Exotic Derivative Valuation Based on Simulations: A Case Study of Rainbow lookback Options Involving Apple and Google

Charaf ZGUIOUAR, Majd RABBAJ, Meriem BENDIR
Université Panthéon-Sorbonne Paris 1 - Ecole de l'Economie - Master II Finance, Technology, Data

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1 Abstract

The derivative markets have witnessed a surge in importance within the global financial landscape, driven by the need for risk management, speculation, and portfolio diversification. In this dynamic environment, exotic options have emerged as crucial financial instruments, serving dual roles in both hedging and speculation strategies.

Exotic options, including but not limited to rainbow options, have garnered significant attention due to their versatility in addressing diverse financial risk scenarios. These options, often associated with high-profile stocks such as those of Google and Apple, introduce a new dimension to derivative trading by incorporating multiple underlying assets. Rainbow options, in particular, belong to this class of exotic derivatives and offer a spectrum of investment strategies that extend far beyond the capabilities of traditional single-asset options.

The challenge lies in accurately valuing these complex instruments, especially when confronted with the intricacies of multiple interconnected assets. The interconnected among these assets, coupled with factors such as correlations and combined volatility, adds layers of complexity to the valuation process. Precise modeling becomes imperative in this context, given the profound impact of derivative pricing on investment decisions and risk management strategies.

In today's highly interconnected global economy, the pricing of derivatives, including rainbow options, is not merely a matter of academic interest but a practical necessity. Fluctuations in one corner of the world can reverberate throughout the financial system, affecting the valuation of derivative contracts across markets. As a result, the need for accurate valuation models has become paramount to ensure the stability and resilience of the financial system.

2 Case Study Focus Rainbow Option Involving Apple and Google

This presentation will focus on a specific type of exotic derivative known as a 'rainbow option', involving two of the most prominent stocks in the technology sector: Apple Inc. (AAPL) and Google Inc. (Google, GOOGL). The choice of AAPL and GOOGL for this case study is deliberate, given their significant market presence and the dynamic behavior of their stock prices. This case study aims to shed light on the practical aspects of valuing such complex financial instruments using advanced simulation methods.

3 Limitations of Conventional Methods

Traditional econometric models, such as the Black-Scholes framework, face challenges in pricing complex derivatives like rainbow options involving companies like Google and Apple due to:

- Static Correlation Assumptions: These models assume static correlations between assets, which may not reflect the dynamic nature of asset relationships, particularly in the rapidly changing tech sector.

- Neglect of Company-Specific Variables: Traditional models often overlook company-specific factors that impact stock prices, such as product cycles and competitive positioning.

To improve pricing accuracy for such options, advanced econometric approaches are essential. These methods should dynamically model correlations among assets and incorporate company-specific data to better reflect the complexities of multi-asset options within dynamic market environments.

The Need for More Coherent Methods: Given these limitations, there is a clear need for more coherent and sophisticated methods for pricing rainbow options involving tech giants like Google and Apple. These methods should account for dynamic correlations, changing market conditions, and company-specific factors that influence stock prices. Advanced computational finance techniques and models that can adapt to the intricacies of exotic derivatives and the unique characteristics of the tech sector are essential. Developing and applying such methods is crucial to ensure accurate pricing, effective risk management, and informed investment decisions in the context of these complex financial instruments and dynamic market environments.

4 Model and assumptions parameters:

4.1 Data:

Historical market data for AAPL and GOOGL, spanning from 2008 to 2023, provides the basis for our analysis. We calculate the daily logarithmic returns, offering a more stabilized view of the market trends.

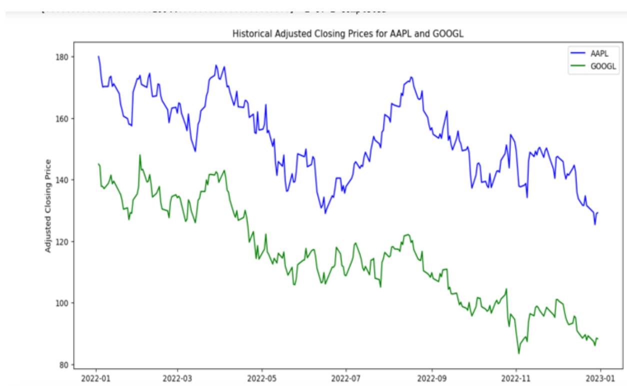

Figure 1: Historical adjusted prices for Apple and Google

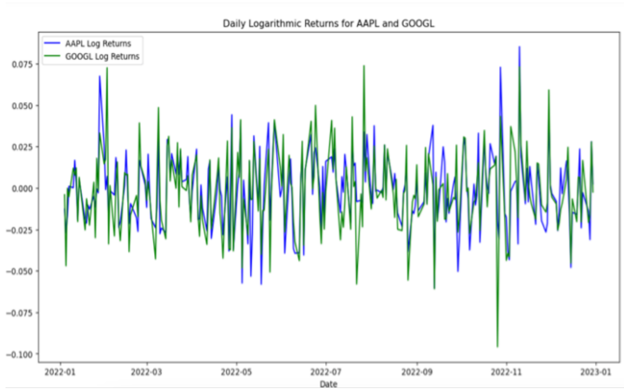

Figure 2: Daily Log return of Apple and Google

4.2 Parameters:

- **Number of Simulations**: $n = 10000$ (Monte Carlo simulations)
- **Time Steps**: $t = 252$ (Daily steps over the year)
- **Strike Price**: $S = 100$ (Strike price of the option)
- **Underlying Assets**: Symbols for the assets - AAPL, GOOGL
- **Maturity**: $M = 1$ year (Time to maturity)
- **Risk-Free Rate**: $r = 0.05$ (Risk-free interest rate)
- **Time Step in Years**: $\Delta t = \frac{M}{t}$
- **Correlation Matrix**: The correlation between the stocks is extrapolated from historical data, forming a critical input in our simulation model. This matrix reflects how closely the stocks move in relation to each other.

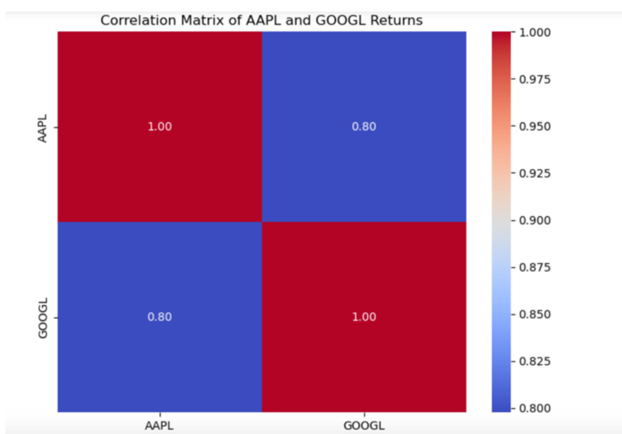

Figure 3: Correlation matrix

Volatility and Risk-Free Rate: We derive the annualized volatilities of both stocks from historical returns. The risk-free rate, a key element in option pricing, is set at 0.05, in line with current market condition Market Volatility: A key driver of option pricing, is considered for both AAPL and GOOGL. The model accounts for fluctuations in their market prices, which directly impacts the option's valuation

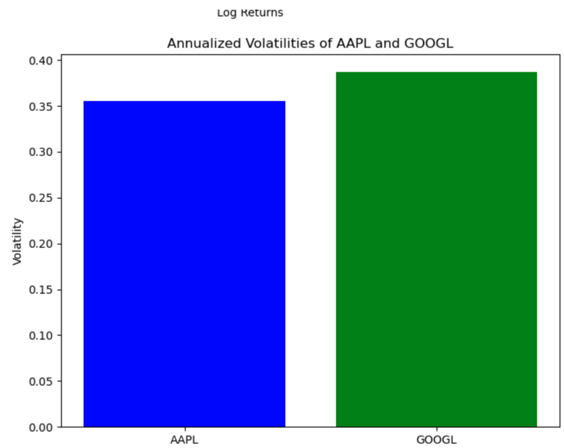

Figure 4: Annualized volatilities

Using Python the process involves generating correlated asset price paths, calculating the option payoffs for each simulation, and then averaging these payoffs, discounted at the risk-free rate, to estimate the option price.

5 Monte Carlo Simulations with Cholesky Decomposition:

The core of our methodology is grounded in Monte Carlo simulations, a robust stochastic process that allows for the exploration of a wide range of outcomes based on probabilistic models. This technique is particularly suited for valuing complex financial derivatives like rainbow options, where traditional analytical methods fall short.

Simulation Framework: Our approach utilizes a Monte Carlo framework to simulate the future price paths of Apple (AAPL) and Google (GOOGL) stocks, which form the underlying assets for the rainbow option.

Cholesky Decomposition: To accurately model the joint behavior of these two stocks, we employ Cholesky decomposition. This mathematical tool helps in generating correlated random variables, which is essential to reflect the real-world correlation between AAPL and GOOGL stocks in our simulations.

For a correlation matrix Σ, Cholesky decomposition allows us to find a lower triangular matrix L such that:

$$\Sigma = LL^T$$

This matrix L can be used to transform uncorrelated standard normal random variables into correlated random variables that reflect the correlation structure of AAPL and GOOGL stocks.

Simulating Asset Price Paths:

Given a time step Δt, the risk-free rate r, and volatilities σ_i for each stock i, the asset prices are simulated using the geometric Brownian motion (GBM) model:

$$S_{i,t+\Delta t} = S_{i,t} \exp\left(\left(r - \frac{1}{2}\sigma_i^2\right)\Delta t + \sigma_i\sqrt{\Delta t}Z_{i,t}\right)$$

where $S_{i,t}$ is the price of asset i at time t, and $Z_{i,t}$ are the correlated standard normal random variables for each asset, generated using the Cholesky decomposition L and independent random variables $X_{i,t}$:

$$Z_t = LX_t$$

5.1 Python script :

We transalte this into a python script w perform Cholesky decomposition on the correlation matrix to obtain a lower triangular matrix. This matrix will be used to generate correlated random variables that reflect the actual historical correlation between the stock prices.

We simulates the price paths for AAPL and GOOGL stocks using a multivariate geometric Brownian motion **(GBM)**. The simulation considers the drift (risk-free rate) and diffusion (volatility) for each stock, incorporating the correlation between the two stocks.

At maturity, the maximum stock price between AAPL and GOOGL is taken for each simulation path, and the payoff is calculated as the maximum of this value minus the strike price, with a floor at zero.

The average of the discounted payoffs across all simulations gives the estimated price of the rainbow option and finally we Computes the standard error of the payoffs to assess the precision of the estimate and establishes a 95% confidence interval around the estimated option price.

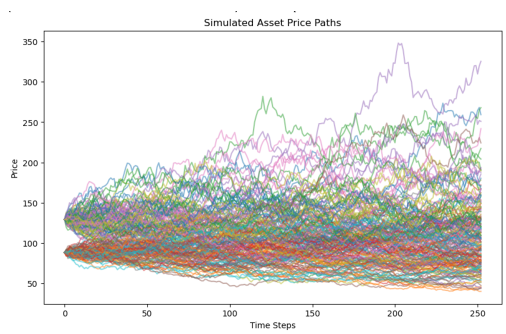

Figure 5: Simulated Asset prices path

This plot shows simulates future price paths for the underlying assets (AAPL and GOOGL) used to price a rainbow option. The multitude of lines represents different potential future scenarios for the assets' prices, as predicted by a stochastic model, likely Geometric Brownian Motion (GBM), which takes into account the drift (expected return), volatility, and correlation between the assets.

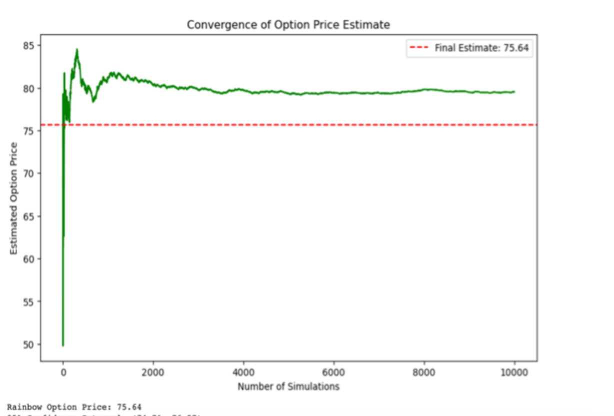

Figure 6: Convergence option price Estimate

Convergence of Option Price Estimate shows the convergence of the estimated option price as the number of simulations increases. The horizontal dashed line indicates the final estimated price of the option. This helps to assess the stability and reliability of the simulation; the price should stabilize as the number of simulations grows, indicating that the estimate is converging.

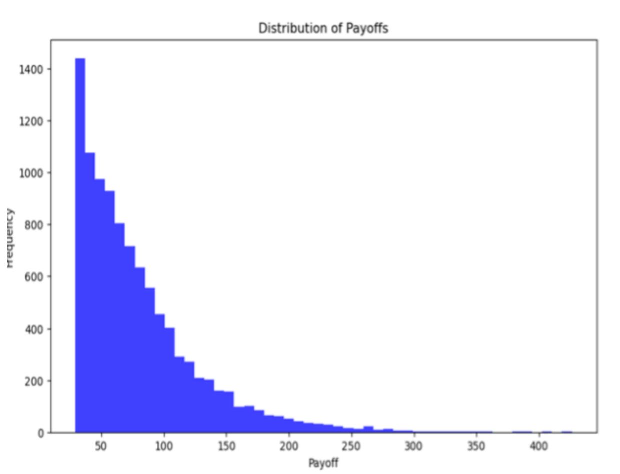

Figure 7: Distribution payoffs

Distribution of Payoffs: with this histogramme of the distribution of payoffs from the simulated price paths at the option's maturity. The shape of the distribution provides insight into the potential outcomes for the option's value, allowing for a better understanding of the risk and potential reward associated with the option.

The figure 8 a valuable tool in financial analysis, there is a positive linear relationship between the final price and the payoff for both stocks. As the price increases, so does the payoff, which is typical of call options at higher prices, the spread of payoffs widens, indicating that the option likely has a strike price. Below the strike price, the option has no value (no payoff), but above it, the payoff increases with the asset's price. Comparison between AAPL and GOOGL: The scatter plot allows for a direct comparison between AAPL and GOOGL payoffs at various price points. It helps assess which stock's payoffs are higher for the same final price, potentially due to differences in option terms.

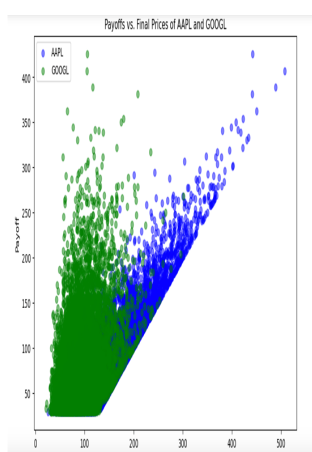

Figure 8: Payoffs vs Final prices of Apple and Google

6 Neural Networks (NNs):

NNs are a class of machine learning algorithms modeled after the human brain's structure. They are excellent at recognizing patterns in data, making them ideal for forecasting financial market behaviors.

The process iterates over multiple rounds (epochs), each time improving the model's predictions by adjusting the weights and biases.

1. **Forward Propagation:**
   - Compute layer outputs: $a^{[l]} = g^{[l]}(W^{[l]}a^{[l-1]} + b^{[l]})$
2. **Loss Function:**
   - Mean Squared Error for predictions: $L = \frac{1}{m}\sum(y - \hat{y})^2$
3. **Backpropagation:**
   - Calculate gradients: $\frac{\partial L}{\partial W^{[l]}}, \frac{\partial L}{\partial b^{[l]}}$
4. **Weight Update:**
   - Adjust parameters: $W^{[l]} := W^{[l]} - \alpha\frac{\partial L}{\partial W^{[l]}}, b^{[l]} := b^{[l]} - \alpha\frac{\partial L}{\partial b^{[l]}}$

Here, $g^{[l]}$ is the activation function (like ReLU or sigmoid), $W^{[l]}$ and $b^{[l]}$ are the weights and biases, $a^{[l]}$ are the activations, y is the actual price, $\hat{y}$ is the predicted price, m is the number of samples, and α is the learning rate. This is a simplified overview but captures the essence of the NN's operation in your context.

6.1 Python script

we translate this into a Python script can be summarized as :

1. **Data Processing:** - Historical stock prices for AAPL and GOOGL are downloaded for the most recent year. - Training data is generated by randomly sampling from the historical prices and calculating option payoffs based on a predetermined strike price.

2. **Neural Network (NN) Model:** A neural network is constructed with two hidden layers of 64 neurons each, using ReLU activation, and a single output neuron with linear activation. The NN is trained to predict option payoffs based on input features derived from stock prices, using mean squared error as the loss function and the Adam optimizer.

3. **Model Evaluation and Explanation**: The convergence of the NN during training is visualized by plotting the training loss over epochs. SHAP (SHapley Additive exPlanations) values are computed to interpret the NN model's predictions, explaining the impact of each feature on the prediction.

Functions are provided to visualize the weights of the NN model , giving insights into the learning process and feature relevance.

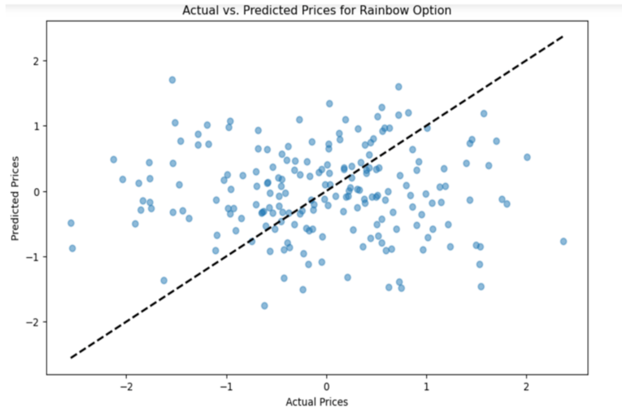

Figure 9: Convergence option price Estimate

Here in this scatter plot comparing the actual prices of the rainbow options against the prices predicted by the neural network. Each point represents an option, with its actual price on the x-axis and the predicted price on the y-axis. The dashed line indicates where the predicted prices equal the actual prices. Points close to this line indicate accurate predictions.

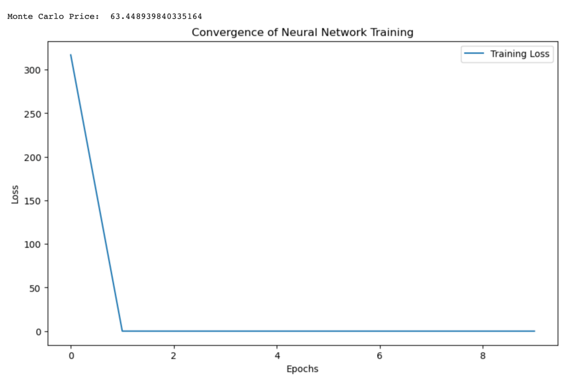

Figure 10: Convergence Neuronal Training

The convergence plot of the neural network training, showing the decrease in training loss as the number of epochs increases. This indicates that the model is learning from the training data over time. The sharp decline suggests that the model is quickly improving its predictions during the initial epochs.

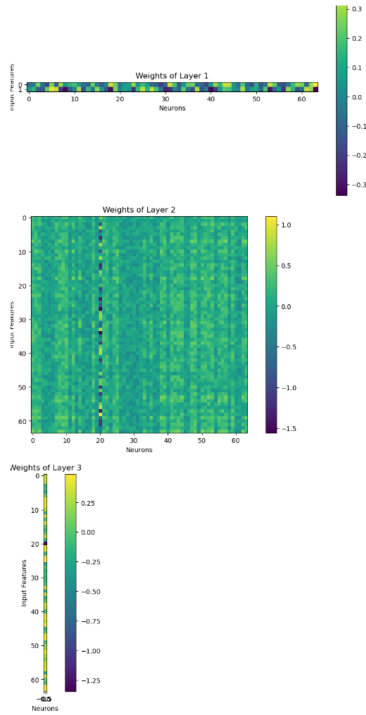

Figure 11: Convergence Neuronal Training

.**Weights of Neural Network Layers**: These heatmaps visualize the weights learned by each layer in a neural network. Darker or brighter colors indicate larger absolute values for the weights, which can hint at the importance of certain connections between neurons, In the context of predicting a rainbow option's payoff, which depends on the maximum of two asset prices (Google and Apple in your case), the neural network would have been trained on historical data to find patterns that would help it estimate future prices and thus the option payoff. The heatmaps would then provide insight into how the neural network has structured its internal representation of the problem based on the data it was trained on.

1. Weights of Layer 1: This heatmap shows the weights connecting the input features to the first hidden layer of the neural network. Each column represents a neuron, and each row represents an input feature. The colors represent the strength and sign of the weights (e.g., purple might be a large negative weight, green might represent weights close to zero, and yellow a large positive weight).

2. Weights of Layer 2: Similar to the first heatmap, but this one represents the weights connecting the first hidden layer to the second hidden layer. The patterns in this heatmap can be more complex, as they represent higher-level features learned from the input data.

3. Weights of Layer 3: This heatmap shows the weights connecting the second hidden layer to the third hidden layer. Again, the colors indicate the value of the weights, and the complexity increases with each subsequent layer in the network.

7 The Neural Network and Monte Carlo methods

The Neural Network and Monte Carlo methods have produced the following prices for the Call Rainbow Option maturing in one year

Neural Network Predicted Price: 66.36482

Monte Carlo Simulation Price: 63.448939840335164

These prices represent the fair value of the option based on different methods. The slight discrepancy between the two prices is normal due to the different underlying models and assumptions. The Neural Network price reflects the model's learned patterns from historical data, while the Monte Carlo price is derived from a probabilistic model simulating many possible future scenarios.

8 Pricing a lookback option

A Lookback Option is a type of exotic option that allows the holder to 'look back' over the time the option was active and choose a favorable exercise price. This research focuses on pricing a Lookback Option on Apple Inc. (AAPL) stock using Monte Carlo simulation methods.

8.1 Methodology

Monte Carlo simulations are employed to price the Lookback Option. The approach involves generating a large number of simulated paths for the underlying asset price (AAPL), based on historical volatility and price data obtained from Yahoo Finance. Key parameters include:

- Underlying asset: AAPL
- Number of simulations: 10,000
- Daily steps: 252
- Strike price: \$100
- Maturity: 1 year
- Risk-free rate: 5%
- Premium: \$5

8.2 Results

The Monte Carlo simulation provided insights into the potential evolution of AAPL's stock price and the distribution of possible payoffs for the Lookback Option. The results include:

- A histogram representing the distribution of payoffs.
- A plot showing sample paths of the stock price.

These visualizations aid in understanding the variability and potential outcomes of the option's value over time.

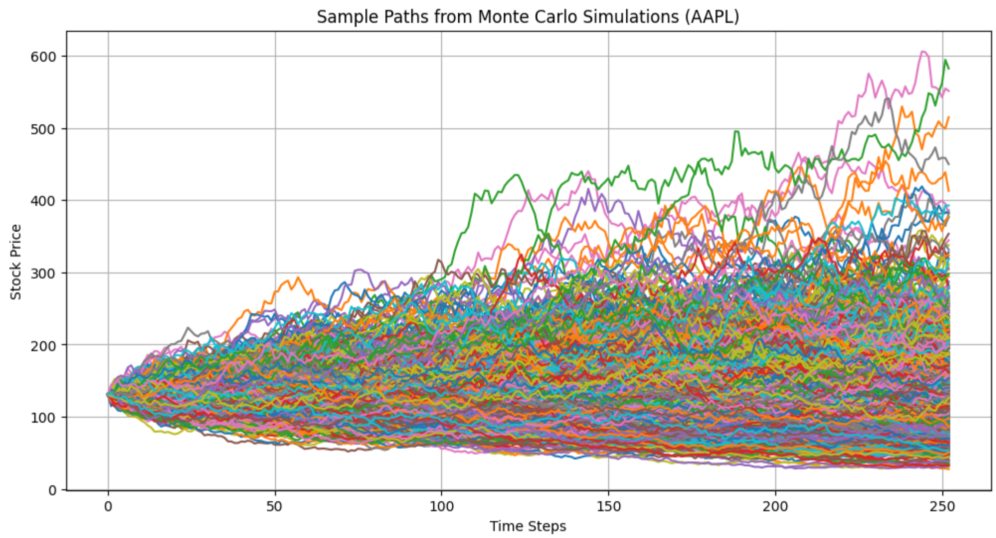

Figure 12: A plot showing sample paths of the stock price

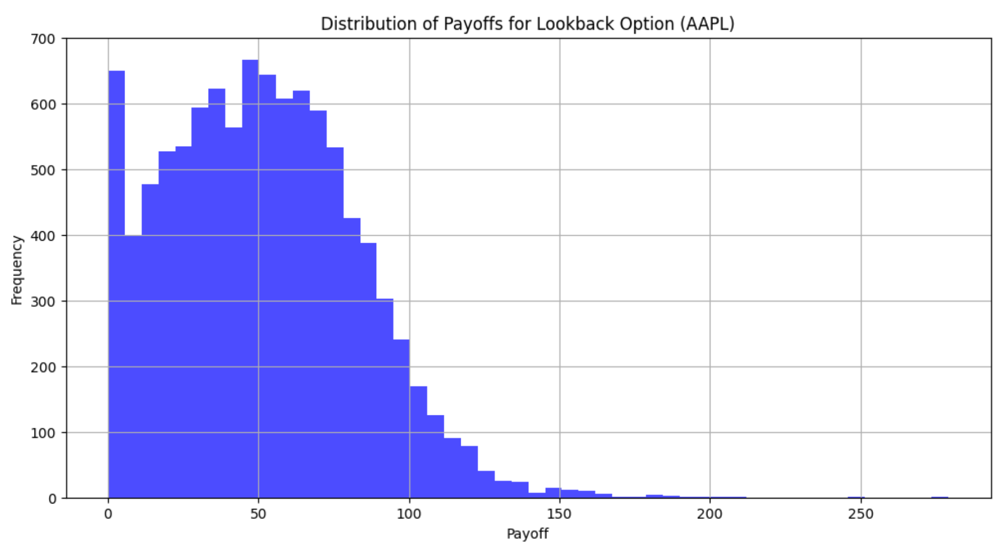

Figure 13: Distribution of Payoffs for Lookback Option

8.3 The lookback option price

The Lookback Option pricing simulation yielded a final price of **43.88** US dollars, considering the specified parameters and premium. This price represents the cost of the option, accounting for the flexibility it offers in choosing the exercise price.

9 conclusion of the key findings :

The study has quantitatively assessed rainbow options for AAPL and GOOGL using Neural Network and Monte Carlo simulations, yielding close yet distinct valuations. The Neural Network capitalized on historical data to infer potential future prices, while the Monte Carlo approach provided a statistical estimate by simulating numerous future scenarios. The slight variance in results between the models emphasizes the need for a diversified approach to option pricing in complex market environments.

Technically, the study advances the field by integrating machine learning with stochastic processes to address the limitations of classical models like Black-Scholes in a multi-asset option context. While historical data provides a solid foundation for predictive analytics, future models could benefit from incorporating real-time market dynamics and alternative stochastic elements to refine accuracy.

The research underscores the importance of robust valuation models for regulatory frameworks aiming to uphold market integrity. As financial instruments grow in complexity, the necessity for advanced computational methods in both financial practice and regulatory oversight becomes increasingly evident. This study's methodology and insights could thus inform both future academic inquiry and practical financial engineering developments.