \mathscr{F}\mathbf{A})$ and $\mathbf{B} = (|\mathbf{B}|, \mathscr{F}\mathbf{B})$, the categories $\mathscr{F}(\mathbf{A} \otimes \mathbf{B})$ and $\mathscr{F}(\mathbf{B} \otimes \mathbf{A})$ are isomorphic which implies that the component of the symmetry $\sigma_{|\mathbf{A}|, |\mathbf{B}|} : |\mathbf{A}| \times |\mathbf{B}| \nrightarrow |\mathbf{B}| \times |\mathbf{A}|$ is in $\mathbf{FinProf}(\mathbf{A} \otimes \mathbf{B}, \mathbf{B} \otimes \mathbf{A})$.*

Proof. Immediate. $\square$

Showing that the associator has components in **FinProf** is difficult to prove directly so we make use of the duality between the tensor and the internal hom to do it.

Lemma 2.3.10. *For finiteness structures* $\mathbf{A}$ *and* $\mathbf{B}$*, define the finiteness structure* $\mathbf{A} \multimap \mathbf{B}$ *as* $(|\mathbf{A}|^{\mathrm{op}} \times |\mathbf{B}|, \mathscr{F}(\mathbf{A} \multimap \mathbf{B}))$ *where* $\mathscr{F}(\mathbf{A} \multimap \mathbf{B})$ *is the full subcategory of finite presheaves* $\widehat{|\mathbf{A}|^{op} \times |\mathbf{B}|}_{\mathrm{fin}}$ *that verify Definition 2.2.11.*

Proof. We prove that $\mathbf{A} \multimap \mathbf{B}$ is indeed a finiteness structure. We first show that for $X \in \mathscr{F}\mathbf{A}$ and $Y' \in \mathscr{F}\mathbf{B}^{\perp}$, $X \times Y' \in \mathscr{F}(\mathbf{A} \multimap \mathbf{B})^{\perp}$. Indeed, for $F \in \mathscr{F}(\mathbf{A} \multimap \mathbf{B})$, we have:

$$\langle X \times Y' \mid F \rangle = \int^{a \in |\mathbf{A}|, b \in |\mathbf{B}|} X(a) \times Y'(b) \times F(a,b) \cong \langle Y' \mid FX \rangle \in \mathbf{FinSet}.$$

Now, let $W \in \mathscr{F}(\mathbf{A} \multimap \mathbf{B})^{\perp\perp}$, we want to show that $W \in \mathscr{F}(\mathbf{A} \multimap \mathbf{B})$, i.e. that for all $X \in \mathscr{F}\mathbf{A}$, $WX \in \mathscr{F}\mathbf{B}$. Let $Y' \in \mathscr{F}\mathbf{B}^{\perp}$, $\langle Y' \mid WX \rangle \cong \langle X \times Y' \mid W \rangle \in \mathbf{FinSet}$ by the previous remark. $\square$

Lemma 2.3.11. *For finiteness structures* $\mathbf{A}$ *and* $\mathbf{B}$*, the categories* $\mathscr{F}(\mathbf{A} \otimes \mathbf{B})$ *and* $\mathscr{F}(\mathbf{A} \multimap \mathbf{B}^{\perp})^{\perp}$ *are isomorphic.*

Proof. We prove that $\mathscr{F}(\mathbf{A} \otimes \mathbf{B})^{\perp} \cong \mathscr{F}(\mathbf{A} \multimap \mathbf{B}^{\perp})$. Let $F : \mathbf{A} \nrightarrow \mathbf{B}^{op}$, we have:

$$\begin{aligned} F \in \mathscr{F}(\mathbf{A} \otimes \mathbf{B})^{\perp} &\Leftrightarrow \forall X \in \mathscr{F}(\mathbf{A}), \forall Y \in \mathscr{F}(\mathbf{B}) \langle F, X \times Y \rangle \in \mathbf{FinSet} \\ &\Leftrightarrow \forall X \in \mathscr{F}(\mathbf{A}), \forall Y \in \mathscr{F}(\mathbf{B}) \langle FX, Y \rangle \in \mathbf{FinSet} \\ &\Leftrightarrow \forall X \in \mathscr{F}(\mathbf{A}), FX \in \mathscr{F}(\mathbf{B})^{\perp} \\ &\Leftrightarrow F \in \mathscr{F}(\mathbf{A} \multimap \mathbf{B}^{\perp}) \end{aligned}$$

$\square$

Lemma 2.3.12. *For finiteness structures* $\mathbf{A}$*,* $\mathbf{B}$ *and* $\mathbf{C}$*, the categories* $\mathscr{F}((\mathbf{A} \otimes \mathbf{B}) \multimap \mathbf{C})$ *and* $\mathscr{F}(\mathbf{A} \multimap (\mathbf{B} \multimap \mathbf{C}))$ *are isomorphic.*

Proof. Let $F : |\mathbf{A}| \times |\mathbf{B}| \nrightarrow_{\mathsf{f}} |\mathbf{C}|$ be in $\mathscr{F}((\mathbf{A} \otimes \mathbf{B}) \multimap \mathbf{C})$ and denote by $\overline{F} : |\mathbf{A}| \nrightarrow_{\mathsf{f}} |\mathbf{B}|^{\mathrm{op}} \times |\mathbf{C}|$ the corresponding profunctor obtained from the isomorphism $\mathbf{Prof}(|\mathbf{A}| \times |\mathbf{B}|, |\mathbf{C}|) \cong \mathbf{Prof}(|\mathbf{A}|, |\mathbf{B}|^{\mathrm{op}} \times |\mathbf{C}|)$. Let $X \in \mathscr{F}(\mathbf{A})$, we want to show that $\overline{F}X$ is in $\mathscr{F}(\mathbf{B} \multimap \mathbf{C})$, i.e. for all $Y \in \mathscr{F}(\mathbf{B})$, $\overline{F}(X)(Y) \in \mathscr{F}(\mathbf{C})$. We have that $X \times Y$ is in $\mathscr{F}(\mathbf{A} \otimes \mathbf{B})$ so that $F \circ (X \times Y) \cong \overline{F}(X)(Y)$ is in $\mathscr{F}(\mathbf{C})$.

For the other direction, let $G : |\mathbf{A}| \nrightarrow_{\mathsf{f}} |\mathbf{B}|^{\mathrm{op}} \times |\mathbf{C}|$ be in $\mathscr{F}(\mathbf{A} \multimap (\mathbf{B} \multimap \mathbf{C}))$ and denote by $\overline{G}$ the corresponding profunctor in $\mathbf{Prof}(|\mathbf{A}| \times |\mathbf{B}|, |\mathbf{C}|)$. We show that $\overline{G}^{\perp} \in \mathscr{F}(\mathbf{C}^{\perp} \multimap (\mathbf{A} \otimes \mathbf{B})^{\perp})$. Let $Z \in \mathscr{F}(\mathbf{C})^{\perp}$, we want $\overline{G}^{\perp} Z \in \mathscr{F}(\mathbf{A} \otimes \mathbf{B})^{\perp}$ i.e. for all $X \in \mathscr{F}\mathbf{A}$ and $Y \in \mathscr{F}\mathbf{B}$, $\langle \overline{G}^{\perp} Z, X \times Y \rangle \in \mathbf{FinSet}$. Since $\langle \overline{G}^{\perp} Z \mid X \times Y \rangle \cong \langle G(X)(Y) \mid Z \rangle$, we obtain the desired result. $\square$

Corollary 2.3.13. *For finiteness structures* $\mathbf{A}$*,* $\mathbf{B}$ *and* $\mathbf{C}$*, the component of the associator* $\alpha_{|\mathbf{A}|,|\mathbf{B}|,|\mathbf{C}|} : (|\mathbf{A}| \times |\mathbf{B}|) \times |\mathbf{C}| \nrightarrow |\mathbf{A}| \times (|\mathbf{B}| \times |\mathbf{C}|)$ *given by:*

$$((a_1, b_1, c_1), (a_2, b_2, c_2)) \mapsto |\mathbf{A}|(a_2, a_1) \times |\mathbf{B}|(b_2, b_1) \times |\mathbf{C}|(c_2, c_1)$$

is a finiteness profunctor in $\mathbf{FinProf}((\mathbf{A} \otimes \mathbf{B}) \otimes \mathbf{C}, \mathbf{A} \otimes (\mathbf{B} \otimes \mathbf{C}))$.

Proof. It suffices to show that the categories $\mathscr{F}((\mathbf{A} \otimes \mathbf{B}) \otimes \mathbf{C})$ and $\mathscr{F}(\mathbf{A} \otimes (\mathbf{B} \otimes \mathbf{C}))$ are isomorphic. By Lemmas 2.3.11 and 2.3.12, we have

$$\begin{aligned} \mathscr{F}((\mathbf{A} \otimes \mathbf{B}) \otimes \mathbf{C}) &\cong \mathscr{F}((\mathbf{A} \otimes \mathbf{B}) \multimap \mathbf{C}^{\perp})^{\perp} \cong \mathscr{F}(\mathbf{A} \multimap (\mathbf{B} \multimap \mathbf{C}^{\perp}))^{\perp} \\ &\cong \mathscr{F}(\mathbf{A} \multimap (\mathbf{B} \multimap \mathbf{C}^{\perp})^{\perp\perp})^{\perp} \cong \mathscr{F}(\mathbf{A} \otimes (\mathbf{B} \otimes \mathbf{C})). \quad \square \end{aligned}$$

Proposition 2.3.14. FinProf *a $\star$-autonomous bicategory.*

Proof. The duality $(-)^{\perp} : \mathbf{A} \mapsto \mathbf{A}^{\perp} = (|\mathbf{A}|^{\mathrm{op}}, \mathscr{F}\mathbf{A}^{\perp})$ induces a full and faithful functor by Lemma 2.2.12. For finiteness structures $\mathbf{A} = (|\mathbf{A}|, \mathscr{F}\mathbf{A})$, $\mathbf{B} = (|\mathbf{B}|, \mathscr{F}\mathbf{B})$ and $\mathbf{C} = (|\mathbf{C}|, \mathscr{F}\mathbf{C})$, by Lemma 2.3.1, there is a pseudo-natural family of adjoint equivalences

$$\mathbf{FinProf}(\mathbf{A} \otimes \mathbf{B}, \mathbf{C}^{\perp}) \simeq \mathbf{FinProf}(\mathbf{A}, (\mathbf{B} \otimes \mathbf{C})^{\perp}). \qquad \square$$

The interpretation of the $\parr$ connective is defined by dualizing the tensor $\mathbf{A} \parr \mathbf{B} = (\mathbf{A}^{\perp} \otimes \mathbf{B}^{\perp})^{\perp}$. In the compact closed bicategory **Prof**, the two connectives have the same interpretation whereas in **FinProf**, adding the orthogonality eliminates this degeneracy. The inclusion $\mathscr{F}(\mathbf{A} \otimes \mathbf{B}) \hookrightarrow \mathscr{F}(\mathbf{A} \parr \mathbf{B})$ always hold which implies that we can interpret the mix rule in **FinProf**. It can be derived from the set inclusion

$$\{X \times Y \mid X \in \mathscr{F}\mathbf{A}^{\perp} \text{ and } Y \in \mathscr{F}\mathbf{B}^{\perp}\} \hookrightarrow \{X \times Y \mid X \in \mathscr{F}\mathbf{A} \text{ and } Y \in \mathscr{F}\mathbf{B}\}^{\perp}$$

and the fact that $\mathscr{F}(\mathbf{A} \parr \mathbf{B})$ has object set $\{X \times Y \mid X \in \mathscr{F}(\mathbf{A})^{\perp} \text{ and } Y \in \mathscr{F}(\mathbf{B})^{\perp}\}^{\perp}$.

The other inclusion does not hold in general: consider the presheaf $P : ((\mathcal{S}\,\mathbf{1})^{\mathrm{op}} \times \mathcal{S}\,\mathbf{1})^{\mathrm{op}} \to \mathbf{Set}$ given by $(n, m) \mapsto \mathcal{S}\,\mathbf{1}(m, n)$ corresponding to the identity profunctor $\mathcal{S}\,\mathbf{1} \nrightarrow_{\mathsf{f}} \mathcal{S}\,\mathbf{1}$. P is in $\mathscr{F}(!\,\mathbf{1} \multimap !\,\mathbf{1}) \cong \mathscr{F}((!\,\mathbf{1})^{\perp} \parr !\,\mathbf{1})$ but it is not in $\mathscr{F}((!\,\mathbf{1})^{\perp}\otimes !\,\mathbf{1})$. Indeed, let $Q : (\mathcal{S}\,\mathbf{1})^{\mathrm{op}}\times\mathcal{S}\,\mathbf{1}) \to \mathbf{Set}$ be dually given by $(n,m)\mapsto \mathcal{S}\,\mathbf{1}(n,m)$, it verifies that for all $X\in\mathscr{F}(!\,\mathbf{1})^{\perp}$ and $Y\in\mathscr{F}(!\,\mathbf{1})$,

$$\langle X\times Y \mid Q\rangle = \int^{n,m} X(n)\times Y(m)\times \mathcal{S}\,\mathbf{1}(n,m) \cong \langle X \mid Y\rangle \in \mathbf{FinSet}$$

which implies that Q is in $\mathscr{F}((!\,\mathbf{1})^{\perp}\otimes !\,\mathbf{1})^{\perp}$. However, the set $\langle P \mid Q\rangle = \int^{n,m}\mathcal{S}\,\mathbf{1}(m,n)\times\mathcal{S}\,\mathbf{1}(n,m)\cong\int^{n}\mathcal{S}\,\mathbf{1}(n,n)$ is not in $\mathbf{FinSet}$.

2.3.3 Exponential structure

We show in this section that the pseudo-comonad structure described in Section 1.4.5 can be refined to the setting of finiteness structures.

Definition 2.3.15. For a finiteness structure $\mathbf{A} = (|\mathbf{A}|, \mathscr{F}\mathbf{A})$, we define $!(\mathbf{A},\mathscr{F}\mathbf{A}) := (\mathcal{S}\,|\mathbf{A}|, \mathscr{F}!\mathbf{A})$ where $\mathscr{F}!\mathbf{A}$ is the full subcategory of $\widehat{\mathcal{S}\,|\mathbf{A}|}_{\mathrm{fin}}$ with object set $\{X^{\mathcal{S}} \mid X\in\mathscr{F}\mathbf{A}\}^{\perp\perp}$.

Recall that for a presheaf $X : |\mathbf{A}|^{\mathrm{op}}\to\mathbf{Set}$ (seen as a species $\mathcal{S}\,\mathbf{0}\nrightarrow|\mathbf{A}|$), its lifting $X^{\mathcal{S}} : (\mathcal{S}\mathbf{A})^{\mathrm{op}}\to\mathbf{Set}$ is given by

$$\langle a_1,\ldots,a_n\rangle\mapsto \mathcal{S}X\circ\mathrm{dig}_{\mathbf{0}}(\langle a_1,\ldots,a_n\rangle)\cong\prod_{i\in\underline{n}}X(a_i).$$

In particular, if X is a finite presheaf, then so is $X^{\mathcal{S}}$.

We first start by showing that a finiteness species $F : !\mathbf{A}\nrightarrow_{\mathsf{f}}\mathbf{B}$ can be characterized by its analytic functor $\widetilde{F} = \mathbf{Lan}_{s_{|\mathbf{A}|}}F : \widehat{|\mathbf{A}|}\to\widehat{|\mathbf{B}|}$ preserving the finiteness structure.

Lemma 2.3.16. *For finiteness structures $\mathbf{A}$ and $\mathbf{B}$, a species $F : |!\mathbf{A}|\nrightarrow_{\mathsf{f}}|\mathbf{B}|$ is in $\mathscr{F}(!\mathbf{A}\multimap\mathbf{B})$ (viewed as a finite presheaf $(|!\mathbf{A}|^{\mathrm{op}}\times|\mathbf{B}|)^{\mathrm{op}}\to\mathbf{Set}$) if and only if for all $X\in\mathscr{F}(\mathbf{A})$, $\widetilde{F}X$ is in $\mathscr{F}(\mathbf{B})$.*

Proof. Assume that F is in $\mathscr{F}(!\mathbf{A}\multimap\mathbf{B})$ and let X be in $\mathscr{F}(\mathbf{A})$. Since $X^{\mathcal{S}}$ is in $\mathscr{F}(!\mathbf{A})$, we have that $FX^{\mathcal{S}}\cong\widetilde{F}X\in\mathscr{F}(\mathbf{B})$.

For the other direction, it suffices to show that if for all $X\in\mathscr{F}(\mathbf{A})$, $FX^{\mathcal{S}}$ is in $\mathscr{F}(\mathbf{B})$, then $F^{\perp}(\mathscr{F}(\mathbf{B})^{\perp})\hookrightarrow(\mathscr{F}(!\mathbf{A})^{\perp})$. Let Y be in $(\mathscr{F}\mathbf{B})^{\perp}$ and $X\in\mathscr{F}\mathbf{A}$, since $\langle F^{\perp}Y \mid X^{\mathcal{S}}\rangle\cong\langle FX^{\mathcal{S}}\mid Y\rangle\in\mathbf{FinSet}$, we obtain the desired result. $\square$

We obtain as a corollary that for a finiteness structure $\mathbf{A} = (|\mathbf{A}|, \mathscr{F}\mathbf{A})$, $(\mathscr{F}!\mathbf{A})^{\perp}$ is isomorphic to the full subcategory of finite copresheaves $P : |!\mathbf{A}|\to\mathbf{Set}$ (or equivalently finite profunctors $|!\mathbf{A}|\nrightarrow_{\mathsf{f}}\mathbf{1}$) such that $\widetilde{P}(\mathscr{F}\mathbf{A})\hookrightarrow\mathbf{FinSet}$.

Example. In particular, $\mathscr{F}(!\,\mathbf{1})^\perp$ is isomorphic to species whose analytic functor maps finite sets to finite sets. In other words, $F : \mathcal{S}\,\mathbf{1} \to \mathbf{Set}$ must verify that for all $S \in \mathbf{FinSet}$, $\sum\limits_{n\in\mathbb{N}} F(n) \times_{\mathfrak{S}_n} S^n$ is finite.

Similarly to relational finiteness spaces, we can see here that the fixpoint operator cannot be interpreted in **FinProf**. Indeed, consider the species of binary trees $B : \mathcal{S}\,\mathbf{1} \nrightarrow \mathbf{1}$, it is a solution of the fixpoint equation $B = 1 + X \cdot B^2$ where $1 : \mathcal{S}\,\mathbf{1} \nrightarrow \mathbf{1}$ is the species $(u, \star) \mapsto \mathcal{S}\,\mathbf{1}(\langle\rangle, u)$ whose analytic functor $\mathbf{Set} \to \mathbf{Set}$ is the constant $S \mapsto \{\star\}$ and $X : \mathcal{S}\,\mathbf{1} \nrightarrow \mathbf{1}$ is the species $(u, \star) \mapsto \mathcal{S}\,\mathbf{1}(\langle\star\rangle, u)$ whose analytic functor $\mathbf{Set} \to \mathbf{Set}$ is the identity $S \mapsto S$ (see [12] for more details). Both 1 and X are finiteness species since their analytic functors restrict to $\mathbf{FinSet} \to \mathbf{FinSet}$. The species of binary trees however has analytic functor $\mathbf{Set} \to \mathbf{Set}$ given by $S \mapsto \sum_{n\in\mathbb{N}} C_n \times S^n$ where C_n is the nth Catalan number so this functor can not be restricted as a functor $\mathbf{FinSet} \to \mathbf{FinSet}$.

Lemma 2.3.17. *For finiteness structures* $\mathbf{A}$ *and* $\mathbf{B}$*, if* $F : \mathbf{A} \nrightarrow_{\mathsf{f}} \mathbf{B}$ *is a finiteness profunctor, then* $\mathcal{S}F : |!\mathbf{A}| \nrightarrow |!\mathbf{B}|$ *is in* $\mathbf{FinProf}(!\mathbf{A}, !\mathbf{B})$*.*

Proof. We show that $(\mathcal{S}F)(\mathscr{F}!\mathbf{B}^\perp) \hookrightarrow \mathscr{F}!\mathbf{A}^\perp$. Let P be in $\mathscr{F}!\mathbf{B}^\perp$, i.e. for all Y in $\mathscr{F}\mathbf{B}$, $\widetilde{P}Y$ is in **FinSet**.

$$\begin{aligned}(\mathcal{S}F)(P) \in \mathscr{F}!\mathbf{A}^\perp &\Leftrightarrow \forall X \in \mathscr{F}\mathbf{A}, \int^{\substack{u\in|!\mathbf{A}|,\\ v\in|!\mathbf{B}|}} \mathcal{S}F(u,v) \times P(v) \times X^{\mathcal{S}}(u) \in \mathbf{FinSet}\\ &\Leftrightarrow \forall X \in \mathscr{F}\mathbf{A}, \int^{v\in|!\mathbf{B}|} P(v) \times (\mathcal{S}F \circ X^{\mathcal{S}})(v) \in \mathbf{FinSet}\\ &\Leftrightarrow \forall X \in \mathscr{F}\mathbf{A}, \int^{v\in|!\mathbf{B}|} P(v) \times (F \circ X)^{\mathcal{S}}(v) \in \mathbf{FinSet}\end{aligned}$$

Since FX is in $\mathscr{F}\mathbf{B}$, $(FX)^{\mathcal{S}} \in \mathscr{F}!\mathbf{B}$ which implies the desired result. □

We now show that the pseudo-comonad structure in **Prof** can be refined to **FinProf**.

Lemma 2.3.18. *For a finiteness structure* $\mathbf{A} = (|\mathbf{A}|, \mathscr{F}\mathbf{A})$*, the component of the counit pseudo-natural transformation* $\mathrm{der}_{|\mathbf{A}|} : |!\mathbf{A}| \nrightarrow |\mathbf{A}|$ *is in* $\mathbf{FinProf}(!\mathbf{A}, \mathbf{A})$*.*

Proof. Since $|!\mathbf{A}|$ is locally finite, $\mathrm{der}_{|\mathbf{A}|}$ is a finite profunctor. By Lemma 2.2.12, it remains to show that $\mathrm{der}_{|\mathbf{A}|}^\perp((\mathscr{F}\mathbf{A})^\perp) \hookrightarrow (\mathscr{F}!\mathbf{A})^\perp$ i.e. that for all

$X' \in (\mathscr{F}\mathbf{A})^{\perp}$ and $X \in \mathscr{F}\mathbf{A}$, $(\mathrm{der}^{\perp}_{|\mathbf{A}|})X' \perp X^{\mathcal{S}}$.

$$\begin{aligned}
\langle (\mathrm{der}^{\perp}_{|\mathbf{A}|})X' \mid X^{\mathcal{S}} \rangle &= \int^{u \in |!\mathbf{A}|} X^{\mathcal{S}}(u) \times \int^{a \in |\mathbf{A}|} |!\mathbf{A}| \left(\langle a \rangle \mid u\right) \times X'(a) \\
&\cong \int^{a,a' \in |\mathbf{A}|} X(a') \times |\mathbf{A}| \left(a, a'\right) \times X'(a) \\
&\cong \int^{a \in |\mathbf{A}|} X(a) \times X'(a) \in \mathbf{FinSet} \qquad \square
\end{aligned}$$

Lemma 2.3.19. *For a small category* $\mathbb{A}$ *and a presheaf* $X : \mathbb{A}^{\mathrm{op}} \to \mathbf{Set}$*,* $\mathrm{dig}_{\mathbb{A}} \circ X^{\mathcal{S}} \cong X^{\mathcal{S}\mathcal{S}}$

Proof. We have:

$$\begin{aligned}
\mathrm{dig}_{\mathbb{A}} \circ X^{\mathcal{S}} &= \mathrm{dig}_{\mathbb{A}} \circ \mathcal{S}X \circ \mathrm{dig}_{\mathbf{0}} \cong \mathcal{S}\mathcal{S}X \circ \mathrm{dig}_{\mathcal{S}\,\mathbf{0}} \circ \mathrm{dig}_{\mathbf{0}} \\
&\cong \mathcal{S}\mathcal{S}X \circ \mathcal{S}\mathrm{dig}_{\mathbf{0}} \circ \mathrm{dig}_{\mathbf{0}} \cong X^{\mathcal{S}\mathcal{S}}
\end{aligned}$$

the first isomorphism follows from the pseudo-naturality of dig and the last from the pseudo-comonad axioms. □

Lemma 2.3.20. *For a finiteness structure* $\mathbf{A} = (|\mathbf{A}|, \mathscr{F}\mathbf{A})$*, the component of the comultiplication pseudo-natural transformation* $\mathrm{dig}_{|\mathbf{A}|} : |!\mathbf{A}| \nrightarrow |!!\mathbf{A}|$ *is in* $\mathbf{FinProf}(!\mathbf{A}, !!\mathbf{A})$*.*

Proof. Since $|!\mathbf{A}|$ is locally finite, $\mathrm{dig}_{|\mathbf{A}|}$ is a finite profunctor. We show that $(\mathrm{dig}^{\perp}_{|\mathbf{A}|})(\mathscr{F}!!\mathbf{A})^{\perp} \hookrightarrow (\mathscr{F}!\mathbf{A})^{\perp}$. For a presheaf X in $\mathscr{F}\mathbf{A}$ considered as a species $!\,\mathbf{0} \nrightarrow |\mathbf{A}|$, we have $\mathrm{dig}_{|\mathbf{A}|} \circ X^{\mathcal{S}} \cong X^{\mathcal{S}\mathcal{S}}$ by Lemma 2.3.19. Hence, for W in $\mathscr{F}!!\mathbf{A}^{\perp}$ and X in $\mathscr{F}\mathbf{A}$, we have

$$\langle (\mathrm{dig}^{\perp}_{|\mathbf{A}|})W \mid X^{\mathcal{S}} \rangle \cong \langle W \mid \mathrm{dig}_{|\mathbf{A}|} X^{\mathcal{S}} \rangle \cong \langle W \mid X^{\mathcal{S}\mathcal{S}} \rangle.$$

Since $X^{\mathcal{S}\mathcal{S}}$ is in $\mathscr{F}!!\mathbf{A}$, we obtain the desired result. □

2.3.4 Cartesian closed structure

We show in this section that the cartesian closed structure of $\mathbf{Prof}_!$ can be extended to $\mathbf{FinProf}$. For finiteness structures $\mathbf{A}$ and $\mathbf{B}$, the exponential object $\mathbf{A} \Rightarrow \mathbf{B}$ is given by $!\mathbf{A} \multimap \mathbf{B}$. We first show that the Seely adjoint equivalence in $\mathbf{Prof}$ lifts to $\mathbf{FinProf}$.

Lemma 2.3.21. *For finiteness structures* $\mathbf{A}$ *and* $\mathbf{B}$*, the Seely profunctors* $S_{|\mathbf{A}|,|\mathbf{B}|} : \mathcal{S}(|\mathbf{A}| \,\&\, |\mathbf{B}|) \nrightarrow \mathcal{S}\,|\mathbf{A}| \otimes \mathcal{S}\,|\mathbf{B}|$ *and* $I_{|\mathbf{A}|,|\mathbf{B}|} : \mathcal{S}\,|\mathbf{A}| \otimes \mathcal{S}\,|\mathbf{B}| \nrightarrow \mathcal{S}(|\mathbf{A}| \,\&\, |\mathbf{B}|)$ *induce an adjoint equivalence* $!(\mathbf{A} \,\&\, \mathbf{B}) \simeq !\mathbf{A} \otimes !\mathbf{B}$ *in* $\mathbf{FinProf}$*.*

Proof.

- We first show that $S_{|\mathbf{A}|,|\mathbf{B}|} : \mathcal{S}(|\mathbf{A}| \mathbin{\&} |\mathbf{B}|) \nrightarrow \mathcal{S}\,|\mathbf{A}| \otimes \mathcal{S}\,|\mathbf{B}|$ given by

  $$(w,(u,v)) \mapsto \mathcal{S}\,|\mathbf{A}|\,(u,\pi_1 w) \times \mathcal{S}\,|\mathbf{B}|\,(v,\pi_2 w)$$

  is in $\mathbf{FinProf}(!(\mathbf{A}\&\mathbf{B}), !\mathbf{A}\otimes!\mathbf{B})$ i.e. $(S^{\perp}_{|\mathbf{A}|,|\mathbf{B}|})\mathscr{F}(!\mathbf{A}\otimes!\mathbf{B})^{\perp} \hookrightarrow (\mathscr{F}!(\mathbf{A}\&\mathbf{B}))^{\perp}$.

  Let T be in $\mathscr{F}(!\mathbf{A}\otimes!\mathbf{B})^{\perp}$, we want to show that for all $W=(W_1,W_2) \in \mathscr{F}(\mathbf{A} \mathbin{\&} \mathbf{B})$, $\langle S^{\perp}_{|\mathbf{A}|||\mathbf{B}|}(T), W^{\mathcal{S}}\rangle \in \mathbf{FinSet}$. The set $\langle S^{\perp}_{|\mathbf{A}|||\mathbf{B}|}(T), W^{\mathcal{S}}\rangle$ is isomorphic to:

  $$\begin{aligned}&\int^{w\in\mathcal{S}(|\mathbf{A}|\&|\mathbf{B}|)} W^{\mathcal{S}}(w) \times \int^{\substack{u\in\mathcal{S}|\mathbf{A}|,\\ v\in\mathcal{S}|\mathbf{B}|}} \mathcal{S}\,|\mathbf{A}|\,(u,\pi_1 w) \times \mathcal{S}\,|\mathbf{B}|\,(v,\pi_2 w) \times T(u,v)\\ &\cong \int^{u\in\mathcal{S}|\mathbf{A}|,v\in\mathcal{S}|\mathbf{B}|} W_1^{\mathcal{S}}(u) \times W_2^{\mathcal{S}}(v) \times T(u,v)\end{aligned}$$

  Since W is in $\mathscr{F}(\mathbf{A} \mathbin{\&} \mathbf{B})$, W_1 and W_2 are in $\mathscr{F}(\mathbf{A})$ and $\mathscr{F}(\mathbf{B})$ respectively, so that $W_1^{\mathcal{S}}$ and $W_2^{\mathcal{S}}$ are in $\mathscr{F}(!\mathbf{A})$ and $\mathscr{F}(!\mathbf{B})$ respectively. Hence, $T \perp W_1^{\mathcal{S}} \times W_2^{\mathcal{S}}$ as desired.

- We show that $I_{|\mathbf{A}|,|\mathbf{B}|} : \mathcal{S}\,|\mathbf{A}|\otimes\mathcal{S}\,|\mathbf{B}| \nrightarrow \mathcal{S}(|\mathbf{A}|\&|\mathbf{B}|)$ given by $((u,v),w) \mapsto \mathcal{S}\,|\mathbf{A}|\,(\pi_1 w,u) \times \mathcal{S}\,|\mathbf{B}|\,(\pi_2 w,v)$ is in $\mathscr{F}((!\mathbf{A} \otimes !\mathbf{B}) \multimap !(\mathbf{A} \mathbin{\&} \mathbf{B}))$. By Lemma 2.3.12, $\mathscr{F}((!\mathbf{A}\otimes!\mathbf{B}) \multimap !(\mathbf{A}\&\mathbf{B})) \cong \mathscr{F}(!\mathbf{A} \multimap (!\mathbf{B} \multimap !(\mathbf{A}\&\mathbf{B}))$ and using Lemma 2.3.10 twice, it suffices to show that for all $X \in \mathscr{F}\mathbf{A}$ and $Y \in \mathscr{F}\mathbf{B}$, $(I_{|\mathbf{A}|,|\mathbf{B}|}X^{\mathcal{S}})Y^{\mathcal{S}}$ is in $\mathscr{F}!(\mathbf{A}\&\mathbf{B})$. Let Z be $\mathscr{F}(!(\mathbf{A}\&\mathbf{B}))^{\perp}$, the set $\langle (I_{\mathbf{A},\mathbf{B}}X^{\mathcal{S}})Y^{\mathcal{S}} \mid Z\rangle$ is isomorphic to:

  $$\begin{aligned}&\int^{\substack{w\in|!(\mathbf{A}\&\mathbf{B})|,\\ u\in|!\mathbf{A}|,v\in|!\mathbf{B}|}} Z(w) \times |!\mathbf{A}|(\pi_1 w,u) \times |!\mathbf{B}|(\pi_2 w,v) \times X^{\mathcal{S}}(u) \times Y^{\mathcal{S}}(v)\\ &\cong \int^{w\in!(\mathbf{A}\&\mathbf{B})} Z(w) \times (X,Y)^{\mathcal{S}}(w)\end{aligned}$$

  Since $(X,Y)^{\mathcal{S}}$ is in $\mathscr{F}(!(\mathbf{A} \mathbin{\&} \mathbf{B}))$, we obtain the desired result. □

It remains to show that the non-linear evaluation and currying preserve the finiteness structure. The non-linear evaluation $\mathrm{Ev}_{|\mathbf{A}|,|\mathbf{B}|} : \mathcal{S}((|\mathbf{A}| \Rightarrow |\mathbf{B}|) \mathbin{\&} |\mathbf{A}|) \nrightarrow |\mathbf{B}|$ is given by the composite

$$\mathrm{ev}_{\mathcal{S}|\mathbf{A}|,|\mathbf{B}|} \circ (\mathrm{der}_{|\mathbf{A}|\Rightarrow|\mathbf{B}|} \otimes \mathrm{id}) \circ S_{|\mathbf{A}|\Rightarrow|\mathbf{B}|,|\mathbf{A}|}$$

where $\mathrm{ev}_{|\mathbf{A}|,|\mathbf{B}|} : \mathbf{A} \otimes (\mathbf{A} \multimap \mathbf{B}) \nrightarrow \mathbf{B}$ is the linear evaluation coming from the monoidal closed structure in the linear bicategory **FinProf**. As a composite of finiteness profunctors, $\mathrm{Ev}_{|\mathbf{A}|,|\mathbf{B}|}$ is in $\mathbf{FinProf}_{!}((\mathbf{A} \Rightarrow \mathbf{B}) \mathbin{\&} \mathbf{A}, \mathbf{B})$. For a finiteness species P in $\mathbf{FinProf}_{!}(\mathbf{A} \mathbin{\&} \mathbf{B}, \mathbf{C})$, its *currying* $\Lambda(P) \in \mathbf{FinProf}_{!}(\mathbf{A}, \mathbf{B} \Rightarrow \mathbf{C})$ is given by $\lambda(P \circ I_{|\mathbf{A}|,|\mathbf{B}|})$ where

$$\lambda : \mathbf{FinProf}(!\mathbf{A} \otimes !\mathbf{B}, \mathbf{C}) \to \mathbf{FinProf}(!\mathbf{A}, !\mathbf{B} \multimap \mathbf{C})$$

is provided by the monoidal closed structure on **FinProf**.

Theorem 2.3.22. $\mathbf{FinProf}_{!}$ *is cartesian closed.*

Proof. Direct consequence of the remarks above and Lemma 2.3.1. ☐

2.3.5 Differential structure

We show in this section that the differential structure in the bicategory of generalized species extends to **FinProf**. It suffices to show that the codereliction, coweakening and cocontraction pseudo-natural transformations have components in **FinProf** and all the coherence axioms will be immediately verified.

Lemma 2.3.23. *For a finiteness structure* $\mathbf{A} = (|\mathbf{A}|, \mathscr{F}\mathbf{A})$*, the component of codereliction pseudo-natural transformation* $\overline{\mathrm{der}}_{|\mathbf{A}|} : |\mathbf{A}| \nrightarrow \mathcal{S}|\mathbf{A}|$ *given by* $(a, u) \mapsto \mathcal{S}|\mathbf{A}|(u, \langle a \rangle)$ *is a finiteness profunctor* $\mathbf{A} \nrightarrow_{\mathsf{f}} !\mathbf{A}$.

Proof. Since $|\mathbf{A}|$ is locally finite, $\overline{\mathrm{der}}_{|\mathbf{A}|}$ is a finite profunctor. By Lemma 2.2.12, it remains to show that $\overline{\mathrm{der}}^{\perp}_{|\mathbf{A}|}((\mathscr{F}!\mathbf{A})^{\perp}) \hookrightarrow (\mathscr{F}\mathbf{A})^{\perp}$ i.e. that for all $Z \in (\mathscr{F}!\mathbf{A})^{\perp}$ and $X \in \mathscr{F}\mathbf{A}$, $(\overline{\mathrm{der}}^{\perp}_{|\mathbf{A}|})Z \perp X$.

$$\begin{aligned}\langle (\overline{\mathrm{der}}^{\perp}_{|\mathbf{A}|})Z \mid X \rangle &= \int^{u \in |!\mathbf{A}|, a \in |\mathbf{A}|} Z(u) \times |!\mathbf{A}|(u, \langle a \rangle) \times X(a) \\ &\cong \int^{a \in |\mathbf{A}|} Z(\langle a \rangle) \times X(a) \hookrightarrow \int^{u \in |!\mathbf{A}|} Z(u) \times X^{\mathcal{S}}(u) \in \mathbf{FinSet}\end{aligned}$$

The last inclusion follows from the isomorphism $X^{\mathcal{S}}(\langle a \rangle) \cong X(a)$. ☐

Since the components of the coweakening $\overline{\mathrm{w}}_{|\mathbf{A}|} : \mathbf{1} \nrightarrow \mathcal{S}|\mathbf{A}|$ and cocontraction $\overline{\mathrm{c}}_{|\mathbf{A}|} : \mathcal{S}|\mathbf{A}| \times \mathcal{S}|\mathbf{A}| \nrightarrow \mathcal{S}|\mathbf{A}|$ pseudo-natural transformations are obtained from the Seely equivalences and the biproduct structure, it is immediate that they can be extended to **FinProf**. It implies that the deriving pseudo-natural transformation $\delta_{|\mathbf{A}|} : |\mathbf{A}| \nrightarrow \mathcal{S}|\mathbf{A}| \times |\mathbf{A}|$ given by

$$\mathcal{S}\,|\mathbf{A}| \times |\mathbf{A}| \xrightarrow{\mathrm{id} \times \overline{\mathrm{der}}_{|\mathbf{A}|}} \mathcal{S}\,|\mathbf{A}| \times \mathcal{S}\,|\mathbf{A}| \xrightarrow{\overline{\mathrm{c}}_{|\mathbf{A}|}} \mathcal{S}\,|\mathbf{A}|$$

is therefore a finiteness profunctor $!\mathbf{A} \otimes \mathbf{A} \nrightarrow_{\mathsf{f}} !\mathbf{A}$ so that for a finiteness species $F : !\mathbf{A} \nrightarrow \mathbf{B}$ its differential $F \circ \delta_{|\mathbf{A}|} : !\mathbf{A} \otimes \mathbf{A} \nrightarrow_{\mathsf{f}} \mathbf{B}$ is also a finiteness species.

Conclusion and perspectives

We have constructed a new bicategorical model of differential linear logic categorifying the finiteness model first introduced by Ehrhard [32]. The resulting cartesian closed bicategory refines the model of generalized species by Fiore et al. [41]. The objects are endowed with an additional structure which enables to enforce finite computations as morphisms are species that preserve the finiteness structure.

In future work, we aim to prove that our construction can be generalized to the setting of enriched species studied by Gambino and Joyal [48]. In the 1-categorical model of finiteness spaces, we can express various forms of non-determinism depending on the semi-ring of scalars chosen for the series coefficients. In our case, the analogous variation would come from changing the enrichment basis. In particular, for species enriched over vector spaces, our construction will guarantee that computations are always finite dimensional even if we work in an infinite dimensional setting which could lead to interesting applications for the semantics of quantum λ-calculus [94] and stochastic rewriting systems [8].

Chapter 3

Stable Species of Structures

This chapter is based on joint work with Marcelo Fiore and Hugo Paquet.

Stable functions were introduced by Berry in his study of the full abstraction problem for sequential programs [13]. Sequential functions do not form a cartesian closed category so they were approximated by stable functions to obtain cartesian closure. It required to introduce a different ordering between stable functions refining the usual pointwise ordering for Scott-continuous functions. Stability was also independently studied by Girard in the theory of dilators [51] and later in the model of coherence spaces [52]. In more recent developments, stable functions were used in probabilistic semantics where a stability condition was added to morphisms between measurable cones in order to obtain cartesian closure [36].

Girard also studied a categorification of stability with normal functors (set-valued formal power series) [54], later on considered by Hasegawa [61]. Normal functors also fit within the general categorical theory of polynomials [49]. The connection between Girard's normal functors [54] and Berry's stable domain theory [13] was developed by Lamarche [81] and Taylor [110]. Taylor generalized the notion of normal functors between sets to stable functors between groupoids equipped with an additional structure called *creeds* to model intuitionistic linear logic.

While the notion of stability led to many models of linear logic in the setting of 1-categories (dI-domains, coherence spaces, probabilistic stable functions), there is no satisfactory 2-dimensional model of stability. The normal functors of Girard do not constitute a cartesian closed structure (unless the 2-cells are quotiented) and Taylor's creeds do not model classical negation. In this chapter, we show how adding an orthogonality relation on the model of profunctors allows us to solve this issue by restricting the analytic functors induced by species to stable functors.

We work with the bicategory of profunctors restricted to groupoids as the biequivalence between species and analytic functors only holds in this setting [39]. In the first part, we start by recalling the basic properties of stable functions and their analogues for stable functors. In the second part, we introduce an orthogonality on families of subgroups of endomorphisms in a groupoid. The orthogonality induces boolean algebras called *kits* and we show that the bicategory of kits and stabilized profunctors between them are a model of differential linear logic. We then show in the third part that this orthogonality can be translated to an orthogonality on presheaves which allows us to prove that stabilized profunctors and species can be characterized extensionally as linear and stable functors respectively between restricted categories of presheaves closed under double-orthogonality.

3.1 From stable functions to stable functors

Definition 3.1.1 (Berry 1978)**.** For cpos $(A, \leq_A)$ and $(B, \leq_B)$, a Scott-continuous function $f : A \to B$ is *stable* if for all $b \leq_B f(a)$, there exists $a_0 \in A$ such that:

- $b \leq_B f(a_0)$ and $a_0 \leq_A a$;
- for all $a' \leq_A a$, if $b \leq_B f(a')$ then $a_0 \leq_A a'$.

The stability condition corresponds to the operational property that for a given output b for an input a, there exists a unique minimal part a_0 of the input a necessary to compute b. We denote this minimal element a_0 by $m(f, a, b)$.

Example. The canonical non-stable function is the parallel or $\mathbf{Bool} \times \mathbf{Bool} \to \mathbf{Bool}$ given by

$$\begin{aligned}(\mathbf{true}, \bot) &\mapsto \mathbf{true}\\ (\bot, \mathbf{true}) &\mapsto \mathbf{true}\\ (\mathbf{false}, \mathbf{false}) &\mapsto \mathbf{false}\end{aligned}$$

with the remaining outputs being determined by the monotonicity of the function. The parallel or function is not stable since for the output **true** from the input $(\mathbf{true}, \mathbf{true})$, it is not possible to determine whether the left-hand side or the right-hand side of the input was used to compute the output.

The elements $m(f,a,b)$ determine a family of *local left adjoints* to the restriction $f : A/a \to B/f(a)$ between the slices $A/a = \{a' \in A \mid a' \leq_A a\}$ and $B/f(a) = \{b \in B \mid b \leq_B f(a)\}$:

$$\begin{tikzcd}
A/a \arrow[r, bend left, "f"] \arrow[r, phantom, "\top"] & B/f(a) \arrow[l, bend left, "{m(f,a,-)}"]
\end{tikzcd}$$

and having local left adjoints is equivalent to stability.

Lemma 3.1.2. *For cpos $(A, \leq_A)$ and $(B, \leq_B)$, a Scott-continuous function $f : A \to B$ is stable if and only if the restriction $f : A/a \to B/f(a)$ has a left adjoint for all $a \in A$.*

This viewpoint has the advantage of having an immediate generalization from functions between preorders to functors between general categories.

Definition 3.1.3. A functor $T : \mathbb{A} \to \mathbb{B}$ is said to be a *local right (respectively left) adjoint* if for every object $a \in \mathbb{A}$, the induced functor

$$T_a : \mathbb{A}/_a \to \mathbb{B}/_{T(a)}$$

is a right (respectively left) adjoint. Note that a right (respectively left) adjoint is automatically a local right (respectively left) adjoint.

The theory of local right adjoint functors was developed by Diers, Street [30, 108], Carboni and Johnstone [21] and Weber [115, 116]. They also arise in the theory of polynomial functors developed by Gambino and Kock [49, 75] and they are also related to normal functors investigated by Girard and Hasegawa which are finitary local right adjoint functors [54, 61]. Their relevance to stable domain theory was considered by Lamarche [81] and Taylor [110]. Further investigation in the context of the theory of spectra can be found in recent work by Osmond [93].

The categorification of the minimal part of the input for a stable function corresponds to the notion of generic morphism:

Definition 3.1.4. For a functor $T : \mathbb{A} \to \mathbb{B}$, a morphism $g : b \to T(a_0)$ in $\mathbb{B}$ is said to be *generic* if for every commutative square

$$\begin{tikzcd}
b \arrow[r, "f"] \arrow[d, "g"'] & T(a') \arrow[d, "T(k)"] \\
T(a_0) \arrow[r, "T(h)"'] & T(a)
\end{tikzcd}$$

there exists a unique morphism $l : a_0 \to a'$ such that

$$\begin{tikzcd}
& a' \arrow[d, "k"] \\
a_0 \arrow[ur, "l"] \arrow[r, "h"'] & a
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
b \arrow[r, "f"] \arrow[d, "g"'] & T(a') \\
T(a_0) \arrow[ur, "T(l)"'] &
\end{tikzcd}$$

Generic morphisms correspond to Taylor's notion of *candidates* [110]. The terminology generic morphisms is sometimes used for morphisms where the uniqueness of l in Definition 3.1.4 is not required and the terminology *strict generic* is used when l is unique [115].

Definition 3.1.5. A functor $T : \mathbb{A} \to \mathbb{B}$ is said to *admit generic factorizations relative to an object* $b \in \mathbb{B}$ if every morphism $b \to T(a)$ in $\mathbb{A}$ can be factored as:

$$\begin{tikzcd}
b \arrow[r, "g"] & T(a_0) \arrow[r, "T(h)"] & T(a)
\end{tikzcd}$$

with g in $\mathbb{B}$ generic and h in $\mathbb{A}$. A functor $T : \mathbb{A} \to \mathbb{B}$ *admits generic factorizations* if it admits generic factorizations relative to all objects in $b \in \mathbb{B}$.

The correspondence between stable functions and Scott-continuous functions that are local right adjoints becomes an equivalence between functors admitting generic factorizations and local right adjoints which we recall here. Most of the proofs can be found in the literature and we do not claim any novelty in this section. The reader may skip them and go directly to Section 3.2.

Definition 3.1.6 (Definition 6.4 in[115])**.** For a functor $T : \mathbb{A} \to \mathbb{B}$ and a morphism $f : b \to T(a)$ in $\mathbb{B}$, define the *category of factorizations of f*, denoted by $\mathbf{Fact}(f)$ as follows:

1. objects: pairs $g : b \to T(c)$ in $\mathbb{B}$ and $h : c \to a$ in $\mathbb{A}$ such that $T(h) \circ g = f$;

2. morphisms: a morphism from $(g : b \to T(c), h : c \to a)$ to $(g' : b \to T(c'), h' : c' \to a)$ is a morphism $k : c \to c'$ in $\mathbb{A}$ such that $h' \circ k = h$ and $T(k) \circ g = g'$.

$$\begin{tikzcd}
b \arrow[r, "g'"] \arrow[d, "g"'] & T(c') \arrow[d, "T(h')"] \\
T(c) \arrow[ur, "T(k)"] \arrow[r, "T(h)"'] & T(a)
\end{tikzcd}$$

Remark 4. Note that T has generic factorizations if and only if the category $\mathbf{Fact}(f)$ has an initial object for every $f : b \to T(a)$.

Lemma 3.1.7. *Let $T : \mathbb{A} \to \mathbb{B}$ be a functor. For $g : b \to T(a)$ in $\mathbb{B}$ and $f : a \to a'$, $f' : a' \to a''$ in $\mathbb{A}$, the following hold:*

1. *f is a morphism $(g, f'f) \to (T(f) \circ g, f')$ in $\mathbf{Fact}(T(f'f) \circ g)$,*

2. *if $(g, f'f)$ is initial in $\mathbf{Fact}(T(f'f) \circ g)$ then (g, f) is initial in $\mathbf{Fact}(T(f) \circ g)$,*

3. *if (g, f) is initial in $\mathbf{Fact}(T(f) \circ g)$ then, provided that $\mathbf{Fact}(T(f'f) \circ g)$ has initial object, this is given by $(g, f'f)$.*

Proof.

1. Immediate.

2. For $(k : b \to T(c), h : c \to a) \in \mathbf{Fact}(T(f)g)$, we show that $e : a \to c$ in $\mathbb{A}$ is a morphism $(g, f) \to (k, h)$ in $\mathbf{Fact}(T(f)g)$ if and only if it is a morphism $(g, f'f) \to (k, f'h)$ in $\mathbf{Fact}(T(f'f)g)$. The (only if) direction is clear. For the (if) direction, note that f is a morphism $(g, f'f) \to (T(f)g, f')$ and k is a morphism $(k, f'h) \to (T(h)k, f') = T(f)g, f')$. Hence, if e is a morphism $(g, f'f) \to (k, f'h)$ then, by initiality, $f = he$ and
$$(g, f) = (g, he) \xrightarrow{e} (T(e)g, h) = (k, h) \ .$$

3. Let (g, f) be initial in $\mathbf{Fact}(T(f)g)$ and (k, h) be initial in $\mathbf{Fact}(T(f'f)g)$. There exists a unique $l : (k, h) \to (g, f'f)$ in $\mathbf{Fact}(T(f'f)g)$ and, since $T(fl)k = T(f)g$, also a unique $q : (g, f) \to (k, fl)$ in $\mathbf{Fact}(T(f)g)$. Then, $q : (g, f'f) \to (k, f'fl) = (k, h)$ and it follows that ql is an endomorphism on (k, h) and therefore the identity. Moreover, $l : (k, fl) \to (T(l)k, f) = (g, f)$ so that lq is an endomorphism on (g, f) and therefore the identity. Thus, (k, h) and $(g, f'f)$ are isomorphic and we are done.

□

Lemma 3.1.8. *A functor $T : \mathbb{A} \to \mathbb{B}$ is a local right adjoint if and only if for every $f : b \to T(a) \in \mathbb{B}$, $\mathbf{Fact}(f)$ has an initial object.*

Proof.

- $(\Leftarrow)$ Let $f : b \to T(a) \in \mathbb{B}$ and let $(g : b \to T(c), h : c \to a)$ be the initial object of $\mathbf{Fact}(f)$. We want to show that g is generic i.e. that for any commutative square in $\mathbb{B}$ as below, (g, k) is initial in $\mathbf{Fact}(T(k) \circ g)$.

$$\begin{tikzcd} b \arrow[r, "g'"] \arrow[d, "g"'] & T(d) \arrow[d, "T(k')"] \\ T(c) \arrow[r, "T(k)"'] & T(a') \end{tikzcd}$$

  We have by Lemma 3.1.7.2 that (g, id_c) is initial in $\mathbf{Fact}(g)$ which implies by Lemma 3.1.7.3 that (g, k) is initial in $\mathbf{Fact}(T(k) \circ g)$ as desired.

- $(\Rightarrow)$ Straightforward.

□

To construct the local left adjoints, we make use of the characterization of adjunctions in terms of universal arrows:

Definition 3.1.9. Let $T : \mathbb{A} \to \mathbb{B}$ be a functor and b be an object of $\mathbb{A}$. A *universal arrow from b to T* consists of:

1. an object $L(b)$ in $\mathbb{A}$,
2. a morphism $\eta_b : b \to T(L(b))$ in $\mathbb{B}$

such that for every a in $\mathbb{A}$ and $f : b \to T(a)$ in $\mathbb{B}$, there exists a unique morphism $h : L(b) \to a$ such that $T(h) \circ \eta_b = f$.

Theorem 3.1.10. *Let $T : \mathbb{A} \to \mathbb{B}$ be a functor, then the following are equivalent:*

1. *T has a left adjoint functor;*
2. *for every $b \in \mathbb{B}$, there exists a universal arrow $\eta_b : b \to T(L(b))$.*

Lemma 3.1.11. *Let $T : \mathbb{A} \to \mathbb{B}$ be a functor. For a an object of $\mathbb{A}$, the induced functor $T_a : \mathbb{A}/_a \to \mathbb{B}/_{T(a)}$ is a right adjoint if and only if, for every f in $\mathbb{B}/_{T(a)}$, the category* **Fact**(f) *has initial object.*

Proof.

- ($\Rightarrow$) Assume that T_a has a left adjoint $L_a : \mathbb{B}/_{T(a)} \to \mathbb{A}/_a$ and let $f : b \to T(a)$ be in $\mathbb{B}/_{T(a)}$. Define h to be $L_a(f) : c \to a$ in $\mathbb{A}/_a$. The counit of the adjunction $\eta_f : f \to T_a(h)$ in $\mathbb{B}/_{T(a)}$ corresponds to a morphism $\eta_f : b \to T(c)$ in $\mathbb{B}$ verifying $T(h) \circ \eta_f = f$ which implies that (η_f, h) is in $\mathbf{Fact}(f)$. Since η_f is a universal arrow, it entails by Theorem 3.1.10 that (η_f, h) is an initial object in $\mathbf{Fact}(f)$.
- ($\Leftarrow$) Assume that for every f in $\mathbb{B}/_{T(a)}$, the category $\mathbf{Fact}(f)$ has initial object. Let $f : b \to T(a)$ be an object in $\mathbb{B}/_{T(a)}$ and $(g : b \to T(c), h : c \to a)$ be the initial object of $\mathbf{Fact}(f)$. One can easily check that $g : f \to T_a(h)$ is a universal arrow for f in $\mathbb{B}/_{T(a)}$ which implies that T_a has a left adjoint by Theorem 3.1.10.

□

Theorem 3.1.12. *A functor $T : \mathbb{A} \to \mathbb{B}$ admits generic factorizations if and only if it is a local right adjoint.*

Proof. Direct consequence of Lemmas 3.1.8 and 3.1.11. □

Local right adjoints have the property of preserving wide pullbacks which corresponds to the preservation of bounded meets for stable functions.

Theorem 3.1.13. *For meet cpos $(A, \leq_A)$ and $(B, \leq_B)$, and a monotonous function $f : A \to B$, if f is stable, then it preserves bounded meets:*

$$\forall x, y \in A, x \uparrow y \Rightarrow (f(x \wedge y) = f(x) \wedge f(y))$$

Proposition 3.1.14. *Assume that $\mathbb{A}$ has wide pullbacks, if $T : \mathbb{A} \to \mathbb{B}$ is a local right adjoint then T preserves wide pullbacks.*

We recall below the Berry ordering between stable functions which allowed to obtain cartesian closure for the category of stable functions and dI-domains.

Definition 3.1.15. For meet cpos $(A, \leq_A)$ and $(B, \leq_B)$ and two stable functions $f, g : A \to B$, define $f \leq_{\mathrm{st}} g$ as

$$\forall a, a' \in A, a \leq_A a' \Rightarrow (f(a) = f(a') \wedge g(a))$$

The equality $f(a) = f(a') \wedge g(a)$ tells us that the following square

$$\begin{tikzcd}
f(a) \arrow[r, "\leq"] \arrow[d, "\leq"'] \arrow[dr, phantom, "\lrcorner", very near start] & g(a) \arrow[d, "\leq"] \\
f(a') \arrow[r, "\leq"'] & g(a')
\end{tikzcd}$$

is a pullback in the preorder $(B, \leq_B)$ seen as a category. The analogous notion for local right adjoints is that of a cartesian natural transformation.

Note that if $f \leq_{\mathrm{st}} g$ then f is less than g for the pointwise ordering $\leq_{\mathrm{pt}}$. The pointwise ordering only tells us that f approximates g on the output values whereas the stable ordering further tells us that the computations of f approximate those of g. Explicitely, for an input a and $b \leq_B f(a)$, the minimal part of the input a necessary for f to compute b is the same as for g, i.e. we have $m(f, a, b) = m(g, a, b)$ for all $a \in A$ and $b \in B$.

Lemma 3.1.16. *For meet cpos $(A, \leq_A)$ and $(B, \leq_B)$ and two stable functions $f, g : A \to B$, we have $f \leq_{\mathrm{st}} g$ if and only if*

$$(f \leq_{\mathrm{pt}} g \text{ and } \forall a \in A, \forall b \in B, b \leq_B f(a) \Rightarrow m(f, a, b) = m(g, a, b)).$$

In the categorified setting, cartesian transformations preserve and reflect generic elements.

Definition 3.1.17. For functors $S, T : \mathbb{A} \to \mathbb{B}$ and a natural transformation $\alpha : S \Rightarrow T$,

- *α preserves generic elements relative to $b \in \mathbb{B}$* if for all $a \in \mathbb{A}$ and for all $g : b \to S(a)$, if g is generic, then $b \xrightarrow{g} S(a) \xrightarrow{\alpha_a} T(a)$ is generic;
- *α reflects generic elements relative to $b \in \mathbb{B}$* if for all $a \in \mathbb{A}$ and for all $g : b \to S(a)$, if $b \xrightarrow{g} S(a) \xrightarrow{\alpha_a} T(a)$ is generic, then g is generic.

We say that α *preserves (resp. reflects) generic elements* if it preserves (resp. reflects) generic element relative to all $b \in \mathbb{B}$.

Theorem 3.1.18 (Proposition 5.11 in [115])**.** *For functors $S, T : \mathbb{A} \to \mathbb{B}$ and a natural transformation $\alpha : S \Rightarrow T$, if α is cartesian, then it preserves and reflects generic elements.*

We now give the definition of stable functor which we consider in our setting.

Definition 3.1.19. A *stable* functor is a finitary epi-preserving functor that admits generic factorizations.

Preserving filtered colimits is a categorification of Scott-continuity and is also necessary to obtain cartesian closure due to size issues. The property of preserving epimorphisms gives the connection to the theory of generalized species by Fiore et al. Analytic functors between presheaf categories can indeed be characterized as finitary functors that preserve wide quasi-pullbacks (which implies preservation of epimorphisms). The need for this additional condition will become clear as we construct the trace of a stable functor.

In our case, we consider functors between restricted categories of representables and reduce the proofs to showing that the functors admit generic factorizations relative to representables using the following lemmas.

Lemma 3.1.20. *Let $T : \mathbb{A} \to \mathbb{B}$ be a functor.*

1. *For diagrams $A : \mathbb{I} \to \mathbb{A}$ and $B : \mathbb{I} \to \mathbb{B}$, and a natural transformation $g : B \Rightarrow TA : \mathbb{I} \to \mathbb{B}$, if $g_i : B_i \to T(A_i)$ is generic for all $i \in \mathbb{I}$ then so is the induced composite*

$$k = \varinjlim(B) \xrightarrow{\varinjlim(g)} \varinjlim(TA) \xrightarrow{[T\mathrm{inj}_i]_{i\in\mathbb{I}}} T(\varinjlim A).$$

2. *For $b \in \mathbb{B}$, a diagram $A : \mathbb{I} \to \mathbb{A}$, and a cone $g : b \Rightarrow TA : \mathbb{I} \to \mathbb{B}$, if $g_i : b \to T(A_i)$ is generic for all $i \in \mathbb{I}$ then so are the composites*

$$b \xrightarrow{g_i} T(A_i) \xrightarrow{T(\mathrm{inj}_i)} T(\varinjlim(A))$$

   for every $i \in \mathbb{I}$.

Proof.

1. Let $(\mathrm{inj}_i : B_i \to \varinjlim(B))_{i\in\mathbb{I}}$ be a colimiting cocone. Consider $l : \varinjlim(B) \to T(d)$ in $\mathbb{B}$ and $h : \varinjlim(A) \to c$, $f : d \to c$ in $\mathbb{A}$ such that $T(h) \circ k = T(f) \circ l$; equivalently, $T(h) \circ T(\mathrm{inj}_i) \circ g_i = T(f) \circ l \circ \mathrm{inj}_i$ for all $i \in \mathbb{I}$. Since each g_i is generic, there exists, for every $i \in \mathbb{I}$, a unique morphism $e_i : A_i \to d$ such that $h \circ \mathrm{inj}_i = f \circ e_i$ and $T(e_i) \circ g_i = l \circ \mathrm{inj}_i$.

$$\begin{tikzcd}
B_i \arrow[r, "\mathrm{inj}_i"] \arrow[d, "g_i"'] & \varinjlim(B) \arrow[r, "l"] \arrow[d, "k"'] & T(d) \arrow[d, "T(f)"] \\
T(A_i) \arrow[r, "T(\mathrm{inj}_i)"'] \arrow[urr, dotted, "T(e_i)"] & T(\varinjlim A) \arrow[r, "T(h)"'] \arrow[ur, dotted, "T(e)"'] & T(c)
\end{tikzcd}$$

One can show that $(e_i : A_i \to d)_{i\in\mathbb{I}}$ is a cocone of A. So there exists a unique morphism $e : \varinjlim(A) \to d$ such that $e \circ \mathrm{inj}_i = e_i$; from which it follows that $f \circ e = h$. Moreover, $T(e) \circ k = l$ because $T(e) \circ k \circ \mathrm{inj}_i = l \circ \mathrm{inj}_i$ for all $i \in \mathbb{I}$. This establishes existence. As for uniqueness, if $e' : \varinjlim(A) \to d$ has the required property of k with respect to h, f, l then each $e' \circ \mathrm{inj}_i$ has the generic property of g_i with respect to $h \circ \mathrm{inj}_i$, f, $l \circ \mathrm{inj}_i$ making $e' \circ \mathrm{inj}_i = e_i$ and therefore $e' = e$.

2. Let $i \in \mathbb{I}$ and assume that there exists $l : b \to T(d)$, $k : \varinjlim(A) \to c$ and $f : d \to c$ such that $T(k) \circ T(\mathrm{inj}_i) \circ g_i = T(f) \circ l$. Since g_i is generic, there exists a unique $e_i : A_i \to d$ such that $k \circ \mathrm{inj}_i = f \circ e_i$ and $T(e_i) \circ g_i = l$.

$$\begin{tikzcd}
b \arrow[r, "l"] \arrow[d, "g_i"'] & T(d) \arrow[dd, "T(f)"] \\
T(A_i) \arrow[ur, dotted, "T(e_i)"'] \arrow[d, "T(\mathrm{inj}_i)"'] & \\
T(\varinjlim(A)) \arrow[r, "T(k)"'] & T(c)
\end{tikzcd}$$

One can check that $(e_i : A_i \to d)_{i\in\mathbb{I}}$ is a cocone of A. Hence, there exists a unique morphism $e : \varinjlim(A) \to d$ such that $e \circ \mathrm{inj}_i = e_i$ for all

$i \in \mathbb{I}$. Since $(f \circ e_i : a_i \to c)_{i \in \mathbb{I}}$ is a cocone of A, there exists a unique $q : \varinjlim(A) \to c$ such that $q \circ \mathrm{inj}_i = f \circ e_i$ for all $i \in \mathbb{I}$. Both k and $f \circ e$ verify this condition which implies that $k = f \circ e$. We also have that $T(e) \circ T(\mathrm{inj}_i) \circ g_i = T(e_i) \circ g_i = l$. To show that $T(\mathrm{inj}_i) \circ g_i$ is generic, it remains to show that $e : \varinjlim(A) \to d$ is the unique morphism that makes the two diagrams below commute.

$$\begin{tikzcd}
b \arrow[r, "l"] \arrow[d, "T(\mathrm{inj}_i) \circ g_i"'] & T(d) & & d \arrow[d, "f"] \\
T(\varinjlim(A)) \arrow[ur, "T(e)"'] & & \varinjlim(A) \arrow[ur, dotted, "e"] \arrow[r, "k"'] & c
\end{tikzcd}$$

Assume that there exists $e' : \varinjlim(A) \to d$ such that $k = f \circ e'$ and $T(e') \circ T(\mathrm{inj}_i) \circ g_i = l$. We then have that $e' \circ \mathrm{inj}_i = e_i$ for all $i \in \mathbb{I}$ by genericity of the g_i's which implies that $e' = e$ by the universal property of the colimit.

□

Proposition 3.1.21. *Let $T : \mathbb{A} \to \mathbb{B}$ be a functor. For a diagram $B : \mathbb{I} \to \mathbb{B}$, if T admits generic factorizations relative to b_i for all $i \in \mathbb{I}$ then it admits generic factorizations relative to $\varinjlim(B)$ whenever this exists.*

Proof. Assume that $\varinjlim(B)$ exists in $\mathbb{B}$ and let $f : \varinjlim(B) \to T(a)$ be a morphism in $\mathbb{B}$. By hypothesis, each $f_i := f \circ \mathrm{inj}_i : b_i \to T(a)$ can be factored as:

$$b_i \xrightarrow{g_i} T(c_i) \xrightarrow{T(h_i)} T(a)$$

where g_i is generic. Define the diagram $C : \mathbb{I} \to \mathbb{A}$ as follows: $C(i) := c_i$ for $i \in \mathbb{I}$ and for a morphism $e : i \to j \in \mathbb{I}$, $C(e)$ is the unique morphism $c_i \to c_j$ obtained from the genericity of g_i in the square below:

$$\begin{tikzcd}
b_i \arrow[r, "g_j \circ B(e)"] \arrow[d, "g_i"'] & T(c_j) \arrow[d, "T(h_j)"] \\
T(c_i) \arrow[r, "T(h_i)"'] & T(a)
\end{tikzcd}$$

By Lemma 3.1.20.1, $[T(\mathrm{inj}_i)]_{i\in\mathbb{I}} \circ \varinjlim(g) : \varinjlim(B) \to \varinjlim(C)$ is generic. Let $[h] : \varinjlim(C) \to a$ be the mediating morphism induced by the cocone $(h_i : c_i \to a)_{i\in\mathbb{I}}$. Both f and $T([h]) \circ [T(\mathrm{inj}_i)]_{i\in\mathbb{I}} \circ \varinjlim(g)$ verify the conditions of the mediating morphism $\varinjlim(B) \to T(a)$ induced by the cocone $(T(h_i) \circ T(\mathrm{inj}_i) \circ g_i : b_i \to T(a))_{i\in\mathbb{I}}$ which implies that $f = T([h]) \circ [T(\mathrm{inj}_i)]_{i\in\mathbb{I}} \circ \varinjlim(g)$ as desired. $\square$

3.2 Intensional Theory

3.2.1 Orthogonality on groupoids

We attach to groupoids an additional structure, which we call *kit*, consisting of a family of subgroups of endomorphisms for each object in the groupoid. We start by defining the more general notion of *prekit*.

3.2.2 Prekits

Definition 3.2.1. A *prekit* on a groupoid $\mathbb{A}$ is a family $\mathcal{A} = \{\mathcal{A}(a)\}_{a\in\mathbb{A}}$ consisting of sets $\mathcal{A}(a)$ of subgroups of $\mathrm{Endo}_{\mathbb{A}}(a)$ that is closed under conjugation. Explicitely, for all $f : a' \to a$ in $\mathbb{A}$, if $G \leq \mathrm{Endo}(a)$ is in $\mathcal{A}(a)$ then the conjugate subgroup $f^{-1}Gf \leq \mathrm{Endo}(a')$ is in $\mathcal{A}(a')$.

Example. Every groupoid $\mathbb{A}$ has the following canonical choices of prekits: the trivial one $\mathrm{Triv}_{\mathbb{A}}(a) = \{\{\mathrm{id}_a\}\}$, and the maximal one $\mathrm{Endo}_{\mathbb{A}}(a) = \{G \mid G \leq \mathbb{A}(a,a)\}$.

To construct prekits on dual groupoids we consider a notion of subgroup orthogonality.

Definition 3.2.2. For a group G and subgroups $H \leq G$ and $K \leq G^{\mathrm{op}}$, we say that H and K are *orthogonal* if $H \cap K = \{\mathrm{id}\}$, where id is the identity element in G (and G^{op}).

From this we get the following construction:

Lemma 3.2.3. *For a groupoid $\mathbb{A}$ and a prekit $\mathcal{A}$, $\mathbb{A}^{\mathrm{op}}$ may be equipped with its* orthogonal prekit *$\mathcal{A}^{\perp}$ defined for $a \in \mathbb{A}^{\mathrm{op}}$ by:*

$$\mathcal{A}^{\perp}(a) = \{G \leq \mathrm{Endo}_{\mathbb{A}^{\mathrm{op}}}(a) \mid \forall H \in \mathcal{A}(a), G \perp H\}.$$

Note that a morphism $f : a \to a$ is in $\bigcup \mathcal{A}^{\perp}(a)$ if for all $n \in \mathbb{Z}$, $f^n \in \bigcup \mathcal{A}(a)$ implies $f^n = \mathrm{id}_a$.

Proof. For the closure under conjugation observe that for a morphism $f : a' \to a$ and subgroups $G \leq \mathrm{Endo}_{\mathbb{A}}(a)$ and $H \leq \mathrm{Endo}_{\mathbb{A}^{\mathrm{op}}}(a')$, $f^{-1}Gf \perp H$ if and only if $G \perp fHf^{-1}$. $\square$

For a groupoid $\mathbb{A}$, the canonical prekits from Example 3.2.2 are orthogonal:

$$\mathrm{Triv}_{\mathbb{A}}^{\perp} = \mathrm{Endo}_{\mathbb{A}^{\mathrm{op}}} \qquad \mathrm{Endo}_{\mathbb{A}}^{\perp} = \mathrm{Triv}_{\mathbb{A}^{\mathrm{op}}}.$$

3.2.3 Kits.

We define *kits* to be those prekits which are closed under *double orthogonality.*

Definition 3.2.4. A *kit* $\mathcal{A}$ on a groupoid $\mathbb{A}$ is a prekit $\mathcal{A}$ such that $\mathcal{A} = \mathcal{A}^{\perp\perp}$.

Writing $\mathbf{Prekit}(\mathbb{A})$ for the poset of prekits on $\mathbb{A}$ under (pointwise) inclusion, the orthogonality relation on prekits induces a Galois connection

$$\begin{tikzcd}
\mathbf{Prekit}(\mathbb{A})^{\mathrm{op}} \arrow[r, bend left, "(-)^{\perp}"] \arrow[r, phantom, "\perp"] & \mathbf{Prekit}(\mathbb{A}^{\mathrm{op}}) \arrow[l, bend left, "(-)^{\perp}"]
\end{tikzcd}$$

where kits are its fixed points $\mathcal{A} = \mathcal{A}^{\perp\perp}$. We derive these basic properties from the adjunction:

Lemma 3.2.5. *Let $\mathcal{A}, \mathcal{A}'$ be prekits on a groupoid $\mathbb{A}$. Then:*

- $\mathcal{A} \subseteq \mathcal{A}^{\perp\perp}$*, where* $\subseteq$ *is component-wise inclusion.*
- *If* $\mathcal{A} \subseteq \mathcal{A}'$ *then* $\mathcal{A}'^{\perp} \subseteq \mathcal{A}^{\perp}$.

It follows that $\mathcal{A}^{\perp} = \mathcal{A}^{\perp\perp\perp}$*; so, in particular,* $\mathcal{A}^{\perp}$ *is a kit.*

The condition of closure under double orthogonality ensures that certain basic properties always hold for a kit:

Lemma 3.2.6. *For a kit $\mathcal{A}$ on a groupoid $\mathbb{A}$, the following holds for all $a \in \mathbb{A}$:*

1. $\mathcal{A}(a)$ *is down-closed: if* $H' \leq H \in \mathcal{A}(a)$ *then* $H' \in \mathcal{A}(a)$*, and*
2. $\mathcal{A}(a)$ *is closed under directed unions: if* $\{G_i\}_{i \in I}$ *is a directed family in* $\mathcal{A}(a)$ *then* $\bigcup_{i \in I} G_i \in \mathcal{A}(a)$.

Proof.

1. Let $H' \leq H \in \mathcal{G}$ and $G \in \mathcal{G}^{\perp}$, then $H' \cap G \subseteq H \cap G = \{\mathrm{id}\}$ wich implies that $H' \in \mathcal{G}^{\perp\perp} = \mathcal{G}$.
2. Let $\{G_i\}_{i\in I}$ be a directed family in $\mathcal{G}$, then $\bigcup_{i\in I} G_i$ is a subgroup of G and for $H \in \mathcal{G}^{\perp}$, $\left(\bigcup_{i\in I} G_i\right) \cap H = \bigcup_{i\in I}(G_i \cap H) = \{\mathrm{id}\}$ so that $\bigcup_{i\in I} G_i$ is in $\mathcal{G}^{\perp\perp} = \mathcal{G}$ as desired.

□

We consider as an example the possible kits on the group $\mathbb{Z}_6$, the cyclic group of order 6, seen as a groupoid. For an element g of a group G, we denote by $\langle g\rangle$ the *cyclic subgroup* generated by g; that is, $\langle g\rangle = \{g^n \mid n \in \mathbb{Z}\}$. Now, writing g for any generator of $\mathbb{Z}_6$, it can be verified that the only possible kits are the maximal and trivial ones, as well as the subsets $\{\langle g^2\rangle, \{\mathrm{id}\}\}$ and $\{\langle g^3\rangle, \{\mathrm{id}\}\}$. These form a complete Boolean algebra, and in fact this is the case for the set of kits on any groupoid.

There are many intermediate notions between prekits and kits: for example Paul Taylor's notion of creed is a family of subsets $\bigcup \mathcal{A}(a) \subseteq \mathbb{A}(a,a)$, for $a \in \mathbb{A}$ that verifies certain closure conditions. There is however no notion of orthogonality on creeds which implies that they do not model classical negation.

It is sometimes useful to unfold the condition $\mathcal{A} = \mathcal{A}^{\perp\perp}$. In practice, to show that a prekit $\mathcal{A}$ is a kit, we show that for all $a \in \mathbb{A}$, for all $f \in \bigcup \mathcal{A}(a)^{\perp\perp}$, f is in $\bigcup \mathcal{A}(a)$ which corresponds to the *saturation* property defined below:

Definition 3.2.7. A prekit $\mathcal{A}$ on a groupoid $\mathbb{A}$ is *saturated* if it satisfies the following condition: for $a \in \mathbb{A}$ and $f \in \mathrm{Endo}(a)$, if the formula $\Phi_{\bigcup \mathcal{A}}(f)$ given by

$$\forall n \in \mathbb{N}, f^n = \mathrm{id} \vee (\exists m \in \mathbb{N}, f^{nm} \neq \mathrm{id} \wedge f^{nm} \in \bigcup \mathcal{A}(a))$$

holds, then $f \in \bigcup \mathcal{A}(a)$.

Saturation is however not enough to prove $\mathcal{A} = \mathcal{A}^{\perp\perp}$, it only shows that $\bigcup \mathcal{A}(a) = \bigcup \mathcal{A}^{\perp\perp}(a)$ for all $a \in \mathbb{A}$. We need further that $\mathcal{A}(a) = \{H \leq \mathbb{A}(a,a) \mid H \subseteq \bigcup \mathcal{A}(a)\}$ for all $a \in \mathbb{A}$.

Lemma 3.2.8. *For a groupoid $\mathbb{A}$ and a prekit $\mathcal{A}$ on $\mathbb{A}$, $\mathcal{A} = \mathcal{A}^{\perp\perp}$ if and only if*

1. *for all $a \in \mathbb{A}$, $\mathcal{A}(a) = \{H \leq \mathbb{A}(a,a) \mid H \subseteq \bigcup \mathcal{A}(a)\}$*
2. *$\mathcal{A}$ is saturated.*

Proof.

($\Rightarrow$) Assume that $\mathcal{A} = \mathcal{A}^{\perp\perp}$ and let $H \leq \mathbb{A}(a,a)$ be such that $H \subseteq \bigcup \mathcal{A}(a)$ i.e. for all $g \in H$, $\langle g \rangle$ is in $\mathcal{A}(a)$ since $\mathcal{A}(a)$ is downclosed by Lemma 3.2.6. Let K be in $\mathcal{A}^{\perp}(a)$, for any $g \in H \cap K$, we have $\langle g \rangle \cap K = \{\mathrm{id}\}$ which implies that H is in $\mathcal{A}^{\perp\perp}(a) = \mathcal{A}(a)$ so that $\mathcal{A}(a) = \{H \leq \mathbb{A}(a,a) \mid H \subseteq \bigcup \mathcal{A}(a)\}$ as desired. The saturation property is an immediate unfolding of the double orthogonality.

($\Leftarrow$) Since $\mathcal{A}$ is saturated, we have $\bigcup \mathcal{A}(a) = \bigcup \mathcal{A}^{\perp\perp}(a)$ for all $a \in \mathbb{A}$ which implies that $\mathcal{A} = \mathcal{A}^{\perp\perp}$ by 1.

□

3.2.4 Stabilized profunctors

A left group action $\rho : G \times S \to S$ of a group G on a set S may equivalently be presented as a functor from G to **Set**, with the group G viewed as a groupoid with a single object denoted $\star$. This functor takes $\star$ to S and maps a morphism $g : \star \to \star$ in G to its action $\rho(g, -) : S \to S$ on the set S. In this way we obtain an equivalence between the category of left G-actions and the functor category $[G, \mathbf{Set}]$, in which natural transformation are identified with G-*equivariant* functions (functions that commute with the group action). Dually, the category of right G-actions $S \times G \to S$ is equivalent to the category $[G^{\mathrm{op}}, \mathbf{Set}]$ of presheaves on G. This viewpoint extends to groupoids where for a groupoid $\mathbb{A}$, right $\mathbb{A}$-actions are defined as presheaves over $\mathbb{A}$ and left $\mathbb{A}$-actions as co-presheaves over $\mathbb{A}$.

In this setting, profunctors play the role of double-sided group(oid) actions. For groups G and H seen as one object groupoids, a profunctor $P : G \nrightarrow H$ is a set equipped with both a left G-action and a right H-action which are compatible; equivalently, it is a left $H^{\mathrm{op}} \times G$-action. Flat species (combinatorial species whose analytic functors are stable or polynomial) are the ones corresponding to free actions of the symmetric group $\mathfrak{S}_n$ for every $n \in \mathbb{N}$ [12].

For groupoids $\mathbb{A}, \mathbb{B}$, a profunctor $P : \mathbb{A} \nrightarrow \mathbb{B}$ corresponds to a free action if its stabilizers are always trivial i.e. for all $a \in \mathbb{A}$, $b \in \mathbb{B}$, $p \in P(a,b)$, $f \in \mathbb{A}(a,a)$ and $g \in \mathbb{B}(b,b)$,

$$f \cdot p \cdot g = p \quad \Rightarrow \quad (f = \mathrm{id}_a \wedge g = \mathrm{id}_b).$$

The profunctors that we consider here are more general, the stabilizers are not required to be trivial but they are restricted by the kit structure. We use a similar notation as in Chapter 2 and define a *(pre)kit structure* as a pair $\mathbf{A} = (|\mathbf{A}|, \mathcal{A})$ of a small groupoid $|\mathbf{A}|$ and a (pre)kit $\mathcal{A}$. We write $\mathbf{A}^{\perp}$ for the dual structure $(|\mathbf{A}|^{\mathrm{op}}, \mathcal{A}^{\perp})$.

Definition 3.2.9. For kit structures $\mathbf{A} = (|\mathbf{A}|, \mathcal{A})$ and $\mathbf{B} = (|\mathbf{B}|, \mathcal{B})$, a profunctor $P : |\mathbf{A}| \nrightarrow |\mathbf{B}|$ is said to be a *stabilized profunctor* (or *s-profunctor*) if for all $a \in |\mathbf{A}|$, $b \in |\mathbf{B}|$, $p \in P(a,b)$, $f \in |\mathbf{A}|(a,a)$, $g \in |\mathbf{B}|(b,b)$, whenever $f \cdot p \cdot g = p$ then

$$f \in \bigcup \mathcal{A}(a) \Rightarrow g \in \bigcup \mathcal{B}(b) \quad \text{and} \quad g \in \bigcup \mathcal{B}^{\perp}(b) \Rightarrow f \in \bigcup \mathcal{A}^{\perp}(a).$$

Example. For groupoids $\mathbb{A}$ and $\mathbb{B}$, s-profunctors $(\mathbb{A}, \mathrm{Endo}_{\mathbb{A}}) \nrightarrow (\mathbb{B}, \mathrm{Endo}_{\mathbb{B}})$ are equivalent to profunctors $P : \mathbb{A} \nrightarrow \mathbb{B}$ which act freely on $\mathbb{A}$ i.e. for all $a \in \mathbb{A}$, $b \in \mathbb{B}$, $p \in P(a,b)$, $f \in \mathbb{A}(a,a)$, if $f \cdot p = p$ then $f = \mathrm{id}_a$. Dually, s-profunctors $(\mathbb{A}, \mathrm{Triv}) \nrightarrow (\mathbb{B}, \mathrm{Triv})$ correspond to profunctors $\mathbb{A} \nrightarrow \mathbb{B}$ acting freely in $\mathbb{B}$.

Definition 3.2.10. The bicategory **SProf** has objects given by kits, morphisms given by s-profunctors and 2-cells given by natural transformations, with identity and composition structures given as for profunctors.

Proof. We show that **SProf** is indeed a bicategory:

identity: for a kit $\mathbf{A} = (|\mathbf{A}|, \mathcal{A})$, the identity profunctor $\mathrm{id}_{|\mathbf{A}|} : |\mathbf{A}| \nrightarrow |\mathbf{A}|$ given by $(a, a') \mapsto |\mathbf{A}|(a', a)$ is an s-profunctor $\mathbf{A} \nrightarrow \mathbf{A}$ since kits are closed under conjugation.

composition: for s-profunctors $P : \mathbf{A} \nrightarrow \mathbf{B}$ and $Q : \mathbf{B} \nrightarrow \mathbf{C}$, let $a \in |\mathbf{A}|$, $c \in |\mathbf{C}|$, $f \in \mathbb{A}(a,a)$, $h \in \mathbb{C}(c,c)$ and $t \in (Q \circ P)(a,c)$ be such that $f \cdot t \cdot h = t$. The element t is of the form $p \underset{b}{\bowtie} q \in \int^b P(a,b) \times Q(b,c)$[1] for some $b \in |\mathbf{B}|$, $p \in P(a,b)$ and $q \in Q(b,c)$ so $f \cdot t \cdot h = t$ is equivalent to $(f \cdot p) \underset{b}{\bowtie} (q \cdot h) = p \underset{b}{\bowtie} q$ which implies that there exists $g \in \mathbb{B}(b,b)$ such that $f \cdot p \cdot g = p$ and $g^{-1} \cdot q \cdot h = q$. If $f \in \bigcup \mathcal{A}(a)$ then since P is an s-profunctor and $f \cdot p \cdot g = p$, we have $g \in \bigcup \mathcal{B}(b)$ which implies that $g^{-1} \in \bigcup \mathcal{B}(b)$ by closure under powers (g is an element of a group $G \in \mathcal{B}(b)$). Likewise, since Q is an s-profunctor and $g^{-1} \cdot q \cdot h = q$, we have $h \in \bigcup \mathcal{C}(c)$. The other implication $h \in \bigcup \mathcal{C}^{\perp}(c) \Rightarrow f \in \bigcup \mathcal{A}^{\perp}(a)$ is shown similarly. ☐

Remark 5. We obtain immediately from Definition 3.2.9 that for kits $\mathbf{A}$ and $\mathbf{B}$, a profunctor $P : |\mathbf{A}| \to |\mathbf{B}|$ is an s-profunctor $\mathbf{A} \nrightarrow \mathbf{B}$ if and only if its dual $P^{\mathrm{op}} : |\mathbf{B}|^{\mathrm{op}} \nrightarrow |\mathbf{A}|^{\mathrm{op}}$ is an s-profunctor $\mathbf{B}^{\perp} \nrightarrow \mathbf{A}^{\perp}$. The operation $\mathbf{A} \mapsto \mathbf{A}^{\perp}$ extends to a full and faithful functor.

Proposition 3.2.11. *The forgetful functor* **SProf** $\to$ **Prof** *which forgets the kit structure* $(|\mathbf{A}|, \mathcal{A}) \mapsto |\mathbf{A}|$ *is locally faithful and injective on* 1*-cells.*

---

[1] We use here the notation introduced in Section 1.4.2 in Notation 2.

3.2.5 Linear structure

Biproduct structure

For 1-categorical models of stability (coherence spaces, probabilistic coherence spaces, etc.), the linear categories do not have biproducts which implies in particular that they are not models of differential linear logic. In a categorified setting, the biproduct structure in **Prof** can be lifted to **SProf**.

Lemma 3.2.12. *For a family of kit structures* $(\mathbf{A}_i)_{i\in I}$*, we define* $\&_i\mathbf{A}_i$ *as* $(\&_i|\mathbf{A}_i|, \coprod_i \mathcal{A}_i)$*. It is a kit verifying* $(\coprod_{i\in I} \mathcal{A}_i^{\perp})^{\perp} = \coprod_{i\in I} \mathcal{A}_i$*.*

Proof. Immediate. ☐

As a corollary, we obtain that $\oplus_i\mathbf{A}_i := (\&_i\mathbf{A}_i^{\perp})^{\perp}$ is equal to $\&_i\mathbf{A}_i$.

Lemma 3.2.13. *For a family of kit structures* $(\mathbf{A}_i)_{i\in I}$*, the profunctors* $\pi_i : \&_i\,|\mathbf{A}_i| \nrightarrow |\mathbf{A}_i|$ *and* $\mathrm{inj}_i : |\mathbf{A}_i| \nrightarrow \&_i\,|\mathbf{A}_i|$ *are s-profunctors* $\&_i\mathbf{A}_i \nrightarrow \mathbf{A}_i$ *and* $\mathbf{A}_i \nrightarrow \oplus_i\mathbf{A}_i$ *respectively. They induce adjoint equivalences for all* $\mathbf{X} \in$ **SProf***:*

$$\mathbf{SProf}(\mathbf{X}, \&_i\mathbf{A}_i) \simeq \prod_i \mathbf{SProf}(\mathbf{X}, \mathbf{A}_i) \quad \textit{and}$$

$$\mathbf{SProf}(\oplus_i\mathbf{A}_i, \mathbf{X}) \simeq \prod_i \mathbf{SProf}(\mathbf{A}_i, \mathbf{X}).$$

Proof. Let $f : a_i \to a$ be in $\pi_i((i,a_i),a) = |\mathbf{A}_i|\,(a,a_i)$ and assume that there exists $(i,h_i) \in |\&_i\,\mathbf{A}_i|((i,a_i),(i,a_i))$ and $g \in |\mathbf{A}|(a,a)$ such that $(i,h_i)\cdot f\cdot g = f$ i.e. $h_i \circ f \circ g = f$. If (i,h_i) is in $\bigcup \&_i\mathcal{A}_i(i,a_i)$ i.e. $h_i \in \bigcup \mathcal{A}_i(a_i)$ then $g = f^{-1}h_i^{-1}f$ is in $\bigcup \mathcal{A}_i(a_i)$ by closure under powers and conjugation. Dually, if $g \in \bigcup \mathcal{A}_i^{\perp}$, then $h_i = fg^{-1}f^{-1} \in \bigcup \mathcal{A}_i^{\perp}(a_i)$ so that $(i,h_i) \in \bigcup (\&_i\mathcal{A}_i)^{\perp}(i,a_i)$. The proof for $\mathrm{inj}_i \in \mathbf{SProf}(\mathbf{A}_i, \oplus_i\mathbf{A}_i)$ is similar. Using Lemma 2.3.1, the adjoint equivalences above follow from the biproduct structure in **Prof**. ☐

SProf therefore has biproducts with zero object defined by the kit structure $(\mathbf{0}, \varnothing)$ on the empty groupoid $\mathbf{0}$ together with the empty kit.

$*$-autonomous structure

Similarly to the construction presented in chapter 2, the orthogonality eliminates the compact closure degeneracy in **Prof** as **SProf** is only $*$-autonomous.

Definition 3.2.14. For kit structures $\mathbf{A} = (|\mathbf{A}|, \mathcal{A})$ and $\mathbf{B} = (|\mathbf{B}|, \mathcal{B})$, define their tensor product $\mathbf{A} \otimes \mathbf{B}$ as $(|\mathbf{A}| \times |\mathbf{B}|, (\mathcal{A} \times \mathcal{B})^{\perp\perp}))$, where for $a \in |\mathbf{A}|$ and $b \in |\mathbf{B}|$,

$$(\mathcal{A} \times \mathcal{B})(a,b) := (\mathcal{A}(a) \times \mathcal{B}(b)).$$

Note that $\mathcal{A} \otimes \mathcal{B} = (\mathcal{A} \times \mathcal{B})^{\perp\perp}$ is automatically a kit because of the double orthogonal closure.

To prove that **SProf** is a symmetric monoidal bicategory, we show that the tensor product lifts to a pseudo-functor $\mathbf{SProf} \times \mathbf{SProf} \to \mathbf{SProf}$ and that the associator, symmetry, left and right unitors pseudo-natural transformations have components in **SProf**. Due to the double closure $(-)^{\perp\perp}$ in the definition of the tensor, showing associativity is not straightforward and we make use of the relationship between the tensor $\otimes$ and the linear arrow $\multimap$ as in Chapter 2 to prove it.

Lemma 3.2.15. *For kit structures* $\mathbf{A} = (|\mathbf{A}|, \mathcal{A})$ *and* $\mathbf{B} = (|\mathbf{B}|, \mathcal{B})$*, define the kit structure* $\mathbf{A} \multimap \mathbf{B}$ *as* $(|\mathbf{A}|^{\mathrm{op}} \times |\mathbf{B}|, \mathcal{A} \multimap \mathcal{B})$ *where* $(\mathcal{A} \multimap \mathcal{B})(a, b)$ *is the set of all subgroups* $H \leq \mathrm{Endo}(a, b)$ *such that:*

$$\forall (f, g) \in H, f \in \bigcup \mathcal{A}(a) \Rightarrow g \in \bigcup \mathcal{B}(b) \text{ and } g \in \bigcup \mathcal{B}^{\perp}(b) \Rightarrow f \in \bigcup \mathcal{A}^{\perp}(a).$$

Proof. To prove that $\mathbf{A} \multimap \mathbf{B}$ forms a kit structure, we first show that for objects $a \in |\mathbf{A}|, b \in |\mathbf{B}|$ and subgroups $K \in \mathcal{A}(a)$ and $G \in \mathcal{B}^{\perp}(b)$, we have $K \times G \in (\mathcal{A} \multimap \mathcal{B})^{\perp}(a, b)$. Let H be in $(\mathcal{A} \multimap \mathcal{B})(a, b)$ and $(f, g) \in (K \times G) \cap H$. Since $f \in K \in \mathcal{A}(a)$, we have $g \in \mathcal{B}(b)$ by definition of $\mathcal{A} \multimap \mathcal{B}$ which implies that $g = \mathrm{id}$ since $g \in G \in \mathcal{B}^{\perp}(b)$. Likewise, since $g \in G \in \mathcal{B}^{\perp}(b)$, we have $f \in \mathcal{A}^{\perp}(a)$ which implies that $f = \mathrm{id}$ as $f \in K \in \mathcal{A}(a)$.

Now, let H be in $(\mathcal{A} \multimap \mathcal{B})^{\perp\perp}(a, b)$, we show that H is in $(\mathcal{A} \multimap \mathcal{B})(a, b)$. Let (f, g) be in H and assume that $f \in \bigcup \mathcal{A}(a)$ and that there exists $n \in \mathbb{N}$ such that $g^n \in \bigcup \mathcal{B}^{\perp}(b)$. Hence, we have $\langle f^n \rangle \times \langle g^n \rangle \in (\mathcal{A}(a) \times \mathcal{B}^{\perp}(b))$ which implies that $\langle f^n \rangle \times \langle g^n \rangle \in (\mathcal{A} \multimap \mathcal{B})^{\perp}(a, b)$ so that $f^n = \mathrm{id}$ and $g^n = \mathrm{id}$ as desired. We can show similarly that if $g \in \bigcup \mathcal{B}^{\perp}(b)$ then $f \in \bigcup \mathcal{A}^{\perp}(a)$.

For a subgroup $H \leq \mathrm{Endo}(a, b)$ such that $H \subseteq \bigcup (\mathcal{A} \multimap \mathcal{B})(a, b)$, it is immediate that H is in $(\mathcal{A} \multimap \mathcal{B})(a, b)$. Hence, by Lemma 3.2.8, we conclude that $\mathcal{A} \multimap \mathcal{B}$ is a kit. ☐

Corollary 3.2.16. *For kits* $\mathbf{A} = (|\mathbf{A}|, \mathcal{A})$ *and* $\mathbf{B} = (|\mathbf{B}|, \mathcal{B})$*, the categories* $\mathbf{SProf}(\mathbf{A}, \mathbf{B})$ *and* $\mathbf{SProf}(\mathbf{1}, \mathbf{A} \multimap \mathbf{B})$ *are isomorphic.*

Proof. Immediate from the definitions. ☐

Lemma 3.2.17. *If* $P_1 : \mathbf{A}_1 \nrightarrow \mathbf{B}_1$ *and* $P_2 : \mathbf{A}_2 \nrightarrow \mathbf{B}_2$ *are s-profunctors then the profunctor* $P_1 \otimes P_2 : |\mathbf{A}_1| \times |\mathbf{A}_2| \nrightarrow |\mathbf{B}_1| \times |\mathbf{B}_2|$ *given by*

$$(P_1 \otimes P_2)((a_1, a_2), (b_1, b_2)) := P_1(a_1, b_1) \times P_2(a_2, b_2)$$

is in $\mathbf{SProf}(\mathbf{A}_1 \otimes \mathbf{A}_2, \mathbf{B}_1 \otimes \mathbf{B}_2)$.

Proof. Assume that there is $(a_1, a_2) \in |\mathbf{A}_1| \times |\mathbf{A}_2|$, $(b_1, b_2) \in |\mathbf{B}_1| \times |\mathbf{B}_2|$, $(f_1, f_2) \in \mathrm{Endo}(a_1, a_2)$, $(g_1, g_2) \in \mathrm{Endo}(b_1, b_2)$ and $(p_1, p_2) \in P_1(b_1, a_1) \times P_2(b_2, a_2)$ such that

$$(f_1, f_2) \cdot (p_1, p_2) \cdot (g_1, g_2) = (p_1, p_2)$$

i.e. $f_1 \cdot p_1 \cdot g_1 = p_1$ and $f_2 \cdot p_2 \cdot g_2 = p_2$.

- We first show that if $(f_1, f_2) \in \bigcup(\mathcal{A}_1 \otimes \mathcal{A}_2)(a_1, a_2)$ then $(g_1, g_2) \in \bigcup(\mathcal{B}_1 \otimes \mathcal{B}_2)(b_1, b_2)$.

  Recall from Definition 3.2.7 that $(f_1, f_2) \in \bigcup(\mathcal{A}_1 \otimes \mathcal{A}_2)(a_1, a_2) = \bigcup(\mathcal{A}_1 \times \mathcal{A}_2)^{\perp\perp}(a_1, a_2)$ is equivalent to the formula $\Phi_{\bigcup \mathcal{A}_1 \otimes \mathcal{A}_2}(f_1, f_2)$ given by:

  $$\forall n \in \mathbb{N}, (f_1, f_2)^n = (\mathrm{id}, \mathrm{id}) \ \vee$$
  $$\left(\exists m \in \mathbb{N}, (\mathrm{id}, \mathrm{id}) \neq (f_1, f_2)^{nm} \in \bigcup(\mathcal{A}_1 \times \mathcal{A}_2)(a_1, a_2)\right).$$

  Let $n \in \mathbb{N}$ be such that $g_1^n \neq \mathrm{id}$ or $g_2^n \neq \mathrm{id}$. If $(f_1, f_2)^n = (\mathrm{id}, \mathrm{id})$, then $(f_1^n, f_2^n) \in \bigcup(\mathcal{A}_1 \times \mathcal{A}_2)(a_1, a_2)$ which implies that $(g_1^n, g_2^n) \in \bigcup(\mathcal{B}_1 \times \mathcal{B}_2)(b_1, b_2)$ since P_1 and P_2 are s-profunctors. Taking $m = 1$, the formula $\Phi_{\bigcup \mathcal{B}_1 \otimes \mathcal{B}_2}(g_1, g_2)$ holds.

  Assume now that there exists $m \in \mathbb{N}$ such that $(f_1^{nm}, f_2^{nm}) \neq (\mathrm{id}, \mathrm{id})$ and $(f_1^{nm}, f_2^{nm}) \in \bigcup(\mathcal{A}_1 \times \mathcal{A}_2)(a_1, a_2)$. It implies that (g_1^{nm}, g_2^{nm}) is in $\bigcup(\mathcal{B}_1 \times \mathcal{B}_2)(b_1, b_2)$ since P_1 and P_2 are s-profunctors. Assume that $g_1^{nm} = \mathrm{id}$ and $g_2^{nm} = \mathrm{id}$, then $(g_1^{nm}, g_2^{nm}) \in \bigcup(\mathcal{B}_1^{\perp} \times \mathcal{B}_2^{\perp})(b_1, b_2)$ which implies that $(f_1^{nm}, f_2^{nm}) \in \bigcup(\mathcal{A}_1^{\perp} \times \mathcal{A}_2^{\perp})(a_1, a_2)$. Hence, $(f_1^{nm}, f_2^{nm}) = (\mathrm{id}, \mathrm{id})$ which gives us the desired contradiction.

- For the other direction, assume that $(g_1, g_2) \in \bigcup(\mathcal{B}_1 \otimes \mathcal{B}_2)^{\perp}(b_1, b_2) = \bigcup(\mathcal{B}_1 \times \mathcal{B}_2)^{\perp}(b_1, b_2)$, we want to show that $(f_1, f_2) \in \bigcup(\mathcal{A}_1 \times \mathcal{A}_2)^{\perp}(a_1, a_2)$. Let $H_1 \in \mathcal{A}_1(a_1)$, $H_2 \in \mathcal{A}_2(a_2)$ and $n \in \mathbb{N}$ such that $(f_1^n, f_2^n) \in H_1 \times H_2$. We need to show that $(f_1^n, f_2^n) = (\mathrm{id}, \mathrm{id})$. Since P_1 is a s-profunctor, $g_1^n \in \bigcup \mathcal{B}_1(b_1)$, and therefore, by Corollary 3.2.19, we must have $g_2^n \in \bigcup \mathcal{B}_2(b_2)^{\perp}$. Since P_2 is an s-profunctor, we must have $f_2^n \in \mathcal{A}_2(a_2)^{\perp}$, and therefore $f_2^n = \mathrm{id}$. Likewise, we can show that $f_1^n = \mathrm{id}$ so that $(f_1^n, f_2^n) = (\mathrm{id}, \mathrm{id})$ as desired. □

Lemma 3.2.18. *For kits $\mathbf{A} = (|\mathbf{A}|, \mathcal{A})$ and $\mathbf{B} = (|\mathbf{B}|, \mathcal{B})$, the kits $\mathbf{A} \multimap \mathbf{B}$ and $(\mathbf{A} \otimes \mathbf{B}^{\perp})^{\perp}$ are isomorphic.*

Proof. Let $a \in |\mathbf{A}|$, $b \in |\mathbf{B}|$ and $(f,g) \in \bigcup(\mathcal{A} \otimes \mathcal{B}^{\perp})^{\perp}(a,b) = \bigcup(\mathcal{A}(a) \times \mathcal{B}(b)^{\perp})^{\perp}$. Assume that $f \in \bigcup \mathcal{A}(a)$, if there exists $n \in \mathbb{N}$ such that $g^n \in \bigcup \mathcal{B}^{\perp}(b)$, then $(f,g)^n \in \bigcup(\mathcal{A}(a) \times \mathcal{B}(b)^{\perp})$ which implies that $(f,g)^n = (\mathrm{id}, \mathrm{id})$ so that g is in $\bigcup \mathcal{B}(b)$ as desired. Dually, assume that $g \in \bigcup \mathcal{B}^{\perp}(b)$, if there exists $n \in \mathbb{N}$ such that $f^n \in \bigcup \mathcal{A}^{\perp}(a)$ then $(f,g)^n = (\mathrm{id}, \mathrm{id})$ which implies that $f \in \bigcup \mathcal{A}^{\perp\perp}(a) = \bigcup \mathcal{A}(a)$ as desired.

For the other inclusion, let $H \in (\mathcal{A} \multimap \mathcal{B})(a,b)$, $K \in \mathcal{A}(a)$, $G \in \mathcal{B}^{\perp}(b)$ and $(f,g) \in H \cap (K \times G)$. Since f is in $H \in \mathcal{A}(a)$ and $(f,g) \in \bigcup(\mathcal{A} \multimap \mathcal{B})(a,b)$, we have $g \in \bigcup \mathcal{B}(b)$ which implies that $g = \mathrm{id}$ as g is in $G \in \mathcal{B}^{\perp}(b)$. We obtain similarly that $f = \mathrm{id}$ which implies that G is in $(\mathcal{A}(a) \times \mathcal{B}(b)^{\perp})^{\perp}$. □

Corollary 3.2.19. *For kits $\mathbf{A} = (|\mathbf{A}|, \mathcal{A})$ and $\mathbf{B} = (|\mathbf{B}|, \mathcal{B})$, objects $a \in \mathbb{A}$ and $b \in \mathbb{B}$, if $(f,g) \in \bigcup(\mathcal{A}(a) \times \mathcal{B}(b))^{\perp}$, then $f \in \bigcup \mathcal{A}(a)$ implies $g \in \bigcup \mathcal{B}(b)^{\perp}$ and $g \in \bigcup \mathcal{B}(b)$ implies $f \in \bigcup \mathcal{A}(a)^{\perp}$.*

Lemma 3.2.20. *For kits $\mathbf{A} = (|\mathbf{A}|, \mathcal{A})$, $\mathbf{B} = (|\mathbf{B}|, \mathcal{B})$ and $\mathbf{C} = (|\mathbf{C}|, \mathcal{C})$, the kits $(\mathbf{A} \otimes \mathbf{B}) \multimap \mathbf{C}$ and $\mathbf{A} \multimap (\mathbf{B} \multimap \mathbf{C})$ are isomorphic.*

Proof. We show that for objects $a \in |\mathbf{A}|$, $b \in |\mathbf{B}|$ and $c \in |\mathbf{C}|$, we have:

$$\Big((\mathcal{A}(a) \times \mathcal{B}(b))^{\perp\perp} \times \mathcal{C}^{\perp}(c)\Big)^{\perp} \cong \Big((\mathcal{A}(a) \times \mathcal{B}(b)) \times \mathcal{C}^{\perp}(c)\Big)^{\perp} \quad \text{and}$$
$$\Big(\mathcal{A}(a) \times (\mathcal{B}(b) \times \mathcal{C}^{\perp}(c))^{\perp\perp}\Big)^{\perp} \cong \Big(\mathcal{A}(a) \times (\mathcal{B}(b) \times \mathcal{C}^{\perp}(c))\Big)^{\perp}.$$

Since $\mathcal{A}(a) \times \mathcal{B}(b) \subseteq (\mathcal{A}(a) \times \mathcal{B}(b))^{\perp\perp}$, we have $(\mathcal{A}(a) \times \mathcal{B}(b)) \times \mathcal{C}^{\perp}(c) \subseteq (\mathcal{A}(a) \times \mathcal{B}(b))^{\perp\perp} \times \mathcal{C}^{\perp}(c)$ which implies that

$$\Big((\mathcal{A}(a) \times \mathcal{B}(b))^{\perp\perp} \times \mathcal{C}^{\perp}(c)\Big)^{\perp} \subseteq \Big((\mathcal{A}(a) \times \mathcal{B}(b)) \times \mathcal{C}^{\perp}(c)\Big)^{\perp}.$$

For the reverse inclusion, let G be in $\big((\mathcal{A}(a) \times \mathcal{B}(b)) \times \mathcal{C}^{\perp}(c)\big)^{\perp}$ and H in $(\mathcal{A}(a) \times \mathcal{B}(b))^{\perp\perp} \times \mathcal{C}^{\perp}(c)$ i.e. $H = H_1 \times H_2$ with $H_1 \in (\mathcal{A}(a) \times \mathcal{B}(b))^{\perp\perp}$ and $H_2 \in \mathcal{C}^{\perp}(c)$. To show that $G \perp H$, assume that there exists $((f_1, f_2), f_3) \in G \cap H$. Then, $f_3 \in H_2 \in \mathcal{C}^{\perp}(c)$ so by Corollary 3.2.19, we have $(f_1, f_2) \in (\mathcal{A}(a) \times \mathcal{B}(b))^{\perp}$. Since $(f_1, f_2) \in H_1 \in (\mathcal{A}(a) \times \mathcal{B}(b))^{\perp\perp}$, we must have $f_1 = \mathrm{id}$ and $f_2 = \mathrm{id}$. Hence, $(f_1, f_2) \in \mathcal{A}(a) \times \mathcal{B}(b)$ so by Corollary 3.2.19 again, we have $f_3 \in \mathcal{C}^{\perp\perp}(c)$ which implies that $f_3 = \mathrm{id}$ as desired.

The isomorphism $\big(\mathcal{A}(a) \times (\mathcal{B}(b) \times \mathcal{C}^{\perp}(c))^{\perp\perp}\big)^{\perp} \cong \big(\mathcal{A}(a) \times (\mathcal{B}(b) \times \mathcal{C}^{\perp}(c))\big)^{\perp}$ is derived using a similar argument. □

Proposition 3.2.21. *For kits $\mathbf{A} = (|\mathbf{A}|, \mathcal{A})$, $\mathbf{B} = (|\mathbf{B}|, \mathcal{B})$ and $\mathbf{C} = (|\mathbf{C}|, \mathcal{C})$, the component of the associator $\alpha_{|\mathbf{A}|,|\mathbf{B}|,|\mathbf{C}|} : (|\mathbf{A}| \times |\mathbf{B}|) \times |\mathbf{C}| \nrightarrow |\mathbf{A}| \times (|\mathbf{B}| \times |\mathbf{C}|)$ is in $\mathbf{SProf}((\mathbf{A} \otimes \mathbf{B}) \otimes \mathbf{C}, \mathbf{A} \otimes (\mathbf{B} \otimes \mathbf{C}))$.*

Proof. It suffices to show that the kits $(\mathbf{A} \otimes \mathbf{B}) \otimes \mathbf{C}$ and $\mathbf{A} \otimes (\mathbf{B} \otimes \mathbf{C})$ are isomorphic. By Lemmas 3.2.18 and 3.2.20, we have:

$$\begin{aligned}(\mathbf{A} \otimes \mathbf{B}) \otimes \mathbf{C} &\cong \Big((\mathbf{A} \otimes \mathbf{B}) \multimap \mathbf{C}^\perp\Big)^\perp \cong \Big(\mathbf{A} \multimap (\mathbf{B} \multimap \mathbf{C}^\perp)\Big)^\perp \\ &\cong \Big(\mathbf{A} \multimap (\mathbf{B} \otimes \mathbf{C})^\perp\Big)^\perp \cong \mathbf{A} \otimes (\mathbf{B} \otimes \mathbf{C}).\end{aligned}$$ □

Lemma 3.2.22. $(\mathbf{1}, \mathrm{Triv}_{\mathbf{1}})$ *is the tensor unit.*

Proof. For a kit $\mathbf{A}$, the isomorphisms $\mathrm{Triv}_{\mathbf{1}} \times \mathcal{A} \cong \mathcal{A} \cong \mathcal{A} \times \mathrm{Triv}_{\mathbf{1}}$ imply that the kits $\mathbf{1} \otimes \mathbf{A} \cong \mathbf{A} \cong \mathbf{A} \otimes \mathbf{1}$ are also isomorphic. Hence, the components of the left unitor $l_{|\mathbf{A}|} : |\mathbf{A}| \times |\mathbf{1}| \nrightarrow |\mathbf{A}|$ and right unitor $r_{|\mathbf{A}|} : |\mathbf{1}| \times |\mathbf{A}| \nrightarrow |\mathbf{A}|$ are in **SProf**. □

Lemma 3.2.23. *For kits* $\mathbf{A} = (|\mathbf{A}|, \mathcal{A})$ *and* $\mathbf{B} = (|\mathbf{B}|, \mathcal{B})$*,* $\mathcal{A} \otimes \mathcal{B}$ *is isomorphic to* $\mathcal{B} \otimes \mathcal{A}$ *which implies that the component of the symmetry* $\sigma_{|\mathbf{A}|,|\mathbf{B}|} : |\mathbf{A}| \times |\mathbf{B}| \nrightarrow |\mathbf{B}| \times |\mathbf{A}|$ *is in* $\mathbf{SProf}(\mathbf{A} \otimes \mathbf{B}, \mathbf{B} \otimes \mathbf{A})$.

We obtain as a corollary that **SProf** is a symmetric monoidal bicategory.

Proposition 3.2.24. *For kits* $\mathbf{A}, \mathbf{B}$ *and* $\mathbf{C}$ *there is a pseudo-natural adjoint equivalence*

$$\mathbf{SProf}(\mathbf{A} \otimes \mathbf{B}, \mathbf{C}^\perp) \cong \mathbf{SProf}(\mathbf{A}, (\mathbf{B} \otimes \mathbf{C})^\perp).$$

Proof. By Corollary 3.2.16, it suffices to show that $(\mathbf{A} \otimes \mathbf{B}) \multimap \mathbf{C}^\perp \cong \mathbf{A} \multimap (\mathbf{B} \otimes \mathbf{C})^\perp$ which we obtain from Lemmas 3.2.18 and 3.2.20. □

Corollary 3.2.25. *The bicategory* **SProf** *is* $*$*-automonous with dualizing object* $\perp = \mathbf{1}^\perp = \mathbf{1}$.

Proof. Immediate consequence of Remark 5 and Proposition 3.2.24. □

3.2.6 Exponential structure

We show in this section that the pseudo-comonad Σ on **Prof** presented in Section 1.4.5 can be lifted to **SProf**.

Recall that for a sequence $u = \langle a_1, \ldots, a_n \rangle$ in $\mathcal{S}|\mathbf{A}|$, a morphism f in $\mathrm{Endo}(u)$ is a pair $(\sigma, (f_i)_{i \in \underline{n}})$ of a permutation $\sigma \in \mathfrak{S}_n$ and an n-tuple of morphisms $f_i : a_i \to a_{\sigma(i)}$ in $|\mathbf{A}|$. Unless the permutation σ is the identity, the morphisms f_i are not endomorphisms so in order to define $!\mathcal{A}(u)$ from the sets $\mathcal{A}(a_i)$, we generate for a given $f \in \mathrm{Endo}(u)$, endomorphisms of a_i for every $i \in \underline{n}$ as we will see shortly.

Definition 3.2.26. For $\sigma \in \mathfrak{S}_n$ and $i \in \underline{n}$, let $o(\sigma, i)$ be the smallest strictly positive integer such that $\sigma^{o(\sigma,i)}(i) = i$. Equivalently, $o(\sigma, i)$ is the length of the cycle containing i in the disjoint cycle decomposition of the permutation σ. If there is no ambiguity on the permutation, we just write $o(i)$ for $o(\sigma, i)$.

Lemma 3.2.27. *Let n be in $\mathbb{N}$ and σ, τ, ϕ be permutations in $\mathfrak{S}_n$. If $\sigma = \varphi^{-1}\tau\varphi$, then for all $i \in \underline{n}$, $o(\sigma, i) = o(\tau, \varphi(i))$.*

Proof. We have $\sigma^{o(\tau,\varphi(i))}(i) = \varphi^{-1}\tau^{o(\tau,\varphi(i))}\varphi(i) = \varphi^{-1}\varphi(i) = i$. Assume that there exists $0 < j < o(\tau, \varphi(i))$ such that $\sigma^j(i) = i$, then $\tau^j\varphi(i) = \varphi\sigma^j(i) = \varphi(i)$ which contradicts the minimality of $o(\tau, \varphi(i))$. □

Definition 3.2.28. For a sequence $u = \langle a_1, \dots, a_n\rangle$ in $\mathcal{S}|\mathbf{A}|$ and a morphism $f \in \mathcal{S}|\mathbf{A}|(u, u)$, we define for each $i \in \underline{n}$, an endomorphism $\mathrm{loop}_i^f : a_i \to a_i$ as the composite

$$\begin{tikzcd} a_i \arrow[r, "f_i"] & a_{\sigma(i)} \arrow[r, "f_{\sigma(i)}"] & a_{\sigma^2(i)} \cdots \arrow[r, "f_{\sigma^{o(i)-1}(i)}"] & a_i \end{tikzcd}$$

Definition 3.2.29. For a kit structure $\mathbf{A} = (|\mathbf{A}|, \mathcal{A})$, the kit structure $!\mathbf{A}$ is defined as $(\mathcal{S}|\mathbf{A}|, (\mathcal{A}^{\mathcal{S}})^{\perp\perp})$, where for an object $u = \langle a_1, \dots, a_n\rangle \in \mathcal{S}|\mathbf{A}|$, $\mathcal{A}^{\mathcal{S}}(u)$ is given by

$$\{H \leq \mathrm{Endo}(u) \mid \forall f \in H, \forall i \in \underline{n}, \mathrm{loop}_i^f \in \mathcal{A}(a_i)\}.$$

The closure under double orthogonality ensures that $!\mathcal{A} = (\mathcal{A}^{\mathcal{S}})^{\perp\perp}$ is indeed a kit.

Note that for the coherence model of stability, it is not possible to have a model of linear logic where the exponential modality is *non-uniform* i.e. the web does not depend on the coherence structure. The two existing comonad structures (finite cliques and finite multicliques [52, 29]) both rely on the coherence structure, which means that we work in a more restrictive setting where a stable program can only interact with stable environments. Non-uniform coherence spaces have been studied in [19, 20, 18] but we lose the property that they model stable maps. When we categorify stability, we are now able to work in a non-uniform setting, i.e. there is no restriction on the environments a stable program can interact with as the web of $!\mathbf{A}$ does not depend on the kit structure.

Proposition 3.2.30. *For kits $\mathbf{A} = (|\mathbf{A}|, \mathcal{A})$, $\mathbf{B} = (|\mathbf{B}|, \mathcal{B})$ and an s-profunctor $P : \mathbf{A} \nrightarrow \mathbf{B}$, the profunctor $\mathcal{S}P : \mathcal{S}|\mathbf{A}| \nrightarrow \mathcal{S}|\mathbf{B}|$ is an s-profunctor $!\mathbf{A} \nrightarrow !\mathbf{B}$.*

Proof. Consider objects $u = \langle a_1, \ldots, a_n \rangle \in \mathcal{S}|\mathbf{A}|$ and $v = \langle b_1, \ldots, b_n \rangle \in \mathcal{S}|\mathbf{B}|$, morphisms $f = (\sigma, (f_i)_{i \in \underline{n}}) \in \mathrm{Endo}(u)$, $g = (\tau, (g_i)_{i \in \underline{n}}) \in \mathrm{Endo}(v)$, and $p = (\varphi, (p_i)_{i \in \underline{n}})$ be in $\mathcal{S}P(u, v) \cong \sum_{\varphi \in \mathfrak{S}_n} \prod_{i \in \underline{n}} P(a_{\varphi(i)}, b_i)$. Assume that $f \cdot p \cdot g = p$, i.e.

$$\sigma \circ \varphi \circ \tau = \varphi$$

and for all $i \in \underline{n}$,

$$f_{\varphi(\tau(i))} \cdot p_{\tau(i)} \cdot g_i = p_i.$$

We first show that it implies that for all $i \in \underline{n}$, $\mathrm{loop}^f_{\varphi(i)} \cdot p_i \cdot \mathrm{loop}^g_i = p_i$. By Lemma 3.2.27, we have:

$$\begin{aligned}\mathrm{loop}^f_{\varphi(i)} \cdot p_i \cdot \mathrm{loop}^g_i &= f_{\sigma^{o(\sigma,\varphi)-1}(\varphi(i))} \cdots f_{\sigma\varphi(i)} f_{\varphi(i)} \cdot p_i \cdot g_{\tau^{o(\tau,i)-1}(i)} \cdots g_{\tau(i)} \cdot g_i \\ &= f_{\sigma^{o(\tau,i)-1}(\varphi(i))} \cdots f_{\sigma\varphi(i)} f_{\varphi(i)} \cdot p_i \cdot g_{\tau^{o(\tau,i)-1}(i)} \cdots g_{\tau(i)} \cdot g_i\end{aligned}$$

Note that since $\tau^{o(\tau,i)}(i) = i$, $\tau^{o(\tau,i)-1}(i) = \tau^{-1}(i)$ we have $f_{\varphi(i)} \cdot p_i \cdot g_{\tau^{o(\tau,i)-1}(i)} = f_{\varphi\tau(\tau^{-1}(i))} \cdot p_{\tau(\tau^{-1}(i))} \cdot g_{\tau^{-1}(i)} = p_{\tau^{-1}(i)}$. Repeating this process, we obtain the desired result. We now show the following two implications:

$$f \in \bigcup !\mathcal{A}(u) \Rightarrow g \in \bigcup !\mathcal{B}(v) \quad \text{and} \quad g \in \bigcup !\mathcal{B}^{\perp}(v) \Rightarrow f \in \bigcup !\mathcal{A}^{\perp}(u).$$

- Assume that $f \in \bigcup !\mathcal{A}(u)$. By Lemma 3.2.8, it is equivalent to the following formula:

  $$\forall m \in \mathbb{N}, f^m = \mathrm{id} \vee \left(\exists k \in \mathbb{N}, f^{mk} \neq \mathrm{id} \wedge (\forall i \in \underline{n}, \mathrm{loop}^{f^{mk}}_{a_i} \in \mathcal{A}(a_i))\right).$$

  Let $m \in \mathbb{N}$ be such that $g^m \neq \mathrm{id}$, if $f^m = \mathrm{id}$, then $\sigma^m = \mathrm{id}$ and for all $i \in \underline{n}$, $f^m_i = \mathrm{id} \in \mathcal{A}(a_i)$. It implies that $\tau^m = \mathrm{id}$ since $\sigma^m \circ \varphi \circ \tau^m = \varphi$ and for all $i \in \underline{n}$, $\mathrm{loop}^{g^m}_i = g^m_i \in \mathcal{B}(b_i)$ since $f_i \cdot p_i \cdot g_i = p_i$ and P is an s-profunctor. Hence $g \in !\mathcal{B}(v)$ by Lemma 3.2.5.1.s If $f^m \neq \mathrm{id}$, then there exists $k \in \mathbb{N}$ such that $f^{mk} \neq \mathrm{id}$ and for all $i \in \underline{n}$, $\mathrm{loop}^{f^{mk}}_i \in \mathcal{A}(a_i)$. Since P is an s-profunctor and $\mathrm{loop}^{f^{mk}}_i \cdot p_i \cdot \mathrm{loop}^{g^{mk}}_i = p_i$, we have $\mathrm{loop}^{g^{mk}}_i \in \mathcal{B}(b_i)$ for all $i \in \underline{n}$. It remains to show that $g^{mk} \neq \mathrm{id}$. If $g^{mk} = \mathrm{id}$, then $\tau^{mk} = \mathrm{id}$ and $\mathrm{loop}^{g^{mk}}_i = g^{mk}_i = \mathrm{id} \in \mathcal{B}^{\perp}(b_i)$ for all i. It implies that $\sigma^{mk} = \mathrm{id}$ and $\mathrm{loop}^{f^{mk}}_i = f^{mk}_i \in \mathcal{A}^{\perp}(a_i)$ for all $i \in \underline{n}$ as P is an s-profunctor. Hence, we must have $f^{mk}_i = \mathrm{id}$ for all i which implies that $f^{mk} = \mathrm{id}$ since $f^{mk}_i \in \mathcal{A}(a_i)$ by assumption.

- Assume that $g \in \bigcup !\mathcal{B}^{\perp}(v)$ i.e. for all $m \in \mathbb{N}$, if for all $i \in \underline{n}$, $\mathrm{loop}^{g^m}_i \in \mathcal{B}(b_i)$ then $g^m = \mathrm{id}$. Assume that there exists $m \in \mathbb{N}$ such that for all

$i \in \underline{n}$, $\mathrm{loop}_i^{f^m} \in \mathcal{A}(a_i)$. Since $\mathrm{loop}_{\varphi(i)}^{f^m} \cdot p_i \cdot \mathrm{loop}_i^{g^m} = p_i$ for all $i \in \underline{n}$ and P is an s-profunctor, we have $\mathrm{loop}_i^{g^m} \in \mathcal{B}(b_i)$ for all $i \in \underline{n}$ which implies that $g^m = \mathrm{id}$. Hence, $\tau^m = \mathrm{id}$ and for all $i \in \underline{n}$, $\mathrm{loop}_i^{g^m} = g_i^m = \mathrm{id} \in \mathcal{B}^\perp(b_i)$ which entails that $\sigma^m = \mathrm{id}$ and $\mathrm{loop}_i^{f^m} = f_i^m \in \mathcal{A}^\perp(a_i)$ since P is an s-profunctor. We conclude that $f_i^m = \mathrm{id}$ for all $i \in \underline{n}$ so that $f^m = \mathrm{id}$ as desired. ☐

We obtain as a corollary that the operation $\mathbf{A} \mapsto !\mathbf{A}$ extends to a pseudo-functor $! : \mathbf{SProf} \to \mathbf{SProf}$. For the pseudo-comonad structure, we show that the components of the derelection and digging are in $\mathbf{SProf}$.

Lemma 3.2.31. *For a kit $\mathbf{A} = (|\mathbf{A}|, \mathcal{A})$ and a sequence $\langle a \rangle$ of length one in $\mathcal{S}|\mathbf{A}|$, we have*

$$!\mathcal{A}(\langle a \rangle) = \{(\mathrm{id}, \langle f \rangle) \mid f \in \mathcal{A}(a)\}.$$

Lemma 3.2.32. *For a kit $\mathbf{A} = (|\mathbf{A}|, \mathcal{A})$, the profunctor* $\mathrm{der}_{|\mathbf{A}|} : \mathcal{S}|\mathbf{A}| \nrightarrow |\mathbf{A}|$ *given by $(u, a) \mapsto \mathcal{S}|\mathbf{A}|(\langle a \rangle, u)$ is an s-profunctor $!\mathbf{A} \nrightarrow \mathbf{A}$.*

Proof. Note that the set $\mathcal{S}|\mathbf{A}|(\langle a \rangle, u)$ is empty unless the sequence $u = \langle a_1, \ldots, a_n \rangle$ is of length 1, in which case it is isomorphic to $|\mathbf{A}|(a, a_1)$. Let $(\mathrm{id}, \langle h \rangle) \in \mathrm{der}_{|\mathbf{A}|}(a, \langle a_1 \rangle)$ and assume that there exists $(\mathrm{id}, \langle f \rangle) \in \mathrm{Endo}(\langle a_1 \rangle)$ and $g \in \mathrm{Endo}(a)$ such that $(\mathrm{id}, \langle f \rangle) \cdot (\mathrm{id}, \langle h \rangle) \cdot g = (\mathrm{id}, \langle h \rangle)$ i.e. $f \circ h \circ g = h$.

If g is in $\bigcup \mathcal{A}(a)$, then $f = h^{-1} g^{-1} h$ is in $\bigcup \mathcal{A}(a_1)$ which is isomorphic to $\bigcup !\mathcal{A}(\langle a_1 \rangle)$ by Lemma 3.2.31. By a similar argument, we obtain that $(\mathrm{id}, \langle f \rangle) \in \bigcup !\mathcal{A}(\langle a_1 \rangle) \cong \bigcup \mathcal{A}(a_1)$ implies that g is in $\bigcup \mathcal{A}(a)$. ☐

Lemma 3.2.33. *For a kit $\mathbf{A} = (|\mathbf{A}|, \mathcal{A})$ and a sequence $\langle u_1, \ldots, u_n \rangle \in \mathcal{S}\mathcal{S}|\mathbf{A}|$, there is a mapping*

$$\overline{(-)} : \mathrm{Endo}(\langle u_1, \ldots, u_n \rangle) \to \mathrm{Endo}(u_1 \otimes \cdots \otimes u_n)$$

such that for $g \in \mathrm{Endo}(\langle u_1, \ldots, u_n \rangle)$, we have:

$$g \in \bigcup !!\mathcal{A}(\langle u_1, \ldots, u_n \rangle) \quad \Leftrightarrow \quad \overline{g} \in \bigcup !\mathcal{A}(u_1 \otimes \cdots \otimes u_n) \qquad \textit{and}$$
$$g \in \bigcup (!!\mathcal{A})^\perp(\langle u_1, \ldots, u_n \rangle) \quad \Leftrightarrow \quad \overline{g} \in \bigcup (!\mathcal{A})^\perp(u_1 \otimes \cdots \otimes u_n)$$

Proof. Let $g = (\tau, \langle g_i \rangle_{i \in \underline{n}})$ be in $\mathrm{Endo}(\langle u_1, \ldots, u_n \rangle)$ and let n_i be the length of the sequence $u_i = \langle a_1^i, \ldots, a_{n_i}^i \rangle$ for $1 \leq i \leq n$ and define m to be $\sum_{i \in \underline{n}} n_i$. For each i, g_i consists of a permutation $\phi_i : \underline{n_i} \to \underline{n_i}$ and an n_i-tuple of morphisms $\langle e_j^i : a_j^i \to a_{\phi_i(j)}^{\tau(i)} \rangle_{j \in \underline{n_i}}$ where $u_i = \langle a_1^i, \ldots, a_{n_i}^i \rangle$. Define $\overline{g} =$

$(\overline{\tau}, \langle \overline{g}_j \rangle_{j \in \underline{m}}) \in \mathrm{Endo}(u_1 \otimes \cdots \otimes u_n)$ as follows: the permutation $\overline{\tau} : \underline{m} \xrightarrow{\sim} \underline{m}$ is defined by:

$$\overline{\tau} : k \mapsto \phi_{l+1}(k - \sum_{i=1}^{l} n_i) + \sum_{i=1}^{\tau(l+1)-1} n_i \quad \text{if} \quad \sum_{i=1}^{l} n_i < k \leq \sum_{i=1}^{l+1} n_i$$

and for $1 \leq k \leq m$, we define $\overline{g}_k : a_k \to a_{\overline{\tau}(k)}$ as $\overline{g}_k := e^i_j$ where $i := l$ and $j := k - \sum\limits_{i=1}^{l} n_i$ if $\sum\limits_{i=1}^{l} n_i < k \leq \sum\limits_{i=1}^{l+1} n_i$. The idea is that if $\sum\limits_{i=1}^{l} n_i < k \leq \sum\limits_{i=1}^{l+1} n_i$, then k is in the subsequence u_{l+1}.

Note that for a^i_j in $u_1 \otimes \cdots \otimes u_n$, we have $\mathrm{loop}^{\overline{g}}_{a^i_j} = \mathrm{loop}^{\mathrm{loop}^g_{u_i}}_{a^i_j}$ so that g is in $\bigcup \mathcal{A}^{\mathcal{SS}}(\langle u_1, \dots, u_n \rangle)$ if and only if $\overline{g}$ is in $\bigcup \mathcal{A}^{\mathcal{S}}(u_1 \otimes \cdots \otimes u_n)$. Since we further have $\overline{g^n} = (\overline{g})^n$, we have $g \in \bigcup (!!\mathcal{A})^{\perp\perp}(\langle u_1, \dots, u_n \rangle)$ if an only if $\overline{g} \in \bigcup (!\mathcal{A})^{\perp}(u_1 \otimes \cdots \otimes u_n)$. It also implies that the following formulae are also equivalent:

$$\forall n \in \mathbb{N}, g^n = \mathrm{id} \vee (\exists m \in \mathbb{N}, g^{nm} \neq \mathrm{id} \wedge g^{nm} \in \bigcup \mathcal{A}^{\mathcal{SS}}(\langle u_1, \dots, u_n \rangle))$$

$$\forall n \in \mathbb{N}, \overline{g}^n = \mathrm{id} \vee (\exists m \in \mathbb{N}, \overline{g}^{nm} \neq \mathrm{id} \wedge \overline{g}^{nm} \in \bigcup \mathcal{A}^{\mathcal{S}}(u_1 \otimes \cdots \otimes u_n)$$

which implies that $g \in \bigcup !!\mathcal{A}(\langle u_1, \dots, u_n \rangle)$ if an only if $\overline{g} \in \bigcup !\mathcal{A}(u_1 \otimes \cdots \otimes u_n)$ by Definition 3.2.7. ☐

Lemma 3.2.34. *For a kit* $\mathbf{A} = (|\mathbf{A}|, \mathcal{A})$*, the profunctor* $\mathrm{dig}_{|\mathbf{A}|} : \mathcal{S}|\mathbf{A}| \nrightarrow \mathcal{SS}|\mathbf{A}|$ *given by* $(v, \langle u_1, \dots, u_n \rangle) \mapsto \mathcal{S}|\mathbf{A}|(u_1 \otimes \cdots \otimes u_n, v)$ *is an s-profunctor* $!\mathbf{A} \nrightarrow !!\mathbf{A}$.

Proof. Let $h : u_1 \otimes \cdots \otimes u_n \to v$ be in $\mathrm{dig}_{\mathbb{A}}(v, \langle u_1, \dots, u_n \rangle)$, it is of the form $(\rho, \langle h_k \rangle_{k \in \underline{m}})$ where $\rho : \underline{m} \xrightarrow{\sim} \underline{m}$ and $h_k : (u_1 \otimes \cdots \otimes u_n)_k \to v_{\rho(k)}$ for $1 \leq k \leq m$. Assume that there exists $f = (\sigma, \langle f_k \rangle_{k \in \underline{m}}) \in \mathrm{Endo}(v)$ and $g = (\tau, \langle g_i \rangle_{i \in \underline{m}}) \in \mathrm{Endo}(\langle u_1, \dots, u_n \rangle)$ such that $f \cdot h \cdot g = h$. The morphism $\overline{g} : u_1 \otimes \cdots \otimes u_n \to u_1 \otimes \cdots \otimes u_n$ defined in Lemma 3.2.34 verifies $h \cdot g = h \circ \overline{g}$ which implies that $f \cdot h \cdot g = h$ is equivalent to $f \circ h \circ \overline{g} = h$ i.e.

$$\sigma \rho \overline{\tau} = \rho \qquad \text{and} \qquad f_{\rho\overline{\tau}(k)} \circ h_{\overline{\tau}(k)} \circ \overline{g}_k = h_k \quad \text{for all } 1 \leq k \leq m.$$

- Assume that f is in $\bigcup !\mathcal{A}(v)$, then $\overline{g}$ is in $\bigcup !\mathcal{A}(u_1 \otimes \cdots \otimes u_n)$ by closure under conjugation, which implies that g is in $\bigcup !!\mathcal{A}(\langle u_1, \dots, u_n \rangle)$ by Lemma 3.2.34.
- Assume that g is in $\bigcup (!!\mathcal{A})^{\perp}(\langle u_1, \dots, u_n \rangle)$, then $\overline{g}$ is in $\bigcup (!\mathcal{A})^{\perp}(u_1 \otimes \cdots \otimes u_n)$ by Lemma 3.2.34 which implies that f is in $\bigcup (!\mathcal{A})^{\perp}(v)$ by closure under conjugation. ☐

3.2.7 Cartesian closed structure.

In this section, we show that the cartesian closed structure of $\mathbf{Prof}_{\mathcal{S}}$ lifts to **SProf**. Similarly to Chapter 2, we show that the Seely equivalence lifts to **SProf** and the rest of the structure is derived from the monoidal closed structure of **SProf**. The coKleisli bicategory $\mathbf{SProf}_!$ inherits the finite products structure from the linear category **SProf** where they are given by the & construction, and terminal object $(\mathbf{0}, \varnothing)$.

Lemma 3.2.35. *For kits* $\mathbf{A}$ *and* $\mathbf{B}$*, the mapping* $(f, g) \mapsto f \otimes g$ *induces an inclusion*

$$\mathcal{A}^{\mathcal{S}}(u) \times \mathcal{B}^{\mathcal{S}}(v) \subseteq (\mathcal{A} \,\&\, \mathcal{B})^{\mathcal{S}}(u \otimes v)$$

Proof. The inclusion follows from the following observation: for $a \in |\mathbf{A}|$, $\mathrm{loop}^{f\otimes g}_{(1,a)} = (1, \mathrm{loop}^f_a)$ and for $b \in |\mathbf{B}|$, $\mathrm{loop}^{f\otimes g}_{(2,b)} = (2, \mathrm{loop}^g_b)$. ☐

Lemma 3.2.36. *For kits* $\mathbf{A} = (|\mathbf{A}|, \mathcal{A})$, $\mathbf{B} = (|\mathbf{B}|, \mathcal{B})$*, the Seely profunctors* $S_{|\mathbf{A}|,|\mathbf{B}|} : !(|\mathbf{A}| \,\&\, |\mathbf{B}|) \nrightarrow\, !\,|\mathbf{A}| \otimes !\,|\mathbf{B}|$ *and* $I_{|\mathbf{A}|,|\mathbf{B}|} : !\,|\mathbf{A}| \otimes !\,|\mathbf{B}| \nrightarrow\, !(|\mathbf{A}| \,\&\, |\mathbf{B}|)$ *induce an adjoint equivalence* $!(\mathbf{A} \,\&\, \mathbf{B}) \simeq\, !\mathbf{A} \otimes !\mathbf{B}$ *in* **SProf***. We also have* $!(\mathbf{0}, \varnothing) \cong (\mathbf{1}, \mathrm{Triv})$.

Proof. We show that for $(f, g) \in (\mathrm{Endo}(u) \times \mathrm{Endo}(v))$, the mapping $(f, g) \mapsto f \otimes g$ induces an inclusion

$$!\mathcal{A}(u) \times !\mathcal{B}(v) \hookrightarrow\, !(\mathcal{A} \,\&\, \mathcal{B})(u \otimes v).$$

Assume that (f, g) is in $!\mathcal{A}(u) \times !\mathcal{B}(v)$ and that there exists $n \in \mathbb{N}$ such that $(f \otimes g)^n \neq \mathrm{id}$. It implies that $(f, g)^n \neq (\mathrm{id}, \mathrm{id})$, we suppose without loss of generality that $f^n \neq \mathrm{id}$. Since f is in $!\mathcal{A}(u)$, there exists $m \in \mathbb{N}$ such that $f^{nm} \neq \mathrm{id}$ and $f^{nm} \in \mathcal{A}^{\mathcal{S}}(u)$. If $g^{nm} = \mathrm{id}$ then $g^{nm} \in \mathcal{B}^{\mathcal{S}}(v)$ so that $(f, g)^{nm} \in \mathcal{A}^{\mathcal{S}}(u) \times \mathcal{B}^{\mathcal{S}}(v)$ which implies that $(f \otimes g)^{nm} \in (\mathcal{A} \,\&\, \mathcal{B})^{\mathcal{S}}(u \otimes v)$ by Lemma 3.2.35 and $f^{nm} \neq \mathrm{id}$ entails that $(f \otimes g)^{nm} \neq \mathrm{id}$.

If $g^{nm} \neq \mathrm{id}$ then since $g^{nm} \in\, !\mathcal{B}(v)$, there exists $l \in \mathbb{N}$ such that $g^{nml} \neq \mathrm{id}$ and $g^{nml} \in \mathcal{B}^{\mathcal{S}}(v)$. Since $\mathcal{A}^{\mathcal{S}}(u)$ is closed under powers, f^{nml} is in $\mathcal{A}^{\mathcal{S}}(u)$ as well. Hence, by Lemma 3.2.35, $(f \otimes g)^{nml} \in (\mathcal{A} \,\&\, \mathcal{B})^{\mathcal{S}}(u \otimes v)$ and $g^{nml} \neq \mathrm{id}$ implies $(f \otimes g)^{nml} \neq \mathrm{id}$. Hence, $(f \otimes g)$ is in $!(\mathcal{A} \,\&\, \mathcal{B})(u \otimes v)$ as desired.

We now obtain the desired inclusion.

$$(!\mathcal{A} \otimes !\mathcal{B})(u, v) = (!\mathcal{A}(u) \times !\mathcal{B}(v))^{\perp\perp} \hookrightarrow\, !(\mathcal{A} \,\&\, \mathcal{B})^{\perp\perp}(u \otimes v) = !(\mathcal{A} \,\&\, \mathcal{B})(u \otimes v).$$

We show that for $h \in \mathrm{Endo}(w)$, the mapping $h \mapsto (h.1, h.2)$ induces an inclusion

$$!(\mathcal{A} \,\&\, \mathcal{B})(w) \hookrightarrow (!\mathcal{A} \otimes !\mathcal{B})(w.1, w.2).$$

To do so, we show that $(\mathcal{A} \,\&\, \mathcal{B})^{\mathcal{S}}(w) \hookrightarrow !\mathcal{A}(w.1) \times !\mathcal{B}(w.2)$ and the desired inclusion will follow by applying $(-)^{\perp\perp}$ on both sides. Assume that h is in $(\mathcal{A} \,\&\, \mathcal{B})^{\mathcal{S}}(w)$, then since for elements a in $w.1$ and b in $w.2$

$$\mathrm{loop}^{h}_{(1,a)} = (1, \mathrm{loop}^{h.1}_{a}) \quad \text{and} \quad \mathrm{loop}^{h}_{(2,b)} = (2, \mathrm{loop}^{h.2}_{b})$$

we have $(h.1, h.2) \in (\mathcal{A}^{\mathcal{S}}(w.1) \times \mathcal{B}^{\mathcal{S}}(w.2)) \subseteq (!\mathcal{A}(w.1) \times !\mathcal{B}(w.2))$. □

For kit structures $\mathbf{A}$ and $\mathbf{B}$, the internal hom $\mathbf{A} \multimap \mathbf{B}$ is equipped with a linear evaluation morphism $\mathrm{ev}_{|\mathbf{A}|,|\mathbf{B}|} : \mathbf{A} \otimes (\mathbf{A} \multimap \mathbf{B}) \nrightarrow \mathbf{B}$. We define the function space in $\mathbf{SProf}_{!}$ as $\mathbf{A} \Rightarrow \mathbf{B} = !\mathbf{A} \multimap \mathbf{B}$ and the non-linear evaluation $\mathrm{Ev}_{|\mathbf{A}|,|\mathbf{B}|} : (\mathcal{S}(|\mathbf{A}| \Rightarrow |\mathbf{B}|) \,\&\, |\mathbf{A}|) \nrightarrow |\mathbf{B}|$ as the morphism

$$\mathrm{ev}_{\mathcal{S}|\mathbf{A}|,|\mathbf{B}|} \circ (\mathrm{der}_{|\mathbf{A}| \Rightarrow |\mathbf{B}|} \otimes \mathrm{id}) \circ S_{|\mathbf{A}| \Rightarrow |\mathbf{B}|,|\mathbf{A}|}.$$

As a composite of s-profunctors, $\mathrm{Ev}_{|\mathbf{A}|,|\mathbf{B}|}$ is in $\mathbf{SProf}_{!}((\mathbf{A} \Rightarrow \mathbf{B}) \& \mathbf{A}, \mathbf{B})$. For a stable species P in $\mathbf{SProf}_{!}(\mathbf{A} \& \mathbf{B}, \mathbf{C})$, its *currying* $\Lambda(P) \in \mathbf{SProf}_{!}(\mathbf{A}, \mathbf{B} \Rightarrow \mathbf{C})$ is given by $\lambda(P \circ T_{|\mathbf{A}|,|\mathbf{B}|})$ where

$$\lambda : \mathbf{SProf}(!\mathbf{A} \otimes !\mathbf{B}, \mathbf{C}) \to \mathbf{SProf}(!\mathbf{A}, !\mathbf{B} \multimap \mathbf{C})$$

is provided by the monoidal closed structure on $\mathbf{SProf}$.

Theorem 3.2.37. *The bicategory* $\mathbf{SProf}_{!}$ *is cartesian closed.*

3.2.8 Differential structure.

The bicategory of generalized species $\mathbf{Prof}_{\mathcal{S}}$ constitutes a model of differential linear logic but it is not a well-behaved model to study integration or resolution of differential equations. Labelle showed for example that a combinatorial species can have an infinite number of non isomorphic antiderivatives, a unique one or no antiderivatives at all. The study of integration and resolution of differential equation becomes a difficult subject where solutions are often given on a case by case basis [86]. Flat species on the other hand are always integrable and since they correspond to our stable species of type $!\,\mathbf{1} \nrightarrow \mathbf{1}$, we hope to generalize this result to the setting of stable species. In this section, we show that $\mathbf{SProf}_{!}$ is a model of differential linear logic, the study of resolution of differential equations will be the subject of future work.

To show that stable species constitute a model of differential linear logic, it suffices to show that the interpretation of the codereliction rule and two costructural rules (coweakening and cocontraction) can be extended from $\mathbf{Prof}$

to **SProf**. Similarly to Chapter 2, since the components of the coweakening and cocontraction are obtained from the Seely equivalences and the biproduct structure, it only suffices to show it for the codereliction.

Lemma 3.2.38. *For a kit* $\mathbf{A} = (|\mathbf{A}|, \mathcal{A})$*, the profunctor* $\overline{\mathrm{der}}_{|\mathbf{A}|} : |\mathbf{A}| \nrightarrow \mathcal{S}|\mathbf{A}|$ *given by* $(a, u) \mapsto \mathcal{S}|\mathbf{A}|(u, \langle a \rangle)$ *is an s-profunctor* $\mathbf{A} \nrightarrow {!\mathbf{A}}$*.*

Proof. Similar to Lemma 3.2.32. □

3.3 Extensional theory

In the second part of this chapter, we study the extensional aspects of s-profunctors and stable species. A (pre)kit structure $\mathbf{A} = (|\mathbf{A}|, \mathcal{A})$ determines a generalized domain $\mathbf{St}(\mathbf{A}) \hookrightarrow \widehat{|\mathbf{A}|}$ of stabilized presheaves on $|\mathbf{A}|$. S-profunctors and stable species can be characterized extensionally as linear and stable functors respectively between these generalized domains. We obtain as a corollary of the biequivalence between the bicategory of stable species and the 2-category of stable functors (Definition 3.3.15) that the latter is cartesian closed.

3.3.1 Stabilized presheaves

We first describe the generalized domains determined by the prekit structures on groupoids.

For a presheaf $X : |\mathbf{A}|^{\mathrm{op}} \to \mathbf{Set}$, $a \in |\mathbf{A}|$ and $x \in X(a)$, we denote by $\mathrm{Stab}(x)$ the subgroup of $|\mathbf{A}|(a, a)$ consisting of those endomorphisms $f : a \to a$ such that $x \cdot f = x$. We now define ($\mathcal{A}$-)stabilized presheaves as those presheaves whose elements all have stabilizer in $\mathcal{A}$.

Definition 3.3.1. For a prekit structure $\mathbf{A} = (|\mathbf{A}|, \mathcal{A})$, we let $\mathbf{St}(\mathbf{A})$ be the full subcategory of $\widehat{|\mathbf{A}|}$ consisting of presheaves X such that for every $a \in |\mathbf{A}|$ and $x \in X(a)$, $\mathrm{Stab}(x) \in \mathcal{A}(a)$, we call these *stabilized presheaves.*

Note that if $\mathbf{A}$ is a kit structure, it induces an equivalence of categories

$$\mathbf{SProf}(\mathbf{1}, \mathbf{A}) \simeq \mathbf{St}(\mathbf{A}).$$

We now proceed to study properties of the subcategory $\mathbf{St}(\mathbf{A})$ in the more general case of prekits. To generate stabilized presheaves, we consider representables *quotiented* by subgroups in the prekit.

Definition 3.3.2. For an object a of a groupoid $\mathbb{A}$ and $G \leq \mathrm{Endo}(a)$, let $\overline{\mathrm{y}}(a|G) \in \widehat{\mathbb{A}}$ be the quotient of $\mathrm{y}(a)$ under G; that is, the colimit of the composite $G \to \mathbb{A} \hookrightarrow \widehat{\mathbb{A}}$.

Explicitely, the presheaf $\overline{\mathrm{y}}(a|G)$ maps an object a' to the quotient of $\mathrm{y}(a)(a') = \mathbb{A}(a', a)$ under the equivalence relation $\sim_G$ given by $g \sim_G f$ if and only if $gf^{-1} \in G$. For example, $\overline{\mathrm{y}}(a|\mathrm{Triv}(a)) \cong \mathrm{y}(a)$, and $\overline{\mathrm{y}}(a|\mathrm{Endo}(a))(a')$ is a singleton when $a \cong a'$, and $\emptyset$ otherwise.

Lemma 3.3.3. *For a prekit structure* $\mathbf{A} = (|\mathbf{A}|, \mathcal{A})$*,* $a \in |\mathbf{A}|$ *and* $G \in \mathcal{A}(a)$*,* $\overline{\mathrm{y}}(a|G) \in \mathbf{St}(\mathbf{A})$*.*

Proof sketch. The stabilizer of an element $[f]_{\sim_G} \in \overline{\mathrm{y}}(a|G)(a')$ is the conjugate subgroup $f^{-1}Gf$, which is in $\mathcal{A}(a')$ since prekits are closed under conjugation. □

We show that in fact $\mathbf{St}(\mathbf{A})$ is equivalent to the free coproduct completion of the full subcategory of $\widehat{|\mathbf{A}|}$ spanned by quotients of representables of the form $\overline{\mathrm{y}}(a|G)$, for $a \in |\mathbf{A}|$ and $G \in \mathcal{A}(a)$. The first step in showing this is to observe that $\mathbf{St}(\mathbf{A})$ has all coproducts, calculated as in $\widehat{|\mathbf{A}|}$. This holds because for presheaves X and Y, the stabilizer of an element of $X + Y$ is equal to the stabilizer of the corresponding element in X or Y. Next, we show a representation theorem stating that every object of $\mathbf{St}(\mathbf{A})$ can be obtained as a sum of quotients of representables of the form $\overline{\mathrm{y}}(a|G)$, with $a \in |\mathbf{A}|$ and $G \in \mathcal{A}(a)$.

Lemma 3.3.4. *For a prekit structure* $\mathbf{A} = (|\mathbf{A}|, \mathcal{A})$*, assuming the axiom of choice, every presheaf* $X \in \mathbf{St}(\mathbf{A})$ *has a representation*

$$X \cong \sum_{i \in I} \overline{\mathrm{y}}(a_i|G_i)$$

where each $a_i \in |\mathbf{A}|$ *and* $G_i \in \mathcal{A}(a_i)$*.*

Proof. Using the axiom of choice, we choose a representative $a_c \in |\mathbf{A}|$ for each connected component $c \in \pi_0(|\mathbf{A}|)$. For each $x \in X(a)$, we denote by $o(x) := \{x \cdot f \mid f \in |\mathbf{A}|(a, a)\}$, the orbit of x. For $a \in |\mathbf{A}|$, we write $\mathrm{Orbits}(X(a)) := \{o(x) \mid x \in X(a)\}$ for the set of orbits of $X(a)$. Using the axiom of choice again, we chose a representative x_i for each orbit $i \in \mathrm{Orbits}(X(a))$. We define the presheaf Y as:

$$\sum_{c \in \pi_0(|\mathbf{A}|)} \; \sum_{i \in \mathrm{Orbits}(X(a_c))} \overline{\mathrm{y}}(a_c|\mathrm{Stab}(x_i))$$

Let $\alpha : Y \Rightarrow X$ be the natural transformation whose components α_a are given by:

$$(c, i, [f : a \to a_c]) \quad \mapsto \quad x_i \cdot f$$

We first start by showing that α_a is well-defined: assume that $[f] = [g]$ i.e. fg^{-1} is in $\mathrm{Stab}(x_i)$, then $x_i \cdot (fg^{-1}) = x_i$ which implies that $x_i \cdot f = x_i \cdot g$ as desired.

For surjectivity, let x be $X(a)$ and let c be the connected component containing a so there exists a morphism $h : a_c \to a$. Let x_i be the representative of the orbit $o(x \cdot h) \in \mathrm{Orbits}(X(a_c))$ so there exists $g : a_c \to a_c$ such that $x \cdot h = x_i \cdot g$. The element $(c, o(x \cdot h), [gh^{-1}])$ in $Y(a)$ is then a preimage of x as $x_i \cdot gh^{-1} = x$.

For injectivity, let $(c, i, [f : a \to a_c])$ and $(d, j, [g : a \to a_d])$ be two elements of $Y(a)$ such that $x_i \cdot f = x_j \cdot g$. Since $gf^{-1} \in |\mathbf{A}|(a_c, a_d)$, a_c and a_d are in the same connected component which implies that $c = d$ and therefore $a_c = a_d$ since there is one representative chosen for each component. Hence, we have $x_i = x_j \cdot (gf^{-1}) \in X(a_c)$ implies that x_i and x_j are in the same orbit so $i = j$ and $x_i = x_j$ since there is one representative chosen for each orbit. We now have $x_i = x_j \cdot (gf^{-1})$ which implies that $[f] = [g]$.

Naturality of α is immediate so we conclude that it is a natural isomorphism. $\square$

Corollary 3.3.5. *For a kit* $\mathbf{A}$*, using the axiom of choice, every presheaf in* $\mathbf{St}(\mathbf{A})$ *has a representation as a filtered colimit of finitely presentable presheaves of the form* $\coprod_{i \in I} \overline{\mathrm{y}}(a_i|G_i)$ *with the set* I *finite and each group* $G_i \in \mathcal{A}(a_i)$ *finitely generated for* $i \in I$*.*

Proposition 3.3.6. *For a prekit structure* $\mathbf{A} = (|\mathbf{A}|, \mathcal{A})$*, the category* $\mathbf{St}(\mathbf{A})$ *is equivalent to the free coproduct completion of the full subcategory of* $\widehat{|\mathbf{A}|}$ *spanned by the* $\overline{\mathrm{y}}(a|G)$*, for* $a \in |\mathbf{A}|$ *and* $G \in \mathcal{A}(a)$*.*

Proof. Write $\sum_{\mathcal{A}}$ for this completion. Since $\mathbf{St}(\mathbf{A})$ contains the $\overline{\mathrm{y}}(a|G)$ and has all coproducts there is a canonical functor $\sum_{\mathcal{A}} \to \mathbf{St}(\mathbf{A})$. It is full and faithful by a straightforward argument and essentially surjective on objects by Lemma 3.3.4. $\square$

We now focus on categories of stabilized presheaves associated with kits and discuss the additional properties which hold there. A major difference with the prekit case is the existence in $\mathbf{St}(\mathbf{A})$ of all non-empty limits and filtered colimits. For the latter, we use that kits are closed under lubs of directed sets of subgroups.

Lemma 3.3.7. *For a kit structure* $\mathbf{A}$*,* $\mathbf{St}(\mathbf{A})$ *has filtered colimits.*

The case of non-empty limits is immediate given the following:

Lemma 3.3.8. *For a kit structure* $\mathbf{A}$ *and a morphism* $\alpha : X \Rightarrow Y$ *in* $\widehat{|\mathbf{A}|}$*, if* $Y \in \mathbf{St}(\mathbf{A})$ *then* $X \in \mathbf{St}(\mathbf{A})$*.*

Proof idea. For $a \in |\mathbf{A}|$ and $x \in X(a)$, $\mathrm{Stab}(x) \subseteq \mathrm{Stab}(\alpha_a x) \in \mathcal{A}(a)$, and kits are closed under subgroups. □

Corollary 3.3.9. *For a kit structure* $\mathbf{A}$*, the embedding* $\mathbf{St}(\mathbf{A}) \hookrightarrow \widehat{|\mathbf{A}|}$ *creates isomorphisms, coproducts, filtered colimits, epimorphisms and non-empty limits.*

Proof. Let $\{X_i\}_{i\in I}$ be a finite set of presheaves in $\mathbf{St}(\mathbf{A})$. We show that their coproduct in $\widehat{|\mathbf{A}|}$ is an element of $\mathbf{St}(\mathbf{A})$. For $(i, x) \in (\coprod_{i\in I} X_i)(a) = \coprod_{i\in I} X_i(a)$, $\mathrm{Stab}_{\coprod_{i\in I} X_i}(i, x) = \mathrm{Stab}_{X_i}(x)$ which is in $\mathcal{A}(a)$, so we are done.

Let $D : \mathbb{I} \to \mathbf{St}(\mathbf{A})$ be a filtered diagram. Denote by X_i the presheaf $D(i)$ for $i \in \mathbb{I}$ and write X for the colimit of D. Let a be in $|\mathbf{A}|$, for every $x \in X(a)$, there exists $i \in \mathbb{I}$ and $y \in X_i(a)$ such that $(\iota_i)_a(y) = x$, where $\iota_i : X_i \to X$ denotes the cone component. By naturality of ι_i, if $f : a \to a$ is in $\mathrm{Stab}_X(x)$ then $(\iota_i)_a(yf) = (\iota_i)_a(y)$. Since $\mathbb{I}$ is filtered, there exists $j \in \mathbb{I}$ and $\alpha : i \to j$ such that $D(\alpha)_a(y \cdot f) = D(\alpha)_a(y)$. Hence, $f \in \mathrm{Stab}_{X_j}(D(\alpha)_a(y))$ so f is in $\bigcup \mathcal{A}(a)$ as desired.

We show that a morphism $\varepsilon : X \Rightarrow Y$ is an epimorphism in $\mathbf{St}(\mathbf{A})$ if and only if its components $\varepsilon_a : X(a) \to Y(a)$ are surjective for all $a \in |\mathbf{A}|$. Assume that there exists $a \in |\mathbf{A}|$ and $y \in Y(a)$ such that for all $x \in X(a)$, $\varepsilon_a(x) \neq y$. By Lemma 3.3.4, $Y \cong \sum_{i\in I} \overline{\mathrm{y}}(a_i|G_i)$ where each G_i is a group in $\mathcal{A}(a_i)$. Let $j \in I$ be such that y is in the component $\overline{\mathrm{y}}(a_j|G_j)(a)$. We show that for all $b \in |B|$ and $z \in Y(b)$, z is not in the image of ε_b. Assume that there exists $x \in X(b)$ such that $\varepsilon_b(x) = z$, then for any representative $f \in |\mathbf{A}|(a, a_j)$ of the equivalence class y and $g \in |\mathbf{A}|(b, b_j)$ of the equivalence class z, we obtain by naturality of ε that

$$\varepsilon_a(x \cdot (g^{-1}f)) = (\varepsilon_b(x)) \cdot (g^{-1}f) = [g] \cdot (g^{-1}f) = [gg^{-1}f] = [f] = y.$$

Hence, we can factor ε as

$$X \longrightarrow \textstyle\sum_{\substack{i\in I\\ i\neq j}} \overline{\mathrm{y}}(a_i|G_i) \hookrightarrow \textstyle\sum_{i\in I} \overline{\mathrm{y}}(a_i|G_i) \cong Y$$

Let $\iota_Y : Y \to Y + \mathbf{1}$ be the left hand coproduct inclusion and let $\alpha : Y \to Y + \mathbf{1}$ be the natural transformation whose components α_a are given by

$$\sum_{i \in I} \overline{\mathrm{y}}(a_i|G_i)(a) \ni (i, y) \mapsto \begin{cases} \iota_a(i, y) \in \{1\} \times Y(a) & \text{if } i \neq j \\ (2, \star) \in \{2\} \times \mathbf{1}(a) & \text{if } i = j \end{cases}$$

It is immediate that $\iota_Y \varepsilon = \alpha\varepsilon$ but we do not have $\iota_y = \alpha$ contradicting the assumption of ε being an epimorphism. Hence, we must have that ε is pointwise surjective as desired. □

On the other hand, the category $\mathbf{St}(\mathbf{A})$ has a terminal object (an empty limit) only if $\mathcal{A}$ is maximal:

Proposition 3.3.10. *For a kit structure* $\mathbf{A}$*, the following are equivalent.*

1. $\mathbf{St}(\mathbf{A}) = \widehat{|\mathbf{A}|}$.
2. *For all* $a \in |\mathbf{A}|$, $\mathcal{A}(a) = \{G \mid G \leq |\mathbf{A}|(a, a)\}$.
3. *The terminal presheaf is in* $\mathbf{St}(\mathbf{A})$.

Proof.

- $(1 \Rightarrow 2)$ Let a be in $|\mathbf{A}|$ and G be a subgoup of $|\mathbf{A}|(a, a)$. Since $\overline{\mathrm{y}}(a|G)$ is in $\mathbf{St}(\mathbf{A})$ by assumption and $\mathrm{Stab}(\overline{\mathrm{y}}(a|G)) = \{K \mid K \leq G\}$, we obtain that $G \in \mathcal{A}(a)$ as desired.
- $(2 \Rightarrow 3)$ Let $1_{|\mathbf{A}|} : |\mathbf{A}|^{op} \to \mathbf{Set}$ denote the terminal presheaf. For all $a \in |\mathbf{A}|$, $\mathrm{Stab}(1_{|\mathbf{A}|})(a) = \{G \leq |\mathbf{A}|(a, a) \mid a \in |\mathbf{A}|\}$ which implies that $1_{|\mathbf{A}|}$ is in $\mathbf{St}(\mathbf{A})$ by definition.
- $(3 \Rightarrow 1)$ Direct consequence of Lemma 3.3.8.

□

Convention 1. For a kit $\mathbf{A} = (|\mathbf{A}|, \mathcal{A})$, the Yoneda embedding $\mathrm{y}_{|\mathbf{A}|} : |\mathbf{A}| \hookrightarrow \widehat{|\mathbf{A}|}$ factors through the inclusion $\iota_{\mathbf{A}} : \mathbf{St}(\mathbf{A}) \hookrightarrow \widehat{|\mathbf{A}|}$ by an embedding which we will denote $\mathbf{y}_{\mathbf{A}} : |\mathbf{A}| \hookrightarrow \mathbf{St}(\mathbf{A})$.

Lemma 3.3.11. *The embeddings* $\iota_{\mathbf{A}}$ *and* $\mathbf{y}_{\mathbf{A}}$ *are dense, and* $\iota_{\mathbf{A}}$ *is a Lan of* $\mathrm{y}_{|\mathbf{A}|}$ *along* $\mathbf{y}_{\mathbf{A}}$.

$$\begin{tikzcd}
|\mathbf{A}| \arrow[rr, "\mathrm{y}_{|\mathbf{A}|}"] \arrow[dr, hook, "\mathbf{y}_{\mathbf{A}}"'] & {} \arrow[d, phantom, "\Downarrow"] & \widehat{|\mathbf{A}|} \\
& \mathbf{St}(\mathbf{A}) \arrow[ur, dotted, hook, "\iota_{\mathbf{A}}"'] &
\end{tikzcd}$$

Presheaf orthogonality

In this section, we show that the orthogonality on prekits on a groupoid $\mathbb{A}$ translates to an orthogonality between presheaves and co-presheaves over $\mathbb{A}$. This orthogonality allows us to recast the duality between prekits to the usual setting where we control interaction between programs and environments. We ensure here that the stabilizer groups on either side of the interaction between a program and an environment have no elements in common but the identity.

Definition 3.3.12. For a groupoid $\mathbb{A}$, a presheaf $X : \mathbb{A}^{\mathrm{op}} \to \mathbf{Set}$ and a co-presheaf $Y : \mathbb{A} \to \mathbf{Set}$, we say that X and Y are *orthogonal*, written $X \perp_{\mathbb{A}} Y$, if the presheaf $\langle X, Y\rangle$ on $\mathbb{A}$ with object mapping $a \longmapsto X(a) \times Y(a)$ and functorial action defined by $(x, y) \cdot f = (x \cdot f, f^{-1} \cdot y)$ is *free*: all its elements have trivial stabilizer.

Note that this orthogonality is not focused, i.e. for a profunctor $P : \mathbb{A} \nrightarrow \mathbb{B}$, a presheaf $X : \mathbb{A}^{\mathrm{op}} \to \mathbf{Set}$ and a copresheaf $Y : \mathbb{A}^{\mathrm{op}} \to \mathbf{Set}$ the equivalence $X \perp_{\mathbb{A}} Y \circ P \Leftrightarrow P \circ X \perp_{\mathbb{B}} Y$ does not hold in general. This is why we had to prove both forward and backward preservation in the previous section for the linear logic structure unlike the finiteness spaces construction in Chapter 2.

For a groupoid $\mathbb{A}$, the operation sending a prekit $\mathcal{A}$ to the category $\mathbf{St}(\mathbb{A}, \mathcal{A})$ has a right inverse.

Lemma 3.3.13. *For a groupoid* $\mathbb{A}$ *and a full subcategory* $\mathbb{C}$ *of* $\widehat{\mathbb{A}}$*, the family* $\mathrm{Stab}(\mathbb{C})$ *defined by*

$$\mathrm{Stab}(\mathbb{C})(a) = \{G \mid \exists X \in \mathbb{C}, \exists x \in X(a), G = \mathrm{Stab}(x)\}$$

is a prekit on $\mathbb{A}$*. Additionally,* $\mathrm{Stab}(\mathbf{St}(\mathbb{A}, \mathcal{A})) = (\mathbb{A}, \mathcal{A})$ *and* $\mathbb{C} \subseteq \mathbf{St}(\mathbb{A}, \mathrm{Stab}(\mathbb{C}))$.

The following construction may then be performed on subcategories of presheaves: for any full subcategory $\mathbb{C}$ of $\widehat{\mathbb{A}}$, the category $\mathbb{C}^{\perp}$ is the full subcategory of $\widehat{\mathbb{A}^{\mathrm{op}}}$ with objects $\{Y : \mathbb{A} \to \mathbf{Set} \mid \forall X \in \mathbb{C}, X \perp Y\}$.

Let $\mathbf{FS}(\mathbb{A})$ be the (large) poset of full subcategories of $\widehat{\mathbb{A})}$ under inclusion. As with prekits, the orthogonality relation induces a Galois connection

$$\begin{tikzcd}
\mathbf{FS}(\mathbb{A})^{\mathrm{op}} \arrow[r, bend left, "(-)^{\perp}"] \arrow[r, phantom, "\perp"] & \mathbf{FS}(\mathbb{A}^{\mathrm{op}}) \arrow[l, bend left, "(-)^{\perp}"]
\end{tikzcd}$$

whose fixed points are those $\mathbb{C}$ verifiying $\mathbb{C}^{\perp\perp} \cong \mathbb{C}$.

Thus we have two notions of duality, respectively intensional (based on subgroups) and extensional (based on presheaves). These are fundamentally connected via the equations

$$(\mathbf{St}(\mathbb{A}, \mathcal{A}))^{\perp} \cong \mathbf{St}(\mathbb{A}^{\mathrm{op}}, \mathcal{A}^{\perp})) \quad \text{and} \quad \mathrm{Stab}(\mathbb{C})^{\perp} = \mathrm{Stab}(\mathbb{C}^{\perp})$$

from which we derive another characterisation of kits as those prekits $\mathcal{A}$ satisfying $(\mathbf{St}(\mathbb{A}, \mathcal{A}))^{\perp\perp} = \mathbf{St}(\mathbb{A}, \mathcal{A})$. Moreover, our earlier definition of s-profunctor (Definition 3.2.9) may be rephrased in extensional style:

Lemma 3.3.14. *For kit structures $\mathbf{A} = (|\mathbf{A}|, \mathcal{A})$ and $\mathbf{B} = (|\mathbf{B}|, \mathcal{B})$, a profunctor $P : |\mathbf{A}| \nrightarrow |\mathbf{B}|$ is an s-profunctor from $\mathbf{A}$ to $\mathbf{B}$ if and only if*

$$\begin{tikzcd}
\widehat{|\mathbf{A}|} \arrow[r, "P^{\#}"] & \widehat{|\mathbf{B}|} & & \widehat{|\mathbf{B}|^{\mathrm{op}}} \arrow[r, "P_{\#}"] & \widehat{|\mathbf{A}|^{\mathrm{op}}} \\
\mathbf{St}(\mathbf{A}) \arrow[u, hook] \arrow[r, dotted] & \mathbf{St}(\mathbf{B}) \arrow[u, hook] & \textit{and} & \mathbf{St}(\mathbf{B}^{\perp}) \arrow[u, hook] \arrow[r, dotted] & \mathbf{St}(\mathbf{A}^{\perp}) \arrow[u, hook]
\end{tikzcd}$$

That is, $P^{\#} : \widehat{|\mathbf{A}|} \to \widehat{|\mathbf{B}|}$ restricts to a functor $\mathbf{St}(\mathbf{A}) \to \mathbf{St}(\mathbf{B})$ and $P_{\#} : \widehat{|\mathbf{B}|^{\mathrm{op}}} \to \widehat{|\mathbf{A}|^{\mathrm{op}}}$ restricts to a functor $\mathbf{St}(\mathbf{B}^{\perp}) \to \mathbf{St}(\mathbf{A}^{\perp})$ (see Definition 1.4.5 for the notations $P^{\#}$ and $P_{\#}$).

Proof.

- (⇒) Assume that $P : |\mathbf{A}| \nrightarrow |\mathbf{B}|$ is an s-profunctor. For $X \in \mathbf{St}(\mathbf{A})$ and $b \in |\mathbf{B}|$ we want to show that the stabilizer of every element in $P^{\#}X(b) = \int^{a \in |\mathbf{A}|} P(a, b) \times X(a)$ is in $\mathcal{B}(b)$. Let $t = p \underset{a}{\bowtie} x$ be in $P^{\#}X(b)$. For every $g : b \to b$ in $\mathrm{Stab}(t)$, we have $(p \cdot g) \underset{a}{\bowtie} x = (p \underset{a}{\bowtie} x) \cdot g = p \underset{a}{\bowtie} x$, i.e. there exists $f : a \to a$ in $|\mathbf{A}|$ such that $f \cdot p \cdot g = p$ and $x \cdot \alpha = x$. Since X is in $\mathbf{St}(\mathbf{A})$, we must have $f \in \bigcup \mathcal{A}(a)$ and since P is an s-profunctor, we must also have $g \in \bigcup \mathcal{B}(b)$ as desired. One shows that $P_{\#}Y$ is in $\mathbf{St}(\mathbf{A}^{\perp})$ for $Y \in \mathbf{St}(\mathbf{B}^{\perp})$ analogously.

- (⇐) Assume that $P^{\#}(\mathbf{St}(\mathbf{A})) \hookrightarrow \mathbf{St}(\mathbf{B})$. Let $a \in |\mathbf{A}|$, $b \in |\mathbf{B}|$, $p \in P(a, b)$, $f \in |\mathbf{A}|(a, a)$, and $g \in |\mathbf{B}|(b, b)$ be such that $f \cdot p \cdot g = p$. Suppose that $f \in \bigcup \mathcal{A}(a)$ and let $X = \overline{\mathrm{y}}(a|\langle f \rangle) \in \mathbf{St}(\mathbf{A})$. We then have $P^{\#}(X) \in \mathbf{St}(\mathbf{B})$ by hypothesis. Hence, for the element $p \underset{a}{\bowtie} \langle f \rangle = p \underset{a}{\bowtie} \langle f \rangle f = (f \cdot p) \underset{a}{\bowtie} \langle f \rangle$ in $P^{\#}X(b)$, it follows that $(p \underset{a}{\bowtie} \langle f \rangle) \cdot g = (f \cdot p \cdot g) \underset{a}{\bowtie} \langle f \rangle = p \underset{a}{\bowtie} \langle f \rangle$ which implies $g \in \bigcup \mathcal{B}(b)$ as desired.

Assuming $P_{\#}(\mathbf{St}(\mathbf{B}^{\perp})) \hookrightarrow \mathbf{St}(\mathbf{A}^{\perp})$, for $g \in \bigcup \mathcal{B}^{\perp}(b)$, one considers $\overline{\mathrm{y}}(b|\langle g\rangle) \in \mathbf{St}(\mathbf{B}^{\perp})$ and reasons analogously to show that $f \in \bigcup \mathcal{A}^{\perp}(a)$. $\square$

3.3.2 From analytic to stable functors

Recall from Section 1.4.8 that the bicategory of generalized species $\mathbf{Prof}_{\mathcal{S}}$ restricted to groupoids is biequivalent to the 2-category of analytic functors **An**. We show in this section that the orthogonality structure allows us to restrict to stable functors by exhibiting a biequivalence between $\mathbf{SProf}_{!}$ and the 2-category **Stable** defined below:

Definition 3.3.15. We denote by **Stable** the 2-category given by:

- **objects:** kits $\mathbf{A}, \mathbf{B}$;
- **1-cells:** stable functors $\mathbf{St}(\mathbf{A}) \to \mathbf{St}(\mathbf{B})$;
- **1-cells:** cartesian natural transformations.

For kit structures $\mathbf{A}$ and $\mathbf{B}$, a generalized species $P : |!\mathbf{A}| \nrightarrow |\mathbf{B}|$ induces an analytic functor $\widehat{|\mathbf{A}|} \to \widehat{|\mathbf{B}|}$ by taking the left Kan extension of P along the sum functor $s_{|\mathbf{A}|} : |!\mathbf{A}| \to \widehat{|\mathbf{A}|}$ mapping a sequence $\langle a_1, \ldots a_n\rangle$ to $\sum_i \mathrm{y}(a_i)$. Since the category $\mathbf{St}(\mathbf{A})$ contains sums of representables, the functor $s_{|\mathbf{A}|}$ factors through the inclusion $\iota_{\mathbf{A}} : \mathbf{St}(\mathbf{A}) \hookrightarrow \widehat{|\mathbf{A}|}$ by a functor $|!\mathbf{A}| \to \mathbf{St}(\mathbf{A})$ that we denote by $s_{\mathbf{A}}$. We show that a stable species $P : \mathbf{A} \nrightarrow \mathbf{B}$ induces a stable functor $\mathbf{St}(\mathbf{A}) \to \mathbf{St}(\mathbf{B})$ by taking the left Kan extension along the restricted sum functor $s_{\mathbf{A}}$.

We use the universal property of Kan extensions in terms of weighted colimits:

Proposition 3.3.16. *For functors $F : \mathbb{A} \to \mathbb{C}$ and $H : \mathbb{A} \to \mathbb{B}$, there is an isomorphism:*

$$\mathbb{C}\left((\mathbf{Lan}_H F)(b)), c\right) \cong \widehat{\mathbb{A}}\left(\mathbb{B}(H(-), b), \mathbb{C}(F(-), c)\right).$$

Lemma 3.3.17. *Let $I : \mathbb{C} \to \mathbb{D}$ and $J : \mathbb{B} \to \mathbb{E}$ be embeddings. For functors $F : \mathbb{A} \to \mathbb{C}$, $H : \mathbb{A} \to \mathbb{B}$ and $L : \mathbb{B} \to \mathbb{C}$, if $IL \cong (\mathbf{Lan}_{JH}(IF))J$ then $L \cong \mathbf{Lan}_H F$.*

$$\begin{tikzcd}
\mathbb{A} \arrow[rr, "F"] \arrow[dr, "H"'] & & \mathbb{C} \arrow[r, "I"] & \mathbb{D} \\
& \mathbb{B} \arrow[ur, "L"'] \arrow[dr, "J"'] & & \\
& & \mathbb{E} \arrow[uur, "\mathbf{Lan}_{JH}(IF)"'] &
\end{tikzcd}$$

Proof. For all $b \in \mathbb{B}$ and $c \in \mathbb{C}$, we have:

$$\begin{aligned}
\mathbb{C}\,(L(b), c) &\cong \mathbb{D}(IL(b), I(c)) \\
&\cong \mathbb{D}(\mathbf{Lan}_{JH}(IF)(Jb), I(c)) \\
&\cong \widehat{\mathbb{A}}\,(\mathbb{E}(JH(-), J(b)), \mathbb{D}(IF(-), I(c))) \\
&\cong \widehat{\mathbb{A}}\,(\mathbb{B}(H(-), b), \mathbb{C}(F(-), c))
\end{aligned}$$

By Proposition 3.3.16, we obtain the desired result. ☐

Lemma 3.3.18. *For a stable species $P : !\mathbf{A} \nrightarrow \mathbf{B}$, the functor*

$$(\mathbf{Lan}_{s_{|\mathbf{A}|}} P)\iota_{\mathbf{A}} : \mathbf{St}(\mathbf{A}) \to |\widehat{\mathbf{B}}|$$

restricts to a functor $T : \mathbf{St}(\mathbf{A}) \to \mathbf{St}(\mathbf{B})$. P factors through the inclusion $\iota_{\mathbf{B}}$ by a functor $Q : |\mathbf{A}| \to \mathbf{St}(\mathbf{B})$ and we have $T = \mathbf{Lan}_{s_{\mathbf{A}}} Q$.

$$\begin{tikzcd}
{|!\mathbf{A}|} \arrow[rr, "P"] \arrow[dr, "s_{|\mathbf{A}|}"'] & & {|\widehat{\mathbf{B}}|} & & {|!\mathbf{A}|} \arrow[r, "Q"] \arrow[d, "s_{\mathbf{A}}"'] & \mathbf{St}(\mathbf{B}) \arrow[r, "\iota_{\mathbf{B}}"] & {|\widehat{\mathbf{B}}|} \\
& {|\widehat{\mathbf{A}}|} \arrow[ur, "\mathbf{Lan}s_{|\mathbf{A}|}P"'] & & = & \mathbf{St}(\mathbf{A}) \arrow[ur, dotted, "T"'] \arrow[r, "\iota_{\mathbf{A}}"'] & {|\widehat{\mathbf{A}}|} \arrow[ur, "\mathbf{Lan}s_{|\mathbf{A}|}P"'] &
\end{tikzcd}$$

Proof. To prove that $(\mathbf{Lan}_{s_{|\mathbf{A}|}} P)\iota_{\mathbf{A}}$ restricts to a functor $\mathbf{St}(\mathbf{A}) \to \mathbf{St}(\mathbf{B})$, we show that for all $X \in \mathbf{St}(\mathbf{A})$ and $b \in |\mathbf{B}|$, the stabilizer of every element in $\mathbf{Lan}_{s_{|\mathbf{A}|}} PX(b) = \int^{u \in \mathcal{S}|\mathbf{A}|} P(u, b) \times X^{\mathcal{S}}(u)$ is in $\mathcal{B}(b)$. Let $t = p \underset{u}{\bowtie} \overline{x}$ in $\mathbf{Lan}_{s_{|\mathbf{A}|}} PX(b)$. For every $g : b \to b$ in $\mathrm{Stab}(t)$, we have $(p \cdot g) \underset{u}{\bowtie} \overline{x} = p \underset{u}{\bowtie} \overline{x}$, i.e. there exists $f : u \to u$ in $\mathcal{S}|\mathbf{A}|$ such that $f \cdot p \cdot g = p$ and $\overline{x} \cdot f = \overline{x}$. Since X is in $\mathbf{St}(\mathbf{A})$ and f is in $\mathrm{Stab}(X^{\mathcal{S}}(u))$, we must have $f \in \bigcup !\mathcal{A}(u)$ and since P is an s-profunctor, it implies that $g \in \bigcup \mathcal{B}(b)$ as desired. For $u \in |!\mathbf{A}|$, it is immediate that $P(u)$ is in $\mathbf{St}(\mathbf{B})$ and we obtain we obtain that $T \cong \mathbf{Lan}_{s_{\mathbf{A}}} Q$ by Lemma 3.3.17. ☐

We denote by $\widetilde{P}$ the functor $(\mathbf{Lan}_{s_{|\mathbf{A}|}}P)\iota_{\mathbf{A}} \cong \mathbf{Lan}_{s_{\mathbf{A}}}P$ and we show that $\widetilde{(-)}$ defines a functor $\mathbf{SProf}(\mathbf{A},\mathbf{B}) \to \mathbf{Stable}(\mathbf{St}(\mathbf{A}),\mathbf{St}(\mathbf{B}))$

Lemma 3.3.19. *For kit structures* $\mathbf{A},\mathbf{B}$*, a presheaf* $X \in \mathbf{St}(\mathbf{A})$ *and an s-profunctor* $P : !\mathbf{A} \nrightarrow \mathbf{B}$*, if two elements* $p \underset{u}{\bowtie} \overline{x}$ *and* $q \underset{v}{\bowtie} \overline{y}$ *in* $\widetilde{P}X(b)$ *are equal for some* $b \in |\mathbf{B}|$*, there exists a unique* $f : u \to v$ *in* $|!\mathbf{A}|$ *such that* $\overline{y} \cdot f = \overline{x}$ *and* $f \cdot p = q$*.*

Proof. Assume that $p \underset{u}{\bowtie} \overline{x} = q \underset{v}{\bowtie} \overline{y}$ where $p \in P(u,b)$, $q \in P(v,b)$, $\overline{x} \in X^{\mathcal{S}}(u)$ and $\overline{y} \in X^{\mathcal{S}}(v)$. The existence of $f : u \to v$ in $|!\mathbf{A}|$ such that $\overline{y} \cdot f = \overline{x}$ and $f \cdot p = q$ follows from the pointwise definition of the coend $\widetilde{P}X(b)$. For uniqueness, assume that there exists $g : u \to v$ such that $\overline{y} \cdot g = \overline{x}$ and $g \cdot p = q$ as well. The equality $f \cdot p = g \cdot p$ implies that $g^{-1}f \cdot p = p$ and since id_b is in $\bigcup \mathcal{B}^{\perp}(b)$, we have $g^{-1}f \in \bigcup(!\mathcal{A})^{\perp}(u)$ as P is an s-profunctor. On the other hand, $\overline{y} \cdot f = \overline{y} \cdot g$ implies that $g^{-1}f$ is in $\mathrm{Stab}(X^{\mathcal{S}}(u))$ which entails that $g^{-1}f \in \bigcup(!\mathcal{A})(u)$ by Lemma 3.3.13. Hence, $g^{-1}f = \mathrm{id}$ as desired. ☐

Lemma 3.3.20. *For kits* $\mathbf{A} = (|\mathbf{A}|,\mathcal{A})$*,* $\mathbf{B} = (|\mathbf{B}|,\mathcal{B})$ *and an s-profunctor* $P : !\mathbf{A} \nrightarrow \mathbf{B}$*, the functor* $\widetilde{P} : \mathbf{St}(\mathbf{A}) \to \mathbf{St}(\mathbf{B})$ *is stable.*

Proof. By Lemma 1.4.19, the functor $\mathbf{Lan}_{s_{|\mathbf{A}|}}P$ is finitary and by Lemma 1.4.20, $\mathbf{Lan}_{s_{|\mathbf{A}|}}P : \widehat{|\mathbf{A}|} \to \widehat{|\mathbf{B}|}$ preserves quasi-pullbacks and hence epimorphisms by Remark 3. Since the embeddings $\iota_{\mathbf{A}}$ and $\iota_{\mathbf{B}}$ create epimorphisms and filtered colimits, $\widetilde{P}$ is epi-preserving and finitary.

It remains to show that $\widetilde{P}$ admits generic factorizations. By Proposition 3.1.21, it suffices to show that $\widetilde{P}$ admits generic factorizations relative to representables. Consider a morphism $\mathrm{y}b \to \widetilde{P}(X)$ in $\mathbf{St}(\mathbf{B})$, it is of the form $p \underset{u}{\bowtie} \overline{x}$ with $p \in P(u,b)$ and $\overline{x} : s_{|\mathbf{A}|}u \to X$ for some $u \in |!\mathbf{A}|$. We show that

$$\begin{tikzcd}
\mathrm{y}b \arrow[r, "p \underset{u}{\bowtie} \mathrm{id}"] & \widetilde{P}(s_{|\mathbf{A}|}u) \arrow[r, "\widetilde{P}(\overline{x})"] & \widetilde{P}(X)
\end{tikzcd}$$

is a generic factorization for $p \underset{u}{\bowtie} \overline{x}$. The equality $p \underset{u}{\bowtie} \overline{x} = (\widetilde{P}(\overline{x}))(p \underset{u}{\bowtie} \mathrm{id})$ is immediate, it remains to prove that $p \underset{a}{\bowtie} \mathrm{id}$ is generic.

Assume that there exist $Y, Z \in \mathbf{St}(\mathbf{A})$ and morphisms $p' \underset{v}{\bowtie} \overline{y} : \mathrm{y}b \to \widetilde{P}(Y)$ and $\alpha : Y \to Z$ and $\overline{z} : s_{|\mathbf{A}|}u \to Z$ such that the following diagram commutes in $\mathbf{St}(\mathbf{B})$:

$$\begin{tikzcd}
\mathrm{y}b \arrow[r, "p' \underset{v}{\bowtie} \overline{y}"] \arrow[d, "p \underset{u}{\bowtie} \mathrm{id}"'] & \widetilde{P}(Y) \arrow[d, "\widetilde{P}(\alpha)"] \\
\widetilde{P}(s_{|\mathbf{A}|}u) \arrow[r, "\widetilde{P}(\overline{z})"'] & \widetilde{P}(Z)
\end{tikzcd}$$

The equality $\widetilde{P}(\alpha)(p' \underset{v}{\bowtie} \overline{y}) = p \underset{u}{\bowtie} \overline{z}$ is equivalent to $p' \underset{v}{\bowtie} \alpha\overline{y} = p \underset{u}{\bowtie} \overline{x}$ which implies by Lemma 3.3.19 that there exists a unique $f : u \to v$ in $|!\mathbf{A}|$ such that $(\alpha\overline{y}) \cdot f = \overline{z}$ and $f \cdot p = p'$. We then have $\widetilde{P}(\overline{y} \cdot f)(p \underset{u}{\bowtie} \mathrm{id}) = p \underset{u}{\bowtie} \overline{y} \cdot f = f \cdot p \underset{v}{\bowtie} \overline{y} = p' \underset{v}{\bowtie} \overline{y}$. It remains to show uniqueness, assume that there exists $\overline{t} : s_{|\mathbf{A}|}u \to Y$ such that $\alpha\overline{t} = \overline{x}$ and $\widetilde{P}(\overline{t})(p \underset{u}{\bowtie} \mathrm{id}) = p' \underset{v}{\bowtie} \overline{y}$ i.e. $p \underset{u}{\bowtie} \overline{z} = p' \underset{v}{\bowtie} \overline{y}$.

$$\begin{tikzcd}
\mathrm{y}b \arrow[r, "p' \underset{v}{\bowtie} \overline{y}"] \arrow[d, "p \underset{u}{\bowtie} \mathrm{id}"'] & \widetilde{P}(Y) \arrow[d, "\widetilde{P}(\alpha)"] \\
\widetilde{P}(s_{|\mathbf{A}|}u) \arrow[ur, dotted, "\widetilde{P}(\overline{t})"] \arrow[r, "\widetilde{P}(\overline{z})"'] & \widetilde{P}(Z)
\end{tikzcd}$$

It implies that there exists $g : u \to v$ such that $\overline{y} \cdot g = \overline{t}$ and $g \cdot p = p'$. Hence, $(\alpha\overline{y}) \cdot g = \alpha\overline{t} = \overline{z}$ which implies that $f = g$ so that $\overline{t} = \overline{y} \cdot f$ as desired. □

Lemma 3.3.21. *For kit structures* $\mathbf{A} = (|\mathbf{A}|, \mathcal{A})$, $\mathbf{B} = (|\mathbf{B}|, \mathcal{B})$ *and a natural transformation* $\alpha : P \Rightarrow Q : !\mathbf{A} \nrightarrow \mathbf{B}$ *in* $\mathbf{SProf}(P, Q)$*, the natural transformation* $\widetilde{\alpha} : \widetilde{P} \to \widetilde{Q} : \mathbf{St}(\mathbf{A}) \to \mathbf{St}(\mathbf{B})$ *is cartesian.*

Proof. Let $\gamma : X \to Y$ in $\mathbf{St}(\mathbf{A})$, we want to show that the square below is a pullback in **Set** for all $b \in |\mathbf{B}|$:

$$\begin{tikzcd}
\widetilde{P}(X)(b) \arrow[r, "\widetilde{\alpha}_X"] \arrow[d, "\widetilde{P}(\gamma)"'] & \widetilde{Q}(X)(b) \arrow[d, "\widetilde{Q}(\gamma)"] \\
\widetilde{P}(Y)(b) \arrow[r, "\widetilde{\alpha}_Y"'] & \widetilde{Q}(Y)(b)
\end{tikzcd}$$

We show that for all $(p \underset{u_1}{\bowtie} \overline{y}) \in \widetilde{P}(Y)(b)$ and $(q \underset{u_2}{\bowtie} \overline{x}) \in \widetilde{Q}(X)(b)$ such that

$$\alpha_{u_1}(p) \underset{u_1}{\bowtie} \overline{x} = \widetilde{\alpha}_Y(p \underset{u_1}{\bowtie} \overline{x}) = \widetilde{Q}(\gamma)(q \underset{u_2}{\bowtie} \overline{x}) = q \underset{u_2}{\bowtie} \gamma(\overline{x})$$

there exists a unique $t \in \widetilde{P}(X)(b)$ such that $\widetilde{P}(\gamma)(t) = p \underset{u_1}{\bowtie} \overline{y}$ and $\widetilde{\alpha}_X(t) = q \underset{u_2}{\bowtie} \overline{x}$. The equality $\alpha_{u_1}(p) \underset{u_1}{\bowtie} \overline{y} = q \underset{u_2}{\bowtie} \gamma(\overline{x})$ implies that there exists $f : u_1 \to u_2$ in $\mathcal{S}|\mathbf{A}|$ such that $q = f \cdot \alpha(p)$ and $\gamma(\overline{x}) \cdot f = \gamma(\overline{x}) \circ s_{|\mathbf{A}|}(f) = \overline{y}$. Define t to be $p \underset{u_1}{\bowtie} (\overline{x} \cdot f) \in \widetilde{P}(X)(b)$, we then obtain that

$$\widetilde{P}(\gamma)(t) = p \underset{u_1}{\bowtie} \gamma(\overline{x} \cdot f) = p \underset{u_1}{\bowtie} \gamma(\overline{x}) \cdot f = p \underset{u_1}{\bowtie} \overline{y}$$

and

$$\widetilde{\alpha}_X(t) = \alpha(p) \underset{u_1}{\bowtie} \overline{x} \cdot f = f \cdot \alpha(p) \underset{u_2}{\bowtie} \overline{x} = q \underset{u_2}{\bowtie} \overline{x}.$$

Assume now that there exists $v = p_0 \underset{u_0}{\bowtie} \overline{x_0} \in \widetilde{P}(X)(b)$ such that $\widetilde{P}(\gamma)(v) = p \underset{u_1}{\bowtie} \overline{y}$ and $\widetilde{\alpha}_X(v) = q \underset{u_2}{\bowtie} \overline{x}$. We then have that $p_0 \underset{u_0}{\bowtie} \gamma(\overline{x_0}) = p \underset{u_1}{\bowtie} \overline{y}$ and $\alpha(p_0) \underset{u_0}{\bowtie} \overline{x_0} = q \underset{u_2}{\bowtie} \overline{x}$. Hence, there exists $g : u_0 \to u_1$ in $\mathcal{S}|\mathbf{A}|$ such that $q = \alpha(p_0) \cdot g$ and $\overline{x} \cdot g = \overline{x_0}$, and there exists $h : u_1 \to u_0$ in $\mathcal{S}|\mathbf{A}|$ such that $p_0 = h \cdot p$ and $\gamma(\overline{x_0}) \cdot h = \overline{y}$. Therefore,

$$gh \cdot \alpha(p) = g \cdot \alpha(h \cdot p) = g \cdot \alpha(p_0) = q$$

and

$$\gamma(\overline{x}) \cdot gh = \gamma(\overline{x} \cdot g) \cdot h = \gamma(\overline{x_0}) \cdot h = y.$$

Therefore, ghf^{-1} is in $\mathrm{Stab}(Y^{\mathcal{S}}(u_1))$ which implies that $ghf^{-1} \in \bigcup !\mathcal{A}(u_1)$ since $Y \in \mathbf{St}(\mathbf{A})$. We also have $ghf^{-1} \cdot q = q$. Since Q is a q-profunctor and id_b is in $\bigcup \mathcal{B}^{\perp}(b)$, we obtain that ghf^{-1} is in $\bigcup !\mathcal{A}^{\perp}(u_1)$ which implies that $f = gh$. Hence, we have

$$t = p \underset{u_1}{\bowtie} (\overline{x} \cdot gh) = p \underset{u_1}{\bowtie} (\overline{x_0} \cdot h) = (h \cdot p) \underset{u_0}{\bowtie} \overline{x_0} = p_0 \underset{u_0}{\bowtie} \overline{x_0} = v$$

as desired. □

We now define a trace functor $\mathbf{Tr} : \mathbf{Stable}(\mathbf{St}(\mathbf{A}), \mathbf{St}(\mathbf{B})) \to \mathbf{SProf}(!\mathbf{A}, \mathbf{B})$ analogous to the trace operator for a stable function. To reconstruct an s-species from a stable functor, we use the property of that they admit generic factorizations. Similar constructions are done for normal functors or analytic functors where generic elements correspond to the normal forms studied by Girard and Hasegawa [54, 61] or to the compact elements considered by Fiore [39].

Lemma 3.3.22. *For kit structures* $\mathbf{A} = (|\mathbf{A}|, \mathcal{A})$*,* $\mathbf{B} = (|\mathbf{B}|, \mathcal{B})$*, if a functor* $T : \mathbf{St}(\mathbf{A}) \to \mathbf{St}(\mathbf{B})$ *is stable then the profunctor* $\mathbf{Tr}(T) : \mathcal{S}|\mathbf{A}| \nrightarrow |\mathbf{B}|$ *given by:*

$$(u, b) \mapsto \{t : b \to T(s_{|\mathbf{A}|}(u)) \mid t \text{ generic}\}$$

is an s-profunctor $!\mathbf{A} \nrightarrow \mathbf{B}$*.*

Proof. We first need to show that $\mathbf{Tr}(T)$ is indeed a profunctor, i.e. for $t : b \to T(s_{|\mathbf{A}|}(u))$ generic and morphisms $f \in \mathcal{S}|\mathbf{A}|(u, u)$, $g \in |\mathbf{B}|(b, b)$, $f \cdot t \cdot g = T(s_{|\mathbf{A}|}(f)) \circ t \circ g$ is also generic which holds since generic elements are stable under precomposition or postcomposition by isomorphisms. We now prove that $\mathbf{Tr}(T)$ is an s-profunctor. Assume that $f \cdot t \cdot g = t$ i.e. the following square commutes in $\mathbf{St}(\mathbf{B})$:

$$\begin{tikzcd}
b \arrow[r, "t"] & T(s_{|\mathbf{A}|}(u)) \arrow[d, "T(s_{|\mathbf{A}|}f)"] \\
b \arrow[u, "g"] \arrow[r, "t"'] & T(s_{|\mathbf{A}|}(u))
\end{tikzcd}$$

- Assume that $f \in \bigcup !\mathcal{A}(u)$, using Lemma 3.2.8, it implies that for all $n \in \mathbb{N}$, $f^n = \mathrm{id}$ or there exists $m \in \mathbb{N}$ and $\mathrm{id} \neq f^{nm} \in \bigcup \mathcal{A}^{\mathcal{S}}(u)$. We make use here of the equality:

  $$\mathcal{A}^{\mathcal{S}}(u) = \{G \mid \exists X \in \mathbf{St}(\mathbf{A}), \exists \gamma \in X^{\mathcal{S}}(u), G = \mathrm{Stab}(\gamma)\}.$$

  which is a consequence of the translation in Lemma 3.3.13 between the orthogonality for kits and presheaves. We want to show that g is in $\bigcup \mathcal{B}(b) = \bigcup \mathcal{B}^{\perp\perp}(b)$ i.e. for all $n \in \mathbb{N}$, $g^n = \mathrm{id}$ or there exists $m \in \mathbb{N}$ such that $\mathrm{id} \neq g^{nm} \in \bigcup \mathcal{B}(b)$. Equivalently, we show that for all $n \in \mathbb{N}$, $g^n = \mathrm{id}$ or there exist $m \in \mathbb{N}$ and $Y \in \mathbf{St}(\mathbf{B})$ with $\mathrm{id} \neq g^{nm} \in \mathrm{Stab}(Y(b))$ by Lemma 3.3.13.

  Let $n \in \mathbb{N}$ and assume that $g^n \neq \mathrm{id}$, if $f^n = \mathrm{id}$, then $t \cdot g^n = t$ which implies that g^n is in $\mathrm{Stab}(T(s_{|\mathbf{A}|}(u))$. Since $T(s_{|\mathbf{A}|}(u))$ is in $\mathbf{St}(\mathbf{B})$, we can take $m := 1$ and obtain the desired result. If there exist $m \in \mathbb{N}$ and $X \in \mathbf{St}(\mathbf{A})$ such that $\mathrm{id} \neq f^{nm} \in \mathrm{Stab}(X^{\mathcal{S}}(u))$, then there exists $\gamma : s_{|\mathbf{A}|}(u) \to X$ in $\mathbf{St}(\mathbf{A})$ (or equivalently $\gamma \in X^{\mathcal{S}}(u)$) such that $\gamma \cdot f^{nm} = \gamma$ which implies that the following diagram commutes in $\mathbf{St}(\mathbf{B})$:

$$\begin{tikzcd}
b \arrow[r, "t"] & T(s_{|\mathbf{A}|}(u)) \arrow[dd, "T(s_{|\mathbf{A}|} f^{nm})"] \arrow[dr, "T(\gamma)"] & \\
 & & T(X) \\
b \arrow[uu, "g^{nm}"] \arrow[r, "t"'] & T(s_{|\mathbf{A}|}(u)) \arrow[ur, "T(\gamma)"'] &
\end{tikzcd}$$

Hence, g^{nm} is in $\mathrm{Stab}(T(X)(b))$ and since $T(X)$ is in $\mathbf{St}(\mathbf{B})$, it only remains to show that $g^{nm} \neq \mathrm{id}$. Assume that $g^{nm} = \mathrm{id}$, then $f^{nm} \cdot t = t$, so the following diagram commutes in $\mathbf{St}(\mathbf{B})$:

$$\begin{tikzcd}
b \arrow[r, "t"] \arrow[d, "t"'] & T(s_{|\mathbf{A}|}(u)) \arrow[d, "T(\gamma)"] \\
T(s_{|\mathbf{A}|}(u)) \arrow[ur, "T(s_{|\mathbf{A}|} f^{nm})" description] \arrow[r, "T(\gamma)"'] & T(X)
\end{tikzcd}$$

since t is generic, it implies that $f^{nm} = \mathrm{id}$ so $g^{nm} \neq \mathrm{id}$ as desired.

- Assume now that $g \in \bigcup \mathcal{B}^{\perp}(b)$, we want to show that $f \in \bigcup ?\mathcal{A}^{\perp}(u)$ i.e. for all n, if there exists $X' \in \mathbf{St}(\mathbf{A})$ such that f^n is in $\mathrm{Stab}(X'^{\mathcal{S}}(u))$, then $f^n = \mathrm{id}$. Assume that there exists $X' \in \mathbf{St}(\mathbf{A})$ such that f^n is in $\mathrm{Stab}(X'^{\mathcal{S}}(u))$, i.e. there exists $\gamma' : s_{|\mathbf{A}|}(u) \to X'$ in $\mathbf{St}(\mathbf{A})$ such that $\gamma' \cdot f^n = \gamma'$ which implies that the following diagram commutes in $\mathbf{St}(\mathbf{B})$:

$$\begin{tikzcd}
b \arrow[r, "t"] & T(s_{|\mathbf{A}|}(u)) \arrow[dd, "T(s_{|\mathbf{A}|} f^{n})"'] \arrow[dr, "T(\gamma')"] & \\
 & & T(X') \\
b \arrow[uu, "g^{n}"] \arrow[r, "t"'] & T(s_{|\mathbf{A}|}(u)) \arrow[ur, "T(\gamma')"'] &
\end{tikzcd}$$

Hence, $g^n \in \mathrm{Stab}(T(X')(b))$ which implies that $g^n = \mathrm{id}$ so that the following diagram commutes:

$$\begin{tikzcd}
b \arrow[r, "t"] \arrow[d, "t"'] & T(s_{|\mathbf{A}|}(u)) \arrow[d, "T(\gamma')"] \\
T(s_{|\mathbf{A}|}(u)) \arrow[r, "T(\gamma')"'] \arrow[ur, "T(s_{|\mathbf{A}|}(f^n))" description] & T(X')
\end{tikzcd}$$

Since t is generic, we obtain that $f^n = \mathrm{id}$ as desired. □

Lemma 3.3.23. *Let* $\mathbf{A}, \mathbf{B}$ *be kit structures and* $T, S : \mathbf{St}(\mathbf{A}) \to \mathbf{St}(\mathbf{B})$ *be stable functors. For a cartesian transformation* $\gamma : T \Rightarrow S$*, the transformation* $\mathbf{Tr}(\gamma) : \mathbf{Tr}(T) \Rightarrow \mathbf{Tr}(S)$ *whose components* $\mathbf{Tr}(\gamma)_{u,b} : \mathbf{Tr}(T)(u,b) \to \mathbf{Tr}(T)(v,b)$ *are given by:*

$$(t : b \to T(s_{|\mathbf{A}|}(u)) \mapsto (\gamma_{s_{|\mathbf{A}|(u)}}(t) : b \to S(s_{|\mathbf{A}|}))$$

is well-defined and natural.

Proof. Since cartesian transformations preserve and reflect generic elements, $\gamma_{s_{|\mathbf{A}|(u)}}(t)$ is generic if t is generic. Naturality of $\mathbf{Tr}(\gamma)$ follows immediately from the naturality of γ. □

Lemma 3.3.24. *For a stable species* $P : !\mathbf{A} \nrightarrow \mathbf{B}$*, the transformation* $\eta_P : P \Rightarrow \mathbf{Tr}(\widetilde{P})$ *whose components* $(\eta_P)_{u,b} : P(u,b) \to \mathbf{Tr}(\widetilde{P})(u,b)$ *are given by*

$$p \mapsto (p \underset{u}{\bowtie} \mathrm{id}) : b \to \widetilde{P}(s_{|\mathbf{A}|}u)$$

is a natural isomorphism.

Proof. We first need to show that $(p \underset{u}{\bowtie} \mathrm{id}) : b \to \widetilde{P}(s_{|\mathbf{A}|}u)$ is generic for the map $(\eta_P)_{u,b}$ to be well-defined. Assume that the diagram below commutes in $\mathbf{St}(\mathbf{B})$ for some morphisms $\alpha : s_{|\mathbf{A}|}u \to X$, $\beta : Y \to X$ and $\delta : b \to \widetilde{P}(Y)$:

$$\begin{tikzcd}
b \arrow[r, "\delta"] \arrow[d, "(p \underset{u}{\bowtie} \mathrm{id})"'] & \widetilde{P}(Y) \arrow[d, "\widetilde{P}(\beta)"] \\
\widetilde{P}(s_{|\mathbf{A}|}u) \arrow[r, "\widetilde{P}(\alpha)"'] & \widetilde{P}(X)
\end{tikzcd}$$

The morphism $\delta : b \to \widetilde{P}(Y)$ corresponds to an element $(p' \underset{u'}{\bowtie} \rho)$ where $p' \in P(u', b)$ and $\rho : s_{|\mathbf{A}|}(u') \to Y$. Since the diagram commutes, we have $p \underset{u}{\bowtie} \alpha = p' \underset{u'}{\bowtie} \beta\rho$ i.e. there exists $f : u \to u' \in \mathcal{S}|\mathbf{A}|$ such that $f \cdot p = p'$ and $\beta\rho s_{|\mathbf{A}|}(f) = \alpha$. Define $\xi : s_{|\mathbf{A}|}(u) \to Y$ to be $\rho s_{|\mathbf{A}|}(f)$. We then have $\beta\xi = \alpha$ and $\widetilde{P}(\xi)(p \underset{u}{\bowtie} \mathrm{id}) = p \underset{u}{\bowtie} \rho s_{|\mathbf{A}|}(f) = f \cdot p \underset{u'}{\bowtie} \rho = p' \underset{u'}{\bowtie} \rho$ as desired.

For uniqueness, assume that there exists $\nu : s_{|\mathbf{A}|}(u) \to Y$ such that $\beta\nu = \alpha$ and $\widetilde{P}(\nu)(p \underset{u}{\bowtie} \mathrm{id}) = p \underset{u}{\bowtie} \nu = p' \underset{u'}{\bowtie} \rho$. It implies that there exists $g : u \to u'$ in $\mathcal{S}|\mathbf{A}|$ such that $g \cdot p = p'$ and $\rho s_{|\mathbf{A}|}(g) = \nu$. Since $\mathrm{id}_b \in \bigcup \mathcal{B}(b)^{\perp}$ and $gf^{-1} \cdot p' = p'$, we have $gf^{-1} \in \bigcup !\mathcal{A}(u')^{\perp}$. Now, $\beta\rho s_{|\mathbf{A}|}(g) = \beta\rho s_{|\mathbf{A}|}(f)$ implies that gf^{-1} is in $\mathrm{Stab}(X^{\mathcal{S}}(u'))$. Since $X \in \mathbf{St}(\mathbf{A})$, we obtain that $f = g$ as desired. Hence, $(\eta_P)_{u,b}$ is well-defined, it remains to show that it is injective and surjective.

For injectivity, if there are p and p' in $P(u, b)$ such that $p \underset{u}{\bowtie} \mathrm{id} = p' \underset{u}{\bowtie} \mathrm{id}$ then there exists $f : u \to u'$ such that $f \cdot p = p'$ and $f\mathrm{id} = \mathrm{id}$ which implies that $p = p'$. For surjectivity, let $p \underset{v}{\bowtie} \gamma : b \to \widetilde{P}(s_{|\mathbf{A}|}u)$ be a generic map. Since the diagram below commutes, there exists a unique $\xi : s_{|\mathbf{A}|}u \to s_{|\mathbf{A}|}v$ in $\mathbf{St}(\mathbf{A})$ such that $\gamma\xi = \mathrm{id}$ and $p \underset{v}{\bowtie} \xi\gamma = p \underset{v}{\bowtie} \mathrm{id}$.

$$\begin{tikzcd}
b \arrow[r, "(p \underset{v}{\bowtie} \mathrm{id})"] \arrow[d, "(p \underset{v}{\bowtie} \gamma)"'] & \widetilde{P}(s_{|\mathbf{A}|}v) \arrow[d, "\widetilde{P}(\gamma)"] \\
\widetilde{P}(s_{|\mathbf{A}|}u) \arrow[r, "\widetilde{P}(\mathrm{id})"'] & \widetilde{P}(s_{|\mathbf{A}|}u)
\end{tikzcd}$$

Since $p \underset{v}{\bowtie} \mathrm{id}$ is generic, we can apply the same reasoning and obtain that γ has a left inverse as well and is therefore an isomorphism. By Lemma 1.4.15, there exists $f : u \to u'$ in $|\mathbf{A}|$ such that $\gamma = s_{|\mathbf{A}|}(f)$. Hence, $p \underset{v}{\bowtie} \gamma = f^{-1} \cdot p \underset{u}{\bowtie} \mathrm{id}$ which implies that $(\eta_P)_{u,b}$ is surjective as desired. $\square$

Lemma 3.3.25. *For a kit $\mathbf{A} = (|\mathbf{A}|, \mathcal{A})$ and presheaves X and Y in $\mathbf{St}(\mathbf{A})$, if Y is a coproduct of representables $\sum_{i \in I} \mathrm{y}(a_i)$ and there is a monomorphism $m : X \hookrightarrow Y$, then $X \cong \sum_{j \in J} \mathrm{y}(a_j)$ where $J \subseteq I$.*

Proof. By Lemma 3.3.4, X is isomorphic to $\sum_{j \in J} \overline{\mathrm{y}}(b_j | G_j)$ where each $b_j \in |\mathbf{A}|$ and $G_j \in \mathcal{A}(b_j)$. Since m is monic, X is a free action, i.e. for all $a \in |\mathbf{A}|$, $x \in X(a)$ and $f : a \to a$ such that $x \cdot f = x$, we must have $f = \mathrm{id}_a$. Indeed, $x \cdot f = x$ implies $(m_a(x)) \cdot f = m_a(x)$ which entails that $f = \mathrm{id}_a$ since Y is a free action. Hence, all the groups G_j are trivial and $X \cong \sum_{j \in J} \mathrm{y}(a_j)$. Now, since $\widehat{|\mathbf{A}|}(X, Y)$ is isomorphic to

$$\sum_{\sigma : J \to I} \prod_{j \in J} |\mathbf{A}|(b_j, a_{\sigma(j)})$$

$m : X \to Y$ corresponds to a pair $(\sigma, (f_j)_j)$ of a function $\sigma : I \to J$ and a family of isomorphisms $f_j : b_j \to a_{\sigma(j)}$. Since m is monic, σ is injective and we obtain the desired result. $\square$

Lemma 3.3.26. *Let* $\mathbf{A} = (|\mathbf{A}|, \mathcal{A})$ *and* $\mathbf{B} = (|\mathbf{B}|, \mathcal{B})$ *be kits and* $T : \mathbf{St}(\mathbf{A}) \to \mathbf{St}(\mathbf{B})$ *be a stable functor. For every* X *in* $\mathbf{St}(\mathbf{A})$ *and* $t : b \to T(X)$ *in* $\mathbf{St}(\mathbf{B})$, *there exists* $u \in \mathcal{S}|\mathbf{A}|$, $g : b \to T(s_{|\mathbf{A}|}(u))$ *generic and* $\overline{x} : s_{|\mathbf{A}|}(u) \to X$ *such that* $t = T(\overline{x})g$.

Proof. Since T is stable, t can be factored as $b \xrightarrow{g} T(Y) \xrightarrow{T(f)} T(X)$ where g is generic. By Corollary 3.3.5, $Y \cong \varinjlim_{i \in I} \sum_{j \in J_i} \overline{\mathrm{y}}(a_{ij}|G_{ij})$ where each G_{ij} is a finitely generated group in $\mathcal{A}(a_{ij})$ and I is filtered. Since T is finitary, we have $T(Y) \cong \varinjlim_{i \in I} T(\sum_{j \in J_i} \overline{\mathrm{y}}(a_{ij}|G_{ij}))$.

Since filtered colimits are computed pointwise in $\mathbf{St}(\mathbf{B})$, it implies that g can be factored as

$$b \xrightarrow{g'} T(\sum_{j \in J} \overline{\mathrm{y}}(a_j|G_j)) \xrightarrow{T(\iota)} \varinjlim_{i \in I} T(\sum_{j \in J_i} \overline{\mathrm{y}}(a_{ij}|G_{ij})).$$

For each group G_j, the projection morphism $q_j : \mathrm{y}a_j \to \overline{\mathrm{y}}(a_j|G_j))$ is an epimorphism which implies that $q := \sum_{j \in J} q_j : \sum_{j \in J} \mathrm{y}a_j \to \sum_{j \in J} \overline{\mathrm{y}}(a_j|G_j)$ is also an epimorphism. Since T is epi-preserving, we can factor g' as

$$b \xrightarrow{g''} T(\sum_{j \in J} \mathrm{y}a_j) \xrightarrow{q} T(\sum_{j \in J} \overline{\mathrm{y}}(a_j|G_j)).$$

Since g is generic, there exists a unique $h : Y \to \sum_{j \in J} \mathrm{y}a_j$ such that $\iota q h = \mathrm{id}$ and $T(h)g = g''$.

$$\begin{tikzcd}
b \arrow[r, "g''"] \arrow[d, "g"'] & T(\sum_{j \in J} \mathrm{y}a_j) \arrow[d, "T(\iota q)"] \\
T(Y) \arrow[ur, dotted, "T(h)"] \arrow[r, "T(\mathrm{id})"'] & T(Y)
\end{tikzcd}$$

Since h is split monic, by Lemma 3.3.25, $Y \cong \sum_{k\in K} \mathrm{y}a_k$ where $K \subseteq J$. $\square$

Lemma 3.3.27. *For a stable functor* $T : \mathbf{St}(\mathbf{A}) \to \mathbf{St}(\mathbf{B})$*, the transformation* $\varepsilon_T : \widetilde{\mathbf{Tr}\,T} \Rightarrow T$ *whose components* $(\varepsilon_T)_{X,b}$ *given by*

$$(t \underset{u}{\bowtie} \gamma : s_{|\mathbf{A}|}u \to X) \mapsto T(\gamma)t : b \to T(X)$$

is a natural isomorphism.

Proof. We first show that the map $(\varepsilon_T)_{X,b}$ is well-defined. For elements $t \underset{u}{\bowtie} \gamma$ and $t' \underset{u'}{\bowtie} \gamma'$ in $\widetilde{\mathbf{Tr}\,T}(X,b)$, if $t \underset{u}{\bowtie} \gamma = t' \underset{u'}{\bowtie} \gamma'$, then there exists a unique $f : u \to u'$ in $\mathcal{S}|\mathbf{A}|$ such that $f \cdot t = T(s_{|\mathbf{A}|}f)t = t'$ and $\gamma'(s_{|\mathbf{A}|}f) = \gamma$. Hence, $T(\gamma)t = T(\gamma')T(s_{|\mathbf{A}|}f)t = T(\gamma')t'$.

For injectivity, assume that there exists $t \underset{u}{\bowtie} \gamma$ and $t' \underset{u'}{\bowtie} \gamma'$ in $\widetilde{\mathbf{Tr}\,T}(X,b)$ such that $T(\gamma)t = T(\gamma')t'$. Since t is generic, there exists a unique $\alpha : s_{|\mathbf{A}|}u \to s_{|\mathbf{A}|}u'$ such that $\gamma'\alpha = \gamma$ and $T(\alpha)t = t'$. Likewise, since t' is generic, there exists a unique $\beta : s_{|\mathbf{A}|}u' \to s_{|\mathbf{A}|}u$ such that $\gamma\beta = \gamma'$ and $T(\beta)t' = t$. Using again the genericity of t and t', we obtain that $\alpha\beta = \mathrm{id}$ and $\beta\alpha = \mathrm{id}$. Hence, by Lemma 1.4.15, there exists $f : u \to u'$ in $\mathcal{S}|\mathbf{A}|$ such that $\alpha \cong s_{|\mathbf{A}|}f$ which implies that $t \underset{u}{\bowtie} \gamma = t \underset{u}{\bowtie} \gamma'(s_{|\mathbf{A}|}f) = f\dot{t} \underset{u'}{\bowtie} \gamma' = t' \underset{u'}{\bowtie} \gamma'$ as desired.

For surjectivity, let $t : b \to T(X)$ be a morphism in $\mathbf{St}(\mathbf{B})$. By Lemma 3.3.26, there exists $u \in \mathcal{S}|\mathbf{A}|$, $g : b \to T(s_{|\mathbf{A}|}u)$ generic and $\gamma : s_{|\mathbf{A}|}u \to X$ such that $t = T(\gamma)g$. Hence, $t \underset{u}{\bowtie} g \in \widetilde{\mathbf{Tr}\,T}(X,b)$ is a preimage for t. $\square$

Theorem 3.3.28. *For kit structures* $\mathbf{A} = (|\mathbf{A}|, \mathcal{A})$ *and* $\mathbf{B} = (|\mathbf{B}|, \mathcal{B})$*, there is an adjoint equivalence as follows*

$$\begin{tikzcd}
\mathbf{SProf}(!\mathbf{A},\mathbf{B}) \arrow[r, bend left, "\widetilde{(-)}"] \arrow[r, phantom, "\perp\,\simeq"] & \mathbf{Stable}(\mathbf{St}(\mathbf{A}),\mathbf{St}(\mathbf{B})) \arrow[l, bend left, "\mathbf{Tr}"]
\end{tikzcd}$$

Proof. The functors $\widetilde{(-)}$ and $\mathbf{Tr}$ are well-defined by Lemmas 3.3.20, 3.3.21, 3.3.22 and 3.3.23. They form an equivalence by Lemmas 3.3.24 and 3.3.27. It remains to show that the unit and counit verify the triangle identities: for $P \in \mathbf{SProf}(!\mathbf{A},\mathbf{B})$ and $p \underset{u}{\bowtie} \gamma \in \widetilde{P}(X,b)$,

$$\varepsilon_{\widetilde{P}}(\widetilde{\eta_P}(p \underset{u}{\bowtie} \gamma) = \varepsilon_{\widetilde{P}}((p \underset{u}{\bowtie} \mathrm{id}) \underset{u}{\bowtie} \gamma) = \widetilde{P}(\gamma)(p \underset{u}{\bowtie} \mathrm{id}) = p \underset{u}{\bowtie} \mathrm{id}.$$

For $T \in \mathbf{Stable}(\mathbf{St}(\mathbf{A}),\mathbf{St}(\mathbf{B}))$ and $t : b \to T(s_{|\mathbf{A}|}u)$ in $\mathbf{Tr}(T)$, we have

$$\mathbf{Tr}(\varepsilon_T)(\eta_{\mathbf{Tr}(T)}(t)) = \mathbf{Tr}(\varepsilon_T)(t \underset{u}{\bowtie} \mathrm{id}) = T(\mathrm{id})t = t. \qquad \square$$

As a corollary, we obtain that $\mathbf{SProf}(!\mathbf{A}, \mathbf{1}) \cong \mathbf{St}(?\mathbf{A}^{\perp})$ is equivalent to $\mathbf{Stable}(\mathbf{St}(!\mathbf{A}), \mathbf{Set})$.

3.3.3 Linear functors

Girard's model of coherence space revealed a subclass of stable functions, called linear functions, which allowed to decompose the function space operator into two more primitive connectives: the exponential modality and the linear implication. We define in this section a sub-2-category **Lin** of **Stable** of linear functors allowing to lift the decomposition $\mathbf{A} \Rightarrow \mathbf{B} \cong\, !\mathbf{A} \multimap \mathbf{B}$ into

$$\mathbf{Stable}(\mathbf{A}, \mathbf{B}) \simeq \mathbf{Lin}(!\mathbf{A}, \mathbf{B}).$$

A stable function between cliqued spaces is said to be linear if it preserves compatible unions. Cliqued space are indeed not closed under arbitrary unions so linear functions are only required to preserve the existing ones. In the categorified setting, this property translates to being a local left adjoint.

Definition 3.3.29. A functor is said to be *linear* if it is stable and a local left adjoint.

Categories of stabilized presheaves do not have all colimits in general, so linear functors $L : \mathbf{St}(\mathbf{A}) \to \mathbf{St}(\mathbf{B})$ do not preserve all colimits, they do however have a right adjoint on each slice which translates to the following weaker property: for a diagram $D : J \to \mathbf{St}(\mathbf{A})$, if $\varinjlim_{j \in J} D_j$ exists in $\mathbf{St}(\mathbf{A})$, and $\varinjlim_{j \in J} F(D_j)$ exists in $\mathbf{St}(\mathbf{B})$, then $F(\varinjlim_{j \in J} D_j) \cong \varinjlim_{j \in J} F(D_j)$.

Definition 3.3.30. We denote by **Lin** the sub-2-category of **Stable** whose objects are kits, 1-cells are linear functors $\mathbf{St}(\mathbf{A}) \to \mathbf{St}(\mathbf{B})$ and 2-cells are cartesian natural transformation between them.

In the rest of the chapter, we exhibit a biequivalence between the bicategory of s-profunctors **SProf** and the 2-category of linear functors **Lin**.

Lemma 3.3.31. *For kit structures* $\mathbf{A}, \mathbf{B}$*, a presheaf* $X \in \mathbf{St}(\mathbf{A})$ *and an s-profunctor* $P : \mathbf{A} \nrightarrow \mathbf{B}$*, if two elements* $p \underset{a}{\bowtie} x$ *and* $p' \underset{a'}{\bowtie} x'$ *in* $P^{\#}X$ *are equal, there exists a unique* $f : a \to a'$ *in* $|\mathbf{A}|$ *such that* $x' \cdot f = x$ *and* $f \cdot p = p'$*.*

Proof. Similar to Lemma 3.3.19. □

Lemma 3.3.32. *Let* $\mathbf{A} = (|\mathbf{A}|, \mathcal{A})$ *and* $\mathbf{B} = (|\mathbf{B}|, \mathcal{B})$ *be kits. For a profunctor* $P : |\mathbf{A}| \nrightarrow |\mathbf{B}|$*, if the functor* $P^{\#}\iota_{\mathbf{A}} : \mathbf{St}(\mathbf{A}) \to \widehat{|\mathbf{B}|}$ *factors through* $\iota_{\mathbf{B}}$ *by a functor* $L : \mathbf{St}(\mathbf{A}) \to \mathbf{St}(\mathbf{B})$*, then there exists a functor* $Q : |\mathbf{A}| \to \mathbf{St}(\mathbf{B})$ *such that* P *factors through* $\iota_{\mathbf{B}}$ *by* Q *and* $L \cong \mathbf{Lan}_{\mathbf{y}_{\mathbf{A}}} Q$*.*

$$\begin{tikzcd}
{|\mathbf{A}|} \arrow[rr, "P"] \arrow[rdd, "\mathrm{y}_{|\mathbf{A}|}"'] & & \widehat{|\mathbf{B}|} & & {|\mathbf{A}|} \arrow[rr, "Q"] \arrow[rd, "\mathbf{y_A}"'] & & \mathbf{St}(\mathbf{B}) \arrow[r, "\iota_{\mathbf{B}}"] & \widehat{|\mathbf{B}|} \\
& & & = & & \mathbf{St}(\mathbf{A}) \arrow[ru, dotted, "L"'] \arrow[rd, "\iota_{\mathbf{A}}"'] & & \\
& \widehat{|\mathbf{A}|} \arrow[ruu, "P^{\#}"'] & & & & & \widehat{|\mathbf{A}|} \arrow[ruu, "P^{\#}"'] &
\end{tikzcd}$$

Proof. The restricted functor $L : \mathbf{St}(\mathbf{A}) \to \mathbf{St}(\mathbf{B})$ such that $P^{\#}\iota_{\mathbf{A}} = \iota_{\mathbf{B}}L$ is obtained from Lemma 3.3.14. By Proposition 1.4.2 and Lemma 3.3.11, $P \cong P^{\#}\mathrm{y}_{|\mathbf{A}|} = P^{\#}\iota_{\mathbf{A}}\mathbf{y_A} = \iota_{\mathbf{B}}L\mathbf{y_A}$. Let $Q := L\mathbf{y_A} : |\mathbf{A}| \to \mathbf{St}(\mathbf{B})$, we obtain that $L \cong \mathbf{Lan}_{\mathbf{y_A}}Q$ by Lemma 3.3.17. $\square$

Notation 3. For an s-profunctor $P \in \mathbf{SProf}(\mathbf{A}, \mathbf{B})$, we denote by $\overline{P} : \mathbf{St}(\mathbf{A}) \to \mathbf{St}(\mathbf{B})$ the functor $\mathbf{Lan}_{\mathbf{y_A}}P \cong P^{\#}\iota_{\mathbf{A}}$.

We proceed to show that $\overline{(-)}$ induces a functor from $\mathbf{SProf}(\mathbf{A}, \mathbf{B})$ to $\mathbf{Lin}(\mathbf{St}(\mathbf{A}), \mathbf{St}(\mathbf{B}))$.

Lemma 3.3.33. *For kits $\mathbf{A} = (|\mathbf{A}|, \mathcal{A})$ and $\mathbf{B} = (|\mathbf{B}|, \mathcal{B})$, if a profunctor $P : |\mathbf{A}| \nrightarrow |\mathbf{B}|$ is in $\mathbf{SProf}(\mathbf{A}, \mathbf{B})$, then $\overline{P} : \mathbf{St}(\mathbf{A}) \to \mathbf{St}(\mathbf{B})$ is stable.*

Proof. Since $P^{\#} : \widehat{|\mathbf{A}|} \to \widehat{|\mathbf{B}|}$ is cocontinuous by Theorem 1.4.7 and the embeddings $\iota_{\mathbf{A}} : \mathbf{St}(\mathbf{A}) \hookrightarrow \widehat{|\mathbf{A}|}$, $\iota_{\mathbf{B}} : \mathbf{St}(\mathbf{B}) \hookrightarrow \widehat{|\mathbf{B}|}$ create filtered colimits and epimorphisms by Corollary 3.3.9, the functor $\overline{P} \cong P^{\#}\iota_{\mathbf{A}} : \mathbf{St}(\mathbf{A}) \to \mathbf{St}(\mathbf{B})$ is finitary and epi-preserving.

It remains to show that $\overline{P}$ admits generic factorizations. By Proposition 3.1.21, it suffices to show that $\overline{P}$ admits generic factorizations relative to representables. Consider a morphism $\mathrm{y}b \to \overline{P}(X)$ in $\mathbf{St}(\mathbf{B})$, it is of the form $p \underset{a}{\bowtie} x$ with $p \in P(a, b)$ and $x \in X(a)$ for some $a \in |\mathbf{A}|$. We show that

$$\mathrm{y}b \xrightarrow{p \underset{a}{\bowtie} \mathrm{id}} \overline{P}(\mathrm{y}a) \xrightarrow{\overline{P}(x)} \overline{P}(X)$$

is a generic factorisation for $p \underset{a}{\bowtie} x$. The equality $p \underset{a}{\bowtie} x = \overline{P}(x)(p \underset{a}{\bowtie} \mathrm{id})$ is immediate, it remains to prove that $p \underset{a}{\bowtie} \mathrm{id}$ is generic.

Assume that there exist $Y, Z \in \mathbf{St}(\mathbf{A})$ and morphisms $p' \underset{a'}{\bowtie} y' : \mathrm{y}b \to \overline{P}(Y)$ and $\alpha : Y \to Z, z : \mathrm{y}a \to Z$ such that the following diagram commutes in $\mathbf{St}(\mathbf{B})$:

$$\begin{tikzcd}
\mathrm{y}b \arrow[r, "p' \underset{a'}{\bowtie} y'"] \arrow[d, "p \underset{a}{\bowtie} \mathrm{id}"'] & \overline{P}(Y) \arrow[d, "\overline{P}(\alpha)"] \\
\overline{P}(\mathrm{y}a) \arrow[r, "\overline{P}(z)"'] & \overline{P}(Z)
\end{tikzcd}$$

The equality $\overline{P}(\alpha)(p' \underset{a'}{\bowtie} y') = p \underset{a}{\bowtie} z$ is equivalent to $p' \underset{a'}{\bowtie} \alpha y' = p \underset{a}{\bowtie} z$ which implies by Lemma 3.3.31 that there ezists a unique $f : a \to a'$ in $|\mathbf{A}|$ such that $(\alpha y') \cdot f = z$ and $f \cdot p = p'$. We then have $\overline{P}(y' \cdot f)(p \underset{a}{\bowtie} \mathrm{id}) = p \underset{a}{\bowtie} y' \cdot f = f \cdot p \underset{a'}{\bowtie} y' = p' \underset{a'}{\bowtie} y'$. It remains to show uniqueness, assume that there exists $y : \mathrm{y}a \to Y$ such that $\alpha y = z$ and $\overline{P}(y)(p \underset{a}{\bowtie} \mathrm{id}) = p' \underset{a'}{\bowtie} y'$ i.e. $p \underset{a}{\bowtie} y = p' \underset{a'}{\bowtie} y'$. It implies that there exists $g : a \to a'$ such that $y' \cdot g = y$ and $g \cdot p = p'$. Hence, $(\alpha y') \cdot g = \alpha y = z$ which implies that $f = g$ so that $y = y' \cdot f$ as desired.

$$\begin{tikzcd}
\mathrm{y}b \arrow[r, "p' \underset{a'}{\bowtie} y'"] \arrow[d, "p \underset{a}{\bowtie} \mathrm{id}"'] & \overline{P}(Y) \arrow[d, "\overline{P}(\alpha)"] \\
\overline{P}(\mathrm{y}a) \arrow[ur, dotted, "\overline{P}(y)"] \arrow[r, "\overline{P}(z)"'] & \overline{P}(Z)
\end{tikzcd}$$

$\square$

Lemma 3.3.34. *For kits* $\mathbf{A} = (|\mathbf{A}|, \mathcal{A})$ *and* $\mathbf{B} = (|\mathbf{B}|, \mathcal{B})$*, if a profunctor* $P : |\mathbf{A}| \nrightarrow |\mathbf{B}|$ *is in* $\mathbf{SProf}(\mathbf{A}, \mathbf{B})$*, then* $\overline{P} : \mathbf{St}(\mathbf{A}) \to \mathbf{St}(\mathbf{B})$ *is in* $\mathbf{Lin}(\mathbf{St}(\mathbf{A}), \mathbf{St}(\mathbf{B}))$*.*

Proof. Since $\overline{P} : \mathbf{St}(\mathbf{A}) \to \mathbf{St}(\mathbf{B})$ is stable by Lemma 3.3.33, we need to show that for all $X \in \mathbf{St}(\mathbf{A})$, the induced functor $\mathbf{St}(\mathbf{A})/_X \to \mathbf{St}(\mathbf{B})/_{\overline{P}(X)}$ has a right adjoint L_X. For $\alpha : X' \to X$ in $\mathbf{St}(\mathbf{A})$ and $\beta : Y \to \overline{P}(X)$ in $\mathbf{St}(\mathbf{B})$, we require the following correspondence:

$$\begin{tikzcd}
\overline{P}(X') \arrow[r] \arrow[dr, "\overline{P}(\alpha)"'] & Y \arrow[d, "\beta"] & & X' \arrow[r] \arrow[dr, "\alpha"'] & L_X(Y) \arrow[d, "L_X(\beta)"] \\
& \overline{P}(X) & \Leftrightarrow & & X
\end{tikzcd}$$

Note that since $\overline{P} = \mathbf{Lan}_{\mathbf{y_A}} P$, the following sets are isomorphic by Proposition 1.4.2:

$$\begin{aligned}\mathbf{St}(\mathbf{B})(\overline{P}(X'), Y) &\cong |\widehat{\mathbf{A}}| \left(|\widehat{\mathbf{A}}|(\mathbf{y_A}(-), X'), \mathbf{St}(\mathbf{B})(P(-), Y) \right) \\ &\cong |\widehat{\mathbf{A}}| \left(X', \mathbf{St}(\mathbf{B})(P(-), Y) \right).\end{aligned}$$

Hence, the left triangle commutes in $\mathbf{St}(\mathbf{B})$ if and only if the following square commutes in $|\widehat{\mathbf{A}}|$:

$$\begin{tikzcd}
X' \arrow[r] \arrow[d, "\alpha"'] & \mathbf{St}(\mathbf{B})(P(-), Y) \arrow[d, "\mathbf{St}(\mathbf{B})(P(-), \beta)"] \\
X \arrow[r] & \mathbf{St}(\mathbf{B})(P(-), \overline{P}(X))
\end{tikzcd}$$

Let $L_X(Y)$ be given by the following pullback:

$$\begin{tikzcd}
L_X(Y) \arrow[r] \arrow[d, "L_X(\beta)"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{St}(\mathbf{B})(P(-), Y) \arrow[d, "\mathbf{St}(\mathbf{B})(P(-), \beta)"] \\
X \arrow[r] & \mathbf{St}(\mathbf{B})(P(-), P^{\#}(X))
\end{tikzcd}$$

Explicitely, for $a \in |\mathbf{A}|$, $L_X(Y)(a)$ is given by:

$$\{(x, \gamma) \in X(a) \times \mathbf{St}(\mathbf{B})(P(a), Y) \mid \forall b \in |\mathbf{B}|, \forall p \in P(a, b), p \underset{a}{\bowtie} x = \beta\gamma p\}$$

and $L_X(\beta)(x, \gamma) = x$. We need check that the presheaf $L_X(Y)$ is in $\mathbf{St}(\mathbf{A})$. For $(x, \gamma) \in L_X(Y)(a)$ and $f \in \mathrm{Endo}(a)$, since $(x, \gamma) \cdot f = (x \cdot f, \gamma \circ P(f))$, if $(x, \gamma) \cdot f = (x, \gamma)$ then $x \cdot f = x$ and therefore that $f \in \bigcup \mathcal{A}(a)$. □

Lemma 3.3.35. *For kit structures* $\mathbf{A} = (|\mathbf{A}|, \mathcal{A})$*,* $\mathbf{B} = (|\mathbf{B}|, \mathcal{B})$ *and a natural transformation* $\alpha : P \Rightarrow Q$ *in* $\mathbf{SProf}(P, Q)$*, the natural transformation* $\overline{\alpha} : \overline{P} \to \overline{Q} : \mathbf{St}(\mathbf{A}) \to \mathbf{St}(\mathbf{B})$ *is cartesian.*

Proof. Let $\gamma : X \to Y$ in $\mathbf{St}(\mathbf{A})$, we want to show that the square below is a pullback in $\mathbf{Set}$ for all $b \in |\mathbf{B}|$.

$$\begin{tikzcd}
\overline{P}(X)(b) \arrow[r, "\overline{\alpha}_X"] \arrow[d, "\overline{P}(\gamma)"'] & \overline{Q}(X)(b) \arrow[d, "\overline{Q}(\gamma)"] \\
\overline{P}(Y)(b) \arrow[r, "\overline{\alpha}_Y"'] & \overline{Q}(Y)(b)
\end{tikzcd}$$

We show that for all $(p \underset{a_1}{\bowtie} y) \in \overline{P}(Y)(b)$ and $(q \underset{a_2}{\bowtie} x) \in \overline{Q}(X)(b)$ such that

$$\alpha(p) \underset{a_1}{\bowtie} x = \overline{\alpha}_Y(p \underset{a_1}{\bowtie} x) = \overline{Q}(\gamma)(q \underset{a_2}{\bowtie} x) = q \underset{a_2}{\bowtie} \gamma(x)$$

there exists a unique $u \in \overline{P}(X)(b)$ such that $\overline{P}(\gamma)(u) = p \underset{a_1}{\bowtie} y$ and $\overline{\alpha}_X(u) = q \underset{a_2}{\bowtie} x$. The equality $\alpha(p) \underset{a_1}{\bowtie} y = q \underset{a_2}{\bowtie} \gamma(x)$ implies that there exists $f : a_1 \to a_2$ in $\mathbb{A}$ such that $q = f \cdot \alpha(p)$ and $\gamma(x) \cdot f = y$. Define u to be $p \underset{a_1}{\bowtie} (x \cdot f) \in \overline{P}(X)(b)$, we then obtain that

$$\overline{P}(\gamma)(u) = p \underset{a_1}{\bowtie} \gamma(x \cdot f) = p \underset{a_1}{\bowtie} \gamma(x) \cdot f = p \underset{a_1}{\bowtie} y$$

and

$$\overline{\alpha}_X(u) = \alpha(p) \underset{a_1}{\bowtie} x \cdot f = f \cdot \alpha(p) \underset{a_2}{\bowtie} x = q \underset{a_2}{\bowtie} x.$$

Assume now that there exists $v = p_0 \underset{a_0}{\bowtie} x_0 \in \overline{P}(X)(b)$ such that $\overline{P}(\gamma)(v) = p \underset{a_1}{\bowtie} y$ and $\overline{\alpha}_X(v) = q \underset{a_2}{\bowtie} x$. We then have that $p_0 \underset{a_0}{\bowtie} \gamma(x_0) = p \underset{a_1}{\bowtie} y$ and $\alpha(p_0) \underset{a_0}{\bowtie} x_0 = q \underset{a_2}{\bowtie} x$. Hence, there exists $g : a_0 \to a_1$ in $\mathbb{A}$ such that $q = \alpha(p_0) \cdot g$ and $x \cdot g = x_0$, and there exists $h : a_1 \to a_0$ in $\mathbb{A}$ such that $p_0 = h \cdot p$ and $\gamma(x_0) \cdot h = y$. Therefore,

$$\begin{gathered} gh \cdot \alpha(p) = g \cdot \alpha(h \cdot p) = g \cdot \alpha(p_0) = q \quad \text{and} \\ \gamma(x) \cdot gh = \gamma(x \cdot g) \cdot h = \gamma(x_0) \cdot h = y. \end{gathered}$$

Therefore, ghf^{-1} is in $\mathrm{Stab}(Y(a_1))$ which implies that $ghf^{-1} \in \bigcup \mathcal{A}(a_1)$. We also have $ghf^{-1} \cdot q = q$. Since Q is a q-profunctor and id_b is in $\bigcup \mathcal{B}^{\perp}(b)$, we obtain that ghf^{-1} is in $\bigcup \mathcal{A}^{\perp}(a)$ which implies that $f = gh$. Hence, we have

$$u = p \underset{a_1}{\bowtie} (x \cdot gh) = p \underset{a_1}{\bowtie} (x_0 \cdot h) = (h \cdot p) \underset{a_0}{\bowtie} x_0 = p_0 \underset{a_0}{\bowtie} x_0 = v$$

as desired. □

The linear trace $\mathbf{tr} : \mathbf{Lin}(\mathbf{St}(\mathbf{A}), \mathbf{St}(\mathbf{B})) \to \mathbf{SProf}(\mathbf{A}, \mathbf{B})$ is given by the mapping $L \mapsto L\mathbf{y}_{\mathbf{A}}$ for a linear functor $L : \mathbf{St}(\mathbf{A}) \to \mathbf{St}(\mathbf{B})$. We proceed to show that $\mathbf{tr}$ is a well-defined functor.

Lemma 3.3.36. *Let $\mathbf{A} = (|\mathbf{A}|, \mathcal{A})$, $\mathbf{B} = (|\mathbf{B}|, \mathcal{B})$ be kit structures and $L : \mathbf{St}(\mathbf{A}) \to \mathbf{St}(\mathbf{B})$ a linear functor. For every X in $\mathbf{St}(\mathbf{A})$ and $l : b \to L(X)$ in $\mathbf{St}(\mathbf{B})$, there exists $a \in |\mathbf{A}|$, $g : b \to L(\mathbf{y}_{\mathbf{A}}(a))$ generic and $x : \mathbf{y}_{\mathbf{A}}(a) \to X$ such that $l = L(x)g$.*

Proof. Since L is linear, l can be factored as $b \xrightarrow{g} L(Y) \xrightarrow{L(f)} L(X)$ where g is generic. By Corollary 3.3.5, $Y \cong \coprod_{i\in I} \overline{\mathrm{y}}(a_i|G_i)$ where each G_i is a finitely generated group in $\mathcal{A}(a_i)$. Since L preserves coproducts, we obtain that $L(Y) \cong \coprod_{i\in I} L(\overline{\mathrm{y}}(a_i|G_i))$.

Since colimits are computed pointwise in $\mathbf{St}(\mathbf{B})$, it implies that there exists a and G such that g can be factored as

$$b \xrightarrow{g'} L(\overline{\mathrm{y}}(a|G)) \xrightarrow{L(\iota)} \coprod_{i\in I} L(\overline{\mathrm{y}}(a_i|G_i)).$$

Since $\overline{\mathrm{y}}(a|G)$ is the colimit of $(g : \mathbf{y_A}(a) \to \mathbf{y_A}(a))_{g\in G}$ in $\mathbf{St}(\mathbf{A})$ and L is linear , $L(\overline{\mathrm{y}}(a|G))$ is the colimit of $(L(g) : L(\mathbf{y_A}(a)) \to L(\mathbf{y_A}(a)))_{g\in G}$ in $\mathbf{St}(\mathbf{B})$. Hence, there exists $g'' : b \to L(\mathbf{y_A}(a))$ such that $L(q)g'' = g'$.

Since g is generic, there exists $h : Y \to \mathbf{y_A}(a)$ such that $\iota g' h = \mathrm{id}$ and $L(h)g = g''$.

$$\begin{tikzcd}
b \arrow[r, "g''"] \arrow[d, "g"'] & L(\mathbf{y_A}(a)) \arrow[d, "L(\iota g')"] \\
L(Y) \arrow[r, "L(\mathrm{id})"'] \arrow[ur, dotted, "L(h)"] & L(Y)
\end{tikzcd}$$

Since h is split monic, by Lemma 3.3.25, Y is isomorphic to a $\mathbf{y_A}(a)$. $\square$

Corollary 3.3.37. *For kit structures* $\mathbf{A} = (|\mathbf{A}|, \mathcal{A})$ *and* $\mathbf{B} = (|\mathbf{B}|, \mathcal{B})$*, if a functor* $L : \mathbf{St}(\mathbf{A}) \to \mathbf{St}(\mathbf{B})$ *is linear, then for every* $l : b \to L(\mathbf{y_A}(a))$ *in* $\mathbf{St}(\mathbf{B})$*,* l *is generic.*

Proof. By Lemma 3.3.36, there exists $g : b \to L(\mathbf{y_A}(a_0))$ generic and $\alpha : \mathbf{y_A}(a_0) \to \mathbf{y_A}(a)$ such that $l = L(\alpha)g$. Since α is an isomorphism, we obtain that l is generic as well. $\square$

Lemma 3.3.38. *For kit structures* $\mathbf{A} = (|\mathbf{A}|, \mathcal{A})$ *and* $\mathbf{B} = (|\mathbf{B}|, \mathcal{B})$*, if a functor* $L : \mathbf{St}(\mathbf{A}) \to \mathbf{St}(\mathbf{B})$ *is linear then the profunctor* $\mathbf{tr}(L) : |\mathbf{A}| \nrightarrow |\mathbf{B}|$ *given by* $L \circ \mathbf{y_A}$ *is an s-profunctor.*

Proof. Let $l : b \to L(\mathbf{y_A}(a))$ be in $\mathbf{tr}(L)(a, b)$ and $f \in |\mathbf{A}|(a, a)$, $g \in |\mathbf{B}|(b, b)$ such that $f \cdot l \cdot g = l$, i.e. the following square commutes in $\mathbf{St}(\mathbf{B})$:

$$\begin{tikzcd}
b \arrow[r, "l"] & L(\mathbf{y}_{\mathbf{A}}(a)) \arrow[d, "L(\mathbf{y}_{\mathbf{A}}f)"] \\
b \arrow[u, "g"] \arrow[r, "l"'] & L(\mathbf{y}_{\mathbf{A}}(a))
\end{tikzcd}$$

Assume that $f \in \bigcup \mathcal{A}(a)$, then $\overline{\mathrm{y}}(a|\langle f\rangle) \in \mathbf{St}(\mathbf{A})$. Let $q : \mathbf{y}_{\mathbf{A}}(a) \to \overline{\mathrm{y}}(a|\langle f\rangle)$ be the quotient map, we then have that $q \circ \mathbf{y}_{\mathbf{A}}f = q$ in $\mathbf{St}(\mathbf{A})$ which implies that the following diagram commutes in $\mathbf{St}(\mathbf{B})$:

$$\begin{tikzcd}
b \arrow[r, "l"] & L(\mathbf{y}_{\mathbf{A}}(a)) \arrow[dd, "L(\mathbf{y}_{\mathbf{A}}f)"'] \arrow[dr, "L(q)"] & \\
 & & L(\overline{\mathrm{y}}(a|\langle f\rangle)) \\
b \arrow[uu, "g"] \arrow[r, "l"'] & L(\mathbf{y}_{\mathbf{A}}(a)) \arrow[ur, "L(q)"'] &
\end{tikzcd}$$

Hence, $g \in \mathrm{Stab}(L(q)l)$ which implies that $g \in \bigcup \mathcal{B}(b)$.

Assume now that $g \in \bigcup \mathcal{B}^{\perp}(b)$, we want to show that for all n, if f^n is in $\bigcup \mathcal{A}(a)$, then $f^n = \mathrm{id}_a$. If $f^n \in \bigcup \mathcal{A}(a)$, then $\overline{\mathrm{y}}(a|\langle f^n\rangle) \in \mathbf{St}(\mathbf{A})$ and the following diagram commutes in $\mathbf{St}(\mathbf{B})$:

$$\begin{tikzcd}
b \arrow[r, "l"] & L(\mathbf{y}_{\mathbf{A}}(a)) \arrow[dd, "L(f^n)"'] \arrow[dr, "L(q')"] & \\
 & & L(\overline{\mathrm{y}}(a|\langle f^n\rangle)) \\
b \arrow[uu, "g^n"] \arrow[r, "l"'] & L(\mathbf{y}_{\mathbf{A}}(a)) \arrow[ur, "L(q')"'] &
\end{tikzcd}$$

where $q' : \mathbf{y}_{\mathbf{A}}(a) \to \overline{\mathrm{y}}(a|\langle f^n\rangle)$ is the quotient map in $\mathbf{St}(\mathbf{A})$. Hence, $g^n \in \mathrm{Stab}(L(q')l)$ which implies that $g^n = \mathrm{id}_b$ so that the following diagram commutes:

$$\begin{tikzcd}
b \arrow[r, "l"] \arrow[d, "l"'] & L(\mathbf{y}_{\mathbf{A}}(a)) \arrow[d, "L(q')"] \\
L(\mathbf{y}_{\mathbf{A}}(a)) \arrow[ur, "L(f^n)"] \arrow[r, "L(q')"'] & L(\overline{\mathrm{y}}(a|\langle f^n\rangle))
\end{tikzcd}$$

Since l is generic by Corollary 3.3.37, we obtain that $f^n = \mathrm{id}_a$ as desired. $\square$

Note that since $\mathbf{SProf}(\mathbf{A}, \mathbf{B})$ is a full subcategory of $\mathbf{Prof}(|\mathbf{A}|, |\mathbf{B}|)$, for a cartesian transformation $\alpha : L \Rightarrow L'$ in $\mathbf{Lin}(\mathbf{St}(\mathbf{A}), \mathbf{St}(\mathbf{B}))$, we automatically have $\alpha \mathbf{y_A}$ in $\mathbf{SProf}(\mathbf{A}, \mathbf{B})(L\mathbf{y_A}, L', \mathbf{y_A})$.

Lemma 3.3.39. *For kit structures* $\mathbf{A} = (|\mathbf{A}|, \mathcal{A})$*,* $\mathbf{B} = (|\mathbf{B}|, \mathcal{B})$ *and an s-profunctor* $P : \mathbf{A} \nrightarrow \mathbf{B}$*, there is an isomorphism* $\eta : P \cong \mathbf{tr}\,\overline{P}$*.*

Proof. Since $\mathbf{y_A}$ is fully faithful, P is isomorphic to $(\mathbf{Lan}_{\mathbf{y_A}} P)\mathbf{y_A} = \mathbf{tr}\,\overline{P}$ by Proposition 1.4.2. □

Lemma 3.3.40. *For kit structures* $\mathbf{A} = (|\mathbf{A}|, \mathcal{A})$*,* $\mathbf{B} = (|\mathbf{B}|, \mathcal{B})$ *and a linear functor* $L : \mathbf{St}(\mathbf{A}) \to \mathbf{St}(\mathbf{B})$*, there is an isomorphism* $\varepsilon : \overline{\mathbf{tr}(L)} \cong L$*.*

Proof. Let X be in $\mathbf{St}(\mathbf{A})$ and $b \in |\mathbf{B}|$. An element of $\overline{\mathbf{tr}(L)}(X)(b)$ is of the form $l \underset{a}{\bowtie} x$ where $l : b \to L(\mathbf{y_A}(a))$ and $x \in X(a)$, we define $\varepsilon_{X,b} : \overline{\mathbf{tr}(L)}(X, b) \to L(X, b)$ by $l \underset{a}{\bowtie} x \mapsto L(x)l$. For $l \underset{a}{\bowtie} x$ and $l' \underset{a'}{\bowtie} x'$ in $\overline{\mathbf{tr}(L)}(X)(b)$, $l \underset{a}{\bowtie} x = l' \underset{a'}{\bowtie} x'$ is equivalent to the existence of a unique $f : a \to a'$ such that $x' \cdot \alpha = x$ and $L(f)l = l'$ which implies that $\varepsilon_{X,b}$ is well-defined and injective. Let $l : b \to L(X)$ be in $\mathbf{St}(\mathbf{B})$, by Lemma 3.3.36, there exists $a \in |\mathbf{A}|$, $g : b \to L(\mathbf{y_A}(a))$ generic and $x : \mathbf{y_A}(a) \to X$ such that $l = L(x)g$. Now, $\varepsilon_{X,b}(g \underset{a}{\bowtie} x) = L(x)g$ which implies that $\varepsilon_{X,b}$ is surjective as desired. Naturality of α follows immediately from the functoriality of L. □

Theorem 3.3.41. *For kit structures* $\mathbf{A} = (|\mathbf{A}|, \mathcal{A})$ *and* $\mathbf{B} = (|\mathbf{B}|, \mathcal{B})$*, there is an adjoint equivalence as follows*

$$\begin{tikzcd}
\mathbf{SProf}(\mathbf{A}, \mathbf{B}) \arrow[r, bend left, "\overline{(-)}"] \arrow[r, phantom, "\perp\,\simeq"] & \mathbf{Lin}(\mathbf{St}(\mathbf{A}), \mathbf{St}(\mathbf{B})) \arrow[l, bend left, "\mathbf{tr}"]
\end{tikzcd}$$

Proof. The functors $\overline{(-)}$ and $\mathbf{tr}$ are well-defined by Lemmas 3.3.34, 3.3.35 and 3.3.38. They form an equivalence by Lemmas 3.3.39 and 3.3.40. It remains to show that the unit and counit verify the triangle identities: for $P \in \mathbf{SProf}(\mathbf{A}, \mathbf{B})$ and $p \underset{a}{\bowtie} x \in \overline{P}(X, b)$,

$$\varepsilon_{\overline{P}}(\overline{\eta_P}(p \underset{a}{\bowtie} x) = \varepsilon_{\overline{P}}((p \underset{a}{\bowtie} \mathrm{id}) \underset{a}{\bowtie} x) = \widetilde{P}(x)(p \underset{a}{\bowtie} \mathrm{id}) = p \bowtie \mathrm{id}.$$

For $L \in in\mathbf{Lin}(\mathbf{St}(\mathbf{A}), \mathbf{St}(\mathbf{B}))$ and $l : b \to L(\mathbf{y_A}(a))$ in $\mathbf{tr}(L)$, we have

$$\mathbf{tr}(\varepsilon_L)(\eta_{\mathbf{tr}(L)}(l)) = \mathbf{tr}(\varepsilon_L)(l \underset{a}{\bowtie} \mathrm{id}) = L(\mathrm{id})l = l.$$ □

Corollary 3.3.42. *For a kit structure* $\mathbf{A}$*, we have an adjoint equivalence* $\mathbf{St}(\mathbf{A}^\perp) \simeq \mathbf{Lin}(\mathbf{St}(\mathbf{A}), \mathbf{Set})$*.*

Conclusion

To understand the relationship between normal or polynomial functors and generalized species, we adopt the viewpoint of considering profunctors as groupoids actions. It allows us to add more control on the stabilizers induced by these actions through the kit structures. We obtain a 2-categorical model of stability that is also a model of differential linear logic.

For future work, since our model corresponds to a generalization of flat species, we believe that it will be a better behaved setting to obtain a theory of integration. Paul Taylor showed that his creed construction [110] induces a categorical model of polymorphism, we also hope to carry a similar construction in our case.

Chapter 4

Glueing Symmetric and Cartesian Species of Structures

4.1 Introduction

We study in this chapter the free coproduct completion pseudo-comonad $\mathcal{C}$ (corresponding to the finite Fam-construction) on **Prof** which models non-linear operations such as duplication and erasure.

Our motivation is two-fold: firstly, when we take $\mathcal{C}$ as a pseudo-comonad to interpret the exponential modality, we obtain a model of linear logic that generalizes the Scott model. The obtained model of $\mathcal{C}$-species gives a different perspective on how to categorify Scott-continuity: directed suprema now correspond to sifted colimits and Scott-continuous functions correspond to strongly finitary functors. These correspondences are summarized in the table below:

a preorder $A = (\lvert A\rvert, \leq_A)$	a small category $\mathbb{A}$
a monotonous function $f : A \to B$	a functor $F : \mathbb{A} \to \mathbb{B}$
a down-closed subset $x \subseteq \lvert A\rvert$	a presheaf $X : \mathbb{A}^{op} \to \mathbf{Set}$
an ideal relation $R \subseteq A \times B$	a profunctor $F : \mathbb{A} \nrightarrow \mathbb{B}$
inclusion of relations	a natural transformation
a directed supremum	a sifted colimit
a Scott-continuous function	a strongly finitary functor

Secondly, since $\mathcal{S}$-species categorify the relational model and $\mathcal{C}$-species categorify the Scott-model, the next step is to connect them using a construction in the spirit of Ehrhard's extensional collapse [34]. In future work, we aim to explore the intersection type counterpart of this construction in the profunctorial setting.

In the setting of algebraic theories and operads, symmetric operads are monads in the category of combinatorial species $[\mathcal{S}\,\mathbf{1}, \mathbf{Set}]$ with the Day convolution product and a Lawvere theory or cartesian operad is a monad in the category $[\mathbf{FinSet}, \mathbf{Set}] \simeq [\mathcal{C}\,\mathbf{1}, \mathbf{Set}]$ with the substitution product. This analogy extends to the many-sorted case where symmetric many-sorted operads correspond to monads in the bicategory of $\mathcal{S}$-species [41]. Similarly, monads for $\mathcal{C}$-species correspond to many-sorted cartesian operads. $\mathcal{C}$-species are also related to the cartesian closed bicategory of cartesian profunctors studied by Fiore and Joyal [43] where $\mathcal{C}$-species can be obtained by restricting to free cartesian categories.

We start by recalling Ehrhard's orthogonality construction between **Rel** and **ScottL** in [34]. In the second part of the chapter, we show that the bicategory of profunctors with the free finite coproduct pseudo-comonad $\mathcal{C}$ constitutes a model of linear logic that generalizes the Scott model. Lastly, we construct a bicategory providing the first step to connect the obtained $\mathcal{C}$-species with the generalized $\mathcal{S}$-species model.

4.2 Extensional collapse as a double glueing construction

The relational model and the Scott model presented in Sections 1.1.4 and 1.1.5 are tightly related. For a closed linear logic formula A, we denote by $[\![A]\!]_s$ its interpretation in **ScottL** and by $[\![A]\!]_r$ its interpretation in **Rel**. We can show by induction on the structure of A that the set $[\![A]\!]_r$ is equal to the underlying set of the preorder $[\![A]\!]_s$, i.e. if $[\![A]\!]_s$ is equal to the preorder $P = (|P|, \leq_P)$, then $[\![A]\!]_r$ is equal to $|P|$.

For a proof π of $\vdash A$, its interpretation in **Rel**, which we denote $[\![\pi]\!]_r$, is an element of $\mathbf{Rel}(\mathbf{1}, [\![A]\!]_r) \cong \mathcal{P}(|P|)$ or equivalently a subset of $|P|$. The interpretation of π in **ScottL** is a down-closed subset of $|P|$ as $[\![\pi]\!]_s \in \mathbf{ScottL}(\mathbf{1}, P) \cong \downarrow(P)$. The two sets $[\![\pi]\!]_s$ and $[\![\pi]\!]_r$ are related by the following equality

$$[\![\pi]\!]_s = \downarrow([\![\pi]\!]_r)$$

i.e. for all $a \in [\![\pi]\!]_s$, there exists $a' \in [\![\pi]\!]_r$ such that $a \leq_P a'$. In particular, it implies that $[\![\pi]\!]_s \neq \varnothing$ if and only if $[\![\pi]\!]_r \neq \varnothing$. This equivalence is crucially used to translate normalization theorems from the non-idempotent intersection typing system associated to **Rel** to the idempotent one associated to **ScottL** [33]. It is however not possible to prove this property by induction on the structure of the proof. This is due to the fact that for preorders $A = (|A|, \leq_A)$, $B = (|B|, \leq_B)$, $C = (|C|, \leq_C)$ and relations

$R \subseteq |A| \times |B|, S \subseteq |B| \times |C|$, we do not have $\downarrow(S) \circ \downarrow(R) = \downarrow(S \circ R)$ in general.

In order to prove this property, Ehrhard constructs a new category **Pop** (preorders with projections) whose objects are preorders equipped with a predicate relating the interpretation of proofs in the two models in a compositional way. Formally, the category **Pop** is obtained from an orthogonality construction on the product category $\mathbf{Rel} \times \mathbf{ScottL}$. Let (S, P) be an object of the product category $\mathbf{Rel} \times \mathbf{ScottL}$, we have $\mathbf{Rel}(\mathbf{1}, S) \times \mathbf{ScottL}(\mathbf{1}, P) \cong \mathcal{P}(S) \times \downarrow(P)$ and dually $\mathbf{Rel}(S, \mathbf{1}) \times \mathbf{ScottL}(P, \mathbf{1}) \cong \mathcal{P}(S) \times \uparrow(P)$. We define an orthogonality relation $\perp_{(S,P)} \subseteq (\mathbf{Rel}(\mathbf{1}, S) \times \mathbf{ScottL}(\mathbf{1}, P)) \times (\mathbf{Rel}(S, \mathbf{1}) \times \mathbf{ScottL}(P, \mathbf{1}))$ as follows:

$$(x, l) \perp_{(S,P)} (x', l') \quad :\Leftrightarrow \quad (x \cap x' \neq \emptyset \Leftrightarrow l \cap l' \neq \emptyset).$$

Using the equivalence

$$(x \cap x' \neq \emptyset \Leftrightarrow l \cap l' \neq \emptyset) \quad \Leftrightarrow \quad (x' \circ x = \mathrm{id} \Leftrightarrow l' \circ l = \mathrm{id}).$$

we can see that it is in fact a focused orthogonality (Definition 1.2.8) and therefore automatically verifies all the axioms in Definition 1.2.5. The induced tight orthogonality category $\mathbb{T}$ (Definition 1.2.7) is described by:

$$\begin{tikzcd}
 & & \mathbb{T} \arrow[ddll] \arrow[dd] \arrow[ddrr] & & \\
 & \Rightarrow & & \Rightarrow & \\
\mathbf{Set} & & \mathbf{Rel} \times \mathbf{ScottL} \arrow[ll, "\mathbf{Rel}(\mathbf{1}, -) \times \mathbf{ScottL}(\mathbf{1}, -)"'] \arrow[rr, "\mathbf{Rel}(-, \mathbf{1}) \times \mathbf{ScottL}(-, \mathbf{1})"'] & & \mathbf{Set}^{\mathrm{op}}
\end{tikzcd}$$

- objects: tuples (S, P, X) where $S \in \mathbf{Set}$, $P \in \mathbf{ScottL}$ and X is a subset of $\mathcal{P}(S) \times \downarrow(P)$ verifying $X = X^{\perp\perp}$;
- morphisms: a morphism from (S, P, X) to (S', P', X') is a pair of relations $(R_1, R_2) \in \mathbf{Rel}(S, S') \times \mathbf{ScottL}(P, P')$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{P}(S) \times \downarrow(P) \arrow[r, "R_{1_\star} \times R_{2_\star}"] & \mathcal{P}(S') \times \downarrow(P') \\
X \arrow[u, hook] \arrow[r, dotted] & X' \arrow[u, hook]
\end{tikzcd}$$

Note that we only require forward preservation since the orthogonality is focused.

Recall that the objective is to prove that for a proof π of a linear logic formula A, we have $[\![\pi]\!]_s = {\downarrow}([\![\pi]\!]_r)$ given that $[\![A]\!]_r = |[\![A]\!]_s|$. We therefore consider the subcategory of $\mathbb{T}$ containing objects (S, P, X) verifying $S = |P|$ and $X \subseteq \{(x, l) \mid l = {\downarrow}(x)\}$. For morphisms, we further restrict to pairs of relations (R_1, R_2) such that $R_2 = {\downarrow}(R_1)$.

The orthogonality between pairs $(x, l) \in \mathcal{P}(S) \times {\downarrow}(P)$ and $(x', l') \in \mathcal{P}(S) \times {\uparrow}(P)$ can now be reduced to an orthogonality $\perp_P \subseteq \mathbf{Rel}(\mathbf{1}, |P|) \times \mathbf{Rel}(|P|, \mathbf{1})$ given by

$$x \perp_P x' \quad :\Leftrightarrow \quad (x \cap x' \neq \emptyset \Leftrightarrow {\downarrow}(x) \cap {\uparrow}(x') \neq \emptyset). \tag{4.1}$$

Hence, we can simplify the presentation of the obtained subcategory as follows:

Definition 4.2.1. The *category of preorders with projections* **Pop** is defined as

- objects are pairs (P, D) where P is a preorder and D is a subset of $\mathcal{P}(|P|)$ satisfying $D = D^{\perp\perp}$;
- a morphism from (P, D) to (P', D') is a relation R in $\mathbf{Rel}(|P|, |P'|)$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{P}(|P|) \times {\downarrow}(P) \arrow[r, "R_\star \times ({\downarrow}(R))_\star"] & \mathcal{P}(|P'|) \times {\downarrow}(P') \\
D \arrow[u, hook] \arrow[r, dotted] & D' \arrow[u, hook]
\end{tikzcd}$$

  where the inclusions $D \hookrightarrow \mathcal{P}(|P|) \times {\downarrow}(P)$ and $D' \hookrightarrow \mathcal{P}(|P'|) \times {\downarrow}(P')$ are given by $x \mapsto (x, {\downarrow}(x))$.

Remark 6. Equivalently, a relation R is in $\mathbf{Pop}((P, D), (P', D'))$ if for all $x \in D$,

$$Rx \in D' \qquad \text{and} \qquad {\downarrow}(Rx) = {\downarrow}(R){\downarrow}(x).$$

Theorem 4.2.2 ([34])**.** *The category* **Pop** *is a model of linear logic. There are two forgetful functors* $U_s : \mathbf{Pop} \to \mathbf{ScottL}$ *and* $U_r : \mathbf{Pop} \to \mathbf{Rel}$

$$\begin{tikzcd}
& \mathbf{Pop} \arrow[ld, "U_s"'] \arrow[d] \arrow[rd, "U_r"] & \\
\mathbf{Set} & \mathbf{Rel} \times \mathbf{ScottL} \arrow[l] \arrow[r] & \mathbf{ScottL}
\end{tikzcd}$$

where an object (P, D) *in* $\mathbf{Pop}$ *is mapped by* U_r *to* $|P| \in \mathbf{Rel}$ *and to* $P \in \mathbf{ScottL}$ *by* U_s*. A relation* $R \in \mathbf{Pop}((P, D), (P, D'))$ *is mapped by* U_r *to itself and is mapped to* $\downarrow(R)$ *by* U_s*. Both functors preserve the linear logic structure.*

The functor U_r is faithful and the functor U_s is full, the idea is that the category **Pop** identifies a fragment of **Rel** that contains all the interpretation of proofs of linear logic that can be mapped compositionally to **ScottL**.

The co-Kleisli category $\mathbf{ScottL}_!$ is well-pointed whereas the category $\mathbf{Rel}_!$ is not and Ehrhard showed that this construction exhibits the Scott model as the extensional collapse of the relational model [34].

4.3 Categorifying Scott semantics

Domain theory provides a mathematical structure to study computability with a notion of approximation of information. The elements of a domain represent partial stages of computation and the order relation represents increasing computational information. When taking a categorical approach to domain theory, preorders are generalized to categories and a morphism $f : x \to y$ is now an explicit name to represent the fact that y contains more computational information than x. This approach was extensively studied by Winskel among others and has proved in many ways fruitful in the theory of concurrent computation [23, 120]. This analogy can be formalized in the setting of enriched categories. A preorder $A = (|A|, \leq_A)$ corresponds to a category enriched over the two element lattice $\mathbb{2} = (\{\emptyset \leq \mathbf{1}\}, \wedge, \mathbf{1})$ where for every $a, a' \in |A|$, the homset $A(a, a')$ is equal to $\{\star\}$ if $a \leq_A a'$ and is empty otherwise. A $\mathbb{2}$-functor between preorders A and B is simply an order-preserving function $f : |A| \to |B|$ and the presheaf category of a preorder $[A^{op}, \mathbb{2}]$ corresponds to the set of down-closed subsets of A ordered with by inclusion. An ideal relation between preorders A and B (a relation up-closed in A and down-closed in B) corresponds to a monotone function $A \to [B^{op}, \mathbb{2}]$. Using the cartesian closed structure, it can be identified with a monotone map $A \times B^{op} \to \mathbb{2}$ which gives the direct correspondence with $\mathbb{2}$-profunctors.

Following this analogy, Cattani and Winskel showed that the bicategory of profunctors with the finite colimit completion pseudo-comonad $\mathcal{F}$ forms a model of linear logic that generalizes intuitions from the Scott model [23]. In their model, filtered colimits generalize directed suprema and Scott-continuous functions correspond to finitary functors.

In Section 4.3.1, we show that the model of profunctors with the finite coproduct pseudo-comonad $\mathcal{C}$ is a model of linear logic which is a generalization of the qualitative Scott model with **Rel**. The connection is formalized by exhibiting a change of base pseudo-functor that commutes with the linear logic structure (Section 4.3.4). We prove in Section 4.3.3 that morphisms in the associated co-Kleisli bicategory correspond to the notion of functors preserving sifted colimits by providing a biequivalence between the two structures. Lastly, we show in Section 4.3.5 that every recursive type equation built from linear logic connectives has a least fixed point solution, and we exhibit a fixed point operator on terms which allows for the study of recursively defined terms.

4.3.1 The bicategory of $\mathcal{C}$-species

The free finite coproduct pseudo-comonad

We take the same compact closed structure for the linear bicategory described in Section 1.4 (see [23] and [41] for more details). The remaining ingredients to obtain a model of linear logic are a pseudo-comonad structure and Seely equivalences satisfying the coherence conditions for a linear exponential pseudo-comonad.

Definition 4.3.1. For a small category $\mathbb{A}$, define $\mathcal{C}\mathbb{A}$ to be the category whose objects are finite sequences $\langle a_1, \ldots, a_n \rangle$ of objects of $\mathbb{A}$ and a morphism between two sequences $\langle a_1, \ldots, a_n \rangle$ and $\langle b_1, \ldots, b_m \rangle$ consists of a pair $(\sigma, (f_i)_{i \in \underline{n}})$ of a function $\sigma : \underline{n} \to \underline{m}$ and a family of morphisms $f_i : a_i \to b_{\sigma(i)}$ in $\mathbb{A}$ for $i \in \underline{n}$. Equivalently, the hom-sets can be described by:

$$\mathcal{C}\mathbb{A}(\langle a_1, \ldots, a_n \rangle, \langle b_1, \ldots, b_m \rangle) = \prod_{i \in \underline{n}} \sum_{j \in \underline{m}} \mathbb{A}(a_i, b_j).$$

We recall below a classical result:

Lemma 4.3.2. *For two finite sequences u and v in $\mathcal{C}\mathbb{A}$, the concatenation (denoted by $u \oplus v$) provides a coproduct structure for $\mathcal{C}\mathbb{A}$ and the empty sequence $\langle\rangle$ is initial. $\mathcal{C}\mathbb{A}$ is the free finite coproduct completion of $\mathbb{A}$, i.e. for any functor $F : \mathbb{A} \to \mathbb{B}$ where $\mathbb{B}$ is a category with finite coproducts, there exists a unique (up to natural isomorphism) functor $\widehat{F} : \mathcal{C}\mathbb{A} \to \mathbb{B}$ that preserves* *finite coproducts and makes the following diagram commute: s the following diagram commute:*

$$\begin{tikzcd}
\mathbb{A} \arrow[rr, "\eta_{\mathbb{A}}"] \arrow[dr, "F"'] & & \mathcal{C}\mathbb{A} \arrow[dl, dotted, "\widehat{F}"] \\
& \mathbb{B} &
\end{tikzcd}$$

where $\eta_{\mathbb{A}}$ is the functor $a \mapsto \langle a \rangle$.

Note. To obtain the free symmetric monoidal completion $\mathcal{S}\mathbb{A}$, it suffices to take the subcategory of $\mathcal{C}\mathbb{A}$ where we restrict σ in Definition 4.3.1 to be a bijection.

The endofunctor $\mathcal{C} : \mathbf{Cat} \to \mathbf{Cat}$ can be equipped with a 2-monad structure. In order to obtain a pseudo-comonad on **Prof**, one needs to start with the dual construction of the free finite product 2-monad $\mathcal{P} : \mathbf{Cat} \to \mathbf{Cat}$ which takes a small category $\mathbb{A}$ to $\mathcal{P}(\mathbb{A}) = (\mathcal{C}(\mathbb{A}^{op}))^{op}$. In [42], Fiore et al. show that the 2-monad $\mathcal{P}$ lifts to a pseudo-monad on **Prof**. Taking its dual, we obtain the pseudo-comonad of finite coproducts on **Prof** which we briefly describe below.

For a profunctor $P : \mathbb{A} \nrightarrow \mathbb{B}$ between small categories $\mathbb{A}$ and $\mathbb{B}$, $\mathcal{C}P : \mathcal{C}\mathbb{A} \nrightarrow \mathcal{C}\mathbb{B}$ is given by:

$$\mathcal{C}P : (u, v) \mapsto \prod_{j \in |v|} \int^{a_j \in \mathbb{A}} P(a_j, v_j) \times \mathcal{C}\mathbb{A}(\langle a_j \rangle, u)$$

The counit and comultiplication pseudo-natural transformations have the following components:

$$\begin{array}{cc}
\mathrm{der}_{\mathbb{A}} : \mathcal{C}\mathbb{A} \nrightarrow \mathbb{A} & \mathrm{dig}_{\mathbb{A}} : \mathcal{C}\mathbb{A} \nrightarrow \mathcal{C}^2\mathbb{A} \\
(u, a) \mapsto \mathcal{C}\mathbb{A}(\langle a \rangle, u) & (u, \langle u_1, \ldots, u_n \rangle) \mapsto \mathcal{C}\mathbb{A}(u_1 \oplus \cdots \oplus u_n, u)
\end{array}$$

A morphism $F : \mathcal{C}\mathbb{A} \nrightarrow \mathbb{B}$ in the co-Kleisli bicategory $\mathbf{Prof}_{\mathcal{C}}$ is called a $\mathcal{C}$-*species* and its lifting or promotion $F^{\mathcal{C}} : \mathcal{C}\mathbb{A} \nrightarrow \mathcal{C}\mathbb{B}$ is given by:

$$F^{\mathcal{C}}(u, v) = \mathcal{C}F \circ \mathrm{dig}_{\mathbb{A}}(u, v) = \prod_{j \in |v|} F(u, v_j)$$

The composite in $\mathbf{Prof}_{\mathcal{C}}$ of two $\mathcal{C}$-species $F : \mathcal{C}\mathbb{A} \nrightarrow \mathbb{B}$ and $G : \mathcal{C}\mathbb{B} \nrightarrow \mathbb{C}$ is then given by the profunctorial composition $G \circ F^{\mathcal{C}} : \mathcal{C}\mathbb{A} \nrightarrow \mathbb{C}$.

Lemma 4.3.3. *There is a Seely adjoint equivalence of categories $\mathcal{C}(\mathbb{A} \& \mathbb{B}) \simeq \mathcal{C}\mathbb{A} \otimes \mathcal{C}\mathbb{B}$.*

Proof. We follow the same idea as for $\mathcal{S}$-species in Lemma 1.4.9. Define $I_{\mathbb{A},\mathbb{B}} : \mathcal{C}\mathbb{A} \otimes \mathcal{C}\mathbb{B} \to \mathcal{C}(\mathbb{A} \,\&\, \mathbb{B})$ as follows:

$$I_{\mathbb{A},\mathbb{B}} : (u, v) \mapsto \mathcal{C}(\mathrm{inj}_1)(u) \oplus \mathcal{C}(\mathrm{inj}_2)(v) \in \mathcal{C}(\mathbb{A} \,\&\, \mathbb{B})$$

where $\mathrm{inj}_1 : \mathbb{A} \to \mathbb{A} \,\&\, \mathbb{B}$ and $\mathrm{inj}_2 : \mathbb{B} \to \mathbb{A} \,\&\, \mathbb{B}$ are the coprojections maps. Consider now the functor $p_1 : \mathbb{A} \,\&\, \mathbb{B} \to \mathcal{C}\mathbb{A}$ defined by $p_1(1, a) := \langle a \rangle$ and $p_1(2, b) := \langle\rangle$. This functor induces a functor $\overline{p_1} : \mathcal{C}(\mathbb{A} \,\&\, \mathbb{B}) \to \mathcal{C}\mathbb{A}$ (using the universal property of the free finite coproduct completion) that is a retract of $\mathcal{C}(\mathrm{inj}_1) : \mathcal{C}\mathbb{A} \to \mathcal{C}(\mathbb{A} \,\&\, \mathbb{B})$. We define similarly a functor $\overline{p_2} : \mathcal{C}(\mathbb{A} \,\&\, \mathbb{B}) \to \mathcal{C}\mathbb{B}$ that is a retract of $\mathcal{C}(\mathrm{inj}_2) : \mathcal{C}\mathbb{B} \to \mathcal{C}(\mathbb{A} \,\&\, \mathbb{B})$. For $w \in \mathcal{C}(\mathbb{A} \,\&\, \mathbb{B})$, we denote by $w.1 \in \mathcal{C}\mathbb{A}$ its image by $\overline{p_1}$ and by $w.2 \in \mathcal{C}\mathbb{B}$ its image by $\overline{p_2}$. $S_{\mathbb{A},\mathbb{B}} : \mathcal{C}(\mathbb{A} \,\&\, \mathbb{B}) \to \mathcal{C}\mathbb{A} \otimes \mathcal{C}\mathbb{B}$ is then defined to be the functor $w \mapsto (w.1, w.2) \in \mathcal{C}\mathbb{A} \otimes \mathcal{C}\mathbb{B}$.

$$\begin{tikzcd} \mathcal{C}(\mathbb{A} \,\&\, \mathbb{B}) \arrow[r, bend left, "S_{\mathbb{A},\mathbb{B}}"] \arrow[r, phantom, "\top"] & \mathcal{C}\mathbb{A} \otimes \mathcal{C}\mathbb{B} \arrow[l, bend left, "I_{\mathbb{A},\mathbb{B}}"] \end{tikzcd}$$

We now exhibit two natural isomorphisms $\eta : \mathrm{Id}_{\mathcal{C}\mathbb{A}\otimes\mathcal{C}\mathbb{B}} \Rightarrow S_{\mathbb{A},\mathbb{B}} \circ I_{\mathbb{A},\mathbb{B}}$ and $\varepsilon : I_{\mathbb{A},\mathbb{B}} \circ S_{\mathbb{A},\mathbb{B}} \Rightarrow \mathrm{Id}_{\mathcal{C}(\mathbb{A}\&\mathbb{B})}$. For $(u, v) \in \mathcal{C}\mathbb{A} \otimes \mathcal{C}\mathbb{B}$, we have that

$$((\mathcal{C}(\mathrm{inj}_1)(u) \oplus \mathcal{C}(\mathrm{inj}_2)(v)).1, (\mathcal{C}(\mathrm{inj}_1)(u) \oplus \mathcal{C}(\mathrm{inj}_2)(v)).2) = (u, v)$$

so η is just the identity. Let $w \in \mathcal{C}(\mathbb{A} \,\&\, \mathbb{B})$, ε_w is the reshuffling isomorphism from $\mathcal{C}(\mathrm{inj}_1)(w.1) \oplus \mathcal{C}(\mathrm{inj}_2)(w.2)$ to w. The adjunction is obtained by seeing that for $(u, v) \in \mathcal{C}\mathbb{A} \otimes \mathcal{C}\mathbb{B}$ and $w \in \mathcal{C}(\mathbb{A} \,\&\, \mathbb{B})$ there is a natural isomorphism:

$$\mathcal{C}(\mathbb{A} \,\&\, \mathbb{B})(\mathcal{C}(\mathrm{inj}_1)(u) \oplus \mathcal{C}(\mathrm{inj}_2)(v), w) \cong \mathcal{C}\mathbb{A}(u, w.1) \times \mathcal{C}\mathbb{B}(v, w.2). \qquad \square$$

Recall from Section 1.4.7 that Fiore et al. exhibited $\mathcal{S}$-species as a model of differential linear logic which can be seen as a categorification of the differential relational model [41]. We show below that similarly to the Scott model with preorders, $\mathbf{Prof}_{\mathcal{C}}$ is not a model of differential linear logic.

Lemma 4.3.4. $\mathbf{Prof}_{\mathcal{C}}$ *is not a model of differential linear logic.*

Proof. If $\mathbf{Prof}_{\mathcal{C}}$ were a model of differential linear logic, there would exist a pseudo-natural transformation $\overline{\mathrm{der}} : \mathrm{Id}_{\mathbf{Prof}} \to \mathcal{C}$ interpreting the codereliction rule. One of the required coherence axioms for the codereliction is $\mathrm{der} \circ \overline{\mathrm{der}} \cong \mathrm{Id}_{\mathbf{Prof}}$. For all $\mathbb{A} \in \mathbf{Cat}$ and $a, a' \in \mathbb{A}$, it implies that

$$\int^{u \in \mathcal{C}\mathbb{A}} \overline{\mathrm{der}}_{\mathbb{A}}(a, u) \times \mathcal{C}\mathbb{A}(\langle a' \rangle, u) \cong \mathbb{A}(a', a)$$

so that $\overline{\mathrm{der}}_{\mathbb{A}}(a, \langle a' \rangle) \cong A(a', a)$. Another required coherence diagrams for the codereliction map is that for any object $\mathbb{A}$, $\mathrm{w}_{\mathbb{A}} \circ \overline{\mathrm{der}}_{\mathbb{A}} = \mathbb{0}_{\mathbb{A}}$ where $\mathrm{w}_{\mathbb{A}} : \mathcal{C}\mathbb{A} \nrightarrow \mathbf{1}$ is the weakening map given by $u \mapsto \mathcal{C}\mathbb{A}(\langle\rangle, u)$ and $\mathbb{0}_{\mathbb{A}} : \mathbb{A} \nrightarrow \mathbf{1}$ is the empty profunctor. For $a \in \mathbb{A}$, we have:

$$w_{\mathbb{A}} \circ \overline{\mathrm{der}}_{\mathbb{A}}(a) = \int^{u \in \mathcal{C}\mathbb{A}} \mathcal{C}\mathbb{A}(\langle\rangle, u) \times \overline{\mathrm{der}}_{\mathbb{A}}(a, u) \cong \overline{\mathrm{der}}_{\mathbb{A}}(a, \langle\rangle)$$

Since there is a map $\langle\rangle \to \langle a \rangle$ in $\mathcal{C}\mathbb{A}$, it induces a function from $\overline{\mathrm{der}}_{\mathbb{A}}(a, \langle a \rangle)$ to $\overline{\mathrm{der}}_{\mathbb{A}}(a, \langle\rangle)$. The set $\overline{\mathrm{der}}_{\mathbb{A}}(a, \langle a \rangle) \cong \mathbb{A}(a, a)$ is not empty as it contains id_a so the set $\overline{\mathrm{der}}_{\mathbb{A}}(a, \langle\rangle)$ cannot be empty which contradicts our hypothesis. $\square$

Recall that the orthogonality construction presented in Section 4.2 exhibits the well-pointed Scott model as the extensional collapse of the non-well-pointed relational model. In the categorified setting, the situation is however more subtle. In the case of $\mathcal{S}$-species, since the sum functor $s_{\mathbb{A}} : \mathcal{S}\mathbb{A} \to \widehat{\mathbb{A}}$ is not fully faithful, the functor giving the correspondence between $\mathcal{S}$-species and analytic functors:

$$\mathbf{Lan}_{s_{\mathbb{A}}} : \mathbf{Prof}_{\mathcal{S}}(\mathbb{A}, \mathbb{B}) \to [\widehat{\mathbb{A}}, \widehat{\mathbb{B}}]$$

is not fully faithful. If we extend the functor $s_{\mathbb{A}}$ to the category $\mathcal{C}\mathbb{A}$, we obtain a fully faithful functor which entails that $\mathbf{Lan}_{s_{\mathbb{A}}} : \mathbf{Prof}_{\mathcal{C}}(\mathbb{A}, \mathbb{B}) \to [\widehat{\mathbb{A}}, \widehat{\mathbb{B}}]$ is now fully faithful as a corollary of Proposition 1.4.2.

4.3.2 The cartesian closed structure

The cartesian structure in $\mathbf{Prof}_{\mathcal{C}}$ is inherited from the linear bicategory **Prof**. For small categories $\mathbb{A}$ and $\mathbb{B}$, the exponential object $\mathbb{A} \Rightarrow \mathbb{B}$ is defined as $(\mathcal{C}\mathbb{A})^{op} \times \mathbb{B}$. The evaluation map $\mathrm{Ev}_{\mathbb{A},\mathbb{B}} : \mathcal{C}\left((\mathbb{A} \Rightarrow \mathbb{B}) \,\&\, \mathbb{A}\right) \nrightarrow \mathbb{B}$ is obtained from the same construction as in Section 1.4.6 and is explicitly calculated as follows:

$$\mathcal{C}\left((\mathbb{A} \Rightarrow \mathbb{B}) \,\&\, \mathbb{A}\right) \times \mathbb{B}^{op} \ni (w, b) \mapsto \mathcal{C}(\mathbb{A} \Rightarrow \mathbb{B})(\langle (w.2, b) \rangle, w.1).$$

Likewise, for $G : \mathcal{C}(\mathbb{X} \,\&\, \mathbb{A}) \nrightarrow \mathbb{B}$, its currying $\Lambda(G) : \mathcal{C}\mathbb{X} \nrightarrow (\mathcal{C}\mathbb{A} \multimap \mathbb{B})$ is defined by

$$\Lambda(G) : (z, (u, b)) \mapsto F(\mathcal{C}(i_1)(z) \oplus \mathcal{C}(i_2)(u), b).$$

Proposition 4.3.5. $\mathbf{Prof}_{\mathcal{C}}$ *is cartesian closed.*

Proof. Consider now two profunctors $F : \mathcal{C}\mathbb{X} \nrightarrow (\mathbb{A} \Rightarrow \mathbb{B})$ and $G : \mathcal{C}(\mathbb{X} \,\&\, \mathbb{A}) \nrightarrow \mathbb{B}$, we exhibit the following natural ismorphisms:

$$\eta_F : F \overset{\sim}{\Longrightarrow} \lambda(\mathrm{Ev}_{\mathbb{A},\mathbb{B}} \circ (F \,\&\, \mathbb{A})) \qquad \beta_G : \mathrm{Ev}_{\mathbb{A},\mathbb{B}} \circ (\lambda(G) \,\&\, \mathbb{A}) \overset{\sim}{\Longrightarrow} G$$

For $(z, (u, b)) \in \mathcal{C}\mathbb{X} \times (\mathbb{A} \Rightarrow \mathbb{B})^{op}$, we have:

$$\begin{aligned}\lambda(\mathrm{Ev}_{\mathbb{A},\mathbb{B}} \circ (F \,\&\, \mathbb{A}))(z, (u, b)) &\cong (\mathrm{Ev}_{\mathbb{A},\mathbb{B}} \circ (F \,\&\, \mathbb{A}))(\mathcal{C}i_1 z \oplus \mathcal{C}i_2 u, b) \\ &\cong F((\mathcal{C}i_1 z \oplus \mathcal{C}i_2 u).1, (\mathcal{C}i_1 z \oplus \mathcal{C}i_2 u).2, b)) \cong F(z, (u, b))\end{aligned}$$

and for $(w, b) \in \mathcal{C}(\mathbb{X} \,\&\, \mathbb{A}) \times \mathbb{B}^{op}$, we obtain:

$$\begin{aligned}&\mathrm{Ev}_{\mathbb{A},\mathbb{B}}(\lambda(G) \,\&\, \mathbb{A})(w, b) = \lambda(G)(w.1, (w.2, b)) \\ &\cong G((\mathcal{C}(i_1)(w.1) \oplus \mathcal{C}(i_2)(w.2)), b) \cong G(w, b)\end{aligned} \qquad \square$$

4.3.3 Strongly finitary functors

In the case of analytic functors for $\mathcal{S}$-species (restricted to groupoids), one can characterize them as functors preserving filtered colimits and wide quasi-pullbacks [39]. Cattani and Winskel showed that $\mathcal{F}$-species correspond to the notion of finitary functors, i.e. functors preserving filtered colimits [23]. Filtered colimits are the classical way of generalizing directed suprema in Scott's topology, and they are characterized as colimits which commute with finite limits in **Set**. In this section, we focus on a larger class of colimits, called *sifted colimits* which are colimits that commute with finite products in **Set**. A large part of the theory of locally finitely presentable categories and finitely presentable objects has analogues for sifted colimits.

Definition 4.3.6. An object a in a category $\mathbb{A}$ is said to be *strongly finitely presentable* if $\mathbb{A}(a, -) : \mathbb{A} \to \mathbf{Set}$ preserves sifted colimits.

The full subcategory of these objects in $\mathbb{A}$ is denoted by $\mathbb{A}_{\mathrm{sfp}}$. For a preorder, finitely and strongly presentable objects coincide with the compact elements and in the category **Set**, the two notions coincide with finite sets [5]. A category $\mathbb{A}$ is *strongly locally finitely presentable* if it is cocomplete, $\mathbb{A}_{\mathrm{sfp}}$ is a small category and every object of $\mathbb{A}$ is a sifted colimit of a diagram in $\mathbb{A}_{\mathrm{sfp}}$.

Lemma 4.3.7. *For a small category $\mathbb{A}$, the presheaf category $\widehat{\mathbb{A}}$ is strongly finitely presentable and every presheaf is a sifted colimit of finite coproducts of representables.*

Proof. Let $\mathbb{A}$ be a small category, then $\widehat{\mathbb{A}}$ is strongly finitely presentable and the strongly finitely presentable objects are the regular projective presheaves [5]. In presheaf categories, the full subcategory of coproducts of representables is a regular projective cover [96]. Hence every presheaf is sifted colimit of coproducts of representables. Since every coproduct is a filtered colimit of finite coproducts, we obtain the desired result. $\square$

Functors preserving sifted colimits are called *strongly finitary functors.* On **Set**, finitary and strongly finitary functors coincide [5].

Definition 4.3.8. The 2-category **Sift** has small categories as objects and a morphism between two categories $\mathbb{A}$ and $\mathbb{B}$ is a strongly finitary functor $\widehat{\mathbb{A}} \to \widehat{\mathbb{B}}$. The 2-cells between two such functors are natural transformations.

We proceed to show that there is a biequivalence between the bicategory $\mathbf{Prof}_{\mathcal{C}}$ and the 2-category **Sift**.

Lemma 4.3.9. *For a $\mathcal{C}$-species $F : \mathcal{C}\mathbb{A} \nrightarrow \mathbb{B}$, $\mathbf{Lan}_{s_{\mathbb{A}}}(F) : \widehat{\mathbb{A}} \to \widehat{\mathbb{B}}$ preserves sifted colimits.*

Proof. Let $\mathcal{D} : I \to \widehat{\mathbb{A}}$ be a sifted diagram, we have:

$$\begin{aligned}
\mathbf{Lan}_{s_{\mathbb{A}}} F(\varinjlim_{i \in I} \mathcal{D}(i))(b) &= \int^{u = \langle a_1, \ldots, a_n \rangle} F(u,b) \times \widehat{\mathbb{A}}(s_{\mathbb{A}}(u), \varinjlim_{i \in I} \mathcal{D}(i)) \\
&\cong \int^{u} F(u,b) \times \prod_{j=1}^{n} \widehat{\mathbb{A}}(\mathsf{y}(a_j), \varinjlim_{i \in I} \mathcal{D}(i)) \\
&\cong \int^{u} F(u,b) \times \prod_{i=j}^{n} \varinjlim_{i \in I} \mathcal{D}(i)(a_j) \\
&\cong \int^{u} F(u,b) \times \varinjlim_{i \in I} \prod_{j=1}^{n} \mathcal{D}(i)(a_j) \\
&\cong \int^{u} F(u,b) \times \varinjlim_{i \in I} \left(\widehat{\mathbb{A}}(s_{\mathbb{A}}(u), \mathcal{D}(i)) \right) \\
&\cong \int^{u} \varinjlim_{i \in I} \left(F(u,b) \times \widehat{\mathbb{A}}(s_{\mathbb{A}}(u), \mathcal{D}(i)) \right) \\
&\cong \varinjlim_{i \in I} \left(\int^{u} F(u,b) \times \widehat{\mathbb{A}}(s_{\mathbb{A}}(u), \mathcal{D}(i)) \right)
\end{aligned}$$

Since sifted colimits commute with finite products, it allows us to obtain the third isomorphism. We then make use of the facts that $(F(u,b) \times -)$ is a left adjoint, and hence colimit-preserving, and that the coend is a colimit and hence commutes with colimits. $\square$

Lemma 4.3.10. *For small categories $\mathbb{A}$ and $\mathbb{B}$, there is an adjoint equivalence between the categories:*

$$\begin{tikzcd}
\mathbf{Prof}_{\mathcal{C}}(\mathbb{A},\mathbb{B}) \arrow[r, bend left, "\mathbf{Lan}_{s_{\mathbb{A}}}(-)"] \arrow[r, phantom, "\perp"] & \mathbf{Sift}(\mathbb{A},\mathbb{B}) \arrow[l, bend left, "- \circ s_{\mathbb{A}}"]
\end{tikzcd}$$

Proof. Since $s_{\mathbb{A}}$ is fully faithful, for any $\mathcal{C}$-species F in $\mathbf{Prof}_{\mathcal{C}}(\mathbb{A},\mathbb{B})$ there is a natural isomorphism $\alpha_F : F \Rightarrow (\mathbf{Lan}_{s_{\mathbb{A}}}(F)) \circ s_{\mathbb{A}}$. Hence, for a natural transformation $\beta : F_1 \Rightarrow F_2$ in $\mathbf{Prof}_{\mathcal{C}}(\mathbb{A},\mathbb{B})$, its image by $\mathbf{Lan}_{s_{\mathbb{A}}}(-)$ is the unique natural transformation $\gamma : \mathbf{Lan}_{s_{\mathbb{A}}}(F_1) \Rightarrow \mathbf{Lan}_{s_{\mathbb{A}}}(F_2)$ such that $\gamma s_{\mathbb{A}} \alpha_{F_1} = \beta \alpha_{F_2}$ which provides us with a natural isomorphism $\eta : \mathrm{Id}_{\mathbf{Prof}_{\mathcal{C}}(\mathbb{A},\mathbb{B})} \Rightarrow (\mathbf{Lan}_{s_{\mathbb{A}}}(-)) \circ s_{\mathbb{A}}$ by Proposition 1.4.2.

Let $P : \widehat{\mathbb{A}} \to \widehat{\mathbb{B}}$ be a functor that preserves sifted colimits. We want to exhibit a natural isomorphism

$$\mathbf{Lan}_{s_{\mathbb{A}}}(P \circ s_{\mathbb{A}})(X) \cong P(X)$$

By Lemma 4.3.7, X is sifted colimit of finite coproducts of representables, i.e. there exists a sifted diagram $D : I \to \mathcal{C}\mathbb{A}$ such that $X \cong \varinjlim_{i \in I} s_{\mathbb{A}}(D(i))$:

$$\begin{aligned}
\mathbf{Lan}_{s_{\mathbb{A}}}(P \circ s_{\mathbb{A}})(X) &= \int^{u = \langle a_1, \ldots, a_n \rangle} P(s_{\mathbb{A}}(u)) \times \widehat{\mathbb{A}}(s_{\mathbb{A}}(u), X) \\
&\cong \int^{u} P(s_{\mathbb{A}}(u)) \times \prod_{j=1}^{n} \widehat{\mathbb{A}}(\mathrm{y}(a_j), \varinjlim_{i \in I} s_{\mathbb{A}} D(i)) \\
&\cong \int^{u} P(s_{\mathbb{A}}(u)) \times \varinjlim_{i \in I} \prod_{j=1}^{n} \widehat{\mathbb{A}}(\mathrm{y}(a_j), s_{\mathbb{A}} D(i)) \\
&\cong \int^{u} P(s_{\mathbb{A}}(u)) \times \varinjlim_{i \in I} \widehat{\mathbb{A}}(s_{\mathbb{A}}(u), s_{\mathbb{A}} D(i)) \\
&\cong \varinjlim_{i \in I} \int^{u} P(s_{\mathbb{A}}(u)) \times \mathcal{C}\mathbb{A}(s_{\mathbb{A}}(u), s_{\mathbb{A}} D(i)) \\
&\cong \varinjlim_{i \in I} P(s_{\mathbb{A}}(D(i))) \cong P(X)
\end{aligned}$$

which entails the existence of a natural isomorphism $\varepsilon : \mathbf{Lan}_{s_{\mathbb{A}}}(- \circ s_{\mathbb{A}}) \Rightarrow \mathrm{Id}_{\mathbf{Sift}(\mathbb{A},\mathbb{B})}$ as desired. The adjunction

$$[\mathcal{C}\mathbb{A}, \widehat{\mathbb{B}}](F, P \circ s_{\mathbb{A}}) \cong [\widehat{\mathbb{A}}, \widehat{\mathbb{B}}](\mathbf{Lan}_{s_{\mathbb{A}}} F, P).$$

is a direct consequence of the universal property of left Kan extensions. ☐

Proposition 4.3.11. *The bicategory* $\mathbf{Prof}_{\mathcal{C}}$ *is biequivalent to the* 2*-category* **Sift***.*

Proof. We prove that the pseudofunctor $H : \mathbf{Prof}_{\mathcal{C}} \to \mathbf{Sift}$ defined below is a biequivalence. For $\mathbb{A}$ and $\mathbb{B}$ small categories, we define $H(\mathbb{A}) := \mathbb{A}$ and

$$\begin{aligned} H_{\mathbb{A},\mathbb{B}} : \mathbf{Prof}_{\mathcal{C}}(\mathbb{A},\mathbb{B}) &\to \mathbf{Sift}(\mathbb{A},\mathbb{B}) \\ F : \mathcal{C}\mathbb{A} \nrightarrow \mathbb{B} &\mapsto \mathbf{Lan}_{s_{\mathbb{A}}}(F) : \widehat{\mathbb{A}} \to \widehat{\mathbb{B}} \end{aligned}$$

Since $\mathbf{Prof}_{\mathcal{C}}$ and **Sift** have the same objects, it follows immediately that $H : \mathbf{Prof}_{\mathcal{C}} \to \mathbf{Sift}$ is essentially surjective. Lemma 4.3.10 entails that $H_{\mathbb{A},\mathbb{B}}$ is an adjoint equivalence of categories. ☐

4.3.4 From Prof to ScottL

In this section, we formalize the connection between the categorical approach and the preorder model as a change of base for enriched categories. A category enriched over $\mathbb{2} = (\{\emptyset \leq \mathbf{1}\}, \wedge, \mathbf{1})$ is a preorder and a $\mathbb{2}$-profunctor between two preorders $A = (|A|, \leq_A)$ and $B = (|B|, \leq_B)$ corresponds to a relation in $\mathbf{ScottL}(A, B)$. The functor $M : \mathbf{Set} \to \mathbb{2}$ defined by

$$X \mapsto \begin{cases} \emptyset & \text{if } X = \emptyset \\ \mathbf{1} & \text{otherwise} \end{cases}$$

is monoidal and therefore induces a lax functor Ψ from $\mathbf{Prof}_{\mathbf{Set}}$ (just denoted by **Prof**) to $\mathbf{Prof}_{\mathbb{2}} = \mathbf{ScottL}$ [27]. In this section, we give an explicit description of this change of base lax functor $\Psi : \mathbf{Prof} \to \mathbf{ScottL}$ and show that it is in fact a functor that preserves all the structure of linear logic. The viewpoint of enriched categories enables us to work in a unified setting where both models coexist and the change of base becomes a functor that connects the preorder world and the categorified world in a way that preserves the structure of linear logic.

On objects, Ψ sends a small category $\mathbb{A}$ to the following preorder:

$$(\mathrm{Ob}(\mathbb{A}), \leq_A) \quad \text{where} \quad a \leq_A a' \quad :\Leftrightarrow \quad \mathbb{A}(a, a') \neq \emptyset$$

For a profunctor $F : \mathbb{A} \nrightarrow \mathbb{B}$, $\Psi_{\mathbb{A},\mathbb{B}}(F)$ is given by

$$\Psi_{\mathbb{A},\mathbb{B}}(F) := \{(a, b) \mid F(a, b) \neq \emptyset\}.$$

Lemma 4.3.12. *For every* $\mathbb{A}, \mathbb{B}$*,* $\Psi_{\mathbb{A},\mathbb{B}} : \mathbf{Prof}(\mathbb{A},\mathbb{B}) \to \mathbf{ScottL}(\Psi(\mathbb{A}), \Psi(\mathbb{B}))$ *is functorial.*

Proof. We first need to check that $\Psi_{\mathbb{A},\mathbb{B}}(F)$ is indeed in $\mathbf{ScottL}(\Psi(\mathbb{A}),\Psi(\mathbb{B}))$, i.e. that for all $(a,b) \in \Psi_{\mathbb{A},\mathbb{B}}(F)$, $(a',b') \leq_{\mathbb{A}^{op}\times\mathbb{B}} (a,b)$ implies $(a',b') \in \Psi_{\mathbb{A},\mathbb{B}}(F)$. If $(a,b) \in \Psi_{\mathbb{A},\mathbb{B}}(F)$, then $F(a,b) \neq \emptyset$ so there exists an element $s \in F(a,b)$. The inequality $(a',b') \leq_{\mathbb{A}^{op}\times\mathbb{B}} (a,b)$ implies that there exist morphisms $f : a \to a'$ in $\mathbb{A}$ and $g : b' \to b$ in $\mathbb{B}$. Hence, $F(f,g)(s) \in F(a',b')$ which is not empty as desired. When we consider **ScottL** as a bicategory, morphisms in $\mathbf{ScottL}(\Psi(\mathbb{A}),\Psi(\mathbb{B}))$ are just inclusions of relations so we only need to show that if there exists a natural transformation $\alpha : F \Rightarrow G$ in $\mathbf{Prof}(\mathbb{A},\mathbb{B})$, then $\Psi_{\mathbb{A},\mathbb{B}}(F) \subseteq \Psi_{\mathbb{A},\mathbb{B}}(G)$. For $(a,b) \in \Psi_{\mathbb{A},\mathbb{B}}(F)$, if there exists an element $s \in F(a,b)$ then $\alpha_{(a,b)}(s) \in G(a,b)$ which implies that $(a,b) \in \Psi_{\mathbb{A},\mathbb{B}}(G)$ as desired. ☐

Proposition 4.3.13. *Ψ is a strict functor that preserves the linear logic structure.*

Proof.

- For profunctors $F : \mathbb{A} \nrightarrow \mathbb{B}$ and $G : \mathbb{B} \nrightarrow \mathbb{C}$, the following equalities hold:

$$\begin{aligned}&\Psi_{\mathbb{A},\mathbb{C}}(G \circ_{\mathbf{Prof}} F) = \{(a,c) \mid \int^{b\in\mathbb{B}} F(a,b)\times G(b,c) \neq \emptyset\}\\ &= \{(a,c) \mid \exists b \in \mathrm{Ob}(\mathbb{B}), F(a,b) \neq \emptyset \text{ and } G(b,c) \neq \emptyset\}\\ &= \{(a,c) \mid \exists b \in \mathrm{Ob}(\mathbb{B}), (a,b) \in \Psi_{\mathbb{A},\mathbb{B}}(F) \text{ and } (b,c) \in \Psi_{\mathbb{B},\mathbb{C}}(G)\}\\ &= \Psi_{\mathbb{B},\mathbb{C}}(G) \circ_{\mathbf{ScottL}} \Psi_{\mathbb{A},\mathbb{B}}(F)\end{aligned}$$

- We only show that Ψ commutes with the pseudo-comonad structure, the other cases being similar. For a small category $\mathbb{A}$, $!\Psi(\mathbb{A})$ is the preorder whose underlying set is equal to the object set of $\mathcal{C}\mathbb{A}$ so $!\Psi$ and $\Psi\mathcal{C}$ coincide on objects. For a profunctor $F : \mathbb{A} \nrightarrow \mathbb{B}$, we have:

$$\begin{aligned}!\Psi_{\mathbb{A},\mathbb{B}}(F) &= \{(\langle a_1,\ldots,a_n\rangle,\langle b_1,\ldots,b_m\rangle) \mid \forall j \in \underline{m}, \exists i \in \underline{n}, (a_i,b_j) \in \Psi(F)\}\\ &= \{(\langle a_1,\ldots,a_n\rangle,\langle b_1,\ldots,b_m\rangle) \mid \forall j \in \underline{m}, \exists i \in \underline{n}, F(a_i,b_j) \neq \emptyset\}\\ &= \{(\langle a_1,\ldots,a_n\rangle,\langle b_1,\ldots,b_m\rangle) \mid \prod_{j\in\underline{m}}\sum_{i\in\underline{n}} F(a_i,b_j) \neq \emptyset\}\\ &= \{(\langle a_1,\ldots,a_n\rangle,\langle b_1,\ldots,b_m\rangle) \mid \mathcal{C}F(\langle a_1,\ldots,a_n\rangle,\langle b_1,\ldots,b_m\rangle) \neq \emptyset\}\\ &= \Psi(\mathcal{C}F)\end{aligned}$$

  The following equalities also hold for the dereliction and the digging pseudo-natural transformations:

$$
\begin{aligned}
\Psi(\varepsilon_{\mathbb{A}}) &= \{(u,a) \mid \varepsilon_{\mathbb{A}}(u,a) \neq \emptyset\} = \{(u,a) \mid \sum_{i \in |u|} \mathbb{A}(a, u_i) \neq \emptyset\} \\
&= \{(u,a) \mid \forall i \in |u|, a \leq_{\Psi(\mathbb{A})} u_i\} = \varepsilon_{\Psi(\mathbb{A})} \\
\Psi(\delta_{\mathbb{A}}) &= \{(u, \langle u_1, \dots, u_n \rangle) \mid \mathcal{C}\mathbb{A}(u_1 \oplus \cdots \oplus u_n, u) \neq \emptyset\} \\
&= \{(u, \langle u_1, \dots, u_n \rangle) \mid u_1 \oplus \cdots \oplus u_n \leq_{\Psi(\mathcal{C}\mathbb{A})} u\} = \delta_{\Psi \mathcal{C}\mathbb{A}}
\end{aligned}
$$

□

4.3.5 Fixed points of types

Recursive domain equations play a central role in denotational semantics. A classical example is Scott's D_∞ construction providing an extensional model of the untyped λ-calculus. In $\mathbf{Prof}_{\mathcal{C}}$, we show that full subcategory inclusion is a partial order relation on objects such that all linear logic constructions define Scott-continuous maps on this partially ordered class. It entails that we can give solutions to any recursive type equation constituted of linear logic connectives and we exhibit in this section an example of a 2-dimensional model of pure λ-calculus in $\mathbf{Prof}_{\mathcal{C}}$.

Definition 4.3.14. For small categories $\mathbb{A}$ and $\mathbb{B}$, we write $\mathbb{A} \sqsubseteq \mathbb{B}$ if $\mathbb{A}$ is a full subcategory of $\mathbb{B}$, i.e. $\mathrm{Ob}(\mathbb{A}) \subseteq \mathrm{Ob}(\mathbb{B})$ and for all a and a' in $\mathrm{Ob}(\mathbb{A})$, $\mathbb{A}(a, a') = \mathbb{B}(a, a')$.

One can easily check that $\sqsubseteq$ defines a partial order relation on the class of small categories. We denote by $\mathbf{Cat}_{\sqsubseteq}$ the obtained partially ordered class and show the following lemma:

Lemma 4.3.15. *$\mathbf{Cat}_{\sqsubseteq}$ is closed under directed colimits.*

Proof. Let $D : I \to \mathbf{Cat}_{\sqsubseteq}$ be a directed diagram. We denote by $\bigvee_{i \in I} D_i$ the category whose set of objects is $\bigcup_{i \in I} \mathrm{Ob}(D_i)$ so that for any $a, b \in \mathrm{Ob}(\bigvee_{i \in I} D_i)$, there exist $i, j \in I$ such that $a \in \mathrm{Ob}(D_i)$ and $b \in \mathrm{Ob}(D_j)$. Since I is directed, there exists $k \in I$ such that $a, b \in \mathrm{Ob}(D_k)$ so we define $\bigvee_{i \in I} D_i(a, b)$ to be $D_k(a, b)$. □

Lemma 4.3.16. *All the linear logic constructions are Scott-continuous with respect to the order $\sqsubseteq$.*

Proof. The proof is routine, we only exhibit the dual and exponential cases:

- **Dual:** It is noteworthy to observe that the dual is monotonous with respect to this order. For $\mathbb{A} \sqsubseteq \mathbb{B}$, we have that $\mathrm{Ob}(\mathbb{A}^{op}) = \mathrm{Ob}(\mathbb{A}) \subseteq \mathrm{Ob}(\mathbb{B}) = \mathrm{Ob}(\mathbb{B}^{op})$ and for any $a, a' \in \mathbb{A}^{op}$, $\mathbb{A}^{op}(a, a') = \mathbb{A}(a', a) = \mathbb{B}(a', a) = \mathbb{B}^{op}(a, a')$ which entails that $\mathbb{A}^{op} \sqsubseteq \mathbb{B}^{op}$. Let $D : I \to \mathbf{Cat}_{\sqsubseteq}$ be a directed diagram, we want to show that $\left(\bigvee_{i\in I} D_i\right)^{op} = \bigvee_{i\in I} D_i^{op}$. It is immediate to show that these two categories have the same objects and for $a, a' \in \bigvee_{i\in I} D_i^{op}$, there exists $k \in I$ such that $a, a' \in \mathrm{Ob}(D_k)$ so that:

  $$\bigvee_{i\in I} D_i^{op}(a, a') = D_k^{op}(a, a') = D_k(a', a) = (\bigvee_{i\in I} D_i)(a', a) = \\ (\bigvee_{i\in I} D_i)^{op}(a, a').$$

- **Exponential:** For $\mathbb{A} \sqsubseteq \mathbb{B}$, $\mathrm{Ob}(\mathcal{C}\mathbb{A}) = \{\langle a_1, \dots, a_n\rangle \mid a_i \in \mathrm{Ob}(\mathbb{A})\} \subseteq \{\langle b_1, \dots, b_n\rangle \mid b_i \in \mathrm{Ob}(\mathbb{B})\} = \mathrm{Ob}(\mathcal{C}\mathbb{B})$ and for u, v in $\mathrm{Ob}(\mathcal{C}\mathbb{A})$:

  $$\mathcal{C}\mathbb{A}(u, v) = \prod_{i\in|u|}\sum_{j\in|v|} \mathbb{A}(u_i, v_j) = \prod_{i\in|u|}\sum_{j\in|v|} \mathbb{B}(u_i, v_j) = \mathcal{C}\mathbb{B}(u, v)$$

  which entails that $\mathcal{C}\mathbb{A} \sqsubseteq \mathcal{C}\mathbb{B}$ as desired. Let $D : I \to \mathbf{Cat}_{\sqsubseteq}$ be a directed diagram, we want to show that $\mathcal{C}(\bigvee_{i\in I} D_i) = \bigvee_{i\in I} \mathcal{C}D_i$. For the object sets, we haveL

  $$\begin{aligned} \mathrm{Ob}(\mathcal{C}(\bigvee_{i\in I} D(i))) &= \bigcup_{n\in\mathbb{N}} \mathrm{Ob}(\bigvee_{i\in I} D(i))^n = \bigcup_{n\in\mathbb{N}} (\bigcup_{i\in I} \mathrm{Ob}(D_i))^n \\ &= \bigcup_{n\in\mathbb{N}} \bigcup_{i\in I} (\mathrm{Ob}(D(i)))^n = \bigcup_{i\in I} \bigcup_{n\in\mathbb{N}} (\mathrm{Ob}(D_i))^n \\ &= \mathrm{Ob}\left(\bigvee_{i\in I} \mathcal{C}D_i\right) \end{aligned}$$

  The third equality follows from the fact that directed unions commute with finite products. Consider now two elements $u := \langle x_1, \dots, x_n\rangle$ and $v := \langle y_1, \dots, y_m\rangle$ in $\bigvee_{i\in I} \mathcal{C}(D_i)$. Since I is directed, there exists $k \in I$ such that $u, v \in \mathrm{Ob}(\mathcal{C}(D_k))$, we therefore obtain:

  $$(\bigvee_{i\in I} \mathcal{C}(D_i)(u, v) = \mathcal{C}(D_k)(u, v) = \prod_{l\in\underline{n}}\sum_{r\in\underline{m}} D_k(x_l, y_r) = \\ \prod_{l\in\underline{n}}\sum_{r\in\underline{m}}\bigvee_{i\in I} D_k(x_l, y_r) = \mathcal{C}(\bigvee_{i\in I} D(i))(u, v)$$

The last equality follows from the fact that $D_k \sqsubseteq \bigvee_{i\in I} D_i$. $\square$

Example. By the previous lemma, any recursive type equation on $\mathbf{Cat}_{\sqsubseteq}$ built from linear logic connectives has a least fixed point. Let $\mathbf{N}$ be the least fixed point solution of $\mathbf{N} = \mathbf{1} \oplus \mathbf{N}$, it can be explicitly described as the category $\mathbf{N} = \bigoplus_{i\in\mathbb{N}} \mathbf{1}$. Consider now $\mathbf{D}$ to be the least fixed point solution of $\mathbf{D} = (\mathcal{C}(\mathbf{N} \multimap \mathbf{D}))^{op}$. Using the Seely equivalence in Lemma 4.3.3, we can first note that $\mathbf{D}$ verifies the following equivalence:

$$\begin{aligned}\mathbf{D} &= (\mathcal{C}(\mathbf{N} \multimap \mathbf{D}))^{op} \simeq (\mathcal{C}((\mathbf{1} \oplus \mathbf{N}) \multimap \mathbf{D}))^{op} \simeq (\mathcal{C}((\mathbf{1} \multimap \mathbf{D}) \mathbin{\&} (\mathbf{N} \multimap \mathbf{D})))^{op} \\ &\simeq ((\mathcal{C}(\mathbf{D})) \otimes \mathcal{C}(\mathbf{N} \multimap \mathbf{D}))^{op} = (\mathcal{C}\mathbf{D})^{op} \parr \mathbf{D} = (\mathbf{D} \Rightarrow \mathbf{D})\end{aligned}$$

The category $\mathbf{D}$ provides an extensional reflexive object for the pure λ-calculus in the cartesian closed bicategory $\mathbf{Prof}_{\mathcal{C}}$. We make explicit its structure below by first giving the application and lambda profunctors:

$$\mathrm{Ap} : \mathcal{C}(\mathbf{D} \Rightarrow \mathbf{D}) \nrightarrow \mathbf{D} \qquad \lambda : \mathcal{C}\mathbf{D} \nrightarrow (\mathbf{D} \Rightarrow \mathbf{D})$$

as follows: for $W \in \mathcal{C}(\mathbf{D} \Rightarrow \mathbf{D})$ and $d \in \mathbf{D}^{op}$, let $k \in \mathbb{N}$ be the smallest index such that $W \in \mathcal{C}(\mathbf{D}_k \Rightarrow \mathbf{D}_k)$ and $d \in \mathbf{D}_k^{op}$. Since $\mathbf{D}_k^{op} = (\mathcal{C}((\mathbf{1} \oplus \mathbf{N}) \multimap \mathbf{D}_{k-1})) \cong (\mathcal{C}(\mathbf{D}_{k-1}) \mathbin{\&} (\mathbf{N} \multimap \mathbf{D}_{k-1})))$, we use the Seely equivalence and obtain $d.1 \in \mathcal{C}(\mathbf{D}_{k-1}) \sqsubseteq \mathcal{C}(\mathbf{D}_k)$ and $d.2 \in \mathcal{C}(\mathbf{N} \multimap \mathbf{D}_{k-1}) = \mathbf{D}_k^{op}$. We now define Ap as the profunctor taking (W, d) to $\mathcal{C}(\mathbf{D}_k \Rightarrow \mathbf{D}_k)(\langle (d.1, d.2)\rangle, W)$.

To define $\lambda(u, (v, d))$ for $u \in \mathcal{C}\mathbf{D}$ and $(v, d) \in (\mathbf{D} \Rightarrow \mathbf{D})^{op}$, we first let l to be the smallest index such that $u \in \mathcal{C}(\mathbf{D})_l$, $v \in \mathcal{C}(\mathbf{D}_l)$ and $d \in \mathbf{D}_l^{op} \sqsubseteq \mathbf{D}_{l+1}^{op} = \mathcal{C}((\mathbf{1} \oplus \mathbf{N}) \multimap \mathbf{D}_l) \cong \mathcal{C}(\mathbf{D}_l \mathbin{\&} (\mathbf{N} \multimap \mathbf{D}_l))$. Considering the diagram below,

$$\begin{tikzcd}
\mathcal{C}(\mathbf{D})_l \arrow[r, "\mathcal{C}(i_1)"'] & \mathcal{C}(\mathbf{D}_l \mathbin{\&} (\mathbf{N} \multimap \mathbf{D}_l)) \arrow[d, equal] & \mathcal{C}(\mathbf{N} \multimap \mathbf{D}_l) \arrow[l, "\mathcal{C}(i_2)"] \\
 & \mathbf{D}_{l+1}^{op} &
\end{tikzcd}$$

we obtain that $\mathcal{C}(i_1)(u) \oplus \mathcal{C}(i_2)(d)$ is an element of $\mathbf{D}_{l+1}^{op}$, so we define $\lambda(u, (v, d))$ to be $\mathcal{C}(\mathbf{D}_{l+1})(\mathcal{C}(i_1)(v) \oplus \mathcal{C}(i_2)(d), u)$. We then obtain:

$$\begin{aligned}\lambda \circ \mathrm{Ap}(W, (v, d)) &= \int^{u\in\mathcal{C}\mathbf{D}} \lambda(u, (v, d)) \times \mathrm{Ap}^{\mathcal{C}}(W, u) \\ &= \int^{u} \mathcal{C}\mathbf{D}(\mathcal{C}(i_1)(v) \oplus \mathcal{C}(i_2)(d), u) \times \mathrm{Ap}^{\mathcal{C}}(W, u) \\ &\cong \mathrm{Ap}(W, \mathcal{C}(i_1)(v) \oplus \mathcal{C}(i_2)(d)) \\ &= \mathcal{C}(\mathbf{D} \Rightarrow \mathbf{D})(\langle (v, d)\rangle, W) \\ &= \mathrm{Id}_{\mathbf{D}\Rightarrow\mathbf{D}}(W, (v, d))\end{aligned}$$

The second to last equality follows from the fact that $(\mathcal{C}(i_1)(v) \oplus \mathcal{C}(i_2)(d)).1 = v$ and $(\mathcal{C}(i_1)(v) \oplus \mathcal{C}(i_2)(d)).2 = d$. We also obtain the following isomorphism:

$$\begin{aligned}
\mathrm{Ap} \circ \lambda(u, d) &= \int^{W \in \mathcal{C}(\mathbf{D} \Rightarrow \mathbf{D})} \mathrm{Ap}(W, d) \times \lambda^{\mathcal{C}}(u, W) \\
&= \int^{W} \mathcal{C}(\mathbf{D} \Rightarrow \mathbf{D})(\langle (d.1, d.2) \rangle, W) \times \lambda^{\mathcal{C}}(u, W) \cong \lambda(u, (d.1, d.2)) \\
&= \mathcal{C}\mathbf{D}(\mathcal{C}(i_1)(d.1) \oplus \mathcal{C}(i_2)(d.2), u) \cong \mathcal{C}\mathbf{D}(\langle d \rangle, u) = \mathrm{Id}_{\mathbf{D}}(u, d)
\end{aligned}$$

The second to last equality follows from the fact that d is isomorphic to $\mathcal{C}(i_1)(d.1) \oplus \mathcal{C}(i_2)(d.2)$ in $\mathcal{C}(\mathbf{D})$.

4.3.6 Fixed point operator for terms

A fundamental property of Scott-continuous functions is that they admit a least fixed point which allows for the study of recursively defined programs.

Theorem 4.3.17 (e.g. [106])**.** *Let $\mathbb{C}$ be a category with ω-colimits together with an initial object 0 and let $F : \mathbb{C} \to \mathbb{C}$ be an endofunctor that preserves ω-chains. Then F has an initial algebra obtained by taking the colimit of the following diagram:*

$$\begin{tikzcd}
0 \arrow[r, "i"'] & F(0) \arrow[r, "F(i)"'] & F^2(0) \arrow[r, "F^2(i)"'] & \cdots
\end{tikzcd}$$

where i is the unique map from the initial object to $F(0)$.

Lemma 4.3.18 (e.g. [106])**.** *Let $F : \mathbb{C} \to \mathbb{C}$ be an endofunctor and $a : F(c) \to c$ an initial algebra. Then a is an isomorphism.*

Definition 4.3.19. Let $\mathscr{B}$ be a cartesian closed bicategory and A an object of $\mathscr{B}$. A *fixpoint operator for an object A* in $\mathscr{B}$ is a 1-cell $\mathbf{fix}_A \in \mathscr{B}(A \Rightarrow A, A)$ together with an invertible 2-cell α:

$$\begin{tikzcd}
A \Rightarrow A \arrow[d, "\langle Id_{A \Rightarrow A}, \mathbf{fix}_A \rangle"'] \arrow[dr, "\mathbf{fix}_A"] \arrow[dr, phantom, "\overset{\sim}{\Longrightarrow}_{\alpha}"' near start] & \\
(A \Rightarrow A) \,\&\, A \arrow[r, "\mathrm{Ev}_{A,A}"'] & A
\end{tikzcd}$$

For $f \in A \Rightarrow A$, we obtain that $\mathrm{Ev}_{A,A}\langle f, \mathbf{fix}_A(f) \rangle \overset{\sim}{\Longrightarrow} \mathbf{fix}_A(f)$.

For a small category $\mathbb{A}$, $\mathbf{fix}_{\mathbb{A}} \in \mathbf{Prof}_{\mathcal{C}}(\mathbb{A} \Rightarrow \mathbb{A}, \mathbb{A})$ is obtained as the initial algebra of the following functor:

$$\begin{aligned} \mathcal{Y}_{\mathbb{A}} : \mathbf{Prof}_{\mathcal{C}}(\mathbb{A} \Rightarrow \mathbb{A}, \mathbb{A}) &\to \mathbf{Prof}_{\mathcal{C}}(\mathbb{A} \Rightarrow \mathbb{A}, \mathbb{A}) \\ F &\mapsto \mathrm{Ev} \circ \langle Id, F \rangle \end{aligned}$$

We identify $\mathbf{Prof}_{\mathcal{C}}(\mathbb{A} \Rightarrow \mathbb{A}, \mathbb{A})$ with the presheaf category of $(\mathbb{A} \Rightarrow \mathbb{A}) \Rightarrow \mathbb{A}$ whose initial object is the empty presheaf. Since for any morphism $H : \mathcal{C}\mathbb{X} \nrightarrow \mathbb{Y}$ in $\mathbf{Prof}_{\mathcal{C}}$, $\mathbf{Lan}_{s_{\mathbb{X}}}(H) : \widehat{X} \to \widehat{Y}$ preserves ω-colimits (as a particular case of sifted colimits), we show that $\mathcal{Y}_{\mathbb{A}}$ can be obtained as the left Kan extension of a $\mathcal{C}$-species in $\mathbf{Prof}_{\mathcal{C}}((\mathbb{A} \Rightarrow \mathbb{A}) \Rightarrow \mathbb{A}, (\mathbb{A} \Rightarrow \mathbb{A}) \Rightarrow \mathbb{A})$ which entails the existence of $\mathbf{fix}_{\mathbb{A}}$ by Theorem 4.3.17.

Consider the profunctor $\mathcal{Z}_{\mathbb{A}} \in \mathbf{Prof}_{\mathcal{C}}(((\mathbb{A} \Rightarrow \mathbb{A}) \Rightarrow \mathbb{A}) \,\&\, (\mathbb{A} \Rightarrow \mathbb{A}), \mathbb{A})$ defined by the following composition:

$$\begin{tikzcd}
((\mathbb{A} \Rightarrow \mathbb{A}) \Rightarrow \mathbb{A}) \,\&\, (\mathbb{A} \Rightarrow \mathbb{A}) \arrow[d, "Id \,\&\, \langle Id, Id \rangle"'] & & \\
((\mathbb{A} \Rightarrow \mathbb{A}) \Rightarrow \mathbb{A}) \,\&\, (\mathbb{A} \Rightarrow \mathbb{A}) \,\&\, (\mathbb{A} \Rightarrow \mathbb{A}) \arrow[d, "\mathrm{Ev}_{\mathbb{A} \Rightarrow \mathbb{A}, \mathbb{A}} \,\&\, Id"'] & & \\
\mathbb{A} \,\&\, (\mathbb{A} \Rightarrow \mathbb{A}) \arrow[r, "\langle \pi_2, \pi_1 \rangle"] & (\mathbb{A} \Rightarrow \mathbb{A}) \,\&\, \mathbb{A} \arrow[r, "\mathrm{Ev}_{\mathbb{A}, \mathbb{A}}"] & \mathbb{A}
\end{tikzcd}$$

By currying, we obtain a profunctor $\lambda(\mathcal{Z}_{\mathbb{A}})$ in $\mathbf{Prof}_{\mathcal{C}}((\mathbb{A} \Rightarrow \mathbb{A}) \Rightarrow \mathbb{A}, (\mathbb{A} \Rightarrow \mathbb{A}) \Rightarrow \mathbb{A})$ whose left Kan extension along $s_{(\mathbb{A} \Rightarrow \mathbb{A}) \Rightarrow \mathbb{A}}$ is isomorphic to $\mathcal{Y}_{\mathbb{A}}$ as desired. Explicitely, $\mathcal{Y}_{\mathbb{A}}$ is given by:

$$\mathcal{Y}_{\mathbb{A}} : (F, (U, a)) = \int^{u \in \mathcal{C}\mathbb{A}} F^{\mathcal{C}}(U, u) \times \mathcal{C}(\mathbb{A} \Rightarrow \mathbb{A})(\langle (u, a) \rangle, U)$$

We can now obtain $\mathbf{fix}_{\mathbb{A}} : \mathcal{C}(\mathbb{A} \Rightarrow \mathbb{A}) \nrightarrow \mathbb{A}$ by computing $\varinjlim_{n \in \omega} \mathcal{Y}^n_{\mathbb{A}}(\mathbb{0})$.

Examples.

- In the theory of combinatorial species, the species of lists is a solution of the equation $L = 1 + X \cdot L$ where 1 is the species whose analytic functor $\mathbf{Set} \to \mathbf{Set}$ is given by $S \mapsto \{\star\}$ and X is the singleton species whose analytic functor is the identity endofunctor on $\mathbf{Set}$. It follows the intuition that a list is either empty or an element followed by a list. In the case of $\mathbf{Prof}_{\mathcal{C}}$, we can define for every small category $\mathbb{A}$ a

$\mathcal{C}$-species of lists $L_{\mathbb{A}} : \mathcal{C}\mathbb{A} \nrightarrow \mathbb{A}$. $L_{\mathbb{A}}$ is obtained as the least fixpoint of the operator:

$$\begin{aligned} E_{\mathbb{A}} : \mathbf{Prof}_{\mathcal{C}}(\mathbb{A}, \mathbb{A}) &\to \mathbf{Prof}_{\mathcal{C}}(\mathbb{A}, \mathbb{A}) \\ (F, (u, a)) &\mapsto \mathbf{1}_{\mathbb{A}}(u, a) + X_{\mathbb{A}}(u, a) \times F(u, a) \\ &\quad = \mathcal{C}\mathbb{A}(\langle\rangle, u) + \mathcal{C}\mathbb{A}(\langle a\rangle, u) \times F(u, a) \end{aligned}$$

where $\mathbf{1}_{\mathbb{A}}(u, a)$ is the constant species $(u, a) \mapsto \mathcal{C}\mathbb{A}(\langle\rangle, u) \simeq \{\star\}$ and $X_{\mathbb{A}}$ is the singleton species $(u, a) \mapsto \mathcal{C}\mathbb{A}(\langle a\rangle, u)$. Note that if we take $\mathbb{A}$ to be the category $\mathbf{1}$, we obtain the species 1 and X mentionned above. Explicitly, the $\mathcal{C}$-species of lists $L_{\mathbb{A}} : \mathcal{C}\mathbb{A} \nrightarrow \mathbb{A}$ maps (u, a) to $\sum_{n\in\mathbb{N}} \mathcal{C}\mathbb{A}(\langle a\rangle, u)^n$ which entails that $\mathbf{Lan}_{s_{\mathbb{A}}}(L_{\mathbb{A}}) : \widehat{\mathbb{A}} \to \widehat{\mathbb{A}}$ is given by

$$(X, a) \mapsto \sum_{n\in\mathbb{N}} (X(a))^n.$$

- Using a similar reasoning, we can obtain a $\mathcal{C}$-species of binary trees, which is a solution of the equation $B = 1 + X \cdot B^2$. For a small category $\mathbb{A}$, if we compute the least fixpoint of the operator:

  $$\begin{aligned} H_{\mathbb{A}} : \mathbf{Prof}_{\mathcal{C}}(\mathbb{A}, \mathbb{A}) &\to \mathbf{Prof}_{\mathcal{C}}(\mathbb{A}, \mathbb{A}) \\ (F, (u, a)) &\mapsto \mathcal{C}\mathbb{A}(\langle\rangle, u) + \mathcal{C}\mathbb{A}(\langle a\rangle, u) \times F(u, a) \times F(u, a) \end{aligned}$$

  we obtain the $\mathcal{C}$-species $B_{\mathbb{A}} : \mathcal{C}\mathbb{A} \nrightarrow \mathbb{A}$ given by

  $$(u, a) \mapsto \sum_{n\in\mathbb{N}} C_n \times \mathcal{C}\mathbb{A}(\langle a\rangle, u)^n$$

  where C_n is the nth Catalan number.

4.4 Glueing $\mathcal{S}$-species and $\mathcal{C}$-species

In the categorified setting, we replace sets by groupoids and preorders by categories so that the forgetful functor **Pord** → **Set** mapping a preorder to its underlying set $P \mapsto |P|$ now corresponds to taking the *core* of category:

Definition 4.4.1. For a small category $\mathbb{A}$, its *core*, denoted by $\mathbf{c}\,\mathbb{A}$, is the maximal subgroupoid of $\mathbb{A}$.

The core functor $\mathbf{c} : \mathbf{Cat} \to \mathbf{Gpd}$ is the right adjoint functor of the inclusion $\iota : \mathbf{Gpd} \to \mathbf{Cat}$.

$$\begin{tikzcd}
\mathbf{Gpd} \arrow[r, bend left, "\iota"] \arrow[r, phantom, "\perp"] & \mathbf{Cat} \arrow[l, bend left, "\mathbf{c}"]
\end{tikzcd}$$

Analogously, the inclusion functor **Set** → **Pord** mapping a set S to the preorder $(S, =)$ is left adjoint to the forgetful functor **Pord** → **Set**.

Similarly to Section 4.2, the semantics of linear logic in the model of $\mathcal{S}$-species (restricted to groupoids) and in the model of $\mathcal{C}$-species are related. For a closed linear logic formula A, we denote by $\llbracket A \rrbracket_{\mathcal{S}}$ its interpretation in **ProfG** (the bicategory of profunctors restricted to groupoids) with the $\mathcal{S}$ pseudo-comonad and by $\llbracket A \rrbracket_{\mathcal{C}}$ its interpretation in **Prof** with the $\mathcal{C}$ pseudo-comonad. We can show by induction on the structure of A that the following equality holds:

$$\llbracket A \rrbracket_{\mathcal{S}} = \mathbf{c} \llbracket A \rrbracket_{\mathcal{C}}.$$

This equality is immediate for all linear connectives and for the exponential, it suffices to notice that for a category $\mathbb{A}$, we have $\mathcal{S}\, \mathbf{c}\, \mathbb{A} = \mathbf{c}\, \mathcal{C} \mathbb{A}$.

For a proof π of $\vdash A$, we write $\llbracket \pi \rrbracket_{\mathcal{S}}$ for its interpretation in **ProfG** with the $\mathcal{S}$-pseudo-comonad and by $\llbracket \pi \rrbracket_{\mathcal{C}}$ its interpretation in **Prof** with the $\mathcal{C}$-pseudo-comonad.

Notation 4. Let $\mathbb{A}$ be a small category, the inclusion $\iota_{\mathbb{A}} : \mathbf{c}\, \mathbb{A} \to \mathbb{A}$ induces an adjunction in **Prof** where the left and right adjoints $L_{\mathbb{A}} : \mathbf{c}\, \mathbb{A} \nrightarrow \mathbb{A}$ and $R_{\mathbb{A}} : \mathbb{A} \nrightarrow \mathbf{c}\, \mathbb{A}$ are given by $(a, a') \mapsto \mathbb{A}(a', \iota_{\mathbb{A}}(a))$ and $(a, a') \mapsto \mathbb{A}(\iota_{\mathbb{A}}(a'), a)$ respectively. The internal monad in **Prof** induced by this adjunction will be denoted by $(\downarrow_{\mathbb{A}} : \mathbf{c}\, \mathbb{A} \nrightarrow \mathbf{c}\, \mathbb{A}, \eta_{\mathbb{A}}, \mu_{\mathbb{A}})$.

The morphism $\llbracket \pi \rrbracket_{\mathcal{S}}$ is in $\mathbf{ProfG}(\mathbf{1}, \llbracket A \rrbracket_{\mathcal{S}})$, i.e. it is a profunctor $\mathbf{1} \nrightarrow \mathbf{c} \llbracket A \rrbracket_{\mathcal{C}}$ and $\llbracket \pi \rrbracket_{\mathcal{C}}$ is a profunctor $\mathbf{1} \nrightarrow \llbracket A \rrbracket_{\mathcal{C}}$. The equality $\llbracket \pi \rrbracket_s = \downarrow(\llbracket \pi \rrbracket_r)$ between the interpretations of π in **Rel** and **ScottL** is now replaced by an inclusion and a retraction between the profunctors:

$$\begin{tikzcd}
& \mathbf{c}\llbracket A \rrbracket_{\mathcal{C}} \arrow[dr, "L_{\llbracket A \rrbracket}", "|" marking] & \\
\mathbf{1} \arrow[ur, "\llbracket \pi \rrbracket_{\mathcal{S}}", "|" marking] \arrow[rr, bend right, "\llbracket \pi \rrbracket_{\mathcal{C}}"', "|" marking] & \Downarrow\Uparrow & \llbracket A \rrbracket_{\mathcal{C}}
\end{tikzcd}$$

Similarly to the 1-categorical case, this property cannot be proved by induction on the structure of π since for arbitrary profunctors $P : \mathbf{c}\, \mathbb{A} \nrightarrow$

$\mathbf{c}\,\mathbb{B}$ and $Q : \mathbf{c}\,\mathbb{B} \nrightarrow \mathbf{c}\,\mathbb{C}$, we do not necessarily have a retraction from the profunctor

$$\begin{tikzcd}
\mathbb{A} \arrow[r, "R_{\mathbb{A}}", "\shortmid"{marking}] & \mathbf{c}\,\mathbb{A} \arrow[r, "P", "\shortmid"{marking}] & \mathbf{c}\,\mathbb{B} \arrow[r, "\downarrow_{\mathbb{B}}", "\shortmid"{marking}] & \mathbf{c}\,\mathbb{B} \arrow[r, "Q", "\shortmid"{marking}] & \mathbf{c}\,\mathbb{C} \arrow[r, "L_{\mathbb{C}}", "\shortmid"{marking}] & \mathbb{C}
\end{tikzcd}$$

to the profunctor

$$\begin{tikzcd}
\mathbb{A} \arrow[r, "R_{\mathbb{A}}", "\shortmid"{marking}] & \mathbf{c}\,\mathbb{A} \arrow[r, "P", "\shortmid"{marking}] & \mathbf{c}\,\mathbb{B} \arrow[r, "Q", "\shortmid"{marking}] & \mathbf{c}\,\mathbb{C} \arrow[r, "L_{\mathbb{C}}", "\shortmid"{marking}] & \mathbb{C}
\end{tikzcd}$$

Recall that in the 1-categorical case, for a preorder P and a pair $(x, y) \in \mathbf{Rel}(\mathbf{1}, |P|) \times \mathbf{Rel}(|P|, \mathbf{1})$, x and y are orthogonal if the following equivalence holds

$$x \cap y \neq \emptyset \Leftrightarrow \downarrow(x) \cap \uparrow(y) \neq \emptyset.$$

In the bicategorical case, for a category $\mathbb{A}$ and a pair $(X, Y) \in \mathbf{ProfG}(\mathbf{1}, \mathbf{c}\,\mathbb{A}) \times \mathbf{ProfG}(\mathbf{c}\,\mathbb{A}, \mathbf{1})$, the orthogonality now carries a witness 2-cell

$$\lambda : Y \downarrow_{\mathbb{A}} X \Rightarrow YX$$

that is retraction of the inclusion $YX \Rightarrow Y \downarrow_{\mathbb{A}} X$ induced by $\eta_{\mathbb{A}} : \mathrm{id}_{\mathbf{c}\,\mathbb{A}} \Rightarrow \downarrow_{\mathbb{A}}$. Since we work with bicategories, the two composites $(Y \downarrow_{\mathbb{A}})X$ and $Y(\downarrow_{\mathbb{A}} X)$ are isomorphic through the associator $a : (Y \downarrow_{\mathbb{A}})X \overset{\sim}{\Rightarrow} Y(\downarrow_{\mathbb{A}} X)$ but not equal and we need to carefully keep track of these isomorphisms. The triangle identity axiom for bicategories tells us that $\lambda : (Y \downarrow_{\mathbb{A}})X \Rightarrow YX$ is a retract of the inclusion:

$$\begin{tikzcd}
YX \arrow[r, "r^{-1} * \mathrm{Id}"] & (Y\mathrm{id})X \arrow[r, "(\mathrm{Id} * \eta_{\mathbb{A}}) * \mathrm{Id}"] & (Y \downarrow_{\mathbb{A}})X
\end{tikzcd}$$

if and only if λa^{-1} is a retract of the inclusion

$$\begin{tikzcd}
YX \arrow[r, "\mathrm{Id} * l^{-1}"] & Y(\mathrm{id}X) \arrow[r, "\mathrm{Id} * (\eta_{\mathbb{A}} * \mathrm{Id})"] & Y(\downarrow_{\mathbb{A}} X).
\end{tikzcd}$$

In Section 4.2, we make use of the isomorphisms $\mathbf{Rel}(\mathbf{1}, S) \cong \mathcal{P}(S)$ and $\mathbf{ScottL}(\mathbf{1}, P) \cong \downarrow(P)$ for a set S and a preorder P to work with subsets instead of relations. In our case, we want to work with (co-)presheaves instead of profunctors so we replace the pseudo-functor $\mathbf{Prof}(\mathbf{1}, -) : \mathbf{Prof} \to \mathbf{CAT}$ with the pseudo-functor $(-)^{\#} : \mathbf{Prof} \to \mathbf{CAT}$ mapping a category $\mathbb{A}$ to the presheaf category $\widehat{\mathbb{A}} \simeq \mathbf{Prof}(\mathbf{1}, \mathbb{A})$, a profunctor $P : \mathbb{A} \nrightarrow \mathbb{B}$ to

$P^{\#} = \mathbf{Lan}_{\mathrm{y}} P : \widehat{\mathbb{A}} \to \widehat{\mathbb{B}}$ and the image of a 2-cell $\alpha \in \mathbf{Prof}(\mathbb{A}, \mathbb{B})(P, Q)$ is obtained from the universal property of the Kan extension.

We denote by $\langle - \mid - \rangle : \widehat{\mathbf{c}\,\mathbb{A}} \times \widehat{\mathbf{c}\,\mathbb{A}^{\mathrm{op}}} \to \mathbf{Set}$ the functor

$$(X, Y) \mapsto \int^{a \in \mathbf{c}\,\mathbb{A}} X(a) \times Y(a)$$

corresponding to the horizontal composition

$$* : \mathbf{Prof}(\mathbf{1}, \mathbf{c}\,\mathbb{A}) \times \mathbf{Prof}(\mathbf{c}\,\mathbb{A}, \mathbf{1}) \to \mathbf{Prof}(\mathbf{1}, \mathbf{1}).$$

We abuse notation and denote by a the natural isomorphism

$$\langle - \mid \downarrow^{\#}_{\mathbb{A}^{\mathrm{op}}} (-) \rangle \xRightarrow{a} \langle \downarrow^{\#}_{\mathbb{A}} (-) \mid - \rangle$$

corresponding to the associator in $\mathbf{Prof}$. In this setting, for $(X, Y) \in \widehat{\mathbf{c}\,\mathbb{A}} \times \widehat{\mathbf{c}\,\mathbb{A}^{\mathrm{op}}}$, a 2-cell $\lambda : \langle \downarrow^{\#}_{\mathbb{A}} X \mid Y \rangle \Rightarrow \langle X \mid Y \rangle$ is a retract of the inclusion

$$\langle X \mid Y \rangle \xrightarrow{\langle \varphi_0 * \mathrm{Id} | \mathrm{Id} \rangle} \langle \mathrm{id}^{\#} X \mid Y \rangle \xrightarrow{\langle \eta^{\#}_{\mathbb{A}} * \mathrm{Id} | \mathrm{Id} \rangle} \langle \downarrow^{\#}_{\mathbb{A}} X \mid Y \rangle$$

if and only if $\lambda a_{X,Y}$ is a retract of the inclusion

$$\langle X \mid Y \rangle \xrightarrow{\langle \mathrm{Id} | \varphi_0 * \mathrm{Id} \rangle} \langle X \mid \mathrm{id}^{\#} Y \rangle \xrightarrow{\langle \mathrm{Id} | \eta^{\#}_{\mathbb{A}^{\mathrm{op}}} * \mathrm{Id} \rangle} \langle X \mid \downarrow^{\#}_{\mathbb{A}^{\mathrm{op}}} Y \rangle.$$

4.4.1 Towards orthogonality bicategories

While the notion of orthogonality bicategories has yet to be spelled out, we present in this chapter a construction that brings us closer to the general definition as the 2-cells in the glued bicategory that we construct in the next section are restricted to the ones preserving the retractions.

We start by sketching what we believe is the general construction: for a bicategory $\mathscr{B}$ that is $*$-autonomous with monoidal units $\mathbf{1}$ and $\perp$, the orthogonality relation $\perp_c \subseteq \mathbb{C}(\mathbf{1}, c) \times \mathbb{C}(c, \perp)$ ranging over the objects c of a category $\mathbb{C}$ is replaced with a functor $\perp_b \to \mathscr{B}(\mathbf{1}, b) \times \mathscr{B}(b, \perp)$ where b ranges over the object of $\mathscr{B}$. This functor should induce an adjunction between the slice bicategories

$$\begin{tikzcd}
\mathbf{CAT}/\mathscr{B}(\mathbf{1}, b) \arrow[r, bend left, "(-)^{\perp}"] \arrow[r, phantom, "\perp"] & (\mathbf{CAT}/\mathscr{B}(b, \perp))^{\mathrm{op}} \arrow[l, bend left, "(-)^{\perp}"]
\end{tikzcd}$$

for every object $b \in \mathscr{B}$.

Definition 4.4.2. For a bicategory $\mathscr{B}$ and an object $b \in \mathscr{B}$, the *lax slice bicategory* $\mathscr{B}/b$ consists of:

- 0-cells are pairs (a, f) of an object $a \in \mathscr{B}$ and a 1-cell $f : a \to b$ in $\mathscr{B}$;
- 1-cells from $(a_1, f_1) \to (a_2, f_2)$ are pairs (g, α) of a 1-cell $g \in \mathscr{B}(a_1, a_2)$ and a 2-cell $\alpha : f_1 \Rightarrow f_2 g$ in $\mathscr{B}$;
- for 1-cells $(g, \alpha), (h, \beta) : (a_1, f_1) \to (a_2, f_2)$, a 2-cell from (g, α) to (h, β) is a 2-cell γ in $\mathscr{B}(a_1, a_2)(g, h)$ such that

$$\begin{tikzcd}
a_1 \arrow[rr, bend left, "h"] \arrow[rr, bend right, "g"'] \arrow[rr, phantom, "\Uparrow \gamma"] \arrow[dr, bend right, "f_1"'] & & a_2 \arrow[dl, bend left, "f_2"] & & a_1 \arrow[rr, bend left, "h"] \arrow[dr, bend right, "f_1"'] & & a_2 \arrow[dl, bend left, "f_2"] \\
& b \arrow[u, phantom, "\Uparrow \alpha"] & & = & & b \arrow[u, phantom, "\Uparrow \beta"] &
\end{tikzcd}$$

Colax slice bicategories are obtained by reversing the direction of the 2-cells, *slice bicategories* is when the 2-cell α is an isomorphism and *strict slice bicategories* is when α is the identity.

In this chapter, we only consider slice 1-categories (or strict slice bicategories) but we expect that the general construction will deal with slice bicategories.

We are interested in the objects $\mathbb{A} \xrightarrow{F} \mathscr{B}(\mathbf{1}, b)$ that are fixed points of this adjunction above, so the objects of the orthogonality bicategory will be pairs $(b, (\mathbb{A}, F : \mathbb{A} \to \mathscr{B}(\mathbf{1}, b)))$ of an object $b \in \mathscr{B}$ and an object $(\mathbb{A}, F)$ in the slice bicategory $\mathbf{CAT}/\mathscr{B}(\mathbf{1}, b)$ such that the unit of the adjunction is an isomorphism $(\mathbb{A}, F) \cong (\mathbb{A}, F)^{\perp\perp}$. We sketch below what we expect is the linear logic structure in the orthogonality bicategory from the one in $\mathscr{B}$.

Assuming that the pseudo-functor $\mathscr{B}(\mathbf{1}, -) : \mathscr{B} \to \mathbf{CAT}$ is monoidal with pseudo-natural transformation $m_{b_1,b_2} : \mathscr{B}(\mathbf{1}, b_2) \to \mathscr{B}(\mathbf{1}, b_1 \otimes b_2)$, for two objects $(\mathbb{A}_1, F_1 : \mathbb{A}_1 \to \mathscr{B}(\mathbf{1}, b_1))$ and $(\mathbb{A}_1, F_1 : \mathbb{A}_1 \to \mathscr{B}(\mathbf{1}, b_2))$ their tensor will be given by:

$$\Big(\mathbb{A}_1 \times \mathbb{A}_2 \xrightarrow{F_1 \times F_2} \mathscr{B}(\mathbf{1}, b_1) \times \mathscr{B}(\mathbf{1}, b_2) \xrightarrow{m_{b_1,b_2}} \mathscr{B}(\mathbf{1}, b_1 \otimes b_2)\Big)^{\perp\perp}$$

The linear hom is then given by $((\mathbb{A}_1, F_1) \otimes (\mathbb{A}_2, F_2)^{\perp})^{\perp}$. The cartesian product $(\mathbb{A}_1, F_1) \mathbin{\&} (\mathbb{A}_2, F_2)$ is given by:

$$\mathbb{A}_1 \times \mathbb{A}_2 \xrightarrow{F_1 \times F_2} \mathscr{B}(\mathbf{1}, b_1) \times \mathscr{B}(\mathbf{1}, b_2) \xrightarrow{\langle -,- \rangle} \mathscr{B}(\mathbf{1}, b_1 \mathbin{\&} b_2)$$

where $\langle -, - \rangle$ is the pairing functor. The coproduct is then obtained by dualization: $(\mathbb{A}_1, F_1) \oplus (\mathbb{A}_2, F_2) = \big((\mathbb{A}_1, F_1)^{\perp} \mathbin{\&} (\mathbb{A}_2, F_2)^{\perp}\big)^{\perp}$.

For the exponential structure, we assume that there is a pseudo-distributive law $\kappa_b : \mathscr{B}(\mathbf{1}, b) \to \mathscr{B}(\mathbf{1}, !b)$ between the $!$ pseudo-comonad in $\mathscr{B}$ and the identity comonad in **CAT**. For an object $(\mathbb{A}, F : \mathbb{A} \to \mathscr{B}(\mathbf{1}, b))$, its exponential is given by:

$$\Big(\mathbb{A} \xrightarrow{F} \mathscr{B}(\mathbf{1}, b) \xrightarrow{\kappa_b} \mathscr{B}(\mathbf{1}, !b)\Big)^{\perp\perp}$$

4.4.2 Orthogonality construction

We start by defining the analogue of the orthogonality relation 4.1 in Section 4.2 to our setting:

Definition 4.4.3. For a small category $\mathbb{A}$, define $\perp\!\!\!\perp_{\mathbb{A}}$ to be the category consisting of:

- objects: triples (X, Y, γ) where $X \in \widehat{\mathbf{c}\,\mathbb{A}} \simeq \mathbf{ProfG}(\mathbf{1}, \mathbf{c}\,\mathbb{A})$, $Y \in \widehat{\mathbf{c}\,\mathbb{A}^{\mathrm{op}}} \simeq \mathbf{ProfG}(\mathbf{c}\,\mathbb{A}, \mathbf{1})$ and $\gamma : \langle \downarrow^{\#}_{\mathbb{A}} X \mid Y \rangle \to \langle X \mid Y \rangle$ is a natural transformation that is a retract of the canonical inclusion $\langle X \mid Y \rangle \hookrightarrow \langle \downarrow^{\#}_{\mathbb{A}} X \mid Y \rangle$:

  $$\langle X \mid Y \rangle \xrightarrow{\langle \varphi_0 * \mathrm{Id} | \mathrm{Id} \rangle} \langle \mathrm{id}^{\#} X \mid Y \rangle \xrightarrow{\langle \eta^{\#}_{\mathbb{A}} * \mathrm{Id} | \mathrm{Id} \rangle} \langle \downarrow^{\#}_{\mathbb{A}} X \mid Y \rangle$$

  in **Set**.

- a morphism from (X, Y, γ) to (X', Y', γ') is a pair (α, β) where $\alpha \in \widehat{\mathbf{c}\,\mathbb{A}}(X, X')$ and $\beta \in \widehat{\mathbf{c}\,\mathbb{A}^{\mathrm{op}}}(Y, Y')$ are natural transformations satisfying $\gamma' \circ \langle \downarrow^{\#}_{\mathbb{A}} \alpha \mid \beta \rangle = \langle \alpha \mid \beta \rangle \circ \gamma$.

$$\begin{tikzcd}
\langle \downarrow^{\#}_{\mathbb{A}} X \mid Y \rangle \arrow[r, "\langle \downarrow^{\#}_{\mathbb{A}} \alpha \mid \beta \rangle"] \arrow[d, "\gamma"'] & \langle \downarrow^{\#}_{\mathbb{A}} X' \mid Y' \rangle \arrow[d, "\gamma'"] \\
\langle X \mid Y \rangle \arrow[r, "\langle \alpha \mid \beta \rangle"'] & \langle X' \mid Y' \rangle
\end{tikzcd}$$

We denote by $U_{\mathbb{A}}$ the forgetful functor

$$\perp\!\!\!\perp_{\mathbb{A}} \longrightarrow \widehat{\mathbf{c}\,\mathbb{A}} \times \widehat{\mathbf{c}\,\mathbb{A}^{\mathrm{op}}}$$

given by $(X, Y, \gamma) \mapsto (X, Y)$.

Proof. We show that $\perp\!\!\!\perp_{\mathbb{A}}$ is indeed a category. For (X, Y, γ) in $\perp\!\!\!\perp_{\mathbb{A}}$, since $\langle \downarrow^{\#}_{\mathbb{A}} \mathrm{Id} \mid \mathrm{Id} \rangle = \mathrm{Id}$ and $\langle \mathrm{Id}, \mathrm{Id} \rangle = \mathrm{Id}$, we immediately have that the pair $(\mathrm{Id}, \mathrm{Id})$ verifies $\gamma \circ \langle \downarrow^{\#}_{\mathbb{A}} \mathrm{Id} \mid \mathrm{Id} \rangle = \langle \mathrm{Id} \mid \mathrm{Id} \rangle \circ \gamma = \gamma$.

Using a similar argument for morphisms $(\alpha, \beta) : (X, Y, \gamma) \to (X', Y', \gamma')$ and $(\alpha', \beta') : (X', Y', \gamma') \to (X'', Y'', \gamma'')$ in $\perp\!\!\!\perp_{\mathbb{A}}$, we have:

$$\begin{aligned}
\gamma'' \circ (\langle \downarrow^{\#}_{\mathbb{A}} \alpha'\alpha \mid \beta'\beta \rangle &= \gamma'' \circ \langle \downarrow^{\#}_{\mathbb{A}} \alpha' \mid \beta' \rangle \circ \langle \downarrow^{\#}_{\mathbb{A}} \alpha \mid \beta \rangle \\
&= \langle \alpha' \mid \beta' \rangle \circ \gamma' \circ \langle \downarrow^{\#}_{\mathbb{A}} \alpha \mid \beta \rangle \\
&= \langle \alpha' \mid \beta' \rangle \circ \langle \alpha \mid \beta \rangle \circ \gamma \\
&= \langle \alpha'\alpha \mid \beta'\beta \rangle \circ \gamma
\end{aligned}$$

so that $(\alpha'\alpha, \beta'\beta)$ is a morphism from (X, Y, γ) to (X'', Y'', γ'') in $\perp\!\!\!\perp_{\mathbb{A}}$. □

4.4.3 Adjunction induced by the orthogonality

The content of this section benefited from early discussions with Federico Olimpieri.

Definition 4.4.4. For a small category $\mathbb{A}$, we define a functor

$$(-)^{\perp} : \mathbf{CAT}/\widehat{\mathbf{c}\,\mathbb{A}} \to \left(\mathbf{CAT}/\widehat{\mathbf{c}\,\mathbb{A}^{\mathrm{op}}}\right)^{\mathrm{op}}$$

as follows:

1. For an object $(\mathbb{D}, F : \mathbb{D} \to \widehat{\mathbf{c}\,\mathbb{A}})$ in the slice category $\mathbf{CAT}/\widehat{\mathbf{c}\,\mathbb{A}}$, define the pair $(\mathbb{D}^{\perp}, F^{\perp} : \mathbb{D}^{\perp} \to \widehat{\mathbf{c}\,\mathbb{A}^{\mathrm{op}}})$ in $\mathbf{CAT}/\widehat{\mathbf{c}\,\mathbb{A}^{\mathrm{op}}}$ by:

   - objects of $\mathbb{D}^{\perp}$: pairs (Y, λ) where Y is a co-presheaf in $\widehat{\mathbf{c}\,\mathbb{A}^{\mathrm{op}}}$ and λ is a 2-cell in $\mathbf{CAT}$

$$\begin{tikzcd}
 & \widehat{\mathbf{c}\,\mathbb{A}} \arrow[r, "\downarrow^{\#}_{\mathbb{A}}"] & \widehat{\mathbf{c}\,\mathbb{A}} \arrow[dr, "\langle - \mid Y \rangle"] & \\
\mathbb{D} \arrow[ur, bend left, "F"] \arrow[drr, bend right, "F"'] & & \Downarrow \lambda & \mathbf{Set} \\
 & & \widehat{\mathbf{c}\,\mathbb{A}} \arrow[ur, bend right, "\langle - \mid Y \rangle"'] &
\end{tikzcd}$$

whose components λ_d are retracts of the canonical inclusion induced by $\eta_{\mathbb{A}}$. It means that the following pasting diagram is equal to the identity 2-cell:

$$\begin{tikzcd}
 & \widehat{\mathbf{c}\,\mathbb{A}} \arrow[r, bend left=70, "\mathrm{id}"] \arrow[r, phantom, shift left=3, "\Downarrow \varphi_0"] \arrow[r, "\mathrm{id}^{\#}" description] \arrow[r, phantom, shift right=3, "\Downarrow \eta^{\#}_{\mathbb{A}}"] \arrow[r, bend right=70, "\downarrow^{\#}_{\mathbb{A}}"'] & \widehat{\mathbf{c}\,\mathbb{A}} \arrow[rd, bend left, "\langle - \mid Y\rangle"] & \\
\mathbb{D} \arrow[ru, bend left, "F"] \arrow[rd, bend right, "F"'] & & \Downarrow \lambda & \mathbf{Set} \arrow[r, phantom, "="] & \mathrm{Id} \\
 & \widehat{\mathbf{c}\,\mathbb{A}} \arrow[rru, bend right, "\langle - \mid Y\rangle"'] & &
\end{tikzcd}$$

or equivalently that for all $d \in \mathbb{D}$, $(F(d), Y, \lambda_d)$ is an object of $\perp\!\!\!\perp_{\mathbb{A}}$.

- morphisms of $\mathbb{D}^{\perp}$: a morphism from (Y, λ) to (Z, γ) is a natural transformation $\beta \in \widehat{\mathbf{c}\,\mathbb{A}^{\mathrm{op}}}(Y, Z)$ such that the following pasting diagrams are equal:

$$\begin{tikzcd}
 & \widehat{\mathbf{c}\,\mathbb{A}} \arrow[r, "\downarrow^{\#}_{\mathbb{A}}"] & \widehat{\mathbf{c}\,\mathbb{A}} \arrow[rd, bend left=50, "\langle - \mid Y\rangle"] \arrow[rd, bend right=30, "\langle - \mid Y'\rangle"'] \arrow[rd, phantom, "\Downarrow \langle - \mid \beta\rangle"] & & & \widehat{\mathbf{c}\,\mathbb{A}} \arrow[r, "\downarrow^{\#}_{\mathbb{A}}"] & \widehat{\mathbf{c}\,\mathbb{A}} \arrow[rd, bend left, "\langle - \mid Y\rangle"] & \\
\mathbb{D} \arrow[ru, bend left, "F"] \arrow[rd, bend right, "F"'] & \Downarrow \lambda' & & \mathbf{Set} \arrow[r, phantom, "="] & \mathbb{D} \arrow[ru, bend left, "F"] \arrow[rd, bend right, "F"'] & & \Downarrow \lambda & \mathbf{Set} \\
 & \widehat{\mathbf{c}\,\mathbb{A}} \arrow[rru, bend right, "\langle - \mid Y'\rangle"'] & & & & \widehat{\mathbf{c}\,\mathbb{A}} \arrow[rru, bend left=30, "\langle - \mid Y\rangle"] \arrow[rru, bend right=50, "\langle - \mid Y'\rangle"'] \arrow[rru, phantom, "\Downarrow \langle - \mid \beta\rangle"] & &
\end{tikzcd}$$

  i.e. for all d in $\mathbb{D}$, (Id, β) is in $\perp\!\!\!\perp_{\mathbb{A}}((F(d), Y, \lambda_d), (F(d), Z, \gamma_d))$.

- The functor $F^{\perp}$ is given by $(Y, \lambda) \mapsto Y$.

2. For a morphism $H : (\mathbb{D}_1, F_1) \to (\mathbb{D}_2, F_2)$ in $\mathbf{CAT}/\widehat{\mathbf{c}\,\mathbb{A}}$, the functor $H^{\perp} : (\mathbb{D}_2, F_2)^{\perp} \to (\mathbb{D}_1, F_1)^{\perp}$ maps an object (Y, λ) in $\mathbb{D}_2^{\perp}$ to $(Y, \lambda H)$ in $\mathbb{D}_1^{\perp}$ and a morphism β in $\mathbb{D}_2^{\perp}((Y, \lambda), (Z, \gamma))$ to itself.

$$\begin{tikzcd}
& & \widehat{\mathbf{c}\,\mathbb{A}} \arrow[r, "\downarrow^{\#}_{\mathbb{A}}"] & \widehat{\mathbf{c}\,\mathbb{A}} \arrow[dr, "\langle - \mid Y \rangle"] & \\
\mathbb{D}_1 \arrow[r, "H"] \arrow[urr, bend left, "F_1"] \arrow[drr, bend right, "F_1"'] & \mathbb{D}_2 \arrow[ur, "F_2"] \arrow[dr, "F_2"'] & \Downarrow \lambda & & \mathbf{Set} \\
& & \widehat{\mathbf{c}\,\mathbb{A}} \arrow[urr, "\langle - \mid Y \rangle"'] & &
\end{tikzcd}$$

Proof. We show that $(-)^{\perp}$ is a well-defined functor.

- We first show that $\mathbb{D}^{\perp}$ is indeed a category. Consider two morphisms $\beta \in \mathbb{D}^{\perp}((Y, \lambda), (Z, \gamma))$ and $\gamma \in \mathbb{D}^{\perp}((Z, \gamma), (T, \delta))$, since $\langle - \mid \gamma\beta \rangle = \langle - \mid \gamma \rangle \langle - \mid \beta \rangle$, we obtain immediately that $\gamma\beta$ is in $\mathbb{D}^{\perp}((Y, \lambda), (T, \delta))$. Closure under identities follows from the equality $\langle - \mid \mathrm{Id} \rangle = \mathrm{Id}$.
- For a morphism $H : (\mathbb{D}_1, F_1) \to (\mathbb{D}_2, F_2)$ in $\mathbf{CAT}/\widehat{\mathbf{c}\,\mathbb{A}}$, we first need to show that $H^{\perp} : (\mathbb{D}_2, F_2)^{\perp} \to (\mathbb{D}_1, F_1)^{\perp}$ is in the slice category $\mathbf{CAT}/\widehat{\mathbf{c}\,\mathbb{A}^{\mathrm{op}}}$ i.e. that $F_1^{\perp} H^{\perp} = F_2^{\perp}$. For $(Y, \lambda) \in \mathbb{D}_2^{\perp}$, $F_1^{\perp} H^{\perp}(Y, \lambda) = Y = F_2^{\perp}(Y)$ and for a morphism $\beta \in \mathbb{D}_2^{\perp}$, $F_1^{\perp} H^{\perp}(\beta) = \beta = F_2^{\perp}(\beta)$ as desired. We also need to show that $H^{\perp}$ is well-defined. If $(Y, \lambda) \in \mathbb{D}_2^{\perp}$ i.e. the pasting diagram in black is equal to the identity, then $(Y, \lambda H) \in \mathbb{D}_1^{\perp}$ by precomposing with H.

$$\begin{tikzcd}
& & \widehat{\mathbf{c}\,\mathbb{A}} \arrow[r, bend left=50, "\mathrm{id}"] \arrow[r, "\mathrm{id}^{\#}" description] \arrow[r, bend right=50, "\downarrow^{\#}_{\mathbb{A}}"'] \arrow[r, phantom, shift left=3, "\Downarrow \varphi^{0}"] \arrow[r, phantom, shift right=3, "\Downarrow \eta^{\#}_{\mathbb{A}}"] & \widehat{\mathbf{c}\,\mathbb{A}} \arrow[dr, "\langle - \mid Y \rangle"] & & \\
\color{blue}\mathbb{D}_1 \arrow[r, dashed, blue, "H"] \arrow[urr, dashed, blue, bend left, "F_1"] \arrow[drr, dashed, blue, bend right, "F_1"'] & \mathbb{D}_2 \arrow[ur, "F_2"] \arrow[dr, "F_2"'] & \Downarrow \lambda & & \mathbf{Set} \; = & \mathrm{Id} \\
& & \widehat{\mathbf{c}\,\mathbb{A}} \arrow[urr, "\langle - \mid Y \rangle"'] & & &
\end{tikzcd}$$

which is immediate from the assumption $(Y, \lambda) \in \mathbb{D}_2^{\perp}$.

Likewise, if β is in $\mathbb{D}_2^{\perp}((Y, \lambda), (Z, \gamma))$, then the pasting diagrams in black are equal which implies that $\beta \in \mathbb{D}_1^{\perp}((Y, \lambda H), (Z, \gamma H))$ by precomposing with H.

$$\begin{tikzcd}
 & & \widehat{\mathbf{c}\,\mathbb{A}} \arrow[r, "\downarrow^{\#}_{\mathbb{A}}"] & \widehat{\mathbf{c}\,\mathbb{A}} \arrow[dr, bend left, "\langle - \mid Y\rangle"] \arrow[dr, bend right, "\langle - \mid Y'\rangle"'] \arrow[dr, phantom, "\Downarrow \langle - \mid \beta\rangle"] & & & & & \widehat{\mathbf{c}\,\mathbb{A}} \arrow[r, "\downarrow^{\#}_{\mathbb{A}}"] & \widehat{\mathbf{c}\,\mathbb{A}} \arrow[dr, "\langle - \mid Y\rangle"] & \\
\mathbb{D}_1 \arrow[r, dashed, blue, "H"] \arrow[urr, dashed, blue, bend left, "F_1"] \arrow[drrr, dashed, blue, bend right, "F_1"'] & \mathbb{D}_2 \arrow[ur, "F_2"] \arrow[drr, "F_2"'] & & \Downarrow \lambda' & \mathbf{Set} & = & \mathbb{D}_1 \arrow[r, dashed, blue, "H"] \arrow[urr, dashed, blue, bend left, "F_1"] \arrow[drrr, dashed, blue, bend right, "F_1"'] & \mathbb{D}_2 \arrow[ur, "F"] \arrow[drr, "F"'] & & \Downarrow \lambda & \mathbf{Set} \\
 & & & \widehat{\mathbf{c}\,\mathbb{A}} \arrow[ur, "\langle - \mid Y'\rangle"'] & & & & & & \widehat{\mathbf{c}\,\mathbb{A}} \arrow[ur, bend left, "\langle - \mid Y\rangle"] \arrow[ur, bend right, "\langle - \mid Y'\rangle"'] \arrow[ur, phantom, "\Downarrow \langle - \mid \beta\rangle"] &
\end{tikzcd}$$

- It is immediate that $(-)^{\perp}$ preserves identities and composition. $\square$

Definition 4.4.5. We define analogously a functor

$$(-)^{\perp} : \mathbf{CAT}/\widehat{\mathbf{c}\,\mathbb{A}^{\mathrm{op}}} \to \left(\mathbf{CAT}/\widehat{\mathbf{c}\,\mathbb{A}}\right)^{\mathrm{op}}$$

as follows:

1. For a pair $(\mathbb{E}, G : \mathbb{E} \to \widehat{\mathbf{c}\,\mathbb{A}^{\mathrm{op}}})$ in the slice category $\mathbf{CAT}/\widehat{\mathbf{c}\,\mathbb{A}^{\mathrm{op}}}$, define the pair $(\mathbb{E}^{\perp}, G^{\perp} : \mathbb{E}^{\perp} \to \widehat{\mathbf{c}\,\mathbb{A}})$ by:
    - objects of $\mathbb{E}^{\perp}$: pairs (X, ρ) where X is a presheaf in $\widehat{\mathbf{c}\,\mathbb{A}}$ and ρ is a 2-cell in **CAT**

$$\begin{tikzcd}
 & \widehat{\mathbf{c}\,\mathbb{A}^{\mathrm{op}}} \arrow[r, "\downarrow^{\#}_{\mathbb{A}^{\mathrm{op}}}"] & \widehat{\mathbf{c}\,\mathbb{A}^{\mathrm{op}}} \arrow[dr, "\langle X \mid - \rangle"] & \\
\mathbb{E} \arrow[ur, "G"] \arrow[drr, "G"'] & & \Downarrow \rho & \mathbf{Set} \\
 & & \widehat{\mathbf{c}\,\mathbb{A}^{\mathrm{op}}} \arrow[ur, "\langle X \mid - \rangle"'] &
\end{tikzcd}$$

    such that for all $e \in \mathbb{E}$, $(X, G(e), \rho_e)$ is an object of $\perp\!\!\!\perp_{\mathbb{A}}$.
    - morphisms of $\mathbb{E}^{\perp}$: a morphism from (X, ρ) to (X', ρ') is a natural transformation $\alpha \in \widehat{\mathbf{c}\,\mathbb{A}^{\mathrm{op}}}(X, X')$ such that the following pasting diagrams are equal:

$$\begin{tikzcd}
& \widehat{\mathbf{c}\,\mathbb{A}^{\mathrm{op}}} \arrow[r, "{\downarrow_{\mathbb{A}^{\mathrm{op}}}}^{\#}"] & \widehat{\mathbf{c}\,\mathbb{A}^{\mathrm{op}}} \arrow[dr, bend left, "\langle X \mid - \rangle"] \arrow[dr, "\langle X \mid - \rangle"'] \arrow[dr, phantom, "\Downarrow \langle \alpha \mid - \rangle"] & \\
\mathbb{E} \arrow[ur, "G"] \arrow[dr, "G"'] & \Downarrow \rho' & & \mathbf{Set} \\
& \widehat{\mathbf{c}\,\mathbb{A}^{\mathrm{op}}} \arrow[urr, "\langle X' \mid - \rangle"'] & &
\end{tikzcd}
=
\begin{tikzcd}
& \widehat{\mathbf{c}\,\mathbb{A}^{\mathrm{op}}} \arrow[r, "\downarrow^{\#}_{\mathbb{A}^{\mathrm{op}}}"] & \widehat{\mathbf{c}\,\mathbb{A}^{\mathrm{op}}} \arrow[dr, "\langle X \mid - \rangle"] & \\
\mathbb{E} \arrow[ur, "G"] \arrow[dr, "G"'] & & \Downarrow \rho & \mathbf{Set} \\
& \widehat{\mathbf{c}\,\mathbb{A}^{\mathrm{op}}} \arrow[urr, "\langle X \mid - \rangle"] \arrow[urr, bend right, "\langle X' \mid - \rangle"'] \arrow[urr, phantom, "\Downarrow \langle \alpha \mid - \rangle"] & &
\end{tikzcd}$$

- The functor $G^{\perp}$ is given by $(X, \rho) \mapsto X$.

2. For a morphism $H : (\mathbb{E}_1, G_1) \to (\mathbb{E}_2, G_2)$ in $\mathbf{CAT}/\widehat{\mathbf{c}\,\mathbb{A}^{\mathrm{op}}}$, the functor $H^{\perp} : (\mathbb{E}_2, G_2)^{\perp} \to (\mathbb{E}_1, G_1)^{\perp}$ maps an object (X, ρ) in $\mathbb{E}_2^{\perp}$ to $(X, \rho H)$ in $\mathbb{E}_1^{\perp}$ and a morphism α in $\mathbb{E}_2^{\perp}((X, \rho), (X', \rho'))$ to itself.

This definition allows the retracts to be natural in both variables as we show below:

Lemma 4.4.6. *For a small category $\mathbb{A}$ and an object $(\mathbb{D}, F : \mathbb{D} \to \widehat{\mathbf{c}\,\mathbb{A}}$ in $\mathbf{CAT}/\widehat{\mathbf{c}\,\mathbb{A}}$, the construction in Definition 4.4.4 induces a natural transformation $r^{\mathbb{D}}$ in* **CAT**

$$\begin{tikzcd}
& \widehat{\mathbf{c}\,\mathbb{A}} \times \widehat{\mathbf{c}\,\mathbb{A}^{\mathrm{op}}} \arrow[r, "\mathrm{id} \times \downarrow^{\#}_{\mathbb{A}^{\mathrm{op}}}"] & \widehat{\mathbf{c}\,\mathbb{A}} \times \widehat{\mathbf{c}\,\mathbb{A}^{\mathrm{op}}} \arrow[dr, "\langle -, - \rangle"] & \\
\mathbb{D} \times \mathbb{D}^{\perp} \arrow[ur, "F \times F^{\perp}"] \arrow[dr, "F \times F^{\perp}"'] & & \Downarrow r^{\mathbb{D}} & \mathbf{Set} \\
& \widehat{\mathbf{c}\,\mathbb{A}} \times \widehat{\mathbf{c}\,\mathbb{A}^{\mathrm{op}}} \arrow[urr, "\langle -, - \rangle"'] & &
\end{tikzcd}$$

whose components $r^{\mathbb{D}}_{d,(Y,\lambda)} : \langle F(d) \mid \downarrow^{\#}_{\mathbb{A}^{\mathrm{op}}} Y \rangle \to \langle F(d) \mid Y \rangle$ are given by

$$\langle F(d) \mid \downarrow^{\#}_{\mathbb{A}^{\mathrm{op}}} Y \rangle \xrightarrow{a_{Fd,Y}} \langle \downarrow^{\#}_{\mathbb{A}} F(d) \mid Y \rangle \xrightarrow{\lambda_d} \langle F(d) \mid Y \rangle.$$

Symmetrically, for an object $(\mathbb{E}, G : \mathbb{E} \to \widehat{\mathbf{c}\,\mathbb{A}^{\mathrm{op}}})$ in $\mathbf{CAT}/\widehat{\mathbf{c}\,\mathbb{A}^{\mathrm{op}}}$, the construction in Definition 4.4.5 induces a natural transformation

$$\begin{tikzcd}
& \widehat{\mathbf{c}\,\mathbb{A}} \times \widehat{\mathbf{c}\,\mathbb{A}^{\mathrm{op}}} \arrow[rr, "\downarrow^{\#}_{\mathbb{A}} \times \mathrm{id}"] & & \widehat{\mathbf{c}\,\mathbb{A}} \times \widehat{\mathbf{c}\,\mathbb{A}^{\mathrm{op}}} \arrow[dr, bend left, "\langle -,- \rangle"] & \\
\mathbb{E}^{\perp} \times \mathbb{E} \arrow[ur, bend left, "G^{\perp} \times G"] \arrow[drr, bend right, "G^{\perp} \times G"'] & & \Downarrow r^{\mathbb{E}} & & \mathbf{Set} \\
& & \widehat{\mathbf{c}\,\mathbb{A}} \times \widehat{\mathbf{c}\,\mathbb{A}^{\mathrm{op}}} \arrow[urr, bend right, "\langle -,- \rangle"'] & &
\end{tikzcd}$$

whose components $r^{\mathbb{E}}_{(X,\rho),e} : \langle \downarrow^{\#}_{\mathbb{A}} X \mid G(e) \rangle \to \langle X \mid G(e) \rangle$ *are given by*

$$\langle \downarrow^{\#}_{\mathbb{A}} X \mid G(e) \rangle \xrightarrow{a^{-1}_{X,G(e)}} \langle X \mid \downarrow^{\#}_{\mathbb{A}^{\mathrm{op}}} G(e) \rangle \xrightarrow{\rho_e} \langle X \mid G(e) \rangle.$$

Proof. Let $f : c \to d$ be a morphism in $\mathbb{D}$ and $\beta : (Y, \lambda) \to (Z, \gamma)$ be a morphism in $\mathbb{D}^{\perp}$, we want to show that the following diagram commutes:

$$\begin{tikzcd}
\langle F(c) \mid \downarrow^{\#}_{\mathbb{A}^{\mathrm{op}}} Y \rangle \arrow[r, "r^{\mathbb{D}}_{c,(Y,\lambda)}"] \arrow[d, "\langle F(f) \mid \downarrow^{\#}_{\mathbb{A}^{\mathrm{op}}} \beta \rangle"'] & \langle F(c) \mid Y \rangle \arrow[d, "\langle F(f) \mid \beta \rangle"] \\
\langle F(d) \mid \downarrow^{\#}_{\mathbb{A}^{\mathrm{op}}} Z \rangle \arrow[r, "r^{\mathbb{D}}_{d,(Z,\gamma)}"'] & \langle F(d) \mid Z \rangle
\end{tikzcd}$$

It amounts to showing that the following diagram commutes:

$$\begin{tikzcd}
\langle F(c) \mid \downarrow^{\#}_{\mathbb{A}^{\mathrm{op}}} Y \rangle \arrow[r, "a_{Fc,Y}"] \arrow[d, "\langle \mathrm{Id} \mid \downarrow^{\#}_{\mathbb{A}^{\mathrm{op}}} \beta \rangle"'] & \langle \downarrow^{\#}_{\mathbb{A}} F(c) \mid Y \rangle \arrow[r, "\lambda_c"] \arrow[d, "\langle \mathrm{Id} \mid \beta \rangle"'] & \langle F(c) \mid Y \rangle \arrow[d, "\langle \mathrm{Id} \mid \beta \rangle"] \\
\langle F(c) \mid \downarrow^{\#}_{\mathbb{A}^{\mathrm{op}}} Z \rangle \arrow[r, "a_{Fc,Y}"] \arrow[d, "\langle F(f) \mid \mathrm{Id} \rangle"'] & \langle \downarrow^{\#}_{\mathbb{A}} F(c) \mid Z \rangle \arrow[r, "\gamma_c"] \arrow[d, "\langle \downarrow^{\#}_{\mathbb{A}} F(f) \mid \mathrm{Id} \rangle"'] & \langle F(c) \mid Z \rangle \arrow[d, "\langle F(f) \mid \mathrm{Id} \rangle"] \\
\langle F(d) \mid \downarrow^{\#}_{\mathbb{A}^{\mathrm{op}}} Z \rangle \arrow[r, "a_{Fd,Z}"'] & \langle F(d) \mid \downarrow^{\#}_{\mathbb{A}^{\mathrm{op}}} Z \rangle \arrow[r, "\gamma_d"'] & \langle F(d) \mid Z \rangle
\end{tikzcd}$$

The two squares in blue on the left commute from the naturality of a, the upper right square in purple commutes from the definition of morphisms in $\mathbb{D}^{\perp}$ and the right lower square in green from the naturality of γ. The proof for $r^{\mathbb{E}}$ is similar. $\square$

Corollary 4.4.7. *For a small category* $\mathbb{A}$ *and an object* $(\mathbb{D}, F : \mathbb{D} \to \widehat{\mathbf{c}\,\mathbb{A}})$ *in* $\mathbf{CAT}/\widehat{\mathbf{c}\,\mathbb{A}}$, $(F(d), r^{\mathbb{D}}_{d,-}))$ *is an object in* $\mathbb{D}^{\perp\perp}$ *for every* $d \in \mathbb{D}$.

Proposition 4.4.8. *The two functors in Definitions 4.4.4 and 4.4.5 induce a contravariant adjunction:*

$$\begin{tikzcd}
\mathbf{CAT}/\widehat{\mathbf{c}\,\mathbb{A}} \arrow[r, bend left, "(-)^{\perp}"] \arrow[r, phantom, "\perp"] & \left(\mathbf{CAT}/\widehat{\mathbf{c}\,\mathbb{A}^{\mathrm{op}}}\right)^{\mathrm{op}} \arrow[l, bend left, "(-)^{\perp}"]
\end{tikzcd}$$

Proof. We use the characterization of adjunctions in terms of universal morphisms (see Theorem 3.1.10). For $(\mathbb{D}, F : \mathbb{D} \to \widehat{\mathbf{c}\,\mathbb{A}})$, we define a functor $\eta_{\mathbb{D}} : \mathbb{D} \to \mathbb{D}^{\perp\perp}$ whose action on objects is given by $d \mapsto (F(d), r^{\mathbb{D}}_{d,(-)})$ where $r^{\mathbb{D}}_{d,(-)}$ is obtained from Lemma 4.4.6. On morphisms, for $f : c \to d$ in $\mathbb{D}$, $\eta_{\mathbb{D}}(f)$ is defined as $F(f)$. To check that it is a morphism in $\mathbb{D}^{\perp\perp}((F(c), r^{\mathbb{D}}_{c,(-)}), (F(d), r^{\mathbb{D}}_{d,(-)}))$, the following diagram must commute for every $(Y, \lambda) \in \mathbb{D}^{\perp}$:

$$\begin{tikzcd}
\langle \downarrow^{\#}_{\mathbb{A}} F(c) \mid Y \rangle \arrow[r, "r^{\mathbb{D}}_{c,(Y,\lambda)}"] \arrow[d, "\langle \downarrow^{\#}_{\mathbb{A}} F(f) \mid \beta \rangle"'] & \langle F(c) \mid Y \rangle \arrow[d, "\langle F(f) \mid \mathrm{Id} \rangle"] \\
\langle \downarrow^{\#}_{\mathbb{A}} F(d) \mid Y \rangle \arrow[r, "r^{\mathbb{D}}_{d,(Y,\lambda)}"'] & \langle F(d) \mid Y \rangle
\end{tikzcd}$$

which holds by naturality of $r^{\mathbb{D}}$. It is clear that $\eta_{\mathbb{D}}$ is a morphism in the slice category $\mathbf{CAT}/\widehat{\mathbf{c}\,\mathbb{A}}$ from $(\mathbb{D}, F)$ to $(\mathbb{D}^{\perp\perp}, F^{\perp\perp})$ since $F^{\perp\perp}\eta_{\mathbb{D}} = F$. We now show that $\eta_{\mathbb{D}}$ is a universal arrow from $(\mathbb{D}, F)$ to $(-)^{\perp} : \mathbf{CAT}/\widehat{\mathbf{c}\,\mathbb{A}} \to \left(\mathbf{CAT}/\widehat{\mathbf{c}\,\mathbb{A}^{\mathrm{op}}}\right)^{\mathrm{op}}$ i.e. for all $(\mathbb{E}, G) \in \mathbf{CAT}/\widehat{\mathbf{c}\,\mathbb{A}^{\mathrm{op}}}$ and $H : (\mathbb{D}, F) \to (\mathbb{E}^{\perp}, G^{\perp})$ in $\mathbf{CAT}/\widehat{\mathbf{c}\,\mathbb{A}}$, there exists a unique morphism $K : (\mathbb{E}, G) \to (\mathbb{D}^{\perp}, F^{\perp})$ in $\mathbf{CAT}/\widehat{\mathbf{c}\,\mathbb{A}^{\mathrm{op}}}$ such that $K^{\perp}\eta_{\mathbb{D}} = H$.

$$\begin{tikzcd}
\mathbb{D} \arrow[rr, bend left, "H"] \arrow[dr, "\eta_{\mathbb{D}}"'] \arrow[ddr, bend right, "F"'] & & \mathbb{E}^{\perp} \arrow[ddl, bend left, "G^{\perp}"] \\
& \mathbb{D}^{\perp\perp} \arrow[ur, dotted, "K^{\perp}"'] \arrow[d, "F^{\perp\perp}"] & \\
& \widehat{\mathbf{c}\,\mathbb{A}} &
\end{tikzcd}$$

For $e \in \mathbb{E}$, define $K(e)$ as $(G(e), r^{\mathbb{E}}_{H(-),e})$, it is indeed an element of $\mathbb{D}^{\perp}$ since for all $d \in \mathbb{D}$, $(F(d), G(e), r^{\mathbb{E}}_{H(d),e}) = (G^{\perp}H(d), G(e), r^{\mathbb{E}}_{H(d),e}) \in \perp\!\!\!\perp_{\mathbb{A}}$ by definition of $r^{\mathbb{E}}$.

$$\begin{tikzcd}
& & \widehat{\mathbf{c}\,\mathbb{A}} \arrow[r, "\downarrow^{\#}_{\mathbb{A}}"] & \widehat{\mathbf{c}\,\mathbb{A}} \arrow[dr, "\langle - \mid G(e)\rangle"] & \\
\mathbb{D} \arrow[r, "H"] \arrow[urr, bend left, "F"] \arrow[drrr, bend right, "F"'] & \mathbb{E}^{\perp} \arrow[ur, bend left, "G^{\perp}"] \arrow[drr, bend right, "G^{\perp}"'] & \Downarrow r^{\mathbb{E}}_{H(-),e} & & \mathbf{Set} \\
& & & \widehat{\mathbf{c}\,\mathbb{A}} \arrow[ur, "\langle - \mid G(e)\rangle"'] &
\end{tikzcd}$$

For a morphism $g : e \to e'$ in $\mathbb{E}$, define $K(g)$ as $G(f)$ which is a morphism from $(G(e), r^{\mathbb{E}}_{H(-),e})$ to $(G(e'), r^{\mathbb{E}}_{H(-),e'})$ in $\mathbb{D}^{\perp}$ by naturality of $r^{\mathbb{E}}$. Since $F^{\perp}G = K$, K is indeed a morphism from $(\mathbb{E}, G) \to (\mathbb{D}^{\perp}, F^{\perp})$ in the slice category $\mathbf{CAT}/\widehat{\mathbf{c}\,\mathbb{A}^{\mathrm{op}}}$.

We now show that $K^{\perp}\eta_{\mathbb{D}} = H$. Consider $d \in \mathbb{D}$, $H(d) \in \mathbb{E}^{\perp}$ is of the form (X, ρ) where $X \in \widehat{\mathbf{c}\,\mathbb{A}}$ and ρ is a natural transformation $\langle X \mid\downarrow^{\#}_{\mathbb{A}^{\mathrm{op}}} G(-)\rangle \Rightarrow \langle X \mid G(-)\rangle : \mathbb{E} \to \mathbf{Set}$. Since $G^{\perp}H = F$, we must have $X = F(d)$. Now, $K^{\perp}\eta_{\mathbb{D}}(d) = K^{\perp}(F(d), r^{\mathbb{D}}_{d,-}) = (F(d), r^{\mathbb{D}}_{d,K(-)})$ and for $e \in \mathbb{E}$,

$$r^{\mathbb{D}}_{d,K(e)} = r^{\mathbb{E}}_{H(d),e}a = \rho_e a^{-1} a = \rho_e$$

by definition of $r^{\mathbb{D}}, r^{\mathbb{E}}$ and K. For a morphism $f : c \to d$ in $\mathbb{D}$, $K^{\perp}\eta_{\mathbb{D}}(f) = K^{\perp}F(f) = F(f) = G^{\perp}H(f) = H(f)$ which implies that $K^{\perp}\eta_{\mathbb{D}} = H$ holds as desired.

It remains to show uniqueness, assume that there exists $L : (\mathbb{E}, G) \to (\mathbb{D}^{\perp}, F^{\perp})$ in $\mathbf{CAT}/\widehat{\mathbf{c}\,\mathbb{A}^{\mathrm{op}}}$ such that $L^{\perp}\eta_{\mathbb{D}} = H$. Since $F^{\perp}L = G$, for e in $\mathbb{E}$,

$L(e)$ is of the form $(G(e), \lambda)$ where λ is a natural transformation $\langle \downarrow^{\#}_{\mathbb{A}} F(-) \mid G(e) \rangle \Rightarrow \langle F(-) \mid G(e) \rangle : \mathbb{D} \to \mathbf{Set}$ and for a morphism $g : e \to e'$ in $\mathbb{E}$, $L(g) = G(g)$. It remains to show that for all $d \in \mathbb{D}$, $\lambda_d = r^{\mathbb{E}}_{H(d),e}$ to prove that $L = K$. If $H(d) = (F(d), \rho)$, then $L^{\perp}\eta_{\mathbb{D}}(d) = H(d)$ is equivalent to $(F(d), r^{\mathbb{D}}_{d,L(-)}) = (F(d), \rho)$ i.e. for all e in $\mathbb{E}$, $r^{\mathbb{D}}_{d,L(e)} = \rho_e$. Since $r^{\mathbb{D}}_{d,L(e)} = \lambda_d a^{-1}$, we obtain that $\lambda_d = \rho_e a = r^{\mathbb{E}}_{H(d),e}$ as desired. □

4.4.4 Bicategory CProf

This section is devoted to the construction of the glued bicategory **CProf** that connects $\mathcal{S}$-species and $\mathcal{C}$-species. The linear logic structure is left for future work.

Definition 4.4.9. A *closed structure* is a pair $\mathbf{A} = (|\mathbf{A}|, (\mathcal{D}(\mathbf{A}), F_{\mathbf{A}}))$ of a small category $|\mathbf{A}|$ and an object $(\mathcal{D}(\mathbf{A}), F_{\mathbf{A}}) \in \mathbf{CAT}/\widehat{\mathbf{c}\,|\mathbf{A}|}$ that is a fixpoint for the adjunction described in 4.4.8, i.e. the unit is an isomorphism.

Definition 4.4.10. A 1-cell between closed structures $\mathbf{A}$ and $\mathbf{B}$ is a tuple $P = (|P|, \overline{P}, \nu, \xi)$ where

- $|P|$ is profunctor in $\mathbf{ProfG}(\mathbf{c}\,|\mathbf{A}|, \mathbf{c}\,|\mathbf{B}|)$,
- $\overline{P}$ is a functor $\mathcal{D}(\mathbf{A}) \to \mathcal{D}(\mathbf{B})$ in $\mathbf{CAT}$,
- ν and ξ are natural transformations

$$\begin{tikzcd}
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[r, "|P|^{\#}"] & \widehat{\mathbf{c}\,|\mathbf{B}|} \\
\mathcal{D}(\mathbf{A}) \arrow[u, "F_{\mathbf{A}}"] \arrow[r, "\overline{P}"'] \arrow[ur, phantom, "\Downarrow \nu"] & \mathcal{D}(\mathbf{B}) \arrow[u, "F_{\mathbf{B}}"']
\end{tikzcd}$$

$$\begin{tikzcd}
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[r, "|P|^{\#}"] & \widehat{\mathbf{c}\,|\mathbf{B}|} \arrow[r, "\downarrow^{\#}_{|\mathbf{B}|}"] & \widehat{\mathbf{c}\,|\mathbf{B}|} \\
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[u, "\downarrow^{\#}_{|\mathbf{A}|}"] & \Downarrow \xi & \widehat{\mathbf{c}\,|\mathbf{B}|} \arrow[u, "\downarrow^{\#}_{|\mathbf{B}|}"'] \\
\mathcal{D}(\mathbf{A}) \arrow[u, "F_{\mathbf{A}}"] \arrow[rr, "\overline{P}"'] & & \mathcal{D}(\mathbf{B}) \arrow[u, "F_{\mathbf{B}}"']
\end{tikzcd}$$

such that ν is an isomorphism and ξ verifies the coherence axioms:

1. retraction axiom: we require ξ to be a retract of the canonical inclusion i.e. the following pasting diagram is equal to the identity 2-cell:

$$\begin{tikzcd}[column sep=large]
 & & \mathcal{D}(\mathbf{B}) \arrow[dr, bend left, "F_{\mathbf{B}}"] \arrow[d, phantom, "\Downarrow \nu^{-1}"] & & & \\
\mathcal{D}(\mathbf{A}) \arrow[urr, bend left, "\overline{P}"] \arrow[r, "F_{\mathbf{A}}"] \arrow[drr, bend right, "\overline{P}"'] & \widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[r, "\mathrm{id}^{\#}" description] \arrow[r, bend left=60, "\mathrm{id}"] \arrow[r, bend right=60, "\downarrow^{\#}_{|\mathbf{A}|}"'] \arrow[r, phantom, shift left=3, "\Downarrow \varphi^{0}"] \arrow[r, phantom, shift right=3, "\Downarrow \eta^{\#}_{|\mathbf{A}|}"] & \widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[r, "|P|^{\#}"] & \widehat{\mathbf{c}\,|\mathbf{B}|} \arrow[r, "\downarrow^{\#}_{|\mathbf{B}|}"] \arrow[d, phantom, "\Downarrow \xi"] & \widehat{\mathbf{c}\,|\mathbf{B}|} & = \quad \mathrm{Id} \\
 & & \mathcal{D}(\mathbf{B}) \arrow[r, "F_{\mathbf{B}}"'] & \widehat{\mathbf{c}\,|\mathbf{B}|} \arrow[ur, bend right, "\downarrow^{\#}_{|\mathbf{B}|}"'] & &
\end{tikzcd}$$

2. commutation axiom: we also require ξ to commute with respect to $\mu_{|\mathbf{B}|} : \downarrow_{|\mathbf{B}|}\downarrow_{|\mathbf{B}|} \to \downarrow_{|\mathbf{B}|}$ as follows:

$$\begin{tikzcd}
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[r, "|P|^{\#}"] & \widehat{\mathbf{c}\,|\mathbf{B}|} \arrow[r, "\downarrow^{\#}_{|\mathbf{B}|}"] & \widehat{\mathbf{c}\,|\mathbf{B}|} \arrow[r, "\downarrow^{\#}_{|\mathbf{B}|}"] & \widehat{\mathbf{c}\,|\mathbf{B}|} \\
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[u, "\downarrow^{\#}_{|\mathbf{A}|}"] & \Downarrow \xi & \widehat{\mathbf{c}\,|\mathbf{B}|} \arrow[u, "\downarrow^{\#}_{|\mathbf{B}|}"] \arrow[ur, "(\downarrow_{|\mathbf{B}|}\downarrow_{|\mathbf{B}|})^{\#}" description] \arrow[ur, bend right=60, "\downarrow^{\#}_{|\mathbf{B}|}"'] \arrow[ur, phantom, shift left=4, "\Downarrow \varphi_{2}"] \arrow[ur, phantom, shift right=4, "\Downarrow \mu^{\#}_{|\mathbf{B}|}"] & \\
\mathcal{D}(\mathbf{A}) \arrow[u, "F_{\mathbf{A}}"] \arrow[rr, "\overline{P}"'] & & \mathcal{D}(\mathbf{B}) \arrow[u, "F_{\mathbf{B}}"'] &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[r, "|P|^{\#}"] & \widehat{\mathbf{c}\,|\mathbf{B}|} \arrow[r, "\downarrow^{\#}_{|\mathbf{B}|}"] \arrow[rr, bend right=40, "(\downarrow_{|\mathbf{B}|}\downarrow_{|\mathbf{B}|})^{\#}" description] \arrow[rr, bend right=80, "\downarrow^{\#}_{|\mathbf{B}|}"'] \arrow[rr, phantom, bend right=20, "\Downarrow \varphi_{2}"] \arrow[rr, phantom, bend right=60, "\Downarrow \mu^{\#}_{|\mathbf{B}|}"] & \widehat{\mathbf{c}\,|\mathbf{B}|} \arrow[r, "\downarrow^{\#}_{|\mathbf{B}|}"] & \widehat{\mathbf{c}\,|\mathbf{B}|} \\
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[u, "\downarrow^{\#}_{|\mathbf{A}|}"] & \Downarrow \xi & & \widehat{\mathbf{c}\,|\mathbf{B}|} \arrow[u, "\downarrow^{\#}_{|\mathbf{B}|}"'] \\
\mathcal{D}(\mathbf{A}) \arrow[u, "F_{\mathbf{A}}"] \arrow[rrr, "\overline{P}"'] & & & \mathcal{D}(\mathbf{B}) \arrow[u, "F_{\mathbf{B}}"']
\end{tikzcd}$$

Recall that in the case of the category **Pop**, for objects (P, D) and (P', D') in **Pop**, a relation R is in $\mathbf{Pop}((P, D), (P', D'))$ if and only if for all $x \in D$,

$$Rx \in D' \qquad \text{and} \qquad \downarrow(Rx) = \downarrow(R)\downarrow(x).$$

In our setting, the first condition corresponds to the existence of a functor $\overline{P} : \mathcal{D}(\mathbf{A}) \to \mathcal{D}(\mathbf{B})$ and an isomorphism $\nu : F_{\mathbf{B}}\overline{P} \cong |P|^{\#} F_{\mathbf{A}}$. The second condition is replaced with an explicit retraction ξ to the inclusion $\downarrow^{\#}_{|\mathbf{B}|} F_{\mathbf{B}}\overline{P} \hookrightarrow \downarrow^{\#}_{|\mathbf{B}|} |P|^{\#} \downarrow^{\#}_{|\mathbf{A}|} F_{\mathbf{A}}$ analogous to the inclusion $\downarrow(Rx) \hookrightarrow \downarrow(R)\downarrow(x)$. The need for the commutation axiom will become clear as we deal with the unitors.

Since our 1-cells carry explicit 2-cell isomorphisms and retractions, the 2-cells in the bicategory **CProf** are required to preserve them in the following sense:

Definition 4.4.11. For closed structures $\mathbf{A}$ and $\mathbf{B}$ and 1-cells

$$P_1 = (|P_1|, \overline{P_1}, \nu_1, \xi_1) : \mathbf{A} \nrightarrow \mathbf{B} \text{ and } P_2 = (|P_2|, \overline{P_2}, \nu_2, \xi_2) : \mathbf{A} \nrightarrow \mathbf{B},$$

a 2-cell from P_1 to P_2 is a pair $\alpha = (|\alpha|, \overline{\alpha})$ of a 2-cell $|\alpha| \in \mathbf{ProfG}(|P_1|, |P_2|)$ and a 2-cell $\overline{\alpha} \in \mathbf{CAT}(\overline{P_1}, \overline{P_2})$ satisfying the following coherence axioms:

$$\begin{tikzcd}
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[rr, bend left, "|P_1|^{\#}"] \arrow[rr, bend right, "|P_2|^{\#}"'] \arrow[rr, phantom, "\Downarrow |\alpha|^{\#}"] & & \widehat{\mathbf{c}\,|\mathbf{B}|} & & \widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[rr, bend left, "|P_1|^{\#}"] & & \widehat{\mathbf{c}\,|\mathbf{B}|} \\
& \Downarrow \nu_2 & & = & & \Downarrow \nu_1 & \\
\mathcal{D}(\mathbf{A}) \arrow[uu, "F_{\mathbf{A}}"] \arrow[rr, bend right, "\overline{P_2}"'] & & \mathcal{D}(\mathbf{B}) \arrow[uu, "F_{\mathbf{B}}"'] & & \mathcal{D}(\mathbf{A}) \arrow[uu, "F_{\mathbf{A}}"] \arrow[rr, bend left, "\overline{P_1}"] \arrow[rr, bend right, "\overline{P_2}"'] \arrow[rr, phantom, "\Downarrow \overline{\alpha}"] & & \mathcal{D}(\mathbf{B}) \arrow[uu, "F_{\mathbf{B}}"']
\end{tikzcd}$$

$$\begin{tikzcd}
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[r, bend left, "|P_1|^{\#}"] \arrow[r, bend right, "|P_2|^{\#}"'] \arrow[r, phantom, "\Downarrow |\alpha|^{\#}"] & \widehat{\mathbf{c}\,|\mathbf{B}|} \arrow[r, "\downarrow^{\#}_{|\mathbf{B}|}"] & \widehat{\mathbf{c}\,|\mathbf{B}|} & & \widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[r, bend left, "P_1^{\#}"] & \widehat{\mathbf{c}\,|\mathbf{B}|} \arrow[r, "\downarrow^{\#}_{|\mathbf{B}|}"] & \widehat{\mathbf{c}\,|\mathbf{B}|} \\
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[u, "\downarrow^{\#}_{|\mathbf{A}|}"] & \Downarrow \xi_2 & \widehat{\mathbf{c}\,|\mathbf{B}|} \arrow[u, "\downarrow^{\#}_{|\mathbf{B}|}"'] & = & \widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[u, "\downarrow^{\#}_{|\mathbf{A}|}"] & \Downarrow \xi_1 & \widehat{\mathbf{c}\,|\mathbf{B}|} \arrow[u, "\downarrow^{\#}_{|\mathbf{B}|}"'] \\
\mathcal{D}(\mathbf{A}) \arrow[u, "F_{\mathbf{A}}"] \arrow[rr, bend right, "\overline{P_2}"'] & & \mathcal{D}(\mathbf{B}) \arrow[u, "F_{\mathbf{B}}"'] & & \mathcal{D}(\mathbf{A}) \arrow[u, "F_{\mathbf{A}}"] \arrow[rr, bend left, "\overline{P_1}"] \arrow[rr, bend right, "\overline{P_2}"'] \arrow[rr, phantom, "\Downarrow \overline{\alpha}"] & & \mathcal{D}(\mathbf{B}) \arrow[u, "F_{\mathbf{B}}"']
\end{tikzcd}$$

We let **CProf** be the bicategory whose 0-cells are closed structures, 1-cells are given in Definition 4.4.10 and 2-cells are given in Definition 4.4.11. We proceed to show that it forms a bicategory.

Lemma 4.4.12. *For closed structures* $\mathbf{A}$ *and* $\mathbf{B}$, $\mathbf{CProf}(\mathbf{A}, \mathbf{B})$ *forms a category.*

Proof.

- Vertical composition: for 2-cells $\alpha : P_1 \Rightarrow P_2$ and $\beta : P_2 \Rightarrow P_3$ in $\mathbf{CProf}(\mathbf{A}, \mathbf{B})$, their vertical composite $\beta\alpha : P_1 \Rightarrow P_3$ is defined as $(|\beta|\,|\alpha|\,, \overline{\beta}\,\overline{\alpha})$ where the first component $|\beta|\,|\alpha|$ is the vertical composition in $\mathbf{ProfG}$ and the second component $\overline{\beta}\overline{\alpha}$ is the vertical composition in $\mathbf{CAT}$. Since $(|\beta|\,|\alpha|)^{\#} = |\beta|^{\#}\,|\alpha|^{\#}$, if $\alpha \in \mathbf{CProf}(\mathbf{A}, \mathbf{B})(P_1, P_2)$ and $\beta \in \mathbf{CProf}(\mathbf{A}, \mathbf{B})(P_2, P_3)$, then the ν_i's verify:

$$\begin{tikzcd}
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[r, bend left=60, "|P_1|^{\#}"] \arrow[r, dashed, "|P_2|^{\#}" description] \arrow[r, bend right=60, "|P_3|^{\#}"'] \arrow[r, phantom, shift left=3.5ex, "\Downarrow |\alpha|^{\#}"] \arrow[r, phantom, shift right=3.5ex, "\Downarrow |\beta|^{\#}"] & \widehat{\mathbf{c}\,|\mathbf{B}|} & & \widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[r, bend left=60, "|P_1|^{\#}"] \arrow[r, dashed, "|P_2|^{\#}" description] \arrow[r, phantom, shift left=3.5ex, "\Downarrow |\alpha|^{\#}"] & \widehat{\mathbf{c}\,|\mathbf{B}|} & & \widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[r, bend left=60, "|P_1|^{\#}"] & \widehat{\mathbf{c}\,|\mathbf{B}|} \\
\mathcal{D}(\mathbf{A}) \arrow[u, "F_{\mathbf{A}}"] \arrow[r, bend right=60, "\overline{P_3}"'] \arrow[ur, phantom, "\Downarrow \nu_3"] & \mathcal{D}(\mathbf{B}) \arrow[u, "F_{\mathbf{B}}"'] & = & \mathcal{D}(\mathbf{A}) \arrow[u, "F_{\mathbf{A}}"] \arrow[r, "\overline{P_2}"] \arrow[r, bend right=60, "\overline{P_3}"'] \arrow[r, phantom, shift right=3.5ex, "\Downarrow \overline{\beta}"] \arrow[ur, phantom, "\Downarrow \nu_2"] & \mathcal{D}(\mathbf{B}) \arrow[u, "F_{\mathbf{B}}"'] & = & \mathcal{D}(\mathbf{A}) \arrow[u, "F_{\mathbf{A}}"] \arrow[r, bend left=60, "\overline{P_1}"] \arrow[r, dashed, "\overline{P_2}" description] \arrow[r, bend right=60, "\overline{P_3}"'] \arrow[r, phantom, shift left=3.5ex, "\Downarrow \overline{\alpha}"] \arrow[r, phantom, shift right=3.5ex, "\Downarrow \overline{\beta}"] \arrow[ur, phantom, "\Downarrow \nu_1"] & \mathcal{D}(\mathbf{B}) \arrow[u, "F_{\mathbf{B}}"']
\end{tikzcd}$$

and we can show similarly that the following equality holds:

$$\begin{tikzcd}
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[r, bend left=60, "|P_1|^{\#}"] \arrow[r, dashed, "|P_2|^{\#}" description] \arrow[r, bend right=60, "|P_3|^{\#}"'] \arrow[r, phantom, shift left=3.5ex, "\Downarrow |\alpha|^{\#}"] \arrow[r, phantom, shift right=3.5ex, "\Downarrow |\beta|^{\#}"] & \widehat{\mathbf{c}\,|\mathbf{B}|} \arrow[r, "\downarrow^{\#}_{|\mathbf{B}|}"] & \widehat{\mathbf{c}\,|\mathbf{B}|} & & \widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[r, bend left=60, "P_1^{\#}"] & \widehat{\mathbf{c}\,|\mathbf{B}|} \arrow[r, "\downarrow^{\#}_{|\mathbf{B}|}"] & \widehat{\mathbf{c}\,|\mathbf{B}|} \\
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[u, "\downarrow^{\#}_{|\mathbf{A}|}"] \arrow[rr, phantom, "\Downarrow \xi_3"] & & \widehat{\mathbf{c}\,|\mathbf{B}|} \arrow[u, "\downarrow^{\#}_{|\mathbf{B}|}"'] & = & \widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[u, "\downarrow^{\#}_{|\mathbf{A}|}"] \arrow[rr, phantom, "\Downarrow \xi_1"] & & \widehat{\mathbf{c}\,|\mathbf{B}|} \arrow[u, "\downarrow^{\#}_{|\mathbf{B}|}"'] \\
\mathcal{D}(\mathbf{A}) \arrow[u, "F_{\mathbf{A}}"] \arrow[rr, bend right=30, "\overline{P_3}"'] & & \mathcal{D}(\mathbf{B}) \arrow[u, "F_{\mathbf{B}}"'] & & \mathcal{D}(\mathbf{A}) \arrow[u, "F_{\mathbf{A}}"] \arrow[rr, bend left=30, "\overline{P_1}"] \arrow[rr, "\overline{P_2}" description] \arrow[rr, bend right=30, "\overline{P_3}"'] \arrow[rr, phantom, shift left=2.5ex, "\Downarrow \overline{\alpha}"] \arrow[rr, phantom, shift right=2.5ex, "\Downarrow \overline{\beta}"] & & \mathcal{D}(\mathbf{B}) \arrow[u, "F_{\mathbf{B}}"']
\end{tikzcd}$$

which implies that $\beta\alpha$ is in $\mathbf{CProf}(\mathbf{A}, \mathbf{B})(P_1, P_3)$ as desired.

- Identity: for a 1-cell $P = (|P|\,, \overline{P}, \nu, \xi) \in \mathbf{CProf}(\mathbf{A}, \mathbf{B})$, the identity 2-cell Id_P is defined as $(\mathrm{Id}_{|P|}, \mathrm{Id}_{\overline{P}})$ where $\mathrm{Id}_{|P|}$ is the identity 2-cell in

ProfG and $\mathrm{Id}_{\overline{P}}$ is the identity 2-cell in **CAT**. Since $(\mathrm{Id}_{|P|})^{\#} = \mathrm{Id}_{|P|^{\#}}$, the following equalities trivially hold:

$$\begin{tikzcd}
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[r, bend left, "|P|^{\#}"] \arrow[r, bend right, "|P|^{\#}"'] \arrow[r, phantom, "="] & \widehat{\mathbf{c}\,|\mathbf{B}|} \\
\mathcal{D}(\mathbf{A}) \arrow[u, "F_{\mathbf{A}}"] \arrow[r, bend right, "\overline{P}"'] \arrow[ur, phantom, "\Downarrow \nu"] & \mathcal{D}(\mathbf{B}) \arrow[u, "F_{\mathbf{B}}"']
\end{tikzcd}
=
\begin{tikzcd}
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[r, bend left, "|P|^{\#}"] & \widehat{\mathbf{c}\,|\mathbf{B}|} \\
\mathcal{D}(\mathbf{A}) \arrow[u, "F_{\mathbf{A}}"] \arrow[r, bend left, "\overline{P}"] \arrow[r, bend right, "\overline{P}"'] \arrow[r, phantom, "="] \arrow[ur, phantom, "\Downarrow \nu"] & \mathcal{D}(\mathbf{B}) \arrow[u, "F_{\mathbf{B}}"']
\end{tikzcd}$$

$$\begin{tikzcd}
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[r, bend left, "|P|^{\#}"] \arrow[r, bend right, "|P|^{\#}"'] \arrow[r, phantom, "="] & \widehat{\mathbf{c}\,|\mathbf{B}|} \arrow[r, "\downarrow^{\#}_{|\mathbf{B}|}"] & \widehat{\mathbf{c}\,|\mathbf{B}|} \\
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[u, "\downarrow^{\#}_{|\mathbf{A}|}"] & \Downarrow \xi & \widehat{\mathbf{c}\,|\mathbf{B}|} \arrow[u, "\downarrow^{\#}_{|\mathbf{B}|}"'] \\
\mathcal{D}(\mathbf{A}) \arrow[u, "F_{\mathbf{A}}"] \arrow[rr, bend right, "\overline{P}"'] & & \mathcal{D}(\mathbf{B}) \arrow[u, "F_{\mathbf{B}}"']
\end{tikzcd}
=
\begin{tikzcd}
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[r, bend left, "P^{\#}"] & \widehat{\mathbf{c}\,|\mathbf{B}|} \arrow[r, "\downarrow^{\#}_{|\mathbf{B}|}"] & \widehat{\mathbf{c}\,|\mathbf{B}|} \\
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[u, "\downarrow^{\#}_{|\mathbf{A}|}"] & \Downarrow \xi & \widehat{\mathbf{c}\,|\mathbf{B}|} \arrow[u, "\downarrow^{\#}_{|\mathbf{B}|}"'] \\
\mathcal{D}(\mathbf{A}) \arrow[u, "F_{\mathbf{A}}"] \arrow[rr, bend left, "\overline{P}"] \arrow[rr, bend right, "\overline{P}"'] \arrow[rr, phantom, "="] & & \mathcal{D}(\mathbf{B}) \arrow[u, "F_{\mathbf{B}}"']
\end{tikzcd}$$

so that Id_P is in $\mathbf{CProf}(\mathbf{A}, \mathbf{B})(P, P)$ as desired. $\square$

Horizontal composition

For closed structures $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and 1-cells $P_1 = (|P_1|, \overline{P_1}, \nu_1, \xi_1) : \mathbf{A} \nrightarrow \mathbf{B}$ and $P_2 = (|P_2|, \overline{P_2}, \nu_2, \xi_2) : \mathbf{B} \nrightarrow \mathbf{C}$ in **CProf**, their horizontal composite is defined as

$$P_2 P_1 := (|P_2|\,|P_1|, \overline{P_2}\,\overline{P_1}, \nu_2 \star \nu_1, \xi_2 \star \xi_1)$$

where $|P_2|\,|P_1|$ is the horizontal composition in **ProfG**, $\overline{P_2}\overline{P_1}$ is the horizontal composition in **CAT** and the natural transformations $\nu_2 \star \nu_1, \xi_2 \star \xi_1$ are given by:

$$\nu_2 \star \nu_1 := \begin{tikzcd}
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[r, "|P_1|^{\#}"] \arrow[rr, bend left=50, "(|P_2|\,|P_1|)^{\#}"] & \widehat{\mathbf{c}\,|\mathbf{B}|} \arrow[r, "|P_2|^{\#}"] \arrow[u, phantom, "\Downarrow \varphi_2^{-1}"] & \widehat{\mathbf{c}\,|\mathbf{C}|} \\
\mathcal{D}(\mathbf{A}) \arrow[u, "F_{\mathbf{A}}"] \arrow[r, "\overline{P_1}"'] \arrow[ur, phantom, "\Downarrow \nu_1"] & \mathcal{D}(\mathbf{B}) \arrow[u, "F_{\mathbf{B}}"'] \arrow[r, "\overline{P_2}"'] \arrow[ur, phantom, "\Downarrow \nu_2"] & \mathcal{D}(\mathbf{C}) \arrow[u, "F_{\mathbf{C}}"']
\end{tikzcd}$$

$$\xi_2 \star \xi_1 := \begin{tikzcd}
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[r, "P_1^{\#}"] \arrow[rrr, bend left=50, "(P_2P_1)^{\#}"] \arrow[rr, phantom, shift left=6, "\Downarrow \varphi_2^{-1}"] & \widehat{\mathbf{c}\,|\mathbf{B}|} \arrow[r, dashed, "\mathrm{id}^{\#}" description] \arrow[r, bend left=60, "\mathrm{id}"] \arrow[r, bend right=60, "\downarrow^{\#}_{|\mathbf{B}|}"'] \arrow[r, phantom, shift left=3, "\Downarrow \varphi_0"] \arrow[r, phantom, shift right=3, "\Downarrow \eta^{\#}_{|\mathbf{B}|}"] & \widehat{\mathbf{c}\,|\mathbf{B}|} \arrow[r, "P_2^{\#}"] & \widehat{\mathbf{c}\,|\mathbf{C}|} \arrow[r, "\downarrow^{\#}_{|\mathbf{C}|}"] & \widehat{\mathbf{c}\,|\mathbf{C}|} \\
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[u, "\downarrow^{\#}_{|\mathbf{A}|}"] \arrow[rr, phantom, "\Downarrow \xi_1"] & & \widehat{\mathbf{c}\,|\mathbf{B}|} \arrow[u, "\downarrow^{\#}_{|\mathbf{B}|}"'] \arrow[rr, phantom, "\Downarrow \xi_2"] & & \widehat{\mathbf{c}\,|\mathbf{C}|} \arrow[u, "\downarrow^{\#}_{|\mathbf{C}|}"'] \\
\mathcal{D}(\mathbf{A}) \arrow[u, "F_{\mathbf{A}}"] \arrow[rr, "\overline{P_1}"'] & & \mathcal{D}(\mathbf{B}) \arrow[u, "F_{\mathbf{B}}"'] \arrow[rr, "\overline{P_2}"'] & & \mathcal{D}(\mathbf{C}) \arrow[u, "F_{\mathbf{C}}"']
\end{tikzcd}$$

Proof. We need to verify that this operation is well-defined, i.e. that $\nu_2 \star \nu_1$ and $\xi_2 \star \xi_1$ verify the axioms in Definition 4.4.10.

1. retraction axiom: substituting $\nu_2 \star \nu_1$ and $\xi_2 \star \xi_1$ into the first pasting diagram in Definition 4.4.10, we obtain the following diagram:

$$\begin{tikzcd}
& & \mathcal{D}(\mathbf{B}) \arrow[rr, "\overline{P_2}"] \arrow[dr, "F_{\mathbf{B}}"] \arrow[d, phantom, "\Downarrow \nu_1^{-1}"] & & \mathcal{D}(\mathbf{C}) \arrow[dr, "F_{\mathbf{C}}"] \arrow[d, phantom, "\Downarrow \nu_2^{-1}"] & & \\
\mathcal{D}(\mathbf{A}) \arrow[urr, "\overline{P_1}"] \arrow[ddrr, "\overline{P_1}"'] \arrow[r, "F_{\mathbf{A}}"] & \widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[r, "\mathrm{id}^{\#}" description] \arrow[r, bend left=60, "\mathrm{id}"] \arrow[r, bend right=60, "\downarrow^{\#}_{|\mathbf{A}|}"'] \arrow[r, phantom, shift left=3, "\Downarrow \varphi^0"] \arrow[r, phantom, shift right=3, "\Downarrow \eta^{\#}_{|\mathbf{A}|}"] & \widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[r, "|P_1|^{\#}"] \arrow[d, phantom, "\Downarrow \xi_1"] & \widehat{\mathbf{c}\,|\mathbf{B}|} \arrow[r, "\mathrm{id}^{\#}" description] \arrow[r, bend left=60, "\mathrm{id}"] \arrow[r, bend right=60, "\downarrow^{\#}_{|\mathbf{B}|}"'] \arrow[r, phantom, shift left=3, "\Downarrow \varphi^0"] \arrow[r, phantom, shift right=3, "\Downarrow \eta^{\#}_{|\mathbf{B}|}"] & \widehat{\mathbf{c}\,|\mathbf{B}|} \arrow[r, "|P_2|^{\#}"] \arrow[d, phantom, "\Downarrow \xi_2"] & \widehat{\mathbf{c}\,|\mathbf{C}|} \arrow[r, "\downarrow^{\#}_{|\mathbf{A}|}"] & \widehat{\mathbf{c}\,|\mathbf{C}|} \\
& & & \widehat{\mathbf{c}\,|\mathbf{B}|} \arrow[ur, "\downarrow^{\#}_{|\mathbf{B}|}"'] & & & \\
& & \mathcal{D}(\mathbf{B}) \arrow[ur, "F_{\mathbf{B}}"'] \arrow[rr, "\overline{P_2}"'] & & \mathcal{D}(\mathbf{C}) \arrow[uurr, "F_{\mathbf{C}}"'] & &
\end{tikzcd}$$

The left part of the pasting diagram colored in yellow is equal to the identity 2-cell since P_1 is in **CProf** and the right part colored in red is equal to the identity since P_2 is in **CProf**.

2. commutation axiom: substituting $\xi_2 \star \xi_1$ into the second pasting diagram in Definition 4.4.10, we obtain the following diagram:

$$\begin{tikzcd}
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[r, "P_1^{\#}"] \arrow[rrr, bend left=50, "(P_2P_1)^{\#}"] & \widehat{\mathbf{c}\,|\mathbf{B}|} \arrow[r, "\mathrm{id}^{\#}" description] \arrow[r, bend left=60, "\mathrm{id}"] \arrow[r, bend right=60, "\downarrow^{\#}_{|\mathbf{B}|}"'] \arrow[r, phantom, shift left=3, "\Downarrow \varphi_0"] \arrow[r, phantom, shift right=3, "\Downarrow \eta^{\#}_{|\mathbf{B}|}"] \arrow[u, phantom, "\Downarrow \varphi_2^{-1}"] & \widehat{\mathbf{c}\,|\mathbf{B}|} \arrow[r, "P_2^{\#}"] & \widehat{\mathbf{c}\,|\mathbf{C}|} \arrow[r, "\downarrow^{\#}_{|\mathbf{C}|}"] & \widehat{\mathbf{c}\,|\mathbf{C}|} \arrow[r, "\downarrow^{\#}_{|\mathbf{C}|}"] \arrow[dr, phantom, "\Downarrow \varphi_2"] & \widehat{\mathbf{c}\,|\mathbf{C}|} \\
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[u, "\downarrow^{\#}_{|\mathbf{A}|}"] \arrow[r, phantom, "\Downarrow \xi_1"] & {} & \widehat{\mathbf{c}\,|\mathbf{B}|} \arrow[u, "\downarrow^{\#}_{|\mathbf{B}|}"'] \arrow[r, phantom, "\Downarrow \xi_2"] & {} & \widehat{\mathbf{c}\,|\mathbf{C}|} \arrow[u, "\downarrow^{\#}_{|\mathbf{C}}"'] \arrow[ur, "(\downarrow_{|\mathbf{C}|}\downarrow_{|\mathbf{C}|})^{\#}" description] \arrow[ur, bend right=60, "\downarrow^{\#}_{|\mathbf{C}|}"'] \arrow[r, phantom, "\Downarrow \mu^{\#}_{|\mathbf{C}|}"] & {} \\
\mathcal{D}(\mathbf{A}) \arrow[u, "F_{\mathbf{A}}"] \arrow[rr, "\overline{P_1}"'] & & \mathcal{D}(\mathbf{B}) \arrow[u, "F_{\mathbf{B}}"'] \arrow[rr, "\overline{P_2}"'] & & \mathcal{D}(\mathbf{C}) \arrow[u, "F_{\mathbf{C}}"'] &
\end{tikzcd}$$

Since P_2 is in **CProf**, the diagram above is equal to:

$$\begin{tikzcd}
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[r, "P_1^{\#}"] \arrow[rrr, bend left=60, "(P_2P_1)^{\#}"] \arrow[rrr, phantom, bend left=40, "\Downarrow \varphi_2^{-1}"] & \widehat{\mathbf{c}\,|\mathbf{B}|} \arrow[r, bend left=50, "\mathrm{id}"] \arrow[r, dashed, "\mathrm{id}^{\#}" description] \arrow[r, bend right=50, "\downarrow^{\#}_{|\mathbf{B}|}"'] \arrow[r, phantom, bend left=25, "\Downarrow \varphi_0"] \arrow[r, phantom, bend right=25, "\Downarrow \eta^{\#}_{|\mathbf{B}|}"] & \widehat{\mathbf{c}\,|\mathbf{B}|} \arrow[r, "P_2^{\#}"] & \widehat{\mathbf{c}\,|\mathbf{C}|} \arrow[r, "\downarrow^{\#}_{|\mathbf{C}|}"] \arrow[rr, bend right=20, "(\downarrow_{|\mathbf{C}|}\downarrow_{|\mathbf{C}|})^{\#}" description] \arrow[rr, bend right=60, "\downarrow^{\#}_{|\mathbf{C}|}"'] \arrow[rr, phantom, bend right=10, "\Downarrow \varphi_2"] \arrow[rr, phantom, bend right=40, "\Downarrow \mu^{\#}_{|\mathbf{C}|}"] & \widehat{\mathbf{c}\,|\mathbf{C}|} \arrow[r, "\downarrow^{\#}_{|\mathbf{C}|}"] & \widehat{\mathbf{c}\,|\mathbf{C}|} \\
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[u, "\downarrow^{\#}_{|\mathbf{A}|}"] & \Downarrow \xi_1 & \widehat{\mathbf{c}\,|\mathbf{B}|} \arrow[u, "\downarrow^{\#}_{|\mathbf{B}|}"'] & \Downarrow \xi_2 & & \widehat{\mathbf{c}\,|\mathbf{C}|} \arrow[u, "\downarrow^{\#}_{|\mathbf{C}|}"'] \\
\mathcal{D}(\mathbf{A}) \arrow[u, "F_{\mathbf{A}}"] \arrow[rr, "\overline{P_1}"'] & & \mathcal{D}(\mathbf{B}) \arrow[u, "F_{\mathbf{B}}"'] \arrow[rrr, "\overline{P_2}"'] & & & \mathcal{D}(\mathbf{C}) \arrow[u, "F_{\mathbf{C}}"']
\end{tikzcd}$$

which implies the desired equality. ☐

Identity 1-cell

For a closed structure $\mathbf{A}$, the identity 1-cell is defined as

$$\mathrm{id}_{\mathbf{A}} := (\mathrm{id}_{\mathbf{c}\,|\mathbf{A}|}, \mathrm{id}_{\mathcal{D}\mathbf{A}}, \nu_{\mathbf{A}}, \xi_{\mathbf{A}})$$

where $\mathrm{id}_{\mathbf{c}\,|\mathbf{A}|}$ is the identity profunctor in $\mathbf{ProfG}(\mathbf{c}\,|\mathbf{A}|, \mathbf{c}\,|\mathbf{A}|)$, $\mathrm{id}_{\mathcal{D}\mathbf{A}} : \mathcal{D}(\mathbf{A}) \to \mathcal{D}(\mathbf{A})$ is the identity functor in $\mathbf{CAT}$ and the 2-cells $\nu_{\mathbf{A}}$ and $\xi_{\mathbf{A}}$ are defined as:

$$\nu_{\mathbf{A}} := \begin{tikzcd}
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[r, bend left=50, "\mathrm{id}^{\#}"] \arrow[r, bend right=50, "\mathrm{id}"'] \arrow[r, phantom, "\Downarrow \varphi_0^{-1}"] & \widehat{\mathbf{c}\,|\mathbf{A}|} \\
\mathcal{D}(\mathbf{A}) \arrow[u, "F_{\mathbf{A}}"] \arrow[r, "\mathrm{id}"'] \arrow[ur, phantom, "="] & \mathcal{D}(\mathbf{A}) \arrow[u, "F_{\mathbf{A}}"']
\end{tikzcd}
\qquad
\xi_{\mathbf{A}} := \begin{tikzcd}
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[r, bend left=50, "\mathrm{id}^{\#}"] \arrow[r, bend right=50, "\mathrm{id}"'] \arrow[r, phantom, "\Downarrow \varphi_0^{-1}"] & \widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[r, "\downarrow^{\#}_{|\mathbf{A}|}"] & \widehat{\mathbf{c}\,|\mathbf{A}|} \\
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[u, "\downarrow^{\#}_{|\mathbf{A}|}"] \arrow[urr, "(\downarrow_{|\mathbf{A}|}\downarrow_{|\mathbf{A}|})^{\#}" description] \arrow[rr, "\mathrm{id}"'] \arrow[ur, phantom, "\Downarrow \varphi_2"] & \Downarrow \mu^{\#}_{|\mathbf{A}|} & \widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[u, "\downarrow^{\#}_{|\mathbf{A}|}"'] \\
\mathcal{D}(\mathbf{A}) \arrow[u, "F_{\mathbf{A}}"] \arrow[rr, "\mathrm{id}"'] & = & \mathcal{D}(\mathbf{A}) \arrow[u, "F_{\mathbf{A}}"']
\end{tikzcd}$$

Proof. We need to show that it is indeed a 1-cell in $\mathbf{CProf}$.

1. retraction axiom: substituting $\nu_{\mathbf{A}}$ and $\xi_{\mathbf{A}}$ into the first coherence diagram of Definition 4.4.10, we obtain the following pasting diagram:

$$\begin{tikzcd}[column sep=large]
\mathcal{D}(\mathbf{A}) \arrow[r, "F_{\mathbf{A}}"] & \widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[r, "\mathrm{id}^\#" description] \arrow[r, bend left=60, "\mathrm{id}"] \arrow[r, bend left=30, phantom, "\Downarrow \varphi_0"] \arrow[r, bend right=60, "\downarrow^\#_{|\mathbf{A}|}"'] \arrow[r, bend right=30, phantom, "\Downarrow \eta^\#_{|\mathbf{A}|}"] \arrow[rrr, bend right=40, "(\downarrow_{|\mathbf{A}|}\downarrow_{|\mathbf{A}|})^\#" description] \arrow[rrr, bend right=90, "\downarrow^\#_{|\mathbf{A}|}"'] \arrow[rrr, bend right=65, phantom, "\Downarrow \mu^\#_{|\mathbf{A}|}"] & \widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[r, "\mathrm{id}^\#" description] \arrow[r, bend left=60, "\mathrm{id}"] \arrow[r, bend left=30, phantom, "\Downarrow \varphi_0"] \arrow[r, bend right=60, "\mathrm{id}"'] \arrow[r, bend right=30, phantom, "\Downarrow \varphi_0^{-1}"] \arrow[r, bend right=90, phantom, "\Downarrow \varphi_2"] & \widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[r, "\downarrow^\#_{|\mathbf{A}|}"] & \widehat{\mathbf{c}\,|\mathbf{A}|}
\end{tikzcd}$$

and we want to show that it is equal to the identity. Using the naturality of φ_2, we have $\varphi_2(\mathrm{Id} * \eta^\#_{|\mathbf{A}|}) = (\mathrm{Id} * \eta_{|\mathbf{A}|})^\# \varphi_2$ so the pasting diagram above is equal to:

$$\begin{tikzcd}[column sep=large]
\mathcal{D}(\mathbf{A}) \arrow[r, "F_{\mathbf{A}}"] & \widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[r, "\mathrm{id}^\#" description] \arrow[r, bend left=60, "\mathrm{id}"] \arrow[r, bend left=30, phantom, "\Downarrow \varphi_0"] \arrow[rr, bend right=30, "(\mathrm{id} \downarrow_{|\mathbf{A}|})^\#" description] \arrow[rr, bend right=15, phantom, "\Downarrow \varphi_2"] \arrow[rr, bend right=50, phantom, "\Downarrow (\mathrm{Id} * \eta_{|\mathbf{A}|})^\#"] \arrow[rr, bend right=70, "(\downarrow_{|\mathbf{A}|}\downarrow_{|\mathbf{A}|})^\#" description] \arrow[rr, bend right=90, phantom, "\Downarrow \mu^\#_{|\mathbf{A}|}"] \arrow[rr, bend right=110, "\downarrow^\#_{|\mathbf{A}|}"'] & \widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[r, "\downarrow^\#_{|\mathbf{A}|}"] & \widehat{\mathbf{c}\,|\mathbf{A}|}
\end{tikzcd}$$

Using the axioms of an internal monad (Definition 1.3.9), we have $\mu_{|\mathbf{A}|}(\mathrm{Id} * \eta_{|A|}) = r$ so the pasting diagram is now equal to:

$$\begin{tikzcd}[column sep=large]
\mathcal{D}(\mathbf{A}) \arrow[r, "F_{\mathbf{A}}"] & \widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[r, "\mathrm{id}^{\#}"] \arrow[r, bend left=60, "\mathrm{id}"] \arrow[r, phantom, bend left=30, "\Downarrow \varphi_0"] \arrow[rr, bend right=30, "(\mathrm{id} \downarrow_{|\mathbf{A}|})^{\#}"'] \arrow[rr, bend right=70, "\downarrow^{\#}_{|\mathbf{A}|}"'] \arrow[rr, phantom, bend right=15, "\Downarrow \varphi_2"] \arrow[rr, phantom, bend right=50, "\Downarrow r^{\#}"] & \widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[r, "\downarrow^{\#}_{|\mathbf{A}|}"] & \widehat{\mathbf{c}\,|\mathbf{A}|}
\end{tikzcd}$$

This last diagram is equal to the identity 2-cell from the right unity axiom for pseudo-functors (Definition 1.3.2).

2. commutation axiom: substituting $\xi_{\mathbf{A}}$ into the second pasting diagram in Definition 4.4.10, we obtain the following diagram:

$$\begin{tikzcd}[column sep=large]
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[r, bend left=50, "\mathrm{id}^{\#}"] \arrow[r, bend right=50, "\mathrm{id}"'] \arrow[r, phantom, "\Downarrow \varphi_0^{-1}"] & \widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[r, "\downarrow^{\#}_{|\mathbf{A}|}"] & \widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[r, "\downarrow^{\#}_{|\mathbf{A}|}"] & \widehat{\mathbf{c}\,|\mathbf{A}|} \\
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[u, "\downarrow^{\#}_{|\mathbf{A}|}"] \arrow[urr, "(\downarrow_{|\mathbf{A}|}\downarrow_{|\mathbf{A}|})^{\#}"'] \arrow[rr, "\mathrm{id}"'] \arrow[ur, phantom, "\Downarrow \varphi_2"] \arrow[rr, phantom, bend left=20, "\Downarrow \mu^{\#}_{|\mathbf{A}|}"] & & \widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[u, "\downarrow^{\#}_{|\mathbf{A}|}"] \arrow[ur, "(\downarrow_{|\mathbf{A}|}\downarrow_{|\mathbf{A}|})^{\#}"'] \arrow[ur, bend right=50, "\downarrow^{\#}_{|\mathbf{A}|}"'] \arrow[ur, phantom, bend left=30, "\Downarrow \varphi_2"] \arrow[ur, phantom, bend right=25, "\Downarrow \mu^{\#}_{|\mathbf{A}|}"] & \\
\mathcal{D}(\mathbf{A}) \arrow[u, "F_{\mathbf{A}}"] \arrow[rr, "\mathrm{id}"'] & & \mathcal{D}(\mathbf{A}) \arrow[u, "F_{\mathbf{A}}"'] &
\end{tikzcd}$$

We have:

$$\begin{aligned}
\mu^{\#}_{|\mathbf{A}|}\, \varphi_2\, (\mathrm{Id} * \mu^{\#}_{|\mathbf{A}|})(\mathrm{Id} * \varphi_2) &= \mu^{\#}_{|\mathbf{A}|}\, (\mathrm{Id} * \mu_{|\mathbf{A}|})^{\#} \varphi_2\, (\mathrm{Id} * \varphi_2) && (4.2)\\
&= \mu^{\#}_{|\mathbf{A}|}\, (\mathrm{Id} * \mu_{|\mathbf{A}|})^{\#}\, a^{\#}\, \varphi_2\, (\varphi_2 * \mathrm{Id}) && (4.3)\\
&= \mu^{\#}_{|\mathbf{A}|}\, (\mu_{|\mathbf{A}|} * \mathrm{Id})^{\#}\, \varphi_2\, (\varphi_2 * \mathrm{Id}) && (4.4)\\
&= \mu^{\#}_{|\mathbf{A}|}\, \varphi_2\, (\mathrm{Id} * \mu^{\#}_{|\mathbf{A}|})\, (\varphi_2 * \mathrm{Id}) && (4.5)
\end{aligned}$$

The first equality follows from the naturality of φ_2, the second from the associativity axiom for pseudo-functors (Definition 1.3.2), the third from the axioms of internal monads (Definition 1.3.9) and the last one from the naturality of φ_2 again. The pasting diagram above is therefore equal to:

$$\begin{tikzcd}
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[r, "\mathrm{id}^{\#}"] & \widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[r, "\downarrow^{\#}_{|\mathbf{A}|}"] \arrow[rr, bend right=40, "(\downarrow_{|\mathbf{A}|}\downarrow_{|\mathbf{A}|})^{\#}" description] & \widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[r, "\downarrow^{\#}_{|\mathbf{A}|}"] \arrow[d, phantom, "\Downarrow \varphi_2"] & \widehat{\mathbf{c}\,|\mathbf{A}|} \\
& \Downarrow \varphi_2 & \Downarrow \mu^{\#}_{|\mathbf{A}|} & \Downarrow \mu^{\#}_{|\mathbf{A}|} \\
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[uu, "\downarrow^{\#}_{|\mathbf{A}|}"] \arrow[rrr, "\mathrm{id}"'] \arrow[rrruu, bend right=20, "\downarrow^{\#}_{|\mathbf{A}|}", "(\downarrow_{|\mathbf{A}|}\downarrow_{|\mathbf{A}|})^{\#}"'] & & & \widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[uu, "\downarrow^{\#}_{|\mathbf{A}|}"'] \\
& & = & \\
\mathcal{D}(\mathbf{A}) \arrow[uu, "F_{\mathbf{A}}"] \arrow[rrr, "\mathrm{id}"'] & & & \mathcal{D}(\mathbf{A}) \arrow[uu, "F_{\mathbf{A}}"']
\end{tikzcd}$$

which implies the desired result. $\square$

Horizontal composition of 2-cells

For 2-cells $\alpha \in \mathbf{CProf}(\mathbf{A}, \mathbf{B})(P_1, Q_1)$ and $\beta \in \mathbf{CProf}(\mathbf{B}, \mathbf{C})(P_2, Q_2)$, define their horizontal composite as

$$\beta * \alpha := (|\beta| * |\alpha|, \overline{\beta} * \overline{\alpha})$$

where $|\beta| * |\alpha|$ is the horizontal composition in **ProfG** and $\overline{\beta} * \overline{\alpha}$ is the horizontal composition in **CAT**.

Proof. We need to check that $\beta * \alpha$ is a 2-cell in **CProf**, i.e. that the following diagrams are equal:

$$\begin{tikzcd}[column sep=large, row sep=large]
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[r, bend left=40, "(Q_1P_1)^{\#}"] \arrow[r, bend right=40, "(Q_2P_2)^{\#}"'] \arrow[r, phantom, "\Downarrow (\beta * \alpha)^{\#}"] & \widehat{\mathbf{c}\,|\mathbf{C}|} & & \widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[r, bend left=40, "(Q_1P_1)^{\#}"] \arrow[dr, phantom, "\Downarrow (\tau_1 \star \nu_1)"] & \widehat{\mathbf{c}\,|\mathbf{C}|} \\
\mathcal{D}(\mathbf{A}) \arrow[u, "F_{\mathbf{A}}"] \arrow[r, bend right=40, "\overline{Q_2P_2}"'] \arrow[ur, phantom, "\Downarrow (\tau_2 \star \nu_2)"] & \mathcal{D}(\mathbf{C}) \arrow[u, "F_{\mathbf{C}}"'] & = & \mathcal{D}(\mathbf{A}) \arrow[u, "F_{\mathbf{A}}"] \arrow[r, bend left=40, "\overline{Q_1P_1}"] \arrow[r, bend right=40, "\overline{Q_2P_2}"'] \arrow[r, phantom, "\Downarrow \overline{\beta} * \overline{\alpha}"] & \mathcal{D}(\mathbf{C}) \arrow[u, "F_{\mathbf{C}}"']
\end{tikzcd}$$

We unfold $\tau_2 \star \nu_2$ in the first diagram and use the equality $\varphi_2^{-1}(\beta * \alpha)^{\#} = (\beta^{\#} * \alpha^{\#})\varphi_2^{-1}$ from the naturality of φ_2 and obtain that the first diagram is equal to:

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[rr, bend left=70, "(Q_1P_1)^{\#}"] \arrow[rr, bend left=35, "(Q_2P_2)^{\#}"] \arrow[rr, phantom, shift left=9ex, "\Downarrow (\beta * \alpha)^{\#}"] \arrow[rr, phantom, shift left=3ex, "\Downarrow \varphi_2^{-1}"] \arrow[r, bend right, "P_2^{\#}"'] \arrow[dr, phantom, "\Downarrow \nu_2"] & \widehat{\mathbf{c}\,|\mathbf{B}|} \arrow[r, bend right, "Q_2^{\#}"'] \arrow[dr, phantom, "\Downarrow \tau_2"] & \widehat{\mathbf{c}\,|\mathbf{C}|} \\
\mathcal{D}(\mathbf{A}) \arrow[u, "F_{\mathbf{A}}"] \arrow[r, bend right, "\overline{P_2}"'] & \mathcal{D}(\mathbf{B}) \arrow[u, "F_{\mathbf{B}}"'] \arrow[r, bend right, "\overline{Q_2}"'] & \mathcal{D}(\mathbf{C}) \arrow[u, "F_{\mathbf{C}}"']
\end{tikzcd}
=
\begin{tikzcd}[column sep=huge, row sep=huge]
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[rr, bend left=70, "(Q_1P_1)^{\#}"] \arrow[rr, phantom, shift left=6ex, "\Downarrow \varphi_2^{-1}"] \arrow[r, bend left, "P_1^{\#}"] \arrow[r, phantom, "\Downarrow \alpha^{\#}"] \arrow[r, bend right, "P_2^{\#}"'] \arrow[dr, phantom, "\Downarrow \nu_2"] & \widehat{\mathbf{c}\,|\mathbf{B}|} \arrow[r, bend left, "Q_1^{\#}"] \arrow[r, phantom, "\Downarrow \beta^{\#}"] \arrow[r, bend right, "Q_2^{\#}"'] \arrow[dr, phantom, "\Downarrow \tau_2"] & \widehat{\mathbf{c}\,|\mathbf{C}|} \\
\mathcal{D}(\mathbf{A}) \arrow[u, "F_{\mathbf{A}}"] \arrow[r, bend right, "\overline{P_2}"'] & \mathcal{D}(\mathbf{B}) \arrow[u, "F_{\mathbf{B}}"'] \arrow[r, bend right, "\overline{Q_2}"'] & \mathcal{D}(\mathbf{C}) \arrow[u, "F_{\mathbf{C}}"']
\end{tikzcd}$$

We now use the fact that the 2-cell α is in $\mathbf{CProf}(\mathbf{A},\mathbf{B})(P_1, P_2)$ and β is in $\mathbf{CProf}(\mathbf{B},\mathbf{C})(Q_1, Q_2)$ to obtain that the diagram above is equal to:

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[rr, bend left=70, "(Q_1P_1)^{\#}"] \arrow[rr, phantom, shift left=6ex, "\Downarrow \varphi_2^{-1}"] \arrow[r, bend left, "P_1^{\#}"] \arrow[dr, phantom, "\Downarrow \nu_1"] & \widehat{\mathbf{c}\,|\mathbf{B}|} \arrow[r, bend left, "Q_1^{\#}"] \arrow[dr, phantom, "\Downarrow \tau_1"] & \widehat{\mathbf{c}\,|\mathbf{C}|} \\
\mathcal{D}(\mathbf{A}) \arrow[u, "F_{\mathbf{A}}"] \arrow[r, bend left, "\overline{P_1}"] \arrow[r, phantom, "\Downarrow \overline{\alpha}"] \arrow[r, bend right, "\overline{P_2}"'] & \mathcal{D}(\mathbf{B}) \arrow[u, "F_{\mathbf{B}}"'] \arrow[r, bend left, "\overline{Q_1}"] \arrow[r, phantom, "\Downarrow \overline{\beta}"] \arrow[r, bend right, "\overline{Q_2}"'] & \mathcal{D}(\mathbf{C}) \arrow[u, "F_{\mathbf{C}}"']
\end{tikzcd}
=
\begin{tikzcd}[column sep=huge, row sep=huge]
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[r, bend left, "(Q_1P_1)^{\#}"] \arrow[d, phantom, shift left=14ex, "\Downarrow (\tau_1 \star \nu_1)"] & \widehat{\mathbf{c}\,|\mathbf{C}|} \\
\mathcal{D}(\mathbf{A}) \arrow[u, "F_{\mathbf{A}}"] \arrow[r, bend left, "\overline{Q_1P_1}"] \arrow[r, phantom, "\Downarrow \overline{\beta} * \overline{\alpha}"] \arrow[r, bend right, "\overline{Q_2P_2}"'] & \mathcal{D}(\mathbf{C}) \arrow[u, "F_{\mathbf{C}}"']
\end{tikzcd}$$

which implies the desired equality. We also need to show:

$$\begin{tikzcd}
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[r, bend left, "(|Q_1|\,|P_1|)^{\#}"] \arrow[r, bend right, "(|Q_2|\,|P_2|)^{\#}"'] \arrow[r, phantom, "\Downarrow (|\beta| * |\alpha|)^{\#}"] & \widehat{\mathbf{c}\,|\mathbf{C}|} \arrow[r, "\downarrow^{\#}_{|\mathbf{C}|}"] & \widehat{\mathbf{c}\,|\mathbf{C}|} \\
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[u, "\downarrow^{\#}_{|\mathbf{A}|}"] & \Downarrow \chi_2 \star \xi_2 & \widehat{\mathbf{c}\,|\mathbf{C}|} \arrow[u, "\downarrow^{\#}_{|\mathbf{C}|}"'] \\
\mathcal{D}(\mathbf{A}) \arrow[u, "F_{\mathbf{A}}"] \arrow[rr, bend right, "\overline{Q_2 P_2}"'] & & \mathcal{D}(\mathbf{C}) \arrow[u, "F_{\mathbf{C}}"']
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[r, bend left, "(|Q_1|\,|P_1|)^{\#}"] & \widehat{\mathbf{c}\,|\mathbf{C}|} \arrow[r, "\downarrow^{\#}_{|\mathbf{C}|}"] & \widehat{\mathbf{c}\,|\mathbf{C}|} \\
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[u, "\downarrow^{\#}_{|\mathbf{A}|}"] & \Downarrow \chi_1 \star \xi_1 & \widehat{\mathbf{c}\,|\mathbf{C}|} \arrow[u, "\downarrow^{\#}_{|\mathbf{C}|}"'] \\
\mathcal{D}(\mathbf{A}) \arrow[u, "F_{\mathbf{A}}"] \arrow[rr, bend left, "\overline{Q_1 P_1}"] \arrow[rr, bend right, "\overline{Q_2 P_2}"'] \arrow[rr, phantom, "\Downarrow \overline{\beta} * \overline{\alpha}"] & & \mathcal{D}(\mathbf{C}) \arrow[u, "F_{\mathbf{C}}"']
\end{tikzcd}$$

Unfolding the left-hand pasting diagram, we obtain:

$$\begin{tikzcd}
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[rrr, bend left=60, "(|P_2|\,|P_1|)^{\#}"] \arrow[rrr, bend left=45, phantom, "\Downarrow (|\beta| * |\alpha|)^{\#}"] \arrow[rrr, bend left=30, "(|Q_2|\,|P_2|)^{\#}"] \arrow[rr, bend left=20, phantom, "\Downarrow \varphi_2^{-1}"] \arrow[r, "|P_2|^{\#}"] & \widehat{\mathbf{c}\,|\mathbf{B}|} \arrow[r, bend left, "\mathrm{id}"] \arrow[r, dashed, "\mathrm{id}^{\#}" description] \arrow[r, bend right, "\downarrow^{\#}_{|\mathbf{B}|}"'] & \widehat{\mathbf{c}\,|\mathbf{B}|} \arrow[r, "|Q_2|^{\#}"] & \widehat{\mathbf{c}\,|\mathbf{C}|} \arrow[r, "\downarrow^{\#}_{|\mathbf{C}|}"] & \widehat{\mathbf{c}\,|\mathbf{C}|} \\
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[u, "\downarrow^{\#}_{|\mathbf{A}|}"] & \Downarrow \varphi_0 \quad \Downarrow \eta^{\#}_{|\mathbf{B}|} & \widehat{\mathbf{c}\,|\mathbf{B}|} \arrow[u, "\downarrow^{\#}_{|\mathbf{B}|}"'] & & \widehat{\mathbf{c}\,|\mathbf{C}|} \arrow[u, "\downarrow^{\#}_{|\mathbf{C}|}"'] \\
 & \Downarrow \xi_2 & & \Downarrow \chi_2 & \\
\mathcal{D}(\mathbf{A}) \arrow[uu, "F_{\mathbf{A}}"] \arrow[rr, bend right, "\overline{P_2}"'] & & \mathcal{D}(\mathbf{B}) \arrow[uu, "F_{\mathbf{B}}"'] \arrow[rr, bend right, "\overline{Q_2}"'] & & \mathcal{D}(\mathbf{C}) \arrow[uu, "F_{\mathbf{C}}"']
\end{tikzcd}$$

Using the naturality of φ_2, the diagram above is equal to:

$$\begin{tikzcd}[column sep=large, row sep=large]
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[r, bend left, "|P_1|^{\#}"] \arrow[r, bend right, "|P_2|^{\#}"'] \arrow[r, phantom, "\Downarrow |\alpha|^{\#}"] \arrow[rrr, bend left=60, "(|P_2|\,|P_1|)^{\#}"] \arrow[rrr, phantom, bend left=40, "\Downarrow \varphi_2^{-1}"] & \widehat{\mathbf{c}\,|\mathbf{B}|} \arrow[r, bend left, "\mathrm{id}"] \arrow[r, "\mathrm{id}^{\#}" description] \arrow[r, bend right, "\downarrow^{\#}_{|\mathbf{B}|}"'] \arrow[r, phantom, shift left=3, "\Downarrow \varphi_0"] \arrow[r, phantom, shift right=3, "\Downarrow \eta^{\#}_{|\mathbf{B}|}"] & \widehat{\mathbf{c}\,|\mathbf{B}|} \arrow[r, bend left, "|Q_1|^{\#}"] \arrow[r, bend right, "|Q_2|^{\#}"'] \arrow[r, phantom, "\Downarrow |\beta|^{\#}"] & \widehat{\mathbf{c}\,|\mathbf{C}|} \arrow[r, "\downarrow^{\#}_{|\mathbf{C}|}"] & \widehat{\mathbf{c}\,|\mathbf{C}|} \\
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[u, "\downarrow^{\#}_{|\mathbf{A}|}"] \arrow[rr, phantom, "\Downarrow \xi_2"] & & \widehat{\mathbf{c}\,|\mathbf{B}|} \arrow[u, "\downarrow^{\#}_{|\mathbf{B}|}"'] \arrow[rr, phantom, "\Downarrow \chi_2"] & & \widehat{\mathbf{c}\,|\mathbf{C}|} \arrow[u, "\downarrow^{\#}_{|\mathbf{C}|}"'] \\
\mathcal{D}(\mathbf{A}) \arrow[u, "F_{\mathbf{A}}"] \arrow[rr, bend right, "\overline{P_2}"'] & & \mathcal{D}(\mathbf{B}) \arrow[u, "F_{\mathbf{B}}"'] \arrow[rr, bend right, "\overline{Q_2}"'] & & \mathcal{D}(\mathbf{C}) \arrow[u, "F_{\mathbf{C}}"']
\end{tikzcd}$$

We now use the fact that the 2-cell α is in $\mathbf{CProf}(\mathbf{A}, \mathbf{B})(P_1, P_2)$ and β is in $\mathbf{CProf}(\mathbf{B}, \mathbf{C})(Q_1, Q_2)$ to obtain that the diagram above is equal to:

$$\begin{tikzcd}[column sep=large, row sep=large]
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[r, "|P_1|^{\#}"] \arrow[rrr, bend left=60, "(|P_2|\,|P_1|)^{\#}"] \arrow[rrr, phantom, bend left=40, "\Downarrow \varphi_2^{-1}"] & \widehat{\mathbf{c}\,|\mathbf{B}|} \arrow[r, bend left, "\mathrm{id}"] \arrow[r, "\mathrm{id}^{\#}" description] \arrow[r, bend right, "\downarrow^{\#}_{|\mathbf{B}|}"'] \arrow[r, phantom, shift left=3, "\Downarrow \varphi_0"] \arrow[r, phantom, shift right=3, "\Downarrow \eta^{\#}_{|\mathbf{B}|}"] & \widehat{\mathbf{c}\,|\mathbf{B}|} \arrow[r, "|Q_1|^{\#}"] & \widehat{\mathbf{c}\,|\mathbf{C}|} \arrow[r, "\downarrow^{\#}_{|\mathbf{C}|}"] & \widehat{\mathbf{c}\,|\mathbf{C}|} \\
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[u, "\downarrow^{\#}_{|\mathbf{A}|}"] \arrow[rr, phantom, "\Downarrow \xi_1"] & & \widehat{\mathbf{c}\,|\mathbf{B}|} \arrow[u, "\downarrow^{\#}_{|\mathbf{B}|}"'] \arrow[rr, phantom, "\Downarrow \chi_1"] & & \widehat{\mathbf{c}\,|\mathbf{C}|} \arrow[u, "\downarrow^{\#}_{|\mathbf{C}|}"'] \\
\mathcal{D}(\mathbf{A}) \arrow[u, "F_{\mathbf{A}}"] \arrow[rr, bend left, "\overline{P_1}"] \arrow[rr, bend right, "\overline{P_2}"'] \arrow[rr, phantom, "\Downarrow \overline{\alpha}"] & & \mathcal{D}(\mathbf{B}) \arrow[u, "F_{\mathbf{B}}"'] \arrow[rr, bend left, "\overline{Q_1}"] \arrow[rr, bend right, "\overline{Q_2}"'] \arrow[rr, phantom, "\Downarrow \overline{\beta}"] & & \mathcal{D}(\mathbf{C}) \arrow[u, "F_{\mathbf{C}}"']
\end{tikzcd}$$

which induces the desired equality. □

Associator

Consider 1-cells $P_1 \in \mathbf{CProf}(\mathbf{A}, \mathbf{B})$, $P_2 \in \mathbf{CProf}(\mathbf{B}, \mathbf{C})$ and $P_3 \in \mathbf{CProf}(\mathbf{C}, \mathbf{D})$, we define the component of the associator a_{P_3,P_2,P_1} as

$$a_{P_3,P_2,P_1} := (a_{|P_3|,|P_2|,|P_1|}, \mathrm{Id}) : (P_3P_2)P_1 \Rightarrow P_3(P_2P_1)$$

where $a_{|P_3|,|P_2|,|P_1|}$ is the associator in $\mathbf{ProfG}((|P_3|\,|P_2|)\,|P_1|\,, |P_3|\,(|P_2|\,|P_1|))$ and Id is the identity 2-cell $(\overline{P_3}\,\overline{P_2})\overline{P_1} = \overline{P_3}(\overline{P_2}\,\overline{P_1})$ since $\mathbf{CAT}$ is a 2-category.

Proof. We show that a_{P_3,P_2,P_1} is indeed in $\mathbf{CProf}(\mathbf{A}, \mathbf{D})((P_3P_2)P_1, P_3(P_2P_1))$. We first show that the following diagrams are equal:

$$\begin{tikzcd}
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[r, bend left, "((|P_3|\,|P_2|)\,|P_1|)^{\#}"] \arrow[r, bend right, "(|P_3|\,(|P_2|\,|P_1|))^{\#}"'] \arrow[r, phantom, "\Downarrow a^{\#}"] & \widehat{\mathbf{c}\,|\mathbf{D}|} & & \widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[r, bend left, "((|P_3|\,|P_2|)\,|P_1|)^{\#}"] & \widehat{\mathbf{c}\,|\mathbf{D}|} \\
 & & = & & \\
\mathcal{D}(\mathbf{A}) \arrow[uu, "F_{\mathbf{A}}"] \arrow[r, bend right, "\overline{P_3}\,\overline{P_2}\,\overline{P_1}"'] \arrow[uur, phantom, "\Downarrow \nu_3 \star (\nu_2 \star \nu_1)"] & \mathcal{D}(\mathbf{D}) \arrow[uu, "F_{\mathbf{D}}"'] & & \mathcal{D}(\mathbf{A}) \arrow[uu, "F_{\mathbf{A}}"] \arrow[r, bend left, "\overline{P_3}\,\overline{P_2}\,\overline{P_1}"] \arrow[r, bend right, "\overline{P_3}\,\overline{P_2}\,\overline{P_1}"'] \arrow[r, phantom, "="] \arrow[uur, phantom, "\Downarrow (\nu_3 \star \nu_2) \star \nu_1"] & \mathcal{D}(\mathbf{D}) \arrow[uu, "F_{\mathbf{D}}"']
\end{tikzcd}$$

Unfolding the 2-cell $\nu_3 \star (\nu_2 \star \nu_1)$, we obtain that the left-hand diagram is equal to:

$$\begin{tikzcd}
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[r, "P_1^{\#}"'] \arrow[rr, bend left, "(|P_2|\,|P_1|)^{\#}"] \arrow[rr, phantom, bend left=20, "\Downarrow \varphi_2^{-1}"] \arrow[rrr, bend left=50, "(|P_3|\,(|P_2|\,|P_1|))^{\#}"] \arrow[rrr, phantom, bend left=35, "\Downarrow \varphi_2^{-1}"] \arrow[rrr, bend left=80, "((|P_3|\,|P_2|)\,|P_1|)^{\#}"] \arrow[rrr, phantom, bend left=65, "\Downarrow a^{\#}"] & \widehat{\mathbf{c}\,|\mathbf{B}|} \arrow[r, "P_2^{\#}"'] & \widehat{\mathbf{c}\,|\mathbf{C}|} \arrow[r, "P_3^{\#}"'] & \widehat{\mathbf{c}\,|\mathbf{D}|} \\
\mathcal{D}(\mathbf{A}) \arrow[u, "F_{\mathbf{A}}"] \arrow[r, "\overline{P_1}"'] \arrow[ur, phantom, "\Downarrow \nu_1"] & \mathcal{D}(\mathbf{B}) \arrow[u, "F_{\mathbf{B}}"'] \arrow[r, "\overline{P_2}"'] \arrow[ur, phantom, "\Downarrow \nu_2"] & \mathcal{D}(\mathbf{C}) \arrow[u, "F_{\mathbf{C}}"'] \arrow[r, "\overline{P_3}"'] \arrow[ur, phantom, "\Downarrow \nu_3"] & \mathcal{D}(\mathbf{D}) \arrow[u, "F_{\mathbf{D}}"']
\end{tikzcd}$$

Using the associativity axiom for pseudo-functors (Definition 1.3.2), we obtain that the diagram above is equal to:

$$
\begin{tikzcd}[column sep=large]
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[r, "P_1^{\#}"'] \arrow[rrr, bend left=50, "((|P_3|\,|P_2|)\,|P_1|)^{\#}"] \arrow[r, phantom, bend left=40, "\Downarrow \varphi_2^{-1}"] & \widehat{\mathbf{c}\,|\mathbf{B}|} \arrow[r, "P_2^{\#}"'] \arrow[rr, bend left=30, "(|P_3|\,|P_2|)^{\#}"] \arrow[rr, phantom, bend left=15, "\Downarrow \varphi_2^{-1}"] & \widehat{\mathbf{c}\,|\mathbf{C}|} \arrow[r, "P_3^{\#}"'] & \widehat{\mathbf{c}\,|\mathbf{D}|} \\
\mathcal{D}(\mathbf{A}) \arrow[u, "F_{\mathbf{A}}"] \arrow[r, "\overline{P_1}"'] \arrow[ur, phantom, "\Downarrow \nu_1"] & \mathcal{D}(\mathbf{B}) \arrow[u, "F_{\mathbf{B}}"'] \arrow[r, "\overline{P_2}"'] \arrow[ur, phantom, "\Downarrow \nu_2"] & \mathcal{D}(\mathbf{C}) \arrow[u, "F_{\mathbf{C}}"'] \arrow[r, "\overline{P_3}"'] \arrow[ur, phantom, "\Downarrow \nu_3"] & \mathcal{D}(\mathbf{D}) \arrow[u, "F_{\mathbf{D}}"']
\end{tikzcd}
=
\begin{tikzcd}[column sep=huge]
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[r, bend left, "((|P_3|\,|P_2|)\,|P_1|)^{\#}"] & \widehat{\mathbf{c}\,|\mathbf{D}|} \\
\mathcal{D}(\mathbf{A}) \arrow[u, "F_{\mathbf{A}}"] \arrow[r, bend left, "\overline{P_3}\,\overline{P_2}\,\overline{P_1}"] \arrow[r, bend right, "\overline{P_3}\,\overline{P_2}\,\overline{P_1}"'] \arrow[r, phantom, "="] \arrow[ur, phantom, "\Downarrow (\nu_3 \star \nu_2) \star \nu_1"] & \mathcal{D}(\mathbf{D}) \arrow[u, "F_{\mathbf{D}}"']
\end{tikzcd}
$$

which implies the desired equality. We show similarly that the following diagrams are equal

$$
\begin{tikzcd}[column sep=large]
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[r, bend left, "((|P_3|\,|P_2|)\,|P_1|)^{\#}"] \arrow[r, bend right, "(|P_3|\,(|P_2|\,|P_1|))^{\#}"'] \arrow[r, phantom, "\Downarrow a^{\#}"] & \widehat{\mathbf{c}\,|\mathbf{D}|} \arrow[r, "\downarrow^{\#}_{|\mathbf{D}|}"] & \widehat{\mathbf{c}\,|\mathbf{D}|} \\
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[u, "\downarrow^{\#}_{|\mathbf{A}|}"] \arrow[rr, phantom, "\Downarrow \xi_3 \star (\xi_2 \star \xi_1)"] & & \widehat{\mathbf{c}\,|\mathbf{D}|} \arrow[u, "\downarrow^{\#}_{|\mathbf{D}|}"'] \\
\mathcal{D}(\mathbf{A}) \arrow[u, "F_{\mathbf{A}}"] \arrow[rr, bend right, "\overline{P_3}\,\overline{P_2}\,\overline{P_1}"'] & & \mathcal{D}(\mathbf{D}) \arrow[u, "F_{\mathbf{D}}"']
\end{tikzcd}
=
\begin{tikzcd}[column sep=large]
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[r, bend left, "((|P_3|\,|P_2|)\,|P_1|)^{\#}"] & \widehat{\mathbf{c}\,|\mathbf{D}|} \arrow[r, "\downarrow^{\#}_{|\mathbf{D}|}"] & \widehat{\mathbf{c}\,|\mathbf{D}|} \\
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[u, "\downarrow^{\#}_{|\mathbf{A}|}"] \arrow[urr, phantom, "\Downarrow (\xi_3 \star \xi_2) \star \xi_1"] & & \widehat{\mathbf{c}\,|\mathbf{D}|} \arrow[u, "\downarrow^{\#}_{|\mathbf{D}|}"'] \\
\mathcal{D}(\mathbf{A}) \arrow[u, "F_{\mathbf{A}}"] \arrow[rr, bend left, "\overline{P_3}\,\overline{P_2}\,\overline{P_1}"] \arrow[rr, bend right, "\overline{P_3}\,\overline{P_2}\,\overline{P_1}"'] \arrow[rr, phantom, "="] & & \mathcal{D}(\mathbf{D}) \arrow[u, "F_{\mathbf{D}}"']
\end{tikzcd}
$$

and obtain that a_{P_3,P_2,P_1} is indeed in $\mathbf{CProf}(\mathbf{A},\mathbf{D})((P_3P_2)P_1, P_3(P_2P_1))$. □

Left and right unitors

For a 1-cell $P = (|P|, \overline{P}, \nu_P, \xi_P)$ in $\mathbf{CProf}(\mathbf{A},\mathbf{B})$, we define the left and right unitors $l_P : \mathrm{id}_{\mathbf{B}} P \Rightarrow P$ and $r_P : P\mathrm{id}_{\mathbf{A}} \Rightarrow P$ as follows:

$$l_P := (l_{|P|}, \mathrm{Id}_{\overline{P}}) \qquad \text{and} \qquad r_P := (r_{|P|}, \mathrm{Id}_{\overline{P}})$$

where $l_{|P|} \in \mathbf{ProfG}(\mathrm{id}_{\mathbf{c}|\mathbf{B}|}\,|P|\,, |P|)$ and $r_{|P|} \in \mathbf{ProfG}(|P|\,\mathrm{id}_{\mathbf{c}|\mathbf{A}|}, |P|)$ are the components of the left and right associator in $\mathbf{ProfG}$ and $\mathrm{Id}_{\overline{P}}$ corresponds to the component of the left and right associator in $\mathbf{CAT}$ since it is a 2-category.

Proof. For l_P to be a 2-cell in $\mathbf{Prof}(\mathrm{id}_{\mathbf{B}}P, P)$, we first need to show that the following diagrams are equal:

$$\begin{tikzcd}[row sep=huge, column sep=huge]
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[r, bend left, "(\mathrm{id}_{\mathbf{c}|\mathbf{B}|}\,|P|)^{\#}"] \arrow[r, bend right, "|P|^{\#}"'] \arrow[r, phantom, "\Downarrow l^{\#}"] & \widehat{\mathbf{c}\,|\mathbf{B}|} \\
\mathcal{D}(\mathbf{A}) \arrow[u, "F_{\mathbf{B}}"] \arrow[r, bend right, "\overline{P}"'] \arrow[r, phantom, "\Downarrow \nu_P"] & \mathcal{D}(\mathbf{B}) \arrow[u, "F_{\mathbf{A}}"']
\end{tikzcd}
=
\begin{tikzcd}[row sep=huge, column sep=huge]
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[r, bend left, "(\mathrm{id}_{\mathbf{c}|\mathbf{B}|}\,|P|)^{\#}"] & \widehat{\mathbf{c}\,|\mathbf{B}|} \\
\mathcal{D}(\mathbf{A}) \arrow[u, "F_{\mathbf{A}}"] \arrow[r, bend left, "\overline{P}"] \arrow[r, bend right, "\overline{P}"'] \arrow[r, phantom, "="] \arrow[ur, phantom, "\Downarrow \nu_{\mathbf{B}} \star \nu_P"] & \mathcal{D}(\mathbf{B}) \arrow[u, "F_{\mathbf{B}}"']
\end{tikzcd}$$

Unfolding $\nu_{\mathbf{B}} \star \nu_P$ in the left-hand diagram, we obtain the following diagram:

$$\begin{tikzcd}[row sep=huge, column sep=huge]
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[r, "|P|^{\#}"] \arrow[rr, bend left=60, "(\mathrm{id}_{\mathbf{c}|\mathbf{B}|}\,|P|)^{\#}"] \arrow[rr, phantom, bend left=30, "\Downarrow \varphi_2^{-1}"] & \widehat{\mathbf{c}\,|\mathbf{B}|} \arrow[r, bend left, "\mathrm{id}^{\#}_{\mathbf{c}|\mathbf{B}|}"] \arrow[r, bend right, "\mathrm{id}"'] \arrow[r, phantom, "\Downarrow \varphi_0^{-1}"] & \widehat{\mathbf{c}\,|\mathbf{B}|} \\
\mathcal{D}(\mathbf{A}) \arrow[u, "F_{\mathbf{A}}"] \arrow[r, bend right, "\overline{P}"'] \arrow[ur, phantom, "\Downarrow \nu_P"] & \mathcal{D}(\mathbf{B}) \arrow[u, "F_{\mathbf{B}}"'] \arrow[r, bend right, "\mathrm{id}"'] \arrow[ur, phantom, "="] & \mathcal{D}(\mathbf{B}) \arrow[u, "F_{\mathbf{B}}"']
\end{tikzcd}
=
\begin{tikzcd}[row sep=huge, column sep=huge]
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[r, bend left, "(\mathrm{id}_{\mathbf{c}|\mathbf{B}|}\,|P|)^{\#}"] \arrow[r, bend right, "|P|^{\#}"'] \arrow[r, phantom, "\Downarrow l^{\#}"] & \widehat{\mathbf{c}\,|\mathbf{B}|} \\
\mathcal{D}(\mathbf{A}) \arrow[u, "F_{\mathbf{B}}"] \arrow[r, bend right, "\overline{P}"'] \arrow[r, phantom, "\Downarrow \nu_P"] & \mathcal{D}(\mathbf{B}) \arrow[u, "F_{\mathbf{A}}"']
\end{tikzcd}$$

The equality follows from the left unity axiom for pseudo-functors (Definition 1.3.2). It remains to show the following equality:

$$\begin{tikzcd}
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[r, bend left, "(\mathrm{id}_{\mathbf{B}}\,|P|)^{\#}"] \arrow[r, bend right, "|P|^{\#}"'] \arrow[r, phantom, "\Downarrow l^{\#}"] & \widehat{\mathbf{c}\,|\mathbf{B}|} \arrow[r, "\downarrow^{\#}_{|\mathbf{B}|}"] & \widehat{\mathbf{c}\,|\mathbf{B}|} & & \widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[r, bend left, "(\mathrm{id}_{\mathbf{B}}\,|P|)^{\#}"] & \widehat{\mathbf{c}\,|\mathbf{B}|} \arrow[r, "\downarrow^{\#}_{|\mathbf{B}|}"] & \widehat{\mathbf{c}\,|\mathbf{B}|} \\
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[u, "\downarrow^{\#}_{|\mathbf{A}|}"] & \Downarrow \xi_P & \widehat{\mathbf{c}\,|\mathbf{B}|} \arrow[u, "\downarrow^{\#}_{|\mathbf{B}|}"'] & = & \widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[u, "\downarrow^{\#}_{|\mathbf{A}|}"] & \Downarrow \xi_{\mathbf{B}} \star \xi_P & \widehat{\mathbf{c}\,|\mathbf{B}|} \arrow[u, "\downarrow^{\#}_{|\mathbf{B}|}"'] \\
\mathcal{D}(\mathbf{A}) \arrow[u, "F_{\mathbf{A}}"] \arrow[rr, bend right, "\overline{P}"'] & & \mathcal{D}(\mathbf{B}) \arrow[u, "F_{\mathbf{B}}"'] & & \mathcal{D}(\mathbf{A}) \arrow[u, "F_{\mathbf{A}}"] \arrow[rr, bend left, "\overline{P}"] \arrow[rr, bend right, "\overline{P}"'] \arrow[rr, phantom, "="] & & \mathcal{D}(\mathbf{B}) \arrow[u, "F_{\mathbf{B}}"']
\end{tikzcd}$$

Unfolding $\xi_{\mathbf{B}} \star \xi_P$, we obtain that the second diagram is equal to:

$$\begin{tikzcd}
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[r, "|P|^{\#}"] \arrow[rrr, bend left=50, "(\mathrm{id}_{\mathbf{B}} P)^{\#}"] & \widehat{\mathbf{c}\,|\mathbf{B}|} \arrow[r, bend left, "\mathrm{id}"] \arrow[r, "\mathrm{id}^{\#}" description] \arrow[r, bend right, "\downarrow^{\#}_{|\mathbf{B}|}"'] & \widehat{\mathbf{c}\,|\mathbf{B}|} \arrow[r, bend left, "\mathrm{id}^{\#}"] \arrow[r, "\mathrm{id}"'] & \widehat{\mathbf{c}\,|\mathbf{B}|} \arrow[r, "\downarrow^{\#}_{|\mathbf{B}|}"] & \widehat{\mathbf{c}\,|\mathbf{B}|} \\
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[u, "\downarrow^{\#}_{|\mathbf{A}|}"] & \Downarrow \xi_P & \widehat{\mathbf{c}\,|\mathbf{B}|} \arrow[u, "\downarrow^{\#}_{|\mathbf{B}|}"'] \arrow[urr, "(\downarrow_{|\mathbf{A}|}\downarrow_{|\mathbf{B}|})^{\#}"] \arrow[rr, "\mathrm{id}"'] & & \widehat{\mathbf{c}\,|\mathbf{B}|} \arrow[u, "\downarrow^{\#}_{|\mathbf{B}|}"'] \\
\mathcal{D}(\mathbf{A}) \arrow[u, "F_{\mathbf{A}}"] \arrow[rr, "\overline{P}"'] & & \mathcal{D}(\mathbf{B}) \arrow[u, "F_{\mathbf{B}}"'] \arrow[rr, "\mathrm{id}"'] & = & \mathcal{D}(\mathbf{B}) \arrow[u, "F_{\mathbf{B}}"']
\arrow[from=1-2, to=1-3, phantom, bend left=25, "\Downarrow \varphi_0"]
\arrow[from=1-2, to=1-3, phantom, bend right=25, "\Downarrow \eta^{\#}_{|\mathbf{B}|}"]
\arrow[from=1-3, to=1-4, phantom, bend left=25, "\Downarrow \varphi_0^{-1}"]
\arrow[from=1-1, to=1-3, phantom, bend left=40, "\Downarrow \varphi_2^{-1}"]
\arrow[from=2-3, to=1-4, phantom, "\Downarrow \varphi_2"]
\arrow[from=2-3, to=2-5, phantom, bend left=20, "\Downarrow \mu^{\#}_{|\mathbf{B}|}"]
\end{tikzcd}$$

Since P is in **CProf**, it satisfies the commutation axiom in Definition 4.4.10 so that the diagram above is equal to:

$$\begin{tikzcd}[column sep=large]
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[r, "|P|^{\#}"] \arrow[rrr, bend left=60, "(\mathrm{id}_{\mathbf{B}}\,|P|)^{\#}"] \arrow[r, phantom, bend left=50, "\Downarrow \varphi_2^{-1}"] & \widehat{\mathbf{c}\,|\mathbf{B}|} \arrow[r, bend left=60, "\mathrm{id}"] \arrow[r, bend left=25, "\mathrm{id}^{\#}" description] \arrow[r, "\downarrow^{\#}_{|\mathbf{B}|}"'] \arrow[r, phantom, bend left=40, "\Downarrow \varphi_0"] \arrow[r, phantom, bend left=12, "\Downarrow \eta^{\#}_{|\mathbf{B}|}"] \arrow[rrr, bend right=20, "(\downarrow_{|\mathbf{B}|}\downarrow_{|\mathbf{B}|})^{\#}" description] \arrow[rrr, bend right=50, "\downarrow^{\#}_{|\mathbf{B}|}"'] & \widehat{\mathbf{c}\,|\mathbf{B}|} \arrow[r, "\mathrm{id}" description] \arrow[r, bend left=50, "\mathrm{id}^{\#}"] \arrow[r, phantom, bend left=25, "\Downarrow \varphi_0^{-1}"] \arrow[r, phantom, bend right=30, "\Downarrow \varphi_2"] & \widehat{\mathbf{c}\,|\mathbf{B}|} \arrow[r, "\downarrow^{\#}_{|\mathbf{B}|}"] \arrow[d, phantom, "\Downarrow \mu^{\#}_{|\mathbf{B}|}"] & \widehat{\mathbf{c}\,|\mathbf{B}|} \\
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[u, "\downarrow^{\#}_{|\mathbf{A}|}"] \arrow[r, phantom, "\Downarrow \xi_P"] & {} & & & \widehat{\mathbf{c}\,|\mathbf{B}|} \arrow[u, "\downarrow^{\#}_{|\mathbf{B}|}"'] \\
\mathcal{D}(\mathbf{A}) \arrow[u, "F_{\mathbf{A}}"] \arrow[rrrr, "\overline{P}"'] & & & & \mathcal{D}(\mathbf{B}) \arrow[u, "F_{\mathbf{B}}"']
\end{tikzcd}$$

The following equalities hold:

$$\mu^{\#}_{|\mathbf{B}|}\,\varphi_2\,(\mathrm{Id} * \eta^{\#}_{|\mathbf{B}|})(\mathrm{Id} * \varphi_0) = \mu^{\#}_{|\mathbf{B}|}\,(\mathrm{Id} * \eta_{|\mathbf{B}|})^{\#}\varphi_2\,(\mathrm{Id} * \varphi_0) \tag{4.6}$$

$$= r^{\#}\varphi_2\,(\mathrm{Id} * \varphi_0) \tag{4.7}$$

$$= \mathrm{Id} \tag{4.8}$$

the first one follows from the naturality of φ_2, the second from the internal monad axioms and the third from the right unity axiom for pseudo-functors. Hence, the diagram above is now equal to:

$$\begin{tikzcd}[column sep=large]
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[r, "|P|^{\#}"] \arrow[rr, bend left=60, "(\mathrm{id}_{\mathbf{B}}\,|P|)^{\#}"] \arrow[r, phantom, bend left=50, "\Downarrow \varphi_2^{-1}"] & \widehat{\mathbf{c}\,|\mathbf{B}|} \arrow[r, "\mathrm{id}" description] \arrow[r, bend left=50, "\mathrm{id}^{\#}"] \arrow[r, phantom, bend left=25, "\Downarrow \varphi_0^{-1}"] & \widehat{\mathbf{c}\,|\mathbf{B}|} \arrow[r, "\downarrow^{\#}_{|\mathbf{B}|}"] & \widehat{\mathbf{c}\,|\mathbf{B}|} \\
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[u, "\downarrow^{\#}_{|\mathbf{A}|}"] & \Downarrow \xi_P & & \widehat{\mathbf{c}\,|\mathbf{B}|} \arrow[u, "\downarrow^{\#}_{|\mathbf{B}|}"'] \\
\mathcal{D}(\mathbf{A}) \arrow[u, "F_{\mathbf{A}}"] \arrow[rrr, "\overline{P}"'] & & & \mathcal{D}(\mathbf{B}) \arrow[u, "F_{\mathbf{B}}"']
\end{tikzcd}$$

The left unity axiom for pseudo-functors implies that $l^{\#} = (\varphi_0^{-1} * \mathrm{Id})\,\varphi_2^{-1}$ so the pasting diagram above is equal to:

$$\begin{tikzcd}
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[r, bend left, "(\mathrm{id}_{\mathbf{B}}\,|P|)^{\#}"] \arrow[r, bend right, "|P|^{\#}"'] \arrow[r, phantom, "\Downarrow l^{\#}"] & \widehat{\mathbf{c}\,|\mathbf{B}|} \arrow[r, "\downarrow^{\#}_{|\mathbf{B}|}"] & \widehat{\mathbf{c}\,|\mathbf{B}|} \\
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[u, "\downarrow^{\#}_{|\mathbf{A}|}"] & \Downarrow \xi_P & \widehat{\mathbf{c}\,|\mathbf{B}|} \arrow[u, "\downarrow^{\#}_{|\mathbf{B}|}"'] \\
\mathcal{D}(\mathbf{A}) \arrow[u, "F_{\mathbf{A}}"] \arrow[rr, bend right, "\overline{P}"'] & & \mathcal{D}(\mathbf{B}) \arrow[u, "F_{\mathbf{B}}"']
\end{tikzcd}$$

which implies that l_P is a 2-cell in **CProf** as desired. We now proceed to show that r_P is a 2-cell in $\mathbf{ProfG}(P\mathrm{id}_{\mathbf{A}}, P)$. The proof for the equality

$$\begin{tikzcd}
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[r, bend left, "(|P|\,\mathrm{id}_{\mathbf{c}|\mathbf{A}|})^{\#}"] \arrow[r, bend right, "|P|^{\#}"'] \arrow[r, phantom, "\Downarrow r^{\#}"] & \widehat{\mathbf{c}\,|\mathbf{B}|} \\
\Downarrow \nu_P & \\
\mathcal{D}(\mathbf{A}) \arrow[uu, "F_{\mathbf{B}}"] \arrow[r, bend right, "\overline{P}"'] & \mathcal{D}(\mathbf{B}) \arrow[uu, "F_{\mathbf{A}}"']
\end{tikzcd}
=
\begin{tikzcd}
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[r, bend left, "(|P|\,\mathrm{id}_{\mathbf{c}|\mathbf{A}|})^{\#}"] & \widehat{\mathbf{c}\,|\mathbf{B}|} \\
\Downarrow \nu_P \star \nu_{\mathbf{A}} & \\
\mathcal{D}(\mathbf{A}) \arrow[uu, "F_{\mathbf{A}}"] \arrow[r, bend left, "\overline{P}"] \arrow[r, bend right, "\overline{P}"'] \arrow[r, phantom, "="] & \mathcal{D}(\mathbf{B}) \arrow[uu, "F_{\mathbf{B}}"']
\end{tikzcd}$$

is similar to the left unitor. It remains to show the following equality:

$$\begin{tikzcd}
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[r, bend left, "(|P|\,\mathrm{id}_{\mathbf{c}|\mathbf{A}|})^{\#}"] \arrow[r, bend right, "|P|^{\#}"'] \arrow[r, phantom, "\Downarrow r^{\#}"] & \widehat{\mathbf{c}\,|\mathbf{B}|} \arrow[r, "\downarrow^{\#}_{|\mathbf{B}|}"] & \widehat{\mathbf{c}\,|\mathbf{B}|} \\
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[u, "\downarrow^{\#}_{|\mathbf{A}|}"] & \Downarrow \xi_P & \widehat{\mathbf{c}\,|\mathbf{B}|} \arrow[u, "\downarrow^{\#}_{|\mathbf{B}|}"'] \\
\mathcal{D}(\mathbf{A}) \arrow[u, "F_{\mathbf{A}}"] \arrow[rr, bend right, "\overline{P}"'] & & \mathcal{D}(\mathbf{B}) \arrow[u, "F_{\mathbf{B}}"']
\end{tikzcd}
=
\begin{tikzcd}
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[r, bend left, "(|P|\,\mathrm{id}_{\mathbf{c}|\mathbf{A}|})^{\#}"] & \widehat{\mathbf{c}\,|\mathbf{B}|} \arrow[r, "\downarrow^{\#}_{|\mathbf{B}|}"] & \widehat{\mathbf{c}\,|\mathbf{B}|} \\
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[u, "\downarrow^{\#}_{|\mathbf{A}|}"] & \Downarrow \xi_P \star \xi_{\mathbf{A}} & \widehat{\mathbf{c}\,|\mathbf{B}|} \arrow[u, "\downarrow^{\#}_{|\mathbf{B}|}"'] \\
\mathcal{D}(\mathbf{A}) \arrow[u, "F_{\mathbf{A}}"] \arrow[rr, bend left, "\overline{P}"] \arrow[rr, bend right, "\overline{P}"'] \arrow[rr, phantom, "="] & & \mathcal{D}(\mathbf{B}) \arrow[u, "F_{\mathbf{B}}"']
\end{tikzcd}$$

Unfolding $\xi_P \star \xi_{\mathbf{A}}$ in the right-hand diagram, we obtain:

$$\begin{tikzcd}
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[r, "\mathrm{id}^{\#}"] \arrow[r, bend right, "\mathrm{id}"'] \arrow[r, phantom, shift right=2, "\Downarrow \varphi_0^{-1}"] \arrow[rrr, bend left=60, "(P\mathrm{id}_{\mathbf{c}|\mathbf{A}|})^{\#}"] \arrow[rr, phantom, bend left=40, "\Downarrow \varphi_2^{-1}"] & \widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[r, dashed, "\mathrm{id}^{\#}" description] \arrow[r, bend left, "\mathrm{id}"] \arrow[r, bend right] \arrow[r, phantom, shift left=2, "\Downarrow \varphi_0"] \arrow[r, phantom, shift right=2, "\Downarrow \eta^{\#}_{|\mathbf{A}|}"] & \widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[r, "P^{\#}"] & \widehat{\mathbf{c}\,|\mathbf{B}|} \arrow[r, "\downarrow^{\#}_{|\mathbf{B}|}"] & \widehat{\mathbf{c}\,|\mathbf{B}|} \\
 & & & & \widehat{\mathbf{c}\,|\mathbf{B}|} \arrow[u, "\downarrow^{\#}_{|\mathbf{B}|}"'] \\
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[uu, "\downarrow^{\#}_{|\mathbf{A}|}"] \arrow[uurr, bend right, "(\downarrow_{|\mathbf{A}|}\downarrow_{|\mathbf{A}|})^{\#}" description] \arrow[rr, "\mathrm{id}"'] \arrow[uur, phantom, "\Downarrow \varphi_2"] & & \widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[uu, "\downarrow^{\#}_{|\mathbf{A}|}"'] \arrow[uu, phantom, shift right=4, "\Downarrow \mu^{\#}_{|\mathbf{A}|}"] \arrow[uurr, phantom, "\Downarrow \xi_P"] & & \\
\mathcal{D}(\mathbf{A}) \arrow[u, "F_{\mathbf{A}}"] \arrow[rr, "\mathrm{id}"'] & & \mathcal{D}(\mathbf{A}) \arrow[u, "F_{\mathbf{A}}"'] \arrow[rr, "\overline{P}"'] & & \mathcal{D}(\mathbf{B}) \arrow[uu, "F_{\mathbf{B}}"']
\end{tikzcd}$$

We use the equalities

$$\begin{aligned}
\mu^{\#}_{|\mathbf{A}|}\,\varphi_2\,(\eta^{\#}_{|\mathbf{A}|} * \mathrm{Id})(\varphi_0 * \mathrm{Id}) &= \mu^{\#}_{|\mathbf{A}|}\,(\eta_{|\mathbf{A}|} * \mathrm{Id})^{\#}\varphi_2\,(\varphi_0 * \mathrm{Id}) \\
&= l^{\#}\varphi_2\,(\varphi_0 * \mathrm{Id}) \\
&= \mathrm{Id}
\end{aligned}$$

to reduce the diagram above to:

$$\begin{tikzcd}
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[r, bend left, dashed, "\mathrm{id}^{\#}"] \arrow[r, bend right, "\mathrm{id}"'] \arrow[r, phantom, "\Downarrow \varphi_0^{-1}"] \arrow[rr, bend left=60, "(P\mathrm{id}_{\mathbf{c}|\mathbf{A}|})^{\#}"] \arrow[rr, phantom, bend left=40, "\Downarrow \varphi_2^{-1}"] & \widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[r, "P^{\#}"] & \widehat{\mathbf{c}\,|\mathbf{B}|} \arrow[r, "\downarrow^{\#}_{|\mathbf{B}|}"] & \widehat{\mathbf{c}\,|\mathbf{B}|} \\
 & \widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[u, "\downarrow^{\#}_{|\mathbf{A}|}"'] \arrow[rr, phantom, "\Downarrow \xi_P"] & & \widehat{\mathbf{c}\,|\mathbf{B}|} \arrow[u, "\downarrow^{\#}_{|\mathbf{B}|}"'] \\
 & \mathcal{D}(\mathbf{A}) \arrow[u, "F_{\mathbf{A}}"'] \arrow[rr, "\overline{P}"'] & & \mathcal{D}(\mathbf{B}) \arrow[u, "F_{\mathbf{B}}"']
\end{tikzcd}$$

Since $r^{\#} = (\mathrm{Id} * \varphi_0^{-1})\varphi_2^{-1}$, we obtain the desired result. □

Theorem 4.4.13. **CProf** *is a bicategory.*

Proof. The remaining ingredient for **CProf** to be a bicategory is to show that the pentagon and triangle identities (Definition 1.3.1) hold and they are immediate since the coherence 2-cells in **CProf** are in $\mathbf{ProfG} \times \mathbf{CAT}$. $\square$

We now describe the two forgetful functors $U_{\mathcal{S}} : \mathbf{CProf} \to \mathbf{ProfG}$ and $U_{\mathcal{C}} : \mathbf{CProf} \to \mathbf{Prof}$ and leave for future work the proof that they preserve the linear logic structure. The forgetful functor $U_{\mathcal{S}} : \mathbf{CProf} \to \mathbf{ProfG}$ is straightforward: it maps a closed structure $\mathbf{A}$ to $\mathbf{c}\,|\mathbf{A}|$, a 1-cell $P = (|P|\,, \overline{P}, \nu, \xi) \in \mathbf{CProf}(\mathbf{A}, \mathbf{B})$ to the profunctor $|P| \in \mathbf{ProfG}(\mathbf{c}\,|\mathbf{A}|, \mathbf{c}\,|\mathbf{B}|)$ and a 2-cell $\alpha = (|\alpha|\,, \overline{\alpha})$ to $|\alpha|$. It is immediate that it is a strict functor between the two bicategories.

We now define a lax functor $U_{\mathcal{C}} : \mathbf{CProf} \to \mathbf{Prof}$ verifying the property that the 2-cells

$$\varphi_2 : U_{\mathcal{C}}(Q)U_{\mathcal{C}}(P) \Rightarrow U_{\mathcal{C}}(PQ)$$

are retracts of the inclusion $U_{\mathcal{C}}(PQ) \Rightarrow U_{\mathcal{C}}(Q)U_{\mathcal{C}}(P)$.

For a closed structure $\mathbf{A}$, we define $U_{\mathcal{C}}(\mathbf{A})$ to be $|\mathbf{A}|$ and for a 1-cell $P \in \mathbf{CProf}(\mathbf{A}, \mathbf{B})$, we define $U_{\mathcal{C}}(P)$ as $L_{|\mathbf{B}|}\,|P|\,R_{|\mathbf{A}|}$:

$$\begin{tikzcd}
{|\mathbf{A}|} \arrow[r, "R_{|\mathbf{A}|}", "\shortmid" marking] & \mathbf{c}\,|\mathbf{A}| \arrow[r, "|P|", "\shortmid" marking] & \mathbf{c}\,|\mathbf{B}| \arrow[r, "L_{|\mathbf{B}|}", "\shortmid" marking] & {|\mathbf{B}|}
\end{tikzcd}$$

The image of a 2-cell $\alpha \in \mathbf{CProf}(\mathbf{A}, \mathbf{B})(P, Q)$ is given by the whiskering:

$$\begin{tikzcd}
{|\mathbf{A}|} \arrow[r, "R_{|\mathbf{A}|}", "\shortmid" marking] & \mathbf{c}\,|\mathbf{A}| \arrow[r, bend left=60, "|P|", "\shortmid" marking, ""{name=U, below}] \arrow[r, bend right=60, "|Q|"', "\shortmid" marking, ""{name=D, above}] & \mathbf{c}\,|\mathbf{B}| \arrow[r, "L_{|\mathbf{B}|}", "\shortmid" marking] & {|\mathbf{B}|}
\arrow[from=U, to=D, Rightarrow, "|\alpha|"]
\end{tikzcd}$$

For a closed structure $\mathbf{A}$, we define $\varphi_0 : \mathrm{id}_{|\mathbf{A}|} \Rightarrow U_{\mathcal{C}}(\mathrm{id}_{\mathbf{A}})$ as the following 2-cell in **Prof**:

$$\begin{tikzcd}
{|\mathbf{A}|} \arrow[r, "R_{|\mathbf{A}|}", "\shortmid" marking] \arrow[rr, bend right=40, "R_{|\mathbf{A}|}"', "\shortmid" marking, ""{name=R, above}] \arrow[rrr, bend right=70, "\mathrm{id}_{|\mathbf{A}|}"', "\shortmid" marking, ""{name=I, above}] & \mathbf{c}\,|\mathbf{A}| \arrow[r, "\mathrm{id}_{\mathbf{c}\,|\mathbf{A}|}", "\shortmid" marking] \arrow[phantom, d, "\Uparrow l^{-1}"] & \mathbf{c}\,|\mathbf{A}| \arrow[r, "L_{|\mathbf{A}|}", "\shortmid" marking] & {|\mathbf{A}|} \\
 & & \Uparrow \eta_{|\mathbf{A}|} &
\end{tikzcd}$$

For 1-cells $P \in \mathbf{CProf}(\mathbf{A},\mathbf{B})$ and $Q \in \mathbf{CProf}(\mathbf{B},\mathbf{C})$, to define the 2-cell $\varphi_2 : U_{\mathcal{C}}(Q)U_{\mathcal{C}}(P) \Rightarrow U_{\mathcal{C}}(PQ)$, we make use of the following lemma:

Lemma 4.4.14. *For a closed structure* $\mathbf{A}$*, there is a morphism* $H_{\mathbf{A}} : \widehat{\mathbf{c}\,|\mathbf{A}|} \to \mathcal{D}(\mathbf{A})$ *in the slice category* $\mathbf{CAT}/\widehat{\mathbf{c}\,|\mathbf{A}|}$*:*

$$\begin{tikzcd}
\widehat{\mathbf{c}\,|\mathbf{A}|} \arrow[rr, "H_{\mathbf{A}}"] \arrow[dr, bend right, "\downarrow^{\#}_{|\mathbf{A}|}"'] & & \mathcal{D}(\mathbf{A})^{\perp\perp} \cong \mathcal{D}(\mathbf{A}) \arrow[dl, bend left, "F^{\perp\perp}"] \\
& \widehat{\mathbf{c}\,|\mathbf{A}|} &
\end{tikzcd}$$

Proof. For a presheaf X in $\widehat{\mathbf{c}\,|\mathbf{A}|}$, define $H_{\mathbf{A}}(X)$ to be the pair $(\downarrow^{\#}_{|\mathbf{A}|} X, \rho)$ where for $(Y,\lambda) \in \mathcal{D}(\mathbf{A})^{\perp}$, the component $\rho_{(Y,\lambda)} : \langle \downarrow^{\#}_{|\mathbf{A}|} X \,|\downarrow^{\#}_{|\mathbf{A}^{\mathrm{op}}|} Y\rangle \to \langle \downarrow^{\#}_{|\mathbf{A}|} X \mid Y\rangle$ is given by:

$$\langle \downarrow^{\#}_{|\mathbf{A}|} X \,|\downarrow^{\#}_{|\mathbf{A}^{\mathrm{op}}|} Y\rangle \xrightarrow{a} \langle \downarrow^{\#}_{|\mathbf{A}|}\downarrow^{\#}_{|\mathbf{A}|} X \mid Y\rangle \xrightarrow{\langle \mu^{\#}_{|\mathbf{A}|} * \mathrm{Id} | \mathrm{Id}\rangle} \langle \downarrow^{\#}_{|\mathbf{A}|} X \mid Y\rangle.$$

For a natural transformation $\alpha \in \widehat{\mathbf{c}\,|\mathbf{A}|}$, define $H_{\mathbf{A}}(\alpha)$ to be $\downarrow^{\#}_{|\mathbf{A}|} \alpha$. □

We want to construct a retract

$$\begin{tikzcd}
& \mathbf{c}\,|\mathbf{A}| \arrow[r, "|P|", "\shortmid"{marking}] & \mathbf{c}\,|\mathbf{B}| \arrow[r, "\downarrow_{|\mathbf{B}|}", "\shortmid"{marking}] & \mathbf{c}\,|\mathbf{B}| \arrow[r, "|Q|", "\shortmid"{marking}] & \mathbf{c}\,|\mathbf{C}| \arrow[dr, "L_{\mathbf{C}}", "\shortmid"{marking}] & \\
|\mathbf{A}| \arrow[ur, "R_{|\mathbf{A}|}", "\shortmid"{marking}] \arrow[dr, "R_{|\mathbf{A}|}"', "\shortmid"{marking}] & & & \Downarrow \varphi_2 & & |\mathbf{C}| \\
& \mathbf{c}\,|\mathbf{A}| \arrow[rr, "|P|"', "\shortmid"{marking}] & & \mathbf{c}\,|\mathbf{B}| \arrow[r, "|Q|"', "\shortmid"{marking}] & \mathbf{c}\,|\mathbf{C}| \arrow[ur, "L_{|\mathbf{C}|}"', "\shortmid"{marking}] &
\end{tikzcd}$$

which is equivalent to constructing a retract

$$L^{\#}_{|\mathbf{C}|}\,|Q|^{\#} \downarrow^{\#}_{|\mathbf{B}|} |P|^{\#}\, R^{\#}_{|\mathbf{A}|}(\mathrm{y}_{|\mathbf{A}|}(a)) \Rightarrow L^{\#}_{|\mathbf{C}|}\,|Q|^{\#}\,|P|^{\#}\, R^{\#}_{|\mathbf{A}|}(\mathrm{y}_{|\mathbf{A}|}(a))$$

for every a in $|\mathbf{A}|$. Using Lemma 4.4.14 above, there is an isomorphism

$$|P|^{\#}\, R^{\#}_{|\mathbf{A}|}(\mathrm{y}_{|\mathbf{A}|}(a)) \cong F_{\mathbf{B}}\overline{P}H_{\mathbf{A}}(\mathrm{y}_{\mathbf{c}|\mathbf{A}|}(a))$$

induced by ν_P. Since $\overline{P}H_{\mathbf{A}}(\mathrm{y}_{\mathbf{c}|\mathbf{A}|}(a))$ is in $\mathcal{D}(\mathbf{B})$, the retract ξ_Q induces a retract

$$L^{\#}_{|\mathbf{C}|}\,|Q|^{\#}\downarrow^{\#}_{|\mathbf{B}|}\,F_{\mathbf{B}}\overline{P}H_{\mathbf{A}}(\mathrm{y}_{\mathbf{c}|\mathbf{A}|}(a)) \Rightarrow L^{\#}_{|\mathbf{C}|}\,|Q|^{\#}\,F_{\mathbf{B}}\overline{P}H_{\mathbf{A}}(\mathrm{y}_{\mathbf{c}|\mathbf{A}|}(a)).$$

Using the isomorphism ν_P again, we obtain an isomorphism

$$L^{\#}_{|\mathbf{C}|}\,|Q|^{\#}\,F_{\mathbf{B}}\overline{P}H_{\mathbf{A}}(\mathrm{y}_{\mathbf{c}|\mathbf{A}|}(a)) \cong L^{\#}_{|\mathbf{C}|}\,|Q|^{\#}\,|P|^{\#}\,R^{\#}_{|\mathbf{A}|}(\mathrm{y}_{|\mathbf{A}|}(a))$$

which implies the desired result.

Conclusion

We have presented three orthogonality constructions on bicategories. The first one generalizes Ehrhard's finiteness spaces to the bicategory of profunctors by enforcing finite interactions. In this construction, we have worked on a focused orthogonality on the subclass of finitely presented objects. This construction opens the way for a lot of variation in terms of the chosen class of objects: for example, restricting the interactions to absolutely presentable objects could yield to a model of totality in the spirit of the one studied by Loader [87].

In the second construction, the orthogonality restricts the analytic functors induced by generalized species to stable functors. Since the linear logic structure is preserved, we are able to obtain a cartesian closed bicategory of stable functors providing a connection with Girard's normal functors and polynomial functors. As the species corresponding to stable functors are better behaved in terms of integration, we hope that our construction sets the first step towards obtaining a theory of integration and resolution of differential equations for these stabilized species. We also aim to carry out this construction in the setting of enriched species over vector spaces where the finiteness construction described above will ensure that the computations always yield finite dimensional vector spaces.

In the last construction, our objective is to construct an orthogonality bicategory that connects the symmetric $\mathcal{S}$-species and the cartesian $\mathcal{C}$-species in a compositional way. Since the orthogonality contains witnessing 2-cells for the interaction between the two models, the 2-cells in the glueing bicategory are therefore restricted to the ones that preserve these retractions. It provides with a first step towards understanding the general construction of orthogonality bicategories where the orthogonality further allows us to control the reductions between programs.

As we are getting closer to general orthogonality bicategories, the proofs become very tedious since we need to keep track of all the coherence 2-cells for the bicategorical structure. Proving for example that it forms a symmetric monoidal bicategory to interpret the linear logic tensor now requires a large amount of diagrammatic calculations. We are aiming to investigate orthogonality constructions on double categories as symmetric monoidal double categories contain less coherence data and we can use recent results by Hansen and Shulman to show that the underlying bicategory of a symmetric double category is also symmetric monoidal under certain conditions [105, 59]. Since symmetric operads correspond to monads in the bicategory of $\mathcal{S}$-species and monads for $\mathcal{C}$-species correspond to cartesian operads or clones, we hope that our construction will also provide a connection between the two notions. The relationship between profunctors and intersection types has also recently been explored by Olimpieri where the non-idempotent intersection type system corresponds to the free symmetric monoidal pseudo-monad and the idempotent case corresponds to the cartesian pseudo-monad [92]. Our future goal is to connect the two type systems using our glueing bicategory.

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