Submitted to MFPS 2023

Fixpoint constructions in focused orthogonality models of linear logic

Marcelo Fiore[a,1,2] Zeinab Galal[b,1,4,3] Farzad Jafarrahmani[b,4,5]

[a] *University of Cambridge*

[b] *Sorbonne University*

Abstract

Orthogonality is a notion based on the duality between programs and their environments used to determine when they can be safely combined. For instance, it is a powerful tool to establish termination properties in classical formal systems. It was given a general treatment with the concept of orthogonality category, of which numerous models of linear logic are instances, by Hyland and Schalk. This paper considers the subclass of focused orthogonalities.

We develop a theory of fixpoint constructions in focused orthogonality categories. Central results are lifting theorems for initial algebras and final coalgebras. These crucially hinge on the insight that focused orthogonality categories are relational fibrations. The theory provides an axiomatic categorical framework for models of linear logic with least and greatest fixpoints of types.

We further investigate domain-theoretic settings, showing how to lift bifree algebras, used to solve mixed-variance recursive type equations, to focused orthogonality categories.

Keywords: Orthogonality; linear logic; categorical models; fixpoint constructions; inductive, coinductive, and recursive types.

⋆ We are grateful to Thomas Ehrhard and Lionel Vaux Auclair for valuable discussions on this work.

[1] Research partially supported by EPSRC grant EP/V002309/1.

[2] Email: marcelo.fiore@cl.cam.ac.uk

[3] Email: zeinab.galal@lip6.fr

[4] Research partially supported by the RealiSe Emergence project.

[5] Email: farzad.jafarrahmani@lip6.fr

Introduction

Linear logic with fixpoints

Propositional linear logic lacks datatypes with iteration or recursion principles. This is usually remedied by extending it to the second order, thus defining a logical system in which Girard's System F [19] can be embedded. Even if such a system is very expressive in terms of computable functions, its algorithmic expressiveness is poor: for instance, it is not possible to write a term that computes the predecessor function in one (or a uniformly bounded) number of reduction steps.

Girard first considered an extension of linear logic with fixpoints of formulas in an unpublished note [17]. However, the first comprehensive proof-theoretic investigation of such a system was given by Baelde [2] who introduced and studied $\mu\mathsf{MALL}$, an extension of multiplicative additive linear logic with induction and coinduction principles, with motivations coming from proof-search and system verification. Linear logic exponentials were not considered in $\mu\mathsf{MALL}$; they could be somewhat encoded with inductive and coinductive types though without their denotational interpretation satisfying the required Seely isomorphisms.

Ehrhard and Jafarrahmani [9] introduced a system $\mu\mathsf{LL}$ extending $\mu\mathsf{MALL}$ with exponentials and studied it from the Curry-Howard-Lambek perspective. Their notion of categorical model of $\mu\mathsf{LL}$ is an extension of the standard notion of Seely category for classical linear logic with suitable initial algebras and final coalgebras. Specifically, they presented a totality model of $\mu\mathsf{LL}$ that is an instance of a general categorical construction developed in this paper. In the totality model, least and greatest fixpoints are calculated by lifting initial algebras and final coalgebras from the relational model. Here, by viewing it as a special case of a *focused orthogonality* construction, we are able to develop a general methodology for constructing models of linear logic with fixpoints.

Orthogonality and glueing for models of linear logic

Logical relations [38,31,37] are by now a standard tool in the theory of programming languages to certify program properties that cannot be obtained by naive induction arguments. The basic idea is to associate to each type a predicate that is preserved by the operations on the type. Such predicates depend on the program property that one is interested in proving (termination, type safety, parametricity, etc.) and their use provides a powerful proof method.

Orthogonality methods originate from the semantics of linear logic and are particularly well-suited for languages modelling classical negation [15]. The general principle is to restrict attention to pairs of terms and contexts in a *pole* $\perp\!\!\!\perp \subseteq \text{Terms} \times \text{Contexts}$ that contains *correct computations*. For a set of terms $T \subseteq \text{Terms}$, one can then consider the set of all contexts $T^{\perp} \subseteq \text{Contexts}$ that yield a correct computation when combined with any term in T. Dually, for a set of contexts $C \subseteq \text{Contexts}$, one can consider the set of all terms $C^{\perp} \subseteq \text{Terms}$ that yield a correct computation when combined with any context in C. These constructions yield a duality between subsets of terms and subsets of contexts, and one associates to each type a subset of terms T that is equal to its double dual $T^{\perp\perp}$. Such dualities between terms and environments (or player and opponent) form the basis of game semantics [22] and of Krivine's classical realizability [25].

Logical relations have a categorical abstraction given by Artin-Wraith glueing [43], while orthogonality constructions are obtained via Hyland-Schalk double glueing [23]. Here, we will be particularly interested in a well-behaved subclass of orthogonality categories arising from poles and referred to as *focused orthogonality categories*. These, we will recast as relational fibrations and therefrom develop a general categorical theory that lifts initial algebras and final coalgebras to focused orthogonality categories, and therefore provides models of linear logic with least and greatest fixpoints.

Structure of the paper

- We start by recalling the notion of orthogonality category by Hyland and Schalk in Section 1.
- In Section 2, we develop a theory of fixpoint constructions for relational fibrations by lifting initial algebras and final coalgebras to the Grothendieck category of a relational fibration.
- We show in Section 3 that focused orthogonality models are instances of relational fibrations. This provides us with a general categorical construction to obtain models of linear logic with least and greatest fixpoints. A variety of examples is considered in Section 4.
- Finally, in Section 5, we show how to lift bifree algebras to focused orthogonality categories in an axiomatic domain-theoretic setting.

1 Preliminaries on orthogonality categories

From a categorical viewpoint, logical relations can be presented using glueing constructions, also called Artin-Wraith glueing, sconing, or Freyd covering [1,43,14]. These allow the lifting of monoidal (or cartesian) closed structure to glueing categories. Orthogonality methods fit into the more general framework of double-glueing constructions by Hyland, Schalk, and Tan [39,23] which is tailored to $\star$-autonomous categories. The general idea is to associate two predicates with each type: one for the type and another one for its dual.

For a $\star$-autonomous category $\mathscr{C}$ with monoidal units 1 and $\perp$, an *orthogonality relation* $\perp$ is a family of subsets

$$\perp_c \;\subseteq\; \mathscr{C}(1,c) \times \mathscr{C}(c,\perp)$$

indexed by objects $c \in \mathscr{C}$ and verifying some compatibility conditions with respect to the linear logic structure [23]. For a subset $X \subseteq \mathscr{C}(1,c)$, its *orthogonal* $X^\perp \subseteq \mathscr{C}(c,\perp)$ is given by $X^\perp := \{\, y : c \to \perp \mid \forall\, x \in X.\, x \perp_c y \,\}$ with the idea that $X^\perp$ contains the environments (or counter-terms) that yield a correct computation (with respect to the chosen orthogonality relation) when combined with any term in X. Dually, for a subset $Y \subseteq \mathscr{C}(c,\perp)$, its orthogonal $Y^\perp \subseteq \mathscr{C}(1,c)$ is given by $Y^\perp := \{\, x : 1 \to c \mid \forall\, y \in Y.\, x \perp_c y \,\}$. We restrict attention subsets of global elements that are double orthogonally closed and define the *orthogonality category* $\mathcal{O}_\perp(\mathscr{C})$ to have objects given by pairs (c, X) with $X = X^{\perp\perp} \subseteq \mathscr{C}(1,c)$ and morphisms $(c,X) \to (d,Y)$ given by morphisms $f : c \to d$ in $\mathscr{C}$ such that:

$$\forall\, x \in X.\, fx \in Y \quad \text{ and } \quad \forall\, y \in Y^\perp.\, yf \in X^\perp \;. \tag{1}$$

Provided that some conditions on $\mathscr{C}$ and the orthogonality relation $\{\perp_c\}_{c\in\mathscr{C}}$ hold, if $\mathscr{C}$ is a model of classical linear logic then so is the induced orthogonality category, and the forgetful functor preserves the linear logic structure strictly [23].

In this paper, we will restrict to the special case where the orthogonality relation arises from a distinguished subset $\perp\!\!\!\perp \subseteq \mathscr{C}(1, \perp)$, referred to as a pole, as follows:

$$\perp\!\!\!\perp_c \;:=\; \{\, (x, y) \in \mathscr{C}(1, c) \times \mathscr{C}(c, \perp) \mid y\,x \in \perp\!\!\!\perp \,\} \;.$$

Such orthogonality relations are called *focused* and for them the two conditions in (1) above are equivalent [23]. This property will crucially allow us to subsume focused orthogonality categories within a fibrational setting and, from the theory of fixpoint constructions for relational fibrations of the following section, we will obtain models of linear logic with least and greatest fixpoints.

2 Fixpoint constructions in relational fibrations

This section develops a general method to lift initial algebras and final coalgebras form the base category of a relational fibration to its total category.

2.1 Relational fibrations

We start by recalling some basic properties of relational fibrations.

Definition 2.1 A *$\mathscr{C}$-indexed poset* is a contravariant functor from a category $\mathscr{C}$ to the category **Poset** of posets and monotone functions between them.

For an indexed poset $\mathcal{R} : \mathscr{C}^{\mathrm{op}} \to \mathbf{Poset}$, a morphism $f : c \to d$ in $\mathscr{C}$, and $S \in \mathcal{R}(d)$, it is customary to write $f^*(S)$ for $\mathcal{R}(f)(S) \in \mathcal{R}(c)$. For $R \in \mathcal{R}(c)$, we moreover write $f : R \supset S$ for $R \leqslant f^*(S)$.

Definition 2.2 The *Grothendieck category* $\mathcal{G}_{\mathscr{C}}(\mathcal{R})$ of a $\mathscr{C}$-indexed poset $\mathcal{R} : \mathscr{C}^{\mathrm{op}} \to \mathbf{Poset}$ has objects given by pairs $\{\, c \,|\, R \,\}$ with $c \in \mathscr{C}$ and $R \in \mathcal{R}(c)$, and morphisms $f : \{\, c \,|\, R \,\} \to \{\, d \,|\, S \,\}$ given by morphisms $f : c \to d$ in $\mathscr{C}$ such that $f : R \supset S$ in $\mathcal{R}(c)$. Identities and composition are given as in $\mathscr{C}$.

The forgetful functor $U : \mathcal{G}_{\mathscr{C}}(\mathcal{R}) \to \mathscr{C}$ is a Grothendieck fibration with partially-ordered fibers. Note that, for every $c \in \mathscr{C}$, $R \leqslant R'$ in $\mathcal{R}(c)$ if and only if $\mathrm{id}_c : \{\, c \,|\, R \,\} \to \{\, c \,|\, R' \,\}$ in $\mathcal{G}_{\mathscr{C}}(\mathcal{R})$.

We refer to U as the *relational fibration* of the $\mathscr{C}$-indexed poset $\mathcal{R}$. The cartesian lifting of $f : c \to d$ in $\mathscr{C}$ with respect to $\{\, d \,|\, S \,\} \in \mathcal{G}_{\mathscr{C}}(\mathcal{R})$ is the morphism $f : \{\, c \,|\, f^*S \,\} \to \{\, d \,|\, S \,\}$ in $\mathcal{G}_{\mathscr{C}}(\mathcal{R})$.

Definition 2.3 For a $\mathscr{C}$-indexed poset $\mathcal{R}$, we say that an endofunctor $\overline{F}$ on $\mathcal{G}_{\mathscr{C}}(\mathcal{R})$ is a *lifting* of an endofunctor F on $\mathscr{C}$ whenever the following diagram commutes

$$\begin{tikzcd}
\mathcal{G}_{\mathscr{C}}(\mathcal{R}) \arrow[r, "\overline{F}"] \arrow[d, "U"'] & \mathcal{G}_{\mathscr{C}}(\mathcal{R}) \arrow[d, "U"] \\
\mathscr{C} \arrow[r, "F"'] & \mathscr{C}
\end{tikzcd}$$

We let $\overline{F}$ be a lifting of F as above for the rest of the section.

For $\{\,c\,|\,R\,\} \in \mathcal{G}_{\mathscr{C}}(\mathcal{R})$, we write $\overline{F}_c(R)$ for the element in $\mathcal{R}(Fc)$ given by $\overline{F}\{\,c\,|\,R\,\}$; in other words, we let $\overline{F}\{\,c\,|\,R\,\} = \{\,Fc \mid \overline{F}_c(R)\,\}$. Therefore, for all $f : \{\,c\,|\,R\,\} \to \{\,d\,|\,S\,\}$ in $\mathcal{G}_{\mathscr{C}}(\mathcal{R})$, since $\overline{F}(f) = F(f)$, we have that $\overline{F}_c(R) \leqslant (Ff)^*\big(\overline{F}_d(S)\big)$ in $\mathcal{R}(Fc)$. This has the following direct consequences.

Lemma 2.4 (i) *For all $c \in \mathscr{C}$, the function $\overline{F}_c : \mathcal{R}(c) \to \mathcal{R}(Fc)$ is monotone.*

(ii) *For all $f : c \to d$ in $\mathscr{C}$ and $S \in \mathcal{R}(d)$, $\overline{F}_c\big(f^*(S)\big) \leqslant (Ff)^*\big(\overline{F}_d(S)\big)$ in $\mathcal{R}(Fc)$.*

2.2 Initial-algebra lifting theorem

By Lemma 2.4(i), every coalgebra $\gamma : c \to Fc$ induces the monotone operator

$$\mathcal{R}(c) \xrightarrow{\overline{F}_c} \mathcal{R}(Fc) \xrightarrow{\gamma^*} \mathcal{R}(c) \ .$$

A lifting of the F-coalgebra γ to an $\overline{F}$-coalgebra amounts to providing a post fixpoint of $\gamma^* \circ \overline{F}_c$; that is, an $R \in \mathcal{R}(c)$ such that $R \leqslant \gamma^*\big(\overline{F}_c(R)\big)$. On the other hand, a lifting of an F-algebra $\delta : Fd \to d$ amounts to providing an $S \in \mathcal{R}(d)$ such that $\overline{F}_d(S) \leqslant \delta^*(S)$.

We now consider homomorphisms from a coalgebra $\gamma : c \to Fc$ to an algebra $\delta : Fd \to d$ as given by morphisms $f : c \to d$ such that the following diagram commutes:

$$\begin{tikzcd} Fc \arrow[r, "Ff"] & Fd \arrow[d, "\delta"] \\ c \arrow[u, "\gamma"] \arrow[r, "f"'] & d \end{tikzcd}$$

Lemma 2.5 *For a coalgebra $\gamma : c \to Fc$, the least pre-fixpoint $\nabla_\gamma \in \mathcal{R}(c)$ of the monotone operator $\gamma^* \circ \overline{F}_c$, whenever it exists, provides a lifting of γ such that for all liftings $S \in \mathcal{R}(d)$ of an algebra $\delta : Fd \to d$, every homomorphism $c \to d$ from γ to δ lifts to a morphism $\{\,c\,|\,\nabla_\gamma\,\} \to \{\,d\,|\,S\,\}$.*

Proof We have $\gamma : \{\,c\,|\,\nabla_\gamma\,\} \to \{\,Fc \mid \overline{F}_c(\nabla_\gamma)\,\}$ because ∇_γ is a fixpoint of $\gamma^* \circ \overline{F}_c$.

Let $\delta : Fd \to d$ and $S \in \mathcal{R}(d)$ be such that $\overline{F}_d(S) \leqslant \delta^*(S)$ and let $f : c \to d$ be an homomorphism from γ to δ.

By Lemma 2.4(ii) and the assumption on S, we have

$$\overline{F}_c\big(f^*(S)\big) \leqslant (Ff)^*\big(\overline{F}_d(S)\big) \leqslant (Ff)^*\big(\delta^*(S)\big)$$

and hence

$$(\gamma^* \circ \overline{F}_c)\big(f^*(S)\big) \leqslant \big(\delta \circ (Ff) \circ \gamma\big)^*(S) = f^*(S) \ ;$$

that is, $f^*(S)$ is a pre-fixpoint of $\gamma^* \circ \overline{F}_c$. Therefore, $\nabla_\gamma \leqslant f^*(S)$ as required. $\square$

Theorem 2.6 *For an initial F-algebra $\alpha : Fa \to a$, if the monotone operator $(\alpha^{-1})^* \circ \overline{F}_a$ on $\mathcal{R}(a)$ has a least pre-fixpoint ∇_α then $\alpha : \{\,Fa \mid \overline{F}_a(\nabla_\alpha)\,\} \to \{\,a\,|\,\nabla_\alpha\,\}$ is an initial $\overline{F}$-algebra.*

Proof For every $\overline{F}$-algebra $\delta : \{\, Fd \mid \overline{F}_d(S) \,\} \to \{\, d \,|\, S \,\}$ the unique homomorphism $u(\delta) : a \to d$ from $\alpha : Fa \to a$ to $\delta : Fd \to d$ is a homomorphism from $\alpha^{-1} : a \to Fa$ to $\delta : Fd \to d$ and, by Lemma 2.5, it is also the unique homomorphism from $\alpha : \{\, Fa \mid \overline{F}_a(\nabla_\alpha) \,\} \to \{\, a \,|\, \nabla_\alpha \,\}$ to $\delta : \{\, Fd \mid \overline{F}_d(S) \,\} \to \{\, d \,|\, S \,\}$. $\square$

Let $\alpha : Fa \to a$ be an initial algebra and, for an algebra $\delta : Fd \to d$, let $u(\delta) : a \to d$ be the unique homomorphism from α to δ. In the situation of the theorem, initial algebras α satisfy the following property:

$$\text{for every algebra } \delta : Fd \to d \text{ and } S \in \mathcal{R}(d), \text{ if } \delta : \overline{F}_d(S) \supset S \text{ then } u(\delta) : \nabla_\alpha \supset S \ .$$

This provides an abstract general form of *induction principle*. Indeed, in particular, one has:

$$\text{for every } S \in \mathcal{R}(a), \text{ if } \overline{F}_a(S) \leqslant \alpha^*(S) \text{ then } \nabla_\alpha \leqslant S \ .$$

As advocated by Hermida and Jacobs [21, Definition 3.1], the standard induction principle is recovered when ∇_α is the greatest element $\top_a$ of $\mathcal{R}(a)$, in which case one has:

$$\text{for every algebra } \delta : Fd \to d \text{ and } S \in \mathcal{R}(d), \text{ if } \delta : \overline{F}_d(S) \supset S \text{ then } u(\delta)^*(S) = \top_a$$

and, in particular, that:

$$\text{for every } S \in \mathcal{R}(a), \text{ if } \overline{F}_a(S) \leqslant \alpha^*(S) \text{ then } S = \top_a \ .$$

An *ipo* (inductive poset) is a poset with a least element and joins of directed subsets. Such posets are particularly relevant to us here because of Pataraia's constructive theorem that every monotone endofunction on an ipo has a least pre-fixpoint [30].

Definition 2.7 A $\mathscr{C}$*-indexed ipo* is a $\mathscr{C}$-indexed poset $\mathcal{R}$ such that $\mathcal{R}(c)$ is an ipo for all $c \in \mathscr{C}$.

Corollary 2.8 *For every $\mathscr{C}$-indexed ipo $\mathcal{R}$, every endofunctor F on $\mathscr{C}$, and every endofunctor $\overline{F}$ on $\mathcal{G}_{\mathscr{C}}(\mathcal{R})$ lifting it, initial F-algebras lift to initial $\overline{F}$-algebras.*

2.3 Final-coalgebra lifting theorem

Definition 2.9 A $\mathscr{C}$-indexed poset $\mathcal{R}$ is said to have *existential quantification* whenever, for all $f : a \to b$ in $\mathscr{C}$, the monotone function $f^* : \mathcal{R}(b) \to \mathcal{R}(a)$ has a left adjoint, for which we write $f_! : \mathcal{R}(a) \to \mathcal{R}(b)$.

Lemma 2.10 *For a $\mathscr{C}$-indexed poset $\mathcal{R}$ with existential quantification, let F be an endofunctor on $\mathscr{C}$ and $\overline{F}$ be an endofunctor on $\mathcal{G}_{\mathscr{C}}(\mathcal{R})$ lifting it.*

For a coalgebra $\delta : d \to Fd$, the greatest post-fixpoint $\Delta_\delta \in \mathcal{R}(d)$ of the monotone operator $\delta^* \circ \overline{F}_d$, whenever it exists, provides a lifting of δ such that for all liftings $R \in \mathcal{R}(c)$ of a coalgebra $\gamma : c \to Fc$, every homomorphism $c \to d$ from γ to δ lifts to a morphism $\{\, c \,|\, R \,\} \to \{\, d \,|\, \Delta_\delta \,\}$.

Proof We have $\delta : \{\, d \,|\, \Delta_\delta \,\} \to \{\, Fd \mid \overline{F}_d(\Delta_\delta) \,\}$ because Δ_δ is a fixpoint of $\delta^* \circ \overline{F}_d$.

Let $\gamma : c \to Fc$ and $R \in \mathcal{R}(c)$ be such that $R \leqslant \gamma^*\big(\overline{F}_c(R)\big)$ and let $f : c \to d$ be an homomorphism from γ to δ.

We prove $R \leqslant f^*(\Delta_\delta)$ by equivalently showing $f_!(R) \leqslant \Delta_\delta$ establishing that $f_!(R)$ is a post-fixpoint of $\delta^* \circ \overline{F}_d$. Indeed,

$$\begin{aligned} R &\leqslant \gamma^*\big(\overline{F}_c(R)\big) && \text{, by assumption} \\ &\leqslant \gamma^*(\overline{F}_c(f^*(f_!(R)))) && \text{, as } f_! \dashv f^* \\ &\leqslant \gamma^*((Ff)^*(\overline{F}_d(f_!(R)))) && \text{, by Lemma 2.4(ii)} \\ &= f^*(\delta^*(\overline{F}_d(f_!(R)))) && \text{, as } f : (c,\gamma) \to (d,\delta) \end{aligned}$$

and therefore $f_!(R) \leqslant \delta^*(\overline{F}_d(f_!(R)))$. $\square$

Definition 2.11 A *co-ipo* is a poset whose opposite is an ipo. A *$\mathscr{C}$-indexed co-ipo* is a $\mathscr{C}$-indexed poset $\mathcal{R}$ such that $\mathcal{R}(c)$ is a co-ipo for all $c \in \mathscr{C}$.

By the dual of Pataria's theorem [30], that monotone endofunctions on co-ipos have greatest post-fixpoints, we obtain the following.

Corollary 2.12 *For every $\mathscr{C}$-indexed co-ipo with existential quantification $\mathcal{R}$, every endofunctor F on $\mathscr{C}$, and every endofunctor $\overline{F}$ on $\mathcal{G}_{\mathscr{C}}(\mathcal{R})$ lifting it, final F-coalgebras lift to final $\overline{F}$-coalgebras.*

We note that the above may be also established under slightly weaker hypothesis than existential quantification.

Lemma 2.13 *For a $\mathscr{C}$-indexed poset $\mathcal{R}$, let F be an endofunctor on $\mathscr{C}$ and $\overline{F}$ be an endofunctor on $\mathcal{G}_{\mathscr{C}}(\mathcal{R})$ lifting it. For a coalgebra $\delta : d \to Fd$, such that $\mathcal{R}(d)$ is a co-ipo let $\Delta_\delta \in \mathcal{R}(d)$ be the greatest post-fixpoint of the monotone operator $\delta^* \circ \overline{F}_d$.*

For a coalgebra $\gamma : \{\, c \,|\, R \,\} \to \{\, Fc \,|\, \overline{F}_c(R) \,\}$ and a homomorphism $f : c \to d$ from γ to δ, if $\{\, S \in \mathcal{R}(d) \mid R \leqslant f^*(S) \,\}$ is a sub co-ipo of $\mathcal{R}(d)$ then f lifts to a morphism $\{\, c \,|\, R \,\} \to \{\, d \,|\, \Delta_\delta \,\}$.

Proof It suffices to show that $\{\, S \in \mathcal{R}(d) \mid R \leqslant f^*(S) \,\}$ is invariant under $\delta^* \circ \overline{F}_d$. Indeed, if $R \leqslant f^*(S)$, then $\overline{F}_c(R) \leqslant \overline{F}_c\big(f^*(S)\big) \leqslant (Ff)^*\big(\overline{F}_d(S)\big)$ and $R \leqslant \gamma^*\big(\overline{F}_c(R)\big) \leqslant \gamma^*\big((Ff)^*(\overline{F}_d(S))\big) = f^*\big(\delta^*\big(\overline{F}_d(S)\big)\big)$. $\square$

3 Focused orthogonality fibrationally

We study focused orthogonality categories representing them in terms of Grothendieck categories of indexed complete lattices with existential quantification. This, together with the study of the previous section, provides an axiomatic theory of fixpoint constructions in focused orthogonality models of linear logic.

3.1 Focused orthogonality categories

We expand upon the exposition of focused orthogonality given in Section 1. A *pole* in a category $\mathscr{C}$ is a subset $\perp\!\!\!\perp \subseteq \mathscr{C}(\mathsf{s},\mathsf{t})$ for a pair of distinguished objects s and t. To obtain a model of intuitionistic linear logic one takes s to be the monoidal unit 1, while in the classical setting one further takes t to be its dual $\bot$. The *focused orthogonality* induced by a pole $\perp\!\!\!\perp$ is the family of relations $\{ \perp\!\!\!\perp_c \subseteq \mathscr{C}(\mathsf{s},c) \times \mathscr{C}(c,\mathsf{t}) \}_{c \in \mathscr{C}}$ given by

$$x \perp\!\!\!\perp_c u \iff (u \circ x) \in \perp\!\!\!\perp \quad .$$

The defining property of focused orthogonalities is being *reciprocal* [20]; in the sense that, for all morphisms $x : \mathsf{s} \to c$, $f : c \to d$, and $u : d \to \mathsf{t}$ in $\mathscr{C}$,

$$(f \circ x) \perp\!\!\!\perp_d u \iff x \perp\!\!\!\perp_c (u \circ f) \quad . \tag{2}$$

This plays a crucial role in the following section.

For a subset $X \subseteq \mathscr{C}(\mathsf{s},c)$, its *orthogonal* $X^\perp \subseteq \mathscr{C}(c,\mathsf{t})$ is given as below on the left

$$X^\perp := \{ u : c \to \mathsf{t} \mid \forall\, x \in X.\, x \perp u \} \quad , \quad U^\perp := \{ x : \mathsf{s} \to c \mid \forall\, u \in U.\, x \perp u \}$$

while, dually, for a subset $U \subseteq \mathscr{C}(c,\mathsf{t})$, its orthogonal $U^\perp \subseteq \mathscr{C}(\mathsf{s},c)$ is given as above on the right. As it is standard, these definitions induce a Galois connection between $\mathcal{C}(c) := \big(\mathcal{P}\big(\mathscr{C}(\mathsf{s},c)\big), \subseteq\big)$ and $\mathcal{C}^\circ(c) := \big(\mathcal{P}\big(\mathscr{C}(c,\mathsf{t})\big), \supseteq\big)$. The fixpoints of the associated closure operator on $\mathcal{C}(c)$, referred to as *double orthogonally closed* subsets, form complete lattices:

$$\mathcal{D}(c) := \{ X \subseteq \mathscr{C}(\mathsf{s},c) \mid X = X^{\perp\perp} \} \quad . \tag{3}$$

Definition 3.1 The *focused orthogonality category* $\mathcal{O}_{\perp\!\!\!\perp}(\mathscr{C})$ of a category $\mathscr{C}$ with a pole $\perp\!\!\!\perp \subseteq \mathscr{C}(\mathsf{s},\mathsf{t})$ has objects given by pairs (c,X) with $c \in \mathscr{C}$ and $X \in \mathcal{D}(c)$, and morphisms $f : (c,X) \to (d,Y)$ given by morphisms $f : c \to d$ in $\mathscr{C}$ such that

$$\forall\, (x : \mathsf{s} \to c) \in X.\, (f \circ x) \in Y \quad . \tag{4}$$

3.2 Focused orthogonality categories are relational fibrations

We need to introduce some notation.

(i) For a morphism $f : c \to d$ in a category $\mathscr{C}$, we respectively write

$$f\circ := \mathscr{C}(\mathsf{s},f) : \mathscr{C}(\mathsf{s},c) \to \mathscr{C}(\mathsf{s},d) \qquad \text{and} \qquad \circ f := \mathscr{C}(f,\mathsf{t}) : \mathscr{C}(d,\mathsf{t}) \to \mathscr{C}(c,\mathsf{t})$$

for the operations of post and pre composition with f.

(ii) For a function $h : A \to B$ between sets, we write $h^* : \mathcal{P}(B) \to \mathcal{P}(A)$ for the inverse image function and $h_! : \mathcal{P}(A) \to \mathcal{P}(B)$ for its left adjoint. In elementary terms,

$$h^*(T) := \{ a \in A \mid h(a) \in T \} \quad \text{and} \quad h_!(S) := \{ b \in B \mid \exists\, a \in S.\, h(a) = b \} \quad .$$

For the rest of the section, let $f : c \to d$ be a morphism in a category $\mathscr{C}$ with a pole $\perp\!\!\!\perp \subseteq \mathscr{C}(\mathsf{s},\mathsf{t})$. Since, for $x : \mathsf{s} \to c$ in $\mathscr{C}$ and $V \subseteq \mathscr{C}(d,\mathsf{t})$,

$$x \in (f\circ)^*(V^\perp) \iff \forall\, v \in V.\, (f \circ x) \perp v \quad \text{and} \quad x \in \big((\circ f)_!(V)\big)^\perp \iff \forall\, v \in V.\, x \perp (v \circ f)$$

we have

$$\big((\circ f)_!(V)\big)^\perp = (f\circ)^*(V^\perp) \text{ for all } V \subseteq \mathscr{C}(d,\mathsf{t})$$

and obtain the commutative diagram on the left below that recasts reciprocity as a lifting property by duality:

$$\begin{tikzcd} \mathcal{C}^\circ(d) \arrow[r, "(\circ f)_!"] \arrow[d, "(-)^\perp"'] & \mathcal{C}^\circ(c) \arrow[d, "(-)^\perp"] \\ \mathcal{C}(d) \arrow[r, "(f\circ)^*"'] & \mathcal{C}(c) \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C}(d) \arrow[r, "(f\circ)^*"] \arrow[d, "(-)^\perp"'] \arrow[dr, phantom, "\swarrow"] & \mathcal{C}(c) \arrow[d, "(-)^\perp"] \\ \mathcal{C}^\circ(d) \arrow[r, "(\circ f)_!"'] & \mathcal{C}^\circ(c) \end{tikzcd} \tag{5}$$

Then, as $(f\circ)^* : \mathcal{C}(d) \to \mathcal{C}(c)$ lifts along the right adjoints $(-)^\perp$, it also lifts along the forgetful functors from their induced categories of algebras; that is, in this case, it restricts to double orthogonally closed subsets. We spell out the details.

In (5), the diagram on the left has as mate the diagram on the right; that is,

$$\big((f\circ)^*(Y)\big)^\perp \subseteq (\circ f)_!(Y^\perp) \text{ for all } Y \subseteq \mathscr{C}(\mathsf{s},d) \ .$$

Pasting these two diagrams, we obtain a distributive law:

$$\begin{tikzcd} \mathcal{C}(d) \arrow[r, "(f\circ)^*"] \arrow[d, "(-)^{\perp\perp}"'] \arrow[dr, phantom, "\swarrow"] & \mathcal{C}(c) \arrow[d, "(-)^{\perp\perp}"] \\ \mathcal{C}^\circ(d) \arrow[r, "(f\circ)^*"'] & \mathcal{C}^\circ(c) \end{tikzcd}$$

that is, $\big((f\circ)^*(Y)\big)^{\perp\perp} \subseteq (f\circ)^*(Y^{\perp\perp})$ for all $Y \subseteq \mathscr{C}(\mathsf{s},c)$. Thus, $(f\circ)^* : \mathcal{C}(d) \to \mathcal{C}(c)$ restricts to a meet-preserving monotone function $\mathcal{D}(d) \to \mathcal{D}(c)$ between complete lattices.

The above development results in a representation theorem for focused orthogonality categories in terms of Grothendieck categories of indexed complete lattices with existential quantification.

Definition 3.2 A *$\mathscr{C}$-indexed complete lattice* is a $\mathscr{C}$-indexed poset $\mathcal{R}$ such that $\mathcal{R}(c)$ is a complete lattice for all $c \in \mathscr{C}$.

Theorem 3.3 *Every category $\mathscr{C}$ with a pole $\perp\!\!\!\perp$ induces a $\mathscr{C}$-indexed complete lattice with existential quantification $\mathcal{D}$ whose Grothendieck category $\mathcal{G}_{\mathscr{C}}(\mathcal{D})$ is isomorphic to the focused orthogonality category $\mathcal{O}_{\perp\!\!\!\perp}(\mathscr{C})$.*

Proof Consider the indexed family $\{\,\mathcal{D}(c)\,\}_{c\in\mathscr{C}}$ of double orthogonally closed subsets (3) with action $\mathcal{D}(f) := (f\circ)^* : \mathcal{D}(d) \to \mathcal{D}(c)$ for all $f : c \to d$ in $\mathscr{C}$. Note that the condition $X \subseteq \mathcal{D}(f)(Y)$ is equivalent to the statement (4). $\square$

Corollary 3.4 *Let $\mathscr{C}$ be a category with a pole $\perp\!\!\!\perp$. For every endofunctor F on $\mathscr{C}$ and every endofunctor $\overline{F}$ on $\mathcal{O}_{\perp\!\!\!\perp}(\mathscr{C})$ lifting it, F has an initial algebra (resp. final coalgebra) if and only if so does $\overline{F}$.*

4 Models of linear logic with fixpoints

4.1 Categorical models

We provide an alternative presentation of the notion of categorical model of classical linear logic with fixpoints given by Ehrhard and Jafarrahmani [9, Definition 7]. Our approach is adaptable to the intuitionistic setting which we also include.

We restrict attention to the specification of linear logic types; we omit the specification of the logical system, the categorical models of which are well known. The treatment of fixpoint operators requires the consideration of types with variance in contexts of type variables with variance. To this end, we introduce judgements for types of the form $\Gamma \vdash \tau : v$ where v ranges over the set of sorts $\mathsf{V} := \{+, -\}$, Γ ranges over V-sorted contexts, and τ ranges over types. The sorts are used to indicate the intended variance, with $+$ (positive) standing for covariance and $-$ (negative) standing for contravariance; as such the dual sort $\overline{v}$ of a sort v is given by $\overline{+} = -$ and $\overline{-} = +$.

The core judgement rules of the *types of linear logic* are:

$$\frac{}{\Gamma \vdash x : v}\ (x : v \text{ in } \Gamma) \qquad \frac{}{\Gamma \vdash \mathrm{o} : +}\ \big(\mathrm{o} \text{ in } \{1, 0, \top\}\big)$$

$$\frac{\Gamma \vdash \tau_1 : v \quad \Gamma \vdash \tau_2 : v}{\Gamma \vdash \tau_1 \,\mathrm{o}\, \tau_2 : v}\ \big(\mathrm{o} \text{ in } \{\otimes, \oplus, \&\}\big) \qquad \frac{\Gamma \vdash \tau : v}{\Gamma \vdash\, !\tau : v}$$

In classical linear logic, these are extended with:

$$\frac{}{\Gamma \vdash \bot : +} \qquad \frac{\Gamma \vdash \tau_1 : v \quad \Gamma \vdash \tau_2 : v}{\Gamma \vdash \tau_1 \parr \tau_2 : v} \qquad \frac{\Gamma \vdash \tau : v}{\Gamma \vdash\, ?\tau : v} \qquad \frac{\Gamma \vdash \tau : v}{\Gamma \vdash \tau^{\perp} : \overline{v}}$$

while in intuitionistic linear logic they are extended with:

$$\frac{\Gamma \vdash \tau_1 : \overline{v} \quad \Gamma \vdash \tau_2 : v}{\Gamma \vdash \tau_1 \multimap \tau_2 : v}$$

A model of *classical linear logic* (resp. *intuitionistic linear logic*) is a $\star$-autonomous (resp. symmetric monoidal closed bicartesian) category with a linear exponential comonad [34,32,29]. For both classical and intuitionistic models $\mathscr{L}$, every judgement $x_1 : v_1, \ldots, x_n : v_n \vdash \tau : v$ has a standard *interpretation* functor $\mathscr{L}^{v_1} \times \cdots \times \mathscr{L}^{v_n} \to \mathscr{L}^{v}$, where $\mathscr{L}^{+} := \mathscr{L}$ and $\mathscr{L}^{-} := \mathscr{L}^{\mathrm{op}}$. The class of interpretation functors induced by judgements forms a V-sorted concrete clone $\mathcal{L}$ on $\mathscr{L}$.

We recall the notion of *parameterised fixpoint* (see, for instance, Fiore [11, Chapter 6]). A functor $F : \mathscr{D} \times \mathscr{C} \to \mathscr{C}$ is said to have parameterised initial algebras (resp. final coalgebras) whenever, for all $d \in \mathscr{D}$, the endofunctor $F(d, -)$ on $\mathscr{C}$ has an initial algebra (resp. final coalgebra), say $\mu F(d)$ $\big($resp. $\nu F(d)\big)$, in which case the structure canonically extends to a functor $\mu F : \mathscr{D} \to \mathscr{C}$ (resp. $\nu F : \mathscr{D} \to \mathscr{C}$).

A V-sorted concrete clone of functors $\mathcal{F} = \{\, \mathcal{F}_{\sigma,v} \subseteq [\mathscr{C}^{\sigma}, \mathscr{C}^{v}] \,\}_{\sigma \in \mathsf{V}^{\star}, v \in \mathsf{V}}$, where $\mathscr{C}^{(s_1,\ldots,s_n)} := \mathscr{C}^{s_1} \times \cdots \times \mathscr{C}^{s_n}$, is said to have parameterised fixpoints whenever every $F \in \mathcal{F}_{(s_1,\ldots,s_n,v),v}$ has parameterised initial algebras and final coalgebras, and their induced functors μF and νF are in $\mathcal{F}_{(s_1,\ldots,s_n),v}$.

A *model of linear logic with fixpoints* is a model of linear logic $\mathscr{L}$ on which there is a V-sorted concrete clone of functors with parameterised fixpoints containing the V-sorted clone $\mathcal{L}$ on $\mathscr{L}$. This notion of model is in line with the general notion of model for second-order equational presentations [12] and allows for a canonical interpretation of the extension of the linear logic typing judgements with rules for least and greatest fixpoints as follows:

$$\frac{\Gamma, x : \upsilon \vdash \tau : \upsilon}{\Gamma \vdash \varphi\, x.\, \tau : \upsilon} \ (\varphi \text{ in } \{\mu, \nu\})$$

4.2 Focused orthogonality models

Theorem 4.1 (Hyland and Schalk [23]) *For a model of classical linear logic $\mathscr{L}$ with a distributive law $\mathscr{L}(1, -) \to \mathscr{L}(1, !-)$ and a pole $\perp\!\!\!\perp \subseteq \mathscr{L}(1, \perp)$ between the monoidal units, the induced focused orthogonality category $\mathcal{O}_{\perp\!\!\!\perp}(\mathscr{L})$ is a model of classical linear logic and the forgetful functor to $\mathscr{L}$ preserves the structure strictly.*

There is an analogous theorem for models of intuitionistic linear logic for which the reader is referred to Hyland and Schalk [23].

The following result is a consequence of the theorem above and Corollary 3.4.

Theorem 4.2 *Under the hypothesis of Theorem 4.1, if $\mathscr{L}$ is a model of linear logic with fixpoints then so is $\mathcal{O}_{\perp\!\!\!\perp}(\mathscr{L})$ and the forgetful functor to $\mathscr{L}$ preserves the structure strictly.*

4.3 Examples

A variety of models of linear logic are instances of focused orthogonality constructions. We examine examples to which Theorem 4.2 applies and thereby yield models with fixpoints. As not all orthogonality models of linear logic are instances of focused orthogonality constructions, we also discuss the cases of coherence and finiteness spaces to which our results may be applied, even though these models do not arise from focused orthogonalities in the relational model.

Example 4.3 (Phase spaces [18]) Consider a commutative monoid $(M, e, \cdot)$ and a subset $\perp\!\!\!\perp \subseteq M$. For a subset $X \subseteq M$, its orthogonal is defined as $X^{\perp} := \{\, y \in M \mid \forall x \in M, x \cdot y \in \perp\!\!\!\perp \,\}$. Subsets satisfying $X = X^{\perp\perp}$ are called *facts* and provide a complete provability semantics for linear logic. The commutative monoid M can be considered as a compact closed category $\mathscr{M}$ with a single object $\bullet$ (being both 1 and $\perp$). The pole $\perp\!\!\!\perp \subseteq \mathscr{M}(\bullet, \bullet)$ corresponds to a subset of M and one can reformulate the phase model within the setting of focused orthogonality (except for the exponential structure which is defined differently). Therefore, one can interpret least and greatest fixpoints of multiplicative and additive linear logic types in phase semantics to provide a Tarskian sound model of $\mu\mathsf{MALL}$ [5].

The category **Rel** of sets and relations between them is one of the most basic models of linear logic, with many other models arising as refinements of it. Being compact closed, it is a degenerate model. The monoidal units are singletons and there are only two non-trivial focused orthogonalities on **Rel** given by the poles $\{\, \varnothing \,\}$ and $\{\, \{\mathrm{id}\} \,\}$.

Example 4.4 The model of non-uniform totality spaces studied by Ehrhard and Jafarrahmani [9] corresponds to the focused orthogonality category $\mathbf{Rel}_{\perp\!\!\!\perp}$ induced by the pole $\perp\!\!\!\perp = \{\{\mathrm{id}\}\}$. Explicitly, for a set A and a subset $X \subseteq \mathbf{Rel}(1, A) \cong \mathcal{P}(A)$, one has:

$$X^{\perp} = \{\, y \in \mathcal{P}(A) \mid \forall x \in X, y \circ x = \mathrm{id} \,\} = \{\, y \in \mathcal{P}(A) \mid \forall x \in X, x \cap y \neq \varnothing \,\} \ .$$

The induced category is a model of $\mu\mathsf{LL}$ that provides a normalization theorem for proofs.

One can generalize the relational model by considering the category of *weighted relations* (or *matrices*) $\mathbf{Rel}_{\mathbb{S}}$ over a continuous semiring $\mathbb{S}$ [27,26]. Standard examples are the Boolean semiring $(\{\mathbf{t}, \mathbf{f}\}, \vee, \wedge, \mathbf{f}, \mathbf{t})$, the completed natural numbers $\overline{\mathbb{N}} = (\mathbb{N} \cup \{\infty\}, +, \cdot, 0, 1)$, and the completed non-negative reals $\overline{\mathbb{R}_+} = (\mathbb{R}_+ \cup \{\infty\}, +, \cdot, 0, 1)$. Objects of $\mathbf{Rel}_{\mathbb{S}}$ are sets and a morphism from A to B is a function $f : A \times B \to \mathbb{S}$ (also called an $\mathbb{S}$-matrix). The composite of $f : A \times B \to \mathbb{S}$ and $g : B \times C \to \mathbb{S}$ is the function $A \times C \to \mathbb{S}$ given by $(g \circ f)(a, c) := \sum_{b \in B} f(a, b) \cdot g(b, c)$. Since $\mathbf{Rel}_{\mathbb{S}}(1, \perp) \cong \mathbb{S}$, a pole consists of a subset of $\mathbb{S}$.

Example 4.5 The model of probabilistic coherence spaces $\mathbf{PCoh}$ by Danos and Ehrhard [4] can be obtained as a focused orthogonality category on $\mathbf{Rel}_{\overline{\mathbb{R}_+}}$ with pole $[0, 1] \subseteq \overline{\mathbb{R}_+}$. The associated coKleisli category provides a fully abstract model of probabilistic PCF [10] with a fixpoint operator for terms that may be obtained as an application of our results in Section 5 (see Example 5.9).

Hamano [20] considered a continuous extension of $\mathbf{PCoh}$ based on the category of measurable spaces and s-finite transition kernels. Even if not providing monoidal closed structure, his construction involves a focused orthogonality. Indeed, taking the same pole $\perp\!\!\!\perp := [0, 1] \subseteq \overline{\mathbb{R}_+}$, for a measurable space (A, Σ), a measure μ viewed as a morphism $(1, \mathcal{P}1) \to (A, \Sigma)$, and a measurable function f viewed as a morphism $(A, \Sigma) \to (1, \mathcal{P}1)$, one has $\mu \perp\!\!\!\perp_{(A,\Sigma)} f$ iff $\int_A f \, d\mu \leqslant 1$.

Example 4.6 Orthogonality can be also used to relate models. For instance, by considering the qualitative linear logic model $\mathbf{ScottL}$, whose objects are preorders and morphisms are ideal binary relations, the category of preorders and projections introduced by Ehrhard [8] can be obtained as a subcategory of the focused orthogonality model $(\mathbf{ScottL} \times \mathbf{Rel})_{\perp\!\!\!\perp}$ with pole $\perp\!\!\!\perp := \{(\mathrm{id}, \mathrm{id})\}$. A reflexive object in this setting, obtained by lifting reflexive objects from $\mathbf{ScottL}$ and $\mathbf{Rel}$, allows the translation of normalization theorems between idempotent and non-idempotent typing systems [7].

Example 4.7 Coherence spaces, first introduced by Girard [16] to give a denotational semantics for System F, led to the discovery of linear logic through the linear decomposition of stable functions. The category of coherence spaces $\mathbf{Coh}$ can be obtained as an orthogonality construction on $\mathbf{Rel}$: two subsets of a set are orthogonal whenever their intersection has cardinality at most one.

The orthogonality for coherence spaces in $\mathbf{Rel}$ is not focused; however, in $\mathbf{Coh}$ one can consider the refinement of *coherence spaces with totality* that corresponds to the focused orthogonality category $\mathbf{Coh}_{\perp\!\!\!\perp}$ with pole $\perp\!\!\!\perp := \{\{\mathrm{id}\}\} \subseteq \mathbf{Coh}(1, \perp)$.

Example 4.8 The model of (differential) linear logic of finiteness spaces by Ehrhard [6] is based on an orthogonality in $\mathbf{Rel}$ that captures finite computations: two subsets of a set are orthogonal whenever their intersection is finite. This is however not focused and thus one cannot directly apply the results developed in the paper. Tasson and Vaux [40] studied conditions for lifting endofunctors on **Rel** to **Fin** and showed how to calculate least fixpoints for a subclass of linear logic formulas. It remains to be seen whether finiteness spaces provide a model for $\mu\mathsf{LL}$. Instead, one can extend the model by considering a weighted version of finiteness spaces over $\mathbf{Rel}_{\overline{\mathbb{N}}}$ with pole $⫫ := \mathbb{N} \subseteq \overline{\mathbb{N}}$, obtaining a focused model of linear logic with fixpoints.

5 Domain-theoretic models

After Freyd [13], the category-theoretic solution of recursive type equations, where one is interested in fixpoints of recursive types with mixed variance, is based on the notion of algebraic compactness asserting the coincidence of initial algebras and final coalgebras. In domain-theoretic models, this in turn arises from the limit/colimit coincidence in the order-enriched setting [33,42,36].

We write **Cpo** for the category of cpos (ω-chain complete partial orders) and continuous functions between them.

Theorem 5.1 (Fiore [11, Chapter 7]) *Let a* kind *be a* **Cpo***-category with ep-zero (viz. a zero object such that every morphism with it as source is an embedding) and colimits of ω-chains of embeddings.*

Every kind is **Cpo***-algebraically compact; that is, every* **Cpo***-endofunctor on it has a bifree algebra (viz. an initial algebra whose inverse is a final coalgebra).*

In domain-theoretic models of linear logic, fixpoint operators arise naturally and are typically charaterized by the axiom of uniformity.

Definition 5.2 (i) A *fixpoint operator* on a category $\mathscr{D}$ with a terminal object $\top$ is a family of functions $(-)^\dagger : \mathscr{D}(d,d) \to \mathscr{D}(\top, d)$ indexed by the objects d of $\mathscr{D}$ such that $f^\dagger = f \circ f^\dagger$ for all endomorphisms f on d.

(ii) Let $\mathscr{C}$ and $\mathscr{D}$ be categories with terminal object, and let $J : \mathscr{C} \to \mathscr{D}$ be a bijective-on-objects functor preserving terminal objects. A fixpoint operator $(-)^\dagger$ on $\mathscr{D}$ is said to be *J-uniform* if for every $h : c \to d$ in $\mathscr{C}$ and $f : c \to c, g : d \to d$ in $\mathscr{D}$,

$$J(h) \circ f = g \circ J(h) \quad \text{implies} \quad J(h) \circ f^\dagger = g^\dagger \ .$$

The following is a consequence of the study of fixpoint operators by Simpson and Plotkin [35].

Corollary 5.3 *For a kind $\mathscr{D}$ equipped with a* **Cpo***-comonad S on it, the coKleisli category $\mathscr{D}_S$ has a unique J-uniform fixpoint operator for $J : \mathscr{D} \to \mathscr{D}_S$ the cofree functor of the adjoint resolution of S.*

The above results apply, for instance, to the relational model and the coherence space model (for the quantitative and qualitative comonads) of linear logic. We now proceed to show that further examples arise from Grothendieck categories in general and from focused orthogonality in particular.

The following definition and theorem are instances of Section 5 of Cattani, Fiore, and Winskel [3] where the categorified scenario is considered. We will write **Cppo** for the full subcategory of **Cpo** consisting of the pointed cpos (that is, those with bottom element) and let $\mathbf{Cppo}_\perp$ be the subcategory of **Cppo** consisting of the strict (that is, bottom-element preserving) functions.

Definition 5.4 An *admissible $\mathscr{D}$-indexed poset* for a $\mathbf{Cppo}_\perp$-category $\mathscr{D}$ is a **Poset**-functor $\mathcal{R} : \mathscr{D}^{\mathrm{op}} \to \mathbf{Poset}$ such that the poset $\mathcal{R}(d)^{\mathrm{op}}$ is a cppo for all $d \in \mathscr{D}$ and the monotone function $\mathscr{D}(c,d) \to \mathbf{Poset}(\mathcal{R}(d)^{\mathrm{op}}, \mathcal{R}(c)^{\mathrm{op}}) : f \mapsto (f^*)^{\mathrm{op}}$ is strict continuous for all $c, d \in \mathscr{D}$.

Theorem 5.5 *For a kind $\mathscr{D}$ and an admissible $\mathscr{D}$-indexed poset $\mathcal{R}$, the Grothendieck category $\mathcal{G}_{\mathscr{D}}(\mathcal{R})$ is a kind and the forgetful functor to $\mathscr{D}$ preserves the structure strictly.*

Proof (outline) Since $(\bigvee_n f_n)^* = \bigwedge_n (f_n)^* : \mathcal{R}(d) \to \mathcal{R}(c)$ for every ω-chain f in $\mathscr{D}(c,d)$, the Grothendieck category $\mathcal{G}_{\mathscr{D}}(\mathcal{R})$ **Cpo**-enriches. Since, the reindexing $(\perp_{c,d})^* : \mathcal{R}(d) \to \mathcal{R}(c)$ along the bottom element $\perp_{c,d} \in \mathscr{D}(c,d)$ is constantly the top element of $\mathcal{R}(c)$, the ep-zero of $\mathcal{G}_{\mathscr{D}}(\mathcal{R})$ consists of the ep-zero $\perp$ of $\mathscr{D}$ paired with the top element of $\mathcal{R}(\perp)$. The colimiting cone $\langle e_n : (d_n, R_n) \to (d, R) \rangle_n$ of an ω-chain of embeddings $\langle (d_n, R_n) \to (d_{n+1}, R_{n+1}) \rangle_n$ in $\mathcal{G}_{\mathscr{D}}(\mathcal{R})$ consists of the colimiting cone of embeddings $\langle e_n : d_n \to d \rangle_n$ of the ω-chain of embeddings $\langle d_n \to d_{n+1} \rangle_n$ in $\mathscr{D}$ with $R := \bigwedge_n (p_n)^*(R_n) \in \mathcal{R}(d)$ for p_n the projection of e_n. □

Finally, we investigate focused orthogonality in this domain-theoretic setting.

Definition 5.6 An *admissible pole* for a $\mathbf{Cppo}_\perp$-category $\mathscr{D}$ is a pole that is a sub-cppo of $\mathscr{D}(\mathsf{s}, \mathsf{t})$.

Lemma 5.7 *For a $\mathbf{Cppo}_\perp$-category with an admissible pole, the indexed poset of double orthogonally closed sets is admissible.*

Corollary 5.8 *For a kind $\mathscr{D}$ with an admissible pole $\perp\!\!\!\perp$, the focused orthogonality category $\mathcal{O}_{\perp\!\!\!\perp}(\mathscr{D})$ is a kind and the forgetful functor to $\mathscr{D}$ preserves the structure strictly.*

Example 5.9 The weighted relational model $\mathbf{Rel}_{\overline{\mathbb{R}_+}}$ (see Example 4.5) is a kind and the pole $\perp\!\!\!\perp := [0,1] \subseteq \overline{\mathbb{R}_+}$ is admissible. Corollaries 5.8 and 5.3 provide then a uniform fixpoint operator on the coKleisli category of probabilistic coherence spaces which allows us to recover the fixpoint operator for terms of [4].

On the other hand, totality models are tools to provide a denotational account of normalization and therefore do not have fixpoint operators. In particular, note that for the totality models presented in Section 4.3 the underlying orthogonality construction is done with the singleton pole $\{\,\{\mathrm{id}\}\,\}$ which does not contain the empty relation and is therefore not admissible.

Conclusion

Recasting focused orthogonality constructions within a relational fibration framework, we have developed a categorical theory to construct new models of linear logic with fixpoints by means of lifting initial algebras and final coalgebras from the base model to the focused orthogonality one. Our method is widely applicable: it allows to re-explain the totality model of $\mu\mathsf{LL}$ studied by Ehrhard and Jafarrahmani [9] and opens the way for refining a variety of other models besides the relational one. In connection to this, Tsukada and Asada [41] provided a unified framework based on module theory to make the linear algebraic aspect of models of linear logic explicit. In particular, they considered models of intuitionistic linear logic based on categories of R-modules and linear maps for R a Σ-semiring. It would be interesting to investigate fixpoint constructions in these models and thereafter consider refinements of them using our theory for focused orthogonalities. The same applies to their discussion of models of classical linear logic.

Our lifting theorems further extend from relational fibrations to categorical fibrations. In future work, we aim to use these results to obtain a theory of fixpoint constructions for general glueing and double-glueing models. Since double glueing constructions have been extensively used to study full completeness by refining models to contain only morphisms that are the interpretation of proof terms [28], we aim to also use our results to construct fully complete models of linear logic with fixpoints.

While we have considered fixpoint operators in the induced cartesian closed category of domain-theoretic models of linear logic, we also aim to explore lifting theorems for *traces* [24] in the linear base model. Many orthogonality and (double) glueing constructions are indeed done on a compact closed category (which has a canonical trace) and the refinement induced by the orthogonality usually eliminates this degeneracy. Understanding whether one can lift this canonical trace to orthogonality or double-glued categories would provide a new method for constructing traced categories.

References

[1] Artin, M., A. Grothendieck and J. L. Verdier, "Théorie des topos et cohomologie etale des schémas: tome 3," Springer-Verlag, 1973.

[2] Baelde, D., *Least and Greatest Fixed Points in Linear Logic*, ACM Trans. Comput. Log. **13** (2012), pp. 2:1–2:44.

[3] Cattani, G., M. Fiore and G. Winskel, *A theory of recursive domains with applications to concurrency*, in: *Proceedings of the Thirteenth Annual IEEE Symposium on Logic in Computer Science*, 1998, pp. 214–225.

[4] Danos, V. and T. Ehrhard, *Probabilistic coherence spaces as a model of higher-order probabilistic computation*, Information and Computation **209** (2011), pp. 966–991.

[5] De, A., F. Jafarrahmani and A. Saurin, *Phase semantics for linear logic with least and greatest fixed points*, in: *42nd IARCS Annual Conference on Foundations of Software Technology and Theoretical Computer Science (FSTTCS 2022)*, Schloss Dagstuhl-Leibniz-Zentrum für Informatik, 2022.

[6] Ehrhard, T., *Finiteness spaces*, Mathematical Structures in Computer Science **15** (2005), pp. 615–646, 32 pages.

[7] Ehrhard, T., *Collapsing non-idempotent intersection types*, in: *Computer Science Logic (CSL'12)-26th International Workshop/21st Annual Conference of the EACSL*, Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik, 2012.

[8] Ehrhard, T., *The Scott model of Linear Logic is the extensional collapse of its relational model*, Theoretical Computer Science **424** (2012), pp. 20–45, 26 pages.

[9] Ehrhard, T. and F. Jafarrahmani, *Categorical models of Linear Logic with fixed points of formulas*, in: *36th ACM/IEEE Symposium on Logic in Computer Science (LICS 2021)* (2021), pp. 1–13.

[10] Ehrhard, T., C. Tasson and M. Pagani, *Probabilistic coherence spaces are fully abstract for probabilistic PCF*, in: *Proceedings of the 41st ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages*, 2014, pp. 309–320.

[11] Fiore, M. P., "Axiomatic domain theory in categories of partial maps," Distinguished Dissertations in Computer Science, Cambridge University Press, 1996.

[12] Fiore, M. P. and C.-K. Hur, *Second-order equational logic*, in: *Proceedings of the 19th EACSL Annual Conference on Computer Science Logic (CSL 2010)*, Lecture Notes in Computer Science **6247** (2010), pp. 320–335.

[13] Freyd, P., *Algebraically complete categories*, in: *Category Theory*, Springer, 1991, pp. 95–104.

[14] Freyd, P. J. and A. Scedrov, "Categories, allegories," Elsevier, 1990.

[15] Girard, J., *Linear logic*, Theoretical Computer Science **50** (1987), pp. 1–102.

[16] Girard, J.-Y., *The System F of variable types, fifteen years later*, Theoretical Computer Science **45** (1986), pp. 159–192.

[17] Girard, J.-Y., *Fixpoint theorem in linear logic* (1992), email posting to the mailing list linear@cs.stanford.edu.

[18] Girard, J.-Y., *Linear logic: its syntax and semantics*, London Mathematical Society Lecture Note Series (1995), pp. 1–42.

[19] Girard, J.-Y., Y. Lafont and P. Taylor, "Proofs and types," Cambridge Tracts in Theoretical Computer Science **7**, Cambridge University Press, 1989.

[20] Hamano, M., *Double glueing over free exponential: with measure theoretic applications*, arXiv preprint arXiv:2107.07726 (2021).

[21] Hermida, C. and B. Jacobs, *Structural induction and coinduction in a fibrational setting*, Information and Computation **145** (1998), pp. 107–152.

[22] Hyland, M., *Game semantics*, in: A. M. Pitts and P. Dybjer, editors, *Semantics and Logics of Computation*, Publications of the Newton Institute (1997), pp. 131–184.

[23] Hyland, M. and A. Schalk, *Glueing and orthogonality for models of linear logic*, Theoretical Computer Science **294** (2003), pp. 183–231.

[24] Joyal, A., R. Street and D. Verity, *Traced monoidal categories*, Mathematical Proceedings of the Cambridge Philosophical Society **119** (1996), pp. 447–468.

[25] Krivine, J.-L., *Realizability in classical logic*, Panoramas et synthèses **27** (2009), pp. 197–229.

[26] Laird, J., G. Manzonetto, G. McCusker and M. Pagani, *Weighted relational models of typed lambda-calculi*, in: *2013 28th Annual ACM/IEEE Symposium on Logic in Computer Science*, IEEE, 2013, pp. 301–310.

[27] Lamarche, F., *Quantitative domains and infinitary algebras*, Theoretical Computer Science **94** (1992), pp. 37–62.

[28] Loader, R., *Linear logic, totality and full completeness*, in: *Proceedings Ninth Annual IEEE Symposium on Logic in Computer Science*, IEEE, 1994, pp. 292–298.

[29] Mellies, P.-A., *Categorical semantics of linear logic*, Panoramas et syntheses **27** (2009), pp. 15–215.

[30] Pataraia, D., *A constructive proof of Tarski's fixed-point theorem for dcpo's*, presented at the 65th Peripatetic Seminar on Sheaves and Logic, Aarhus (Denmark), November 1997.

[31] Plotkin, G., "Lambda-definability and logical relations," Edinburgh University, 1973.

[32] Schalk, A., *What is a categorical model of linear logic?*, Manuscript, available from http://www.cs.man.ac.uk/~schalk/work.html (2004).

[33] Scott, D., *Continuous lattices*, in: *Toposes, Algebraic Geometry and Logic*, Lecture Notes in Mathematics **274**, Springer, 1972, pp. 97–136.

[34] Seely, R. A. G., *Linear logic, $*$-autonomous categories and cofree coalgebras*, in: J. W. Gray and A. Scedrov, editors, *Categories in Computer Science and Logic*, Contemporary Mathematics **92** (1989), pp. 371–382.

[35] Simpson, A. and G. Plotkin, *Complete axioms for categorical fixed-point operators*, in: *Proceedings Fifteenth Annual IEEE Symposium on Logic in Computer Science (Cat. No. 99CB36332)*, IEEE, 2000, pp. 30–41.

[36] Smyth, M. B. and G. D. Plotkin, *The category-theoretic solution of recursive domain equations*, SIAM Journal on Computing **11** (1982), pp. 761–783.

[37] Statman, R., *Logical relations and the typed λ-calculus*, Information and Control **65** (1985), pp. 85–97.

[38] Tait, W. W., *Intensional interpretations of functionals of finite type I*, Journal of Symbolic Logic **32** (1967), pp. 198–212.

[39] Tan, A. M., "Full completeness for models of linear logic," Ph.D. thesis, University of Cambridge (1998).

[40] Tasson, C. and L. Vaux, *Transport of finiteness structures and applications*, Mathematical Structures in Computer Science **28** (2018), pp. 1061–1096.

[41] Tsukada, T. and K. Asada, *Linear-algebraic models of linear logic as categories of modules over Σ-semirings*, in: C. Baier and D. Fisman, editors, *LICS '22: 37th Annual ACM/IEEE Symposium on Logic in Computer Science, Haifa, Israel, August 2 - 5, 2022* (2022), pp. 60:1–60:13.

[42] Wand, M., *Fixed point constructions in order-enriched categories*, Theoretical Computer Science **8** (1979), pp. 13–30.

[43] Wraith, G., *Artin glueing*, Journal of Pure and Applied Algebra **4** (1974), pp. 345–348.