№3

$$p(x) = \begin{cases} e^{-x/\theta}/\theta, & x \geq 0 \\ 0, & x < 0 \end{cases} \quad \theta > 0.$$

а.)

Несмещённость:

1.) $\tilde{\theta}_1 = \overline{X}$

$$M[\tilde{\theta}_1] = \frac{1}{n}\sum_{i=1}^{n} M[\xi] = \int_0^\infty \frac{x}{\theta} e^{-\frac{x}{\theta}} dx =$$

$$= \theta \int_0^\infty t e^{-t} dt = \theta \Rightarrow \tilde{\theta}_1 - \text{несмещ.}$$

2.) $\tilde{\theta}_2 = \frac{X_{min} + X_{max}}{2}$

$$M[\tilde{\theta}_2] = \frac{1}{2}\left(M[X_{min}] + M[X_{max}]\right) =$$

$$= \frac{1}{2}\left(\int_0^\infty 3x \frac{e^{-\frac{x}{\theta}}}{\theta} e^{-\frac{x}{\theta}\cdot 2} dx + \int_0^\infty 3x \left(1 - e^{-\frac{x}{\theta}}\right)^2 \frac{e^{-\frac{x}{\theta}}}{\theta} dx\right) =$$

$$= \frac{3}{2\theta}\left(\frac{\theta^2}{9} + \theta^2\left(1 - \frac{1}{2} + \frac{1}{9}\right)\right) = \frac{3}{2}\theta \cdot \left(\frac{13}{18}\right) =$$

$$= \theta \cdot \frac{13}{12} \Rightarrow \tilde{\theta}_2 - \text{смещ.}$$

Исправление: $\tilde{\theta}_2' = \frac{12}{13}\left(\frac{X_{min} + X_{max}}{2}\right)$

3.) $\tilde{\theta}_3 = X_{(2)}$

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$$M[\tilde{\theta}_3] = \int_0^\infty 3x\, C_2^1 (1-e^{-\frac{x}{\theta}}) e^{-\frac{x}{\theta}} \frac{e^{-\frac{x}{\theta}}}{\theta} dx =$$

$$= \frac{6}{\theta}\left(\frac{5\theta^2}{36}\right) = \frac{5}{6}\theta \Rightarrow \tilde{\theta}_3 \text{ – смещ.}$$

Исправим: $\tilde{\theta}_3{}' = \frac{6}{5} X_{(2)}$

2. Эффективность:

$$D[\tilde{\theta}_1] = \frac{1}{9} \cdot 3\, D[\xi] = \frac{1}{3} D[\xi] =$$

$$= \frac{1}{3}\left(\int_0^\infty x^2 \frac{e^{-\frac{x}{\theta}}}{\theta} dx - \left(\int_0^\infty x \frac{e^{-\frac{x}{\theta}}}{\theta} dx\right)^2\right) = \frac{1}{3}\theta^2$$

$$D[\tilde{\theta}_2{}'] = D\left[\frac{6}{13}(x_{min} + x_{max})\right] =$$

$$= \left(\frac{6}{13}\right)^2 (D x_{min} + D x_{max} + 2\,\mathrm{cov}(x_{min}, x_{max}))$$

$$D x_{min} = M[x^2_{min}] - M^2[x_{min}] = \int_0^\infty 3x^2 \frac{e^{-\frac{x}{\theta}}}{\theta} e^{-\frac{2x}{\theta}} -$$

$$- \frac{\theta^2}{9} = \frac{2\theta^2}{9} - \frac{\theta^2}{9} = \frac{1}{9}\theta^2$$

$$D x_{max} = M[x^2_{max}] - M^2[x_{max}] = \int_0^\infty 3x^2 (1-e^{-\frac{x}{\theta}})^2 \cdot$$

$$\cdot \frac{e^{-\frac{x}{\theta}}}{\theta} dx - \frac{11^2\theta^2}{6^2} = \frac{49}{36}\theta^2$$

$$\mathrm{cov}(x_{min}, x_{max}) = \iint\limits_{x<y} xy\, p(x,y)\, dx\, dy - \frac{11}{18}\theta^2 =$$

$$= \int_0^\infty \frac{3}{2\theta}\left(2y^2 e^{-\frac{3y}{\theta}} + \theta y \left(3e^{-\frac{3y}{\theta}} + e^{-\frac{y}{\theta}} - 4e^{-\frac{2y}{\theta}}\right)\right) dy -$$

$$- \frac{11}{18}\theta^2 = \frac{3}{2}\theta^2\left(\frac{4}{3^3} + \frac{1}{3} + 1 - 1\right) - \frac{11}{18}\theta^2 = \frac{1}{9}\theta^2$$

Сканировано с CamScanner

$D[\tilde{\theta}_2'] = \frac{61}{169}\theta^2$

$D[\tilde{\theta}_3'] = \frac{36}{25}\theta^2\left(12\left(\frac{1}{8} - \frac{1}{27}\right) - \frac{25}{36}\right) = \frac{13}{25}\theta^2$

$\Downarrow$

θ_1 - наиболее эффективна

б.) Нер-во К-Р

$$D[\tilde{g}(\vec{x}_n)] \geq \frac{g'^2(\theta)}{nI(\theta)}$$

$$g(\theta) = \theta \Rightarrow g'(\theta) = 1$$

$$I(\theta) = -M\left[\frac{\partial^2 \ln p}{\partial\theta^2}\right] = -M\left[\frac{\partial^2(-\frac{x}{\theta} - \ln\theta)}{\partial\theta^2}\right] =$$

$$= M\left[\frac{2x}{\theta^3} - \frac{1}{\theta^2}\right] = \frac{2}{\theta^3}M[\xi] - \frac{1}{\theta^2} = \frac{1}{\theta^2}$$

$$\frac{g'^2(\theta)}{nI(\theta)} = \frac{1}{3}\theta^2$$

$D[\tilde{\theta}] = \frac{1}{3}\theta^2 \Rightarrow \tilde{\theta}_1 = \bar{x}$ - ~~наиболее~~ самая эффективная оценка ~~[illegible]~~